MACHINES, SYSTEMS
AND CATEGORIES

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MACHINES, SYSTEMS AND CATEGORIES

We present the Arbib and Manes theory of machines and behaviors in a category. For discrete, linear machines and automata in a monoidal category, Goguen proved that minimal realization is right adjoint to behavior, as functors between certain categories of machines and behaviors. We show that this result holds in the more general context of Arbib and Manes. As examples we give a complete study of discrete, linear, group and tree machines. We also show that decomposable systems are Arbib machines thus establishing a link between machine and system theory.
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The more we progressed in the study of pure mathematics during our graduate studies, the stronger became our feeling that we were dealing with abstract concepts too far away from practical applications.

We talked out our doubts with some of our instructors in the Mathematics Department, Concordia University, Sir George Williams Campus in Montreal. One of them, Dr. COHEN, suggested to us to read the book "Discrete Mathematics" from BOBROW and ARBIB (our reference [13]).

While doing so, we discovered how so many concepts of modern algebra could be applied to that most earthly and practical branch of applied Mathematics, automata theory. Under the guidance of Professor Cohen, we undertook the fourteen months effort whose result is this thesis. Its purpose is to present how the most abstract branch of algebra, category theory, can be used to help solve problems in machine and control systems theory.

One of the most basic of these problems is, given a function (a behavior), to find a "machine" which will compute (realize) this function in the most "economical" way. This problem is called minimal realization.
The concepts of machine and behavior have been axiomatized by ARBIB and MANES. Technically these notions take form of functors which are moreover related by a condition known as adjointness. This condition allows one to shift freely between two seemingly different contexts according to the nature of the particular problem under consideration.

It turns out that minimal realization is right adjoint to behaviour as functors between certain categories of machines and behaviors. GOUGUEN proved this for discrete and linear machines in 1, for automata in closed symmetric monoidal categories, with countable co-products and canonical cofactorizations, in [2].

In chapter I we show that this very important result holds in the much more general context of machines in categories as defined by ARBIB and MANES (see [3] or [7]). Thus category theory applied to automata theory gives us the following very strong results:

1. For a fairly large class of functions we call A-behaviors (A for Arbib who describes them in [3]), there exists a minimal realization;

2. The way to "build" them is fundamentally unique and we have precise "clues" how to do it;

3. These minimal realizations are uniquely characterized up to isomorphism.
In chapter II we discuss three examples of A-machines: discrete, linear and tree-automata.

In chapter III category theory allows us to "build a bridge" between control theory and automata theory with the concept of decomposable systems as defined by Padulo and Arbib in [4]. In this book, the authors give the conditions for such a system to have a minimal realization and describe this realization. Our contribution is to show that decomposable systems with these suitable conditions are, in fact, A-machines (this is claimed but not proved in [4]).

As examples of decomposable systems we again discuss linear machines in this new context and we give a complete study of group-machines (about which only a few facts are given in [4]).

Decomposable systems not only give us a somewhat shorter way to prove that a function (or a machine) is an A-behavior (or an A-machine) and provide a link between systems and automata, but also allow to define observability in categorical terms. In chapter IV we discuss observability in the context of set theory versus the categorical one of minimality. We then expose the Arbib and Manes theory of state-behavior machines, as described in [9], where a categorical definition of observability (which generalizes
the one seen in chapter III) is given and the usual result:
"A machine is minimal if and only if it is reachable and observable" is proved.

In part V we outline the adjoint machine concept and duality theory of Arbib and Manes as exposed in [9], giving only the few simple proofs not contained in that paper.

All the definitions and results of category theory used in this thesis are listed in the annex.

One word about our notation:

IN. 3 = 3rd definition in the text.
TH 4 = 4th theorem in the text.
IN a 2.5 = part 5 of the 2nd definition in the annex.

\[ f : S \rightarrow S' = f \text{ is a map which sends} \]
\[ s \rightarrow f(s) \quad s \in S \text{ into } f(s) \in S', \forall s \in S. \]

\[
\begin{array}{ccc}
\{\cdot\} & \rightarrow & X^* \\
\gamma \downarrow & & \downarrow \delta^* \\
S & \downarrow & S \\
\sigma \downarrow & & \sigma(\cdot) = \sigma \\
\end{array}
\]

commutes means that \( \sigma = \delta^* \gamma \)

(internal diagrams) and the external diagram shows the actions of the various maps on the elements of the sets involved.
We would like to express our very deep gratitude to Concordia University (Ex George Williams University a few years back). For years it has been the only University, as far as we know, which offered all its regular courses in the evening as well as during the day. Without its pioneering work in adult and continuous education thousands of persons like ourselves could not have realized their expectations.

Finally we must thank our director of thesis, Dr. GERARD ELIE COHEN, for his trust, his encouragements, his help, his patience without which we will not have begun this work, to say nothing of completing it.
CHAPTER I

A-MACHINES, A-BEHAVIORS AND
THEIR MINIMAL REALIZATION

In this chapter we will define A-Machines, A-Behaviors, their respective categories and show that the functor minimal realization is right adjoint to external behavior.
1. MACHINE IN A CATEGORY

Let $\mathcal{C}$ be a category.

**DN 1. Process.**

A process $X$ is a functor $X : \mathcal{C} \to \mathcal{C}$.

**DN 2. $X$ dynamics.**

$\forall s \in \mathcal{C}, \forall \delta \in \mathcal{C}(xs, s), \delta$ is called an $X$ dynamics.

**DN 3. $X$ dynamorphisms**

$\delta : xs \to s, \delta' : xs' \to s'$ are $X$-dynamics

An $X$-dynamorphism $f : \delta \to \delta'$ is a morphism $f \in \mathcal{C}(s, s')$.

$\delta \circ f = f \cdot \delta'$

i.e. the diagram commutes:

$$
\begin{array}{c}
x \downarrow f \\
x \delta \\
xs \\
\delta'
\end{array}
\begin{array}{c}
\delta' \\
\\
s'
\end{array}
\rightarrow
\begin{array}{c}
\delta' \\
\\
s'
\end{array}
$$

**DN 4. Dyn $X$**

Dyn $X$ is the category whose:

- objects are $X$-dynamics,
- morphisms are $X$-dynamorphisms.
\[ \text{DYN} \, X(\delta, \delta') \subseteq \mathcal{C}(s, s') \]

We can define composition and identities in \text{DYN} \, X as in \mathcal{C}. We show \text{DYN} \, X is indeed a category.

Let \( \delta : Xs \to s \), \( \delta' : Xs' \to s' \), \( \delta'' : Xs'' \to s'' \) be \( X \)-dynamics and \( f : \delta \to \delta' \) and \( g : \delta' \to \delta'' \) be \( X \)-dynamorphisms. By \text{DYN} \, 3 both parts of the diagram commutes,

\[
\begin{array}{c}
\text{image diagram}
\end{array}
\]

\[
\begin{array}{ccc}
Xs & \xrightarrow{\delta} & s \\
\downarrow{Xf} & & \downarrow{f} \\
Xs' & \xrightarrow{\delta'} & s'
\end{array}
\]

\[
\begin{array}{ccc}
Xs'' & \xrightarrow{\delta''} & s''
\end{array}
\]

\[
\therefore \text{the whole diagram}
\]

\[
\begin{array}{ccc}
Xs & \xrightarrow{\delta} & s \\
\downarrow{Xg} & & \downarrow{g} \\
Xs & \xrightarrow{\delta'} & s'
\end{array}
\]

\[
\begin{array}{ccc}
Xs'' & \xrightarrow{\delta''} & s''
\end{array}
\]

\[
\begin{array}{c}
\text{X functor implies}
\end{array}
\]

\[
\begin{array}{c}
X \text{ dynorphism} \, \delta \to \delta''
\end{array}
\]

Identity: let \( \text{Id}_\delta = \text{Id}_s \),

\[
\begin{array}{ccc}
Xs & \xrightarrow{\delta} & s \\
\downarrow{X\text{Id}_s} & & \downarrow{\text{Id}_s} \\
Xs & \xrightarrow{\delta} & s
\end{array}
\]

as \( X \) a functor implies \( X \text{Id}_s = \text{Id}_Xs \), \( \text{Id}_s \cdot \delta = \delta \) in \( \mathcal{C} \).

\[
\begin{array}{c}
\text{Id}_s = \text{Id}_s \text{ is an X-dynamorphism which inherits the usual identity properties from } \mathcal{C}
\end{array}
\]
NOTE 1. \( \delta : X \delta \to \delta \) an \( X \)-dynamics, i.e. \( \delta \in \mathcal{C}(X \delta, \delta) \);

is also a \( X \)-dynamorphism \( X \delta \to \delta \);

indeed the diagram \( X(X \delta) \xrightarrow{X \delta} X \delta \)
\( \xrightarrow{\delta} \delta \)
\( X \delta \xrightarrow{\delta} \delta \)

\( X \delta \Rightarrow \delta \) obviously commutes

\[ \therefore \delta \text{ has two different roles: } \delta \in \mathcal{C}(X \delta, \delta) \text{ is an object of } \text{DYN } X \text{ while the same } \delta \in \text{DYN } X (X \delta, \delta) \text{ is a morphism of } \text{DYN } X \neq \text{Id} \delta = \text{Id} \delta. \]

INPUT 5. \( X : \mathcal{C} \to \mathcal{C} \) is an input process if the forgetful functor \( U : \text{DYN } X \to \mathcal{C} \)

on objects \( (\delta : X \delta \to \delta) \xrightarrow{\text{Id}} \delta \)

on morphisms \( (f : \delta \to \delta') \xrightarrow{(f : s \to s')} \delta \to \delta' \)

has a left-adjoint.

In this case \( \forall s \in \mathcal{C}, \exists \left( \xi, \eta \right) \in \text{DYN } X / \delta \) universal from \( s \) to \( U \);

\[ \therefore \forall \left( \delta' : X \delta' \to \delta, f : s \to U \delta = s' \right) \exists \text{ a unique } X \text{-dynamorphism } \psi \in \mathcal{C} \text{ the two diagrams } \]

\[ \begin{array}{c}
X s \xrightarrow{\eta} U \delta \\
\downarrow \psi \\
X s' \xrightarrow{X \psi} U \delta'
\end{array} \text{ commute: } \]

\[ \begin{array}{c}
X \left( U \xi \right) \xrightarrow{\delta} U \delta' \\
\downarrow \psi \\
X s' \xrightarrow{\delta} U \delta' = s'
\end{array} \]
ψ is called the unique dynamorphic extension of f
δₜ is called the free dynamics over s w.r.t. to Uₜ
and we will write \[ Uₜ C = Xₜ \] where \( Xₜ = Uₜ V ; C \rightarrow C' \)
\( \sqrt[1]{U} \).

NOTE 2.(1) Let \( V : C \rightarrow \text{DYN} X \) be the left adjoint of \( U \),

i.e. \( V \dashv U \), then \( V \) is defined by

objects: \( s \mapsto Vₜ = \deltaₜ \)
morphisms: \( f' \mapsto \deltaₜ \mathcal{V}_{f} \)

where \( \mathcal{V}_{f} : Xₜ \rightarrow Xₜ' \) is the unique dynamorphic extension of \( \mathcal{V} \).

i.e. \( \exists \)

\[
\begin{array}{ccc}
S & \xrightarrow{\eta}_S & Xₜ' \\\
\downarrow{f} & & \downarrow{\mathcal{V}_{f}} \\
S' & \xrightarrow{\mathcal{V}_{f}} & Xₜ' \\\
\end{array}
\]

\[
\begin{array}{ccc}
X₁ Xₜ' & \xrightarrow{\deltaₜ} & Xₜ' \\\
\downarrow{\mathcal{V}_{f}} & & \downarrow{\mathcal{V}_{f}} \\
X₁ Xₜ' & \xrightarrow{\deltaₜ} & Xₜ'
\end{array}
\]

commute.

In short \( V \dashv U \)

\( C(s, U\delta' = s') \cong \text{DYN} X (V s = \deltaₜ, \deltaₜ') \)

and \( C(s, U \delta'₉) \cong \text{DYN} X (V s = \deltaₜ, \deltaₜ') \)

\[
\begin{array}{ccc}
S & \xrightarrow{\eta}_S & Xₜ' \\\
\downarrow{f} & & \downarrow{\mathcal{V}_{f}} \\
U \deltaₜ = Xₜ' & \xrightarrow{\psi} & S'
\end{array}
\]

and \( C(s, U \delta'₉) \cong \text{DYN} X (V s = \deltaₜ, \deltaₜ') \)

\[
\begin{array}{ccc}
S & \xrightarrow{\eta}_S & Xₜ' \\\
\downarrow{\mathcal{V}_{f}} & & \downarrow{\mathcal{V}_{f}} \\
U \deltaₜ = Xₜ' & \xrightarrow{\psi} & S'
\end{array}
\]
(2) \( \mathbf{X} = \mathbf{U} \cdot \mathbf{V} : \mathbf{C} \rightarrow \mathbf{C} \) is a functor, as it is a composition of two functors.

**NW 6. Machine in a category**

1. A machine in a category \( \mathbf{C} \) is a septuple \( M = (X, s, y, i, \delta, \sigma, \lambda) \) where
   - \( X \) is an input process,
   - \( s = \text{state object} \in \mathbf{C} \)
   - \( y = \text{output object} \in \mathbf{C} \)
   - \( i = \text{initial state object} \in \mathbf{C} \)
   - \( \delta : X \rightarrow s \), an \( X \)-dynamica, is a \( \mathbf{C} \)-morphism
   - \( \sigma : i \rightarrow s \), initial state morphism, is a \( \mathbf{C} \)-morphism
   - \( \lambda : s \rightarrow y \), output morphism, is a \( \mathbf{C} \)-morphism.

2. Let \( (\delta^+_i : X(X^* i) \rightarrow X^* i, \eta_i = i \rightarrow X^* i) \)
   be universal from \( i \) to \( \mathbf{U} \).
   - the object of inputs \( X^* = X^* i \)
   - the reachability morphism \( \delta^+ : X^* \rightarrow s \)
   is the unique dynamorphic extension of the initial state morphism \( \sigma : i \rightarrow s \).

3. \( M \) is reachable if \( \delta^+ \) is a coequalizer.
NOTE 3. (1) \( \forall \downarrow \mathbf{u} \Rightarrow \exists \delta^+ \) as

\[
\mathcal{G}(i, \mathbf{u}, \delta = s) \not\equiv \text{DYN} \mathbf{X} (\forall \mathbf{v} = \delta_i, \delta)
\]

\[
\begin{array}{ccc}
\mathbf{u} & \xrightarrow{\delta} & \mathbf{x}^* \\
\downarrow & & \downarrow \\
\mathbf{s} & \xrightarrow{\delta} & \mathbf{s}
\end{array}
\]

and the two following diagrams commute:

\[
\begin{array}{ccc}
i & \xrightarrow{\mathcal{U}_i} & \mathbf{x}^* \\
\downarrow & & \downarrow \delta^+ \\
\mathbf{s} & \xrightarrow{\delta^+} & \mathbf{s}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathbf{x}^* & \xrightarrow{\delta_i} & \mathbf{x}^* \\
\downarrow & & \downarrow \\
\mathbf{s} & \xrightarrow{\delta} & \mathbf{s}
\end{array}
\]

2. We can define \( \delta^*: \mathbf{x}^* \mathbf{s} \rightarrow \mathbf{s} \) as the unique
dynamorphic extension of \( \text{Id}_s \) i.e. \( \exists \)

\[
\begin{array}{ccc}
\mathbf{s} & \xrightarrow{\mathcal{M}_s} & \mathbf{x}^*_s \\
\downarrow & & \downarrow \delta^* \\
\mathbf{s} & \xrightarrow{\delta^*} & \mathbf{s}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathbf{x}^*_s & \xrightarrow{\delta_s} & \mathbf{x}^*_s \\
\downarrow & & \downarrow \\
\mathbf{s} & \xrightarrow{\delta} & \mathbf{s}
\end{array}
\]

commutes

3. \( \mathbf{x}^*_\sigma = \forall \sigma \) is the unique dynamorphic extension of

\( \mathcal{M}_s \cdot \sigma \)

i.e. \( \begin{array}{ccc}
i & \xrightarrow{\mathcal{M}_i} & \mathbf{x}^* \\
\downarrow & & \downarrow \\
\mathbf{s} & \xrightarrow{\mathbf{x}^*_\sigma} & \mathbf{s}
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathbf{x}^*_s & \xrightarrow{\delta_s} & \mathbf{x}^*_s \\
\downarrow & & \downarrow \delta_s \\
\mathbf{x}^*_s & \xrightarrow{\delta^*_s} & \mathbf{x}^*_s
\end{array}
\]

commutes
2. **CATEGORY OF MACHINES**

**DN 7. Machine - Morphism**

Let \( M = (X, s, y, i, \delta, \sigma, \lambda) \) and \( M' = (X', s', y', i', \delta', \sigma', \lambda') \) be two machines in a category \( \mathcal{X}^* \) and \( \mathcal{X}'^* \) their object of inputs, \( \delta^* \) and \( \delta'^* \) their reachability morphism respectively. A machine morphism \( M \to M' \) is a triple \( (a, b, c) \) where \( a : \mathcal{X}^* \to \mathcal{X}'^* \), \( b : s \to s' \), \( c : y \to y' \) are morphisms such that the two following diagrams commute:

\[
\begin{align*}
\mathcal{X}^* \xrightarrow{\delta^*} & s \\
\downarrow a & \swarrow b \\
\mathcal{X}'^* \xrightarrow{\delta'^*} & s'
\end{align*}
\quad \text{and} \quad
\begin{align*}
s \xrightarrow{\lambda} & y \\
\downarrow b & \swarrow c \\
\mathcal{X}'^* \xrightarrow{\epsilon'} & \mathcal{X}'
\end{align*}
\]
We call a, b, c respectively the input, state and output components of \((a, b, c)\) or, simply the 1st, 2nd and 3rd components.

In 8. Let \((a, b, c) : M \rightarrow M', (a', b', c') : M' \rightarrow M''\) be a machine morphisms, we define their composition by \((a, b, c) \circ (a', b', c') = (a.a', b.b', c.c')\).

\(C\) is a category \(\Rightarrow \exists\) components of \((a.a', b.b', c.c')\), they are associative and \(\exists\) identities \(\text{Id}_x^*, \text{Id}_s^*, \text{Id}_y^*\).

\(\therefore\) composition of machine-morphisms is defined, associative and identity \(\text{Id}_M = (\text{Id}_x^*, \text{Id}_s^*, \text{Id}_y^*)\) with the usual properties.

Besides In 7 the various parts of the following diagrams commute:

\[
\begin{array}{ccc}
\chi^* & \xrightarrow{\delta^+} & s \\
\downarrow a & & \downarrow b \\
\chi'^* & \xrightarrow{\delta'^+} & s' \\
\downarrow a' & & \downarrow b' \\
\chi''^* & \xrightarrow{\delta''^+} & s'' \\
\end{array}
\quad
\begin{array}{ccc}
s & \xrightarrow{\lambda} & y \\
\downarrow b & & \downarrow c \\
s' & \xrightarrow{\lambda'} & y' \\
\downarrow b' & & \downarrow c \\
s'' & \xrightarrow{\lambda''} & y''
\end{array}
\]
\[ \Rightarrow \delta^+ \circ (a', a) = b' \circ \delta^+ \circ a = (b' \circ b) \circ \delta^+ \]

and \[ \lambda^*(b' \circ b) = c' \circ \lambda^* \circ b = (c' \circ c) \circ \lambda \]
i.e. the whole diagrams commute and \((a, b, c) \circ (a', b', c')\) is a machine morphism.

Finally

\[
\begin{array}{ccc}
\chi^* & \xrightarrow{\delta^+} & s \\
\text{Id}_{\chi^*} & \downarrow & \downarrow \text{Id}_s \\
\chi^* & \xrightarrow{\delta^+} & s
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
s & \xrightarrow{\lambda} & y \\
\text{Id}_s & \downarrow & \downarrow \text{Id}_y \\
s & \xrightarrow{\lambda} & y
\end{array}
\]

obviously commute \(\Rightarrow\) \(\text{Id}_M\) is a machine-morphism.

We are now able to define the category of machines in a category

**DN 9.** \(\text{MAC} \ C\) = category whose objects are machines in the category \(\mathcal{C}\), morphisms are machine-morphisms.

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X
\end{array}
\]

3. **CATEGORY OF BEHAVIOURS**

As seen in DN 7, given \(i \in \mathcal{B}\), \(X : \mathcal{C} \rightarrow \mathcal{B}\) an input process, \(\exists\) an object of input \(\chi^* = X^* i\)

Let \(Y \in \mathcal{B}\). A behavior is a morphism \(\beta : \chi^* \rightarrow Y\)
Category of behavior in category $\mathcal{C} = \text{Beh}\mathcal{C}$.

Objects: behaviors $\beta : \mathcal{X} \rightarrow y$.

Morphism: behavior-morphisms $(a, c) : \beta \downarrow \rightarrow \beta'$.

where $a : \mathcal{X} \rightarrow \mathcal{X'}$, $c : y \rightarrow y'$.

are $\mathcal{C}$ morphism $\alpha : \mathcal{X} \rightarrow y$.

\[ \begin{array}{ccl}
\alpha & \downarrow & \beta' \\
\mathcal{X} & \rightarrow & y' \\
\mathcal{X} & \rightarrow & y \\
\end{array} \]

commutes

i.e. $c \beta = \beta' a$.

Composition: $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b') : \beta \rightarrow \beta''$.

with $(a', b') : \beta' \rightarrow \beta''$.

Identity morphism: $\text{Id}_\beta = (\text{Id}_\mathcal{X}, \text{Id}_y)$.

Like $\text{MAC}\mathcal{C}$, $\text{Beh}\mathcal{C}$ is indeed a category (similar proof).

\[ \times \times \times \]

4. **Categories of a Behaviors and a Machines.**

**DN 11.** Let $\delta : \mathcal{X} c \rightarrow c$ be an $X$-dynamics.

$\delta^{(n)} : \mathcal{X}^n c \rightarrow c$ is defined inductively as follows:

\[ \begin{align*}
\delta^{(0)} &= \text{Id} \\
\delta^{(n+1)} &= \mathcal{X}^{n+1} c \xrightarrow{\mathcal{X} \delta} \mathcal{X}^n c \xrightarrow{\mathcal{X} \delta} \cdots \rightarrow c \\
\end{align*} \]
In [3] Arbib gives a slightly different postulate (1b):
\[ E_\beta \xrightarrow{\alpha} \kappa^* \text{ is called "the" Nerode equivalence for } \beta. \] 
The word "the" is justified by the fact that if \( E \xrightarrow{\alpha} \kappa^* \) and \( E_\gamma \xrightarrow{\alpha'} \kappa^* \) are both Nerode equivalences for \( \beta \) (i.e., \( \beta \) satisfies the four postulates for each of them) then the corresponding Nerode realizations (to be defined later) \( N_\beta \) and \( N_\beta' \) of \( \beta \) are isomorphic. We will discuss this point later on, after TH 1: see Note 7.

**Note 4.** A machine \( M \) realizes a behavior \( \beta : \kappa^* \rightarrow y \) if \( \lambda \cdot \delta^+ = \beta; \lambda \cdot \delta^+ \) is called the external behavior (or response morphism) of \( M \).
DN 14. An **A** machine **M** is a machine in a category \( \mathcal{C} \) with its external behavior \( \lambda \cdot \delta^+ \) is an **A**-behavior.

From now on we will consider only the two following sub-categories of \( \text{MAC} \mathcal{C} \) and \( \text{Beh} \mathcal{C} \):

DN 15. \( \mathcal{M} \) = category of reachable **A** machines and machine-morphisms between them.

\( \mathcal{B} \) = category of **A** behaviors and behavior-morphisms between them.

\[
\begin{align*}
\text{5.} & \quad \text{FUNCTOR } E : \mathcal{M} \rightarrow \mathcal{B} \\
\text{DN 16.} & \quad \text{Functor } E : \mathcal{M} \rightarrow \mathcal{B} \\
& \quad \forall M \in \text{MC} \text{ and } (a, b, c) : M \rightarrow M', \in \mathcal{M} \Rightarrow \\
& \quad E(M) = \lambda \cdot \delta^+ \text{ the external behavior of } M, \\
& \quad E(a, b, c) = (a, c) : EM \rightarrow EM'.
\end{align*}
\]
We show $E$ is indeed a functor.

$\lambda, \delta^+: X^* \to Y$ is an $A$ behavior as $M \in \mathcal{MB}$. $E(a, b, c)$ is a behavior-morphism as by DN 7 the two parts of the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\delta^+} & S \\
\downarrow a & & \downarrow b \\
X' & \xrightarrow{\delta'^+} & S'
\end{array}
\xrightarrow{\lambda} \begin{array}{ccc} 
Y & \\
\downarrow c & \\
Y'
\end{array}
\quad (*)
\]

$\Rightarrow (\lambda', \delta'^+) \cdot a = \lambda \cdot b, \delta^+ = c \cdot (\lambda, \delta^*)$ as required.

Let $(a, b, c) : M \to M', (a', b', c') : M' \to M''$.

$E((a, b, c) \cdot (a', b', c')) = E(a, a', b, b', c, c')$

$\Rightarrow$ IN of $\cdot$ in $\mathcal{MB}$

$= (a, a', c, c')$

$\Rightarrow$ IN of $E$

$= (a, c) \cdot (a', c')$

$\Rightarrow$ IN of $\cdot$ in $\mathcal{B}$

$= E(a, b, c), E(a', b', c')$

$\Rightarrow$ IN of $E$.

Indeed this is a behavior-morphism as the diagram commutes, because its two parts commute by $(*)$.
Finally \( \mathbf{E} (\text{Id}_x^*, \text{Id}_x^*, \text{Id}_y) = (\text{Id}_x^*, \text{Id}_y) \)

i.e. \( \mathbf{E} (\text{Id}_x) = \text{Id}_x^* \), \( \text{Id}_M^* = \text{Id}_M \)

\[
\begin{array}{c}
\times
\end{array}
\]

6. **Functor** \( \mathbf{N} : \mathfrak{B}_x \longrightarrow \mathfrak{M}_x \)

\[
\forall \beta \in \mathfrak{B}, N\beta = (x, s_\beta, y, i, \delta_\beta, \sigma_\beta, \lambda_\beta) \in \mathfrak{M}
\]

is the Nerode realization of \( \beta \), where \( s_\beta, \delta_\beta, \sigma_\beta \)

are defined by \( \mathbf{IN} \, 13 \) an \( \mathfrak{A} \) behavior,

where \( (\delta_i^*, \eta_i) \) is the universal arrow from \( i \) to \( \mathfrak{U} \).

\( \lambda_\beta \) = the unique \( \mathfrak{C} \)-morphism \( \exists \)

\[
\begin{array}{c}
\begin{array}{c}
E_\beta \\
\downarrow \alpha \\
\chi^* \\
\downarrow \beta \\
\delta_\beta \\
\downarrow \gamma \\
\lambda_\beta
\end{array}

\end{array}
\]

commutes,

i.e. \( \beta = \lambda_\beta, \delta_\beta^* \)

\[
\begin{array}{c}
\begin{array}{c}
\exists \lambda_\beta \text{ as } \beta, \alpha = \beta, \gamma \\
\text{IN 12.1(a) with } n = 0
\end{array}

\end{array}
\]

and \( \delta_\beta^* = \text{coeq} (\alpha, \gamma) \), \( \text{IN 12.2} \)

\[
\begin{array}{c}
\begin{array}{c}
N\beta \in \mathfrak{M}
\end{array}

\end{array}
\]

as:

- the two following diagrams commute

\[
\begin{array}{c}
\begin{array}{c}
i \\
\downarrow \sigma_\beta \\
\delta_\beta \\
\downarrow \sigma_\beta^*
\end{array}

\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{IN of } \sigma_\beta
\end{array}

\end{array}
\]
The external behavior of $N_\beta$ is 

$$\lambda_\beta \cdot \delta_\beta^+ = \beta$$

$\lambda_\beta$ is a behavior, $\delta_\beta^+$ is a coequalizer. $N_\beta$ is always reachable.

**DN 19.** We say that $M$ is a reachable realization of $\beta$ if its reachability morphism $\delta^+ = \beta$.

**DN 19. (i)** $M$ is a minimal realization of a behavior $\beta : x^* \rightarrow y$ if $M_\beta$ is a reachable realization of $\beta$ and $\forall M', M'$ a reachable realization of $\beta$, $\exists$ a unique machine morphism $(\text{Id}_x, \beta, \text{Id}_y): M' \rightarrow M$. 
(ii) This is equivalent to say that $\mathcal{M}$ is a minimal realization of $\beta$ if it is a terminal object in the subcategory $\mathcal{M}_\rho$ of reachable realizations of $\beta$.

$\Rightarrow \mathcal{M}$ is unique up to isomorphism.

(iii) $\delta : s' \rightarrow s$ is called a simulation $\mathcal{M}' \rightarrow \mathcal{M}$ if it is the 2nd component of a machine morphism of the form $(\text{Id}_x, h, \text{Id}_y)$ i.e. $\delta^+ = b \cdot \delta''^+$.

$\lambda \cdot b = \lambda'$ \quad IN 7

and $\delta' = b \cdot \sigma'$ \quad IN 6 as

$\sigma = \delta', \eta_i = b \cdot \delta''_1 \cdot \eta_i = b \cdot \sigma'$.

NOTE 6. $\delta : s' \rightarrow s$ is an $\mathcal{X}$-dynamorphism if $\mathcal{X} \delta''$ is epi.

Indeed consider the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & X^* \\
\downarrow \delta'' & & \downarrow \delta''^+ \\
X s' & \xrightarrow{\delta'} & s'
\end{array}
\]

Part (1) and the whole diagram commute (see note 3)

$\Rightarrow b \cdot \delta''^+ \cdot \delta_i = \delta \cdot X b \cdot X \delta''^+$

$\Rightarrow b \cdot \delta' \cdot X \delta''^+ = \delta \cdot X b \cdot X \delta''^+$

$\Rightarrow b \cdot \delta' = \delta \cdot X b$

if $\mathcal{X} \delta''^+$ is epi, which is the case if $\beta'$ is an $\mathcal{A}$-behavior.
THEOREM REALIZATION TH.

\( \beta \in \mathcal{B} \Rightarrow N\beta \) is the minimal realization of \( \beta \).


NOTE 7. Given a behavior \( \beta : \mathcal{X} \rightarrow \mathcal{Y} \), assume it satisfies the four postulates for

\[ E \xrightarrow{\delta^*} \mathcal{X} \xrightarrow{\delta^*} \mathcal{S}, \delta^* = \text{coeq} (\mathcal{X}, \mathcal{Y}) \text{,} \delta^*: \mathcal{X} \mathcal{S} \rightarrow \mathcal{S} \]

and also for \( E' \xrightarrow{\delta'^*} \mathcal{X}' \xrightarrow{\delta'^*} \mathcal{S}' \text{,} \delta'^* = \text{coeq} (\mathcal{X}', \mathcal{Y}') \)

\( \delta'^*: \mathcal{X} \mathcal{S}' \rightarrow \mathcal{S}' \)

\( \therefore \) \( \delta^* \mathcal{X} = \delta'^* \mathcal{X} \Rightarrow \delta^* \mathcal{X}' = \delta'^* \mathcal{X}' \text{ and} \)

\( \delta^* \mathcal{X} = \delta'^* \mathcal{X} \Rightarrow \delta^* \mathcal{X}' = \delta'^* \mathcal{X}' \)

\( \therefore \) \( \delta^* = \text{coeq} (\mathcal{X}, \mathcal{Y}) \Rightarrow \exists \text{ a unique } b: \mathcal{S} \rightarrow \mathcal{S} \exists \delta^* = b \cdot \delta^* \)

and \( \delta'^* = \text{coeq} (\mathcal{X}', \mathcal{Y}') \Rightarrow \exists \text{ a unique } b': \mathcal{S}' \rightarrow \mathcal{S} \exists \delta'^* = b' \cdot \delta'^* \)

\( \therefore \) \( \delta^* = b' \cdot b \cdot \delta^* \text{ and} \delta'^* = b' \cdot b \cdot \delta'^* \Rightarrow b' \cdot b = \text{Id}_{\mathcal{S}} \text{ and} b \cdot b' = \text{Id}_{\mathcal{S}'} \)

\( \Rightarrow \) \( \mathcal{S} \preceq \mathcal{S}' \)

Besides \( \forall f \in \mathcal{F}, f \cdot \mathcal{X} = f \cdot \mathcal{Y} \exists \text{ unique } \psi \in \mathcal{F} = \psi \cdot \delta^* \)

\( \therefore \) \( \psi = f \cdot \mathcal{Y} \cdot \delta^* \)
ϕ and b unique $\implies$ ϕ · b is unique

$\therefore \delta'^+ = \text{coeq} (\alpha, \gamma)$

Similarly $\delta^+ = \text{coeq} (\alpha', \gamma')$.

Now by TH 1, IN 19, and above, the simulation

$$\left( \text{Id}_{x^*}, b, \text{Id}_{\gamma} \right) : N\beta = (X, s, y, i, \delta, \sigma, \lambda)$$

$$\implies N\tilde{\beta} = (X', s', y', i, \delta', \sigma', \lambda')$$

has an inverse $(\text{Id}_{x^*}, b', \text{Id}_{\gamma'})$, $\therefore$ it is an isomorphism

and $N\beta \cong N\tilde{\beta}$.

$\therefore$ the Nerode realization of $\beta$ is unique up to isomorphism.

Given $i, y \in \mathcal{E} / X$, $x$ an input-process, and hence $x^*$, we are basically interested in studying minimal realizations for behaviors $\beta : x^* \longrightarrow y$.

Therefore we can now restrict ourselves to special $\mathcal{A}$ machines and $\mathcal{A}$ behaviors.

**IN 20.** $\mathcal{M}_x$ and $\mathcal{B}_x$ are the subcategories of $\mathcal{M}$ and $\mathcal{B}$ whose objects are machines and behaviors respectively with same initial object $i$ and same input process $x$.

Hence they have the same $x^*$ and their morphisms are of the form $(\text{Id}_{x^*}, b, c)$ and $(\text{Id}_{x^*}, c)$ respectively.
Let \((\text{Id}_{x^*}, c) : \beta \rightarrow \beta' \in \mathcal{B}_x\)

\[
\Rightarrow \beta' = c \cdot \beta \quad \text{as} \quad \frac{\beta}{x^*} \rightarrow \frac{\beta'}{y}
\]

\[
\text{commutes, } \text{IN 8}
\]

Let the Nerode realization of \(\beta'\) and \(\beta\) be respectively:

\(N\beta'(x, s'_p, y', i, \delta'_p, \phi'_p, \lambda'_p)\) with \(\delta'^+_p\) its reachability morphism

\(N\beta = (x, s_p, y, i, \delta_p, \phi_p, \lambda_p)\) with \(\delta^+_p\) its reachability morphism

\[
\therefore M' = (x, s_p, y', i, \delta_p, \phi_p, c \cdot \lambda'_p) \text{ is also a coequalizer reachable realization of } \beta' \text{ as } \delta^+_p \text{ is a coequalizer } \text{ IN 12.2 } \text{and}
\]

\[
(c \cdot \lambda_p), \delta^+_p = c \cdot (\lambda_p \cdot \delta^+_p) = c \cdot \beta = \beta'
\]

\[
\therefore \text{ by TH 1 and IN 19, } \exists \text{ a unique simulation } M' \rightarrow N\beta' \text{ with}
\]

\[
b : s_p \rightarrow s'_p \quad \delta'^+_p : b' \cdot \delta'_p, \phi'_p = b' \cdot \delta_p, \phi_p, c \cdot \lambda_p = \lambda'_p \cdot b'
\]

\[
\Rightarrow \frac{x^* \delta^+_p}{s_p} \rightarrow \frac{s'_p}{y} \quad \frac{s_p \lambda_p}{y} \rightarrow \frac{y'}{s'_p} \quad \frac{s'_p \lambda'_p}{y'} \rightarrow \frac{y'}{s'_p}
\]

\[
\text{commutes.}
\]
\[ (\text{Id}_{\chi^*}, b', c) : N\beta \rightarrow N\beta' \text{ is a machine-morphism} \]

in \( M^\chi \). We can now define the functor
\[ N : \mathcal{B}^\chi \rightarrow M^\chi \]

called the Nerode realization functor.

**Lemma 2.** \( EN = \text{Id}_{\mathcal{B}^\chi} : \mathcal{B}^\chi \rightarrow \mathcal{B}^\chi \)

**Proof:** By **DN 16 and 17** we have
\[ \forall \beta \in \mathcal{B}^\chi, \quad EN \beta = E(N\beta) = \lambda^\rho \cdot \delta^\rho = \beta. \]

By **DN 16 and 21**, \( \forall (\text{Id}_{\chi^*}, c) : \beta \rightarrow \beta' \text{ in } \mathcal{B}^\chi \)

\[ EN(\text{Id}_{\chi^*}, c) = E(\text{Id}_{\chi^*}, b', c) = (\text{Id}_{\chi^*}, c). \]
**7. NATURAL TRANSFORMATION**

\[
\gamma : \text{Id}_M \rightarrow \text{NE} : M \rightarrow M
\]

**DN 22.1.**

\[
\forall M = (x, s, y, i, \delta, \sigma, \lambda), \quad EM = \lambda : \delta^* : \text{DN 15}
\]

\[
NEM = (x, s_{EM}, y, i, \delta_{EM}, \sigma_{EM}, \lambda_{EM})
\]

as defined in DN 17, is called its reduced machine.

\[
\text{Since both } M \text{ and } NEM \text{ are reachable realizations of } EM \quad \text{by TH 1 and DN 19, there is a unique simulation}
\]

\[
b_{EM} : s : \rightarrow s_{EM}
\]

\[
\delta_{EM} = b_{EM} : \delta^*, \quad \sigma_{EM} = b_{EM} : \sigma \quad \text{and } \lambda = \lambda_{EM} : b_{EM}
\]

\[
\Rightarrow \gamma = (\text{Id}_M, b_{EM}, \text{Id}_y) : M \rightarrow NEM \text{ is a machine morphism as both diagrams}
\]

\[
\begin{array}{ccc}
\delta^* & \rightarrow & s^* \\
\downarrow \text{id}_x & & \downarrow b_{EM} \\
\delta_{EM} & \rightarrow & s_{EM}
\end{array}
\]

\[
\begin{array}{ccc}
s & \rightarrow & y \\
\downarrow \text{id}_y & & \downarrow \lambda_{EM} \\
s & \rightarrow & y
\end{array}
\]

\[
\text{commute.}
\]

**LEMMA 2.**

\[
\gamma = \{ \gamma_M : M \rightarrow NEM \mid M \in \mathcal{M} \}
\]

is a natural transformation \( \text{Id}_M \Rightarrow \text{NE} : M \rightarrow M \).
Proof: Let \((\text{Id}_{x^*}, b, c) : \mathcal{M} \rightarrow \mathcal{M}'\)
be a
machine morphism in \(\mathcal{M}_x\)
\[\Rightarrow \delta'^+ = b \cdot \delta^+ \quad \vdash \text{IN 7} \]
\[\text{NE} (\text{Id}_{x^*} b, c) = N (\text{Id}_{x^*}, c) \quad \vdash \text{IN 16} \]
\[= (\text{Id}_{x^*} b', c) \quad \vdash \text{IN 21} \]

\[\eta_M = (\text{Id}_{x^*}, b_M, \text{Id}_{y^*}), \quad \eta_{M'} = (\text{Id}_{x^*}, b_{M'}, \text{Id}_{y^*}) \]
\[\Rightarrow \delta'^+_{EM} = b_{EM} \cdot \delta^+_{EM} \Rightarrow \delta'^+_{EM'} = b_{EM'} \cdot \delta^+ \quad \vdash \text{IN 22} \]
and \(\delta'^+_{EM'} = b' \cdot \delta^+_{EM} \quad \vdash \text{IN 21} \)

\[\Rightarrow b_{EM'}, b = b' \cdot b_{EM} \quad \text{as } \delta^+ \text{ a coequalizer is} \]
epi; i.e. the diagram

\[
\begin{array}{ccc}
\delta & \downarrow b_{EM} & \delta_{EM} \\
\downarrow b & & \downarrow b' \\
\delta' & \downarrow b_{EM'} & \delta'_{EM'} \\
\end{array}
\]
commutes
\[ (\text{Id}_{x'}, b', \text{Id}_{y'}) \cdot (\text{Id}_{x}, b, c) = (\text{Id}_{x'}, b', b, c) \]
\[ = (\text{Id}_{x'}, b', \text{Id}_{y}, c) \]
\[ = (\text{Id}_{x'}, b', c) \cdot (\text{Id}_{x}, b, \text{Id}_{y}) \]

**Theorem 2.** \[ E \leftrightarrow M : \mathcal{M}_x \leftrightarrow \mathcal{B}_x \]

**Lemma 3.** \[ E \eta_M = \text{Id}_{E_M} : E_M \rightarrow E_M \text{ in } \mathcal{B}_x, \forall M \in \mathcal{M}_x \]

**Proof:** \[ \eta_M = (\text{Id}_{x'}, b, \text{Id}_{y}) : M \rightarrow NEM \quad \text{by DN 22} \]
\[ E \eta_M = (\text{Id}_{x}, \text{Id}_{y}) : E_M \rightarrow E \text{ NEM = E M} \]

by DN 16 and Lemma 1, i.e., \[ E \eta_M = \text{Id}_{E_M} \]

Lemma 3 means that reduction preserves behavior \(\forall\) reachable \(A\) machines.
**Lemma 4.** \( \eta_{N \rho} = \text{Id}_{N \rho} \) in \( \mathcal{M}_x \).

**Proof:** \( \forall \beta : x^* \to y \in \mathcal{M}_x \)
\( N \beta = (x, s_\rho, y, i_?, \delta_\rho, \sigma_\rho, \lambda_\rho) \in \mathcal{M}_x \) \( \quad \text{DN 17} \)
\( \eta_{N \rho} = (\text{Id}_{x^*}, b_{EN \beta}, \text{Id}_y) : N \beta \to N E N \beta \) \( \quad \text{DN 22} \)
\( EN \beta = \beta \) \( \quad \text{lemma 1} \)

By DN 22 the two diagrams 1 and 2 commute

\[ \begin{array}{ccc}
   x^* & \xrightarrow{\delta_\rho^+} & s_\rho \\
   \text{Id}_{x^*} \downarrow & & \downarrow \lambda_\rho \\
   x^* & \xrightarrow{\delta_\rho^+} & s_\rho \\
   \delta_\rho^+ = b_{EN \beta} & & \lambda_\rho = \lambda_\rho
\end{array} \]

But

\[ \begin{array}{ccc}
   x^* & \xrightarrow{\delta_\rho^+} & s_\rho \\
   \text{Id}_{x^*} \downarrow & & \downarrow \lambda_\rho \\
   x^* & \xrightarrow{\delta_\rho^+} & s_\rho \\
   \delta_\rho^+ = b_{EN \beta} & & \lambda_\rho = \lambda_\rho
\end{array} \]

also commute

By uniqueness property (\( \therefore \text{TH 1 and DN 19} \)) \( b_{EN \beta} = \text{Id}_{s_\rho} \)

\[ \Rightarrow \eta_{N \rho} = (\text{Id}_{x^*}, \text{Id}_{s_\rho}, \text{Id}_y) = \text{Id}_{N \rho} \]
Proof of TH 2

Let \( \varepsilon : EN \Rightarrow Id_{\mathcal{B}_x} \Rightarrow Id_{\mathcal{B}_x} \) be the identity i.e.

\[
\varepsilon = \{ \varepsilon_B = Id_B : EN_B = B \Rightarrow B \mid B \in \mathcal{B}_x \}
\]

We show that the natural transformations

\( \eta : Id_{\mathcal{M}_x} \Rightarrow NE \) and \( \varepsilon : EN \Rightarrow Id_{\mathcal{B}_x} \)

are respectively the unit and the co-unit of the adjunction \( E \dashv N : \mathcal{B}_x \rightarrow \mathcal{M}_x \).

\[
\forall M \in \mathcal{M}_x , \quad EN EM = EM \quad \Rightarrow \text{Lemma 1}
\]

\[
(EN \eta)_M = EM_M = Id_{EM} \quad \Rightarrow \text{DN a7.5}
\]

\[
(\varepsilon E)_M = E EM = Id_{EM} \quad \Rightarrow \text{DN a7.5}
\]

\[
\varepsilon \equiv \text{DN of } \varepsilon
\]

\[
\Rightarrow \text{commutes } \forall M
\]

\[
\Rightarrow (a) : \quad \Rightarrow \text{commutes}
\]

\[
E \cdot Id_{\mathcal{M}_x} = E \Rightarrow Id_{\mathcal{B}_x} \cdot E = E
\]
\( \forall \beta \in \mathcal{B}_x \), \( N \cdot N \beta = N \beta \quad \text{\textasciitilde lemma 1} \)

\( (\eta N)_{\beta} = \eta_{N_{\beta}} \quad \text{\textasciitilde IN a 7.5} \)

\( = \text{Id}_{N_{\beta}} \quad \text{\textasciitilde lemma 4} \)

\( (N\varepsilon)_{\beta} = N\varepsilon_{\beta} \quad \text{\textasciitilde IN a 7.5} \)

\( = N \cdot \text{Id}_{\beta} \quad \text{\textasciitilde IN of } \varepsilon \)

\( = \text{Id}_{N\beta} \quad \text{\textasciitilde } N \text{ a functor} \)

\[ \begin{array}{c}
  \text{N E N} \beta = N\beta \\
  \text{(commutes,)}
\end{array} \]

\( \begin{array}{c}
\eta N \\
\text{commutes}
\end{array} \)

\[ \begin{array}{c}
\text{Id}_N \cdot N = N \Rightarrow N = N \cdot \text{Id}_N \\
\text{commutes}
\end{array} \]

\( (a) \text{ and } (b) \Rightarrow E \dashv N \text{ and } \eta \text{ and } \varepsilon \text{ are the unit and the co-unit of the adjunction} \)

\( \text{TH 5.4} \)

\[ \begin{array}{c}
\text{x x x}
\end{array} \]
9. **TH 3.**

∀ functor \( F : \mathcal{B} \rightarrow \mathcal{M}_x \), \( F \) is a minimal realization functor (i.e. \( F\beta \) is a minimal realization of \( \beta \), \( \forall \beta \in \mathcal{B}_x \))

\[ E \vdash F : \mathcal{M}_x \rightarrow \mathcal{B}_x \]

**Proof:** By DN.19, \( \forall \beta \in \mathcal{B}_x \), \( N \beta \) and \( F\beta \) are minimal realization of \( \beta \) \( \iff \) a unique \( \varphi : N\beta \rightarrow F\beta \exists \varphi \) is a machine isomorphism (i.e. \( \exists \) unique \( \varphi^{-1} \));

\[ E \vdash N \iff \mathcal{M}_x(\lambda, N\beta) \cong \mathcal{B}_x(\lambda M, \beta) \quad \therefore \text{TH a 5.1} \]

\[ \iff \mathcal{M}_x(\lambda, F\beta) \cong \mathcal{B}_x(\lambda F, \beta) \quad \text{as} \]

\( \forall f : M \rightarrow F\beta \) corresponds a unique \( \varphi^{-1} \), \( f : M \rightarrow N\beta \) i.e. a unique \( E (\varphi^{-1}, f) : EM \rightarrow N\beta \);

and \( \forall g : EM \rightarrow \beta \) corresponds a unique \( h = Ng : EM \rightarrow N\beta \)

i.e. a unique \( \varphi.h : M \rightarrow F\beta \) (\( \eta_M \) given by DN.21)

\[ E \vdash F \quad \therefore \text{TH a 5.1} \]

**TH 3** means that a minimal realization functor \( F \) is exactly a right adjoint to the behavior functor \( E \)

i.e. it is naturally isomorphic to \( N \).

**NOTE 8.**

\( \forall \beta \in \mathcal{B}_x \), \( N\beta \cong RF\beta \iff F \) is naturally isomorphic to \( N \) \( \therefore \) DN a 7.2
The proof of TH 3, suitably modified, yields the following more general result of category theory:

**Proposition 1:** G is a right (left) adjoint of F, H is naturally isomorphic to G

\[ \Rightarrow \text{H is a right (left) adjoint of F.} \]

TH 3 can then be considered as a corollary of proposition 1 as:

\[ F\beta \text{ a minimal realization } \Rightarrow F\beta \simeq N\beta \quad \forall \beta \in G \times \Rightarrow F \text{ naturally isomorphic to N} \]

\[ \Rightarrow E \downarrow F \quad \text{as} \quad E \downarrow N \]

\[ \times \times \times \]
CHAPTER II

EXAMPLES OF A-MACHINES

We will now study three examples of A-Machines: discrete, linear and tree automata.
1 DISCRETE MACHINES.

**Def. 23.** A discrete machine (or automaton) is a sextuple

\[ (X, S, Y, \sigma, \delta, \lambda) \]

where

- \( X \) is the set of inputs,
- \( S \) is the set of states,
- \( Y \) is the set of outputs,
- \( \delta : X \times S \to S \) is the transition function,
- \( \lambda : S \to Y \) is the output function,
- \( \sigma \in S \) is the initial state.

**Example:** We want a decimal computer for adding integers consisting of at most \( n \) digits. We use \( n \) counters modulo 10 which are \( n \) identical machines \( c_{10}^1, c_{10}^2, \ldots \) such that:

- \( X = \{0, 1\} = Y \), \( S = \{0, 1\} \times \mathbb{Z}_{10} \)
- \( \delta(0, (y, z)) = (0, z) \)
- \( \delta(1, (y, z)) = \begin{cases} (1, s(z)) & \text{if } z = 9 \\ (0, s(z)) & \text{otherwise} \end{cases} \)

where \( s(z) = z + 1 \) (mod 10)

\[ \lambda(y, z) = y. \]
For instance let's add 3 to 397; we send in our computer

\[ W = 111 \in \{0, 1\}^*; \] we have:

| \( c_4 \) |
|\------|
| 10 |
| 10 |
| 10 |
| 10 |
| \( c_3 \) |
| 0 |
| 0 |
| 0 |
| 0 |
| \( c_2 \) |
| 0 |
| 0 |
| 0 |
| 0 |
| \( c_1 \) |
| 0 |
| 0 |
| 0 |
| 0 |

initial state

\[ (0, 0) \rightarrow (0, 3) \rightarrow (0, 9) \rightarrow (0, 7) \]

state after 1st input

\[ (0, 0) \rightarrow (0, 3) \rightarrow (0, 9) \rightarrow (0, 8) \]

state after 2nd input

\[ (0, 0) \rightarrow (0, 4) \rightarrow (1, 0) \rightarrow (1, 0) \]

final state

The result at each stage is given by reading in the proper order the 2nd components \( z \) of each state: 397, 398, 399 and, finally, 400.

**Proposition 2.** A discrete machine is an \( A \)-machine

\[ M = (X \times X, S, Y, I, \sigma, \sigma, \lambda) \text{ in } \mathcal{A}, \text{ the category of sets.} \]

**Proof:** Let \( X = \text{set of inputs} \)

\[ X \times : \mathcal{A} \rightarrow \mathcal{A} \text{ is a functor, } \vdash \text{ a process.} \]

\[ S \rightarrow X \times S \]

\[ S \rightarrow X \times S \]

\[ f \mid \downarrow \text{ defined by } \text{Id}_X \times f : (x, s) \]

\[ \downarrow \text{ } \rightarrow (x, f(s)) \]

\[ S' \rightarrow X \times S \]
DYN $X$ is the category of all dynamics i.e. all maps $\delta : X \times S \to S$ and dynamorphisms $f : X \times S \to S'$, where $f : S \to S'$ is a map $\exists \delta'(\text{Id}_X \times f) = f$. $X^*$ is the free monoid whose words are strings of elements of $X$, $\Lambda = \text{empty word}$, and the operation is concatenation.

$X \times -$ is an input-process as $\forall S \in \mathcal{A}$

$\begin{align*}
\gamma_S : S \to X^* S, & \quad \delta_S : X_\times (X^* S) \to X^* \times S \\
\vdash (\Lambda, S), & \quad (x, w, s) \mapsto (xw, s)
\end{align*}$

is universal from $S$ to $U$, the forgetful functor $\text{DYN } X \to \mathcal{A}$.

Indeed $\forall f : S \to S'$, the map $\psi : U \delta_S^* X \times S \to U \delta' = S'$ uniquely defined by

$\psi(x, w, s) = \delta'^*(w, f(s))$ where given $\delta : X \times S \to S$

$\delta^*$ is inductively defined by:

$\begin{align*}
\delta^*(\Lambda, s) &= s \\
\delta^*(xw, s) &= \delta(x, \delta^*(w), s)
\end{align*}$

$\forall x \in X, w \in X^*, s \in S$ makes the two following diagrams commute:
\[ S \xrightarrow{\eta} X^* \xrightarrow{\gamma} (x, \omega, s) \xrightarrow{\delta} (xw, s) \]

And

\[ X \times X^* \xrightarrow{\delta} X^* \]

\[ \text{let } f(s) = \delta((\lambda, f(s))) \]

\[ \delta((x, \delta^*(\omega, f(s)))) = \delta^*(\lambda, f(s)) \]

\[ S = \text{set of states} \]

\[ Y = \text{set of outputs} \]

\[ I = \{s\} \text{ is the initial state object} \]

\[ \sigma : I \rightarrow S \text{ is the initial state morphism} \]

\[ \sigma(s) = \sigma \]

\[ \delta : X \times S \rightarrow S \text{ is the transition function, an X dynamics} \]

\[ \lambda : S \rightarrow Y \text{ is the output morphism} \]

\[ X^* \text{ is the object of inputs} \]

\[ \delta^* \text{ the reachability morphism, is uniquely defined by} \]

\[ \delta^* : X^* \rightarrow S \text{ i.e. by} \]

\[ \delta^*(\lambda) = \sigma \]

\[ \delta^*(\omega, \sigma) = \delta(\lambda, \delta^*(\omega)) \]
It is the unique dynamorphic extension of $\sigma$ as

\[
\begin{align*}
\{\cdot\} \xrightarrow{\gamma} X^* \xrightarrow{\cdot} \{\cdot\} \xrightarrow{\cdot} X^* \\
\downarrow \delta^+ \quad \downarrow \delta^+ \\
S \quad \downarrow \delta^+ \\
\sigma
\end{align*}
\]

\[
\begin{align*}
(x, \delta^+(\omega)) \quad \delta(x, \delta^+(\omega)) = \delta^{x\omega}
\end{align*}
\]

 commute.

\[\therefore M \quad \text{is a machine in category } \Delta \quad \Box \in \mathbb{C} \]

with external behavior \[EM = \lambda \cdot \delta^+ : X^* \rightarrow Y\]

Let \[E_{EM} = \{ (w_1, w_2) / EM(w_1) = EM(w_2) \quad \forall w \in X^* \}\]

This is the Nerode equivalence relation

Let \[\alpha, \gamma : E_{EM} \rightarrow X^* \] be the usual projections

\[
\begin{align*}
\delta^n : x^n \times X^* & \rightarrow X^* \\
(x_n, \ldots, x_0, w) & \rightarrow x_n \ldots x_0 w
\end{align*}
\]
\[ \forall (x_n, \ldots, x_1, (w_1, w_2)) \in X^n \times E_{EM} \]
\[ \lambda \cdot \delta^+ \cdot \phi^{(n)}(x_{x-}) \alpha (x_n, \ldots, x_1, (w_1, w_2)) = \lambda \cdot \delta^+ \cdot \phi^{(n)}(x_n, \ldots, x_1, w) \]
\[ = \lambda \cdot \delta^+ (x_n, \ldots, x_1, w) \]
\[ = \lambda \cdot \delta^+ (x_n, \ldots, x_1, w_2) \]
\[ \therefore (w_1, w_2) \in E_{EM} \]
\[ = \lambda \cdot \delta^+ \cdot \phi^{(n)}(x_n, \ldots, x_1, w_2) \]
\[ = \lambda \cdot \delta^+ \cdot \phi^{(n)}(X_x-)^\gamma (x_n, \ldots, x_1, (w_1, w_2)) \]

Postulate (1'a) is satisfied

Assume \( \exists \ R \) and \( p, p' : R \rightarrow x^* \exists \)
\[ \lambda \cdot \delta^+ \cdot \phi^{(n)}(X_p) = \lambda \cdot \delta^+ \cdot \phi^{(n)}(X_p') \]
\[ \Rightarrow \forall n \in \mathbb{N}, \forall w \in X^*, \exists w = x_n, \ldots, x_1, \forall r \in R \]
\[ \lambda \cdot \delta^+ (w, p(r)) = \lambda \cdot \delta^+ (w, p'(r)) \]
\[ \Rightarrow (p(r), p'(r)) \in E_{EM} \]
\[\exists \varphi : \mathbb{R} \rightarrow E_{EM} \]
\[r \rightarrow (p(r), p'(r))\]
\[p = \alpha \cdot \varphi \quad \text{and} \quad p' = \gamma \cdot \varphi\]
\[\forall \psi \in \psi, \alpha = \psi \cdot \gamma\]
\[\psi \cdot \alpha \cdot \varphi = \psi \cdot \gamma \cdot \varphi\]
\[\psi \cdot p = \psi \cdot p'\]

i.e. postulates \( \text{1b} \) is satisfied.

Let \( S_{EM} = x / E_{EM} \) whose elements are equivalence classes \([w]\) of \( E_{EM} \).

\[\delta^+: X^* \rightarrow S_{EM} \quad \text{is onto} \quad \iff \quad \delta^+ = \text{coeq} (\alpha, \gamma)\]

with

\[\delta^+: X \times S_{EM} \rightarrow S_{EM}\]

\((x, [w]) \mapsto [xw]\)

postulate 2 is satisfied and so is 3 with

\[\delta^-_E : X \times S_{EM} \rightarrow S_{EM}\]

\((x, [w]) \mapsto [xw]\)
Postulate 4 is satisfied as $\delta^+$ onto $\iff$ it is a split-epi in $\mathcal{A}$ and any functor preserves split-epi.

$EM = \lambda \cdot \delta^+$ is an $A$-behavior and $M_\psi$ is an $A$-machine whose Nerode realization is

$$NE M = (X \rightarrow S_{EM}, Y, \{\cdot\}, \delta_{EM}, \sigma_{EM}, \lambda_{EM}, \{x\} \rightarrow S_{EM}, \lambda_{EM}, S_{EM} \rightarrow Y, [\cdot] \rightarrow [\lambda], [x] \rightarrow EM(x))$$

NOTE

For proof of uniqueness of the maps $\psi$, $\delta^+$, etc., see [3], chap. 9.

Given $M_\psi$, an informal description of an algorithm for state-merging (i.e. to find $S_{EM}$), which terminates in at most

$n - 2$ steps if $|S_{EM}| \leq n$, is given in [3] page 152.

\[ x \times x \]
2. **LINEAR MACHINES**

**IN 24.** An R-linear machine \( M \) is an automaton
\[
(X, S, Y, \sigma = \sigma_x, \delta, \lambda) \quad \text{where} \quad \hat{X}, S, Y \text{ are R-modules},
\]
\[
\delta : X \bigoplus S \rightarrow S
\]
and \( \lambda : S \rightarrow Y \text{ are R-linear maps}.
\]

**Example:** Let \( X = S = Y = M \) an R-module,
\[
\delta = \tau_M, \quad \lambda = \text{Id}_M
\]
then our machine is simply the R-module \( M \).

**Note 9.** If \( R \) is a field then \( X, S, Y \) are vector-spaces and \( \delta \) and \( \lambda \) are linear-transformations.

**Proposition 3.** \( M \) is the A-machine \((X \bigoplus - , S, Y, I, \delta, \sigma, \lambda)\) in the category \( \text{Mod}_R \) of R-modules and R-linear maps as morphisms. For details.

**Proof:** Given an R-module \( X \)
\[
\begin{align*}
X \bigoplus - : & \text{R-Mod} \rightarrow \text{R-Mod} \quad \text{is a functor} \\
S & \mapsto X \bigoplus S \\
S & \mapsto X \bigoplus S' \\
\delta & \mapsto \text{Id}_X \bigoplus f \text{ where } \text{Id}_X \bigoplus f : (x, s) \rightarrow (x, f(s))
\end{align*}
\]
DYN $X$ is the category of all dynamics, i.e. of all
$R$-linear maps $\delta : X \oplus S \to S$ and dynamorphisms
$h : (\delta : X \oplus S \to S) \to (\delta' : X \oplus S' \to S')$
where $h : S \to S'$ is a linear map
$\exists \delta' : (\text{Id}_x \oplus h) \cdot \delta = h \cdot \delta$

---

**NOTE 10.1**  \( \forall (x, s) \in X \oplus S, \delta \text{ linear} \)

$$\implies \delta(x, s) = \delta(0_x, s) + \delta(x, 0_s)$$

Let $f : S \to S$ and $g : X \to S$
$s \to \delta(0_x, s)$
$x \to \delta(x, 0_s)$

Again linear $\implies f$ and $g$ are conversely given

two linear maps $f : S \to S$ and $g : X \to S$
$\delta : X \oplus S \to S$
$(x, s) \mapsto f(s) + g(x)$

is certainly linear $\delta : X \oplus S \to S$

is certainly linear $\delta : X \oplus S \to S$ is a dynamics $\iff$

$\delta(x, s) = f(s) + g(x)$ for some linear maps $f : S \to S$ and $g : X \to S$. \(\blacksquare\)
2. Let \( h \) be a dynamorphism \( \delta \rightarrow \delta' \).

\[ h \cdot \delta(x, s) = \delta'(\text{id}_x \circ h)(x, s) = \delta'(x, h(s)) \]

i.e. \( h(\delta(x) + g(s)) = f' \cdot h(s) + g'(x) \)

Putting \( x = O \) we have \( h \cdot f = f' \cdot h \)

and \( s = O' \) we have \( h \cdot g = g' \)

\[ h : S \rightarrow S', \text{ with } (\delta : X \oplus S \rightarrow S) = f + g \]

and \( (\delta' : X \oplus S' \rightarrow S') = f' + g' \), is a dynamorphism

\[ h \cdot f = f' \cdot h \quad \text{and} \quad h \cdot g = g' \]

\[ x^+ = \bigcup_{n \in \mathbb{N}} x_n = \text{countable copower of } x = x_n \quad \forall n \in \mathbb{N} \]

i.e. \( x^+ = \{(x_0, \ldots, x_n, 0, \ldots) | n \in \mathbb{N}\} \) is set of infinite tuples

\[ \exists x_i \in X \quad \text{and} \quad x_i = 0 \quad \text{for all but a finite number of } x_i's. \]

It is an R-module with addition and scalar multiplication defined componentwise by the addition and scalar multiplication in \( X \).
In the string $w = x_0 \ldots x_n \circ \ldots$, $x_n$ is the first non zero input, $x_0$ the last input activating the machine.

$\forall S \in \text{Mod } R /$

\[
\begin{align*}
\gamma_S &: S \longrightarrow x^+ \oplus S^+ \\
& S \longrightarrow (00, \ldots, 00, \ldots)
\end{align*}
\]

\[
\begin{align*}
\delta_S &: x \oplus (x^+ \oplus S^+) \longrightarrow x^+ \oplus S^+ \\
& (x_0, x_1, \ldots, x_n, 0, \ldots, s_0, s_1, 0, \ldots) \longrightarrow (x_0, x_1, \ldots, x_n, 0, \ldots, 0s_0, s_1, 0, \ldots)
\end{align*}
\]

is universal from $S$ to $U$, the forgetful functor

$\text{DYN } X \longrightarrow \text{Mod } R$.

Indeed $\forall \varphi : S \longrightarrow S'$ linear

\[
\psi (x_0, x_1, \ldots, x_n, 0, \ldots, s_0, s_1, 0, \ldots) = \sum f_i \varphi (s_i) + \sum f^{i'} \varphi (s_{i'})
\]

is the unique linear map $\exists$ the following diagrams: commute:

\[
\begin{align*}
S & \xrightarrow{\gamma_S} X^+ \oplus S^+ \\
& \downarrow \psi \\
& \varphi \downarrow \\
& S' \xrightarrow{\psi (s)} f^{i'} \varphi (s) + 0
\end{align*}
\]
\[ (x_1, x_2, \ldots, x_n, 0, \ldots, s, 0, \ldots) \mapsto (x_1, x_2, \ldots, x_n, 0, \ldots, 0, s, 0, \ldots) \]

\[ X \oplus (X \oplus S') \xrightarrow{\delta s} X' \oplus S' \]

\[ \downarrow \quad \downarrow \]

\[ \text{id}_X \oplus \psi \quad \psi \]

\[ X \oplus S' \xrightarrow{\delta'} S' \]

\[ (x_1, \sum f_i^i, \varphi(s_{x}), \sum f_i^i g(x_i)) \mapsto \sum f_i^{i+1} \varphi(s_{x}) + \sum f_i^{i+1} g(x_i) + g(x) \]

\[ S = \text{R-module of states} \]

\[ Y = \text{R-module of outputs} \]

\[ I = \{ 0 \} \text{ is the initial state R-module} \]

\[ \sigma: \{ 0 \} \rightarrow S \text{ is the initial state linear morphism} \]

\[ \sigma: O_s \rightarrow O_s \]

\[ X^+ = \text{the R-module of inputs} \]

\[ \delta^+: X^+ \rightarrow S, \text{ the reachability morphism} \]

\[ (x_1, \ldots, x_n, 0) \mapsto \sum f_i^i g(x_i) \]

is the unique dynamorphique extension of \( \sigma \) as

\[ \mathcal{O} \xrightarrow{\gamma} X^+ \{ 0 \} \xrightarrow{\gamma} X^+ \xrightarrow{\delta^+} S \]

and

\[ O_s \]
\( (x, w = x_0, x_0, \ldots) \overset{\delta_i}{\rightarrow} (x, x_0, x_0, \ldots) \)

\[
\begin{array}{c}
\xrightarrow{\text{Id} \oplus \delta^+} \\
\xrightarrow{X \oplus S} \\
\xrightarrow{\delta} \\
\xrightarrow{\delta^+} \\
\xrightarrow{\text{commute}} \\
\xrightarrow{\Sigma \mathbf{F}^i \cdot g(x_i) + g(x)}
\end{array}
\]

\( (x', \Sigma \mathbf{F}^i \cdot g(x_i)) \overset{\delta^+}{\rightarrow} \Sigma \mathbf{F}^i \cdot g(x_i) + g(x) \)

\( M \) is a machine in Mod \( \mathbb{R} \) with external behavior \( \overline{EM}(w) = \lambda, \delta^+(w) = \Sigma h \cdot \mathbf{F}^i \cdot g(x_i) \)

if we put \( \lambda = h : S \rightarrow Y \).

Let \( \overline{EM} : X^+ \rightarrow Y^{\mathbb{R}} \)

\( (w = x_0, x_0, \ldots) \mapsto (\overline{EM}(w), \overline{EM}(0w), \ldots, \overline{EM}(0^n w), \ldots) \)

with \( 0^n w = 0, 0, x_0, x_0, \ldots \) \( n \) zeros

\( \overline{EM}(w) = (\Sigma h \cdot \mathbf{F}^i \cdot g(x_i), \Sigma h \cdot \mathbf{F}^i \cdot g(x_i), \ldots, \Sigma h \cdot \mathbf{F}^i \cdot g(x_i), \ldots) \)

Then the Nerode equivalence relation for \( EM \) is

\( \overline{EM} = \{ (w_1, w_2) \mid w_1, w_2 \in X^+ \text{ and } \overline{EM}(w_1) = \overline{EM}(w_2) \} \)
as \( \forall \varphi : x^+ \rightarrow Y \) linear

\[
\varphi (w w_i) = \varphi (w w_s) \quad \forall w \in x^+
\]

\[\iff \quad \varphi (w w_i) = \varphi (w) + \varphi (0^{\text{len}(w_i)} w_s) = \varphi (w) + \varphi (0^{\text{len}(w_i)} w_s)
\]

\[\iff \quad \varphi (0^{\text{len}(w_i)} w_s) = \varphi (0^{\text{len}(w_i)} w_s), \text{ where } w = \text{length of } w
\]

\[\iff \quad \varphi (0^{n} w_i) = \varphi (0^{n} w_s) \quad \forall n \in \mathbb{N}
\]

\[S_{EM}^+ = x^+ / E_{EM}
\]

\[\delta_{EM}^+ : x^+ \rightarrow S_{EM}, \quad w \rightarrow [w]
\]

\[\delta_{EM} : x \times S_{EM} \rightarrow S_{EM}, \quad (x, [w]) \rightarrow [xw]
\]

\(\text{EM satisfies the four postulates of \textit{IN} 12}

(see [3] p. 696 to 701)

\[\therefore \text{M is an A-machine and its Nerode realization is:}
\]

\[\text{NEM} = (X \oplus \rightarrow) S_{EM}, Y, \{0\}, S_{EM}, \text{NEM:} \{0\} \rightarrow S_{EM}, \lambda_{EM}, S_{EM} \rightarrow Y
\]

\[0 \rightarrow [0], \quad [w] \rightarrow \text{EM}(w)
\]
NOTE. Again for proof of uniqueness of $\psi$, $\delta^*$, etc. see [3], chap. 9

The realization algorithm for linear machines is given in [3] page 582 to 586 and a partial realization algorithm is given in [3] page 587.

x x x

3. TREE AUTOMATA.

DEF 2.5.1 A multigraded set is a set $\Omega$ together with a function $\gamma : \Omega \rightarrow 2^N$ which assigns to each $\omega \in \Omega$ a finite set of arities $\gamma (\omega) \subseteq \mathbb{N}$

We put $\Omega^* = \gamma^* (\Omega) \subseteq \Omega$. We will use in particular $\Omega^* = \{ \omega \in \Omega | \emptyset \in \gamma (\omega) \}$

2. An $\Omega$-tree is a tree $T \subseteq \mathbb{N}^*$ together with a function $t : T \rightarrow \Omega$ such that if $\omega \in T$ has $n$ successors, then $n \leq \gamma (t (\omega))$.

3. An $\Omega$-algebra $(S, \delta)$ is a set $S$ (the carrier) together with a map $\delta : \omega \mapsto (\delta^n : S^n \rightarrow S)$ for all $\omega \in \Omega$ and all $n \in \gamma (\omega)$. 
4 An $\Omega$-algebra homomorphism is a map $h : S \to S'$ where $(S, \delta)$ and $(S', \delta')$ are $\Omega$-algebras,

- $h(\omega^n(s_1, \ldots, s_n)) = \delta^n_h(h(s_1), \ldots, h(s_n))$ for all $\omega \in \Omega_n$, $\forall n \in \mathbb{N}$

i.e. the diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{\delta^n} & S \\
\downarrow h^{(n)} & & \downarrow h \\
S' & \xrightarrow{\delta'^n} & S'
\end{array}
\]

commutes

where $h^{(n)} : S^n \to S'$

$(s_1, \ldots, s_n) \quad \mapsto \quad (h(s_1), \ldots, h(s_n))$

**DN 26.1** A tree automaton is a quintuple

$M = (S, \Omega, \gamma, \delta, \lambda)$

- $S$ = set of states
- $\Omega$ = multigraded set of inputs, with arity function $\gamma$
- $\gamma$ = set of outputs
- $(S, \delta)$ is an $\Omega$-algebra, $\delta$ is the transition function,
- $\lambda : S \to \gamma$ is a map called the output function.

2 An input-structure for $M$ is an $\Omega$-tree $t : T \to \Omega$

where $\Omega = S \cup \Omega'$ with arity function defined by

\[
\begin{align*}
\gamma(\omega_S) &= \{0\} & \text{if } \omega_S \in S \\
\gamma(\omega_{\Omega'}) &= \gamma(\omega_S) & \text{if } \omega_{\Omega'} \in \Omega
\end{align*}
\]

\[\exists \text{ The RUN of } M \text{ on } t \text{ is the tree } \vec{E} : T \to S' \]

- if $\omega$ a terminal node of $T$ then

\[
\begin{align*}
\vec{E}(\omega) &= t(\omega) & \text{if } t(\omega) \in S \\
\vec{E}(\omega) &= \delta_{t(\omega)}^o(\lambda) \in S' & \text{if } t(\omega) \in \Omega_o
\end{align*}
\]
- if \( w \) has successors \( w_0, \ldots, w_{(n-1)} \) then.
\[
\tilde{E}(w) = \delta^{n}_{\tilde{E}(w)}(\tilde{E}(w_0), \ldots, \tilde{E}(w_{(n-1)}))
\]

4 The evaluation of \( t \) by \( M \) is the output \( \lambda(\tilde{E}(\Lambda)) \)
of the state obtained at the \( \Lambda \) node.

Example: We want a tree automaton and an input-structure to compute \((1 + 3) + (4 \times 2) + 3 = 15\).

Let \( S = Y = IN, \quad \Omega = \{+, x\} \quad \Rightarrow \Omega_{S} = IN \cup \{+, x\}\)

with \( \gamma(n) = 0 \quad \forall n \in IN \quad \text{and} \quad \gamma(+) = \gamma(x) = 2 \)

(\( + \) and \( x \) are the usual addition and multiplication of integers).

We have:
\[
\delta_{n}^{0} : IN^{0} \rightarrow IN
\]
\[
\delta_{\Lambda} : \Lambda \rightarrow n
\]
\[
\delta_{+}^{2} : IN \rightarrow IN
\]
\[
\delta_{x}^{2} : (n, m) \rightarrow n \times m
\]

\[
\lambda = Id_{IN}
\]
We have then:

\[
\begin{align*}
\text{TREE } T \subset N^* & \quad \text{INPUT-STRUCTURE } S \quad \text{RUN of } M \text{ on } t \\
\text{tree } t: T & \rightarrow \Omega_3 \\
\text{Tree } \sigma & \rightarrow 1
\end{align*}
\]

The evaluation of \( t \) by \( M \) is: \( \text{Id}_n(15) = 15 \)

**INPUT PROCESS** \( X : \mathcal{A} \rightarrow \mathcal{A} \). We define \( X \) as follows:

\[
\forall S \in \mathcal{A}, \quad X S = \bigcup_{n \in \omega} S^n x \{ \omega \}
\]

\[
= \left( s_1, \ldots, s_n, \omega \right) / s_1, \ldots, s_n \in S, \omega \in \Omega, n \in \omega \}
\]

\[
\forall \text{ map } h : S \rightarrow S'
\]

\[
X h : XS \rightarrow XS'
\]

\[
\left( s_1, \ldots, s_n, \omega \right) \rightarrow \left( h(s_1), \ldots, h(s_n), \omega \right)
\]

\( h \) a map \( \implies \) \( X h \) is well defined.

Assume we have \( S \xrightarrow{h} S' \xrightarrow{h'} S'' : X (h', h) (s_1, \ldots, s_n, \omega) = (h', h(s_1), \ldots, h'(h(s_n), \omega) \quad \text{def of } X
\]

\[= X h'(h(s_1), \ldots, h(s_n), \omega) \quad \text{def of } \circ \text{ and } X
\]

\[= (X h' \circ h) (s_1, \ldots, s_n, \omega) \quad \text{def of } X \text{ and } \circ
\]

Besides \( X(\text{Id}_n) (s_1, \ldots, s_n, \omega) = (s_1, \ldots, s_n, \omega) = \text{Id}_{X S} (s_1, \ldots, s_n, \omega) \)

\( \therefore \) \( X \) is a functor.
An X-dynamics is a map $\delta^\prime: \bigsqcup_{\omega} S \times \{\omega\} \rightarrow S$

$\forall \omega \in \Omega$ and $\forall n \in \mathcal{V}(\omega)$. To specify $\delta^\prime$ is just to specify $\forall \omega \in \Omega$ and $\forall n \in \mathcal{V}(\omega)$ a map $\delta^n_\omega: S^n \rightarrow S$.

But this is the definition of an $\Omega$-algebra $(S, \delta)$.

$\therefore$ the X-dynamics are precisely the $\Omega$-algebras.

An map $h: S \rightarrow S'$ is an X-dynamorphism

$h: (\delta: XS \rightarrow S) \rightarrow (\delta': XS' \rightarrow S')$ i.e. a map $h: (S, \delta) \rightarrow (S', \delta')$ of $\Omega$-algebras if the diagram

\[
\begin{array}{ccc}
XS & \xrightarrow{\delta} & S \\
\downarrow{Xh} & & \downarrow{h} \\
XS' & \xrightarrow{\delta'} & S'
\end{array}
\]

\[\text{commutes}\]

$(h(s_1), \ldots, h(s_n), \omega) \mapsto \delta^n_\omega(h(s_1), \ldots, h(s_n)) = h(\delta^n_\omega(s_1, \ldots, s_n))$

But this is the definition of an $\Omega$-algebra homomorphism.

$\therefore$ the X-dynamorphisms are precisely the $\Omega$-algebra homomorphisms.

Now we show that $X$ is an input-process i.e. that

$\forall S \in \mathcal{A} \exists X^* S \in \mathcal{A}$, an X-dynamics $\delta_S: X(X^*S) \rightarrow X^*S$ and a map $\gamma_S: S \rightarrow X^*S$

$\exists (\delta_S, \gamma_S)$ is universal over $S$ with respect to $U$, the forgetful functor $\text{MNN} X \rightarrow \mathcal{A}$. 
Let $\Omega_S = \Omega \cup S$ with arity function $\gamma_S : \Omega_S \rightarrow 2^\Omega$ defined by $\gamma_S(\omega_S) = \begin{cases} \gamma(\omega) & \text{if } \omega_S \in \Omega \\ \{\varnothing\} & \text{if } \omega_S \in S \end{cases}$

Let $X^S = \text{set of } \Omega_S \text{-trees}$

\[ X(x^S) = \{(t_1, \ldots, t_n, \omega)/t_1 \ldots t_n \text{ are } \Omega_S \text{-trees and } n \in \gamma(\omega)\} \quad \text{DN of } X \]

Let $\delta_S : X(x^S) \rightarrow x^S$

\[ (t_1, \ldots, t_n, \omega) \rightarrow \Delta \rightarrow \Delta \rightarrow \cdots \rightarrow \Delta \quad \text{is an } x\text{-dynamics as } (x^S, \delta_S) \text{ is an } \Omega \text{-algebra.} \]

Let $\eta_S : S \rightarrow x^S$ (one node tree)

We show by induction that:

\[ \forall (\delta' : xS') \rightarrow s') \in /DYN X/ \quad \text{and } \forall f : S \rightarrow S' \exists \quad \text{a unique map } \psi : x^S \rightarrow s' \quad \text{the two diagrams} \]

\[ S \xrightarrow{\eta_S} x^S \quad x(x^S) \xrightarrow{\delta_S} x^S \]

\[ f \xrightarrow{\psi} \quad \text{and} \quad xS \xrightarrow{\psi} S' \]

\[ \text{commute} \]

(so that $\psi$ is an $x$-dynamorphism $\delta_S \rightarrow \delta'$)

1. Let $t$ be a one node tree. Two cases are possible:

  a) $t(\wedge) = S \in S$

  We want that

  \[ S \xrightarrow{\eta_S} xS \xrightarrow{\delta_S} x^S \xrightarrow{\psi} S' \]

  commutes

  We must have $\psi(t) = f(s)$  \hspace{1cm} (1)
b) \( \mathcal{L} (\Lambda) \ni \omega \in \Omega_0 \) (i.e. \( \omega \in \mathcal{Y}(\omega) \))

We want that

\[
\xymatrix{
(\lambda, \omega) \ar[r] & \omega = t(\Lambda) \\
& x(x^*s) \ar[d] \ar[r]_{\delta_s} & x^*s \\
& x\psi \ar[d] \ar[r] & \psi \ar[d] \\
x's \ar[r]_{\delta'} & s' \\
(\lambda, \omega) \ar[r]_{\delta_{\omega}(\Lambda)} & \delta_{\omega}(\Lambda) 
}
\]

We must have \( \psi(t) = \delta_{\omega}(\Lambda) \in S' \) \( \tag{2} \)

2. Let \( t = \delta_{\omega}(t_1, \ldots, t_n, \omega) \)

We want that

\[
\xymatrix{
& \omega \ar[r] & t \\
& x(x^*s) \ar[d] \ar[r]_{\delta_s} & x^*s \\
& x\psi \ar[d] \ar[r] & \psi \ar[d] \\
x's \ar[r]_{\delta'} & s' \\
(\psi(t_1), \ldots, \psi(t_n), \omega) \ar[r]_{\delta'_{\omega}(\psi(t_1), \ldots, \psi(t_n))} & \delta_{\omega}(\psi(t_1), \ldots, \psi(t_n)) 
}
\]

We must have \( \psi(t) = \delta_{\omega}(\psi(t_1), \ldots, \psi(t_n)) \) \( \tag{3} \)

\[
\vdots
\]

(1), (2) and (3) define inductively a unique dynamorphism \( \psi \) which uses \( f : S \rightarrow S' \) to relabel terminal nodes to form, from an \( \Omega_S \)-tree, a corresponding \( \Omega_{S'} \)-tree and run the \( X \)-dynamics \( \delta' \) on this tree to read out the result from the \( \Lambda \) node.

We show that a tree automaton is a machine in the category \( \mathcal{C} \).
Consider the machine in \( \mathcal{M} \)
\[ M = (X, S, \Omega, I, Y, \sigma, \delta, \lambda) \]
where
- \( X \) = the input-process just defined
- \( S \) = set of states
- \( \Omega \) = multigraded set of inputs
- \( I \) = set of initial states
- \( Y \) = set of outputs.

\((\Omega, \delta)\) is a \( \Omega \)-algebra

\[ \sigma: I \rightarrow S \] is the initial map

\[ \lambda: S \rightarrow Y \] is the output map.

By previous discussion the 2 diagrams

\[ \begin{array}{ccc}
i & \xrightarrow{\gamma} & (t_1, \ldots, t_n, \omega) \\
I & \xrightarrow{\sigma} & X^I = x_i \\
\downarrow \delta & & \downarrow x \downarrow \delta \\
S & \xrightarrow{\sigma} & \sigma(i) \\
& & (\delta(t_1), \ldots, \delta(t_n), \omega) \mapsto \delta^0(\delta(t_1), \ldots, \delta(t_n))
\end{array} \]

commute: with \( \delta^+ \) defined inductively by:

a) \( t(\Lambda) = i \in I \Rightarrow \delta^+(t) = \sigma(i) \)

b) \( t = \delta^0(t_1, \ldots, t_n, \omega) \Rightarrow \delta^+(t) = \delta^0(\delta^+(t_1), \ldots, \delta^+(t_n)) \)

i.e. if \( t' \) is the \( \Omega_S \) tree obtained by relabelling any terminal node labelled \( i \in I \) of an \( \Omega_I \)-tree \( t \) with the state \( \sigma(i) \in S, \delta^+(t) = t'(\Lambda) \), the run of \( M \) on \( t' \);

furthermore the external behavior of \( M \)

\[ EM(t') = \lambda. \delta^+(t) = \lambda(t'(\Lambda)) \]

is the evaluation of \( t' \) by \( M \).
If we put \( I = S, Q = \text{Id}_S \) our machine \( M \) is the tree automaton \((S, \Omega, Y, \delta, \lambda)\) of \( \mathcal{E} \).

Proposition 5. A tree automaton is an \( A \)-machine.

Proof: By above it is a machine in \( A \), we have to show that its external behavior \( EM \) satisfies the four postulates of \( \text{DN 12} \).

If \( \delta: XS \rightarrow S \) is an \( X \)-dynamics, by \( \text{DN 9} \)

\[
\delta^{(n)}: X^n S \rightarrow S \text{ is defined inductively by:}
\]

\[
\begin{align*}
\delta^{(0)} &= \text{Id}_S \\
\delta^{(n+1)} &= X^{n+1} S \xrightarrow{\delta} X^n S \xrightarrow{\delta^{(n)}} S
\end{align*}
\]

In the case at hand we have \( \forall S \in A \)

\[
XS = \{ (s_1, \ldots, s_n, \omega) / s_1, \ldots, s_n \in S \text{ and } \omega \in \gamma(\omega) \}
\]

i.e. any element of \( XS \) may be represented in the form

\[
\omega = (s_1, \ldots, s_n)^{\omega} \text{ or } \omega = (s_n)^{\omega} \text{ i.e. by elements of } X^n S.
\]

By induction on \( n \), we show that any element of \( X^n S \) is a tree of height at most \( n \), where any path of length \( n \) terminates at a node labelled with an element of \( S \), while any path of length less than \( n \) terminates at a node labelled with an element of \( \Omega_b \).

For \( n = 1 \) this is true by our above convention.
Assume it is true for \( n \).
\[ X^n S = X(x^n S) = \{ (t_1, \ldots, t_k, \omega) \} \text{ where} \]
\[ t_1, \ldots, t_k \in X^n S, \text{ i.e. are trees of length at most } n \]
\[ \text{with the required property concerning the labelling of terminal nodes,} \]
\[ \omega \in \Omega_k \]
\[ \therefore (t_1, \ldots, t_k, \omega) \]
\[ \begin{array}{cc}
\omega & \\
\Delta & \\
& t_1 \quad \cdots \quad t_k
\end{array} \]
is obviously a tree of length at most \( n + 1 \) with the required property.

**IN 27.**

We may represent any such tree (whose terminal nodes are labelled by \( s_1, \ldots, s_m \in S \) for a path of length \( n \))

for some \( n \in |N| \), and by \( \omega \in \Omega_k \) for any path of length \( k \leq n \).

by \( \Delta(s_1, \ldots, s_m) \). If \( \Delta \) is a tree obtained by replacing \( k \) of the labels \( s_i \) by \( s_i' \), \( 1 \leq k \leq m \),

we call \( \Delta \) a \( k \)-ary derived operator; in particular \( \Delta \) is unary if \( k = 1 \).

\[ \therefore \text{any element of } X^n(x^I) \text{ is of the form } \Delta(t_1, \ldots, t_n) \]

where the \( t_i \) are \( \bigcup I \)-trees and \( \delta^{(n)} : X^n(x^I) \rightarrow x^I \)

simply maps an \( \bigcup I \)-tree into the \( \bigcup I \)-tree obtained by

"unfurling" the \( \bigcup I \)-trees comprising the terminal nodes

of the \( \bigcup X^k I \)-trees.

**Example:** if \( n = 2 \) we have:

\[ \delta^{(2)} : X^2(x^I) \rightarrow X(x^I) \rightarrow x^I \]

\[ (t_{i_1}, t_{i_2}, \omega) \]

\[ \gamma(t_{m_1}, \ldots, t_{m_s}, \omega_m, \omega) \]

\[ \ni \in v(\omega_1) \quad s \in v(\omega_m) \quad m \in v(\omega) \]

\[ \omega_1 \ldots \omega_m \]

\[ \Delta_{t_{i_1}} \cdots \Delta_{t_{i_2}} \omega \]

\[ \Delta_{t_{m_1}} \cdots \Delta_{t_{m_s}} \omega_m \]
There is little risk of ambiguity in using the same notation for an $\times_1$-tree and the $\times_1$-trees obtained by applying $\delta_1^{(n)}$, to it.

A behavior $\beta$ is a map $\beta : X_1^* \rightarrow Y$.

Let $E_\beta = \{ (t, t') \in X_1^* \times X_1^* | \beta(\Delta(t)) = \beta(\Delta(t')) \}$

$\forall$ unary derived operators $\Delta$

$E_\beta$ is an equivalent relation as $\approx$ is.

Let $\alpha$ and $\gamma : E_\beta \rightarrow X_1^*$ be the usual projections; $\vdash X^\alpha : X^E_\beta \rightarrow X^n (X_1^*)$

$\Delta((t_1', t_1), \ldots, (t_n', t_n)) \rightarrow \Delta(t_1, \ldots, t_m)$

$X^\gamma : X^n E_\beta \rightarrow X^n (X_1^*)$

$\Delta((t_1', t_1), \ldots, (t_n', t_n)) \rightarrow \Delta(t_1', \ldots, t_m)$

$\delta_1^{(n)}$ applied to the trees so obtained "unfurls" them and reads out the corresponding output; $\therefore$ to show that

$\beta \cdot \delta_1^{(n)} (x^\alpha) = \beta \cdot \delta_1^{(n)} (x^\gamma)$

We prove that $\forall i = 1, \ldots, m$, $(t_i, t_i') \in E_\beta$

i.e. $\beta(\Delta(t_i')) = \beta(\Delta(t_i)) \forall$ unary derived operators $\Delta$

$\Rightarrow \beta(\Delta(t_1', \ldots, t_m)) = \beta(\Delta'(t_1', \ldots, t_m))$

$\forall m$-ary derived operators $\Delta'$ ($\ast$)

$\beta(\Delta(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_m)) = \beta(\Delta(t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_m))$

as this elementary translation is in fact a unary derived operator $\Delta_1$ and $\beta(\Delta_1(t_i)) = \beta(\Delta_1(t_i'))$

By using $m$ suitable $\Delta_1's$ we have the desired result.

$\therefore \beta$ satisfies postulate 1(a).
In particular we have that 
\((t_1, t_1') \in E_{\beta} \quad \forall \, i = 1, \ldots, m\) 
\[ \rightarrow \quad (\delta_i \omega^m (t_1, \ldots, t_m), (\delta_i \omega^m (t_1', \ldots, t_m')) \in E_{\beta} \quad \forall \omega \in \Omega \]
where 
\[ (\delta_i \omega^m (t_1, \ldots, t_m) = \quad \begin{array}{c}
\omega \\
\downarrow \\
\delta_i \\
\downarrow \\
\ldots \\
\downarrow \\
t_m \\
\end{array} \]
as (\#) valid \(V\)-ary derived operators 
\[ \rightarrow \quad \text{it is valid for the operator} \quad (\delta_i \omega^m \]
\[ \therefore E_{\beta} \text{ is a congruence.} \]

Assume a set \(R\) and maps \(R \xrightarrow{p} X^I\)
\[ \exists (\beta, (\delta_i \omega^m (x^p)) = \beta, (\delta_i \omega^m (x^{p'})) \]
We have just seen that this condition means that \(V\)-ary derived operator and \(V(r_1, \ldots, r_m)\)
\[ \beta(\Delta(p(r_1), \ldots, p(r_m))) = \beta(\Delta(p'(r_1), \ldots, p'(r_m))) \]
\[ \rightarrow \beta(\Delta(p(r))) = \beta(\Delta(p'(r))) \quad \text{if} \quad m = 1 \]
\[ \rightarrow (p(r), p'(r)) \in E_{\beta} \quad \forall r \in R \]
Let \(\varphi: \, R \rightarrow E_{\beta}\)
\[ r \mapsto \, (p(r), p'(r)) \]
We have \(p = \alpha , \varphi \) and \(p' = \gamma , \varphi \)
\[ \therefore \psi \exists, \alpha = \psi , \gamma \]
\[ \varphi , \gamma = \psi , \gamma , \varphi \]
i.e. \[ \psi , p = \psi , p' \]
\[ \therefore \beta \text{ satisfies postulate 1(b)} \]

If \(E_{\beta} = X^I / E_{\beta} \) = set of equivalence classes \([t]\)
with respect to the congruence \(E_{\beta}\), the usual coeq \((\alpha , \gamma)\)
\(\quad \) in \(O\) is
\[ \delta_\beta^+ : x^I \rightarrow s_\beta \]

\[ t \quad \mapsto \quad [t] \]

\[ \vdots \quad \beta \text{ satisfies postulate 2.} \]

As \( \beta \) is a congruence we can define \( \delta_\beta \) by

\[ (\delta_\beta)_{n} : s^n \rightarrow s \quad n \in \mathcal{V}(\omega) \]

\[ [t_1, \ldots, [t_n] ] \quad \mapsto \quad \sum (\delta_\beta)^{n}_{i} (t_1, \ldots, t_n) \]

This makes \( (s_\beta, \delta_\beta) \) an \( \omega \)-algebra, i.e. \( \delta_\beta \) is an \( \omega \)-dynamics, and \( \delta_\beta \) an \( \omega \)-dynamorphism as

\[ \omega \]

\[ n \in \mathcal{V}(\omega) \]

\[ X(x^I) \xrightarrow{\delta_\beta^+} x^I^+ \]

\[ \Downarrow \delta_\beta^+ \quad \text{commutes.} \]

\[ ( [t_1], \ldots, [t_n], \omega ) \quad \mapsto \quad \sum (\delta_\beta)^{n}_{i} (t_1, \ldots, t_n) \]

\[ \vdots \quad \beta \text{ satisfies postulate 3.} \]

In \( \mathcal{D} \) any coequalizer is a split-epimorphism and any functor \( X : \mathcal{D} \rightarrow \mathcal{D} \) preserves split-epimorphism as

\[ f : f' = \text{Id}_\beta \]

\[ \Rightarrow \quad (Xf) \cdot (Xf') = X(f \cdot f') \]

\[ \vdots \quad \beta \text{ satisfies postulate 4.} \]

\[ \vdots \quad \beta \text{ is an } A \text{-behavior.} \]

Putting \( \beta = E M = \lambda \). \[ \delta^+ \] we have that our tree automaton \( M \) is an \( A \)-machine.
CHAPTER III

DECOMPOSABLE SYSTEMS

In this chapter we will define a "system" in categorical terms and show that it is a particular case of machines in a category, thus establishing a link between automata and system theories. We give two examples of such systems: linear and group machines.
IN 28. A decomposable system in a category \( \mathcal{C} \) is a sextuple 
\((i, x, y, f, g, h)\) where \( x, u \) and \( y \in \mathcal{C} / \mathcal{C} / \) and 
\( f : x \rightarrow y \) is an \( \operatorname{Id}_\mathcal{C} \)-dynamics \( \xi \). 
\( g : u \rightarrow x \) is the initial state morphism \( \kappa \). 
\( h : x \rightarrow y \) is the output morphism \( \varepsilon \). 

We denote the system \((f, g, h)\) for short.

TH 4. \( \mathcal{C} \) a category \( \exists \exists \mathcal{C}^+ \) the countable copower of \( c \forall \mathcal{C} / \mathcal{C} / \) 
\( \Rightarrow \operatorname{Id}_\mathcal{C} \) is an input-process.

Proof: \( \operatorname{Id}_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \) is a functor i.e. a process 
an \( \operatorname{Id}_\mathcal{C} \) -dynamics is any morphism \( f \in \mathcal{C}(x, x) \) \( \forall x \in \mathcal{C} \). 
an \( \operatorname{Id}_\mathcal{C} \) -dynamorphism \( g : (f : x \rightarrow x) \rightarrow (f' : x' \rightarrow x') \) 
is an \( \theta \)-morphism \( g : x \rightarrow x' \). \( \exists \ g. f = f'. g \)

i.e. \( \exists \) \( \operatorname{Id}_\mathcal{C}^x = x \xrightarrow{f} x \)
\( \operatorname{Id}_\mathcal{C}^g = g \xrightarrow{g \text{ commutes}} \)
\( \operatorname{Id}_\mathcal{C}^{x'} = x' \xrightarrow{f'} x' \)

We have to show that the forgetful functor
\( U : \text{DYN} \operatorname{Id}_\mathcal{C} \rightarrow \mathcal{C} \) has a left-adjoint.
\( (f : x \rightarrow x) \xrightarrow{\varphi} x \)
\( (k : f \rightarrow f') \xrightarrow{\varphi} k : x \rightarrow x' \)

By lemma 4.1, \( \forall u \in \mathcal{C} / \mathcal{C} / \), \( \forall \) pair \( (f : x \rightarrow x, g : u \rightarrow x) \)
\( f \in \text{DYN} \operatorname{Id}_\mathcal{C} /, g \in \mathcal{C}(u, x), \exists \delta^+ \) uniquely defined by
\( \delta^+ = f \cdot g \quad \forall \delta \in \text{DYN} \)
There is a unique defined by $\text{in}_n z = \text{in}_{n+1} \forall n \in \mathbb{N}$ (see IN a 13.1).

$\Rightarrow \delta^+ \in \text{Dyn Id}_{\mathcal{C}} (z : u^+ \rightarrow u^+, f : x \rightarrow x)$

and $(z : U \rightarrow Uz = u^+, \text{in}_0 : u \rightarrow Uz = u^+)$

is a universal arrow from $u$ to $U$.

$\Rightarrow \; U$ has a left-adjoint (by TH a 5.2).

**Proposition 6.** A decomposable system $(f, g, h)$ in a category $\mathcal{C}$

$\exists \; U^+ \; \forall \; U \in \mathcal{C}$, is a machine

$S = (\text{Id}_\mathcal{C}, x, u, y, f, g, h)$.

**Proof:** $S$ is a machine in a category $\mathcal{C}$ with:

$\text{Id}_\mathcal{C} = \text{input process 'j'}$ TH 4.

$u = \text{initial state object}$

$x = \text{state object}$

$y = \text{output object}$

$f : x \rightarrow x$ is an $\text{Id}_\mathcal{C}$-dynamics

$g : u \rightarrow x$ is the initial state morphism

$h : x \rightarrow y$ is the output morphism

$(\delta^+_u = z : u^+ \rightarrow u^+, \eta_u = \text{in}_0 : u \rightarrow u^+)$

is the universal arrow from $u$ to $U$, $z$ defined by

$\text{in}_n z = \text{in}_{n+1} \forall n \in \mathbb{N}$; the object of inputs is $u^+ = U \delta_u$,

the reachability morphism is $\delta^+ : u^+ \rightarrow x$ defined by

$\delta^+_n \text{in}_n = f^+_n \text{g} \; \forall n \in \mathbb{N}$;

the external behavior is $E \; S = h \delta^+$

i.e. the unique morphism $u^+ \rightarrow y$ defined by

$E \; S \cdot \text{in}_n = h \cdot f^+_n \cdot g \; \forall n \in \mathbb{N}$. 

\( \text{in}_n \)
In the following a suitable category \( C \) is a category which has countable powers, copowers and coeq. mono factorization \( \forall \) morphisms.

**IN 29.** Observability map.

1. \( S \) is a decomposable system in a suitable category \( C \).
   The observability map of \( S \) is the morphism \( \omega : x \rightarrow y^x \) uniquely defined by \( p_n \cdot \omega = h \cdot y^n \) \( \forall n \in \mathbb{N} \).

\[
\begin{array}{ccc}
\omega & \uparrow \\
\downarrow h \cdot y^n & x & \rightarrow & y^x \\
\left\downarrow \omega \right. & \downarrow \\
y & \left\downarrow p_n \right.
\end{array}
\]

\( \omega \) commutes \( \forall n \in \mathbb{N} \), where \( (y^x, p_n) \) is the countable power of \( y \).

2. \( S \) is observable if \( \omega \) is monic.

**NOTE 11.** By lemma a 4.2, \( h = p_n \cdot \omega \) and \( \omega \) is an \( \text{Id}_y \) dynamorphism: \( (f : x \rightarrow x) \rightarrow (z' : y^x \rightarrow y^x) \).

where \( z' \) is uniquely defined by \( z' \cdot p_n = p_{n+1} \) \( \forall n \in \mathbb{N} \).

**IN 30.** The total external behavior of a decomposable system \( S \) in a suitable category \( C \) is the morphism \( \overline{ES} = \omega \cdot \delta^+ : u^+ \rightarrow y^x \).

**NOTE 12.** \( \delta^+ \) and \( \omega \) are \( \text{Id}_y \) dynamorphism \( \Rightarrow \overline{ES} \) is an \( \text{Id}_y \) dynamorphism: \( (z : u^+ \rightarrow u^+) \rightarrow (z' : y^x \rightarrow y^x) \).

The diagram

\[
\begin{array}{ccc}
u^+ & \xrightarrow{\delta^+} & x \\
\downarrow z & \xrightarrow{\omega} & y \\
u^+ & \xrightarrow{\delta^+} & x \\
\downarrow & \xrightarrow{\omega} & \text{commutes} \\
u^+ & \xrightarrow{\delta^+} & x & \xrightarrow{\omega} & y^x
\end{array}
\]
as $\delta^+$ a $\text{Id}_e$ dynamorphism \((z: u^+ \rightarrow u^+) \rightarrow (f: x \rightarrow x)\)

$\implies$ the left square commutes.

and $\omega$ a $\text{Id}_e$ dynamorphism \((f: x \rightarrow x) \rightarrow (z': y^x \rightarrow y^x)\)

$\implies$ the right square commutes.

\[ \text{2. } ES = \mu. \delta^+ = p_o. \omega. \delta^+ = p_o \cdot ES. \]

**DN 31.** Given a behavior $\beta : u^+ \rightarrow y$ in a suitable category, we uniquely define the corresponding total behavior

$\overline{\beta} : u^+ \rightarrow y^x$ by $p_n \cdot \overline{\beta} = \beta \cdot z^n$

i.e. $\exists \beta \cdot u^+ \rightarrow y^x \quad \overline{\beta} \uparrow \beta$ commutes $\forall n \in \mathbb{N}$.

Again we have $\beta = p_0 \cdot \overline{\beta}$ for $n = 0$.

**Lemma 5.** $f_1 : x_1 \rightarrow x_1, \ f_2 : x_2 \rightarrow x_2$ and $f_3 : x_3 \rightarrow x_3$ are $\text{Id}_e$ dynamics; $\exists \varphi : x_1 \rightarrow x_2, \ \psi : x_2 \rightarrow x_3$

$\exists \psi \cdot \varphi$ is an $\text{Id}_e$ dynamorphism: $f_1 \rightarrow f_2$ and is epi,

$\psi \cdot \varphi$ is an $\text{Id}_e$ dynamorphism: $f_1 \rightarrow f_3 \implies \psi$ is an $\text{Id}_e$ dynamorphism:

**Proof:** We have $\varphi \cdot f_1 = f_2 \cdot \varphi \quad \varphi \cdot f_2$ a $\text{Id}_e$ dynamorphism,

$\psi \cdot \varphi \cdot f_1 = f_2 \cdot \psi \cdot \varphi \quad \psi \cdot \varphi \cdot f_2$ a $\text{Id}_e$ dynamorphism,

$\therefore \psi \cdot f_1 = \psi \cdot \psi \cdot f_2 \cdot \psi = f_3 \cdot \psi \cdot \psi$

$\therefore \psi \cdot f_2 = f_3 \cdot \psi$ as $\varphi$ is epi

i.e. $\psi$ is a $\text{Id}_e$ dynamorphism.
TH 5. \( \beta : u^+ \rightarrow y \) is a behavior in a suitable category

\[ \exists \exists \text{ its total behavior } \beta \text{ is an } \text{Id}_\beta \text{ dynamorphism} \]

\[ (z : u^+ \rightarrow u^+) \rightarrow (z' : y^x \rightarrow y^x) \]

with a coequalizer-mono factorization

\[ \tilde{\beta} = u^+ \xrightarrow{\delta^+_u} x^\beta \xrightarrow{\omega_\beta} y^x \]

\[ \rightarrow \beta \text{ is an } A\text{-behavior}, \]

Proof: 1.a) By DN 11 and proof of proposition 6,

\[ \delta^{(n)}_u = z^n : u^+ \rightarrow u^+, \]

\[ \delta^{(n+1)}_u = \text{Id}_u \]

\[ \delta^{(n+1)}_u = (\text{Id}_z)^n u^+ \xrightarrow{\delta^{(n)}_u u^+} \]

\[ = u^+ \xrightarrow{z^n} u^+ \xrightarrow{\delta^{(n)}_u u^+} u^+ = \delta^{(n)}_u z^n \]

\[ \therefore \delta^{(n)}_u = z^n \forall n \in \mathbb{N}, \text{ by induction.} \]

Let \( \delta^+_\beta = \text{coeq } (\alpha, \gamma) \)

\[ \therefore \beta : \delta^+_u \cdot ((\text{Id}_\gamma)^n_\alpha) = \beta \cdot z^n \cdot \alpha \quad \text{above result} \]

\[ = P_n \cdot \beta \cdot z^n \cdot \alpha \quad \text{DN 31} \]

\[ = P_n \cdot \alpha \cdot \beta \cdot z^n \cdot \alpha \quad \text{hypothesis} \]

\[ = P_n \cdot \alpha \cdot \beta \cdot \gamma \quad \delta^+_\beta = \text{coeq } (\alpha, \gamma) \]

\[ = P_n \cdot \gamma \quad \text{DN 31} \]

\[ = \beta \cdot z^n \cdot \gamma \quad \text{hypothesis} \]

\[ = \beta \cdot \delta^{(n)}_u \cdot ((\text{Id}_\gamma)^n_\alpha) \quad \forall n \in \mathbb{N} \]

\[ \therefore \beta \text{ satisfies postulate 1a.} \]
b) Let \( r \in \mathcal{E} \) and \( \frac{r}{u^+} \). \( \mathcal{E} \) has a \( \delta \) of \( \mathcal{E} \):

\[
\beta, \delta\upsilon^{(n)}(\mathcal{E}) n p = \beta, \delta\upsilon^{(n)}(\mathcal{E}) n p',
\]

i.e., \( \beta, z^n p = \beta, z^n p' \)

\[
\implies r_n, \beta, p = r_n, \beta, p',
\]

by uniqueness in diagram:

\[
\implies \beta, p = \beta, p'
\]

\[
\text{hyposthesis: } \omega, \delta_{\rho}^+ p = \omega, \delta_{\rho}^+ p',
\]

\[
\implies \delta_{\rho}^+ p = \delta_{\rho}^+ p',
\]

\[
\omega, \text{ monic}
\]

\[
\forall \psi \exists \psi', \psi = \psi', \delta_{\rho}^+ \implies \text{coeq (}\alpha, \gamma\text{)} = \delta_{\rho}^+
\]

\[
\exists \text{unique } \psi', \psi = \psi', \delta_{\rho}^+ \implies \psi, p = \psi', \delta_{\rho}^+ p
\]

\[
= \psi, p
\]

and postulate 1b is satisfied.

2) By hypothesis \( \exists x_{\rho} \in \mathcal{E} \) and a \( \mathcal{C} \) morphism

\[
\delta_{\rho}^+ : u^+ \to x_{\rho} \exists \delta_{\rho}^+ = \text{coeq (}\alpha, \gamma\text{)}
\]

postulate 2 is satisfied.

3) \( \tilde{\beta} \) is an \( \text{Id}_{\mathcal{E}} \) dynamorphism (\( z : u^+ \to u^+ \))

\[
\implies \omega, \delta_{\rho}^+ z = \tilde{\beta}, z = \tilde{\beta}, z \implies \omega, \delta_{\rho}^+ z
\]

\[
= \omega, \delta_{\rho}^+ z
\]

\[
= \omega, \delta_{\rho}^+ z
\]

\[
= \omega, \delta_{\rho}^+ z
\]

\[
= \omega, \delta_{\rho}^+ z
\]
\[ \Rightarrow \delta^+_\beta \cdot z \cdot \alpha = \delta^+_\beta \cdot z \cdot \gamma \quad \text{as } \omega_\beta \text{ is monic} \]

\[ \exists \text{ unique } f_\beta : X_\beta \rightarrow X_\beta , \text{ a } \text{Id}_\beta \text{ dynamics} \]

\[ E_\beta \xrightarrow{\alpha} u^+ \xrightarrow{\delta^+\beta} X_\beta \]

\[ \Rightarrow \delta^+_\beta \text{ is a } \text{Id}_\beta \text{ dynamorphism } (z : u^+ \rightarrow u^+) \rightarrow (f_\beta : x_\beta \rightarrow x_\beta) \]

\[ \Rightarrow \text{ postulate 3 is satisfied.} \]

Besides \( \omega_\beta \cdot \delta^+\beta \) and \( \delta^+_\beta \) are \( \text{Id}_\beta \) dynamorphisms,

\( \delta^+_\beta \text{ epi (as a coequalizer)} \Rightarrow \omega_\beta \text{ is a } \text{Id}_\beta \text{ dynamorphism} \)

\[ (f_\beta : X_\beta \rightarrow X_\beta) \rightarrow (z' : y^x \rightarrow y^x) \quad \Rightarrow \text{ lemma 5} \]

4) Postulate 4 is satisfied as \( \text{Id}_\beta \), \( \delta^+_\beta = \delta^+_\beta \).

From proposition 6, TH 1, note 12(1), DN 22 and TH 5, it follows immediately:

**TH 6.1.** A decomposable system \( S = (f, g, h) \) in a suitable category \( C \), \( \exists \) its total external behavior \( ES \) has a coequalizer mono factorization \( ES = u^+ \xrightarrow{\delta^+_ES} \xrightarrow{g} y^x \)

is an A-machine.

2. Its Nerode realization is \( NES = (x_{ES}, u, y, f_{ES}, g_{ES}, h_{ES}) \)

where \( f_{ES} : x_{ES} \rightarrow x_{ES} \) is the unique morphism

\[ \exists \text{ unique } f_{ES} : x_{ES} \rightarrow x_{ES} \quad \text{commutes.} \]
\[ \varepsilon_{ES} = \delta_{ES}^{+} \cdot \eta_{u} = \delta_{ES}^{+} \cdot i_{n_{0}} \]

\( h_{ES} : x_{ES} \rightarrow y \) is the unique morphism

\[ \exists \quad R_{ES} \xrightarrow{\alpha} u^{+} \xrightarrow{\delta_{ES}^{+}} x_{ES} \]

\[ \xrightarrow{R_{ES}^{*}} \quad y \xrightarrow{h_{ES}} \quad y \]

commutes

\[ i.e. \quad ES = h_{ES} \cdot \delta_{ES}^{+} \]

\[ \rho_{o} \cdot \omega_{ES} \cdot \delta_{ES}^{+} = h_{ES} \cdot \delta_{ES}^{+} \quad \text{as} \quad ES = \rho_{o} \cdot \overline{ES} \]

\[ \Rightarrow h_{ES} = \rho_{o} \cdot \omega_{ES} \quad \text{as} \quad \delta_{ES}^{+} \quad \text{is epi} \]

3. NES is the minimal realization of ES.

**NOTE 13.** Similarly if we are given \( \beta : u^{+} \rightarrow y \) \( \exists \)

\( \beta \) is a \( \text{Id}_{g} \) dynamorphism \( z \rightarrow z' \) and has a coequalizer-mono factorization

\[ u^{+} \xrightarrow{\delta_{\beta}^{+}} x_{\beta} \xrightarrow{\omega_{\beta}} y^{x} \]

its Nerode realization is \( N\beta = (x_{\beta}, u, y, f_{\beta}, \varepsilon_{\beta}, h_{\beta}) \)

\[ \times \times \times \]

We will now discuss two examples of decomposable systems:

linear systems and group machines.

2. **LINEAR SYSTEMS**

We have already shown, in the general context of part II, that a linear machine \( M = (X, S, Y, Q_{S}, \delta, \lambda) \) as defined by IN 24 is an A-machine. Now we give the outline of the proof of this fact in the framework of decomposable systems.
Recall that $\delta : X \times S \rightarrow S$ is linear

\[
\delta(x, s) = f(s) + g(x) \quad \forall x \in X \text{ and } s \in S, \quad f \text{ and } g
\]

are linear maps.

Putting $u = X$, $x = S$, $y = Y$, $f = S \rightarrow S$, $g : X \rightarrow X$, $h = \lambda$, $M$ becomes $s \mapsto \delta(x, \lambda)$

a decomposable system $\Sigma = (f, g, h)$ in $R$-Mod.

$R$-Mod is a suitable category: $\forall \Sigma \in R$-Mod/

$X = \bigcup_{n=0}^{\infty} S_n$, the countable copower of $S = S_n$, $\forall n \in \mathbb{N}$,

$S^x = \prod_{n=0}^{\infty} S_n$, the countable power of $S$, i.e. the countable direct sum of the $S_n$'s, $S_n = S \forall n \in \mathbb{N}$, $\forall R$-Module homomorphism $t : S \rightarrow S'$ the usual coequalizer mono factorization

$t = S \xrightarrow{t} \text{Im} t \subseteq S/\ker t \xrightarrow{\text{onto}} S'$ where $t'$ is onto

$s \mapsto t(s) \xrightarrow{\text{onto}} t(s)$

(therefore a coequalizer $\xrightarrow{\text{TH a 3.5}}$ and $t''$ is 1-1 (i.e. monic $\xrightarrow{\text{TH a 2.2}}$).

We define the required $R$-Module homomorphism $x, \text{in}_n,$

$P_n, \delta^+, \omega, E\Sigma, E\Xi$ as follows:

$z : X^+ \rightarrow X^+$

$(x_0, \ldots, x_n, \ldots) \mapsto (0x_0, \ldots, x_n, \ldots)$

$\text{in}_n : X \rightarrow x^+$

$x \mapsto (0, \ldots, 0, x_n, \ldots)$ where $x$ is in the $n+1$st position,

$P_n : \Sigma^x \rightarrow \Sigma$

$(y_0, \ldots, y_n, \ldots) \mapsto y_n$

$\delta^+ : X^+ \rightarrow S$

$(x_0, \ldots, x_n, 0, \ldots) \mapsto \sum_{i=0}^{n} f_i g(x_i)$
\[ \omega : S \xrightarrow{\omega} Y^x \]
\[ \begin{array}{c}
\pi^i \quad (h_0, h.f(s), \ldots, h.f^n(s), \ldots) \\
\overline{E_\Sigma} = X^+ \xrightarrow{\delta^+} X \xrightarrow{\omega} Y^x \\
(x_0 \ldots x_n \ldots) \quad \sum_i h.f^i.g(x_i) \quad \sum_i h.f^{i+1}g(x_i) \ldots \\
\overline{E_\Sigma} = [\overline{E_\Sigma} = h. \delta^+ : X^+ \rightarrow Y] \\
(x_0 \ldots x_n \ldots) \quad \sum_i h.f^i.g(x_i) \\
\end{array} \]

Let \( S_{E_\Sigma} = X^+/\ker E_\Sigma \) i.e. \( \forall \omega \in x_0 \ldots x_n \ldots \subseteq X^+ \)

\[ [\omega] = \omega + \ker E \]

\[ \begin{array}{c}
\omega' \in X^+ / \overline{E_\Sigma} ( \omega ') = \overline{E_\Sigma} (\omega) \\
\iff E_\Sigma (0^n \omega') = E_\Sigma (0^n \omega) \quad \forall n \in \mathbb{N} \\
\end{array} \]

Then \( \overline{E_\Sigma} = X^+ \xrightarrow{\delta^+} S_{E_\Sigma} \xrightarrow{\omega_{E_\Sigma}} Y^x \)

\[ \omega \quad \mapsto \quad \underbrace{\omega^j \quad \rightarrow \quad (E_\Sigma (\omega), E_\Sigma (0\omega), E_\Sigma (00\omega), \ldots)}_{= (\sum_i h.f^i.g(x_i), \sum_i h.f^{i+1}g(x_i), \sum_i h.f^{i+2}g(x_i), \ldots)} \]

is the required coequalizer mono factorization for \( E_\Sigma \) to be an \( A \)-behavior. \( \vdash \) \text{TH 5.}

The external behavior \( E_\Sigma \) of \( \Sigma \) is identical to \( EM \) of \( M \)
and its Nerode realization \( NE_\Sigma = (f_{E_\Sigma}, g_{E_\Sigma}, h_{E_\Sigma}) \) same as \( NE \) of \( M \)
(as defined in chapter II).

Indeed we have:

\[ \begin{array}{c}
f_{E_\Sigma} : S_{E_\Sigma} \xrightarrow{\omega} S_{E_\Sigma} \\
([\omega]) \quad \mapsto \quad [0\omega] \\
\overline{E_\Sigma} = X^+ \xrightarrow{\delta^+} S_{E_\Sigma} \xrightarrow{\omega_{E_\Sigma}} Y^x \quad \sum_i h.f^i.g(x_i) \quad \sum_i h.f^{i+1}g(x_i) \ldots \\
x \quad \xrightarrow{\text{in}_0} \quad x^+ \quad \xrightarrow{\delta^+} \quad S_{E_\Sigma} \xrightarrow{\omega_{E_\Sigma}} \quad [x_0 \ldots] \\
\end{array} \]
\[ h_{E \Sigma} = s_{E \Sigma} \to y \]

\[ S_{E M} = s_{E \Sigma} \]

\[ \delta_{E M} : x \times s_{E M} \to s_{E M} = f_{E \Sigma} + \epsilon_{E \Sigma} \]

\[ (x, [w]) \to [x w] \]

\[ \delta_{E M}^+ = \delta_{E \Sigma}^+ \quad \text{and} \quad \lambda_{E M} = h_{E \Sigma} \]

\[ x \times x \]

3.

GROUP MACHINES.

**Proposition 32.** \( M = (X, S, Y, \delta, \lambda) \) is a group machine if \( X, S, Y \) are groups and \( \delta : X \times S \to S \) with \( X \times S = \) direct product of \( X \) and \( S \) and \( \lambda : S \to Y \) are group-homomorphisms.

**Example:** Let \( X = S = Y = G \) a group

\[ \delta = \circ \] the operation of \( G \), \( \lambda = \text{Id}_G \]

then our machine \( M \) is simply the group \( G \).

**Proposition 7.** \( M \) is an A-machine.

**Proof:** We have to show that:

- the category \( \text{Gr} \) of groups and group homomorphisms is a suitable category;
- \( M \) is a decomposable system in \( \text{Gr} \) with a coequalizer.
- mono factorization.
1. \( \forall G \in \mathfrak{Gr}, \ G^+ = \bigsqcup_{i \in \mathbb{N}} G_i, \ G_i = G \ \forall i \in \mathbb{N}, \)

is called the free product and is defined in the following way:

Its elements are strings of the form
\[(e_{i_0}, i_0)(e_{i_1}, i_1) \ldots (e_{i_n}, i_n), \ e_{i_j} \in G, \ i_j \in \mathbb{N}, \]
with the empty string denoted by \( \Lambda \), and subject to the restrictions:
(i) no \( e_{i_j} = 1_G \) the identity;
(ii) \( \forall j : 0 \leq j \leq n, \ i_j \neq i_{j+1} \)

The product in \( G^+ \) is concatenation, save that if the string so formed does not satisfy conditions (i) and (ii) we apply the following:
(a) replace consecutive elements of the form
\[(g, n) (g', n) \]
by \( (g g', n) \);
(b) delete elements of the form \( (1_G, n) \) until a string meeting conditions (i) and (ii) is obtained.

The injections are \( \forall j : \mathbb{N} \rightarrow G^+ \)
\[\text{in}_j : G \rightarrow G^+ \]
\[g \rightarrow (g, j) \]
\[1_G \rightarrow \Lambda \]

\[ G^\times = \prod_{i \in \mathbb{N}} G_i, \ G_i = G \ \forall i \in \mathbb{N} \]
is the group of countable tuples \( (e, \ldots, e_j, \ldots), \ e \in G \ \forall j \in \mathbb{N}, \)
with the product defined component wise by the product in \( G \). The projections are
\[ P_j(e, \ldots, e_j, \ldots) = e_j \]
\( G^\times \) is called the direct product.
\[ \forall h : G \rightarrow G', \text{ a group homomorphism} \]
\[ \text{Im} h \cong \frac{G}{\text{Ker } h} \text{ is a subgroup of } G', \]

We have the usual factorization:
\[ G \xrightarrow{h'} \text{Im } h \xrightarrow{\text{in}} G' \]
\[ g \xmapsto{\rightarrow} h(g) \xmapsto{\rightarrow} h'(g) \]

where both maps are group homomorphisms, \( h' \) is optimal, therefore a coequalizer (\( \Rightarrow \) Thm 3.5) and in
\( 1 \xrightarrow{1} 1 \), therefore monic (\( \Rightarrow \) Thm a 2.2).

\( Gr \) is a suitable category.

2. \( \delta : X \times S \rightarrow S \) is a group homomorphism.

\[ \delta(x, s) = \delta((1_x, s)(x, l_s)) = \delta((1_x, s)) \delta(x, l_s) = f(s)g(x) \]

where \( f : S \rightarrow S \), \( g : X \rightarrow S \).

\[ s \mapsto \delta((1_x, s)) \quad x \mapsto \delta(x, l_s) \]

\( f \) and \( g \) are group homomorphisms as \( \delta \) is. Conversely, if \( f : S \rightarrow S \) and \( g : X \rightarrow S \) are group homomorphisms then

\[ \delta : X \times S \rightarrow S \text{ is certainly one.} \]

\[ (x, s) \mapsto g(x) \]

Let's consider our group machine as the decomposable system \( (S; X, \gamma, f, g, h = \lambda) \), i.e. the machine
\( (1_{Gr}, S, X, \gamma, f, g, h) \) in \( Gr \).

\[ z : X^+ \rightarrow X^+ \]
\[ (x_0, i_0) \ldots (x_n, i_n) \mapsto (x_0, i_0 + 1) \ldots (x_n, i_{n+1}) \]
is such that \( \forall n \in \mathbb{N}, \forall x \in X \).
\[ z \cdot \text{in}_n(x) = z(x, n) = (x, n+1) = \text{in}_{n+1}(x) \] as required.

The reachability morphism is
\[ \delta^+: x^t \longrightarrow s \]
\[ \Lambda \longrightarrow l_s \]
\[ (x_{i_0}, i_0)(x_{i_n}, i_n) \longrightarrow \prod_{j=0}^{n} f_{i,j} g(x_{i,j}) \]
where \( \prod \) is the product in \( s \).

NOTE. That \( \cdot \) is always used in this paper to indicate map composition, never product in the various groups with which we are dealing. If \( \circ \) is the operation of \( G \), then \( g \circ g' \) is simply written \( gg' \).

As required \[ \forall n \in \mathbb{N} \]
\[ \delta^+ \cdot \text{in}_n(x) = \delta^+(x, n) = r^n g(x) \forall x \in x \]
i.e. \[ \delta^+ \cdot \text{in}_n = r^n g \]
and both following diagram commutes:
\[ x \longrightarrow (x, 0) \]
\[ (x_{i_0}, i_0) \cdots (x_{i_n}, i_n) \longrightarrow (x_{i_0, i_0+1}) \cdots (x_{i_n, i_n+1}) \]
\[ x \xrightarrow{\text{in}_0} x^+ \]
\[ \delta^+ \]
\[ g(x) = r^0 g(x) \]
\[ \prod_{j=0}^{n} f_{i,j} g(x_{i,j}) \]
\[ \longrightarrow \prod_{j=0}^{n} f_{i,j+1} g(x_{i,j}) \]

Let \( \omega : S \longrightarrow \mathbb{R}^k \)
\[ s \longmapsto (h(s), \ldots, h \circ^n (s), \ldots) \]
We have again as required:
\[ p_n \cdot \omega(s) = p_n \cdot \omega(s) = h \cdot \omega(s) \]
\[ \forall s \in S, \]
i.e. \( p_n \cdot \omega = h \cdot \omega \),
\[ \overline{ES} : x^+ \xrightarrow{\delta^+} s \xrightarrow{\omega} y \]
\[ (x_0^+, i_0) \ldots (x_n^+, i_n) \xrightarrow{\prod_i \omega_i(x_i^+)} \prod_j h_{j+1}(x_j^+) \]
\[ \xrightarrow{\prod_j h_{j+1}(x_j^+), \prod_j h_{j+1+1}(x_j^+), \ldots} \]

and \( ES = p_0 \cdot \overline{ES} = \prod_j h_{j+1}(x_j^+) = \lambda \cdot \delta^+ \) as
\[ \lambda = h \) is a group homomorphism.

\[ \text{NOTE 14.1} \]
Recall that, in classical automata theory, \( \delta^+ \) is defined inductively by
\[
\begin{align*}
\delta^+(x^+) &= 1_S \\
\delta^+(x, w) &= \delta(x, \delta^+(w)) & \forall x \in X, \forall w \in X^+
\end{align*}
\]

We show by induction the length of \( w \) that this \( \delta^+ \)
yields the same result as the one we have just obtained.
\[ \forall w = x_0 \ldots x_n \text{ can be written } (x_0, 0) \ldots (x_n, n) \]
making \( X^* \) a subset of \( X^+ \), but not a subgroup as \( X^+ \)
is not a group. Indeed \( X^* \cong (X^+)^* \subset X^+ \) where
\[ (x^*) = \{ \wedge \} \cup \{ (x_0^+, i_0) \ldots (x_n^+, i_n) \mid i_0 < i_1 < \ldots < i_n \} \subset X^+ \]
\[ \theta : X^* \xrightarrow{\wedge} \wedge (x^+) \quad \text{is 1-1 as it has an} \]
\[ \wedge \xrightarrow{\wedge} \wedge \]
\[ (x_0 \ldots x_n) \xrightarrow{\wedge} (x_0, 0) \ldots (x_n, n) \]
Inverse $\delta^{-1}(x^+ : x^*) \longrightarrow x^*$

$\land \longrightarrow \land$

$(x_0, i_0) \cdots (x_n, i_n) \longmapsto x_0 \cdots x_n$

where $\forall j = 0, \ldots, n-1$

$x_j = x_i^+$ \quad if \quad $j = i_k$

$x_j = 1_x$ \quad otherwise

Example: $\theta^{-1}(x_0, 0) (x_2, 2) (x_5, 5)) = x_0 1_x x_2 1_x 1_x x_5 \in x^*$

For $/w/ = 0$, i.e. $w = \land$ this is true by DN.

Assume it is true for $/w/ = n; \forall w 3/w/ = n + 1$

may be written as $w = \chi w'$ for some $\chi \in X$

and $w' = x_0 \cdots x_{n-1} \in x^*$

$\therefore \delta^+(w) = \delta(x, \delta^+(w')) \quad : \quad \text{DN}$

$= \delta(x, \prod_{i=0}^{n-1} f^i \cdot g(x_i))$

$= f(\prod_{i=0}^{n-1} f^i \cdot g(x_i)) g(x) \quad \text{as} \quad \delta(x, s) = f(s) \cdot g(x)$

$= \prod_{i=0}^{n-1} (f^{i+1} \cdot g(x_i)) g(x) \quad \text{as} \quad f \text{is a group homomorphism.}$

$\therefore x = x_0 \cdots x_n \quad x_0 = x_1 \cdots \quad x_{n-1} = x_n$

Q. We show that $\delta^+$ is a group homomorphism.

Case 1. Let $W_1 = (x_{j_0}, i_0) \cdots (x_{j_n}, i_n)$ and

$W_2 = (x_{k_0}, k_0) \cdots (x_{k_m}, k_m) \quad \exists \quad j_{n+1} = k_0$

$\therefore W_1 W_2 = (x_{j_0}, i_0) \cdots (x_{j_n}, i_n) (x_{k_0}, k_0) \cdots (x_{k_m}, k_m)$

$= (x_{j_0}, i_0) \cdots (x_{j_n}, i_n) \cdot (x_{j_{n+1}}, i_{n+1}) \cdots (x_{j_{n+m}}, i_{n+m})$
If we relabelled $x_i^\ell = x_i^{\ell n}$ for $\ell = 0, \ldots, n$
$x_i^{\ell n+\ell+1} = x_i^\ell$ for $\ell = 0, \ldots, m$.

$$\delta^+(W_1, W_2) = \bigotimes_{\ell=0}^{n+m} f^{\ell n+\ell+1} g(x_i^\ell)$$
$$= \bigotimes_{\ell=0}^{n} f^{\ell n} g(x_i^\ell) \bigotimes_{\ell=0}^{m} f^{\ell} g(x_i^{\ell+1})$$
$$= \delta^+(W_1) \delta^+(W_2).$$

**Case 2.** Let $W_1 = (x, n)$, $W_2 = (x', n)$, $x' \neq x^{-1}$

$$\delta^+(W_1, W_2) = f^n g(x x')$$
$$= f^n g(x)^g(x')$$  as $g$ is a homomorphism
$$= f^n (g(x)) f^n (g(x'))$$  as $f$ is a homomorphism
$$= \delta^+(W_1) \delta^+(W_2).$$

**Case 3.** Let $W_1 = (x, n)$, $W_2 = (x^{-1}, n)$

$$\delta^+(W_1, W_2) = \delta^+(1_g, n)$$
$$= \delta^+(1_g, 1_g) = 1_g$$  as $(1_g, n)$ must be deleted.

$$\delta^+(W_1) \delta^+(W_2) = f^n g(x) f^n g(x^{-1})$$
$$= f^n (g(x) (g(x))^{-1})$$  as $f$ and $g$
$$= f^n (1_g) = 1_g.$$

Again we have $\delta^+(W_1, W_2) = \delta^+(W_1) \delta^+(W_2).$
Along similar lines we can show that \( \omega \) is a group homomorphism.

\[ \forall \text{ behavior } \beta : X^+ \to Y, \text{ i.e. } \beta \text{ is a group homomorphism,} \]

\[ \text{let } E_\beta = \{ (w_1, w_2) / w_1, w_2 \in X^+, \beta . z^n(w_1) = \beta . z^n(w_2) \forall n \in \mathbb{N} \} \]

\( E_\beta \) is an equivalence relation as = is and \( \beta \) and \( z^n \)
are well defined. \( \beta \) and \( z^n \), therefore \( z^n \), are group homomorphisms \( \Rightarrow \beta . z^n \) is one

\[ \Rightarrow E_\beta \text{ is a congruence.} \]

Let \( \delta^+, \gamma^+, E_\beta \to X^+ \) be the usual projections,
\( \delta^+, E_\beta \to X^+ \) whose elements are \( E_\beta \) equivalence classes
\( [w] \), is a group with usual product \([w_1][w_2] = [w_1w_2] \)
which is well defined as \( E_\beta \) is a congruence.

\[ \Rightarrow \delta^+: X^+ \to S_\beta^+, \text{ is a group homomorphism.} \]

\[ \omega : S_\beta \to \gamma^+ \]

Besides \( \delta^+ = \text{coeq}(\alpha, \gamma) \) as \( \delta^+.\alpha = \delta^+.\gamma \)
and \( \forall \varphi: X^+ \to S, \varphi.\alpha = \varphi.\gamma \)
\( \exists \varphi: S_\beta \to S \)

\[ [w] \mapsto \varphi(w) \]

uniquely defined \( \varphi = \psi, \delta^+ \)

\[ \omega_\beta : S_\beta \to \gamma^+ \]

\[ [w] \mapsto (\beta(w), \beta . z(w), \beta . z^2(w), \ldots) \]

is a group homomorphism as \( \beta . z^n \) is one \( \forall n \in \mathbb{N} \)
and 1-1, i.e. monic by DN of \( S_\beta ) ( [w] = [w'] \iff \beta . z^n(w) = \beta . z^n(w') ) . \]
Now $\beta = p_{\omega} \cdot \delta_{\rho}^+ = p_{\omega} \cdot \tilde{\beta}$, where $\tilde{\beta} = \omega_{\rho} \cdot \delta_{\rho}^+$ is a coequalizer mono factorization.

The following diagram commute

\[
\begin{array}{ccc}
w & \xrightarrow{\chi^+} & z(w) \\
\downarrow \phi & & \downarrow \delta_{\rho}^+ \\
[w] & \xrightarrow{S} & [z(w)] \xrightarrow{(y_0,y_1,y_2,\ldots)} (y_1,y_2,\ldots)
\end{array}
\]

where $z : Y^x \rightarrow Y^x$ is such that $p_n \cdot z = p_{n+1}$ $\forall n \in \mathbb{N}$.

\[
(\beta(w), \beta \cdot z(w), \beta \cdot z^2(w), \ldots) \xrightarrow{\sim} (\beta \cdot z(w), \beta \cdot z^2(w), \ldots)
\]

$\therefore \beta = \omega_{\rho} \cdot \delta_{\rho}^+$ is an Id$_{Gr}$ dynamorphism $z \xrightarrow{\sim} z'$

$\therefore \beta$ is an $A$-behavior $\therefore$ TH 5.

$\therefore M$ is an $A$-machine with $\beta = EM = \lambda \cdot \delta_{\rho}^+$ a group homomorphism as $\delta_{\rho}^+$ and $\lambda$ are.

$\therefore$ By TH 6, $NEM = (x_{EM}^1, X, Y, f_{EM}^1, \varepsilon_{EM}, h_{EM})$

where $f_{EM} : x_{EM} \rightarrow x_{EM} \triangleright \delta_{EM}^+ : = f_{EM} \cdot \delta_{EM}^+$

\[
[w] \xrightarrow{\phi(w)}
\]

$\varepsilon_{EM} : \xrightarrow{\varepsilon_{EM}} x_{EM} \triangleright \varepsilon_{EM} = \delta_{EM}^+ \cdot in_o$

\[
x \mapsto [(x, 0)]
\]

$h_{EM} : x_{EM} \rightarrow Y \triangleright h_{EM} \cdot \delta_{EM}^+ = EM$

\[
[w] \mapsto EM(w) \quad \text{and} \quad h_{EM} = p_\omega \cdot \omega_{EM}
\]
OBSERVABILITY AND MINIMALITY

We now return to the general concept of machines.

In the context of set theory, observability is defined as follows:

M = (X, S, Y, δ, λ) is an automaton

δ*: X x S → S is defined inductively by

δ*(λ, s) = s ∀ s ∈ S

δ*(wx, s) = (x, δ*(w, s)) ∀ s ∈ S, ∀ w ∈ X∗ the

free monoid of words from S.

M is observable (or reduced) if

ω: S → YX∗ is 1-1

s → λ.δ(λ, s)

i.e. λ.δ(w, s) = λ.δ*(w, s′) ∀ w ∈ X∗ ⇔ s = s′

As already seen if we choose σ ∈ S as initial element,

M is an Ω-machine in Δ with

i = ∅ and σ: i → S

σ(σ) = σ
Recall DN 19. M is a minimal realization of a behavior $\beta$ if M is a terminal object in the subcategory $\mathcal{M}_\beta$ of reachable realizations of $\beta$.

Proposition 9. If M is a reachable automaton

$$M \text{ is observable } \iff M \text{ is minimal}.$$  

Proof: M reachable $\implies \delta^+ : X^* \to S$ is onto ($\iff$ a coequalizer in $\mathcal{A}$) $\implies \forall s \in S \exists \nu \in X^* \exists s = \delta^+(\nu) \implies \delta^+(w, s) = \delta^+(w \nu)$ ($\ast$)

Indeed, by induction on the length $|w|$ of $w$, we have:

$|w| = 0 \implies w = \lambda \implies \delta^+(\lambda, s) = s$ :: IN of $\delta^+$

$= \delta^+(\nu)$ :: above

$= \delta^+(\Lambda \nu)$ :: IN of $\Lambda$.

Assume this result is true for $n = |w|$

$|w'| = n + 1 \implies w' = xw$. for some $x \in X$

$\delta^+(w', s) = \delta^+(xw, s)$ :: $w' = xw$

$= \delta^+(x, \delta^+(w, s))$ :: IN of $\delta^+$

$= \delta^+(x, \delta^+(\nu))$ :: induction assumption

$= \delta^+(xw\nu)$ :: IN of $\delta^+$

$= \delta^+(w'\nu)$ :: $w' = xw$.

($\iff$) Let $b : S \to S_{EM} = X^*/E_{EM}$ (recall $EM = \lambda$, $\delta^+$)

$$s = \delta^+(\nu) \iff [\nu]$$

This map is well defined and 1-1 as:

M observable then by DN 33

$\forall w \in X^*, \lambda. \delta^+(w, s) = \lambda. \delta^+(w, s') \iff s = s'$

i.e. $\lambda. \delta^+(w\nu) = \lambda. \delta^+(w\nu')$ :: $s = s'$
where \( s = \delta^+(v) \), \( s' = \delta^+(v') \)
\[
\therefore [v] = [v'] \iff s = s' \\
\therefore b \text{ is onto as } \forall v \in S \exists \delta^+(v) \in S
\]
\( \therefore b \) is an isomorphism,
\[
\lambda (\Id_x^*, b, \Id_y) : M \longrightarrow \text{NEM is a machine isomorphism, i.e. } M \cong \text{NEM}
\]
\[\Rightarrow M \text{ is terminal in } \mathcal{M}_2 \text{ as NEM is,} \]
\[\Rightarrow M \text{ is minimal } \subseteq \text{ IN 19.} \]

\((\Longleftrightarrow) \text{ M is minimal } \Longleftrightarrow M \text{ is terminal in } \mathcal{M}_2 \text{ as NEM is,} \]
\[\Rightarrow \exists \text{ a machine isomorphism } (\Id_x^*, b', \Id_y) : M \longrightarrow \text{NEM} \]
\[\text{i.e. } b' \cdot \delta = \delta^+ \text{ EM}, \]

and \( b' : S \longrightarrow S \text{ EM is an isomorphism. } \delta^+ \text{ is onto as } M \in /\mathcal{M}_2 \Rightarrow M \text{ reachable } \Rightarrow \delta \text{ a coequalizer (} \Longleftrightarrow \text{ onto in } \mathcal{D} \). \]

Again let \( b : S \longrightarrow S \text{ EM} \)
\[
s = \delta^+(v) \longrightarrow [v] \\
\]
as already seen \( b \) is well-defined as
\[
s = s' \Rightarrow \forall w \in x^*, \lambda . \delta^+(w, s) = \lambda . \delta^+(w, s') \\
\Rightarrow \lambda . \delta^+(w v') = \lambda . \delta^+(w, s') \Rightarrow \text{ (*)} \\
\Rightarrow [v] = [v'] \\
\Rightarrow b(s) = b(s') \\
\Rightarrow \forall v \in x^*, b'. \delta^+(v) = \delta^+ \text{ EM (v) = } [v] = b(\delta^+(v)) \\
\Rightarrow b'. \delta^+ = b . \delta^+ \\
\Rightarrow b' = b \text{ as is onto i.e. epi.} \]
\[ \forall s \in S, s = \delta^*(v) \text{ for some } v \text{ as } \delta^* \text{ onto } \]
\[ = b^{-1}(v, v') \text{ as } b \text{ is an isomorphism.} \]

\[ \forall w \in X^*, \lambda, \delta^*(w, s) = \lambda, \delta^*(w, s') \Rightarrow \]
\[ \lambda, \delta^*(w, v) = \lambda, \delta^*(w, v') \quad (*) \]
\[ \Rightarrow b^{-1}(v, v') = b^{-1}(v', v) \]
\[ \Rightarrow s = s' \]

\[ M \text{ is observable} \quad \text{IN 33} \]

In the case of decomposable systems / IN 29.2 gives us a definition of observability in categorical terms.

We first verify that for linear and group machines IN 29 and IN 33 are indeed equivalent.

Let \( M = (X, S, Y, \delta, \lambda) \) be a linear machine. Recall from previous discussion that this is a linear system \((X, S, Y, f, g, h)\) with \( \delta(x, s) = f(s) \cdot g(x) \) and \( \lambda = h \)

where \( \omega : S \rightarrow Y^+ \)
\[ s \mapsto (h(s), h.f(s), \ldots, h.f^n(s), \ldots) \]

We prove by induction on \( \|w\| \) that \( w = x_0 \ldots x_n \)

\[ \Rightarrow \delta^*(w, s) = f^{n+1}(s) + \sum_{i=0}^{n} f^i.g(x_i) \]

If \( \|w\| = 1 \) i.e. \( w = x_0 \) we have \( \delta^*(x_0, s) = \delta(x_0, \delta^*(\Lambda, s)) \)

\[ \Rightarrow \delta^*(x_0, s) = \delta(x_0, s) \text{ as } \delta^*(\Lambda, s) = s = f^0(s) \]

\[ = f(s) + g(x_0) = f^1(s) + f^0.g(x_0) \text{ as required.} \]
Assume our result true for words of length n.

Let $w = x_0 \ldots x_n$ i.e. $|w| = n + 1$

$\delta^*(w, s) = \delta^*(x_0, \delta^*(x_1 \ldots x_n, s))$ : IN of $\delta^*$

$= \delta^*(x_1 \ldots x_n, s) + g(x_0)$ : $\delta = f + g$

$= \left\{ f^m(s) + \sum_{i=1}^{n+1} f^i g(x_1), s(x_0) \right\} : |x_1 \ldots x_n| = n$

$= \lambda^{n+1}(s) + \sum_{j=0}^{n+1} f^j g(x_1)$ : $f$ ans $g$ linear.

Let $M$ be observable in the sense of DN 33

$\iff \left[ \lambda \cdot \delta^*(w, s) = \lambda \cdot \delta^*(w, s') \forall w \in X^* \iff s = s' \right]$  

$\iff \left[ h \cdot f^{n+1}(s) + \Sigma h \cdot f^i g(x_1) = h \cdot f^{n+1}(s') + \Sigma h \cdot f^i g(x_1) \forall n \in \mathbb{N} \right]$  

and $\lambda \cdot \delta^*(\lambda, s) = \lambda(s) = h(s) = h(s') \iff s = s'$

$\iff \left[ h \cdot f^n(s) = h \cdot f^n(s') \forall n \in \mathbb{N} \iff s = s' \right]$  

$\iff \left[ \omega(s) = (h(s), h.f(s), \ldots) = \omega(s') \iff s = s' \right]$  

$\iff \omega$ is $\mathbb{N}^1$  

$\iff \omega$ is monic in $\mathcal{A}$

$\iff M$ is observable in the sense of DN 29.

For group machines, the proof is similar with $\delta^*(w, s) = f^{n+1}(s) \sum_{j=0}^{n+1} f^j g(x_1)$.

The following corollary of TH 6 gives us the same result, as for general automata.
Corollary: \( S = (f, g, h) \) is the minimal realization of its behavior \( ES \iff S \) is reachable and observable.

**Proof:**

(\( \iff \)) Assume \( S \) is reachable and observable i.e.

\[ ES = \omega : \delta^+ \exists \delta^+ \text{ is a coequalizer and } \omega \text{ monic.} \]

By Th 6 putting \( x_{ES} = x, f_{ES} = f, g_{ES} = g, h_{ES} = h \), we have \( S = NES \) i.e. it is a minimal realization of \( ES \), its behavior.

(\( \implies \)) \( S \) is the minimal realization of \( ES = P_0 : \omega : \delta^+ \implies : \text{DN 19} \) \( S \) is the terminal object in the category of reachable realizations of \( ES \)

\[ \implies \delta^+ \text{ is a coequalizer} \]

and \( \exists (\text{Id}_{u+}, b, \text{Id}_{y}) : S \rightarrow NES \exists b : x \rightarrow x_{ES} \)

is an isomorphism and \( \omega = \omega_{ES} \cdot b \)

\[ \implies \forall f, g \exists \omega . f = \omega . g \]

\[ \implies \omega_{ES} \cdot b . f = \omega_{ES} \cdot b . g \]

\[ \implies b . f = b . g \]

\( \omega_{ES} \) is monic

\[ \implies f = g \]

\( b \) is an isomorphism.

\[ \implies \omega \text{ is monic and } S \text{ is reachable and observable.} \]

\( \therefore \) In all cases at hand we have that:

\( \forall \) reachable machines \( M \), minimal is equivalent to observability (as defined in DN 19 and DN 33 respectively).

\( \therefore \)
The following detailed study of the Arbib and Manes theory of observability for categorical machines as exposed in [9] will show that this result is more generally true.

**IN 34.** A functor \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) is:

1. **An INPUT PROCESS** if the forgetful functor \( U : \text{DYN} \to \mathcal{C} \) has a **LEFT-ADJOINT** (recall of IN 5).

2. **An OUTPUT PROCESS** if \( U \) has a **RIGHT ADJOINT**.

3. **A STATE BEHAVIOR PROCESS** if it is both an input and an output process.

**NOTE 15.** \( \mathcal{F} \) an output process \( \implies \) by TH a 5.3

\[ \forall c \in \mathcal{C} \exists \left( \Delta_c : X(x_c) \to x_c, \Lambda_c : x_c \to c \right) \]

universal from \( U \) to \( c \) i.e. such that

\[ \forall \left( \delta' : x_{c'}, \to c' \right) \exists \text{ a unique } \quad \delta : x_c \to \Delta_c \quad \exists \Lambda_c, \delta = \delta' \]

i.e. \( \exists \) the two following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Lambda_c} & X_c \\
\downarrow{\phi} & & \downarrow{\phi'} \\
\mathcal{C} & \xrightarrow{\Delta_c} & X_{c'}
\end{array}
\]

\[
\begin{array}{ccc}
x_c & \xrightarrow{\Delta_c} & x_c' \\
\uparrow{\phi} & & \uparrow{\phi'} \\
x_{c'} & \xrightarrow{\Lambda_c} & x_{c'}
\end{array}
\]

\[ X : U \cdot V \quad (\text{where, } U \cdot V : \text{Dyn } X \to \mathcal{D}) \]

is a functor as it is a composition of 2 functors.
If \( \sigma = s \), a state object, \( \sigma = \text{Id}_s \), then \( \sigma \) is called the state behavior morphism \( \sigma : s \rightarrow X_s \). The unique morphism \( \sigma \Lambda_s \circ \sigma = \text{Id}_s \).

**DN 35.**

**X : \mathcal{E} \rightarrow \mathcal{C}** is an output process,

\( M = (X, s, y, \delta : X_s \rightarrow s, \lambda : s \rightarrow y) \) has:

1. **response object** = \( \Lambda_Y \)
2. **observability map** \( \omega : s \rightarrow X_{\delta} \) is the unique X dynamorphism \( \exists k = \Lambda_Y \omega \), i.e., the following diagrams commute

\[
\begin{array}{c}
\xymatrix{y 
& X_y 
& X(Xy) 
\downarrow \omega & X_y 
\downarrow \omega \\
& s 
& X_s 
\downarrow \delta & s}
\end{array}
\]

\( \lambda \)

**DN 36.**

**X' : \mathcal{E} \rightarrow \mathcal{C}** is a state behavior process,

\( M = (X, s, i, y, \delta, \sigma, \lambda) \) is a machine in \( \mathcal{C} \).

1. The **total external behavior** of \( M' \) is \( \overline{EM} = \omega \cdot \delta^* : x^* i = x^* \rightarrow X_y \in \text{DYN } X \).

2. Conversely \( M \) is a realization of a given \( \overline{\beta} : x^* \rightarrow X_{\lambda} \in \text{DYN } X \) if \( \overline{EM} = \overline{\beta} \).

3. Given \( \overline{\beta}, \beta = \Lambda_Y \overline{\beta} : x^* \rightarrow y \) is the corresponding behavior and, conversely given a behavior \( \beta : x^* \rightarrow y \exists \) a unique total behavior \( \overline{\beta}, \exists \Lambda_Y \overline{\beta} = \beta \) and \( \overline{\beta} \in \text{DYN } X \).

4. \( M \) is a minimal realization of \( \overline{\beta} \) if \( M \) is a reachable realization of \( \overline{\beta} \) and \( \forall M' \), a reachable realization of \( \overline{\beta} \), \( \exists \) a unique simulation \( (\text{Id}_{x^*}, b, \text{Id}_y) : M' \rightarrow M \).
We will, from now, assume that \( C \) is an \((\mathcal{E}, \mathcal{W})\) category (see DN a 15 and Th a 9).

**Lemma 6.** \( M \) realizes \( \beta \) \( \iff \) \( M \) realizes \( \bar{\beta} \).

**Proof:** 
(\( \implies \)) \( M \) realizes \( \beta \) \( \implies \) \( \beta = EM = \lambda_y \omega \delta^+ \).

But \( \bar{\beta} \) is the unique dynamorphism of \( \beta = \Lambda y, \bar{\beta} \).

\( \therefore \beta = EM \).

(\( \impliedby \)) \( M \) realizes \( \bar{\beta} \) \( \implies \bar{\beta} = \omega, \delta^+ \).

\( \therefore \beta = \Lambda y, \bar{\beta} = \Lambda y, \omega, \delta^+ = EM \).

**Lemma 7.** \( M \) is a minimal realization of \( \beta \) \( \iff \) it is a minimal realization of \( \bar{\beta} \).

**Proof:** Follows immediately from DN 19.1, DN 36.4 and Lemma 6.

Starting from \( \bar{\beta} \), instead of \( \beta \), we now construct a minimal realization of \( \bar{\beta} \), i.e., of \( \beta \). As minimal realization is unique up to isomorphism, we will use for it the same symbols as for the Nerode realization of \( \beta \), see Th 1, \( M_\beta = (X, s, i, x, \delta^+, \lambda) \). Besides Th 3 tells us that these 2 ways of obtaining the minimal realization of \( \beta \) are basically the same (they are "naturally isomorphic").

**DN 37.** \( \mathcal{V} \) is an \((\mathcal{E}, \mathcal{W})\) category, \( X \) a state-behavior process, a machine \( M \) in \( \mathcal{V} \) is reachable if \( \delta^+ \in \mathcal{E} \), observable if \( \omega \in \mathcal{W} \).
NOTE 16. Note that, in DN 37, the definition of reachability is slight by different of the one given in DN 6.3 where \( \delta^+ \) was a coequalizer. It amounts to the same thing if \( \mathcal{C} \) is a coequalizer mono category (i.e. \( \exists \). a coequalizer mono factorization \( \forall \) morphisms in \( \mathcal{C} \)) as \( \mathcal{C} \) is an \( (\mathcal{C}, \mathcal{U}) \) category where \( \mathcal{C} \) is the class of all coequalizers and then, \( \mathcal{U} \) the class of all monics in \( \mathcal{C} \). [J], Gr, R-Mod, among many other usual categories, have that property and so has the "suitable" category used in our study of decomposable systems.

LEMMA 8. \( X \) an input-process \( \mathcal{C} \rightarrow \mathcal{G} \)
i.e. \( \forall c \in \mathcal{C}. / \exists (\delta_c : X.X^c \rightarrow X^c, \eta_c : c \rightarrow X^c) \)
universal from \( c \) to \( \mathcal{G} \) the forgetful functor \( \text{Dyn } X \rightarrow \mathcal{C} \)
\( \eta = \{ \eta_c / c \in \mathcal{C} \} \) and \( d = \{ \delta_c / c \in \mathcal{C} \} \) are natural transformations.

Proof:
(a) \( X \) an input-process \( \rightarrow U \) has a left-adjoint \( V \leftrightarrow \eta : \text{Id}_c \rightarrow U.V = X^* : \mathcal{C} \rightarrow \mathcal{C} \) is natural transformation \( \therefore \) TH a 5.4

(b) \( d : X.X^* \rightarrow X^* : \mathcal{C} \rightarrow \mathcal{C} \) is a natural transformation \( \therefore \) TH a 7.1 as \( \forall c \in \mathcal{C} \)
\( \exists \delta_c : X^c \rightarrow X^c \)
the diagram
\( \begin{array}{ccc}
X.X^c & \xrightarrow{\delta_c} & X^c \\
\downarrow & & \downarrow \\
X^f & \xrightarrow{\delta_{c'}} & X^{c'}
\end{array} \)
commute \( \forall f : c \rightarrow c' \)
\( \therefore \) note 2.1 and 2.2
**Lemma 9.** \( C \) is an \((\mathcal{E}, \mathcal{M})\) category,
\[
f : (\delta : Xs \to s) \to (\delta' : Xs' \to s') \in \text{Dyn} X
\]
\( \exists f = s \overset{\delta}{\underset{m}{\to}} s' \) is its \((\mathcal{E}, \mathcal{M})\)
factorization and either \( X \) or \( X^* \) preserves \( \mathcal{E} \)

\[\Rightarrow \exists \text{ a unique } \delta'' : X \overset{\text{Im}}{\to} \overset{\text{Im}}{\to} s \exists e \text{ and } m \text{ an } X \text{-dynamorphisms } \delta \to \delta'' \text{ and } \delta'' \to \delta' \text{ respectively.} \]

**Proof:**

(a) \( X \) preserves (i.e. \( e \in \mathcal{E} \Rightarrow xe \in \mathcal{E} \)) then our result is immediate

\[
\begin{array}{c}
Xs \overset{\delta}{\to} X \overset{\text{Im} f}{\to} \overset{\text{Im} f}{\to} s' \\
\end{array}
\]

by the \((\mathcal{E}, \mathcal{M})\) diagonalization property as \( f \) a dynamorphism

\[
\begin{array}{c}
\delta \downarrow \delta' \downarrow \delta'' \\
\end{array}
\]

i.e. \( me. \delta = \delta' \cdot Xf \)

(b) \( X^* \) preserves \( \mathcal{E} \).

\[
\forall c \in \mathcal{C} \text{ let } \theta_c : Xc \to Xc \text{ be defined by }
\]

\[
\theta_c = Xc \overset{X\gamma_c}{\to} X(Xc) \overset{\delta_c}{\to} Xc
\]

\[
\eta = \{ \eta_c / c \in \mathcal{C} \} , \quad \delta = \{ \delta_c / c \in \mathcal{C} \}
\]

are natural transformations (\( \vdash \) lemma 8)

\[\Rightarrow \theta = \{ \theta_c / c \in \mathcal{C} \} \text{ is a natural transformation (1).} \]

Recall (note 3.2) that \( \delta^* : X^* s \to s \) is the unique dynamorphic extension of \( \text{Id}_s \), i.e. \( \delta^* \cdot \eta_s \circ \gamma_s = \text{Id}_s \).

It is called the "run morphism".
The diagram
\[
\begin{array}{ccccccccc}
Xs & \xrightarrow{\delta s} & X (X^*_s) & \xrightarrow{\delta^*} & X^*_s \\
\downarrow & & \downarrow & & \downarrow \\
\delta & & \delta^* & & \delta^* \\
\end{array}
\]
commutes

as triangle 1 and square 2 commute by UN of \( \delta^* \).

\[
\therefore \quad \delta = \delta^* \cdot \text{Id}_{Xs} = \delta^* \cdot \delta^* \cdot X \gamma_s = \delta^* \cdot \Theta_g 
\]

(2)

Recall that, by note 2, \( X^* f = UF \) is the unique
dynamorphic extension \( \forall f \) of
\[
\eta_A : f \implies X^* f \gamma_s = \eta_A \cdot f
\]

\[
f' = f \cdot \text{Id}^* s = \text{Id}^* s \cdot f
\]

\[
f' = f \cdot \delta^* \gamma_s = \delta' \cdot \gamma_s \cdot f = \delta' \cdot X^* f \gamma_s
\]

\[
\therefore \quad f \cdot \delta = \delta' \cdot X^* f
\]

by uniqueness in
\[
\begin{array}{cccccc}
s & \xrightarrow{\gamma_s} & X^*_s & \xrightarrow{\delta^*} & X^*_s \\
\downarrow & & \downarrow & & \downarrow \\
f & & f \cdot \gamma_s & & f \cdot \delta^* x f
\end{array}
\]

\[
\therefore \quad \text{m.e.} \quad \delta = \delta' \cdot X^* m \cdot X^* e 
\]

\[
\therefore \quad \exists k : X^* \text{inf} \longrightarrow \text{inf} \quad \exists \text{ the diagram}
\]

\[
\begin{array}{cccccc}
X^*_s & \xrightarrow{X^*_s e} & X^*_s \text{inf} & \xrightarrow{X^*_s m} & X^*_m \\
\downarrow & & \downarrow & & \downarrow \\
\delta & & \delta' & & \delta^* \\
\end{array}
\]

\[
\therefore \quad (\mathcal{E}, \mathcal{M}) \text{ diagonalization}
\]

property as \( X^* e \in \mathcal{E} \).

In the following diagram
(1) \( \implies \) the 2 parts (1) commute
(2) \( \implies \) the 2 parts (2) commute
(3) \( \implies \) part (3) commutes

\[
\therefore \quad \text{the whole diagram commutes .}
\]
and the desired dynamics is \( \delta'' = k \cdot \theta \).

\[ \begin{array}{c}
X_s \xrightarrow{\theta_s} X_e \xrightarrow{\theta_{\text{Im} f}} X_{\text{Im} f} \\
\downarrow \delta \downarrow \delta^* \quad 1 \quad \downarrow \theta_{\text{Im} f} \\
X_s^* \xrightarrow{\delta^*} X_e^* \xrightarrow{1} X_{\text{Im} f}^* \xrightarrow{\theta_{\text{Im} f}} X_{\text{Im} f}^* \\
\downarrow \delta^* \downarrow \delta' \downarrow \delta'' \quad 1 \quad \downarrow \delta'' \\
X_s' \xrightarrow{\delta'} X_{\text{Im} f}' \xrightarrow{\delta''} X_{\text{Im} f}'' \xrightarrow{\delta'} X_{\text{Im} f}'' \\
\downarrow \delta'' \downarrow \delta'' \downarrow \delta'' \downarrow \delta'' \\
1_{\text{Im} f} \xrightarrow{m} \text{Im} f \xrightarrow{m} s' \end{array} \]

\( \delta'' \) is unique as if \( \exists \delta'' \) with the same properties we have \( m \cdot \delta'' = \delta' \cdot \theta_{\text{Im} f} = m \cdot \delta'' \).

\( \Rightarrow \delta'' = \delta'' \) as \( m \) is monic.

\( \delta'' \cdot e = e \cdot \delta \Rightarrow e \) is a dynamorphism \( \delta \rightarrow \delta'' \)

\( \delta' \cdot \theta_{\text{Im} f} = m \cdot \delta'' \Rightarrow m \) is a dynamorphism \( \delta'' \rightarrow \delta' \)

**Lemma 10.** \( \mathcal{C} \) is an \((\mathcal{E}, \mathcal{M})\) category,

\( \mathcal{E} : (\delta : X_s \rightarrow s) \rightarrow (\delta' : X_{s'} \rightarrow s') \in \text{DYN} \mathcal{X} \) and

\( \delta'' : X_{s''} \rightarrow s'' \in \text{Dyn} X'/ \)

\( \mathcal{E} : s' \rightarrow s'' \in \mathcal{E} \quad \exists f \in \text{Dyn} X/ \) is a dynamorphism \( \delta \rightarrow \delta'' \) either \( X \) or \( X^* \) preserves \( \mathcal{E} \)

\( \Rightarrow f \) is a dynamorphism \( \delta' \rightarrow \delta'' \).
Proof: (a) $X$ preserves $E$, consider

$$
\begin{array}{c}
X_e & \xrightarrow{Xf} & X_{s''} \\
\downarrow{\delta} & & \downarrow{\delta''} \\
X_{e'} & \xrightarrow{s'} & X_{s''}
\end{array}
$$

The perimeter and square (1) commute as $e$ and $f \in \text{Dyn } X$

$\Rightarrow f \cdot \delta'. X_{e} = \delta''. X_{f}, X_{e}$

$\Rightarrow f \cdot \delta = \delta''. X_{f}$ as $X_{e} \in E$ is epi

$: x \in \text{Dyn } X$

(b) $X^* \text{ preserves } E \Rightarrow X^* e \in E \text{ (as } e \in E)$

$\forall e : (\delta : X_{e} \rightarrow s) \rightarrow (\delta' : X_{e'} \rightarrow s') \in \text{Dyn } X$

In the following diagram

$$
\begin{array}{c}
\begin{array}{ccc}
\delta & \xrightarrow{\gamma} & \delta' \\
\downarrow & & \downarrow \\
1d_e & \xrightarrow{X^*} & X_{\delta} \\
\downarrow & & \downarrow \\
\delta & \xrightarrow{\delta^*} & \delta
\end{array}
\end{array}
$$

(1) commutes as $\gamma$ is a natural transformation,

(2) and (3) commutes $\therefore \text{RN of } S^*$

$\Rightarrow \delta = \delta \cdot 1d_{\delta} = 1d_{\delta'} \cdot \delta$ $\therefore \text{RN of } 1d$

$= \delta \cdot \delta^* \cdot \eta_{\delta} = \delta^* \cdot \eta_{\delta} \cdot \delta$ $\therefore (2) \text{ and } (3) \text{ commute}$

$= \delta \cdot \delta^* \cdot \eta_{\delta} = \delta^* \cdot \delta \cdot \eta_{\delta}$ $\therefore (1) \text{ commutes}$
\[ \Rightarrow g, \delta^* = \delta^{**} x g \quad \text{by uniqueness of the dynamomorphic extension of } g \text{ in } \]
\[ s \xrightarrow{\gamma_t} x s \]
\[ g \delta^{**} \downarrow \delta^{**}x g \]

\[ \therefore \quad \text{The left-hand side and the perimeter commute in the diagram} \]

\[ x^s \xrightarrow{x^e} x^{s'} \xrightarrow{x^e} x^{s''} \]
\[ \delta'' \downarrow \delta^{**} \quad \text{as } e \text{ and } f.e \in \text{Dyn X} \]

\[ \therefore \quad \delta = \delta^{**} x f \quad x f \]

\[ \therefore \quad \delta'' = \delta^{**} x f \quad x f \quad \text{as } x^e \in E \text{ is epi} \]

Now in

\[ x^s' \xrightarrow{x f} x^{s''} \]
\[ \delta' \downarrow \delta'' \]
\[ \delta^{**} \downarrow \delta^{**}x g \]

(1) commutes by (1) in proof of lemma 9,
(2) and (3) commute by (2) in proof of lemma 9,
(4) commutes by above result

\[ \Rightarrow \quad \text{the perimeter commutes and} \]

\[ f \in \text{Dyn X} \]
Lemma 11. \( \mathcal{C} \) is an \((\mathcal{E}, \mathcal{M})\) category, \( X \) a state-behavior process, \( \beta : X^* \rightarrow X^y \in \text{Dyn } X \)

\( \mathcal{M} \) is a reachable realization of \( \overline{\beta} \), \( \mathcal{M} \) an observable realization of \( \overline{\beta} \), either \( X \) or \( X^* \) preserves \( \mathcal{E} \).

\[ \exists \text{ unique simulation } (\text{id}_{X^y}, b, \text{id}_y) : \mathcal{M} \rightarrow M. \]

Proof.

Define unique \( b : s' \rightarrow s \) by \((\mathcal{E}, \mathcal{M})\) diagonalization property in

\[ \begin{array}{c}
\delta' \rightarrow \omega' \quad \text{as } \omega' \in \mathcal{M} \text{ (M reachable)} \\
\omega' \rightarrow X^y \quad \text{as } \omega' \in \mathcal{M} \text{ (M observable)}
\end{array} \]

\[ \overline{\beta} = \omega' \delta' \omega' \delta'' (\mathcal{M} \text{ and } \mathcal{M}' \text{ realize } \overline{\beta}) \]

\[ \therefore \delta' \in \mathcal{E} \text{ and } \delta'' \text{ is a dynamorphism } \delta' \rightarrow \delta' \]

\[ \delta'' : X^y \rightarrow X^*, \delta' : X^y \rightarrow s', \delta : X \rightarrow s \in \text{Dyn } X \]

\[ b \triangleright b, \delta'' \text{ is a dynamorphism } \delta' \rightarrow \delta, \]

\[ X \text{ or } X^* \text{ preserves } \mathcal{E} \]

\[ \Rightarrow b \text{ is a dynamorphism } \delta \rightarrow \delta' \text{ (\delta'' \text{ lemma 10})} \]

i.e. \((\text{id}_{X^y}, b, \text{id}_y) : \mathcal{M} \rightarrow M \text{ is the unique (as } b \text{ is unique) simulation required.} \]

Theorem 7. Minimal Realization Th. for State Behavior Processes:

\( \mathcal{C} \) is an \((\mathcal{E}, \mathcal{M})\) category,

\( X \) is a state-behavior process \( \exists \) \( X \) or \( X^* \)

preserve \( \mathcal{E} \), \( \mathcal{I} \) and \( \gamma \in \mathcal{C} \); given a behavior

\[ \beta : X^* \rightarrow Y, \overline{\beta} : X^* \rightarrow X^y \] is its corresponding dynamorphism \( \delta' \rightarrow \Delta_y \beta = \Lambda_y \overline{\beta} \)
(1) \( \exists \) a minimal realization of \( \beta \) whose state object is \( \text{Im} \beta \).

(2) \( M \) is a minimal realization of \( \beta \)
\( \iff \) \( M \) is reachable and observable.

**Proof:**

(1) \( \mathcal{C} \) is an \( (\mathcal{E}, \mathcal{U}) \) category

\( \bar{\beta} \) has a unique \( (\mathcal{E}, \mathcal{U}) \) factorization (up to isomorphism)

say \( \bar{\beta} \xrightarrow{X} \xrightarrow{\omega \rho} \xrightarrow{X_y} \). Let \( s_\rho = \text{Im} \beta \).

\( \vdash \) lemma 9: \( \exists \) a unique \( \delta^\rho : X \rho \to s \rho \), \( \delta^\rho \) and \( \omega \rho \in \text{Dyn} X \).

Put \( \sigma^\rho = \delta^\rho \cdot \eta_y \) and \( \lambda^\rho = \Lambda^y \cdot \omega \rho \)

\( \iff \) \( M = (X, X, s_\rho, y, \sigma^\rho, \lambda^\rho) \) is a realization of \( \bar{\beta} \)
\( \iff \) it is a realization of \( \beta \).

It is minimal \( \vdash \) lemma 11.

(2) \( \implies \) \( M \) is a minimal realization of \( \beta \)

\( \implies \) \( M \equiv M_\beta \) \( \vdash \) lemma 19.2

\( \implies \) \( M \) is reachable and observable as \( M_\beta \) is.

(\( \leftarrow \)) \( M \) is reachable and observable

\( \implies \) \( \exists \) an \( (\mathcal{E}, \mathcal{U}) \) factorization

\( \overline{\text{M}} = \beta = \omega, \beta \) unique up to isomorphism

\( \implies \) \( M \equiv M_\beta \), i.e. it is minimal.

**Examples:**

1. Decomposable systems of chapter 3 fit the above description as "suitable categories" have coequalizer mono factorization, i.e. an \( (\mathcal{E}, \mathcal{U}) \) categories. Put \( X = \text{Id} \) and \( \forall \varepsilon, \varepsilon' \neq \varepsilon^+ \) the countable copower of \( \varepsilon \)

\( X \varepsilon = \varepsilon^X \) the countable power of \( \varepsilon \).
\[ \eta_c = \eta_c : c \rightarrow c^+, \quad \delta_c = z \times u^+ \rightarrow u^+ , \]
\[ \Lambda_c = c^x \rightarrow c, \quad \Delta_c = z : c^x \rightarrow c^x \]

\[ \implies \text{Id}_c \text{ is a state-behavior process, which preserves } \mathcal{C}, \]
\[ i = u, \quad \sigma = g, \quad s = x, \quad \delta = f, \quad \lambda = h \quad \text{and the system } (f, g, h) \text{ is a state-behavior machine.} \]

2. **Discrete machines** can also be put in that context.

\[ \mathcal{A} \text{ is an } (\mathcal{E}, \mathcal{M}) \text{ category} \]
\[ X = X \times _\mathcal{M} \mathcal{A} \rightarrow \mathcal{A} \]
\[ \forall \delta \in \mathcal{M}, \quad x^\delta = x^{\mathcal{X}} \quad \{ f / \delta : x^\delta \rightarrow \delta \text{ is a map} \} \]

where \( x^{\mathcal{X}} \) is the free monoid whose words are strings of elements of \( x \);
\[ \Delta_c : x \times x^{\mathcal{X}} \rightarrow x^{\mathcal{X}} \]
\[ (x, r) \quad \mapsto \quad r, \quad R_x : x^{\mathcal{X}} \rightarrow c \]

with \( R_x : x^{\mathcal{X}} \rightarrow x^{\mathcal{X}} \quad \forall x \in X \)

\[ \Lambda_c : c^{\mathcal{X}} \rightarrow c \quad \forall \lambda \in \mathcal{A} \] 

Given \( \delta' : x \times c' \rightarrow c', \quad g : c' \rightarrow c \)
\[ \varphi : c' \rightarrow c^{\mathcal{X}} \quad \text{makes the} \]
\[ c' \quad \mapsto \quad g \cdot \delta' (\_, c') \)

two diagrams commute:

\[ (x, c) \rightarrow (x, c') \rightarrow c' \]

\[ \varphi \downarrow \quad \Delta_c \downarrow \quad \Lambda_c \]

\[ c' \quad \mapsto \quad c^{\mathcal{X}} \quad \mapsto \quad x \times c^{\mathcal{X}} \]

\[ \varphi \downarrow \quad \Lambda_c \downarrow \quad \Delta_c \]

\[ \text{Id}_c \times \varphi \quad \uparrow \quad \varphi \]

\[ \delta' (x, c') \]

\[ \varphi \downarrow \quad \delta' (x, c') \]

\[ \text{Id}_c \times \varphi \quad \uparrow \quad \varphi \]

\[ \delta' (x, c') \]
Recall that if \( \delta : X \times S \rightarrow S \) is a map then:
\[
\delta : X \times S \rightarrow S \quad \text{is defined by}
\begin{align*}
\delta(x, s) &= s \\
\delta(xw, s) &= \delta(x, \delta(w, s))
\end{align*}
\]
\( x \) - is a state behavior process.
Recall \( \omega : S \rightarrow Y \) we have
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\delta^+(\cdot, s)
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\lambda 
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\delta^+(w)
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\lambda \cdot \delta^+(-, \cdot, \delta^+(w)) \quad \lambda \cdot \delta^+(-)
\end{array}
\end{array}
\]
as \( \delta^+ = \delta^+(-, \cdot) \) where \( \sigma \in S \) is the initial state.
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\delta^+(\cdot, \cdot)
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\lambda \cdot \delta^+(\cdot, \cdot)
\end{array}
\end{array}
\]
as required.
Finally in
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\sim \quad \text{as } \quad E_{EM} = \text{im} \omega
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\lambda \cdot \delta^+(w_1) = \lambda \cdot \delta^+(w_2)
\end{array}
\end{array}
\]
as \( E_{EM} = \{(w_1, w_2) / \text{im}(w_1) = \text{im}(w_2) \forall w \in X\} \)
\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\text{as } \quad \text{im}(w_1) = \text{im}(w_2)
\end{array}
\end{array}
\]
\( M \) is reachable if \( \delta^+ \) is onto i.e. \( \delta^+ \in E \).
\( M \) is observable if \( \omega \) is 1-1 i.e. \( \omega \in M \).
As foreseen by TH 3, this way of obtaining a minimal realization is basically the same as the Rosen realization discussed in chapter 2.
ADJOINT MACHINES AND DUALITY

For the sake of completeness we will expose in this chapter the Arbib and Manes theory of adjoint machines and dual of a machine. For missing details and examples see [9].

1. Adjoint machines.

**DEF.** $X : \mathcal{C} \rightarrow \mathcal{U}$ is an **ADJOINT PROCESS** if $X$ has a right adjoint.

**Proposition 2.** $\mathcal{C}$ has countable coproducts which are preserved by $X : \mathcal{C} \rightarrow \mathcal{U}$, a functor.

$\Rightarrow X$ is an input process and

$$X^* = \bigcup_{n \in \mathbb{N}} X^n : \mathcal{C} \rightarrow \mathcal{U}$$

**Proof:** See [9] page 319.

**NOTE.** $\bigcup_{n \in \mathbb{N}} X^n : \mathcal{C} \rightarrow \mathcal{U}$ is defined on object $\alpha \rightarrow \bigcup_{n \in \mathbb{N}} X^n \alpha$. 
and on morphism $f \downarrow \overset{c'}{c} \rightarrow \bigcup x^n$ the unique morphism $\exists$ the following square commutes:

\[
\begin{array}{ccc}
X^n_c & \overset{\text{in}_n}{\longrightarrow} & \bigcup X^n_c \\
X^n f & \downarrow & \exists \bigcup X^n f \\
X^n c' & \overset{\text{in}_n'}{\longrightarrow} & \bigcup X^n c'
\end{array}
\]

Corollary. $\mathcal{C}$ has countable co-products, $X$ has a right-adjoint

$(1)$ $X$ is an input process,
$(2)$ $X^* = \bigcup X^n$.

Proof: TH a 8 and above proposition 9.

**Lemma 12.** $X \rightarrow X^* : \mathcal{C} \rightarrow \mathcal{C}$

$\Rightarrow \mathcal{C}(Xc, c') \cong \mathcal{C}(c', X^* c')$ TH a 5.1

$(f : Xc \rightarrow c') \iff (f' : c \rightarrow X^* c')$

Then we have:

Transposition principle $f : a' \rightarrow a$, $g : Xa \rightarrow b$,

$h : b \rightarrow b'$,

$k = h \cdot g : Xa' \rightarrow b'$

$\Rightarrow k' = X^* h \cdot g : a' \rightarrow X^* b'$

By Din a 4, \( \forall \) category \( \mathcal{C} \exists \) the dual (opposite) category \( \mathcal{C}^{op} \exists /\mathcal{C}/ = /\mathcal{C}^{op}/ \) and \( f^{op} : a \rightarrow b \) corresponds to \( f: b \rightarrow a. \) \( \forall \) functor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}, \mathcal{F} \) defines a functor \( \mathcal{F}^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}. \)

Object \( c \rightarrow \mathcal{F}^{op}(c) = \mathcal{F}_c \)

Morphism \( f^{op} \rightarrow \mathcal{F}^{op}(f) \rightarrow \mathcal{F}_b \)

Corresponding to \( f \rightarrow \mathcal{F}_b \)

i.e. \( \mathcal{F}^{op}(f)^{op} = (\mathcal{F}f)^{op}. \)

\[ \text{TH 8.} \]

\( X \rightarrow X^{op} : \mathcal{C} \rightarrow \mathcal{C}^{op}, \)

\( U : \text{Din} X \rightarrow \mathcal{C} \) and \( U^{op} : \text{Dyn}(X^{op}) \rightarrow \mathcal{C}^{op} \)

are forgetful functors

\( \Rightarrow \) the correspondence \( \delta : Xc \rightarrow c, \delta^{op} : X^{op}c \rightarrow c \)

is an isomorphism of categories i.e.

\( \text{Dyn} X \cong \text{Dyn}(X^{op}) \)

rendering commutative the diagram

\[
\begin{array}{c}
\text{Dyn}(X^{op}) \cong \text{Dyn}(X^{op}) \\
\downarrow U^{op} \quad \downarrow U^{op} \\
\text{Dyn} X^{op} \quad \text{Dyn} X^{op}
\end{array}
\]

\[ \text{Proof:} \]

\( X \rightarrow X^{op} : \mathcal{C} \rightarrow \mathcal{C}^{op} \)

\( \forall c \in \mathcal{C} \)

\( \mathcal{C}(c, Xc) \cong \mathcal{C}(Xc, c) \cong \mathcal{C}(c, Xc) \cong \mathcal{C}(X^c, c) \)

\( (\delta^{op} : c \rightarrow Xc) \leftrightarrow (\delta : Xc \rightarrow c) \leftrightarrow (\delta : Xc \rightarrow Xc) \leftrightarrow (\delta^{op} : Xc \rightarrow c) \)

i.e. \( \forall \delta^{op} \in /\text{Dyn}(X)^{op} / \exists \) unique \( \delta^{op} \in /\text{Dyn}(X^{op}) / \)

and vice-versa, (1)
Consider \( f : c \rightarrow c', \delta : Xc \rightarrow c, \delta' : Xc' \rightarrow c' \)

By lemma l2 \( \exists \) a 1-1 correspondence between
\( \text{Id}_c, \delta', Xf \) and \( X \text{Id}_c, \delta', f = \text{Id}_{Xc}, \delta', f \)

and between \( f \circ \delta, X \text{Id}_c = f \circ \delta, \text{Id}_{Xc} \) and
\( X'f, \delta' \circ \text{Id}_c \).

\[ \therefore f \in \text{DYN} X \Leftrightarrow \delta', Xf = f \circ \delta \quad \therefore \text{DYN} 3 \text{ and DYN} 4 \]
\[ \quad \Leftrightarrow \text{Id}_c, \delta', Xf = f \circ \delta, \text{Id}_{Xc} \]
\[ \quad \Leftrightarrow \text{Id}_{Xc}, \delta', f = X'f \circ \delta' \circ \text{Id}_c \]
\[ \quad \Leftrightarrow \delta', f = X'f, \delta' \]
\[ \quad \Leftrightarrow f^{\text{op}}, \delta'^{\text{op}} = \delta'^{\text{op}}, (X^{\text{op}}) f^{\text{op}} \]
\[ \quad \Leftrightarrow f^{\text{op}} \in \text{DYN} (X^{\text{op}})^{\text{op}} \quad (2) \]

\[ \therefore (1) \text{ and } (2) \Rightarrow (\text{DYN} X)^{\text{op}} \cong \text{DYN} (X^{\text{op}}) \]
Besides \( U^{\text{op}} \delta'^{\text{op}} = c = U' \delta'^{\text{op}} \cup \exists \text{DYN} X \)
and \( U^{\text{op}} f^{\text{op}} = f^{\text{op}} = U' f^{\text{op}} \cup f \in \text{DYN} X \),

\( \therefore \) the given diagram commutes as required.

**Corollary.** \( U \) has a left (right) adjoint \( \Leftrightarrow \)
\( U' \) has a right (left) adjoint,
\( \text{i.e. } X \) is an input (output) process \( \Leftrightarrow \)
\( X' \) is an output (input) process;
\( \therefore X \) is a state-behavior process \( \Leftrightarrow X' \) is .
TH 9. \( \mathcal{U} \) has countable products and coproducts, 
\[ X : \mathcal{U} \rightarrow \mathcal{U} \] is an adjoint process
\[ \Rightarrow X \text{ is a state-behavior process.} \]

Proof: \( \mathcal{U} \) has countable coproduct and \( X \) adjoint
\[ \Rightarrow X \downarrow X^* : \mathcal{U} \rightarrow \mathcal{U} \] \( \overset{\text{DM 38,}}{\Rightarrow} \)
\[ X \text{ is an input process and } X^* = \bigcup_{n \in \mathbb{N}} X_n \]
\[ \overset{\text{Corollary of proposition 9}}{\Rightarrow} X^* \text{ is an output process with} \]
\[ X^*_n = \bigcap_{n \in \mathbb{N}} (X')^n \overset{\text{Corollary of TH 8}}{\Rightarrow} \]
and duality.

Besides \( \mathcal{U} \) has product \( \Rightarrow \mathcal{U}^{\text{op}} \) has coproduct which
are preserved by \( X^{\text{op}} \) as \( X^{\text{op}} \downarrow X^{\text{op}} \) (as \( X \downarrow X^* \)),
\[ \Rightarrow X^{\text{op}} \text{ is an input-process} \]
\[ \Rightarrow X^{\text{op}} \text{ is state behavior} \]
\[ \overset{\text{TH 8}}{\Rightarrow} X^* \text{ is state behavior} \]
\[ \overset{\text{corollary of TH 8}}{\Rightarrow} X \text{ is state behavior} \]

TH 10. \( X : \mathcal{U} \rightarrow \mathcal{U} \) a state-behavior process
\[ \Rightarrow X^*_n \downarrow X^*_n : \mathcal{U} \rightarrow \mathcal{U} \]

Proof: \( X \) a state-behavior process
\[ \Rightarrow V \downarrow U : \mathcal{U} \rightarrow \text{Dyn } X \]
and \( U \downarrow V : \text{Dyn } X \rightarrow \mathcal{U} \), \( \ldots \), by TH a 10,
We have \( U.V = X^*_n \downarrow X^*_n = U.V' \).
∀ c, c' ∈ S we have

\( \psi (X^c_c \rightarrow c') \cong \psi (c \rightarrow X^c_{c'}) \) i.e. a 1-1 correspondence.

or

\[ f \leftrightarrow f^* \]

\[ g \leftrightarrow g^* \]

Hence we have the following adjointness table for state behavior machines.

<table>
<thead>
<tr>
<th>Concept for M</th>
<th>Adjoint concept for M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run morphism</td>
<td>State-behavior morphism</td>
</tr>
<tr>
<td>( \delta^* : X^* s \rightarrow s )</td>
<td>( \varphi_i : s \rightarrow X^*_s ) (see note 15)</td>
</tr>
<tr>
<td>Full response morphism</td>
<td>Observability morphism</td>
</tr>
<tr>
<td>( \lambda : \delta^* = : X^* s \rightarrow y )</td>
<td>( \omega : s \rightarrow X^*_y )</td>
</tr>
<tr>
<td>Reachability morphism</td>
<td>Adjoint reachability morphism</td>
</tr>
<tr>
<td>( \delta^+ : X^*i \rightarrow s )</td>
<td>( \delta^{++} : i \rightarrow X^*_s )</td>
</tr>
<tr>
<td>External behavior</td>
<td>Adjoint external behavior</td>
</tr>
<tr>
<td>( \lambda \delta^+ = EM : X^* \rightarrow y )</td>
<td>( EM' : i \rightarrow X^*_y )</td>
</tr>
</tbody>
</table>

\[ X \times X \times X \]

2. Dual of a machine.

Recall that, by TH 8, if \( X \rightarrow X' \), \( \mathcal{C} 
\Rightarrow \mathcal{C} \)

\( \text{Dyn } X^{\text{op}} \cong \text{Dyn } (X^{\text{op}}') \)

and that \( X'^{\text{op}} \rightarrow X^{\text{op}} ; \mathcal{C}^{\text{op}} \Rightarrow \mathcal{C}^{\text{op}} \)

\( \therefore \) We can define the dual of \( M, M^{\text{op}} \) by the following:
**DN 39.** \( \mathcal{C} \) is a category with countable products and coproducts; \( M = (X, s, i, y, \delta, \sigma, \lambda) \) is an adjoint machine (i.e. \( X \vdash X^* \)).

Then \( M^{\text{op}} = (X^{\text{op}}, s, y, i, \delta^{\text{op}}, \lambda^{\text{op}}, \sigma^{\text{op}}) \).

\( M^{\text{op}} \) is an adjoint machine in \( \mathcal{C}^{\text{op}} \) with initial object \( y \) and output object \( i \). Besides \( (M^{\text{op}})^{\text{op}} = M \).

As \( X \) is an adjoint process then \( X^X = \bigsqcup X^s \)

while \( X^X = \prod X^N \). Thus as we pass from \( \mathcal{C} \) to \( \mathcal{C}^{\text{op}} \)

\[ \bigsqcup X^s \] becomes \( \prod X^s = (X^{\text{op}})^s \); while \( \prod X^N \)

becomes \( \bigsqcup (X^*)^N = (X^{\text{op}})^* \). Since we interchange products and coproducts in opposite categories.

We have the following table.

<table>
<thead>
<tr>
<th>( M ) concept in ( \mathcal{C} )</th>
<th>( M^{\text{op}} ) concept in ( \mathcal{C}^{\text{op}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial state morph. ( i )</td>
<td>Output morph. ( \sigma^{\text{op}} ) ( i ) ( \mapsto ) ( i )</td>
</tr>
<tr>
<td>Output morph. ( \lambda )</td>
<td>Initial state morph. ( \lambda^{\text{op}} ) ( i ) ( \mapsto ) ( i )</td>
</tr>
<tr>
<td>Adjoint process ( X : \mathcal{C} \to \mathcal{C} )</td>
<td>Adjoint process ( X^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{C}^{\text{op}} )</td>
</tr>
<tr>
<td>Dynamics ( \delta : X^s \to s )</td>
<td>Dynamics ( \delta^{\text{op}} : X^{\text{op}}s \to s )</td>
</tr>
<tr>
<td>( X^s = \bigcup X^s )</td>
<td>( (X^{\text{op}})^s = \prod X^s )</td>
</tr>
<tr>
<td>( \delta^s : X^s \to X^s )</td>
<td>( \delta^{\text{op}} : X^{\text{op}}s \to (X^{\text{op}})^s )</td>
</tr>
<tr>
<td>( \gamma^s : s \to X^s )</td>
<td>( \gamma^{\text{op}} : (X^{\text{op}})^s \to s )</td>
</tr>
<tr>
<td>( X^s = \prod (X^s)^N )</td>
<td>( (X^{\text{op}})^s = \bigsqcup (X^s)^N )</td>
</tr>
<tr>
<td>( \Delta^s : X^s \to X^s )</td>
<td>( \Delta^{\text{op}} : X^{\text{op}}(X^{\text{op}})^s \to (X^{\text{op}})^s )</td>
</tr>
<tr>
<td>( \Lambda^s : X^s \to s )</td>
<td>( \Lambda^{\text{op}} : s \to (X^{\text{op}})^s )</td>
</tr>
</tbody>
</table>
Run morph. \( \delta^*: X^s \to s \)

State behavior
\[ \psi_s: s \to X^s \]
reachability \( m \)
\[ \delta^+: x^s \to s \]
Observability \( m \)
\[ \omega_s: s \to X^s \]
External behavior \( EM \)
\[ EM: x^s \to y \]
Adjoint ext. beh.
\[ EM^*: i \to X^s \]
Full response
\[ \omega_s = \lambda. \delta^*: X^s \to y \]
Adj. reachability \( \gamma_n \)
\[ \delta^+: i \to X^s \]
State-behavior morph.
\[ \varphi_s^{op}: s \to (X^{op})^s \]
Run map
\[ \varphi_s^{op}: (X^{op})^s \to s \]
Observability \( m \)
\[ \delta^+: i \to (X^{op})^s \]
Reachability \( m \)
\[ \omega^{op}: (X^{op})^s \to s \]
\[ (EM)^{op}: y \to (X^{op})^s \]
External behavior
\[ E(H^{op}): (X^{op})^s \to i = \sigma^\circ \omega^s \]
Adjoint reachability \( \gamma_n \)
\[ \rho \gamma^{op}: y \to (X^{op})^s \]
Full response
\[ \delta^{op} = \sigma^\circ \varphi^{op}: (X^{op})^s \to i \]

Therefore we have the following principle for adjoint machines:

reachability and observability are dual;
run and state-behavior are dual.

**IN 40.1** A category \( \mathcal{C} \) is self-adjoint for the process \( X \) if \( \exists \) an isomorphism \( \varphi: \mathcal{C} \to \mathcal{C}^{op} \) which is the identity on objects and is reflexive; i.e. \( \forall x: a \to b \) it provides a bijection \( f \) between
\[ (Id_a)^+ = Id_a, (f.g)^+ = g^+ \cdot f^+ \quad \text{and} \quad f^{++} = f \]
2. In such a situation $\mathbf{X}$ is **respectful** if $X \triangleright \psi^{-1} X \psi$.
We write $\psi^{-1} X \psi = \overset{*}{X}$

3. A respectful functor we consider as the dual of a machine $\mathbf{M}$, the machine $\mathbf{M}^\dagger = \psi^{-1} (\mathbf{M}^{op})$, back in $\mathbf{C}$.

We have then the usual result:

**TH 11.**

$\mathbf{U}$ is self-adjoint for a respectful process $\mathbf{X}$, $\mathbf{M} = (X, i, s, y, c, \sigma, \lambda)$ a machine in $\mathbf{U}$.

$\mathbf{M}$ reachable $\leftrightarrow \overset{*}{c} \text{ epi } \overset{\text{monic}}{\leftrightarrow} (\overset{*}{c})^\dagger \text{ observable}$

$\mathbf{M}$ observable $\leftrightarrow \overset{\text{monic}}{\leftrightarrow} (\overset{\text{epi}}{c})^\dagger \text{ reachable}$.

Three of the main concepts of system theory are: reachability, observability and controllability. As we have seen, Arbib and Manes have defined the first two in categorical terms. An interesting topic of further research would be to find a categorical definition for controllability which, in set theoretical terms, consists in:

Do there exist controls steering a given system from any of its states to one particular state (usually the origin)?

This completes the systematic expose of the Arbib and Manes theory of machines in a category, as we know it up to this day. To stress the very wide range of this theory, let us mention that besides the four examples discussed in this work, other systems such as stochastic, topological and metric automata can be put in this context (see [7] and [9]).
Therefore we are strongly inclined to make ours the conclusion of these two authors in [7]:

"Given this vigorous growth [of category theory applied to machines], we may expect the study of machines in a category both to feed back into the study of algebraic structure, per se, and also to have repercussions in many phases of the computer, information and system sciences from programming language studies to control theory."
ANNEX

**In 1.** A category \( \mathcal{C} \) is a class \( /\mathcal{C}/ \) of objects together with
\( \forall a, b \in /\mathcal{C}/, a \) a class \( \mathcal{C}(a, b) \) of morphisms \( a \to b \),
and \( \forall a, b, c \in /\mathcal{C}/ \), an operation, called composition of
morphisms, \( \circ : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c) \)
\((f, g) \mapsto g \circ f\)
such that:
1. \( \forall a \in /\mathcal{C}/ \) identity \( \text{Id}_a \in \mathcal{C}(a, a) \), \( \forall g : b \to a \)
   \( \text{Id}_a \circ g = g \) and \( \forall f : a \to c, f \circ \text{Id}_a = f \).
2. \( \forall a, b, c, d \in /\mathcal{C}/, f : a \to b, g : b \to c, \)
   \( h : c \to d, f \circ (g \circ h) = (f \circ g) \circ h \).

**Notation:** \( f : a \to b \), \( a = \text{dom } f \), \( b = \text{codom } f \).

We can identify \( a \) and \( \text{Id}_a : a \to a \);
then \( /\mathcal{C}/ \subseteq \bigcup_{a,b} \mathcal{C}(a, b) = \mathcal{C} \).

**In 2.** \( i \in /\mathcal{C}/, i \) is initial if \( \# \mathcal{C}(i, o) = 1 \) \( \forall o \in /\mathcal{C}/ \).
\( t \in /\mathcal{C}/, t \) is terminal if \( \# \mathcal{C}(o, t) = 1 \) \( \forall o \in /\mathcal{C}/ \).

**In 3.1** \( m : a \to b \) is monic in \( \mathcal{C} \) if \( \forall f, g : d \to a \)
\( m \cdot f = m \cdot g \implies f = g \) (i.e. \( m \) is left-cancelable).

\( e : a \to b \) is epic in \( \mathcal{C} \) if \( \forall f, g : b \to c \)
\( f \cdot e = g \cdot e \implies f = g \) (i.e. \( e \) is right-cancelable).
3. If \( f \circ g = \text{Id} \) then \( f \) is split-epi (a retraction of \( g \))
   \( g \) is split-monic (a section of \( f \))
   \( h = g \circ f \) is defined and idempotent.

4. \( i : a \rightarrow b \) is an isomorphism if \( \exists \ i^{-1} : b \rightarrow a \)
   \( i^{-1} \circ i = \text{Id}_a, \ i \circ i^{-1} = \text{Id}_b \); we say \( a \) is isomorphic to \( b \) and write \( a \cong b \); \( i^{-1} \) is called the inverse of \( i \).

**TH 1.1.** \( f \) and \( g \) are epi (split-epi, monic, split-monic, iso)
\( \Rightarrow \) \( f \circ g \) is epi (split-epi, monic, split-monic, iso respectively).

2. Split-epi \( \Rightarrow \) epi; split-monic \( \Rightarrow \) monic
   \( f \) iso \( \Leftrightarrow \) \( f \) monic and split-epi.

3. \( g \circ f \) monic; \( \Rightarrow \) \( f \) monic
   \( g \circ f \) epi \( \Rightarrow \) \( g \) epi.

**TH 2.1.** In \( \mathcal{C} \), the category of sets with maps as isomorphisms,
\( f \) onto \( \Leftrightarrow \) split-epi \( \Leftrightarrow \) epi
\( f \) 1-1 \( \Leftrightarrow \) split-monic \( \Leftrightarrow \) monic
\( f \) 1-1 and onto \( \Leftrightarrow \) \( f \) iso.

2. In Gr (category of groups with group homomorphisms),
   E-Mod (category of \( E \)-modules with linear transformation)
   \( f \) onto \( \Leftrightarrow \) epi
   \( f \) 1-1 \( \Leftrightarrow \) monic.
The opposite (dual) category $\mathcal{C}^{\text{op}}$ of $\mathcal{C}$ is defined by:

$\mathcal{C}^{\text{op}} = /\mathcal{C}/$, $\mathcal{C}^{\text{op}}(a, b) \equiv \mathcal{C}(b, a)$, $f^{\text{op}} \cdot g^{\text{op}} = (g \cdot f)^{\text{op}}$

where $f^{\text{op}} \in \mathcal{C}^{\text{op}}(a, b)$ corresponds to $f \in \mathcal{C}(b, a)$.

We write $f^{\text{op}} : a \rightarrow b$ corresponding to $f : b \rightarrow a$

**Duality Principle.**

Let $S$ be a statement valid in a category $\mathcal{C}$ then the corresponding statement $S^{\text{op}}$ in $\mathcal{C}^{\text{op}}$ is called its dual statement $S^{\text{op}}$.

$\forall \mathcal{C}, \mathcal{C}^{\text{op}}$ a category $\Rightarrow S^{\text{op}}$ is a category, and $(S^{\text{op}})^{\text{op}} = S$

$\therefore \forall$ statement $S$ true $\forall$ categories $\Rightarrow S^{\text{op}}$ is also true $\forall$ categories.

**Examples:**

<table>
<thead>
<tr>
<th>$S$</th>
<th>$S^{\text{op}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f : a \rightarrow b$</td>
<td>$f^{\text{op}} : b \rightarrow a$</td>
</tr>
<tr>
<td>$a = \text{dom } f$</td>
<td>$a = \text{cod } f^{\text{op}}$</td>
</tr>
<tr>
<td>$i = \text{Id}_a$</td>
<td>$i^{\text{op}} = \text{Id}_a$</td>
</tr>
<tr>
<td>$h = g \cdot f$</td>
<td>$h^{\text{op}} = f^{\text{op}} \cdot g^{\text{op}}$</td>
</tr>
<tr>
<td>$f$ is monic</td>
<td>$f^{\text{op}}$ is epi</td>
</tr>
<tr>
<td>$f$ is split-monic</td>
<td>$f^{\text{op}}$ is split-epi</td>
</tr>
<tr>
<td>$f$ is iso</td>
<td>$f^{\text{op}}$ is iso</td>
</tr>
<tr>
<td>$t$ is a terminal object</td>
<td>$t$ is an initial object</td>
</tr>
</tbody>
</table>
In 5. \[ b \xrightarrow{g} c \] are a pair of \( C \) morphisms.

1. \( e : a \to b \) is an equalizer of \( f, g \),
   \[ e = \text{equ}(f, g); \]
   a) \( f \cdot e = g \cdot e \)
   b) \( \forall e' : a' \to b \exists f \cdot e' = g \cdot e' \exists \) a unique
   morphism \( \varphi : a' \to a \exists e' = e \cdot \varphi \).

2. \( k : c \to d \) is a coequalizer of \( f, g \),
   \[ k = \text{coeq}(f, g); \]
   a) \( k \cdot f = k \cdot g \)
   b) \( \forall k' : c \to d' \exists k' \cdot f = k' \cdot g \exists \) a unique
   morphism \( \psi : d' \to d \exists k' = \psi \cdot k \).

   i.e. \[ \begin{tikzpicture}
   \node (a) at (0,0) {a};
   \node (b) at (1,0) {b};
   \node (c) at (2,0) {c};
   \node (d) at (3,0) {d};
   \draw[->] (a) -- (b);
   \draw[->] (b) -- (c);
   \draw[->] (c) -- (d);
   \draw[->] (a) to [out=0, in=180] (d);
   \node (e) at (4,0) {commutes};
   \end{tikzpicture} \]

   \[ \text{Equ} \quad \text{Coeq} \]

TH 3:
1. \( e \) an equalizer \( \to \) \( e \) monic.
2. \( k \) a coequalizer \( \to \) \( k \) epi.
3. \( e' \) and \( e \) are equ. of \( f, g \) \( \Rightarrow \) \( a \cong a' \),
   \( k \) and \( k' \) are coeq. of \( f, g \) \( \Rightarrow \) \( d \cong d' \),
   i.e. an equ (a coeq) is unique up to isomorphism.
4. \( \mathcal{A}, \text{Gr, R-Mod} \) have equalizers (coequalizers) i.e.
   \( \forall \) pair of morphisms with common dom and codom.
   has an equ (a coeq).
5. In \( \mathcal{A}, \text{Gr, R-Mod} \), \( f \) onto \( \iff \) \( f \) a coequalizer.
LEMMA 1. \( (14) \) p. 723) Coequalizer-monofactorization (i.e. \( f = m.k, k \) a coeq., \( m \) monic) are unique to isomorphism.

DEF 6.
1. A functor \( F : \mathcal{B} \rightarrow \mathcal{C} \), where \( \mathcal{B} \) and \( \mathcal{C} \) are categories, is a function:
   - on objects \( b \rightarrow F_b \quad \forall b \in \mathcal{B} \)
   - on morphisms \( \frac{b}{b'} \rightarrow \frac{F_b}{F_{b'}} \quad \forall f \in \mathcal{B} \)

   \( \exists F(f.g) = F_f F_g \) and \( F\text{id}_b = \text{id}_{F_b} \)

2. A functor \( F : \mathcal{B} \rightarrow \mathcal{C} \) is faithful if \( F \) restricted to \( \mathcal{B} (b, b') \) is 1-1 \( \forall b, b' \in \mathcal{B} \). A functor \( F : \mathcal{B} \rightarrow \mathcal{C} \) is full if \( \forall b, b' \in \mathcal{B} \) \( F(\mathcal{B}(b, b')) = \mathcal{C}(F_b, F_{b'}) \) i.e. \( F_{\mathcal{B}(b, b')} \) is onto.

3. Composition of functors:
   - \( F : \mathcal{B} \rightarrow \mathcal{C}, G : \mathcal{A} \rightarrow \mathcal{B} \) are functors
   - \( F.G : \mathcal{A} \rightarrow \mathcal{C} \) is a functor \( (F.G(a)) = F(G(a)) \)
   - \( (F.G)(f) = F(G(f)) \)

4. The identity functor \( \text{Id}_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \) is defined
   - \( \text{Id}_\mathcal{C} \circ = \circ) \forall c \in \mathcal{C} \)
   - \( \text{Id}_\mathcal{C} f = f, \forall f \in \mathcal{C} \).
5. The metacategory $\text{CAT}$ is the category whose objects are categories and morphisms are functors.

6. A functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism if either of the following conditions holds:

   (1) $F : b \to Fb$ is a bijection on objects
   $F : f \to Ff$ is a bijection on morphism

   (2) $\exists$ a functor $F^{-1} : \mathcal{D} \to \mathcal{C}$ s.t.
   the diagram
   \[
   \begin{array}{ccc}
   \mathcal{C} & \xrightarrow{\text{Id}_{\mathcal{D}}} & \mathcal{D} \\
   F \downarrow & & \downarrow F^{-1} \\
   \mathcal{C} & \xrightarrow{F^{-1}} & \mathcal{D}
   \end{array}
   \]
   commutes

   We write $\mathcal{C} \cong \mathcal{D}$.

7. A natural transformation $\gamma : F \to G : \mathcal{C} \to \mathcal{D}$, where $F$, $G : \mathcal{C} \to \mathcal{D}$ are functors, is a collection

   $\gamma = \{ \gamma_b : Fb \to Gb \} \quad \forall b \in \mathcal{C}$, $G \gamma_b = \gamma_b \circ F$.

2. A natural equivalence (or natural isomorphism) is a natural transformation $\gamma \exists \forall \gamma_b : \gamma_b$ is an isomorphism.

3. The composite transformation $\eta \cdot \varepsilon : F \to H : \mathcal{C} \to \mathcal{D}$, where $\gamma : G \to H : \mathcal{C} \to \mathcal{D}$ and $\varepsilon : F \to G : \mathcal{C} \to \mathcal{D}$ are natural transformations, is defined by:

   $\eta \cdot \varepsilon = \{ \eta_b \cdot \varepsilon_b : \mathcal{C} \to \mathcal{D} \}$. 
4. Functors: \( \mathcal{B} \to \mathcal{C} \) and corresponding natural transformations from a category called the functor category \( \mathcal{F} \) with identity

\[ \text{Id}_F : F \to F : \mathcal{B} \to \mathcal{C} \] 

\( \exists \) \( \text{Id}_F : F \to F : \mathcal{B} \to \mathcal{C} \)

5. \( \gamma : G \to H : \mathcal{B} \to \mathcal{C} \) a natural transformation

a) \( F : \mathcal{C} \to \mathcal{D} \) a functor

\[ F \gamma : F.G \to F.H : \mathcal{B} \to \mathcal{D} \] is defined by

\[ (F \gamma)_b = F \gamma_b : F.Gb \to F.Hb \]

b) \( F : \mathcal{C} \to \mathcal{B} \) a functor

\[ \gamma F : G.F \to H.F : \mathcal{C} \to \mathcal{C} \] is defined by

\[ (\gamma F)_a = \gamma_{F_a} : G.Fa \to H.Fa \]

**Def 8.** \( F : \mathcal{B} \to \mathcal{C} \) is **LEFT-ADJOINT** of \( G : \mathcal{C} \to \mathcal{B} \)

(\( F \) and \( G \) two functors), or \( G \) is **right-adjoint** of \( F \) if

\( \exists \) a natural isomorphism

\[ \varphi : (F, \cdot) \leftrightarrow (\cdot, G) : \mathcal{B} \times \mathcal{C} \to \mathcal{D} \]

**Notation:** \( F \dashv G : \mathcal{B} \leftrightarrow \mathcal{C} \)

**Th 4.** Any two left-adjoints (or right-adjoints) of a functor \( F \) are naturally isomorphic.
TH 5. \[ F \downarrow G : \mathcal{B} \xrightarrow{\cong} \mathcal{C} \] is equivalent to each of the following:

1. \[ \forall b \in /\mathcal{B}/, \ c \in /\mathcal{C}/, \ (b, Gc) \cong \mathcal{C}(Fb, c). \]

2. \[ \forall b \in /\mathcal{B}/, \exists (c, \eta_b : b \to Gc) \ \text{universal from} \]
\[ b \to G \quad \text{(or free over } b \text{ w.r.t. to } G) \]
\[ \text{i.e. such that } \forall (c', \eta_{b'} : b' \to Gc') \]
\[ \exists \text{unique } \varphi : c \to c' \exists \]
\[ \begin{array}{ccc}
  b & \xrightarrow{\eta_b} & Gc \\
  f & \downarrow & \varphi \\
  & \downarrow & Gc' \\
  \end{array} \]

In that case, given \( G \), its left-adjoint \( F \) is defined by the following:

Let \( (c, \eta_b : b \to Gc) \) be universal from \( b \to G \),
\( (c', \eta_{b'} : b' \to Gc') \) be universal from \( b' \to G \)
then \( F \) sends \( b \to Fb = c \)

\[ \begin{array}{ccc}
  b & \xrightarrow{\eta_b} & Gb \\
  f & \downarrow & \varphi \\
  b' & \xrightarrow{\eta_{b'}} & Gb' \\
  \end{array} \]

\[ \text{I.e.} \quad G \varphi \cdot \eta_b = \eta_{b'} f \]

or written in terminology of (i):

\[ \implies (b, G(Fb' = c')) \cong \mathcal{C}(Fb, Fb') \]
\[ \eta_{b'} \cdot f \longleftrightarrow Ff = \varphi \]
3. Dually $\forall c \in \mathcal{C}$
   $\exists (b \in \mathcal{B}, \varepsilon_c : Fb \to c)$ universal from $F$ to $b$
   is such that $\forall (b', \varepsilon_c' : Fb' \to c)$ $\exists$ unique
   $\varphi' : b' \overrightarrow{\leftarrow} b \exists Fb \to c$ commutes.

   In this case, given $F$, its right-adjoint $G$ is defined by the following:

   Let $(b, \varepsilon_c : Fb \to c)$ be universal from $F$ to $c$
   $(b', \varepsilon_c' : Fb' \to c')$ be universal from $F$
   to $c'$
   Then $G : \text{objects } c \leftrightarrow Go = b$

   morphisms
   $\begin{array}{ccc}
   b' = Gc! & \downarrow \text{unique } \varphi' = Gf' \\
   b = Gc & \downarrow
   \end{array}$

   $\begin{array}{ccc}
   Fb' = FGo' & \xrightarrow{\varepsilon_c'} & c' \\
   F\varphi' & \downarrow & \downarrow \text{commutes, i.e. } f' \cdot \varepsilon_c' = \varepsilon_c \cdot F\varphi' \\
   Fb = FGo & \xrightarrow{\varepsilon_c} & c
   \end{array}$

   or, in the terminology of (1)

   $\mathcal{B}(F(Gc' = b'), c) \cong \mathcal{B} (Go', Go)$

   $f' \cdot \varepsilon_c' \leftrightarrow \varphi' = Gf'$

4. $\exists$ natural transformations $\gamma : \text{Id} \to G.F : \mathcal{B} \to \mathcal{B}$
   and $\varepsilon : F.G \to \text{Id} : F \to \mathcal{V}$, called the unit and
   co-unit of the adjunction, such that:
\[
\begin{array}{c}
\text{Commuting diagrams:} \\
F \eta \xrightarrow{\hom} \epsilon F \\
G \eta \xrightarrow{\hom} \epsilon G \\
\end{array}
\]

In that case, \( \forall b \in \mathcal{B}, (Fb, \eta_b : b \xrightarrow{\hom} G.Fb) \) is universal from \( b \) to \( G \),
\[
\forall c \in \mathcal{C}, (Gc, \epsilon_c : F Go \xrightarrow{\hom} c) \text{ is universal from } F \text{ to } c.
\]

**IN 9.**

1. \( \mathcal{B} \) is a subcategory of \( \mathcal{C} \) if \( \mathcal{B}/ \subseteq \mathcal{C}/ \) and they have the same composition of morphisms.

2. \( \mathcal{B} \subseteq \mathcal{C} \) is full if \( \forall b, b' \in \mathcal{B}, \mathcal{B}(b, b') \equiv \mathcal{C}(b, b') \), i.e. \( \mathcal{B} \) is completely described by its objects.

**IN 10.**

1. Categories \( \mathcal{B} \) and \( \mathcal{C} \) are equivalent if there is a functor \( F : \mathcal{B} \xrightarrow{\hom} \mathcal{C} \) which is full and faithfull (i.e. \( \mathcal{B}(b, b') = \mathcal{C}(Fb, Fb') \), \( \forall b, b' \in \mathcal{B} \) and \( \forall c \in \mathcal{C}, Fb \) for some \( b \in \mathcal{B} \)).

2. \( \mathcal{B} \) is an equivalent subcategory of \( \mathcal{C} \) if the inclusion functor satisfies this condition (i.e. \( \forall c \in \mathcal{C}, \exists (c, c') \in \mathcal{B}(c, c') \)).
TH 6. $\mathcal{C}$ and $\mathcal{E}$ are equivalent if there exist functors $\mathcal{C} \xleftarrow{G} \mathcal{E} \xrightarrow{F} \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are both naturally isomorphic to $\text{Id}_C$ and $\text{Id}_E$ respectively or if there exists a pair of functors $\mathcal{C} \xrightarrow{\eta} \mathcal{E}$ whose unit $\gamma$ and co-unit $\epsilon$ are both natural isomorphisms.

IN 11. $\mathcal{C} \subseteq \mathcal{E}$ is reflective in $\mathcal{E}$ if the inclusion functor has a left-adjoint called the reflector $R$.

TH 7. $\mathcal{C} \subseteq \mathcal{E}$ is reflective if $\forall q \in \mathcal{E}$, and $\gamma_q : q \rightarrow Rq$ in $\mathcal{E}$ for all $q \in \mathcal{E}$. There exists a unique $f : Rq \rightarrow b$ such that $g = f \cdot \gamma_q$.

LEMMA 2. (C11, page 373)

A subcategory $\mathcal{B}$ of $\mathcal{C}$ is reflective if $\mathcal{B}$ is a reflective subcategory of $\mathcal{C}$.

LEMMA 3. $F : \mathcal{C} \xrightarrow{\eta} \mathcal{E}$ and $\mathcal{C} \xleftarrow{\epsilon} \mathcal{B}$ are full subcategories of $\mathcal{C}$ and $\mathcal{E}$ respectively. $F$ factors through $\eta$ and $G$ factors through $\epsilon$. Yielding $F_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{E}$ and $G_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{B}$ respectively.
1. A product of \( \{ c_\alpha \} \) is an object \( c \in \mathcal{C} \) together with a family \( \{ p_\alpha : c \to c_\alpha \} \) of \( \mathcal{C} \)-morphisms \( \Rightarrow \forall \) families \( \{ f_\alpha : x \to c_\alpha \} \) of \( \mathcal{C} \)-morphisms, \( x \in \mathcal{C} \), \( \exists \) unique \( \mathcal{C} \)-morphism \( f : x \to c \) \( \exists \) \( \begin{array}{c} o \\ \downarrow \\ o_\alpha \\ \downarrow p_\alpha \\ c \\ \downarrow f \\ x \end{array} \) \( \text{commutes} \),

we write \( c = \prod c_\alpha \).

2. A co-product of \( \{ c_\alpha \} \) is \( c' \in \mathcal{C} \) with a family \( \{ \text{in}_\alpha : c_\alpha \to c' \} \) \( \Rightarrow \forall \) families \( \{ g_\alpha : c_\alpha \to x \} \) \( \exists \) a unique \( \mathcal{C} \)-morphism \( g : c' \to x \) \( \exists \begin{array}{c} o_\alpha \\ \downarrow \text{in}_\alpha \\ c_\alpha \\ \downarrow g_\alpha \\ x \\ \downarrow g \\ c' \end{array} \) \( \text{commutes} \),

we write \( c' = \coprod c_\alpha \).

**Note:** \( \prod c_\alpha \) and \( \coprod c_\alpha \) are unique up to isomorphism.
1. \( \forall n \in \mathbb{N}, \exists u_n = u \in /c/ \)
the countable copower of \( u \) is \( u^+ = \bigcup_{n \in \mathbb{N}} u_n \)
\( \Rightarrow \exists \) a unique \( \mathbb{S} \)-morphism
\( z : u^+ \rightarrow u^+ \ni z \cdot \text{in}_n = \text{in}_{n+1} \)
i.e. \( \exists u \xrightarrow{\text{in}_n} u^+ \xrightarrow{z} u^+ \)
\( \text{commutes \( \forall n \in \mathbb{N} \).} \)

2. \( \forall n \in \mathbb{N}, \exists \{y_n = y \in /c/ \}
the countable power of \( y \) is \( y^x = \prod_{n \in \mathbb{N}} y_n \)
\( \Rightarrow \exists \) a unique morphism \( z' : y^x \rightarrow y^x \ni p_{n} \cdot z' = p_{n+1} \)
i.e. \( \exists y \xrightarrow{p_n} y^x \xrightarrow{z'} y^x \)
\( \text{commutes \( \forall n \in \mathbb{N} \).} \)

**Lemma 4.**
1. \( \exists u^+ \) the countable copower of \( u \in /c/ \)
\( \Rightarrow \forall \text{pair} (g : u \rightarrow x, f : x \rightarrow x) \) of \( \mathbb{S} \)-morphisms
\( \exists \) a unique \( \mathbb{S} \)-morphism \( \delta^+ : u^+ \rightarrow x \)
\( \xrightarrow{\text{in}_n} u^+ \xrightarrow{\delta^+} x \)
\( \text{and} \)
\( \xrightarrow{g} x \xrightarrow{\delta^+} x \)
\( \text{commute.} \)
\( \delta^+ \) is uniquely defined by \( \delta^+.\text{in}_n = f^n.g \ \forall n \in \mathbb{N} \),
i.e. \( \forall n \in \mathbb{N}, \ x \xrightarrow{\text{in}_n} u^+ \xrightarrow{f^n.g} x \)
\( \delta^+ \text{commutes.} \)
2. \( \exists y^x \) the countable power of \( y \in /\mathcal{C}/ \)

\[ \Rightarrow \forall \text{ pairs } (h: x \rightarrow y, f: x \rightarrow x) \text{ of } \mathcal{C}\text{-morphisms} \]

\[ \exists \text{ a unique } \mathcal{C}\text{-morphism } \omega: x \rightarrow y^x \]

\[ y \xleftarrow{P_n} y^x \xrightarrow{\omega \text{ and } z'} y^x \]

\[ y \xleftarrow{f} x \xrightarrow{x} x \]

\( \omega \) is uniquely defined by

\[ P_n \cdot \omega = h \cdot f^n \text{ i.e. } y \xleftarrow{P_n} y^+ \xrightarrow{\omega} x \]

\[ h \cdot f^n \xleftarrow{\omega} x \text{ commutes } \forall n \in \mathbb{N}. \]

**TH 8.** A functor with a right-adjoint preserves all co-products.

**IN 13.**

1. \( \mathcal{E} \) is a class of epis in category \( \mathcal{C} \), \( \mathcal{M} \) a class of monics in \( \mathcal{B} \), both closed under composition and \( \forall i \) isomorphism in \( \mathcal{B} \), \( i \in \mathcal{E} \) and \( i \in \mathcal{M} \).

2. \( f: a \rightarrow b \) has an \( (\mathcal{E}, \mathcal{M}) \) factorization if

\[ f = a \xrightarrow{s} \inf \xrightarrow{m} b, \text{ where } s \in \mathcal{E}, \ m \in \mathcal{M}. \]

This factorization is said to be **unique** if we also have \( f = a' \xrightarrow{s'} (\inf)' \xrightarrow{m'} b, \) then \( \exists \) an isomorphism \( h: (\inf)' \rightarrow \inf \)

the diagram

\[ \begin{array}{ccc}
\mathcal{E} & \xrightarrow{h} & \mathcal{B} \\
\downarrow m' & & \downarrow m \\
\mathcal{M} & \xrightarrow{\inf} & \mathcal{B}
\end{array} \]

commutes.
3. A category $\mathcal{C}$ is called an $(\mathcal{E}, \mathcal{M})$ category if it is uniquely $(\mathcal{E}, \mathcal{M})$ factorizable, i.e., $\forall p \in \mathcal{C}, \exists$ a unique $(\mathcal{E}, \mathcal{M})$ factorization.

4. $\mathcal{C}$ is said to have the $(\mathcal{E}, \mathcal{M})$ diagonalization property provided that $\forall$ commutative square in $\mathcal{C}$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$,

- $a \xrightarrow{e} b$ with $f \downarrow k \rightarrow \downarrow g$
- $a' \xleftarrow{m} b'$. $k$ is unique as $e$ is epi, or $m$ is monic).

**TH 9.** $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$ category $\iff$ $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$ factorizable and has the $(\mathcal{E}, \mathcal{M})$ diagonalization property.

**NOTE:** $\mathcal{A}$, $\mathcal{R}$-Mod and $\mathcal{G}$ are $(\mathcal{E}, \mathcal{M})$ categories with

- $\mathcal{E} = \{ e \mid e$ is onto$\}$
- $\mathcal{M} = \{ m \mid m$ is $1-1$ $\}$

with $e, m$ maps in $\mathcal{A}$, homo-morphisms in $\mathcal{R}$-Mod or $\mathcal{G}$.

**TH 10.** Adjunctions can be composed: specifically

- $F \dashv G : \mathcal{A} \rightleftharpoons \mathcal{B}$
- $S \dashv T : \mathcal{B} \rightleftharpoons \mathcal{G}$

$S,F \dashv G,T : \mathcal{A} \rightleftharpoons \mathcal{G}$. 
REFERENCES


