MATRIX ANALYSIS OF BOWSTRING ARCHES

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ABSTRACT

Matrix analysis of bowstring arches is presented in this dissertation. The arch is considered as a curved member in the flexibility method, and is discretized into straight members in the stiffness method. The results of the two methods are compared. The flexibility method is a suitable method for the analysis of bowstring arches, however, it is advantageous to use the stiffness method if the cross sections of the structural members vary. There are no thermal stresses in a single span bowstring arch as the result of temperature changes, if the structure is made of one material. A computer program is developed for the flexibility method for practical applications. Preliminary design results obtained by the simplified method, may be used as input data in the computer program. Final analysis is carried out by computer program. In a bowstring arch of a given span, the choice of the geometry of the arch and the selection of suitable proportion of sectional properties of the arch and the girder are the important factors for the optimum design. The final result is not much affected by neglecting individual axial deformation, such as the shortening of arch rib, or the elongations of the tie girder and the hanger rods. However, the result may be significantly changed, when axial deformations in all members of the structural system are neglected.

ACKNOWLEDGMENT

The author wishes to express his special appreciation to Dr. Pen Jeng Fang who gave inspiration and guidance during the course of this study. The help of Dr. Matthew McC. Douglass is also gratefully acknowledged.

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INTRODUCTION

The bowstring arch, also called girder-tied arch, is a structure consisting of an arch rib and a tie girder. The arch and the tie girder are fastened together at the supports and connected to each other between the supports by means of equally spaced vertical suspension rods as shown in Fig.(1-1).

Sometimes in practice, the system is analyzed regarding the girder as a tie to absorb the horizontal force of the arch at the supports. The approach considerably simplifies the analysis, which is similar to the analysis of a two-hinged arch, except that the effect of the tie girder elongation must be accounted for. However, the arch and the tie girder act jointly to resist any loads on the tie girder, resulting in a highly indeterminate structure.

While the analysis of such a structure is usually time-consuming by hand calculation, the development of high-speed electronic computer and matrix methods of structural analysis make it possible to use a rigorous analysis, thus leading to more economical design.

This investigation deals with the analysis of a bowstring arch bridge consisting of steel members. The scope of the study includes specifically the following problems:

1. Analysis of a single span bowstring arch by matrix methods (flexibility and stiffness methods).
2. Proof of absence of thermal stresses.
3. Practical application with example.
4. Computer programs for practical applications.
5. Discussion and comparision of the results of the two methods.

6. Investigation of the effects of member axial deformations, properties of the members, and the geometry of the structure.

CHAPTER 1

THEORETICAL ANALYSIS

1.1 Analysis by Flexibility Method

1.1.1 Derivation of Basic Formulae

The structure to be analyzed is shown in Fig.(2-1), which has a hinged connection at the left support, and a roller support at the right, with hanger rods connecting the arch and the tie girder.

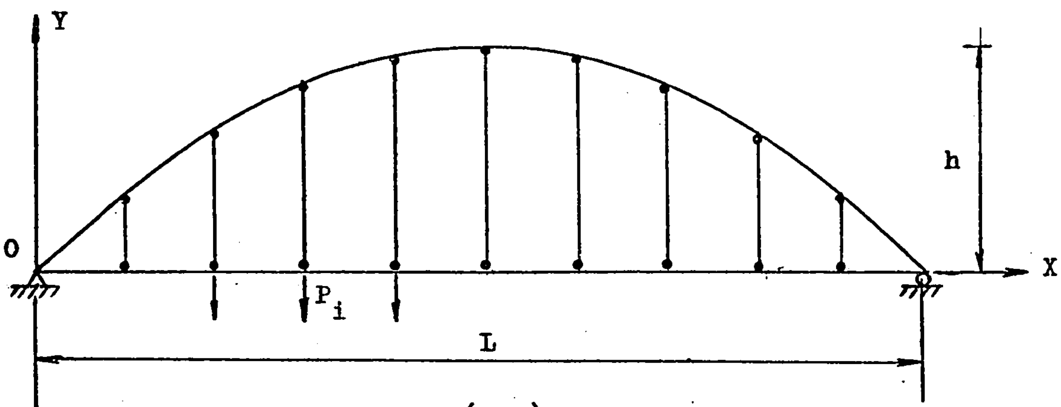


Fig.(1-1)

As the rigidity of the hanger is much smaller than that of the arch and of the tie gider, we may approximate the hanger rod connections to be hinged as shown.

With the above simplification, we consider the structure as a plane frame, being statically determinate externally, but indeterminate internally. The degree of indeterminacy can be determined by the following formulae:

$$n_r = 3j - (m + 2n_h + 1) \qquad (1\text{-}1)$$

where n_r = the number of redundants ;

m = the number of members in the framework;

j = the number of joints;

n_h = the number of hangers, and is equal to n-2

The released structure obtained by cutting all the hanger rods and replacing the rigid joints at the junctures of the arch and girder, by pins, is shown in Fig.(1-2), where the unknown forces Q_i acting at the hinged joints have replaced the hanger rods and the internal moments at the rigid joints.

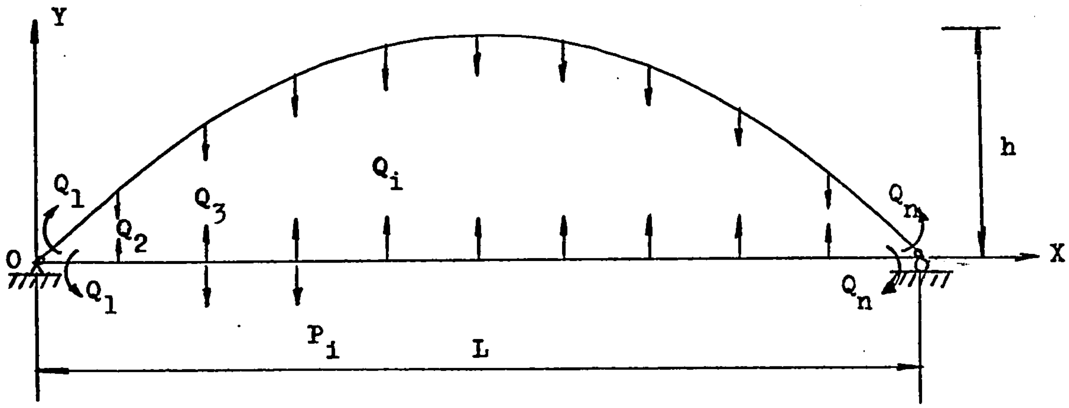


Fig.(1-2)

It is important to note at this point that the released structure is not determinate, but is indeterminate to one degree, hence there will be one less release than the number of redundants. In usual structural analysis, a released structure is statically determinate, whereby the complication of finding the $[F]$ matrix is avoided. In the particular structure under consideration, it is advantageous not to release the horizontal thrust at the right hand end in order to satisfy the boundary conditions. Had it been ortherwise, an extra set of equilibrium equations would be necessary to satisfy the boundary conditions in order to close the gap between the cut of the tie girder and the arch [2].

The following symbols are used in the analysis:

n The number of releases; $n = n_r - 1$

Q_i The redundants;

D_{QLi} The displacements in the released structure corresponding to the redundants and due to the applied load;

F_{ij} The displacements in the released structure corresponding to the redundants and due to unit values of the redundants;

D_{Qi} The actual vertical displacements corresponding to the redundants.

It is convenient to divide the structure into two parts; namely, the arch, and the tie girder. Superscript 'a' refers to the arch, and 'g' to the girder.

The equilibrium condition for the arch may be written as:

$$D^a_{Q1} = D^a_{QL1} + F^a_{11}Q^a_1 + F^a_{12}Q^a_2 + \dots + F^a_{1n}Q^a_n$$

$$D^a_{Q2} = D^a_{QL2} + F^a_{21}Q^a_1 + F^a_{22}Q^a_2 + \dots + F^a_{2n}Q^a_n$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D^a_{Qn} = D^a_{QLn} + F^a_{n1}Q^a_1 + F^a_{n2}Q^a_2 + \dots + F^a_{nn}Q^a_n$$

In matrix form, the above equations become,

$$\begin{bmatrix} D_{Q1} \\ D_{Q2} \\ \cdot\cdot \\ \cdot \\ \cdot \\ D_{Qn} \end{bmatrix}^a = \begin{bmatrix} D_{QL1} \\ D_{QL2} \\ \cdot \\ \cdot \\ \cdot \\ D_{QLn} \end{bmatrix}^a + \begin{bmatrix} F_{11} & F_{12} & F_{13}\cdots & F_{1n} \\ F_{21} & F_{22} & F_{23}\cdots & F_{2n} \\ \cdot & \cdot & \cdot\cdots & \cdot \\ \cdot & \cdot & \cdot\cdots & \cdot \\ \cdot & \cdot & \cdot\cdots & \cdot \\ F_{n1} & F_{n2} & F_{n3} & F_{nn} \end{bmatrix}^a \begin{bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ Q_n \end{bmatrix}^a$$

or $$\left[D_Q\right]^a = \left[D_{QL}\right]^a + \left[F\right]^a \left[Q\right]^a \qquad (1\text{-}2)$$

Similarly for the tie girder,

$$\left[D_Q\right]^g = \left[D_{QL}\right]^g + \left[F\right]^g\left[Q\right]^g \tag{1-3}$$

Let D^h_{Q1}, D^h_{Q2}, ... D^h_{Qn}, denote the elongations of the hangers under the action of the redundants.

$$\begin{bmatrix} D_{Q1} \\ \\ D_{Q2} \\ \vdots \\ D_{Qn} \end{bmatrix}^h = \begin{bmatrix} \frac{Q_1 h_1}{E\ A_1} \\ \frac{Q_2 h_2}{E\ A_2} \\ \vdots \\ \frac{Q_n h_n}{E\ A_n} \end{bmatrix}^h \tag{1-4}$$

where h_1, h_2, ..., h_n, are the lengths of the hangers, $(h_1 = h_n = 0)$, and A_1, A_2, ..., A_n, are the areas of the hanger cross sections.

Consider the whole structure next. The vertical displacement of the girder at the point of redundant action is equal to the sum of the displacements corresponding to the same point in the arch, and the elongation of the corresponding hanger rod, i.e.,

$$\left[D_Q\right]^g = \left[D_Q\right]^a + \left[D_Q\right]^h \tag{1-5}$$

Substituting Equations (1-2), (1-3) and (1-4) into Equation (1-5), we have

$$\left[D_{QL}\right]^g + \left[F\right]^g\left[Q\right]^g = \left[D_{QL}\right]^a + \left[F\right]^a\left[Q\right]^a + \left[D_Q\right]^h$$

Rearranging,

$$\left[D_{QL}\right]^g - \left[D_{QL}\right]^a = \left[F\right]^a\left[Q\right]^a - \left[F\right]^g\left[Q\right]^g + \left[D_Q\right]^h$$

Since,

$$-\left[Q\right]^a = \left[Q\right]^g = -\left[Q\right] \tag{1-6}$$

$$\left[D_{QL}\right]^g - \left[D_{QL}\right]^a = -\left\{\left[F\right]^a + \left[F\right]^g + \left[I\right]\left[B\right]\right\}\left[Q\right] \tag{1-7}$$

Let $$\left[D_{QL}\right] = \left[D_{QL}\right]^{g} - \left[D_{QL}\right]^{a} \tag{1-7a}$$

and $$\left[F\right] = \left[F\right]^{a} + \left[F\right]^{g} + \left[I\right]\left[B\right] \tag{1-7b}$$

where $\left[I\right]$ is a unit matrix,

$$\left[B\right] = \begin{bmatrix} \frac{h_1}{EA_1} \\ \frac{h_2}{EA_2} \\ \vdots \\ \frac{h_n}{EA_n} \end{bmatrix} \tag{1-8}$$

Upon substituting Equations (1-7a) and (1-7b) into (1-7), and solving, the result is

$$\left[Q\right] = - \left[F\right]^{-1} \left[D_{QL}\right] \tag{1-9}$$

The elongation of the hangers is small relative to the deflections of the arch and the tie girder, and can be neglected. Thus, Equation (1-7b) is reduced to

$$\left[F\right] = \left[F\right]^{a} + \left[F\right]^{g} \tag{1-7c}$$

The effects of elongation on the hanger forces can be checked by comparing the $\left[Q\right]$ values obtained when $\left[F\right]$ is determined by Eqs. (1-7b) and (1-7c) respectively.

The vertical deflection of the arch and the tie girder can be obtained by substituting the $\left[Q\right]$ values into Equations (1-2) and (1-3).

The horizontal displacement of the arch and the tie girder does not affect the solution for the redundants; however, such information is sometimes needed for cambering and for the clearance of expansion joint .While the horizontal displacement of the tie girder can be found readily, the calculation of the horizontal displacement of the arch is more involved. The latter will be discussed in the following.

Suppose the structure is subjected to a set of external loads as shown in Fig.(1-1). Assume first that the tie girder is inextensible under the action of the axial force which is induced from the external load. Under this assumption the tied arch approximates a two-hinged arch with moments applied at the two ends. The flexibility matrix of the horizontal displacement of the arch $[FH]^a$ is next found.

Actually the tie girder is elongated under the action of the axial force. The elongation at any point i on the girder can be expressed according to Hooke's law

$$d_i = \frac{T\ X_i}{E\ A_g} \tag{1-10}$$

where T is the axial force, X_i is the distance from left support to point i. Since the structure is hinged at the left end, and supported by a roller at the right end, the horizontal displacement will be from left to right. To satisfy the boundary condition at the right end, the horizontal displacement of the arch and the tie girder must be equal; i. e.,

$$d = \frac{TL}{EA_g}$$

The second step is to find the horizontal displacement of the untied arch, under the action of a horizontal force T' at the right end, which causes the horizontal displacement at the right end of the arch equal to

$$d = \frac{TL}{EA_g}$$

Because $d = \frac{TL}{EA_g}$ is a known value, T' can be found as follows:

since $$\int_0^L \frac{T'Y^2}{EI_a}(ds) = d$$

hence $$T' = d\,EI_a \Big/ \int_0^L Y^2\,(ds) \tag{1-10a}$$

The horizontal displacement at any point of the arch under the second condition can be determined by the unit load theorem.

$$d_{hi} = \int_0^L \frac{Mm_i}{EI_a}(ds) \tag{1-11}$$

where $M = T'Y,$

$m_i = Y - V_iX\ ,\quad 0 \leq X \leq X_i\ ;$

$m_i = Y_i - V_iX,\quad X_i \leq X \leq L\ .$

Finally, the true displacement of the arch will be obtained by combining conditions one and two, as depicted by the following figure.

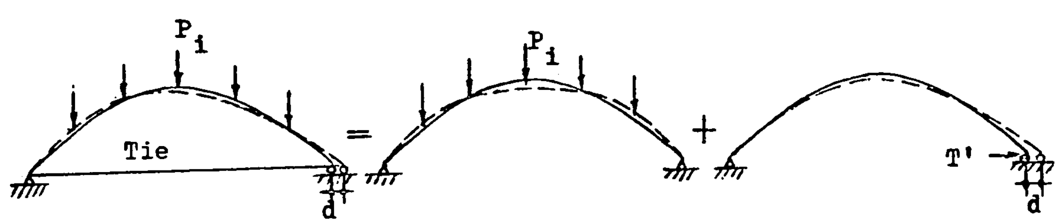


Fig.(1-3)

The total horizontal displacement at any point on the arch can be expressed as

$$[DH]^a = \left[[Q] + [P]^a\right]^T [FH]^a + [d_h]^a \tag{1-12}$$

The horizontal thrust TH in the actual structure can be found by the formula

$$TH = [H][Q] \tag{1-13}$$

where $[H]$ is the matrix of the horizontal thrusts due to unit loads on the released structure corresponding to the redundants.

The axial force and shear force in the arch are, respectively,

$$N_i = V \sin\phi + (TH) \cos\phi \quad , \qquad (1\text{-}14)$$

and

$$V_i = V \cos\phi - (TH) \sin\phi \qquad (1\text{-}15)$$

where V is the vertical shear at section i , ϕ the angle of tangent of the curve with respect to the horizontal axis at point i.

Other actions such as moments can be found from static equilibrium.

The basic formulae having been established, the related matrices shall next be derived.

1.1.2 Matrix $[F]^a$

The flexibility matrix $[F]^a$, for vertical displacements of the arch , can be found by the unit load method.

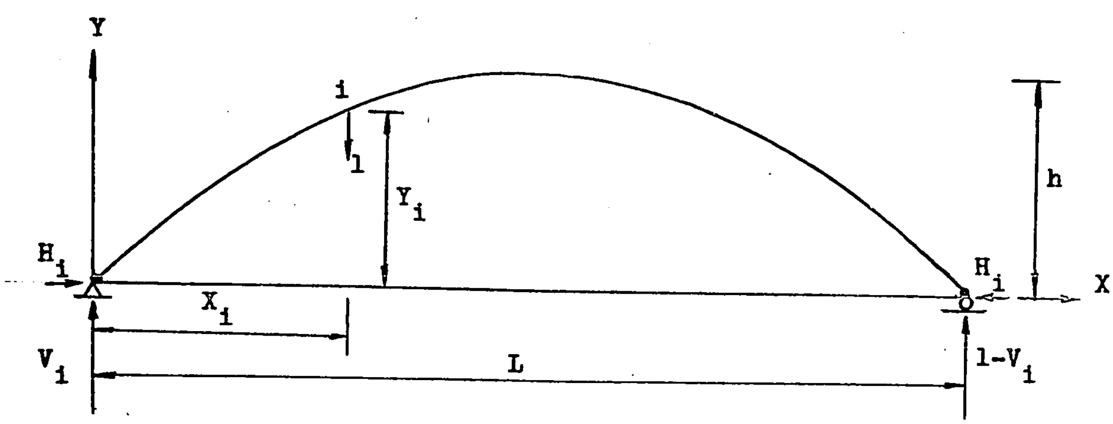


Fig.(1-4)

Since the released structure is indeterminate to one degree, it is necessary first to establish the thrust matrix $[H]$ before the matrix $[F]^a$ is formulated.

When a unit load is applied at any point i on the arch, moment and axial force will be induced in the arch. Applying the strain energy theorem we have,

$$U = \int_0^S \frac{m_i^2\, ds}{2EI_a} + \int_0^S \frac{n_i^2\, ds}{2EA_a} \tag{1-16}$$

where

$$m_i = V_i X - H_i Y\,, \qquad 0 \leq X \leq X_i;$$

$$m_i = V_i X - H_i Y - (X - X_i)\,, \qquad X_i \leq X \leq L\,;$$

$$n_i = V_i \sin\phi + H_i \cos\phi\,, \qquad 0 \leq X \leq X_i;$$

$$n_i = V_i \sin\phi - \sin\phi + H_i \cos\phi\,, \quad X_i \leq X \leq L\,.$$

These can also be written in the form

$$m_i = m_o - H_i Y\,, \qquad n_i = V_o \sin\phi + H_i \cos\phi \tag{1-17}$$

in which m_o is the moment of a simple beam, and V_o is the vertical shear of a simple beam.

By Castigliano's theorem, the horizontal displacement is evaluated by

$$\frac{\partial U}{\partial H} = \int_0^S \frac{m_i \dfrac{dm_i}{dH}\, ds}{EI_a} + \int_0^S \frac{n_i \dfrac{dn_i}{dH}\, ds}{EA_a}$$

$$= H_i \int_0^S \frac{Y^2 ds}{EI_a} - \int_0^S \frac{m_o Y\, ds}{EI_a} + \int_0^S \frac{V_o \sin\phi \cos\phi\, ds}{EA_a}$$

$$+ H_i \int_0^S \frac{\cos^2\phi\, ds}{E\, A_a}$$

Since in the released structure the arch and the girder are tied together, the horizontal displacement of the arch must equal the elongation of the girder under the action of thrust H_i , thus we have

$$\frac{\partial U}{\partial H} = \frac{-H_i L}{EA_g}$$

$$H_i = \frac{\int_0^S \frac{m_o Y\, ds}{EI_a} - \int_0^S \frac{V_o \sin\phi \cos\phi\, ds}{E A_a}}{\int_0^S \frac{Y^2}{EI_a} ds + \int_0^S \frac{\cos^2\phi}{EA_a} ds + \frac{L}{EA_g}} \qquad (1\text{-}18)$$

Neglecting the effect of V_o , we have

$$H_i = \frac{\int_0^S \frac{m_o Y ds}{EI_a}}{\int_0^S \frac{Y^2 ds}{EI_a} + \int_0^S \frac{\cos^2\phi\, ds}{EA_a} + \frac{L}{EA_g}} \qquad (1\text{-}18a)$$

Similarly, we can apply a unit load at any other point to find the corresponding thrust thus generating the matrix $[H]$.

After finding the horizontal thrust due to a unit load corresponding to redundant by Eq.(1-18), the moment and axial force at any point on the arch due to the applied unit load can be defined. The deflection at point i is

$$F_{ii} = \int_0^S \frac{m_i^2\, ds}{EI_a} + \int_0^S \frac{n_i^2\, ds}{EA_a} \qquad (1\text{-}19)$$

To determine the deflection at point j due to the applied unit load at point i, we write

$$F_{ij} = \int_0^S \frac{m_i m_j ds}{EI_a} + \int_0^S \frac{n_i n_j ds}{EA_a} \qquad (1\text{-}20)$$

where m_j and n_j are the moment and axial force, repectively, due to a unit load at point j.

In formulae (1-15) and (1-16), the first term is predominant; the second term represents the influence of the axial force and is usually small, consequently it may be neglected(3).

1.1.3 Matrix $\left[FH\right]^{a}$

As before we can also find the flexibility matrix of horizontal movement of the arch due to the vertical load by the unit load method. As the released structure is indeterminate, we must first determine the horizontal thrust.

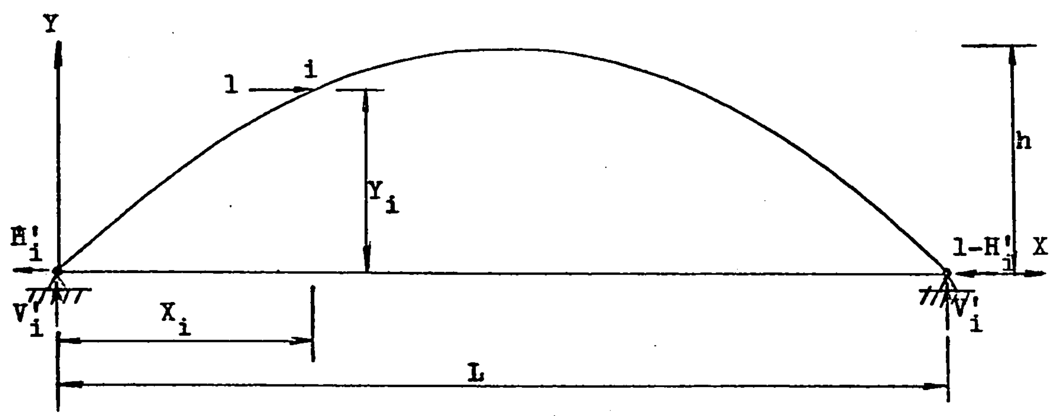


Fig.(1-5)

Applying a unit load on the released structure, as shown in Fig.(1-5) the horizontal thrust, H'_i, will be induced at the left end of the arch, and $(1-H'_i)$ at the right end.

Referring to Article 1, the tie girder is first assumed to be inextensible under the action of the applied load. The horizontal movement at both supports of the arch must be equal to zero. This condition can be expressed by the area moment theorem:

$$d_{hi} = \int_0^S Y \frac{m'_i}{EI_a}(ds) = 0 \tag{1-21}$$

where
$$m_i' = -H_i'Y + V_i'X, \qquad 0 \leq X \leq X_i;$$
$$m_i' = -H_i'Y + V_i'X + (Y - Y_i), \quad X_i \leq X \leq L. \tag{1-21a}$$

Substituting m_i into Equation (1-21)

$$\int_0^{X_i} \frac{Y(-H_iY + V_iX)}{EI_a} ds + \int_{X_i}^{L} \frac{Y(-H_iY + V_iX + (Y-Y_i))}{EI_a} ds = 0$$

or
$$H_i = \frac{\int_0^L XYds + \int_{X_i}^L Y^2ds - Y_i \int_{X_i}^L Yds}{\int_0^L Y^2ds} \tag{1-22}$$

The horizontal thrust due to the applied horizontal unit load on the arch is determined by Equation (1-22). Once H_i is known, the moment m_i on the arch is defined. The general expression for matrix $\left[FH\right]^a$ is similar to Equations (1-18) and (1-19).

$$FH_{ii} = \int_0^L \frac{m_i m_i'}{EI_a} ds$$
$$FH_{ij} = \int_0^L \frac{m_i m_j'}{EI_a} ds \tag{1-22a}$$

where m_i and m_j are as defined in the Equations (1-18) and (1-19), and m_i' , m_j' are defined by Equation (1-21a).

1.1.4 Matrix $\left[F\right]^g$

The flexibility matrix $\left[F\right]^g$ of the vertical displacements of the tie girder for the released structure can also be found by the unit load theorem.

Because the released structure are tied at the junctions of the arch and the tie girder, horizontal thrust will be induced when the arch is loaded. However, this horizontal thrust has only a small influence on the $[F]^g$ matrix and hence may be neglected.

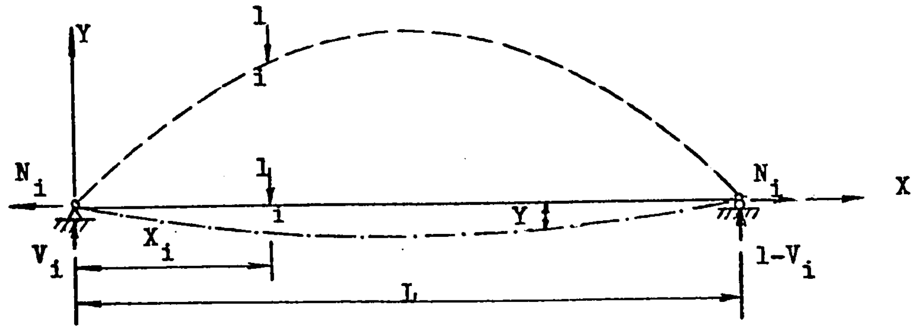


Fig.(1-6)

A unit load is applied on the arch and on the tie girder simultaneously at the corresponding points i lying on the same vertical line. Let ΔY be the general expression of the deflection of the tie girder, then the deflection at point i is given by

$$F_{ii} = \int_0^L \frac{m_i^2\,dx}{EI_g}$$

where
$$m_i = V_i X - N_i(\Delta Y) \quad , \qquad 0 \leq X \leq X_i \; ;$$
$$m_i = V_i X - N_i(\Delta Y) - (X - X_i) \; , \quad X_i \leq X \leq L \; .$$

Since N_i and ΔY are small quantities, $N_i(\Delta Y)$ is also small in comparison with the other terms in the above formula and may be neglected.

$$m_i = V_i X \quad , \qquad 0 \leq X \leq X_i \; ;$$
$$m_i = V_i X - (X - X_i) \quad , \qquad X_i \leq X \leq L \; .$$

Similarly we can write the approximate expression for the deflection at point j due to unit load at i as

$$F_{ij} = \int_0^L \frac{m_i m_j}{EI_g}\, dx$$

A more precise form of $\left[F\right]^g$ can be derived using the formulae given by Timoshenko[8]. Refering to Fig.(1-7), the following expressions are obtained:

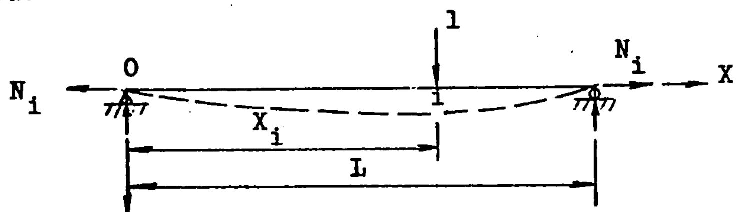


Fig.(1-7)

Deflection $\quad Y = -\dfrac{\sinh P(L-X_i)}{N_i P \sinh PL}\sinh PX + \dfrac{(L-X_i)X}{N_i L},$

$\qquad\qquad\qquad\qquad 0 \leq X \leq X_i, \qquad (1\text{-}23)$

Slope $\quad \dfrac{dY}{dX} = -\dfrac{\sinh P(L-X_i)}{N_i P \sinh PL}\cosh PX + \dfrac{(L-X_i)}{N_i L},$

Deflection $\quad Y = -\dfrac{\sinh PX_i}{N_i P \sinh PL}\sinh P(L-X) + \dfrac{X_i(L-X)}{N_i L},$

$\qquad\qquad\qquad\qquad X_i \leq X \leq L, \qquad (1\text{-}24)$

Slope $\quad \dfrac{dY}{dX} = \dfrac{\sinh PX_i}{N_i P \sinh PL}\cosh P(L-X) - \dfrac{X_i}{N_i L},$

where $\quad P = \sqrt{\dfrac{N_i}{EI}}$

1.1.5 Matrices $\left[D_{QL}\right]^a$ and $\left[D_{QL}\right]^g$

The column matrices $\left[D_{QL}\right]^a$ and $\left[D_{QL}\right]^g$ are the deflections of the arch and the tie girder, respectively, in the released structure due to the applied loads corresponding to the redundants. Finding $\left[D_{QL}\right]^a$ and $\left[D_{QL}\right]^g$ is simple once the matrices $\left[F\right]^a$, and $\left[F\right]^g$ have been determined. Thus

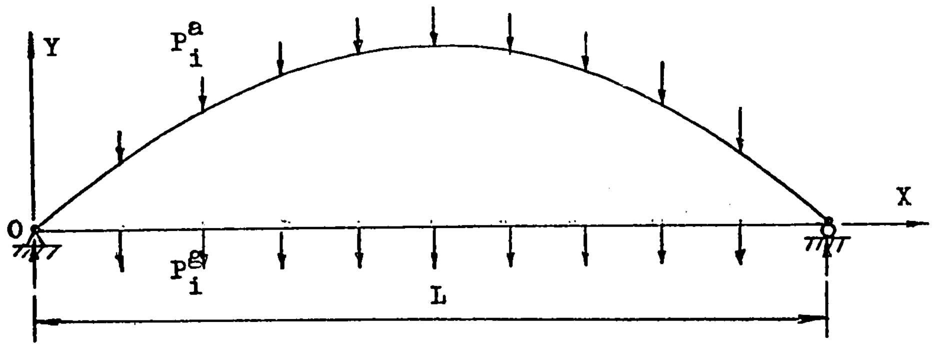


Fig.(1-8)

$$\left[D_{QL}\right]^a = \left[F\right]^a \left[P\right]^a \tag{1-25}$$

$$\left[D_{QL}\right]^g = \left[F\right]^g \left[P\right]^g \tag{1-26}$$

where matrices $\left[P\right]^a$ and $\left[P\right]^g$ are the loads applied to the arch and the tie girder, respectively.

1.2 Analysis by Stiffness Method

While in the flexibility method the unknown quantities are the redundant actions, in the stiffness method the unknowns are the joint displacements in the structure; therefore, in the latter method, the number of unknowns is equal to the number of degrees of freedom of the joints (i.e., degree of kinematic indeterminacy). In a plane frame, the maximum number of degrees of freedom at a joint is 3. Assuming n_f is the number of degrees of freedom, n_j the number of joints, and n_{rs} the number of restraints in the structure, we have the following formula:

$$n_f = 3n_j - n_{rs} \tag{1-27}$$

The total number of degrees of freedom of a bowstring arch is very large. There are three restraints in a single-span bowstring arch. Substituting n_{rs} =3 into the above formula, the number of degrees of freedom of a single span bowstring arch can be readily written as

$$n_f = 3 (n_j - 1) \tag{1-27a}$$

An example is shown in Fig. (1-9)

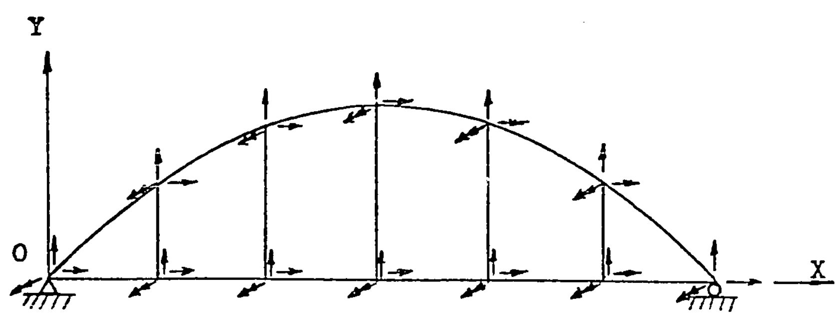


Fig.(1-9)

1.2.1 Basic Formulae

Suppose the structure in the original system is subjected to the external loads P_1, P_2, ... , P_n, as shown in Fig.(1-10), then a set of equilibrium equations for the structure can be written. At each end of a member we shall put a restraint on each corresponding degree of freedom, resulting in fixed end actions being developed at each joint in the structure.

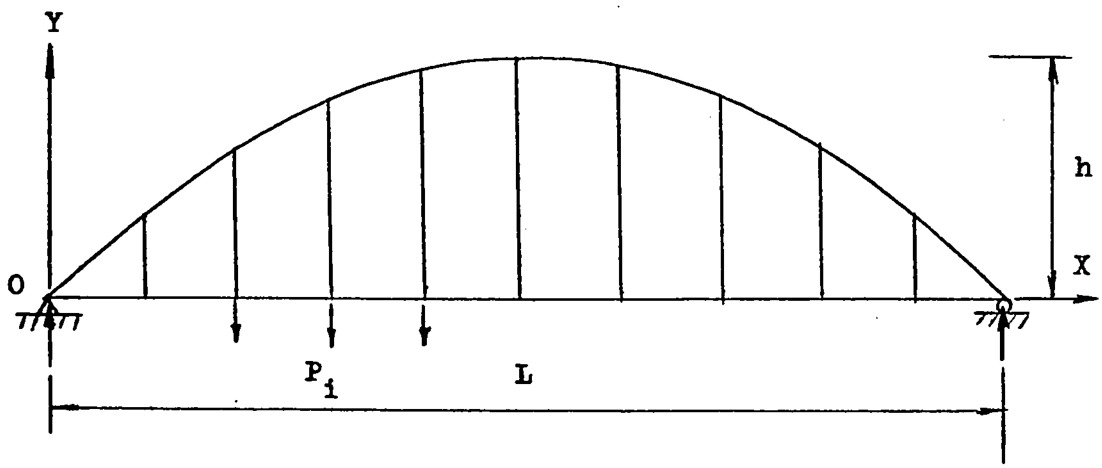


Fig.(1-10)

Let A_{DL1}, A_{DL2}, ..., A_{DLn}, be the forces due to external loads in the restrained structure corresponding to the unknown displacements, D_1, D_2, ..., D_n, denote the unknown displacements, and S_{11}, S_{12}, ..., S_{nn}, the actions in the restrained structure due to unit values of displacements corresponding to the unknown displacements repectively. If A_{D1}, A_{D2},..., A_{Dn}, are the actions in the actual structure corresponding to the unknown displacements, then the equilibrium equation is written as follows:

$$\begin{bmatrix} A_{D1} \\ A_{D2} \\ \vdots \\ A_{Dn} \end{bmatrix} = \begin{bmatrix} A_{DL1} \\ A_{DL2} \\ \vdots \\ A_{DLn} \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \cdots & S_{1n} \\ S_{21} & S_{22} \cdots & S_{2n} \\ \vdots & \vdots \;\vdots\vdots\vdots & \vdots \\ S_{n1} & S_{n2} \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}$$

or

$$\begin{bmatrix} A_D \end{bmatrix} = \begin{bmatrix} A_{DL} \end{bmatrix} + \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \tag{1-28}$$

$$\begin{bmatrix} D \end{bmatrix} = \begin{bmatrix} S \end{bmatrix}^{-1} \left\{ \begin{bmatrix} A_D \end{bmatrix} - \begin{bmatrix} A_{DL} \end{bmatrix} \right\} \tag{1-29}$$

For the external loading as shown in Fig.(1-10),

$$\begin{bmatrix} A_{DL} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

and

$$\begin{bmatrix} D \end{bmatrix} = \begin{bmatrix} S \end{bmatrix}^{-1} \begin{bmatrix} A_D \end{bmatrix} \tag{1-30}$$

where the matrix $\begin{bmatrix} S \end{bmatrix}$ is the stiffness matrix of the structure.

After the joint displacement matrix $\begin{bmatrix} D \end{bmatrix}$ is obtained, the next step is to find the member end actions and the reactions, for which the equilibrium equations are,

$$\begin{bmatrix} A_M \end{bmatrix}^i = \begin{bmatrix} A_{ML} \end{bmatrix}^i + \begin{bmatrix} S_M \end{bmatrix}^i \begin{bmatrix} R \end{bmatrix}^i \begin{bmatrix} D_J \end{bmatrix}^i \tag{1-31}$$

$$\begin{bmatrix} A_R \end{bmatrix} = \begin{bmatrix} A_{RL} \end{bmatrix} + \begin{bmatrix} S_{RD} \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \tag{1-32}$$

in which $\begin{bmatrix} A_M \end{bmatrix}^i$ is the matrix of member end actions in the actual structure; $\begin{bmatrix} A_{ML} \end{bmatrix}^i$ is the member end actions in the restrained structure; $\begin{bmatrix} S_M \end{bmatrix}^i$ is the member stiffness matrix with respect to the member axes; $\begin{bmatrix} A_R \end{bmatrix}$ is the matrix of reactions in the actual structure; $\begin{bmatrix} A_{RL} \end{bmatrix}$ is the matrix matrix of support reactions in the restrained structure due to loads; and $\begin{bmatrix} S_{RD} \end{bmatrix}$ is the matrix of support reactions due to unit values of the joint displacements; $\begin{bmatrix} D_J \end{bmatrix}^i$ is the vector of joint displacements for the ends of member i .

1.2.2 Approximate Method

While the arch is a curved member, it can be analyzed by discretizing the arch into a series of straight members as shown.

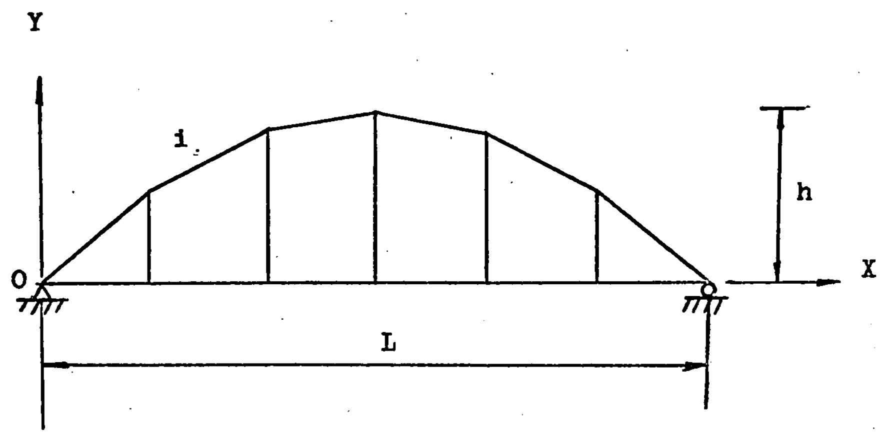


Fig.(1-11)

We shall assume that all the nodes are within the curve of the arch and that they coincide with the panel points. Elemental members are formed by jointing adjacent nodes with straight lines rather than curved segments. The tie girder is also divided into discrete members by the panel points. Such a modification will transform the originally complicated structure into a simplified straight member structure, which can easily be solved by any common method of structural analysis.

Before proceeding with the solution to the problem, construction of various relevant matrices will be discussed briefly.

1.2.3 Member Stiffness Matrix $\left[S_M\right]$

The stiffness matrix of a prismatic beam element with reference to its coordinates can be shown to be as follows[1].

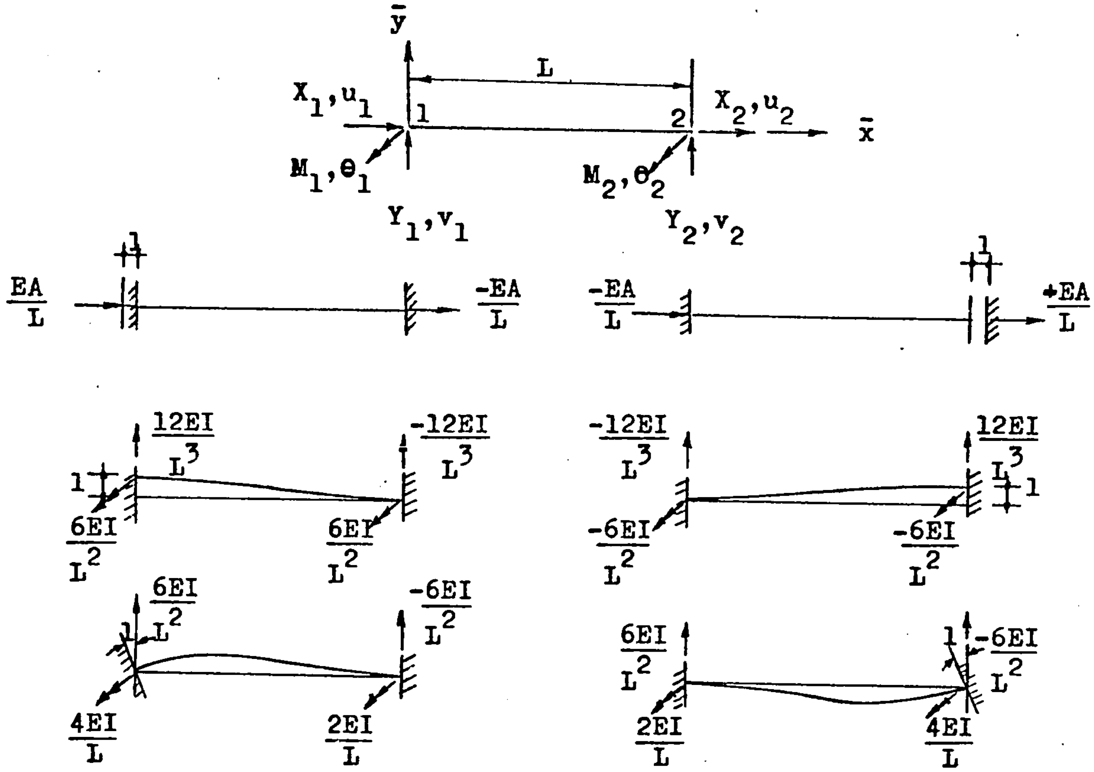


Fig.(1-12)

$$
\left[S_M\right] = \begin{array}{c c}
\begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} & \\
\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\
\frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix} & \begin{array}{c} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{array}
\end{array}
\qquad (1\text{-}33)
$$

The member stiffness matrix with respect to the system axes can be expressed as

$$\left[S_{MD}\right] = \left[R\right]^T \left[S_M\right] \left[R\right] \tag{1-34}$$

where $\left[R\right]^T$ is the transpose of $\left[R\right]$ which has the form

$$\left[R\right] = \left[\begin{array}{ccc|ccc} \cos\phi & \sin\phi & 0 & & & \\ -\sin\phi & \cos\phi & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & \cos\phi & \sin\phi & 0 \\ & 0 & & -\sin\phi & \cos\phi & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \tag{1-35}$$

in which ϕ is the angle between the member axes and system axes. The member axes of tie girder, which is horizontal, conincides with the system axes, hence for the girder

$$\left[S_{MD}\right] = \left[S_M\right] \tag{1-36}$$

The member stiffness matrix of a hanger rod can be determined in the following manner.

The characteristics of the hanger rods are : 1) all hangers are in vertical position, $\phi = 90^\circ$; 2) the structural function of the hanger rod is to transfer the vertical load from the bottom girder to the arch. The hanger is a tension member and its bending rigidity is small in comparison with that of the arch and the tie girder, and can be neglected. From Equation (1-33) we have,

$$\left[S_M\right]^h = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (1\text{-}37)$$

Substituting this expression into Equation (1-34), we have the member stiffness matrix of the hanger in the system coordinates.

$$\left[S_{MD}\right]^h = \left[R\right]^T \left[S_M\right]^h \left[R\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (1\text{-}38)$$

In the next step, the over-all joint stiffness matrix $\left[S_J\right]$ of the entire structure will be formulated.

1.2.4 Construction of the Stiffness Matrix

In the formulation of the system stiffness matrix, the known and unknown displacements of each joint have to be appropriately coded.It is convenient to code the joint with unknown displacements first, numbering first the horizontal translation, and subsequently the vertical translation and the rotation. Fig. (1-13) demonstrates the numbering system.

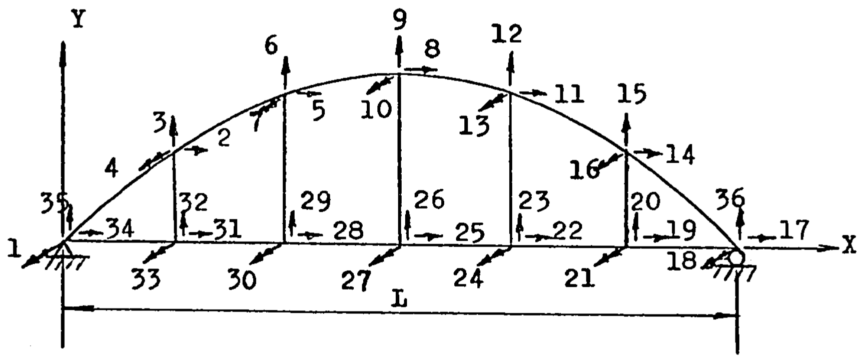
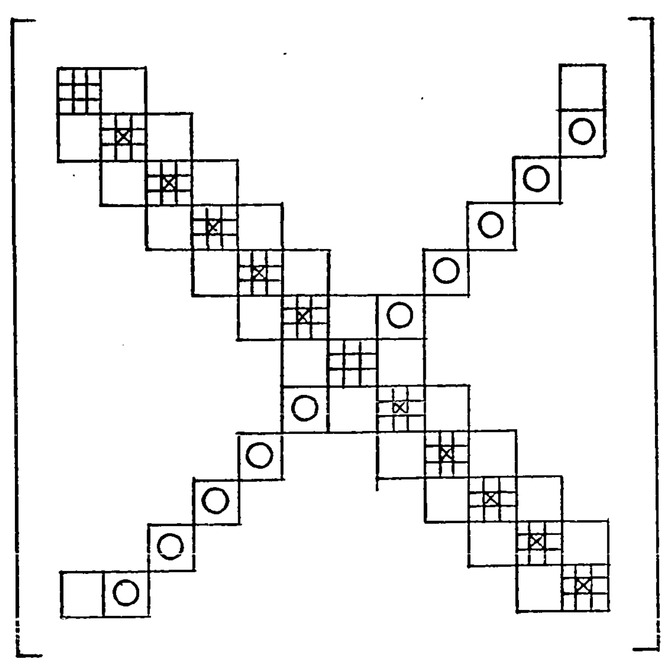


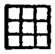
Fig.(1-13)

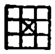
There are three members meeting at each internal joint, and two members at each of support joints. The value of the stiffness at each joint is the sum of the stiffnesses of the three individual members meeting at that joint. By the above numbering system of the structure, the over-all joint stiffness matrix will appear as in the following typical form.

$[S_J]$ =

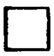


The following notations are used in $[S_J]$ above.

 represents the values of two overlapping members

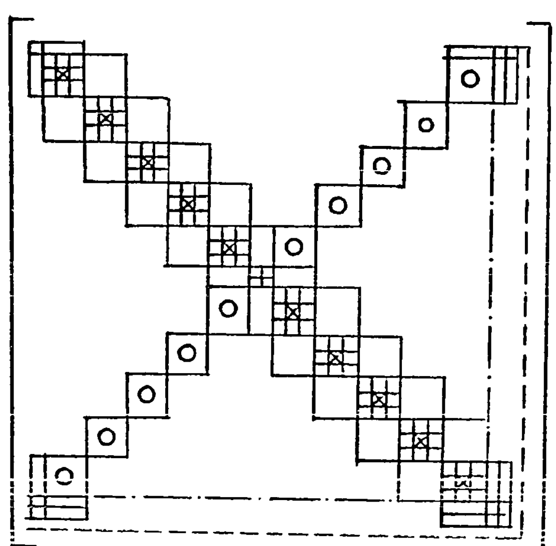
 represents three values of three members overlapping in the central area of the square.

 repressents the right upper corner or left lower corner of the member stiffness matrix of the hanger.

 repressents the right upper corner or left lower corner of the member stiffness matrix of the arch or the tie girder.

The over-all joint stiffness matrix $[S_J]$ contains terms for all possible joint displacements, including those restrained by the supports, while the stiffness matrix $[S]$ excludes those displacements restrained by the supports. To obtain $[S]$, $[S_J]$ can be rearranged by interchanging rows and columns in such a manner that the stiffnesses corresponding to the actual degree of freedom are listed first, and those corresponding to support restraints listed second. Thus the matrix $[S_J]$ can be written as follows:

$[S_J]$ =



or
$$[S_J] = \left[\begin{array}{c|c} S & S_{DR} \\ \hline S_{RD} & S_{RR} \end{array}\right] \tag{1-39}$$

1.3 Thermal Forces

If a structure is restrained, either fully or partially, self-equilibrating stresses will be induced in the structure due to temperature changes. Thermal stresses arise in an indeterminate structure usually because of boundary constraints.

The structure shown in Fig. (1-14) is determinate externally, but indeterminate internally, where no thermal stresses will arise as a result of temperature change. To prove this statement, we need only to verify that the movements due to the change of temperature at any part of the structure are not restrained; i.e., they are not restricting each other.

As the structure is a plane frame, it is implied that movements are in the plane of the structure and have components in the vertical and horizontal directions. If these two movements are not restricting each other, there will not be any stresses as a result of the movements. It is convenient to separate the structure into two parts(arch and tie girder) for the purpose of discussion.

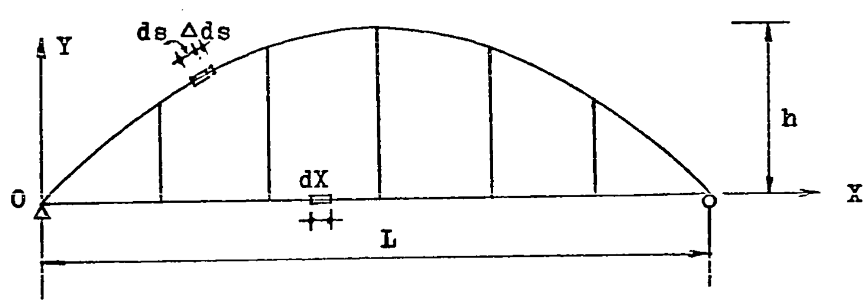


Fig.(1-14)

Consider an element ds of the arch which is subjected to temperature change, whose elongation or contraction equal Δds.

If the temperature expansion coefficent is α(assuming same material for the entire structure) and the change of temperature is ΔT, then

$$\Delta ds = \alpha(\Delta T)\ (ds) \tag{1-40}$$

The horizontal displacement is

$$\Delta U = \cos\phi(\Delta ds) = \alpha(\Delta T)(ds)(\cos\phi) \tag{1-41}$$

The total horizontal displacement of the arch is then

$$U^a = \int_0^S (\Delta U)(ds) = \alpha(\Delta T) \int_0^S \cos\phi\ ds$$

Since $\cos\phi = \frac{dx}{ds}$

$$U^a = \alpha(\Delta T) \int_0^L dx = \alpha(\Delta T)L \tag{1-42}$$

The total horizontal displacement of the tie girder is

$$U^g = \alpha(\Delta T)L$$

hence $U^a = U^g$ (1-43)

The vertical displacement of the arch element can be evaluated as

$$\Delta V = \alpha(\Delta T)(ds)\sin\phi \tag{1-44}$$

Therefore the vertical displacement of the arch at a point X_1 is

$$V^a = \alpha(\Delta T) \int_0^{S_1} \sin\phi\ ds \tag{1-45}$$

Since $\sin\phi = \frac{dy}{ds}$

so that $V^a = \alpha(\Delta T) \int_0^{Y_1} dy = \alpha(\Delta T)Y_1$ (1-46)

The change of length of hanger at the point X_1 is

$$V^h = \alpha(\Delta T)Y_1 \tag{1-46a}$$

It is obvious that

$$v^{a} = v^{h} \tag{1-47}$$

The above analysis of movement of the structure due to temperature change shows

1) that the horizontal movement at any point on the arch is equal to the horizontal movement corresponding to same point on the girder; and

2) that the vertical movement at any hanger connection point on the arch is equal to that hanger elongation or contraction.

One can draw a conclusion that no restraining effects exist within the structural system and hence temperature change does not give rise to thermal stresses, provided that the members of the structure are made of the same kind of material.

CHAPTER 2

PRACTICAL APPLICATION AND EXAMPLE

2.1 Preliminary Design

In this chapter we shall discuss the practical application of the theory that has been discussed previously. As a point of departure some preliminary consideration of the design problems will be given, followed by the establishment of influence lines for live load and dead load.

Many bowstring arch bridges have been designed on the assumption that the tie girder takes axial force only. Such an assumption is approximately true provided certain conditions are met. If the bending moment in the structure is minimum, and the moment of inertia of the tie girder is very small in comparison with that of the arch, then the tie girder will become almost an axial member. The above case occur only if the structure is under dead load action (symmetrical loading) alone. As the bridge is subjected to moving and unsymmetrical loads, considerable bending moments in the structure will arise. The distribution of moments in the arch and on the tie girder is dependent on the relative rigidities of the members, and on the geometry of the arch. The momentless tie girder is not practical in the bridge system, and moreover, the tie girder should share moment resistance from the viewpoint of economical design.

2.1.1 Geometry of Structure

The first step of the design considers the geometry of the structure. The axis of the arch may assume the same shape as the pressure line (axial force) of the arch due to dead load. The effect of this approach is to minimize the bending moment in the arch, whereby the deflection of the structure will also be minimized. While it is difficult to achieve this , the arch curve should be made close to the pressure line of the arch due to dead load. A parabolic curve normally can achieve the above purpose in this type of bridge. Taking the origin of the coordinates of the structural system at the left support, the general expression for the axis of the arch is

$$Y = \frac{4h}{L^2} (LX-X^2) \tag{2-1}$$

where h = the rise of the arch;

L = the span of the arch;

X,Y = the abscissa and ordinate, respectively.

The ratio of rise to span length, h/L, usually ranges from 1/7 to 1/3 . The higher the h/L ratio, the less stable the structure is against overturning wind force.

In practical design, a slight camber of flat parabolic shape is given to the tie girder to suit the roadway geometry or for the purpose of drainage. However, in this presentation, the tie girder will be treated as a straight horizontal member for simplicity.

2.1.2 Preliminary Design of Structural Members

The arch in many bowstring arch bridges has variable cross sections, small at the crown, and increasing toward the supports. The moment of inertia at any section along the axis of the arch can be expressed as follows:

$$I = I_C \sec\phi = I_C \frac{ds}{dX} \tag{2-2}$$

hence

$$ds = \frac{I}{I_C} dX \tag{2-3}$$

where I_C is the moment of inertia at the crown.

A similar expression can be written for the cross-sectional area of the arch. By this transformation, the integral calculation in the computation of stresses can be made with respect to the X-axis instead of the arch axis. The axial force is predominant in the arch, which increases from the crown to the supports, therefore the cross section should also vary accordingly to achieve an economical use of material. In practice,for reasons of neatness and esthestics of the structure, and of simplicity in fabrication and construction, it is preferable to keep the depth of the cross section constant and make only the top and bottom flanges variable.

The loading on the bridge should be discussed at this point before considering the preliminary design. Along the tie-girder, the applied loads are concentrated at the panel points, and the deck slab dead load of relatively small magnitude is uniformly distributed on the tie girder. Theoretically these distributed loads should be transformed into equivalent joint loads[1]. For simplicity the bridge loads are assumed to be concentrated at the panel points of the bottom girder as shown in Fig,(2-1).

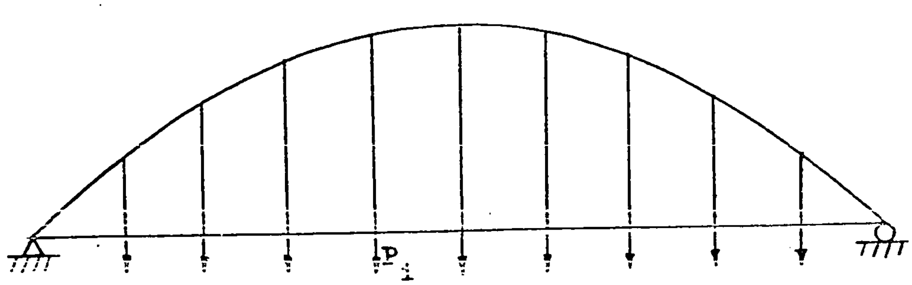


Fig.(2-1)

The following two cases of load transmission may be considered according to the relative rigidities of the arch and the tie girder.

1. If the bending rigidity of the arch is very large compared to the rigidity of the tie girder, the structure becomes similar to the two hinged arch structure. The loads applied at the panel points would be transmitted directly to the arch through the hanger rods. The tie girder would act as a pure tie rod resisting the arch thrust.

2. If the bending rigidity of the arch is very small compared to the rigidity of the tie girder, the structure would be similar to a simply supported beam column with small axial force and moments applied at the supports.

The above two cases are the extreme limits of the system. In the first case, the arch resists bending and axial force, while the tie girder absorbs only axial force. In the second case, the girder is almost the only member that resists the external loads. Hence we can deduce that it is possible to proportion the moments of inertia of the arch and the tie girder to share the role of resisting both bending and axial force in the structure.

It is difficult to determine the exact dimensions of the structure to resist the external loads at the outset. However it is relatively easy to size the members under case 1 condition. Consequently it is convenient to find the dimensions of the structure under case 1 condition first, then subsequently modify these dimensions according to our purpose and experience. The initial size may be used as input data for the computer analysis. By a trial procedure, modification of member sections can be carried out until satisfactory dimensions are reached.

A procedure for determing the initial dimensions of the structure under case 1 condition will be outlined below.

1. Dead Load

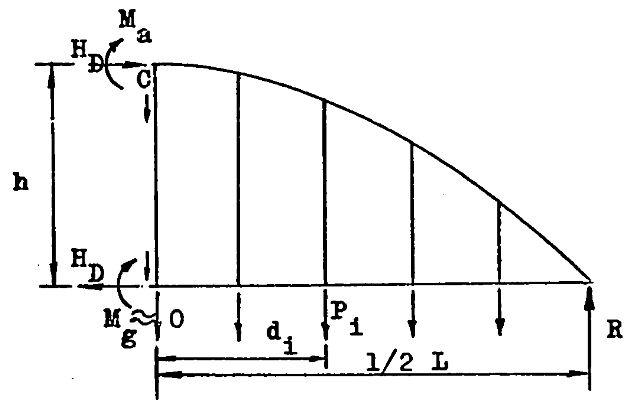


Fig.(2-2)

In the free body diagram shown in Fig. (2-2), the following equilibrium conditions can be written. Taking the sum of moments about point C,

$$M_a + H_D h + \Sigma P_i d_i - R\frac{L}{2} = 0$$

hence

$$H_D = \left(\frac{RL}{2} - \Sigma P_i d_i - M_a\right) / h \tag{2-4}$$

It is known that in an arch design, the pressure line of dead load closely coincides with axis of the arch, so that M_a is small and can be neglected, then H_D can be computed by

$$H_D = \left(\frac{RL}{2} - \Sigma P_i d_i\right)/h \tag{2-5}$$

2. Live Load

If the lane load governs the design, the analysis can be the same as the dead analysis. Otherwise the bridge under truck load action should be analyzed. Now the difficulty is to find in which position the truck will cause maximum stresses in the bridge.However, in the preliminary design, a simplification may be made in such a way that, the truck is placed at midspan of the bridge. Now taking the free-body as in the preceeding case shown in Fig. (2-3), and using the equilibrium condition of moment about point C, we write

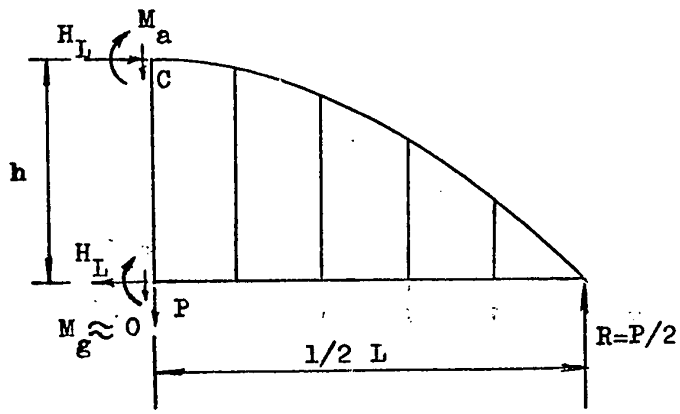


Fig.(2-3)

$$M \quad + H\ h - R\,\frac{L}{2} = 0 \tag{2-5a}$$

H_L can be found approximately by,

$$H_L = (C)P \tag{2-5b}$$

where C, the influence line coefficient of horizontal thrust of a two hinged arch, are given by James Michalos[2]. Then M_a can be defined as

$$M_a = \frac{RL}{2} - H_L h = \frac{RL}{2} - (C)Ph \tag{2-6}$$

The total horizontal thrust in the tie girder is

$$H = H_D + H_L \tag{2-7}$$

The maximum axial force in the arch can be determined by the following formula,

$$N = V\sin\phi + H\cos\phi \tag{2-8}$$

The dimensions of the cross sections of the arch and the tie girder can be computed from the forces obtained from Equations (2-7) and (2-8). This approximation will result in oversizing of the arch and undersizing of the tie girder. Successive modifications must be made to achieve an optimum design.

The first approximate dimensions being determined, the bending moment distribution is next evaluated for the purpose of modification of member sizes.

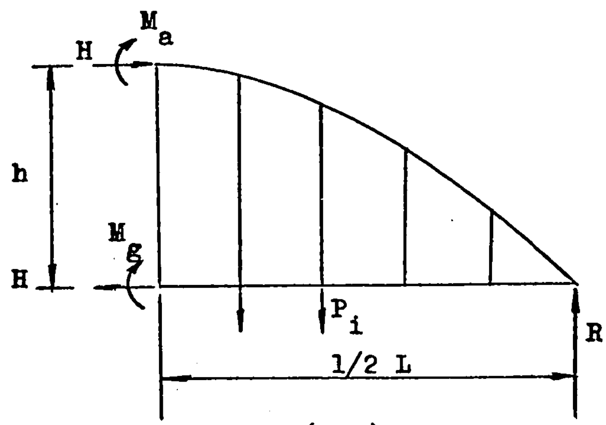


Fig.(2-4)

Referring to the free-body diagram shown in Fig. (2-4), the moment at the cut section is

$$M_o = HY + M_a + M_g \tag{2-9}$$

If the elongation of the hangers is neglected, the deflections of the arch and the tie girder will be the same at each panel point. Furthermore it is assumed that the changes of slope along the arch and along the tie girder are equal, thus by the first Area-Moment principle

$$\frac{M_a ds}{EI_a} = \frac{M_g dx}{EI_g}$$

or
$$\frac{M_a}{M_g} = \frac{I_a \cos\emptyset}{I_g} \tag{2-10}$$

In Equation (2-10) to solve for M_a and M_g, we have

$$M_a = \frac{(M_o - HY)\, I_a \cos\emptyset}{(I_a \cos\emptyset + I_g)} \tag{2-11}$$

$$M_g = \frac{(M_o - HY)\, I_g}{(I_a \cos\emptyset + I_g)} \tag{2-12}$$

Under the above assumptions, Equations (2-11) and (2-12) give approximate values when the structure is subjected a single external load action.

2.2 EXAMPLE

Having obtained the geometry and the approximate dimensions of the arch and the tie girder, further modification to the final design will be made by the use of a computer program. An illustrative example is given in the following.

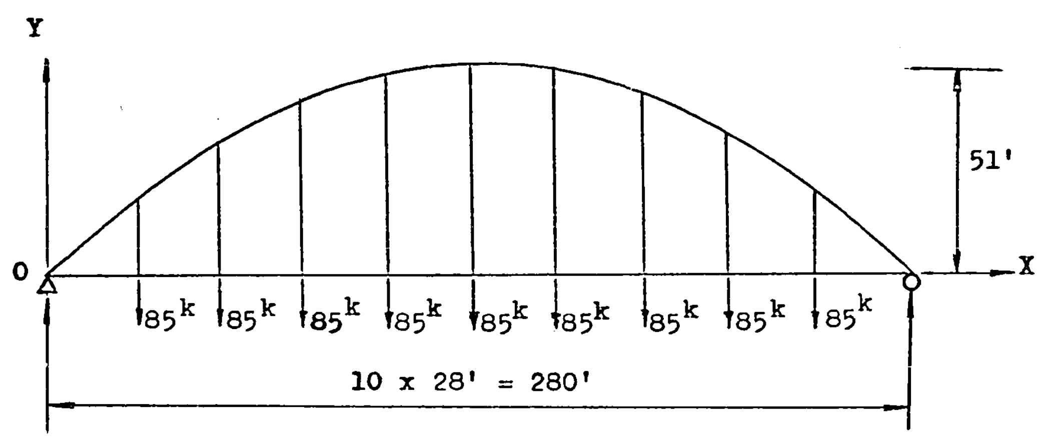


Fig.(2-5)

The geometry of the axis of the arch is

$$Y = \frac{4x51}{280^2}\ (\ 280X - X^2)$$

The nonvarying cross sections of the arch and the tie girder are assumed to be

$I_a = 0.75\ ft^4$ $\qquad$ $A_a = 0.4\ ft^2$

$I_g = 1.50\ ft^4$ $\qquad$ $A_g = 0.5\ ft^2$

and the hanger rods to have 3 inches diameter,

$I_h =$ neglected $\qquad$ $A_h = 0.05\ ft^2$

2.2.1 Dead Load

It is reasonable to assume that the dead load of the bridge is concentrated at the panel points of the bottom girder as shown in Fig.(2-5) and the dead load of the arch is neglected for simplicity. The internal forces due to dead load are shown in Tables (2-1), (2-2) and (2-3)

TABLE (2-1)

Stations / Arch	0.0L	0.1L	0.2L	0.3L	0.4L	0.5L
Axial Force	-694.27	-693.32	-645.99	-612.62	-588.78	-576.53
Moment	44.86	314.85	311.42	346.81	365.88	372.24

TABLE (2-2)

Stations / Tie Girder	0.0L	0.1L	0.2L	0.3L	0.4L	0.5L
Axial force	576.53	576.53	576.53	576.53	576.53	576.53
Moment	44.86	-190.01	-89.47	-55.51	-33.06	-25.44

Table (2-3)

Hangers \ Stations	0.0L	0.1L	0.2L	0.3L	0.4L	0.5L
Axial Force	--	93.7745	82.6225	84.5885	84.4705	84.4559

Note:

Moment $\begin{cases} \text{Unit : k-ft} \\ \text{Sign : compression on top fiber (+)} \end{cases}$

Axial Force $\begin{cases} \text{Unit : k} \\ \text{Sign : compression (-) , tension (+)} \end{cases}$

2.2.2 Live Load

Since the live loads on the bridge are moving loads, the live load stresses can be obtained from influence lines. The influence lines for moments, and axial forces in the arch and the tie girder are shown in Fig.(2-6) to Fig. (2-8). The influence line for the axial force in the tie girder is same as for the arch at the crown, except the signs are opposite.

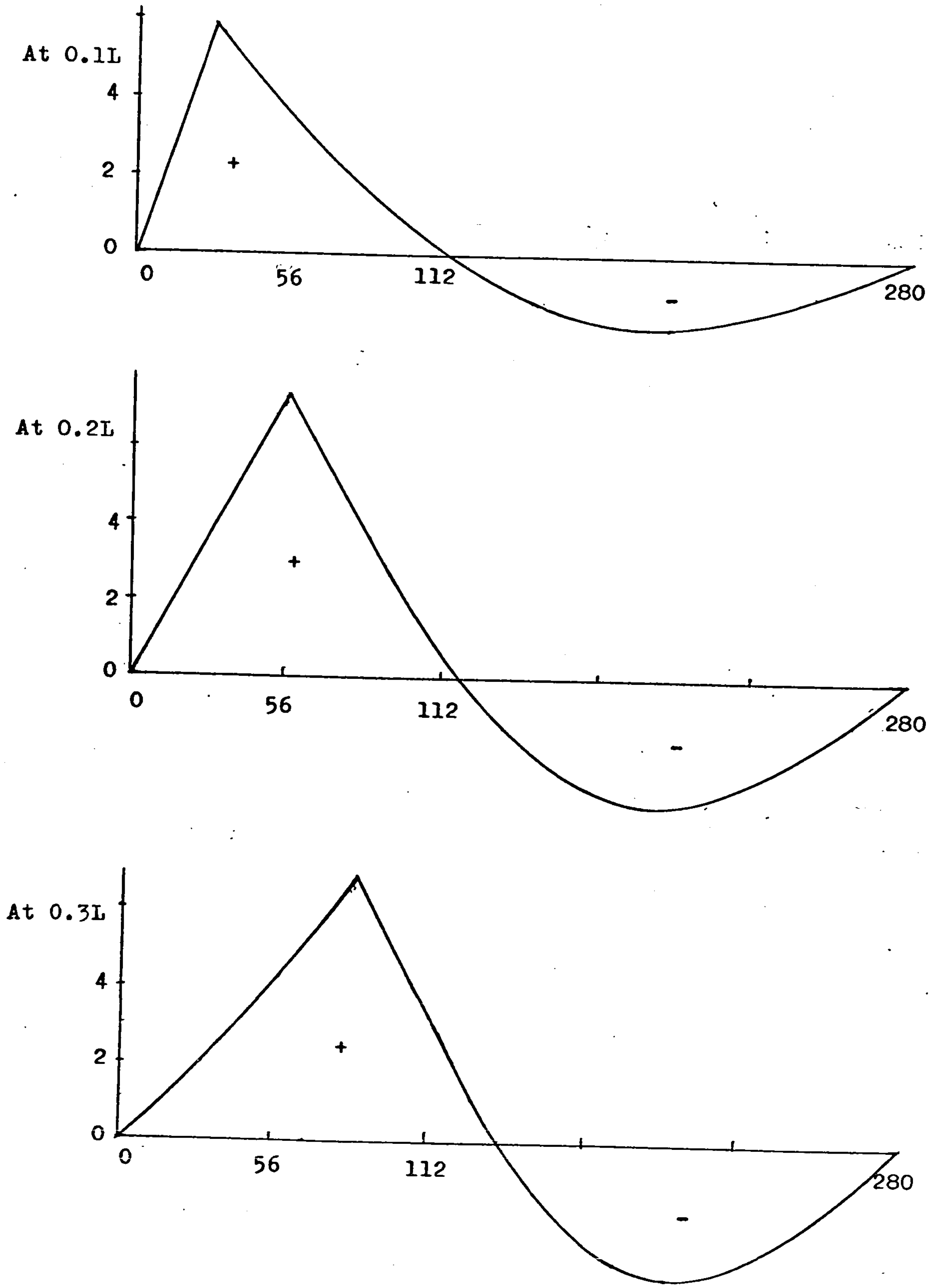
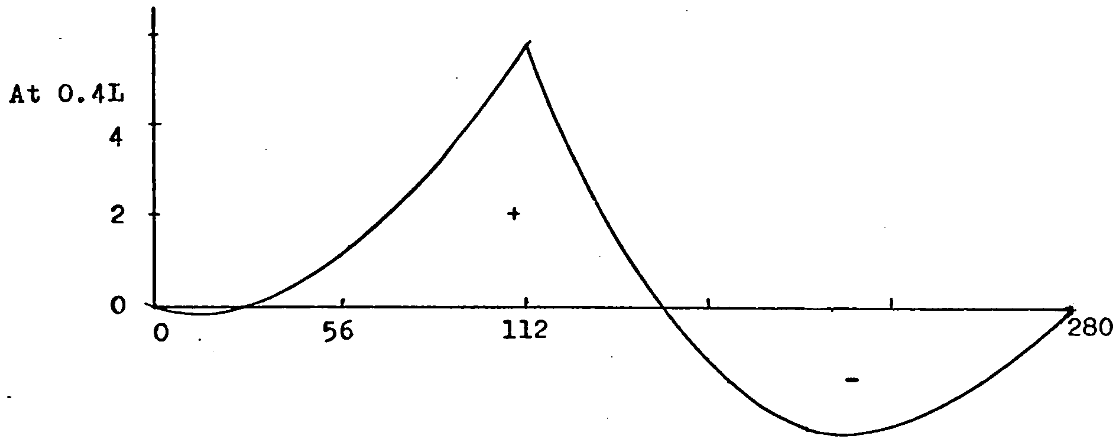
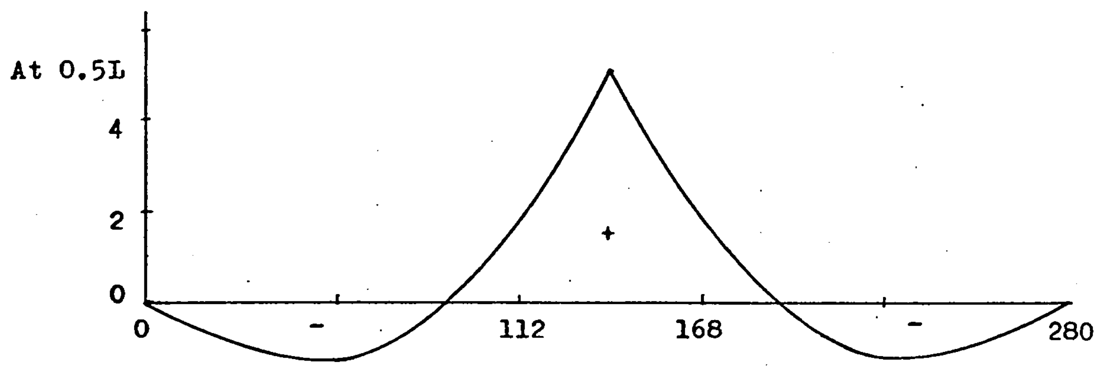


Fig.(2-6)

Influence Lines for Bending Moment in the Arch





The scales for the above diagrams are:

Horizontal; 1/2 in. = 28 ft.

Vertical; 1/4 in. = 1 ft-k

Fig.(2-6) (Cont'd.)

Influence Lines for Bending Moment in the Arch

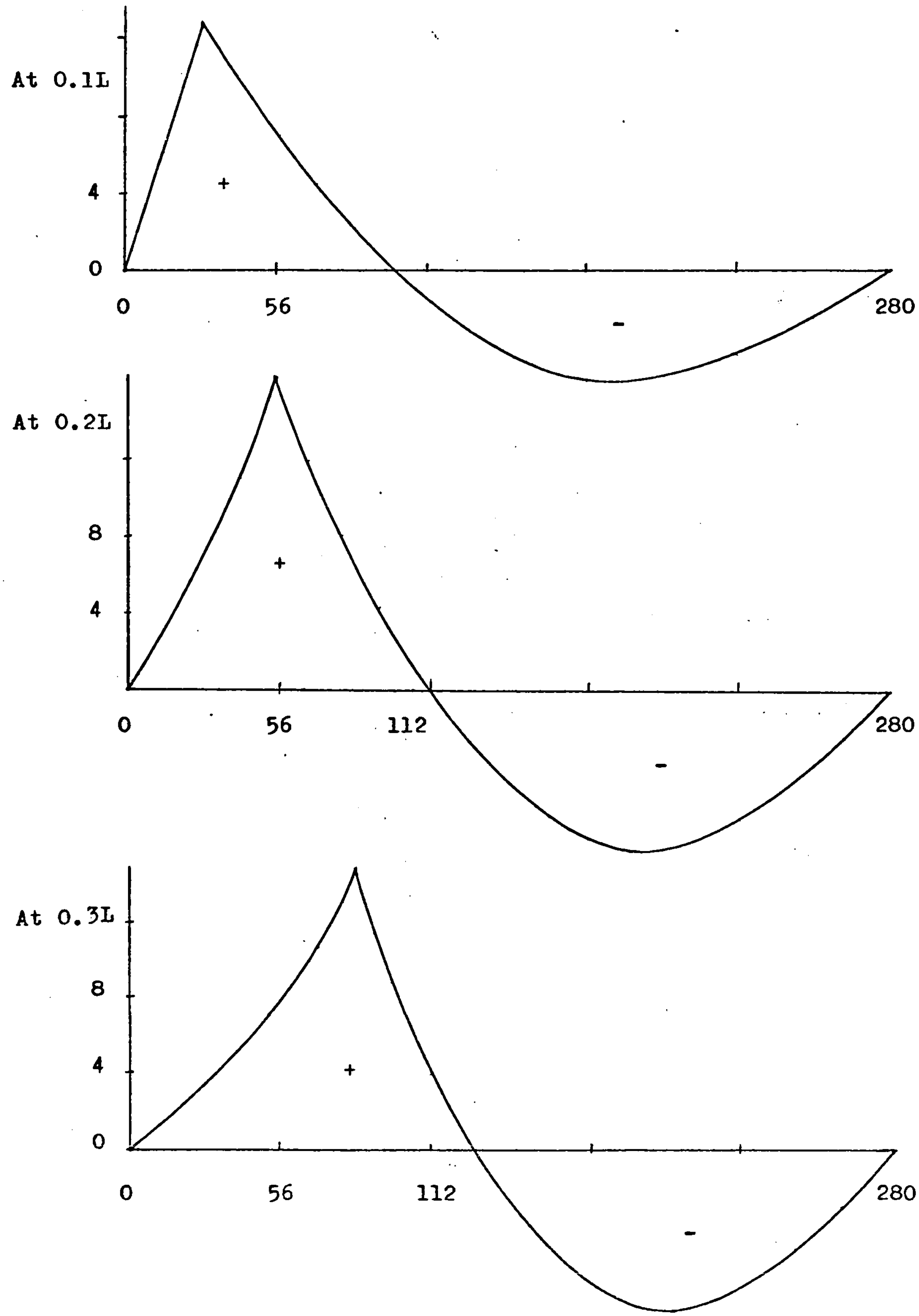
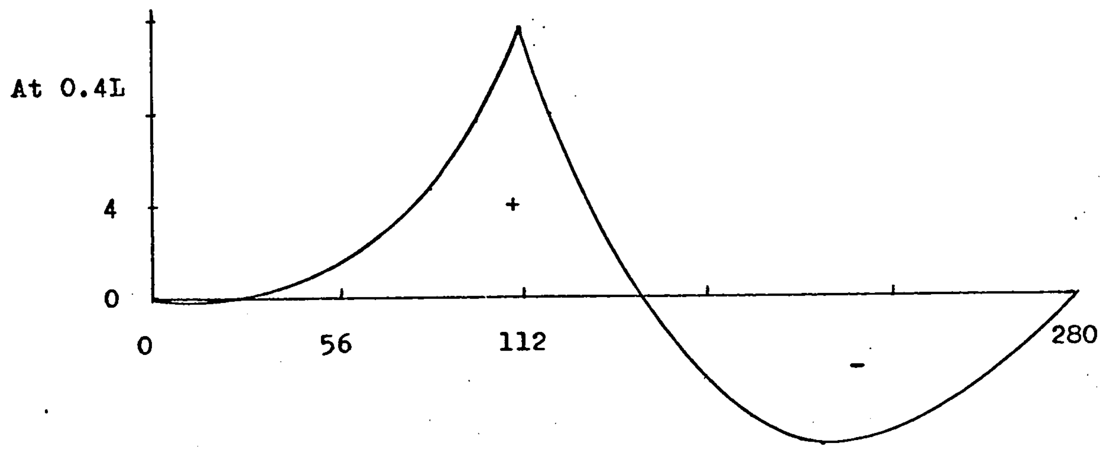
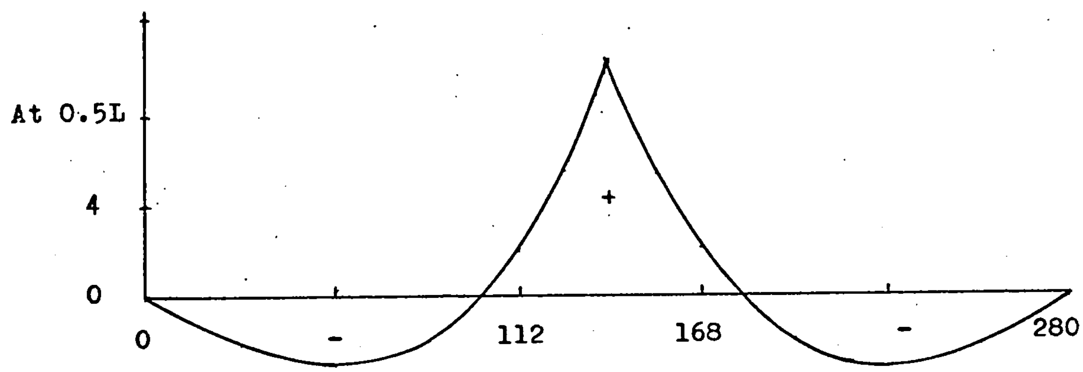


Fig.(2-7)

Influence Lines for Bending Moment in the Tie Girder





The scale of the above diagram are:

Horizontal; 1/2 in. = 28 ft.

Vertical;   1/8 in. = 1 ft-k

Fig.(2-7) (Cont'd)

Influence Lines for Bending Moment in the Tie Girder

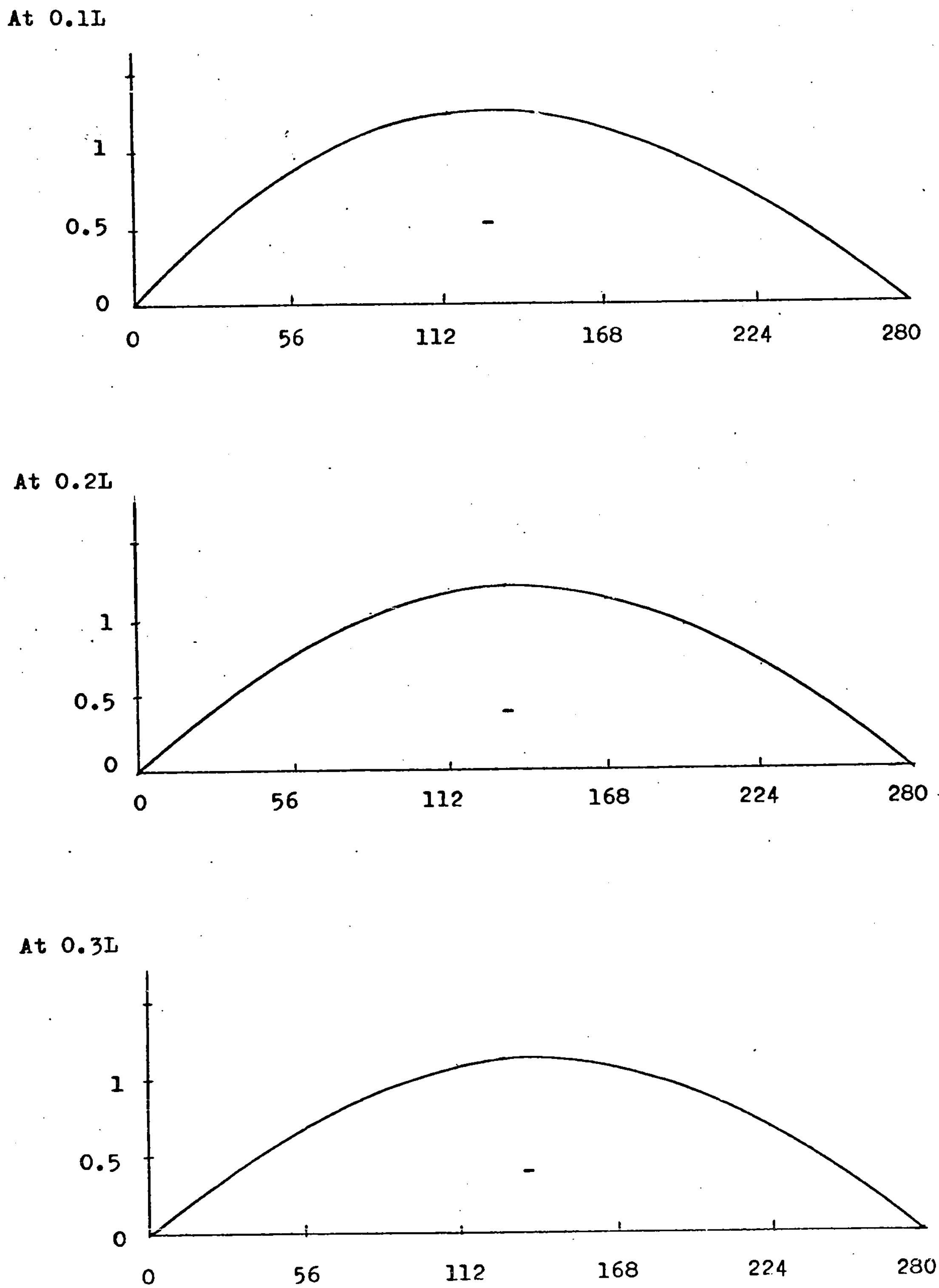
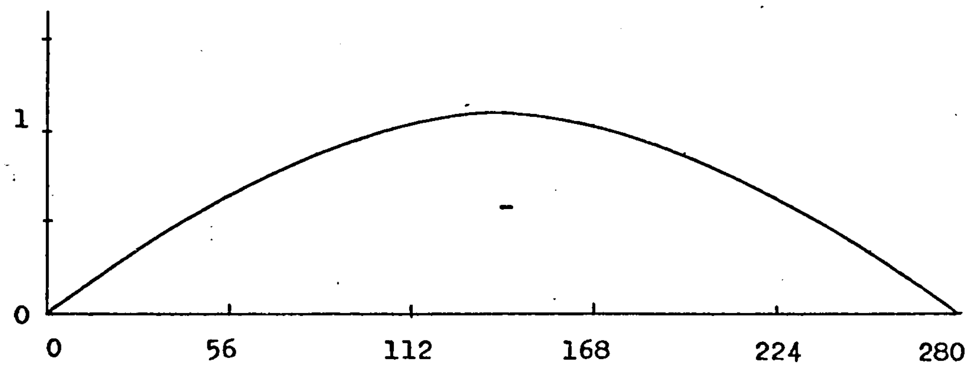


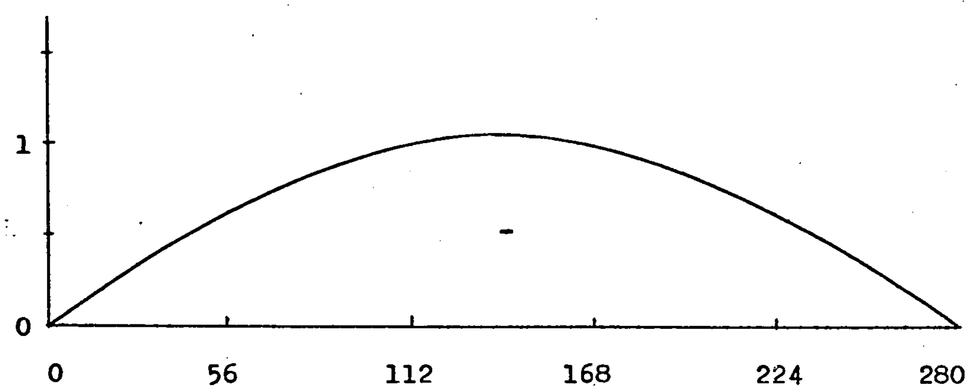
Fig.(2-8)

Influence Lines for Axial force in the Arch

At 0.4L



At 0.5L



Note: Axial force values are at the left hand side of the section under consideration.

Scales:

Horizontal; 1/2 in. = 28 ft.

vertical; 1 in. = 1 kip

Fig.(2-8) (Cont'd)

Influence Lines for Axial Force in the Arch

CHAPTER 3

COMPUTER PROGRAM

3.1 General Description of the Program

The computer program is written for the analysis of a single span bowstring arch bridge by the flexibility method. It consists of the main program and seven subroutines, named THRUST,FLEXA,FLEXG, MAINV,THRUSTH,FLEXAH, and FLEXAR. The first three subroutines furnish the related matrices to the main program for solving redundants.The fourth subroutine MAINV is a matrix inversion subroutine. The last three subroutines do not participate in solving for the redundants, their function is to generate related matrices for the main program to compute the horizontal displacement of the arch.

Subroutine THRUST establishes matrix $[H]$, which gives the horizontal thrusts due to vertical unit loads corresponding to the redundants.

Subroutines FLEXA and FLEXG set up flexibility matrices $[F]^a$ and $[F]^g$ of vertical displacements of the arch and the tie girder, respectively corresponding to the redundants in the released structure.

THRUSTH finds the horizontal thrust due to unit horizontal loads corresponding to redundants in the released structure.

FLEXAH determines the horizontal displacements of a two hinged arch due to unit vertical loads corresponding to the redundants,and FLEXAR computes the horizontal displacement of the arch with a hinge and a roller support, due to a unit horizontal load applied at the roller support.

In this program, it is assumed that the respective cross sections of the arch and the tie girder are constant. It accomodates different materials for the arch and the tie girder.

The geometry of the bowstring arch is standardized in the program. The tie girder is a horizontal straight member, and the axis of the arch is a parabolic curve. The origin of the structural system is located at the left hinged support; the X-axis is positive in the left to right direction, and the Y-axis is positive in the upward direction. The equation of the axis of the arch is

$$Y = \frac{4h}{L^2}\left(LX - X^2 \right)$$

The over all structure of the program is outlined by the flow charts in Appendix A. However, several points in the construction of this program merit special discussion.

A number of integral operations need be carried out in the successive applications of the unit load method to evaluate the deflections at the points where the unit loads are applied. The complexity of the integration is compounded because the analysis involves the curved member, namely, the arch. The expressions of deflections are compiled for each subroutine in Appendix A.

The following six types of integrations are involved in the subroutines:

$$A = \int ds$$

$$B = \int X ds$$

$$C = \int Y ds$$

$$D = \int XY ds$$

$$Z = \int X^2 ds$$

$$F = \int Y^2 ds$$

where

$$ds = \sqrt{1 + (Y')^2}\, dX \;;$$

$$Y = \frac{4h}{L^2} (\, L - 2X \,) \;;$$

$$Y' = \frac{4h}{L^2} (\, L - 2X \,) \;.$$

The above integrations can be found with the aid of the following substitution:

$$W = L - 2X$$

$$dW = -2dX$$

or

$$X = \frac{L-W}{2}$$

$$dX = -\frac{dW}{2}$$

The results of the derivation are given in Appendix A. The actual operation are performed by the subroutines for the pertinent limits of integration involved.

The program is written in FORTRAN IV language, the run time on the CDC 3300 computer at the Sir George Williams University Computer Centre is about 1 minute and 30 seconds.

There are three types of input data. The first type includes the geometry, the number of redundants, the span length of the bridge and the rise of the arch. The second type consists of the properties of structural members, i.e., the moments of inertia of the arch and the tie girder, the cross sectional area of the arch, the tie girder and the hanger rods. The third type is the external loads data.

The output gives the redundants, the moments, the axial forces and the shear force at the panel points of the structure, the vertical and horizontal deflections of the arch and the tie girder, and reactions at the supports.

CHAPTER 4

DISCUSSIONS AND CONCLUSIONS

4.1 Comparison of the Methods

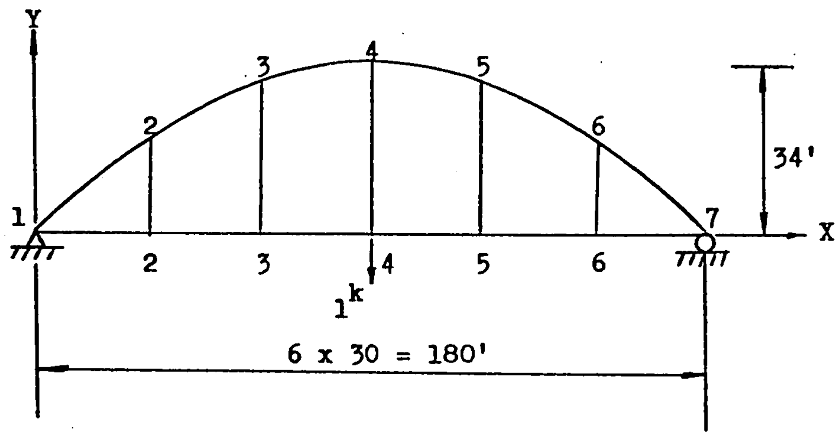
4.1.1 Choice of Methods

The principal differences between the flexibility and stiffness methods are the choice of unknowns and the computational path leading to the calculation of stresses and deformations. The flexibility method seems simpler than the stiffness method in the bowstring arch analysis because the number of kinematical conditions is much greater than the number of statical indeterminacy in a bowstring arch bridge. Hence the stiffness method involves an inversion of a large matrix. Although electronic computers are available, it is still time-consuming and will increase the chances of errors in the structural analysis.

The advantage of the stiffness method is its ease in generating the stiffness matrix of a structure with variable cross sections such as in the arch and the tie girder, while it is comparatively difficult to form the flexibility matrix. However, the effect of variable cross section may be neglected in the flexibility method, if the variation of their moments of inertia is small(3).

4.1.2 Comparison of Numerical Results of the Flexibility Method and the Stiffness Method (Approximate Method)

The configuration of the analyzed structure is shown in Fig. (4-1)



Fig(4-1)

Constant cross section of the arch and of the tie girder is assumed. The cross sectional properties are:

$I_a = 0.5 \text{ ft}^4$ $\qquad$ $I_g = 0.5 \text{ ft}^4$

$A_a = 0.25 \text{ ft}^2$ $\qquad$ $A_g = 0.25 \text{ ft}^2$

The diameter of the hanger rods is 3 inches

I_h = neglected $\qquad$ $A_h = 0.05 \text{ ft}^2$

An external load of 1 kip is applied at the middle panel point of the tie girder. The results of the two methods for this problem are given in Table (4-1) .

Table (4-1)

Items \ Panel Pts.		1	2	3	4	5	6	7
Hanger Force	F.M.	—	0.1516	0.1467	0.5936	0.1467	0.1516	—
	A.M.	—	0.1143	0.1787	0.6035	0.1787	0.1143	—
Moment in Arch	F.M.	0.0895	-1.4230	0.2627	5.2941	0.2627	-1.4230	0.0895
	A.M.	0.4163	-3.6637	-2.4010	2.2642	-2.4010	-3.6637	0.4163
Moment in Girder	F.M.	-0.0895	-2.9438	-1.2496	4.8458	-1.2496	-2.9438	-0.0895
	A.M.	-0.4163	-3.2588	-2.6723	3.2752	-2.6723	-3.2588	-0.4163
Axial F. in Arch	F.M.	-1.1768	-1.1153	-1.0667	-1.0253	-1.0667	-1.1153	-1.1768
	A.M.	-1.2980	-1.2980 -1.2555	-1.1893 -1.2555	-1.1893 -1.1893	-1.2555 -1.1893	-1.2555 -1.2980	-1.2980

Note: 1) Units: Moment , k-ft ; Axial Force , k.

2) F.M. = Flexibility Method ; A.M. = Approximate Method.

From the above results, the difference in the results obtained by the two methods is significant, especially in the bending moments. It should noted that the axial force in the arch is always significant. In the approximate method used, an idealization of the structure is made such that the arch is discretized to consist of straight members. The result is that the axis of the arch changes in a discrete manner so that extra moments are created in the idealized structure. Therefore, the deflection of the structure is also affected.

The accuracy of the solution by the approximate method will improve as the number of straight elements is increased, but the disadvantage is that the number of unknowns will rapidly increase to such an extent that it might be too large for a limited capacity computer to handle.

It follows that the approximate method may not be an ideal method to use in bowstring arch analysis. For a more rigorous analysis it is desirable to derive the stiffness matrix of a curved member to replace that of the straight member in the stiffness method. In this thesis, only the theoretical derivation of the stiffness matrix of a curved member will be given in the appendix, which can be utilized for further investigation.

4.2 Effects of Shortening of Arch and Elongation of Tie Girder and Hanger Rods.

Table(4-2)

Cases	Neglecting Items	Horizontal Thrust (k)	Redundant Q (k)	Moment of Arch at 0.5L (k-ft)	Moment of Girder at .1L (k-ft)
1	Arch Shortening	577.4769	84.4886	320.8851	-178.5847
2	Hanger-rod Elongation	577.4668	84.6057	315.0689	-178.9839
3	Tie girder Elongation	577.1383	84.4564	341.6118	-190.1057
4	Omiting Items 1, 2, 3	578.4097	84.6341	263.7031	-167.3286
5	Considering All Effects	576.5335	84.4559	372.2354	-190.0088

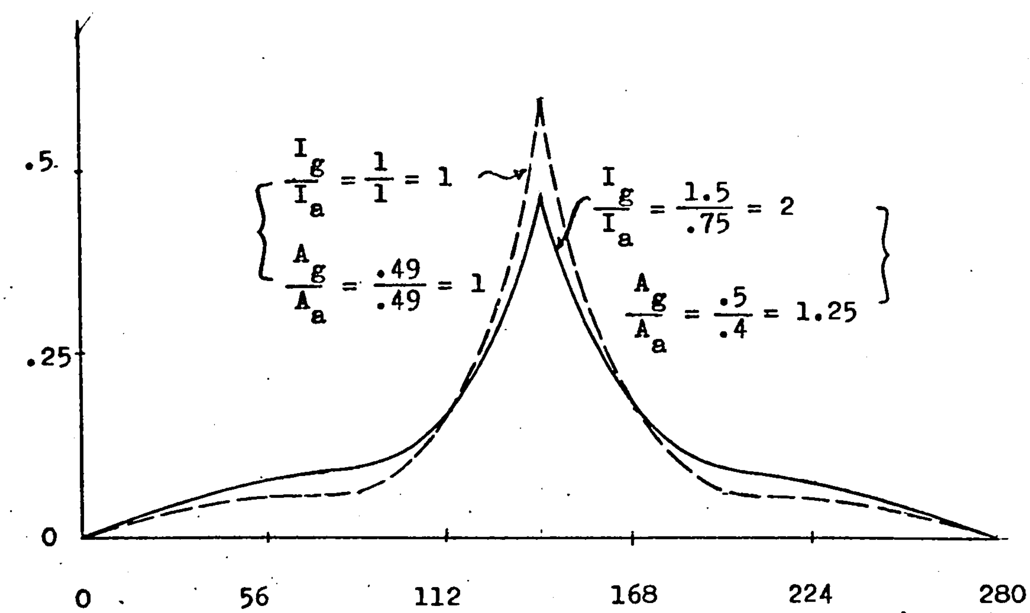
In Table (4-2) is shown a summary of case studies, comparing the effects of neglecting the deformations of arch, tie girder and hanger rods, as reflecting in the values of computed internal forces in the structural system under the external loading shown in Fig.(2-5). It should be pointed out that shear force effect is not included in all of the cases, which is thought to be negligible.

The table indicates that discarding an individual deformation of arch, tie girder, or hanger rods leads to no great change in the computed results. The bending moments appear to have lower values, but the axial forces in the structural members increase slightly. However, if all deformations are not accounted for, the collective effects result in significantly lower values of bending moments,and the gain in axial forces may not offset the decrease in the bending moments.

4.3 Effect of Section Properties of Arch and Tie Girder

4.3.1 Hanger Force

In a bowstring arch of given geometry, the most significant factor influencing the hanger force distribution is the moment of inertia of the arch and the tie girder. If the moment of inertia of the arch constant or decreases, the increase in the moment of inertia of the tie girder would cause decrease in the sum of the hanger forces. Theoretically the sum of hanger forces could approach zero, if the moment of inertia of the tie girder is increased to infinity. However, convergence to this limit is quite slow. Let I_g and I_a be the moment of inertia of the girder and the arch respectively, and ϕ the angle of the tangent with respect to X-axis at the panel point under consideration. The maximum individual hanger force under the action of a single load P at the panel point is close to the value $\frac{I_a \cos\phi}{I_g + I_a \cos\phi}$ P. This is indicated in Fig.(4-2)



Scales:

Horizontal; 1/2 in. = 28 ft.

Vertical; 1 in. = .5 kip

Fig.(4-2) is the influence line of hanger No.5 of the structure shown in Fig.(2-5).

4.3.2 Axial Force in the Arch and Tie Girder

Relative size of the arch and the tie girder affects the distribution of internal forces in the system. Since the axial forces in these two members are transmitted through the bangers, the relation between the stiffnesses of the two members and their axial forcees is similar to that between the stiffnesses of two members and the hanger forces. However, the axial forces in the arch and the tie girder are not sensitive to the change in their relative values of stiffness in a normal range. Table(4-3) shows the influence line values for the arch at 0.5L, the geometry is similar to the structure shown in Fig.(2-5), but with defferent section properties as indicated.

Table(4-3)

Properties \ Points	0.0L (k)	0.1L (k)	0.2L (k)	0.3L (k)	0.4L (k)	0.5L (k)
I_g/I_a=1/1 = 1 A_g/A_a = .5/.5=1	0	0.3376	0.6372	0.8701	1.0172	1.0674
I_g/I_a=1.5/.75=2 A /A =.5/.4=1.25	0	0.3366	0.6358	0.8690	1.0165	1.0669

4.3.3 Bending Moment in the Arch and the Tie Girder

Figs.(4-3) and (4-4) show the moment influence lines at 0.2L for the arch and the tie girder respectively. They are obtained for the system shown in Fig.(2-5), but with different section properties.

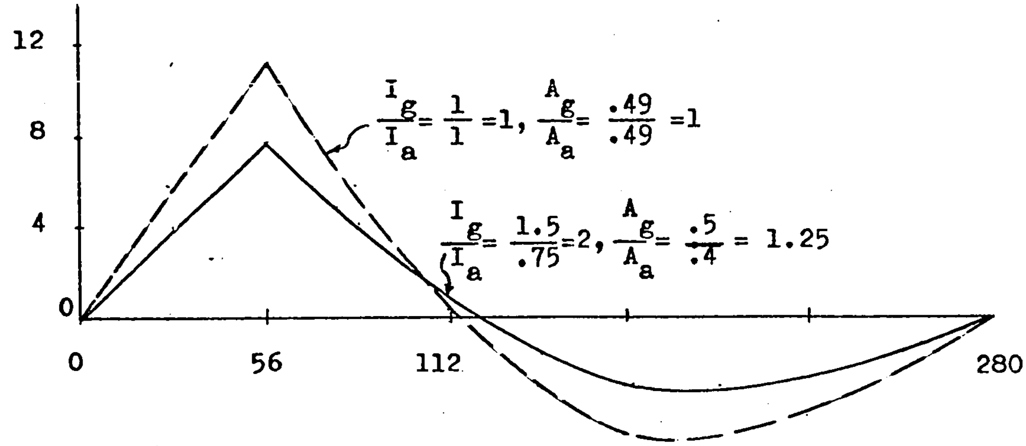


Fig.(4-3)

Influence Line for Bending Moment in Arch at 0.2L

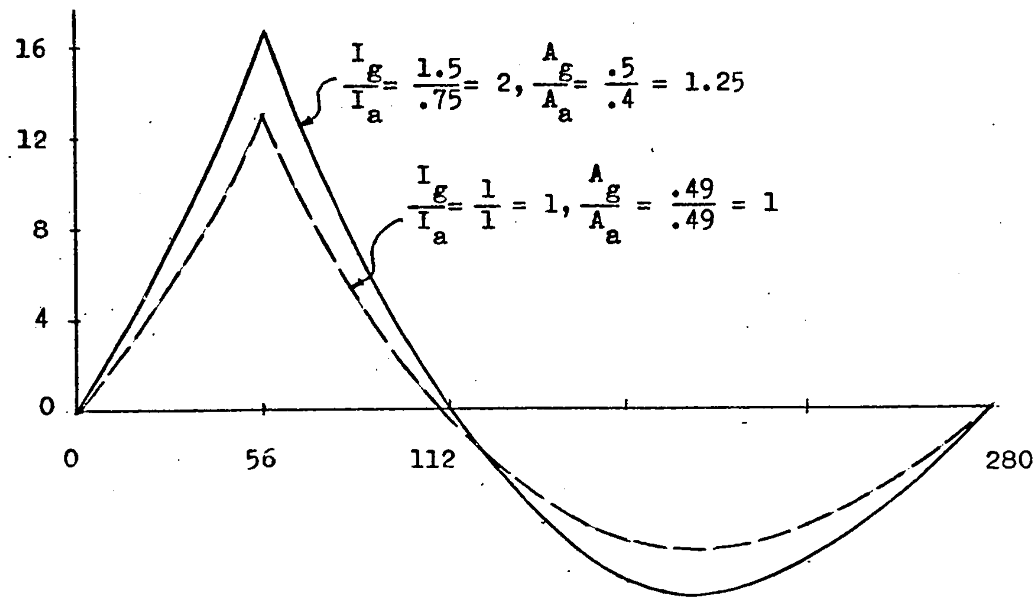


Fig.(4-4)

Influence Line for Bending Moment in Tie Girder at 0.2L

The scales for the above diagrams are:

Horizontal; 1/2 in. = 28 ft.

Vertical; 1/8 in. = 1 ft-k

For a structure with prescribed geometry, the total moment due to a single load at a section (M- HY) appears to be distributed between the arch and the tie girder according to their relative stiffnesses; this fact can be seen from Fig.(4-3), and (4-4).

4.4 Deflection

4.4.1 Comparison of the Results of Flexibility Method and Stiffness Method (Approximate Method)

In Tables (4-4) and (4-5) are shown the results of deflection analysis for the structure under the loading condition shown in Fig. (4-1), which will give a comparison of the two methods.

Table (4-4)

Methods \ Points		1	2 (ft.)	3 (ft.)	4 (ft.)	5 (ft.)	6 (ft.)	7
Vertical Displ. of Arch	Flex. Method	0	.000366	-.000110	-.000660	-.000110	.000366	0
	Approx. Method	0	.000033	-.000406	-.000912	-.000406	.000033	0
Vertical Displ. of Girder	Flex. Method	0	.000353	-.000130	-.000754	-.000130	.000353	0
	Approx. Method	0	.000023	-.000431	-.001007	-.000431	.000023	0

Table (4-5)

Methods \ Points		1	2 (ft.)	3 (ft.)	4 (ft.)	5 (ft.)	6 (ft.)	7 (ft.)
Horiz. Displ. of Arch	Flex. Method	0	-.000183	0.0	.000086	.000172	.000355	.000172
	Approx. Method	0	-.000063	.000065	.000096	.000127	.000255	.000192
Horiz. Displ. of Girder	Flex. Method	0	.000029	.000057	.000085	.000114	.000142	.000171
	Approx. Method	0	.000032	.000064	.000096	.000128	.000160	.000192

It can be observed that considerable discrepancies exist between the results by the different analytical procedures. When the structure is subjected to symmetrical loading, the vertical displacements obtained by the approximate (stiffness) method apparently have higher values. However, higher values are obtained by the flexibility method when the structure is subjected to unsymmetrical loading. Table (4-6) shows the displacements of the arch loaded unsymmetrically by placing a 1 kip load at panel point 5 for the structure shown in Fig.(4-1).

Table (4-6)

Method \ Pts.		1	2 (ft.)	3 (ft.)	4 (ft.)	5 (ft.)	6 (ft.)	7 (ft.)
Vert. Displ. of Arch	F.M.	0	.001363	.001320	-.000109	-.001498	-.001157	0
	A.M.	0	.000309	.000347	-.000410	-.001065	-.000480	0
Horiz. Displ. of Arch	F.M.	0	-.000820	-.000822	-.000661	-.000810	-.000631	.00015
	A.M.	0	-.000226	-.000269	-.000199	-.000306	-.000116	.00015

F.M. = Flexibility Method; A.M. = Approximate Method.

The discrepancies may be attributed to the following reason: Sidesway effect due to unsymmetrical loading would have greater influence on the actual structure than on the idealized system that is used in the approximate stiffness method, leading to greater deflections based on the flexibility method. Under symmetrical loading, the pressure line of the actual structure is closer to the axis of the arch than in the idealized system, so that the actual structure would have less deflections than the idealized system.

4.2.2 Dead Load and Live Load Deflection

The dead load distribution of a bowstring arch is always symmetrical, while the live load is unsymmetrical. The dead load deflection of the structure can be reduced by choosing the curve of the arch axis matching, or close to, the shape of the pressure line of the arch due to dead load. The result of unsymmetrically distributed loads is more significant because of the fact that the pattern of loading induces considerable moments in the arch. This can be seen from the numerical results given in Table (4-7). The data in Table (4-7) is the arch deflection of the structure shown in Fig. (2-5) for symmetrically and unsymmetrically loaded conditions.

Table (4-7)

Station / Load Condition	0.0L	0.1L (ft.)	0.2L (ft.)	0.3L (ft.)	0.4L (ft.)	0.5L (ft.)
85 K acting at all panel points	0	-.002246	.006108	.016015	.023936	.026876
85 K acting at panel point No4 only	0	.126683	.225400	.243314	.147780	-.00505

4.5 Effect of Geometry of the Arch

4.5.1 Bending Moments and Hanger Forces

If the span and cross sectional dimensions of individual members in the bowstring arch bridge are held fixed, variation of the rise to span ratio does not appreciably affect the internal moments in the structure. This is substantiated by the numerical results tabulated in Table (4-8).

It is important to note that for a given rise-span ratio, the sum of the absolute values of bending moments in the section at or near mid-span is almost constant. This implies that the total weight at those sections is not affected very much by the change of bending moments. The hanger forces are also not significantly influenced by the variation of the rise-span ratio, under the above conditions.

Table(4-8) Bending Moments (ft-k)

h/L Ratio \ Location		0.0L	0.1L	0.2L	0.3L	0.4L	0.5L
0.179	Arch	44.47	314.94	312.34	348.44	367.91	374.28
	Girder	-44.47	-189.17	-88.76	-54.99	-32.54	-24.98
0.215	Arch	48.55	316.72	307.26	337.61	354.76	360.28
	Girder	-48.55	-197.92	-96.05	-60.40	-37.94	-30.26
0.250	Arch	52.95	322.59	308.52	335.03	350.87	355.94
	Girder	-52.95	-207.46	-103.84	-66.39	-43.85	-36.13
0.286	Arch	57.68	330.94	313.43	337.16	352.23	357.04
	Girder	-57.68	-217.73	-112.17	-73.00	-50.33	-42.57

Table (4-9) Hanger Forces (kips)

h/L Ratio	0.1L	0.2L	0.3L	0.4L	0.5L
0.179	93.7541	82.6201	84.5956	84.4700	84.4565
0.215	93.9727	82.6353	84.5286	84.4723	84.4515
0.250	94.2187	82.6369	84.4647	84.4710	84.4481
0.286	94.4865	82.6285	84.4110	84.4675	84.4456

4.5.2 Axial Force

There is a definite relationship between the geometry of the system and the magnitude of axial forces. It can be shown that the axial force decreases as the ratio of the rise to the span length increases; hence lighter sections can be used. Conversely, if the ratio decreases, axial forces in the members increase. Therefore, the rise-to-span ratio affects the determination of the sections of the members in the structure. However, the increase or decrease of the rise-to-span ratio also means corresponding increase or decrease of the arch length. Therefore, it can be thought that there exists a rise-to-span ratio that will yield a most economical design.Table (4-10) shows the variation of the axial force in the arch for different rise-to-span ratios.

Table (4-10) Axial Force of Arch (kips)

h/L Ratio \ Location	0.0L	0.1L	0.2L	0.3L	0.4L	0.5L
0.179	703.81	702.88	656.25	623.44	600.02	588.01
0.215	624.69	623.72	570.44	532.33	504.69	490.33
0.250	571.66	570.69	511.68	468.78	437.11	420.43
0.286	534.46	533.51	469.57	422.40	386.93	367.94

4.6 Conclusions

(1) The flexibility method is a suitable method for the analysis of bowstring arch. However, it would be advantageous to use the stiffness method when the cross sections of the structural members are not constant.

(2) Neglecting either arch rib shortening, elongation of tie girder, or the elongation of the hanger rods alone, does not have a great effect on final results. But when all of the deformations are neglected together, the error in the result may be quite significant.

(3) For a given layout of a bowstring arch, the relative size of the arch to the tie girder governs the distribution of the internal forces in the system. If loaded by a single load, the moments in a vertical section through the structure are distributed in proportion to the relative moments of inertia of the arch and the tie girder. The maximum hanger force is approximately close to $P(I_a \cos\phi)/(I_g + I_a \cos\phi)$, where P is the load at a panel point, and ϕ the angle of the tangent with respect to X-axis at the panel point under consideration.

(4) Normally, the vertical displacements due to dead load are less important, if the arch axis is laid out to match the shape of the pressure line curve of dead load in the arch. Live load deflections are often more critical than the dead load deflections.

(5) The actual structure is more flexible with regard to sidesway than the approximate structure when it is subjected to unsymmetrical load actions. However, it is stiffer against vertical deflections than the approximate structure when it is subjected to symmetrical load actions such as dead load.

(6) For a given span of bowstring arch and a loading condition, the magnitude of axial force in the structure is greatly affected by lay-out of the arch. For optimum design, the geometry of the system should be further investigated.

BIBLIOGRAPHY

1. James M. Gere, and William Weaver, Jr., "Analysis of Framed Structures",D. Van Nostrand, 1965.
2. James Michalos, "Theory of Structural Analysis and Design", The Ronald Press Company, 1958.
3. Movses J. Kaldjian, " Prestressed Bowstring Arch",Journal of Structural Division, ASCE, Oct., 1961.
4. Dabrowski, "Prestressed Bowstring Arch",discussion, Journal of Structural Division, ASCE, April, 1962.
5. J.M. Garrelts, "Design of St. Georges Tied Arch Span", Transaction, ASCE, Vol. 69, 1943.
6. L. Balog, "Truss Tied Arch Bridge Has a Long History",Civil Engineering, Dec., 1956.
7. James C. Grassie, "Analysis of Indeterminate Structures", Longmans, Green and Co., 1957.
8. S.Timoshenko, "Strength of Materials",Part II, Van Nostrand, 1956.
9. C.K. Wang, "Statically Indeterminate Structures",McGraw Hill, 1953.
10. Ch. R. Way, "Many Unusual Structures Feature Penn Lincoln Parkway",Civil Engineering, Oct., 1956.
11. Martin "Introduction to Matrix Method of Structural Analysis", McGraw Hill, 1966.
12. Dale L. Morris, "Curved Beam Stiffness Coefficients",Journal of Structural Division, ASCE, May, 1968.
13. Joseph J. Gennaro, "Computer Method in Solid Mechanics", Macmillan Co., N.Y., 1965.

14. S.P.Timoshenko & D. H. Young, "Theory of Structures", McGraw Hill 1965.

15. Harrold I. Laursen, "Structural Analysis",McGraw Hill,1969.

16. P.C. Wang, "Numerical and Matrix Method in Structural Mechanic with Applications to Computers",John Wiley & Sons, 1966.

17. Johns Robinson, "Structural Matrix Analysis for the Engineer", John Wiley & Sons, 1966.

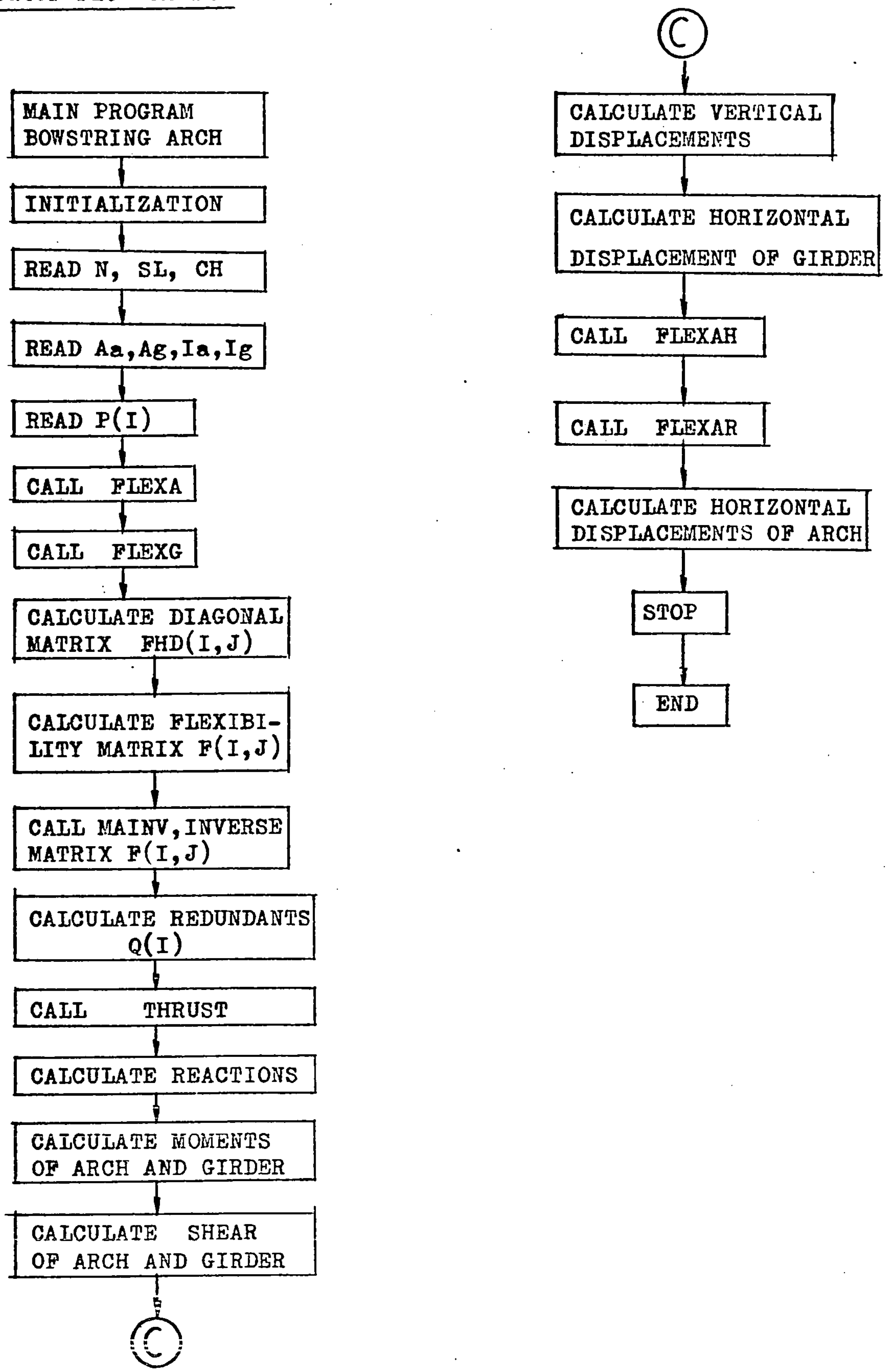
18. Moshe F. Rubinsten, "Matrix Computer Analysis of Structures", Prentice Hall, 1966.

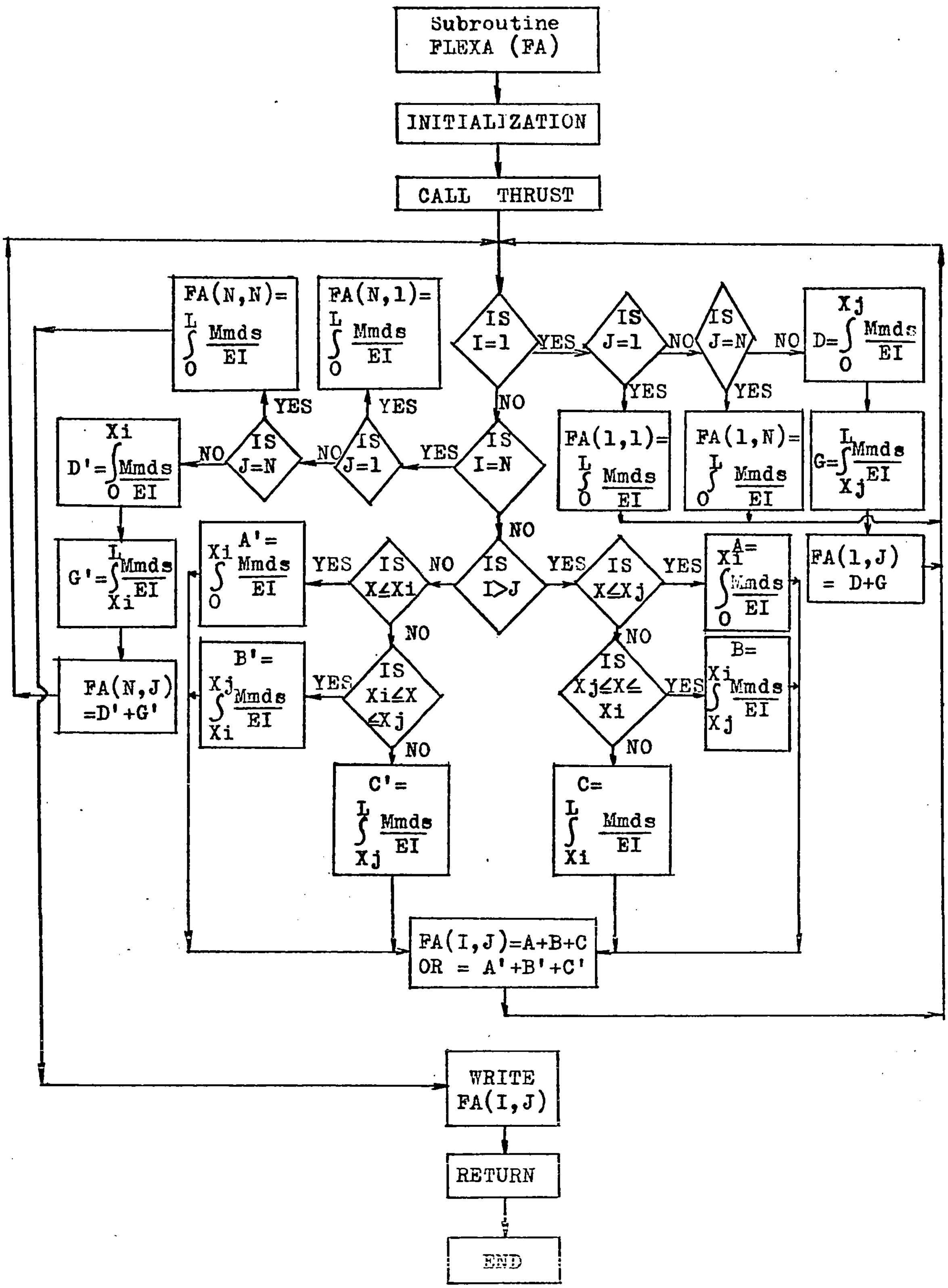
19. R. K. Livesley, "Matrix Methods of Structural Analysis", Macmillan Co., 1964.

APPENDIX A

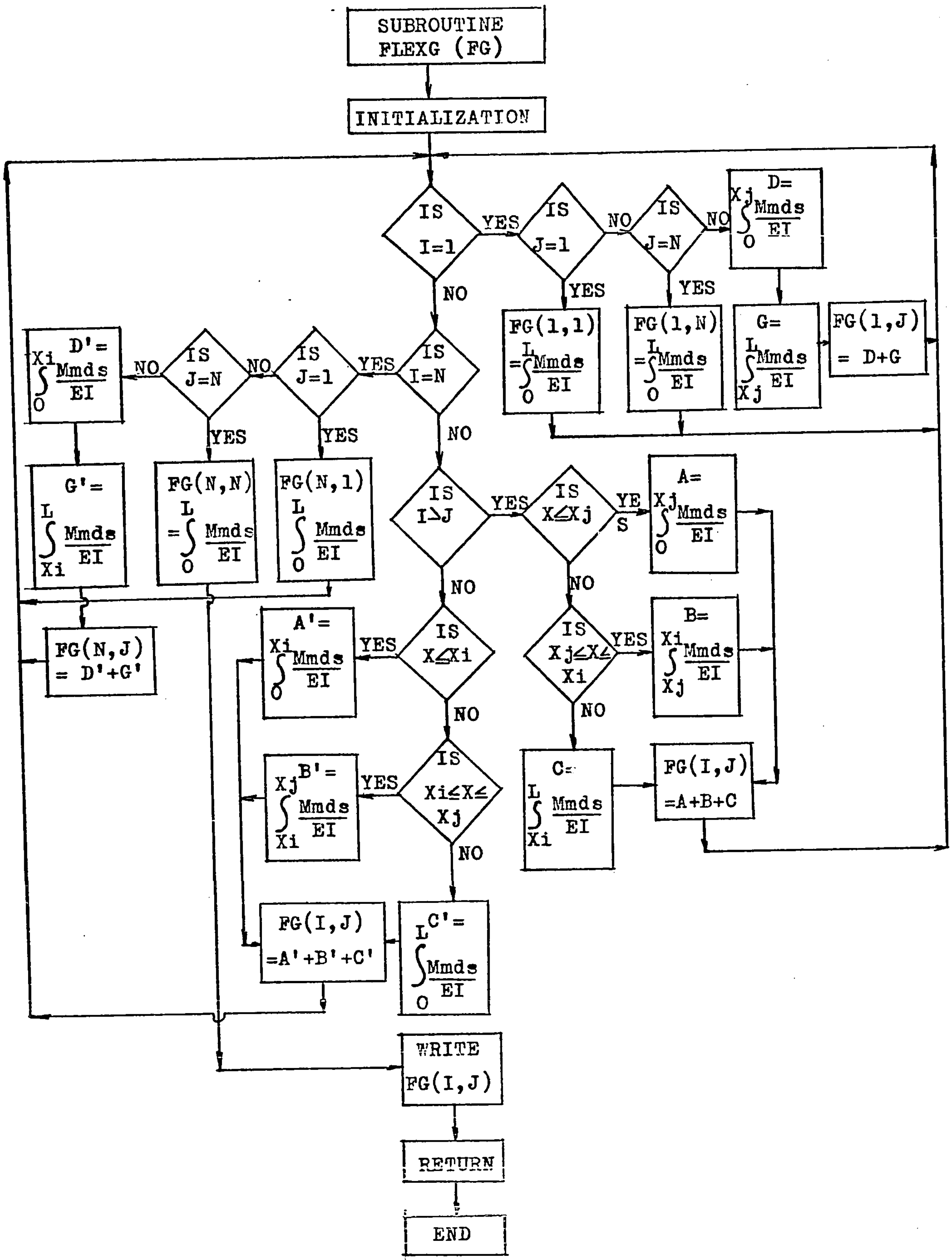
FLOW CHARTS AND FORMULAE

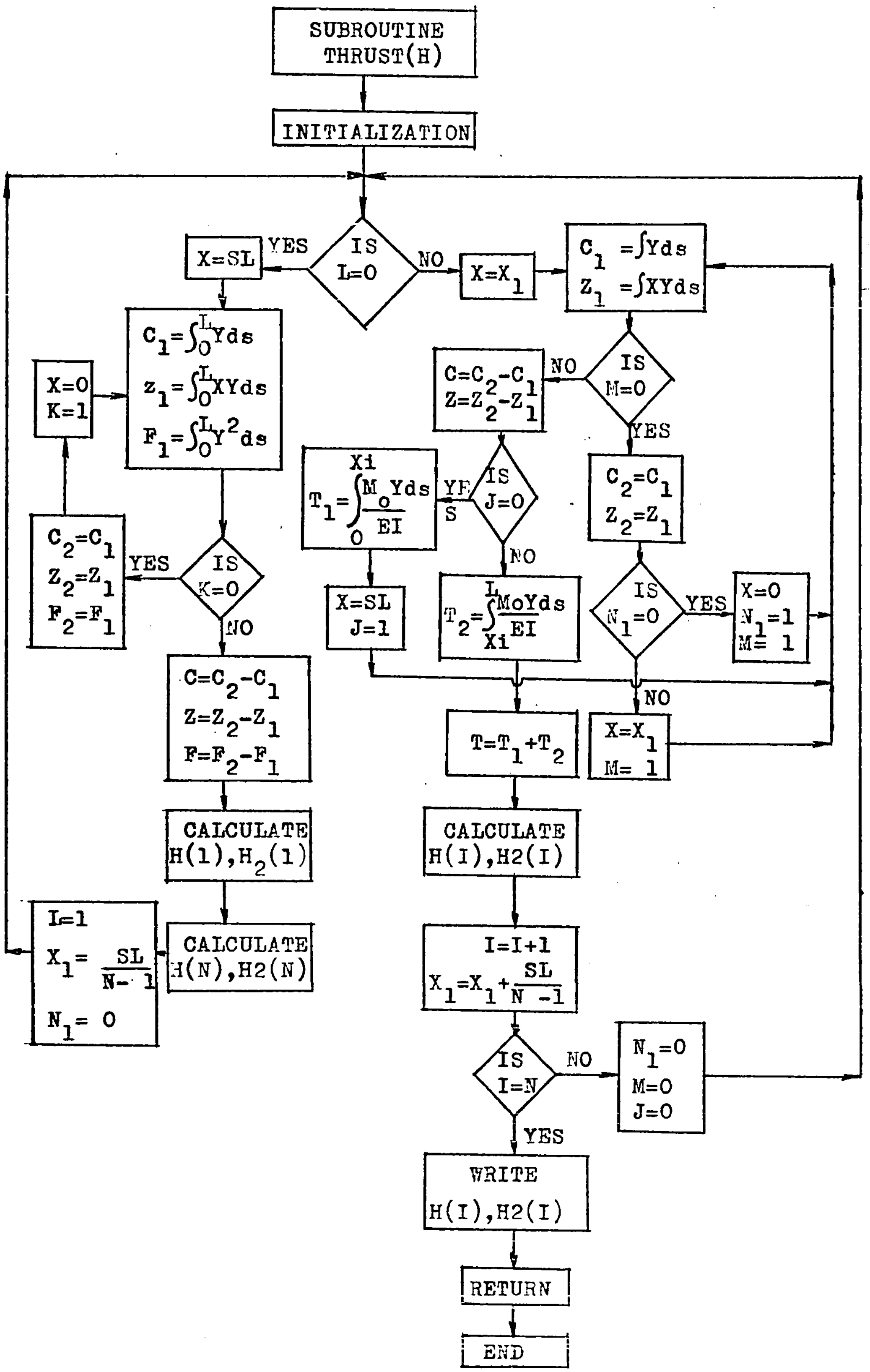
A.1 Computer Flow Charts

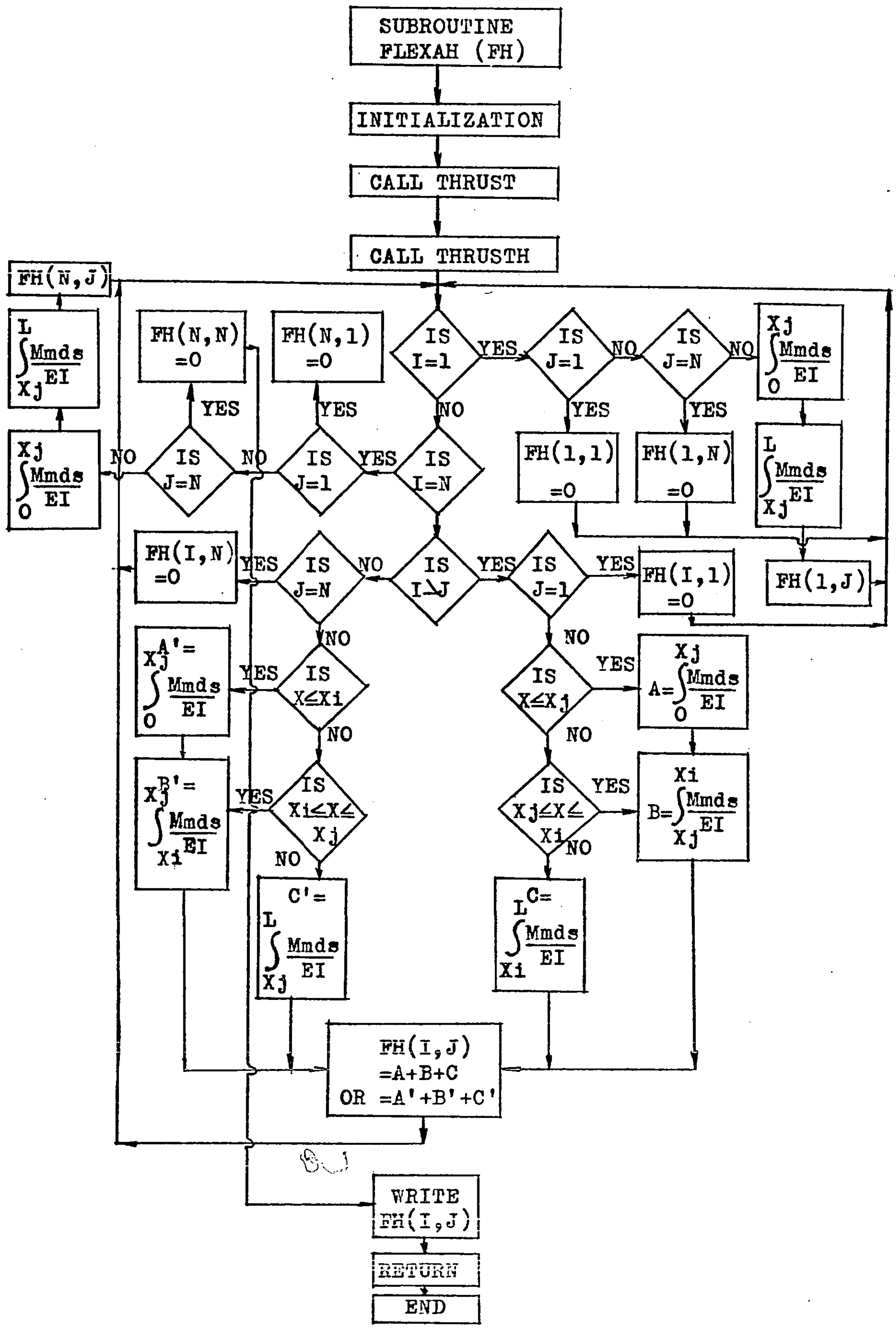


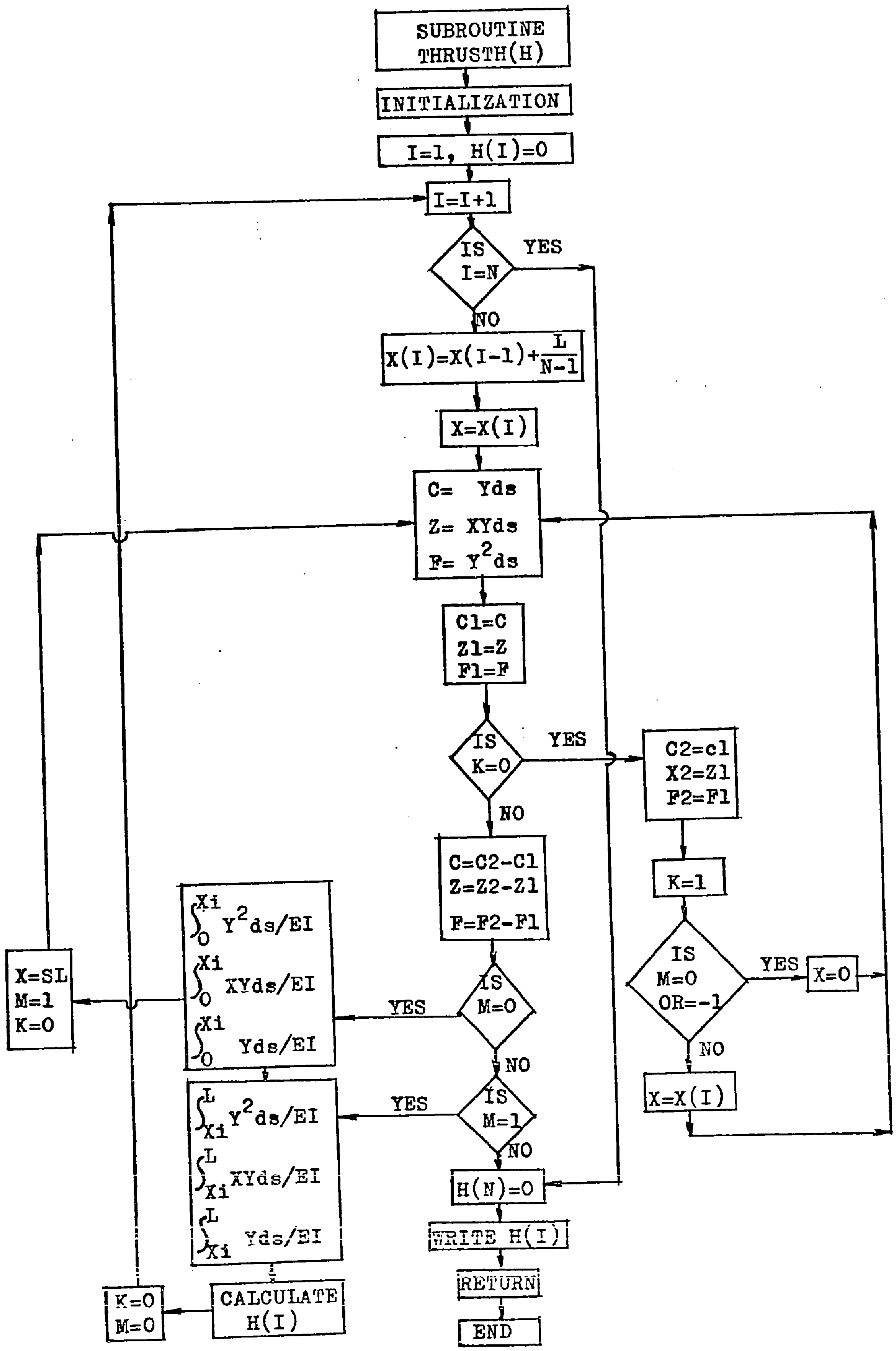


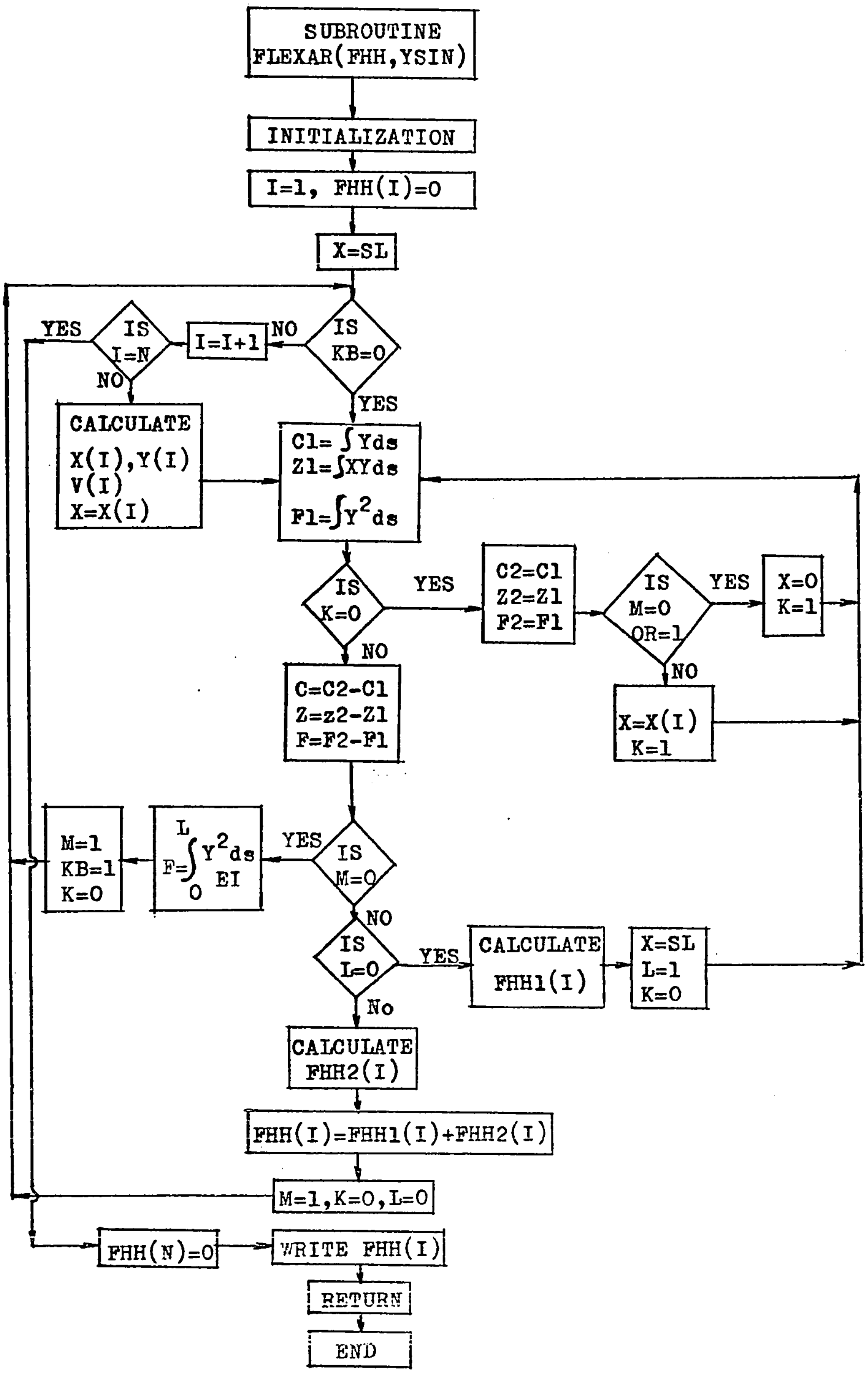
Subroutine FLEXA (FA)
INITIALIZATION
CALL THRUST
IS I=1
YES
IS J=1
NO
IS J=N
NO
YES
YES
NO
IS I=N
YES
IS J=1
NO
IS J=N
NO
YES
YES
FA(N,N)=
FA(N,1)=
FA(1,1)=
FA(1,N)=
D'=
G'=
FA(N,J) =D'+G'
D=
G=
FA(1,J) = D+G
IS I>J
IS X≤Xi
IS Xi≤X ≤Xj
IS X≤Xj
IS Xj≤X≤ Xi
A'=
B'=
C'=
A=
B=
C=
FA(I,J)=A+B+C OR = A'+B'+C'
WRITE FA(I,J)
RETURN
END


SUBROUTINE FLEXG (FG)
INITIALIZATION
IS I=1
YES
NO
IS J=1
IS J=N
D= $\int_0^{Xj} \frac{Mmds}{EI}$
FG(1,1) $= \int_0^L \frac{Mmds}{EI}$
FG(1,N) $= \int_0^L \frac{Mmds}{EI}$
G= $\int_{Xj}^L \frac{Mmds}{EI}$
FG(1,J) = D+G
IS I=N
D'= $\int_0^{Xi} \frac{Mmds}{EI}$
G'= $\int_{Xi}^L \frac{Mmds}{EI}$
FG(N,N) $= \int_0^L \frac{Mmds}{EI}$
FG(N,1) $\int_0^L \frac{Mmds}{EI}$
FG(N,J) = D'+G'
IS I≥J
IS X≤Xj
A= $\int_0^{Xj} \frac{Mmds}{EI}$
IS Xj≤X≤Xi
B= $\int_{Xj}^{Xi} \frac{Mmds}{EI}$
C= $\int_{Xi}^L \frac{Mmds}{EI}$
FG(I,J) =A+B+C
IS X≤Xi
A'= $\int_0^{Xi} \frac{Mmds}{EI}$
IS Xi≤X≤Xj
B'= $\int_{Xi}^{Xj} \frac{Mmds}{EI}$
C'= $\int_0^L \frac{Mmds}{EI}$
FG(I,J) =A'+B'+C'
WRITE FG(I,J)
RETURN
END


SUBROUTINE THRUST(H)
INITIALIZATION
IS L=0
YES
NO
X=SL
X=X1
C1 =∫Yds
Z1 =∫XYds
C1=∫0L Yds
z1=∫0L XYds
F1=∫0L Y2ds
X=0
K=1
IS M=0
C=C2-C1
Z=Z2-Z1
C2=C1
Z2=Z1
IS J=0
Xi
T1=∫0 MoYds/EI
C2=C1
Z2=Z1
F2=F1
IS K=0
X=SL
J=1
T2=∫Xi L MoYds/EI
IS N1=0
X=0
N1=1
M= 1
C=C2-C1
Z=Z2-Z1
F=F2-F1
T=T1+T2
X=X1
M= 1
CALCULATE H(1),H2(1)
CALCULATE H(I),H2(I)
I=1
X1= SL/N- 1
N1= 0
CALCULATE H(N),H2(N)
I=I+1
X1=X1+SL/N -1
IS I=N
N1=0
M=0
J=0
WRITE H(I),H2(I)
RETURN
END


SUBROUTINE FLEXAH (FH)
INITIALIZATION
CALL THRUST
CALL THRUSTH
IS I=1
IS J=1
IS J=N
IS I=N
IS I≥J
IS J=1
IS J=N
IS X≤Xi
IS Xi≤X≤Xj
IS X≤Xj
IS Xj≤X≤Xi
YES
NO
FH(N,J)
FH(N,N) =0
FH(N,1) =0
FH(1,1) =0
FH(1,N) =0
FH(I,N) =0
FH(I,1) =0
FH(1,J)
A'=
B'=
C'=
C=
A=
B=
Mmds
EI
FH(I,J) =A+B+C OR =A'+B'+C'
WRITE FH(I,J)
RETURN
END


SUBROUTINE THRUSTH(H)
INITIALIZATION
I=1, H(I)=0
I=I+1
IS I=N
YES
NO
X(I)=X(I-1)+L/N-1
X=X(I)
C= Yds
Z= XYds
F= Y2ds
C1=C
Z1=Z
F1=F
IS K=0
YES
C2=c1
X2=Z1
F2=F1
K=1
IS M=0 OR=-1
YES
X=0
NO
X=X(I)
NO
C=C2-C1
Z=Z2-Z1
F=F2-F1
IS M=0
YES
∫0Xi Y2ds/EI
∫0Xi XYds/EI
∫0Xi Yds/EI
X=SL
M=1
K=0
NO
IS M=1
YES
∫XiL Y2ds/EI
∫XiL XYds/EI
∫XiL Yds/EI
CALCULATE H(I)
K=0
M=0
NO
H(N)=0
WRITE H(I)
RETURN
END


SUBROUTINE FLEXAR(FHH,YSIN)
INITIALIZATION
I=1, FHH(I)=0
X=SL
IS KB=0
NO
I=I+1
IS I=N
YES
NO
CALCULATE X(I),Y(I) V(I) X=X(I)
YES
C1=∫Yds
Z1=∫XYds
F1=∫Y2ds
IS K=0
YES
C2=C1 Z2=Z1 F2=F1
IS M=0 OR=1
YES
X=0 K=1
NO
X=X(I) K=1
NO
C=C2-C1 Z=z2-Z1 F=F2-F1
IS M=0
YES
F=∫ L 0 Y2ds EI
M=1 KB=1 K=0
NO
IS L=0
YES
CALCULATE FHH1(I)
X=SL L=1 K=0
No
CALCULATE FHH2(I)
FHH(I)=FHH1(I)+FHH2(I)
M=1,K=0,L=0
FHH(N)=0
WRITE FHH(I)
RETURN
END

A.2 List of Basic Formulae in the Subroutines

Subroutine 'THRUST'

1. $$H(1) = \frac{\frac{1}{EIa}\left[\left(C - \frac{Z}{L}\right)\right]_0^L}{DV}$$

2. $$H(I) = \frac{\frac{1}{EIa}\left\{\left[\left(1 - \frac{Xi}{L}\right) Z\right]_0^{Xi} + \left[Xi\ C - \frac{Xi}{L} Z\right]_{Xi}^{L}\right\}}{DV}$$

3. $$H(N) = \frac{\frac{1}{EIa}\left[\frac{Z}{L}\right]_0^L}{DV}$$

where $$DV = \left[\frac{F}{EIa}\right]_0^L + \frac{L^2}{4hEAa}\log\left(\frac{4h}{L} + \sqrt{1 + \left(\frac{4h}{L}\right)^2}\right) + \frac{L}{EAg}$$

Subroutine 'FLEXA'

1. $$FA(1,1) = \frac{1}{EIa}\left[A - \frac{2}{L}B - 2Hi\ C + \frac{D}{L} + \frac{2}{L}Hi\ Z + Hi^2 F\right]_0^L$$

2. $$\begin{aligned} FA(1,J) = \frac{1}{EIa}\Bigg\{ & \left[Vj\ B - Hj\ C - \frac{Vj}{L}D + \left(\frac{Hj}{L} - Hi\ Vj\right) Z + Hi\ Hj\ F\right]_0^{Xj} \\ & + \Bigg[Xj\ A + 2(Vj - 1)B - (Hj + Hi\ Xj)C - \frac{(Vj - 1)}{L} D \\ & + \left(\frac{Hj}{L} - Hi\ Vj + Hi\right)Z + Hi\ Hj\ F\Bigg]_{Xj}^{L}\Bigg\} \end{aligned}$$

3. $$FA(1,N) = \frac{1}{EIa}\left[\frac{B}{L} - Hj\ C - \frac{D}{L^2} + \left(\frac{Hj}{L} - \frac{Hi}{L}\right)Z + Hi\ Hj\ F\right]_0^L$$

4. $$\begin{aligned} FA(I,1) = \frac{1}{EIa}\Bigg\{ & \left[Vi\ B - Hi\ C - \frac{Vi}{L} D + \left(\frac{Hi}{L} - Hj\ Vi\right)Z + Hi\ Hj\ F\right]_0^{Xi} \\ & + \Bigg[Xi\ A + 2(Vi - 1)B - (Hi + Hj\ Xi)\ C - \frac{(Vi - 1)}{L} D \\ & + \left(\frac{Hi}{L} - Hj\ Vi + Hj\right)Z + Hi\ Hj\ F\Bigg]_{Xi}^{L}\Bigg\} \end{aligned}$$

5. a) Case 1, $Xi \geq Xj$

$$\begin{aligned} FA(I,J) = \frac{1}{EIa}\Bigg\{ & \left[Vi\ Vj\ D - (Vi\ Vj + Vj\ Hi)\ Z + Hi\ Hj\ F\right]_0^{Xj} \\ & + \Big[Vi\ Xj\ B - (Vi\ Hj + Vj\ Hi)\ Z + Vi(Vj - 1)\ D - Xj\ HiC \\ & + Hi\ Hj\ F\Big]_{Xj}^{Xi} + \Big[Xi\ Xj\ A + (Xj\ Vi + Xi\ Vj - Xi - Xj)B \\ & - (Xj\ Hi + Xi\ Hj)\ C + Hi\ Hj\ F - (Hj Vi + Hi\ Vj - Hi - Hj)Z \\ & + (Vi - 1)(Vj - 1)\ D\Big]_{Xi}^{L}\Bigg\} \end{aligned}$$

5. b) Case II , $Xj \geq Xi$

$$FA(I,J) = \frac{1}{EIa}\Big\{\Big[Vi\ Vj\ D - (\ Vi\ Hj + Vj\ Hi\ )\ Z + Hi\ Hj\ F\Big]_0^{Xi}$$

$$+\Big[Vi\ Xj\ B - (\ Vi\ Hj + Hi\ Vj - Hj\ )\ Z + Vj\ (\ Vi-1)\ D$$

$$+ Hi\ Hj\ F - C\ Xi\ Hj\Big]_{Xi}^{Xj} + \Big[Xi\ Xj\ A + (\ Xj\ Vi + Xi\ Vj$$

$$-Xi - Xj\ )B - (\ Hj\ Vi + Hi\ Vj - Hi - Hj\ )Z + (\ Vi$$

$$-1)(\ Vj - 1)D - (\ XjHi + Xi\ Hj\ )C + Hi\ Hj\ F\Big]_{Xj}^{L}\Big\}$$

6. $$FA(I,N) = \frac{1}{EIa}\Big\{\Big[\frac{Vi}{L}D - (\ Hi\ Vj + \frac{Hj}{L})Z + Hi\ Hj\ F\Big]_0^{Xi} + \Big[\frac{Xi}{L}B -$$

$$Hj\ Xi\ C + (\ \frac{Hi}{L} + Hj\ Vi - Hj\ )Z + \frac{(\ Vi - 1)}{L}D + HiHjF\Big]_{Xi}^{L}\Big\}$$

7. $$FA(N,J) = \frac{1}{EIa}\Big\{\Big[\frac{Vj}{L}D - (\ Hi\ Vj + \frac{Hj}{L})Z + Hi\ Hj\ F\Big]_0^{Xj} + \Big[\frac{Xj}{L}\ B -$$

$$Hi\ Xj\ C - (\ \frac{Hj}{L} + Hi\ Vj - Hi\ )Z + \frac{(\ Vj - 1\ )}{L}\ D +$$

$$Hi\ Hj\ F\Big]_{Xj}^{L}\Big\}$$

8. $$FA(N,1) = \frac{1}{EIa}\Big[\frac{B}{L} - Hi\ C - \frac{D}{L^2} + (\ \frac{Hi}{L} - \frac{Hj}{L}\ )Z + Hi\ Hj\ F\Big]_0^{L}$$

9. $$FA(N,N) = \frac{1}{EIa}\Big[\frac{D}{L^2} - 2\,\frac{Hi}{L}\ Z + Hi^2F\Big]_0^{L}$$

Subroutine " FLEXG "

1. $$FG(1,1) = -\frac{L}{3EIg}$$

2. $$FG(1,J) = \frac{Vj}{EIg}\Big[\frac{1}{2}\ Xj^2 - \frac{X^3}{3L} + \frac{1}{2}\ (\ Vj - 1\ )(\ L^2 - Xj^2) - \frac{1}{3L}\,(Vj-1)$$

$$(\ L^3 - Xj^3) + Xj(\ L - Xj\ ) - \frac{Xj}{2L}\ (\ L^2 - Xj^2\ )\Big]$$

3. $FG(1,J) = FG(I,1)$, when $I=J$

4. $$FG(1,N) = \frac{L}{6EIg}$$

5. Case 1, $Xi \geq Xj$

$$FG(I,J) = \frac{1}{EIg}\Big[\frac{Vi\ Vj}{3}\ Xi^3 + \frac{1}{3}\ Vj(\ Vi - 1)(\ Xj^3 - Xi^3) + \frac{1}{2}\ Xi\ Xj$$

$$(Xj^2 - Xi^2) + \frac{1}{3}(Vi - 1)(Vj - 1)(L^3 - Xj^3) + \frac{Xj}{2}(Vi$$

$$-1)(L^2 - Xj^2) + \frac{1}{2} Xi(Vj-1)(L^2 - Xj^2) + Xi\ Xj(L - Xj)\Big]$$

5. Case II , $Xj \geqq Xi$

$$FG(I,J) = \frac{1}{EIg}\Big[\frac{1}{3} Vi\ Vj\ Xj^3 + \frac{Vi}{3}(Vj - 1)(Xi^3 - Xj^3) + \frac{Xj\ Vi}{2}(Xi^2$$

$$- Xj^2) + \frac{(Vi - 1)}{3}(Vj - 1)(L^3 - Xi^3) + \frac{Xj}{2}(Vi - 1)(L^2$$

$$- Xi^2) + \frac{Xi}{2}(Vj - 1)(L^2 - Xi^2) + Xi\ Xj(L - Xi)\Big]$$

6. $FG(I,N) = \frac{1}{EIg}\Big[\frac{1}{3} Vi\ Xi^3 + \frac{1}{3L}(Vi - 1)(L^3 - Xi^3) + \frac{Xi}{2L}(L^2 - Xi^2)\Big]$

7. $FG(N,J) = \frac{1}{EIg}\Big[\frac{1}{3} Vj\ Xj^3 + \frac{1}{3L}(Vj - 1)(L^3 - Xj^3) + \frac{Xj}{2L}(L^2 - Xj^2)\Big]$

8. $FG(N,1) = \frac{L}{6EIg}$

9. $FG(N,N) = - \frac{L}{3EIg}$

Subroutine 'THRUSTH'

1. $H(1) = 0$
2. $H(I) = Vi\Big\{[E]_0^L + [F]_{Xi}^L - Yi\ [C]_{Xi}^L\Big\}\Big/[F]_0^L$
3. $H(N) = 0$

Subroutine 'FLEXAH'

1. $FH(1,1) = 0$
2. $FH(1,J) = \frac{1}{EIa}\Big\{\Big[- Vj\ B + Hj\ C + \frac{Vj}{L} D + (Hi\ Hj - \frac{Hj}{L})Z - Hi\ Hj\ F\Big]_0^{Xj}$

$$+\Big[Yj\ A -(Vj + \frac{Yj}{L})B + (Hj - 1 - Hi\ Hj)C + \frac{Vj}{L} D$$

$$+ (Hi\ Vj - \frac{(Hj-1)}{L})Z - Hi(Hj - 1)F\Big]_0^L\Big\}$$

3. $FH(1,N) = 0$
4. $FH(I,1) = 0$
5. a) Case 1 , $Xi \geqq Xj$

$$FH(I,J) = \frac{1}{EIa}\Big\{\Big[(Vi\ Hj + Vj\ Hi\ )\ Z - Vi\ Vj\ D + Hi\ Hj\ F\ \Big]_0^{Xj}$$

$$+\Big[\big[Vi\ (\ Hj-1) + Vj\ Hi\big]\ Z - Vi\ Vj\ D - Hi\ (\ Hj-1)F$$

$$+ Vi\ Yj\ B - Hi\ Yj\ C\ \Big]_{Xj}^{Xi} + \Big[(\ Vi-1)\ Vj\ D - Hi(Hj-1)F$$

$$+\big[(\ Vi\ -1)\ Yj - Vj\ Vi\big]\ B + \big[(\ Hj-1)Xi - Hi\ Hj\ Yj\big]\ C$$

$$+ Xi\ Yj\ A + \big[(\ Vi-1)(Hj-1)+Vj\ Hi\big]\ Z\ \Big]_{Xi}^{L}\Big\}$$

5. b) Case II , $Xj \geq Xi$

$$FH(I,J) = \frac{1}{EIa}\Big\{\Big[(\ Vi\ Hj + Vj\ Hi\ )Z - Vi\ Vj\ D + Hi\ Hj\ F\Big]_0^{Xi}$$

$$+\Big[\big[(\ Vi - 1)\ Hj + Vj\ Hi\big]\ Z - Vj(Vi-1)D - Hi\ Hj\ F$$

$$- Vj\ Xi\ B + Hj\ Xi\ C\Big]_{Xi}^{Xj} + \Big[Xi\ Yj\ A + \big[(\ Vi-1)\ Yj$$

$$- Vj\ Xi\big]\ B + \big[(\ Vi-1)(Hj-1) + Vj\ Hi\ \big]E - (\ Vi- 1$$

$$)\ Vj\ D + \big[(\ Hj-1)\ Xi - Hi\ Yj\big]\ C - Hi(Hj-1)\ F\ \Big]_{Xj}^{L}\Big\}$$

6. $FH(I,N) = 0$

7. $$FH(N,J) = \frac{1}{EIa}\Big\{\Big[-\frac{Vj}{L}\ D + (\ \frac{Hj}{L} + Hi\ Vj\ )\ E - Hi\ Hj\ F\ \Big]_0^{Xj} + \Big[\frac{Yj}{L}B$$

$$- Hi\ Yj\ C - \frac{Vj}{L}D + (\ \frac{Hj}{L} - \frac{1}{L} + Hi\ Vj\ )\ Z + (\ Hi - Hi\ Hj$$

$$)\ F\Big]_{Xj}^{L}\Big\}$$

8. $FH(N,1) = 0$

9. $FH(N,N) = 0$

Subroutine 'FLEXAR'

1. $H(1) = 0$

2. $$H(I) = \frac{1}{EIa}\Big\{\Big[F - Vi\ Z\Big]_0^{Xi} - \Big[Yi\ C - Vi\ Z\Big]_{Xi}^{L}\Big\}$$

3. $$H(N) = \frac{1}{EIa}\Big[F\Big]_0^{L}$$

where

$$A = -\frac{h}{L^2}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2}$$

$$B = -\frac{h}{2L}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} + \left[\frac{h}{3L^2}\sqrt{(U^2+W^2)^3}\right]_{X1}^{X2}$$

$$C = -\frac{h^2}{L^2}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} + \frac{2h^2}{L^4}\left[\frac{W}{4}\sqrt{(U^2+W^2)^3} - \frac{U^2}{8}W\sqrt{U^2+W^2} - \frac{U^4}{8}\log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2}$$

$$D = -\frac{h}{4}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} + \frac{h}{3L}\left[\sqrt{(U^2+W^2)^3}\right]_{X1}^{X2}$$
$$- \frac{h}{2L^2}\left[\frac{W}{4}\sqrt{(U^2+W^2)^3} - \frac{U^2}{8}W\sqrt{U^2+W^2} - \frac{U^4}{8}\log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2}$$

$$Z = -\frac{h^2}{2L}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} + \frac{h^2}{3L^2}\left[\sqrt{(U^2+W^2)^3}\right]_{X1}^{X2}$$
$$+\frac{h^2}{L^3}\left[\frac{W}{4}\sqrt{(U^2+W^2)^3} - \frac{U^2}{8}W\sqrt{U^2+W^2} - \frac{U^4}{8}\log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2}$$
$$-\frac{h^2}{L^4}\left[\left(\frac{W^2}{5} - \frac{2}{15}U^2\right)\sqrt{(U^2+W^2)^3}\right]_{X1}^{X2}$$

$$F = -\frac{h^3}{L^2}\left[W\sqrt{U^2+W^2} + U^2 \log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} + \frac{4h^3}{L^4}\left[\frac{W}{4}\sqrt{(U^2+W^2)^3}\right.$$
$$\left. - \frac{U^2}{8}W\sqrt{U^2+W^2} - \frac{U^4}{8}\log\left(W+\sqrt{U^2+W^2}\right)\right]_{X1}^{X2} - \frac{h^3}{3L^6}\left[W^3\sqrt{(U^2+W^2)^3}\right.$$
$$\left. - 3U^2\left[\frac{W}{4}\sqrt{(U^2+W^2)^3} - \frac{U^2}{8}W\sqrt{U^2+W^2} - \frac{U^4}{8}\log\left(W+\sqrt{U^2+W^2}\right)\right]\right]_{X1}^{X2}$$

$$U = \frac{L^2}{4h}$$

$$W = L - 2X$$

$$Y = \frac{4h}{L^2}\left(LX - X^2\right)$$

Vi , Vj = the reactions at the left support due to unit load actions at point i, and j, corresponding to the redundants respectively.

Xi, Xj, Yi, Yj = the coordinates of point point i, and j, respectively with respect to the structural system.

Hi, Hj = the horizontal thrusts at the left support due to unit load at i and at j corresponding to redundants.

Aa, Ag = the cross sectional areas of the arch and the tie girder respectively.

APPENDIX B

DERIVATION OF CURVED MEMBER STIFFNESS MATRIX

Consider a curved member shown in Fig.(x), in which P is an arbitarary point on the curve.

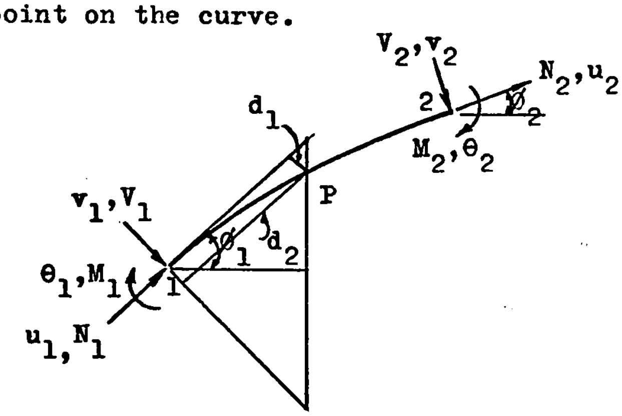


Fig.(x)

The bending strain energy is given by

$$U = \frac{1}{2EI}\int M^2 \, ds$$

The bending moment at point P is

$$M = M_1 + N_1 d_1 - V_1 d_2$$

where $d_1 = (X-X_1)\sin\phi_1 - (Y-Y_1)\cos\phi_1$;

$d_2 = (X-X_1)\cos\phi_1 + (Y-Y_1)\sin\phi_1$;

X_1, Y_1 = the coordinates of the structural system at node 1 ;

X_2, Y_2 = the coordinates of the structural system at node 2 ;

X , Y = the coordinates at point P, and

$ds = \sqrt{1 + (Y')^2}\, dX$

Applying Castigliano's theorem, to the curved beam, we obtain the displacements at node 1,

$$u_1 = \frac{\partial U}{\partial N_1} = \frac{1}{EI} \int_{X_1}^{X} M \frac{\partial M}{\partial N_1} \sqrt{1 + (Y')^2} \; dX$$

$$= \frac{1}{EI} \int_{X_1}^{X} (M_1 + N_1 d_1 - V_1 d_2) \, d_1 \sqrt{1 + (Y')^2} \, dX$$

$$v_1 = \frac{\partial U}{\partial V_1} = \frac{1}{EI} \int_{X_1}^{X} M \frac{\partial M}{\partial V_1} \sqrt{1 + (Y')^2} \; dX$$

$$= -\frac{1}{EI} \int_{X_1}^{X} (M_1 + N_1 d_1 - V_1 d_2) \, d_2 \sqrt{1 + (Y')^2} \, dX$$

$$\theta_1 = \frac{\partial U}{\partial M_1} = \frac{1}{EI} \int_{X_1}^{X} M \frac{\partial M}{\partial M_1} \sqrt{1 + (Y')^2} \; dX$$

$$= \frac{1}{EI} \int_{X_1}^{X} (M_1 + N_1 d_1 - V_1 d_2) \sqrt{1 + (Y')^2} \, dX$$

or

$$u_1 = \frac{1}{EI} \Big\{ \Big[\sin^2\phi_1 (D - 2X_1 B + X_1^2 A) - \sin 2\phi_1 (E - Y_1 B - X_1 C$$

$$+ X_1 Y_1 A) + \cos^2\phi_1 (F - 2Y_1 C + Y_1^2 A) \Big] N_1 - V_1 \Big[\sin^2\phi_1 (E - Y_1 B$$

$$- X_1 C + X_1 Y_1 A) - \frac{1}{2} \sin 2\phi_1 (F - D - 2Y_1 C + 2X_1 B + A(Y_1^2$$

$$- X_1^2)) - \cos^2\phi_1 (E - Y_1 B - X_1 C + X_1 Y_1 A) \Big] + \Big[\sin\phi_1 (B$$

$$- X_1 A) - \cos\phi_1 (C - Y_1 A) \Big] M_1 \Big\}$$

$$v_1 = -\frac{1}{EI} \Big\{ N_1 \Big[\sin^2\phi_1 (E - Y_1 B - X_1 C + X_1 Y_1 A) - \frac{1}{2} \sin 2\phi_1 (F$$

$$- D - 2Y_1 C + 2X_1 B + A(Y_1^2 - X_1^2) - \cos^2\phi_1 (E - Y_1 B - X_1 C + X_1 Y_1 A) \Big]$$

$$-\left[\sin^2\phi_1(F - 2Y_1C + Y_1^2A) + \sin2\phi_1(E - Y_1B - X_1C + X_1Y_1A) + \cos^2\phi_1(D - 2X_1B + X_1^2A)\right]V_1 + \left[\sin\phi_1(C - Y_1A) + \cos\phi_1(B - X_1A)\right]M_1\Big\}$$

$$\theta_1 = \frac{1}{EI}\Big\{\left[\sin\phi_1(B - X_1A) - \cos\phi_1(C - Y_1A)\right]N_1 - \left[\sin\phi_1(C - Y_1A) + \cos\phi_1(B - X_1A)\right]V_1 + AM_1\Big\}$$

where $$A = \int_{X_1}^{X} \sqrt{1 + (Y')^2}\, dx$$

$$B = \int_{X_1}^{X} X\sqrt{1 + (Y')^2}\, dx$$

$$C = \int_{X_1}^{X} Y\sqrt{1 + (Y')^2}\, dx$$

$$D = \int_{X_1}^{X} X^2\sqrt{1 + (Y')^2}\, dx$$

$$E = \int_{X_1}^{X} XY\sqrt{1 + (Y')^2}\, dx$$

$$F = \int_{X_1}^{X} Y^2\sqrt{1 + (Y')^2}\, dx$$

let $$a = \left[\sin^2\phi_1(D - 2X_1B + X_1^2A) - \sin2\phi_1(E - Y_1B - X_1C + X_1Y_1A) + \cos^2\phi_1(F - 2Y_1C + Y_1^2A)\right]$$

$$b = -\left[\sin^2\phi_1(E - Y_1B - X_1C + X_1Y_1A) - \frac{1}{2}\sin2\phi_1(F - D - 2Y_1C + 2X_1B + A(Y_1^2 - X_1^2)) - \cos^2\phi_1(E - Y_1B - X_1C + X_1Y_1A)\right]$$

$$c = \sin\phi_1 (B - X_1 A) - \cos\phi_1 (C - Y_1 A)$$

$$d = \sin^2\phi_1 (F - 2Y_1 C + Y_1^2 A) + \sin 2\phi_1 (E - Y_1 B - X_1 C + X_1 Y_1 A)$$

$$+ \cos^2\phi_1 (D - 2X_1 B + X_1^2 A)$$

$$e = -\left[\sin\phi_1 (C - Y_1 A) + \cos\phi (B - X_1 A)\right]$$

$$f = A$$

The displacements at node 1 may be expressed in matrix form as

$$\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} N_1 \\ V_1 \\ M_1 \end{bmatrix} \qquad \text{(i)}$$

Solving for the force vector,

$$\begin{bmatrix} N_1 \\ V_1 \\ M_1 \end{bmatrix} = EI \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix}$$

or

$$\begin{bmatrix} N_1 \\ V_1 \\ M_1 \end{bmatrix} = \frac{EI}{G} \begin{bmatrix} g & h & i \\ h & j & k \\ i & k & l \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix} \qquad \text{(ii)}$$

Where

$g = df - e^2$; $h = - bf + ce$; $i = be - cd$;

$j = (af - c^2)$; $k = - ae + bc$; $l = ad - b^2$;

$G = adf - ae^2 - b^2 f - c^2 d + 2bce$

The relationship between the displacement and forces at node 1 is thus established. Equation (i) gives the flexibility relationship at node 1,whose flexibility matrix can be inverted to find the stiffness relationship at node 1, and it is given in Eq.(ii). We shall denote the stiffess matrix by S_{11} ; i.e.,

$$\begin{bmatrix} S_{11} \end{bmatrix} = \begin{bmatrix} g & h & i \\ h & j & k \\ i & k & l \end{bmatrix}$$

Now let us consider the forces at node 2, and find out the corresponding displacements at node 1. From Fig.(x), we have the following equilibrium equations.

$$\Sigma H = 0 \ ; \quad N_1\cos\phi_1 + V_1\sin\phi_1 + N_2\cos\phi_2 + V_2\sin\phi_2 = 0$$

$$\Sigma V = 0 \ ; \quad N_1\sin\phi_1 - V_1\cos\phi_1 + N_2\sin\phi_2 - V_2\cos\phi_2 = 0 \qquad \text{(iii)}$$

$$\Sigma M_{@2} = 0 \ ; \ M_1 + N_1\left[(X_2-X_1)\sin\phi_1-(Y_2-Y_1)\cos\phi_1\right] - V_1\left[(X_2-X_1)\cos\phi_1+(Y_2-Y_1)\sin\phi_1\right] + M_2 = 0$$

Equation (iii) can be solved for N_2 and V_2

$$N_2 = \frac{(\cos\phi_1 + \sin\phi_1\tan\phi_2)}{\cos\phi_2 + \sin\phi_2\tan\phi_2} N_1 + \frac{(\sin\phi_1 - \cos\phi_1\tan\phi_2)}{\cos\phi_2 + \sin\phi_2\tan\phi_2} V_1$$

$$V_2 = \sin\phi_1 + \frac{(\cos\phi_1 + \sin\phi_1\tan\phi_2)}{\cot\phi_2 + \tan\phi_2} N_1 - \left(\cos\phi_1 - \frac{\sin\phi_1 - \cos\phi_1\tan\phi_2}{\cot\phi_2 + \tan\phi_2} \right) V_1$$

$$M_2 = -\left[(X_2 - X_1)\sin\phi_1 - (Y_2-Y_1)\cos\phi_1\right]N_1 + \left[(X_2-X_1)\cos\phi_1 + (Y_2 - Y_1)\sin\phi_1\right]V_1 - M_1$$

The above equations in matrix form can be rewritten as

$$\begin{bmatrix} N_2 \\ \\ V_2 \\ \\ M_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\phi_1+\sin\phi_1\tan\phi_2}{\cos\phi_2+\sin\phi_2\tan\phi_2} & \dfrac{\sin\phi_1-\cos\phi_1\tan\phi_2}{\cos\phi_2+\sin\phi_2\tan\phi_2} & 0 \\ \sin\phi_1+\dfrac{\cos\phi_1+\sin\phi_1\tan\phi_2}{\cot\phi_2+\tan\phi_2} & -\cos\phi_1+\dfrac{\sin\phi_1-\cos\phi_1\tan\phi_2}{\cot\phi_2+\tan\phi_2} & 0 \\ -\Delta X\sin\phi_1+\Delta Y\cos\phi_1 & \Delta X\cos\phi_1+\Delta Y\sin\phi_1 & -1 \end{bmatrix} \begin{bmatrix} N_1 \\ \\ V_1 \\ \\ M_1 \end{bmatrix}$$

where $\Delta X = (X_2 - X_1)$, $\Delta Y = (Y_2 - Y_1)$

or
$$\begin{bmatrix} N_2 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} N_1 \\ V_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} S_{11} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} S_{12} \end{bmatrix} \begin{bmatrix} N_1 \\ V_1 \\ M_1 \end{bmatrix}$$

$$\begin{bmatrix} S_{12} \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} S_{11} \end{bmatrix}$$

From reciprocal theorem, we have

$$\begin{bmatrix} S_{12} \end{bmatrix} = \begin{bmatrix} S_{21} \end{bmatrix}$$

So far the only unknown we don't find out is the displacement at node 2 due to the forces at node 2. i.e., $\begin{bmatrix} S_{22} \end{bmatrix}$. Because $\begin{bmatrix} S_{22} \end{bmatrix}$ must be identical with $\begin{bmatrix} S_{11} \end{bmatrix}$, with possible sign changes on the off-diagonal elements.So that we write

$$\begin{bmatrix} S_{22} \end{bmatrix} = \begin{bmatrix} g & -h & i \\ -h & j & -k \\ i & -k & i \end{bmatrix}$$

The stiffness equation for a curved member is

$$\begin{bmatrix} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{bmatrix} = \left[\begin{array}{c|c} [S_{11}] & [S_{12}] \\ \hline [S_{21}] & [S_{22}] \end{array}\right] \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Then the stiffness matrix for a curved member is

$$[S_m] = \left[\begin{array}{c|c} [S_{11}] & [S_{12}] \\ \hline [S_{21}] & [S_{22}] \end{array}\right]$$