Network Reliability, Domination Theory and Reliability Polynomial

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montréal, Québec, Canada

November 1988

ISBN 0-315-49049-7

ABSTRACT

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A probabilistic network consists of a set of vertices and links that fail with some known joint probability distribution.

Assuming statistical independence and common probability of operation, we consider three different types of network operation:

1- The all terminal reliability problem, the network is operational if all the vertices can communicate.
2- The two-terminal reliability problem, the network is operational if two specified nodes communicate.
3- The K-terminal reliability problem, the network is operational if K-specified nodes are connected.

These problems are, in general, NP-hard except for some networks of special structure, where there exist linear and polynomial time algorithms.

In this thesis, we review general methods for network reliability and we investigate the different properties of the domination that plays a key role in network reliability computational complexity. Also we introduce the reliability polynomial and establish the relation between its different coefficients. Finally we investigate some bounds on this polynomial.

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CHAPTER I

1. INTRODUCTION & HISTORICAL REVIEW

1.1 Introduction

Analysis of network reliability is important in computer, communication, power and various networks. Components of a particular network may be subject to random failure and the network may or may not continue to function after some of its components have failed. We wish, as efficiently as possible, to determine the probability that the network is functional.

The network considered is an undirected graph $G = (V, E)$ consisting of a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and a set of connecting edges $E = \{e_1, e_2, \ldots, e_b\}$. Vertices do not fail, but at an instant of interest, an edge e_i has reliability $p_i = 1 - q_i$ independent of the states of the other edges.

Let K be a specified subset of V with at least two vertices. The criterion for success is that vertices in K must be connected. The probability of such an event, $R_k(G)$, is called the K-terminal reliability of G. $R_k(G)$ is the quantity to be computed. Special cases of this K-terminal

problem are those with $|K| = 2$ and $|K| = n$, corresponding to the well-known terminal-pair and all-terminal reliability problems respectively.

In this thesis, the first part of chapter 1 will give a brief review of the central role played by domination theory in network reliability computational complexity. We will introduce the reliability polynomial under its different expressions, in order to prepare the reader for a more detailed analysis.

A list of basic definitions and notations is also included.

In the second part of this chapter, a concise historical background is presented.

The statement of the factoring theorem and the complexity of reliability problems are mentioned at the begining of this part. Then we proceed to introduce, define, and explain the inclusion-exclusion and the sum of disjoint products. The concept of the domination of a graph is then introduced. Based on this concept, the edge factoring theorem is derived. The works of several researchers are mentioned to outline the various means of applications for the edge factoring theorem.

A short introduction of the "minimum domination" is also given; and we proceed to define the chromatic polynomial and introduce the parity which is a new invariant of graphs. Finally we survey the different expressions of the reliability polynomial.

In chapter 2, section 2.1 considers several interesting properties of the domination for graphs. We will cover graphs formed with one cycle only, graphs with positive domination multigraphs and chordal graphs. Section 2.2 will establish the relation between domination and the chromatic polynomial and from this we derive some properties.

Chapter 3, section 3.1 presents the main new result of this thesis which is the relationship that exists between the different expressions of the reliability polynomial. Section 3.2 provides us with the different bounds on the reliability polynomial.

1.1.1 Preliminaries

In this thesis we are concerned only with undirected graphs. Standard terminology can be found in such texts as Harary[1], Swamy and Thulasiraman[2]. However, a few definitions need to be introduced.

Connected graph:	A graph in which there exists at least one path between every pair of vertices. Otherwise, the graph is disconnected.
Simple graph:	A graph that has neither self-loops nor parallel links.
Complete graph:	A simple graph in which there exists a link between every pair of vertices.
Spanning subgraph:	A subgraph containing all vertices of the graph.
Tree:	A connected graph without cycles.
Spanning tree:	A tree of a graph containing all the vertices of the graph.

Formation: A nonempty subset of spanning trees of a graph whose union yields the graph. The formation is termed odd, if the subset consists of an odd number of trees and even otherwise.

Domination: The number of odd formations minus the number of even formations of a graph.

Probability of a graph: The probability that all links and vertices in the graph are good.

Overall reliability: The probability that there exists at least one path between every vertex-pair of the graph such that all links and vertices in these paths are good.

1.1.2 Notation

G : The given probabilistic graph of the system whose overall reliability is to be evaluated.

G_k : Subgraph of G.

$\overline{G}_k$: Connected subgraph of G.

V : Set of vertices of G.

E : Set of edges of G.

$n=|V|$: Number of vertices of G.

$b=|E|$: Number of edges of G.

$|K|$: Number of edges of G_k.

e_i : Edge of G.

$R(G)$: The overall reliability of G.

$d(G)$: Domination of G.

$R_k(G)$: The K-terminal reliability.

$d_k(G)$: The K-terminal domination of G.

$D(G)$: $|d(G)|$

p_e : Probability that edge e is functioning.

q_e : $q_e = (1 - p_e)$.

1.2 Historical review

In network reliability analysis, an important problem is to determine the probability that a specified subset of vertices in an undirected graph is connected. It is well-known that, by using Moskovitz's factoring theorem[3], the reliability of a graph can be expressed in terms of the reliabilities of a graph with one fewer vertex and another with one fewer edge.

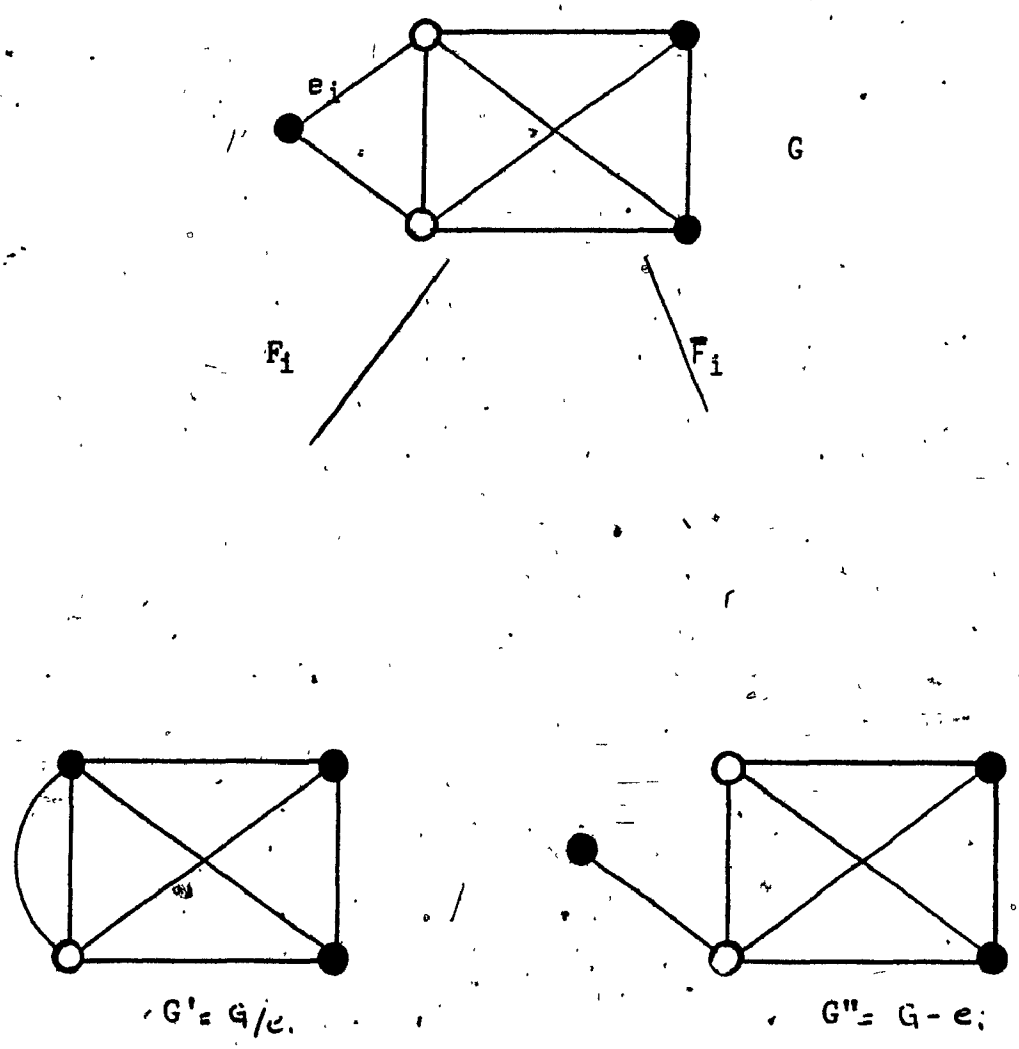
Let $e_i = (u, v)$ be some edge of a graph G and let F_i denotes the event that e_i is connected and $\overline{F}_i$ denote the complementary event. Since R(G) is just a probability, the rules of conditional probability can be applied to obtain

$$(1.1)\quad R(G) = p_i R(G/F_i) + q_i R(G/\overline{F}_i)$$

G/F_i actually defines a new graph in which u and v are known to be connected.

This new induced graph denoted by G/e_i, is obtained by deleting e_i and merging u and v into a single supervertex $W = u \cup v$. Similarly, $G/\overline{F}_i$ defines a new graph denoted by $G-e_i$. Figure (1.1) illustrates how these two graphs are induced. We can write equation (1.1) as

$$(1.2)\quad R(G) = p_i R(G \mid e_i) + (1 - p_i)\, R(G - e_i).$$



Induced Graphs obtained by Factoring on e_i

Figure (1.1)

That this relationship holds was first shown in Moore and Shannon[4] and it is known as the factoring theorem for network reliability.

The computations involved in the recursive application of (1.2) on each subproblem can be represented by a binary structure. Under exhaustive application of (1.2), the binary structure contains 2^b leaves (b the number of edges) and is equivalent to the enumeration of all possible states of G. Brown[5], Mine[6].

Wing and Demetriou[7] used (1.2) directly to compute $R_k(G)$, which is unwieldy even for small networks. Among the 2^b leaves many correspond to failure states of the given graph. By suitable selection of edges, one can avoid generating failed states and reduce the number of leaves in the binary structure. These leaves will then, correspond to trees whose reliability can be readily computed. Misra[8] used the factoring theorem and decomposed the given graph into series-parallel graphs for computing the source-to-terminal reliability. Subsequently Hansler[9] applied (1.2) to the source-to-terminal problem. More recently, Ball[10] and Johnson[11] used the same technique to compute $R_k(G)$ for $|K| = n$ and showed that the number of leaves in the binary structure is at most $(n-1)!$ (n number of vertices of the graphe G).

Figure (1.2) is a well-known example of a computer communication network. A two-terminal reliability problem would be to compute the probability that the distinguished node labeled UCSB can communicate with the distinguished node labeled CMU via some set of arcs or edges. Edges may be subject to failure. Let us assume that the associated success events are independent given those probabilities. A typical network reliability problem is to calculate efficiently the probability that a specific set of nodes can communicate with each other at a given time.

Most network reliability problems are, in the worst case, NP-hard, this is showed in Ball[12], and Provan and Ball[13].

Since, historically, the reliability literature has not placed much emphasis on computational complexity, and because these problems are, in general, NP-hard, many different algorithms based on minimal path-sets and/or cuts sets have been suggested to solve these problems. (A minimal cut set is a minimal set of elements whose failure implies that some distinguished nodes cannot communicate). In general, however, it is neither necessary nor desirable to find the family of minimal paths or cut sets in order to calculate network reliability.

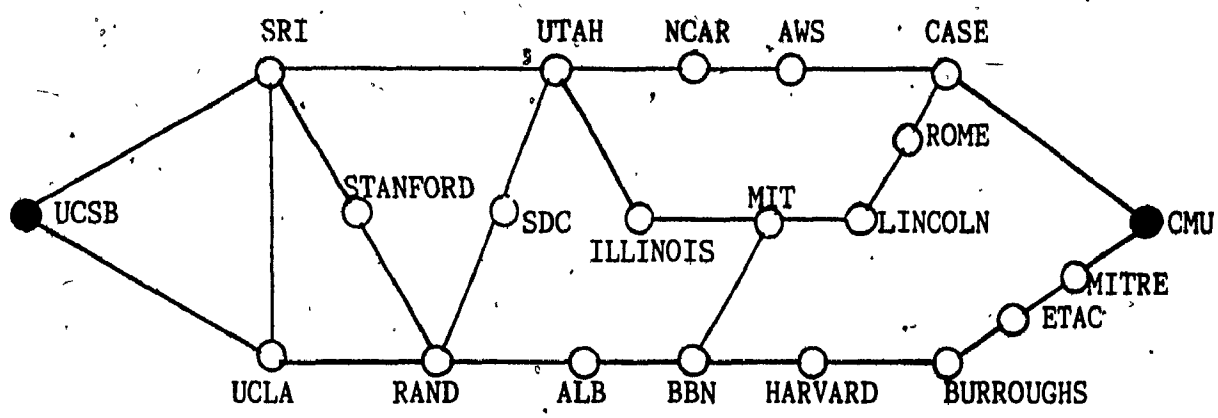


Figure (1.2)

ARPA Computer network

The reliability of a graph G with distinguished nodes K, is the probability $R_k(G)$ that all elements of at least one minimal path-set are working, or one minus the probability that all elements of at least one minimal cut-set have failed. Note that $R_k(G)$ depends on the distinguished node set $K \subseteq V$ (V: set of all the nodes of G) as well as on G. Different methods exist to evaluate this probability. We have already seen in the beginning of this section the pivotal decomposition or the factoring. We can classify briefly the other evaluations in 2 methods. For this purpose let A_i denote the event that all elements in the ith path set are functional and $\bar{A}_i$ denote the complement of this event. Let p be the number of minimal path sets.

1. <u>The Inclusion - Exclusion Method</u>:

$$R_k(G) = P\left(\bigcup_{i=1}^{p} A_i\right) = \sum_{i=1}^{p} P(A_i) - \sum_{i=1}^{n} \sum_{j<1} P[A_iA_j] + \ldots + (-1)^{p-1} P(A_1A_2..A_p)$$

If there are p pathsets, then this calculation involves 2^{p-1} terms. In some cases, two different intersections of A_i's will have the same probability. If one intersection consists of an odd number of A_i's and another intersection consists of an even number of A_i's; these terms will cancel. Satyanarayana and Prabhakar[14] (for k=2) and

Satyanarayana (for any k)[15] give algorithms that generate only the non-cancelling terms. In the reduced inclusion-exclusion expression, there will be a coefficient for the term corresponding to the event $G_p = A_1A_2 \ldots A_p$. (This event will in general, correspond to several different intersections). The coefficient d_k for this term is called the signed domination.

The signed domination of a given graph G is the number of odd formations of G minus the number of even formations of G (A formation of G is a non empty sub-set of k-spanning trees of G whose union yield the graph). The absolute value $D_k(G) = |d_k(G)|$ of this coefficient is called the "domination". As we shall see, this number is a measure of the computational complexity of certain factoring algorithms for undirected networks. The network corresponding to figure (1.3) has 6 formations of which 4 are odd and 2 are even so that $d_k(G) = 2$ in this case.

2. Sum of Disjoint Products:

$$R_k(G) = P(A_1) + P(\overline{A}_1A_2) + \ldots + P(\overline{A}_1\overline{A}_2 \ldots \overline{A}_{p-1}\, A_p)$$

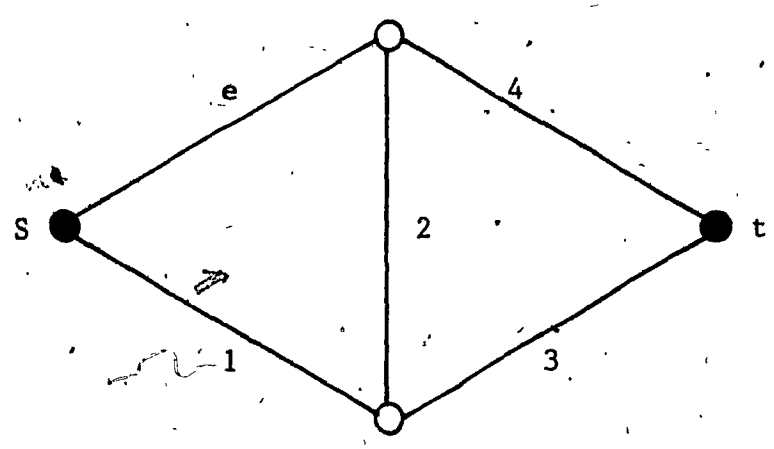


Figure (1.3)

Formations

{1, 3}, {e, 4}, {1, 2, 4}, {e, 2, 3}, {e, 2, 4}, {1, 2, 3}

$d_k(G)$ = Odd Formations - even formations

$d_k(G) = 4 - 2 = 2$

For p path sets there are p terms, but the time needed to generate each term may be exponential in p. Most methods based on Boolean techniques belong to this class. Such methods have been proposed by Abraham[16], Fratta and Montanari[17] and Aggarwal et al.[18] among others.

After this brief survey of these two methods let us focus on the factoring theorem and see one example. Figure (1.4) shows the binary computational tree resulting from using the factoring algorithm for the two-terminal problem corresponding to the graph at the top of figure (1.4) with distinguished nodes s and t, so that K = {s, t}. In the initial step we pivot on edge e forming two subgraphs: G|e corresponding to e working, and G-e, corresponding to e failed. Parallel and series probability reductions are now possible.

A parallel probability reduction replaces two edges, say 2 and 3 in G|e by a single edge with associated probability $p_2 + p_3 - p_2 p_3$. Likewise in G/e, this new edge and edge 5 are in series. The two edges in series are then replaced by a single edge with associated reliability $(p_2 + p_3 - p_2 p_3) p_5$. Pivoting now proceeds on edge 4 resulting in two additional subgraphs, each of which can be reduced to a single edge by series and parallel probability reduction. The "leaves" of the binary tree are the four subgraphs at the bottom of the tree.

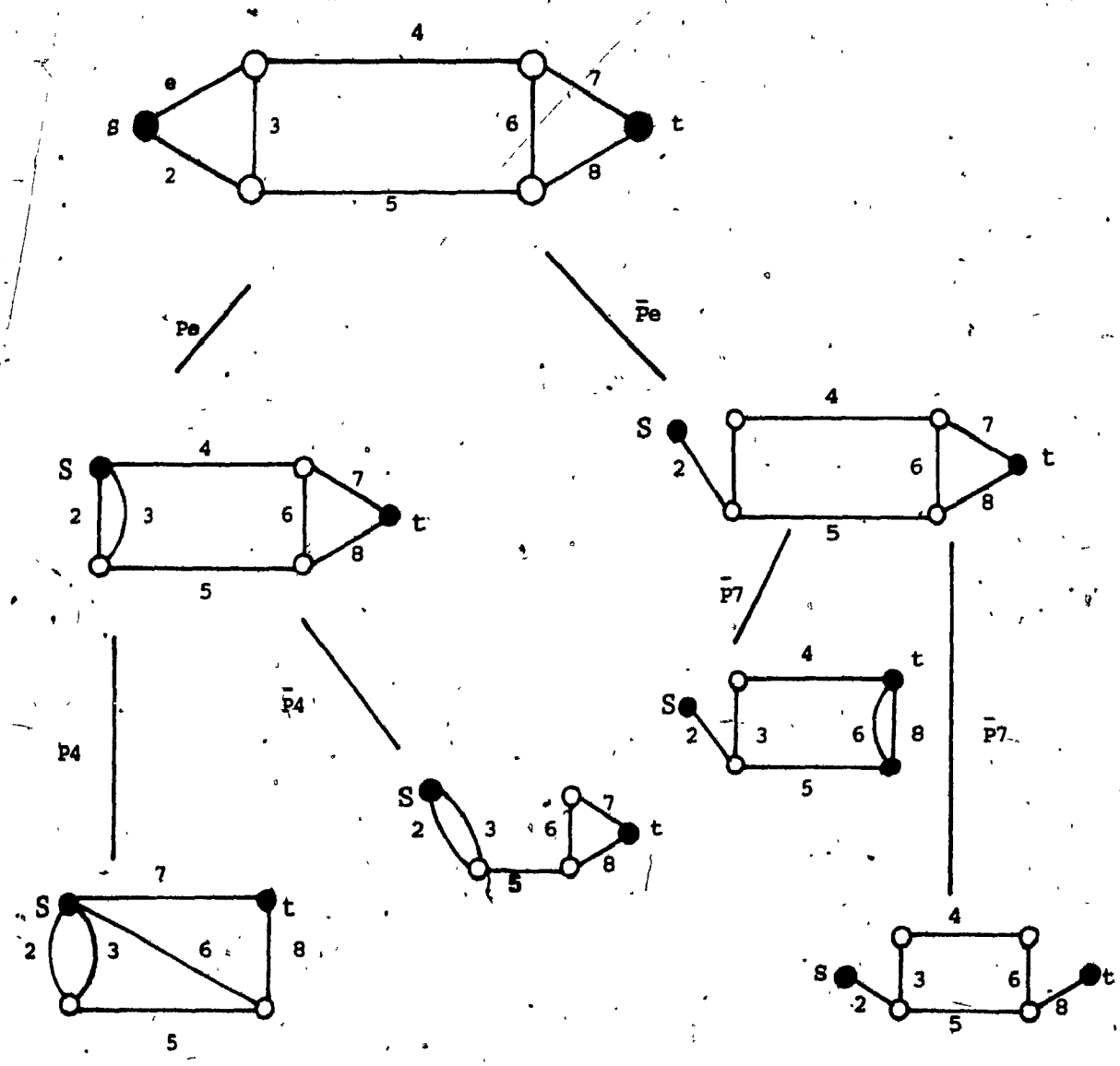


Figure (1.4)

Binary computational tree using the factoring algorithm

If each edge i has probability p of working, it is possible to use the binary computational tree (see fig. 2.4) to show that the system reliability in this case is:

$$R_k(G) = p^2((((((p \amalg p)p) \amalg p)p) \amalg p) +$$

$$p\,(1-p)\,(((p \amalg p)p)\,(p^2 \amalg p)) + p(1-p)\,((p(p \amalg p)) \amalg (p^2))\,p$$

$$+ (1-p)^2\,((p^3 \amalg p)p^2)$$

$$= (p^3 + p^4 + p^5 - 5p^6 + 4p^7 - p^8)$$

$$+ (2p^4 - p^5 - 4p^6 + 4p^7 - p^8)$$

$$+ (3p^4 - 4p^5 - p^6 + 3p^7 - p^8)$$

$$+ (p^3 - 2p^4 + 2p^5 - 3p^6 + 3p^7 - p^8)$$

$$= 2p^3 + 4p^4 - 2p^5 - 13p^6 + 14p^7 - 4p^8$$

where the operator "$\amalg$" corresponds to calculating the reliability of parallel edges i.e.

$$p_i \amalg p_j = p_i + p_j - p_i\,p_j$$

In figure (2.4), $\bar{p}_i = 1 - p_i$. The four graphs at the bottom of the tree are the leaves of the tree and each has domination one since each is series-parallel reducible (Chang[19]).

The domination of the top graph turns out to be $D_k(G) = 4$ (the number of "leaves" at the bottom of the tree), and the tree has $2D(G) - 1 = 7$ nodes, so that the computational running time is proportional to the domination. Satyanarayana and Chang[20] found that, in general, the number of leaves in the binary computational tree using a factoring algorithm with series and parallel probability reductions is at least equal to the domination. Using a simple edge selection strategy, they further showed that it is possible to create a backtrack structure that has a number of leaves exactly equal to the domination $D_k(G)$ of G. Therefore, this edge selection strategy is optimal for factoring algorithms using series parallel probability reductions.

Although the factoring algorithm can, in principle, solve all reliability problems, it is, in the worst case, an exponential time algorithm. For very large networks we need linear or polynomial time algorithms in order to calculate system reliability in "reasonable" computing time. By introducing additional probability reductions, researchers have found such algorithms for both directed and undirected graphs of special structure.

An undirected graph $G=(V,E)$ is said to be basically series-parallel if the graph (without distin-

guished nodes) can be reduced to a single edge by series and parallel replacement. A replacement as opposed to a probability reduction does not involve the probability measure that may be associated with the graph.

For example, Figure (1.5) shows us the steps followed in a series-parallel reduction. The vertices s and t are no longer distinguished; edges e and 1 are replaced by a dotted line using a series replacement as are edges 3 and 4. Finally, the remaining three edges in parallel are replaced by a single edge. No probability calculations are involved. The network in Figure (1.6), on the other hand, is not basically series-parallel.

Satyanarayana and Wood[15] provide linear time algorithms for calculating the k terminal reliability of undirected networks that are basically series-parallel. They introduce the probability reduction called polygon-to-chain reduction to accomplish this.

One would hope that adding polygon-to-chain reductions to the arsenal of reductions would significantly reduce the computational complexity of a factoring algorithm. Since this reduction can be bought for little more than the cost of simple reductions alone, only two facts must be established in order to develop a good factoring algorithm:

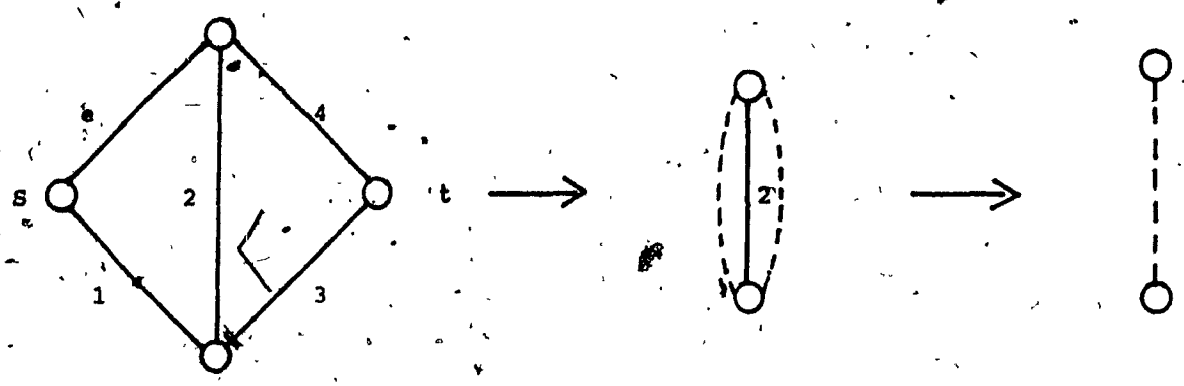


Figure (1.5)

Series and parallel replacement

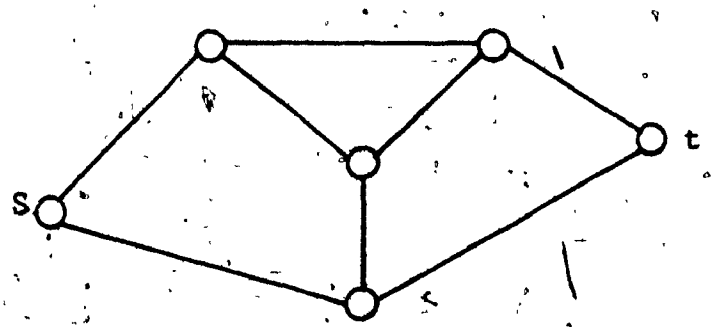


Figure (1.6)

Example network which is not basically series-parallel

1. The selection strategy devised can be implemented efficiently.

2. This strategy will produce fewer leaves than other possible strategies. This work has been done by Wood[21], using the general result on minimal domination of Chang[20].

Chang proved that the optimal algorithm generates a backtrack structure with the number of leaves equal to a combinatorial invariant called the "minimum domination" of G.

The minimal domination M(G) of a graph G is defined by:

$$M(G) = \text{minimum}_{\{K:|K| = 2\}} D_k(G)$$

Where K is a distinguished set of nodes of G. (Johnson points out that M(G) is equivalent to a combinatorial invariant on the graphic matroid of G called the "Crapo beta invariant"). While the domination $D_k(G)$ depends on both the graph G and the distinguished set of nodes K, M(G) obviously depends only on G.

Whereas $D_k(G) = 1$, if and only if, G is reducible to a k-tree by series and parallel probability reductions

(Satyanaryana and Chang), $M(G) = 1$, if and only if, G is basically series-parallel (Wood).

The graph at the top of figure (1.4) has $D_k(G) = 4$ where $K = \{s, t\}$, but $M(G) = 1$ since G is basically series-parallel. Using series and parallel probability reductions and the polygon-to-chain reduction in Table I[2], we can compute $R_k(G)$ in this case (figure (1.4)) without pivoting, so that a linear time algorithm exists for this problem and, in fact, for all such problems where G is basically series-parallel. Figure (1.7) provides an example of a graph where the domination is $D_k(G) = 2^{(|E|-2)/3}$, but $M(G) = 1$.

Procesi-Ciampi[22] proved the minimum domination theorem:

For undirected graph $G = (V, E)$

$$M(G) = M(G|e) + M(G-e)$$

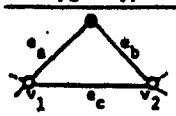
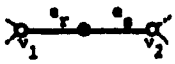
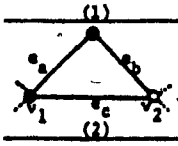
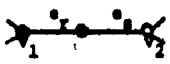
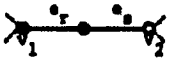
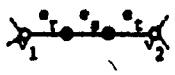
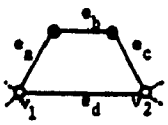
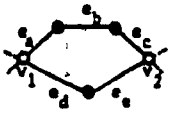
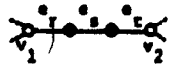
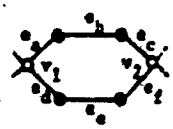
Wood used this result and other properties of minimum domination to evaluate the computational complexity of undirected networks relative to pivoting and polygon-to-chain reduction.

Another figure of interest, is the chromatic polynomial of a given graph and the existing correspondence with the domination theory.

Table I

Polygon-to-Chain Reductions with Unreliable Vertices

Note: Darkened vertices represent K-vertices

Polygon Type	Chain Type	Reduction Formulas	New Edge Reliabilities
 (1)		$\alpha = p_1p_2q_ap_bq_c$ $\quad \phi = q_1p_1p_a$ $\beta = p_1p_2p_aq_bq_c$ $\quad \theta = p_1q_2p_b$ $\delta = p_1p_2p_ap_bp_c\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}\right)$	$p_r = \frac{\phi}{\alpha+\phi}$ $\quad p_s = \frac{\theta}{\beta+\theta}$ $p_1' = \frac{\alpha+\phi}{\alpha+\phi+\theta}$ $\quad p_2' = \frac{\beta+\theta}{\beta+\theta+\phi}$ $\Omega = \frac{(\alpha+\phi+\theta)(\beta+\phi+\theta)}{\phi}$
 (2)		$\alpha = p_2q_ap_bq_c$ $\beta = p_2p_aq_bq_c$ $\quad \phi = q_2p_a$ $\delta = p_2p_ap_bp_c\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}\right)$	$p_r = \frac{\phi}{\alpha+\phi}$ $p_s = \frac{\phi}{\beta+\phi}$ $p_2' = \frac{\beta+\phi}{\alpha+\beta+\phi}$ $\Omega = \frac{(\alpha+\phi)(\beta+\phi+\theta)}{\phi}$
 (3)		$\alpha = p_2(p_aq_bq_cp_d+q_ap_bp_cq_d+q_ap_bq_cp_d)$ $\beta = p_aq_bp_cq_d$ $\quad \phi = q_2p_ap_c$ $\delta = p_ap_bp_cp_d\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}+\frac{q_d}{p_d}\right)$	
 (4)		$\alpha = p_1p_2q_ap_bq_cp_d$ $\quad \phi = q_1p_1p_bp_d$ $\beta = p_1p_2(p_aq_bq_cp_d+q_ap_bp_cq_d)$ $\delta = p_1p_2p_aq_bp_cq_d$ $\quad \theta = p_1q_1p_ap_c$ $\gamma = p_1p_2p_ap_bp_cp_d\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}+\frac{q_d}{p_d}\right)$	$p_r = \frac{\gamma}{\alpha+\gamma}$ $p_s = \frac{\gamma}{\beta+\gamma}$
 (5)	$\lvert K\rvert > 2$  See Note	$\alpha = p_1p_2q_ap_bp_cq_d$ $\quad \phi = q_1p_2p_bp_c$ $\beta = p_1p_2p_aq_bp_cq_d$ $\quad \theta = p_1q_2p_ap_b$ $\delta = p_1p_2p_ap_bq_cq_d$ $\gamma = p_1p_2p_ap_bp_cp_d\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}+\frac{q_d}{p_d}\right)$	$p_r = \frac{\gamma}{\alpha+\gamma}$ $p_s = \frac{\gamma}{\beta+\gamma}$ $p_t = \frac{\gamma}{\delta+\gamma}$ $p_1' = \frac{\alpha+\gamma}{\alpha+\gamma+\theta}$ $p_2' = \frac{\delta+\gamma}{\delta+\gamma+\theta}$
 (6)		$\alpha = p_1p_2q_ap_bp_cq_dp_e$ $\quad \phi = q_1p_2p_bp_cp_e$ $\beta = p_1p_2(p_ap_bp_c(p_dq_e+q_dp_e)$ $+p_b(q_ap_cp_dq_e+p_aq_cq_dp_e))$ $\delta = p_1p_2p_ap_bq_cp_dq_e$ $\quad \theta = p_1q_2p_ap_bp_d$ $\gamma = p_1p_2p_ap_bp_cp_dp_e\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}+\frac{q_d}{p_d}+\frac{q_e}{p_e}\right)$	$\Omega = \frac{(\alpha+\gamma+\theta)(\beta+\gamma)(\delta+\gamma+\theta)}{\gamma^2}$
 (7)		$\alpha = p_1p_2q_ap_bp_cq_dp_ep_f$ $\quad \phi = q_1p_2p_bp_cp_ep_f$ $\beta = p_1p_2(p_aq_bp_c(q_dp_ep_f+p_dq_ep_f+p_dp_eq_f)$ $+p_ap_bq_cp_f(p_dq_e+q_dp_e)$ $+q_ap_bp_cp_d(q_ep_f+p_eq_f))$ $\delta = p_1p_2p_ap_bq_cp_dp_eq_f$ $\quad \theta = p_1q_2p_ap_bp_dp_e$ $\gamma = p_1p_2p_ap_bp_cp_dp_ep_f\left(1+\frac{q_a}{p_a}+\frac{q_b}{p_b}+\frac{q_c}{p_c}+\frac{q_d}{p_d}+\frac{q_e}{p_e}+\frac{q_f}{p_f}\right)$	Note: For $\lvert K\rvert = 2$ chain is  $p_r = (p_b+p_1p_2p_aq_bp_cp_d)/\Omega$ $\Omega = p_b+p_1p_2p_aq_bp_c$ Compare Theorem 3.3f

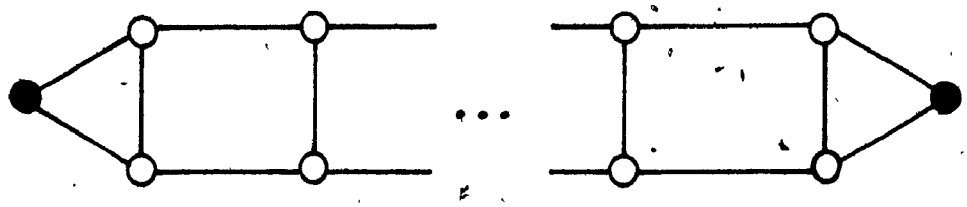


Figure (2.7)

$D_k(G) = 2^{(|E|-2)/3}$

$M(G) = 1$

If $G = (V,E)$ is a graph, a coloring $\underline{h}$ of a graph $G = (V,E)$ is a mapping $h : V \dashrightarrow [h]$. The integers 1, 2, ..., are called colors. A coloring is proper if no two adjacent points of G are assigned the same colour. If h is sufficiently large, G can, in general, be properly colored in many different ways. Birkhoff[23] noticed that the number of distinct proper coloring of a given graph G may be expressed elegantly as a polynomial in h, now well-known as the chromatic polynomial of G and denoted by P(G, h). Clearly, if C_j is the number of proper j-coloring of G in which all j colors are used, then:

$$P(G, h) = \sum_{j=1}^{n} \binom{h}{j} C_j$$

The function P(G, h) is a polynomial of degree n that has integer coefficients that alternate in sign, has leading coefficient 1, and has constant term zero. Also this polynomial is uniquely determined by the following two conditions. Let $G = (V, E)$ be a graph.

1. $P(G,h) = h^n$ if G has no edges

2. If G-x and G/x are the graphs obtained from graph G by deleting and contracting, respectively, an edge x of G then:

$$P(G, h) = P(G-x; h) - P(G/x; h)$$

condition 1 is the discrete graph condition while condition 2 is the pivot condition.

Another invariant of graphs has been introduced by Khalil and Satyanarayana[24] to compute the reliability. This invariant is called parity of a graph.

Let G be a graph and F_i a subset of G containing all subgraphs of G having exactly i edges of G, then the ith parity of G:

$$P_i(G) = \sum_{G_j \in F_i} d_k(G_j)$$

Khalil and Satyanaryana proved the following theorem

$$P_i(G) = P_i(G - e) + P_{i-1}(G|e) - P_{i-1}(G - e)$$

This result is a generalization of the factoring theorem on domination, and it gives a recursive formula for computing $P_i(G)$.

Khalil and Satyanaryana gave us another expression of the reliability polynomial for a given graph G in terms of the parity; if we assume that all edges of G have equal probability, p, and the edge failures are statistically independent, then

$$R(G) = \sum_{i=0}^{b} P_i(G) \; p^i$$

(b: number of edges of G)

As it has been mentioned in this chapter, computing the reliability of a given network is in most cases an N.P. hard problem. Fortunately we have some information on the coefficients of the reability polynomial. So far we have seen parity and domination, let us introduce now, the other expressions of the reliability polynomial.

Colbourn[25] defined the following: let N_i denote the number of operational subgraphs with i edges. By assuming that edges have the same probability p of operation, the reliability polynomial is then

$$R(G) = \sum_{i=0}^{b} N_i p^i \; (1 - p)^{b-i}$$

The expression of R(G) in terms of cutsets is given by:

$$R(G) = 1 - \sum_{i=0}^{b} C_i \; (1 - p)^i \; p^{b-i}$$

where C_i denote the number of i-edge cutsets (leaving b - i operational edges)

Another formulation of R(G), and maybe the most common, is the one using complements of path sets. Let F_i denote the number of sets of i edges for which the b - i remaining edges form a path set then

$$R(G) = \sum_{i=0}^{b} F_i (1 - p)^i p^{b-i}$$

We have in hand several expressions of the reliability polynomial, however, the computation of the different coefficients is in most cases N.P. complete; so instead of computing the reliability polynomial, researchers were more interested in finding bounds on this polynomial.

CHAPTER 2

2. DOMINATION THEORY

2.1 Some properties of the domination

We are concerned only with undirected graphs.

To ensure simultaneous communication among all vertices of a network, at least one path is necessary between every vertex-pair along which all links and vertices are good. It is well-known that any spanning tree of the given network contains exactly one path between every vertex-pair. Therefore, the overall reliability of the given network G can be written as:

$$R(G) = \sum_i E_i \; P_r \; \{T_i\}$$

where T_i is the union of the ith nonempty set of spanning trees of G and E_i is either +1 or -1 depending on whether the number of trees in T_i is either odd or even respectively. The proof of this relation can be found in[26].

Clearly, if $\overline{G}_i$, is any connected spanning subgraph of G,

$$R(G) = \sum_i d(\overline{G}_i) \; P_r \; \{\overline{G}_i\}$$

where $d(\overline{G}_i)$ is the domination of the ith connected spanning subgraph of G.

<u>Theorem</u>:

For any graph G, the domination

$$d(G) = d(G|e) - d(G-e)$$

The proof of this theorem can be found in[27].

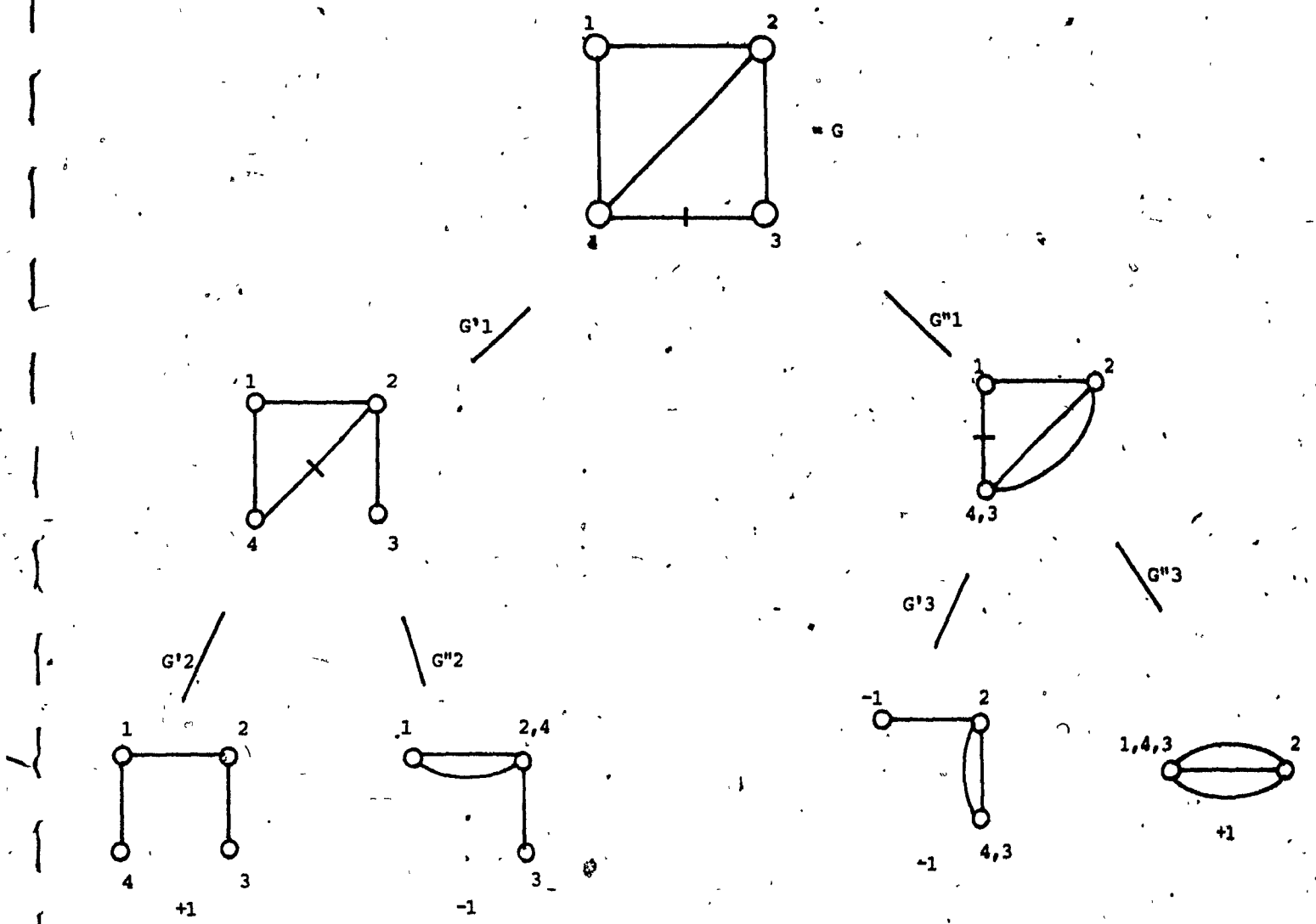
By means of the previous theorem, the domination of any graph can be expressed in terms of the domination of a graph with one fewer vertex and another with one fewer link. The theorem can be applied again to these graphs, and so on. An example of this process is shown in figure (2.1).

<u>Proposition 2.1.1</u>

For any graph containing only one single cycle of length q, $d(G) = -(q-1)$.

<u>Theorem</u>

For any graph G, the sign of the domination d(G) is given by $E = (-1)^{b-n+1}$, where b and n are the number of links and vertices of G. ($b \geq 2$; $n \geq 2$)



$$d(G) = d(G''_1) - d(G'_1)$$

$$= (d(G''_3) - d(G'_3)) - (d(G''_2) - d(G'_2))$$

$$= +1 - (-1) - (-1 - (+1))$$

$$d(G) = +4$$

Figure (2.1)

<u>Proof</u>

The proof is by induction on b and n.

$n = 2,$

$b = 2$

$$d(G) = -1 = (-1)^{2-2+1}$$

$n = 2$

$b = 3$

$$d(G) = +1 = (-1)^{3-2+1}$$

Any connected graph on (n+1) vertices and b=n links which corresponds to a tree, satisfies the theorem.

So let us take $b > n$ such that the theorem is valid for all graphs of (n+1) vertices and b or fewer links.

Consider a graph G with (n+1) vertices and (b+1) links.

$$d(G) = d(G|e) - d(G-e)$$

Since the theorem is true for all G|e and G-e

$$d(G) = (-1)^{b-n+1} D(G|e) - (-1)^{b-n} D(G-e)$$

$$= (-1)^{b-n+1} D(G|e) + (-1)^{b-n+1} D(G-e)$$

$D(G|e)$ and $D(G-e)$ are the absolute values of $d(G|e)$ and $d(G-e)$ respectively

$$d(G) = (-1)^{b-n+1} (D(G|e) + D(G-e))$$

$$d(G) = (-1)^{(b+1)-(n+1)+1} (D(G|e) + D(G-e))$$

The theorem is true for b; b+1; n; n+1 it is true for all graphs.

<u>Corollary</u>

G has a positive domination if b-n + 1 is even.

For the following proposition, let us first state these three facts:

1. The domination of a disconnected graph is zero.
2. The domination of a graph with self-loops is zero.
3. The domination of any tree is equal to +1.

<u>Proposition 2.1.2</u>

$D(G) > 0$ if G is a connected multigraph.

The proof of this proposition is easy by direct use of the definition of a multigraph, and the previous

facts (in a multigraph we don't have loops, but more than one link can join two points).

Corollary

If G is a multigraph and x is a multiple edge of G, then $D(G) = D(G-x)$.

If x is a multiedge then $D(G|x) = 0$ as $D(G) > 0$ (proposition 2.2)

$D(G) = D(G-x)$ (domination theorem)

The last two propositions of this chapter need the following definition:

Definition

A graph G is called a chordal graph if every cycle in G, of length strictly greater than 3, possesses a chord; i.e., an edge joining two nonconsecutive points of the cycle.

Proposition 2.1.3

The domination D(G) of any connected planar chordal graph G, with $n>1$, b edges and k blocks is given by

$$D(G) = 2^{3(n-1)-b-2k}\, 3^{b-2(n-1)+k}$$

<u>Proposition 2.1.4</u>

Let G be any connected planar graph having n points, b edges and k blocks. If G is not chordal then

$$D(G) \geq 2^{3(n-1)-b-2k} \; 3^{b-2(n-1)+k}$$

The proof of these two propositions can be found in[26].

2.2 Graph colouring problem and domination

The combinatorial problem of computing d(G) is analogous to the graph colouring problem.

A proper colouring of a graph is an assignment of colours to the vertices of the graph in such a way that no two adjacent vertices have the same colour. For a fixed positive integer h, the number of proper colourings of a graph G in h or fewer colours is denoted by P(G,h), which is termed the chromatic polynomial of G:

Whitney[28] provided a topological interpretation for the coefficients of the chromatic polynomial which is:

$$P(G, h) = \sum_{S} (-1)^{e(S)} h^{c(S)}$$

where the summation is over the set of spanning subgraphs S of G, e(s) denotes the number of edges of S, and c(s) denotes the number of connected components of S.

Whitney then observed that in most cases there are many pairs of terms of the above summation that cancel each other. For example, if G is the triangle, then the contributions of G and G-x (x an edge of G) are - h and h, respec-

tively. Before stating the theorem that this observation led to, let us state the definition of external and internal activity.

2.2.1 Definition of external activity:

Suppose $G = (V, E)$ is a graph and $(<)$ is a strict linear order on E. An edge x with end points u, v is said to be externally active relative to a set X of edges of G if there is a path P in G between u and v, which uses only edges from $X - \{x\}$ and has the property that $x < y$ for all edges y in P. The number of edges which are externally active relative to X is called the external activity of X in G.

2.2.2 Definition of internal activity:

Let G, X, and $<$ be as in the definition above. An edge $x \in X$ is internally active in X if:

i) x lies on no cycle of X; and
ii) if $y \in E - X$ and $y < x$, then $X - \{x\}$ contains a path connecting end points of y.

The number of edges of x which are internally active is called the internal activity of X.

<u>Theorem</u> (Whitney)

Let $G = (V, E)$ be a graph with n points, let $<$ be a strict linear order on E, and let $m_j(G)$ denote the number of spanning forests of G having j connected components and external activity zero. Then

$$P(G, h) = \sum_{j=1}^{n} (-1)^{n-j} \quad m_j(G)\, h^j$$

Subsequently, Tutte[29] showed that the chromatic polynomial can be expressed in terms of certain spanning trees of G.

<u>Theorem (Tutte)</u>

Let $G = (V, E)$ be a connected graph with n points, let $<$ be a strict linear order on E, and let $t_{j0}(G)$ denote the number of spanning trees of G with internal activity j and external activity 0. Then:

$$P(G, h) = (-1)^{n-1} h \sum_{j=1}^{n-1} t_{j0}(G)\,(1 - h)^j$$

Boesch, Satyanarayana and Suffel[26] proved the following theorem:

Theorem

Let $G = (V, E)$ a connected undirected graph with a nonempty subset $K \subseteq V$, and let $<$ be a strict linear order on E. Then: $D_k(G) = t*_o (G, K)$

($D_k(G)$ K-terminal domination of G and $D_k(G) = |S_o - S_e|$ where

S_o: number of odd formation

S_e: number of even formations)

$t*_o (G, K)$: Number of trees that satisfy the following:

1) they have no externally active edge,

2) if x is an internally active edge in a given tree T, then x is an edge of the unique K-tree contained in T.

An immediate consequence of this theorem is the following corollary, related to the all terminal reliability.

Corollary

For an undirected graph $G = (V, E)$,

$$D(G) = t_0(G).$$

hence using this corollary and Tutte's theorem we obtain the relation between the domination of a graph and its chromatic polynomial:

$$| [P(G, h))/ h] | h = 0 | = D(G)$$

Satyanaryana and Tindell[30] introduced a polynomial $P(G, K, h)$ in h; determined by graph $G = (V, E)$; $K \subseteq V$. Like the classical chromatic polynomial $P(G, h)$, this new polynomial

has integer coefficients that alternate in sign. Furthermore $P(G,K,h) = P(G,h)$ if K is the entire point set of G. This new polynomial has several interesting properties, and in particular, it has been shown that

$$[P(G, K; h)/ h] \mid_{h=0} = D_k(G)$$

<u>Example 2.2.1</u>

Let G be a complete graph on n vertices. Then

$$D(G) = (n-1)!$$

<u>Proof</u>

Since every vertex of G is adjacent to every other one, the numbers of colour-partition are

$$C_1(G) = C_2(G) = \ldots = C_n(G) = 1$$

hence $P(G,h) = h(h-1)(h-2)\ldots(h-n+1)$

$$|P(G,h)/h|_{h=0}| = |(h-1)(h-2)\ldots(h-n+1)|$$

$$= |(-1)(-2)\ldots(-n+1)|$$

$$= (n-1)!$$

Thus

$$D = (n-1)!$$

2.2.3 Definition

The general graph G is quasi-separable if there is a subset K of V(G) (i.e.: V(G): Set of vertices of G) such that the vertex subgraph {K} is a complete graph and the vertex graph {V(G)-K} is disconnected. G is separable if K is empty (in which case G itself is disconnected) or if $|K| = 1$ (in which case we say that the single vertex of K is a cut-vertex).

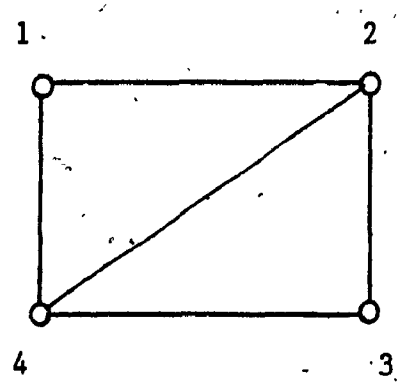
It follows from this definition that in quasi-separable graph we have $V(G) = V(G_1) \cup V(G_2)$ where the vertex graph $\{V(G_1) \cap V(G_2)\}$ is complete, and there are no edges in G joining $V_1 - (V_1 \cap V_2)$ to $V_2 - (V_1 \cap V_2)$.

The smallest graph which is quasi-separable but not separable is shown in figure (2.2). The relevant sets are $V_1 = \{1,2,4\}$; $V_2 = \{2,3,4\}$.

Theorem

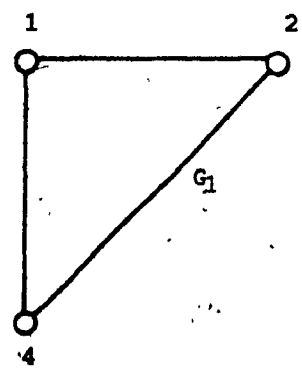
If the graph G is quasi-separable in graphs G_1 and G_2, then

$$P(G,h) = \frac{P(G_1, h) * P(G_2, h)}{P(G_1 \cap G_2, h)}$$



$V_1 = \{1,2,4\}$; $V_2 = \{2,3,4\}$

$V_1 \cap V_2 = \{2,4\}$



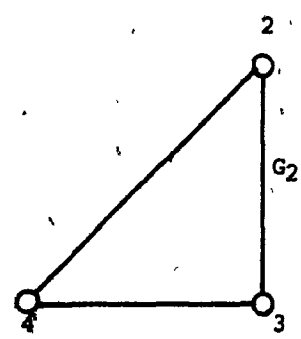


Fig. 2.2

<u>Proof</u>

* If $G_1 \cap G_2 = \emptyset$

we assume $P(G_1 \cap G_2, h) = 1$

In that case G is disconnected, so we can colour the vertices of G_1 and G_2 independently, it follows that

$$P(G, h) = P(G_1, h) * P(G_2, h)$$

* Suppose $G_1 \cap G_2$ is a complete graph K_t, $t \geq 1$. (K vertices and t colours) since G contains this complete graph, G has no vertex-colouring with fewer than t colours. For any natural numbers $h \geq t$ and $h_{(t)} = h(h-1) \ldots (h-t+1)$, $P(G,h)/h(t)$ is the number of ways of extending a given vertex-colouring of $G_1 \cap G_2$ to the whole of G, using at most h colours.

This remark is also valid for both G_1 and G_2.

As there are no edges in G joining $V_1-(V_1 \cap V_2)$ to $V_2-(V_1 \cap V_2)$ the extensions of a vertex-colouring of $G_1 \cap G_2$ to G_1 and G_2 are independent. Hence

$$P(G,h)/h_{(t)} = P(G_1,h)/h_{(t)} * P(G_2,h)/h_{(t)}$$

$$P(G,h) = \frac{P(G_1,h) * P(G_2,h)}{h_{(t)}}$$

$$P(G,h) = \frac{P(G_1,h) * P(G_2,h)}{P(G_1 \cap G_2,h)}$$

Proposition 2.2.1

Let G be quasi-separable in two subgraphs G_1, G_2, and let s be the number of vertices of the complete graph $G_1 \cap G_2$ then:

$$D(G) = \frac{D(G_1)*D(G_2)}{(s-1)!}$$

Proof

The proof of this proposition is a straight application of the previous theorem and example 2.1

$$|P(G,h)/h|h=0| = \frac{|P(G_1,h)/h|h=0| * |P(G_2,h)/h|h=0|}{|P(G_1 \cap G_2,h)/h|h=0|}$$

$$D(G) = \frac{D(G_1) * D(G_2)}{(s-1)!}$$

Proposition 2.2.1 is useful in hand calculation of the domination of small graphs. For example, the domination of the graph in (fig. (2.2)).

$$D(G) = \frac{D(G_1)*D(G_2)}{(2-1)!} = \frac{2 * 2}{1} = 4$$

This follows from proposition 2.2.1.

<u>Proposition 2.2.2</u>

Suppose that G is a connected graph having m blocks $G_1, G_2, \ldots, G_m$. Then $D(G) = D(G_1) * D(G_2) * \ldots * D(G_m)$.

The proof of this proposition follows directly from proposition 2.2.1.

CHAPTER 3

3. RELIABILITY POLYNOMIAL

3.1 Relation between the coefficient of R(G)

In the historical review we have seen the reliability polynomial of a given graph G in its many disguises.

Let us establish the relationship between the different coefficients of this polynomial.

- If N_i is the number of operational subgraph with i edge then

$$R(G) = \sum_{i=0}^{b} N_i \, p^i \, (1 - p)^{b-i}$$

- Let F_i denote the number of sets with i edges for which the b - i remaining edges form a pathset (i.e. a tree)

$$R(G) = \sum_{i=0}^{b} F_i \, (1 - p)^i \, p^{b-i}$$

$$F_i = N_{b-i}$$

. Let C_i denote the number of i-edge cutsets (leaving b-i operational edges)

$$C_{b-i} + N_i = \binom{b}{i}$$

. Let P_i be the ith parity of G

$$R(G) = \sum_{i=0}^{b} P_i \, p^i \tag{3.1.1}$$

3.1.1 <u>Proposition</u>

$$P_{b-k} = \sum_{i=k}^{b} (-1)^{i-k} \binom{i}{i-k} F_i$$

<u>Proof</u>

$$R(G) = \sum_{i=0}^{b} F_i \, (1-p)^i \, p^{b-i}$$

Two polynomial in p are equal if and only if their coefficients are equal. Thus by comparing coefficients of

$$\sum_i P_i \, p^i \qquad \text{and} \qquad \sum_i F_i \, (1-p)^i \, p^{b-i}$$

we have

$$P_{b-k} = \sum_{i=k}^{b} (-1)^{i-k} \binom{i}{i-k} F_i$$

similarly by comparing the coefficients of

$$\sum_{i=0}^{b} N_i p^i (1-p)^{b-i}$$

with the coefficients of (3.1.1) we obtain

$$(3.1.2) \quad P_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{b-i}{k-i} N_i$$

if we use the fact that

$$N_i = \binom{b}{i} - C_{b-i}$$

we have

$$(3.1.3) \quad P_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{b-i}{k-i} \left[\binom{b}{i} - C_{b-i}\right]$$

The following example is a good illustration of the previous relations.

In order to compute the overall reliability polynomial of fig. 3.1 we need to find the different pathsets. In order to establish communication between all vertices, the spanning trees must have at least 3 edges. The set of pathsets is given by:

{abd, abf, adf, bfd, acf, bcd, acd, bcf, abcd, abcf, acdf, bcdf, abdf, abcdf}

$N_0 = N_1 = N_2 = 0$ $\quad N_3 = 8,\ N_4 = 5,\ N_5 = 1$

$F_5 = F_4 = F_3 = 0$ $\quad F_2 = 8,\ F_1 = 5,\ F_0 = 1$

$C_0 = C_1 = 0$ $\quad C_2 = 2,\ C_3 = 10,\ C_4 = 5;\ C_5 = 1$

$P_0 = 0$

$P_1 = P_2 = 0$

(domination of disconnected graph is zero, fact 1)

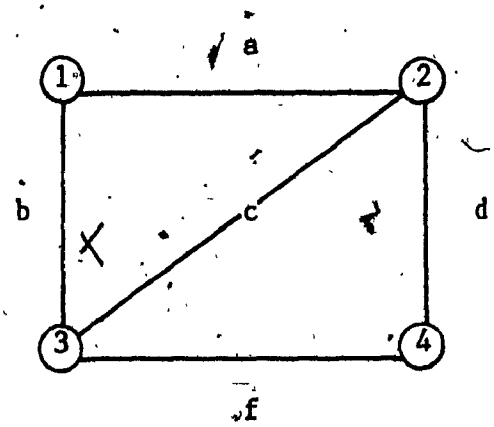
$P_3 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$

$P_4 = -(2 + 2 + 2 + 2 + 3) = -11$

$P_5 = 4$

$$R(G) = \sum_{i=0}^{5} N_i\, p_i\, (1-p)^{5-i}$$

$$R(G) = 4\,p^5 - 11\,p^4 + 8\,p^3$$



All terminal reliability

Figure (3.1)

$$P_5 = \sum_{i=0}^{2} (-1)^{i-0} \binom{i}{i-0} F_i = F_0 - F_1 + F_2 = 4$$

$$P_4 = \sum_{i=1}^{2} (-1)^{i-1} \binom{i}{i-1} F_i = F_1 - 2 F_2 = -11$$

$$P_3 = \sum_{i=2}^{2} (-1)^{i-2} \binom{i}{i-2} F_i = F_2 = 8$$

We also obtain the same results using (3.1.2) and (3.1.3).

3.2 Bounds on R(G)

Let N_m denote the number of minimum cardinality pathset, we have

$$N_i = F_{b-i} = 0 \text{ for } i < m.$$

Let C_c denote the number of minimum cardinality cutset, hence we find $C_i = 0$ for $i < c$, or equivalently

$$N_{b-i} = \left[{b \atop b-i}\right] \text{ for } i < c$$

The simple bounds

Kel'mans[31] noticed that when p is close to zero

$$R(G) \sim N_{n-1}\, p^{n-1} (1 - p)^{b-n+1}$$

and when p is close to 1

$$R(G) \sim 1 - C_c\, p^{b-c} (1 - p)^c$$

$$N_{n-1}\, p^{n-1} (1 - p)^{b-n+1} < R(G) < 1 - C_c\, p^{b-c} (1 - p)^c$$

We note that every i-edge subgraph is either a pathset or a cutset, thus

$$C_{b-i} = \binom{b}{i} - N_i, \quad \text{so we have} \quad 0 \leq N_i \; ; \; C_i \leq \binom{b}{i}$$

The following bounds were first stated by Jacobs[32] and improved by van Slyke and Frank[33],

$$R(G) \geq N_{n-1}\, p^{n-1} (1-p)^{b-n+1} + N_{b-c}\, p^{b-c} (1-p)^{c} + \sum_{i=b-c+1}^{b} \binom{b}{i} p^{i} (1-p)^{b-i}$$

$$R(G) \leq N_{n-1}\, p^{n-1} (1-p)^{b-n+1} + \sum_{i=n}^{b} \binom{b}{i} p^{i} (1-p)^{b-i}$$

These bounds are very weak, but they give a good estimate when p is close to zero or one.

<u>The Bauer, Boesch, Suffel and Tindel bounds (BBST)</u>

Let us first state Sperner's theorem. Let F_i be the number of complements of pathsets $b - i$, then Sperner[30] showed that:

$$F_{i-1} \geq \frac{i}{b - i + 1} F_i$$

Bauer, Boesch, Suffel and Tindel interpreted this result as follows: The fraction of operational subgraphs with i edges over all subgraphs with i edges is nondecreasing as i increases. They used Sperner's theorem to improve the simple bounds.

Assuming that m, c, F_{b-m} and F_c are available, we obtain the BBST bounds:

$$R(G) \geq \sum_{i=0}^{c-1} \binom{b}{i} p^{b-i}(1-p)^i + F_c p^{b-c}(1-p)^c + \sum_{i=c+1}^{b-m} F_{b-m} \frac{\binom{b}{m}}{\binom{b}{b-m}} p^{b-i}(1-p)^i$$

$$R(G) \leq \sum_{i=0}^{c-1} \binom{b}{i} p^{b-i}(1-p)^i + \sum_{i=c}^{b-m-1} F_c \frac{\binom{b}{i}}{\binom{b}{c}} p^{b-i}(1-p)^i + F_{b-m} p^m (1-p)^{b-m}$$

These bounds, have been improved by the Kruskal-Katona bounds[30].

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