Network Synthesis of Complex Impedance and Complex Reactance Functions.

Adel M. K. Hashem

A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfilment of the Requirements

for the Degree of Doctor of Philosophy at

Concordia University

Montréal, Québec, Canada

April 6th, 1993

ISBN 0-315-87266-7

ABSTRACT

Network Synthesis of Complex Impedance and Complex Reactance Functions.

Adel M. K. Hashem

Concordia University, 1993

One of the main problems in system stability analysis address the location of the roots of polynomials with complex coefficients. In this thesis J-fraction expansion is used to test the stability of any complex polynomial.

An extension of the generalized Routh array and the generalized Hurwitz determinant for a polynomial with real coefficients to include a polynomial with complex coefficients is illustrated. Applications of these methods for the stability of multivariable polynomials with real coefficients are illustrated.

The importance of synthesis procedures with real components is well known in network theory. This research addresses an extension to the present methods of circuit theory to include a new type of element, namely, a pure imaginary resistor (jR). A complex impedance $Z(s)$ is synthesized in the s-plane by using four kinds of elements; real resistors (R), imaginary resistors (jR), real inductors (L), and real capacitors (C). Also a complex reactance $X(s)$ is synthesized by using three kinds of elements (jR, L, C) or by using two kinds of elements (jR, L) or (jR, C).

We extend the whole idea of network synthesis of $X(s)$ in a continuous system (s-domain) to synthesise a discrete complex reactance function $X(z)$ in discrete system (z-domain). Discrete complex reactance function could be implemented or synthesized directly by the algebraic equation obtained from $X(z)$. Algebraic equation can be implemented by computer program, digital circuitry, or programmable

integrated circuits. Algebraic equation is one of the many possible realizations of $X(z)$. Other realizations include partial fraction expansion realization (parallel realization), cascade realization, J-fraction expansion, or S-fraction expansion realization.

The implementation of the $X(z)$ function can be done using delay elements which are equivalent to inductors, capacitors or imaginary resistors (energy storage elements), similar to that used with a complex reactance function $X(s)$.

After synthesising $Z(s)$, $X(s)$, and $X(z)$ in continuous and discrete systems respectively, it is logical to extend the above procedures to synthesise complex impedance matrices of two-port networks. We derive the complex impedance matrix $[Z(s)]$ of a two-port network from a stable polynomial with complex coefficients and its alternant polynomial. A complex reactance matrix of a two-port network $[X(s)]$ is constructed from the quasi-real and quasi-imaginary parts of a complex polynomial $P(s)$. These matrices are synthesized by using the J-fraction, the S fraction, and the partial fraction expansions. A complex impedance matrix $[Z(s)]$ is synthesised by using four types of elements (R, jR, L, C). Also, a complex reactance matrix $[X(s)]$ is synthesized by using three types of elements (jR, L, C) or by using two types of elements (jR, L) or (jR, C).

ACKNOWLEDGEMENTS

The author would like to express his sincere appreciation to his supervisor Prof. F.M. Reza for his guidance.

I am deeply grateful to my family, especially my parents and my wife, for their support, patience, and encouragement.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$P(s)$	Complex polynomial
$Z(s)$	Complex impedance function
$X(s)$	Complex reactance function
$H(s)$	Transfer function of the continuous system
$H(z)$	Transfer function of the discrete system
jR	Imaginary resistor
R	Real resistor
L	Real inductor
C	Real capacitor
$High \quad T_c$	High critical temperature superconductor
$[R]$	Load matrix resistor for n-port network
$X(z)$	Discrete complex reactance function
$[Z(s)]$	Complex impedance matrix
$[X(s)]$	Complex reactance matrix
n	Degree of polynomial $P(s)$
$Q(s)$	Alternant polynomial of $P(s)$
λ_i	Roots of polynomial $P(s)$
$Re\ s > 0$	Right half of the s-plane
$Re\ s < 0$	Left half of the s-plane
r_i	First order polynomial
R_i	Remainder polynomial from the division process
$F_1, F_2, F_3, ...$	J-fraction expansion coefficients
$a_1, c_1, d_1, e_1...$	Elements of the first column of the generalized Routh array
A	Hurwitz matrix of a complex polynomial

D_q	Successive principal minors of the determinant of the Hurwitz matrix A
D	Hurwitz determinant of the real polynomial
Δ_q	Successive principal minors of the modified determinant of the Hurwitz matrix A
$D_1, D_3, D_5, ...$	Successive odd principal minors of the determinant of the Hurwitz matrix A
$\Delta_1, \Delta_3, \Delta_5, ...$	Modified successive principal odd minors of the determinant of the Hurwitz matrix A
z	Bilinear transformation
$P(s_1, s_2, ..., s_k)$	Multivariable polynomial with real coefficients
$P(\omega_1, \omega_2, ..., s_k)$	Complex polynomial obtained from multivariable polynomial with real coefficients
$Q(\omega_1, \omega_2, ..., s_k)$	Alternant polynomial obtained from $P(\omega_1, \omega_2, ..., s_k)$
$P(s_1, s_2)$	Two variable polynomial with real coefficients
$P(\omega_1, s_2)$	Complex polynomial in s_2 from $P(s_1, s_2)$
$P(s_1, \omega_2)$	Complex polynomial in s_1 from $P(s_1, s_2)$
$J - fraction$	Mathematical expansion
jG	Imaginary conductance
$P^*(-s)$	Complex polynomial obtained from $P(s)$
$x(s)$	Real reactance function
$p(s)$	Polynomial with real coefficients
$qRe\,P(s)$	Quasi real part of a complex polynomial $P(s)$
$qIm\,P(s)$	Quasi imaginary part of a complex polynomial $P(s)$
$X_1(z), X_2(z), ..$	Complex reactance functions in z-domain
$V(s)$	Voltage vector
$I(s)$	Current vector

$P_1(s)$	Polynomial with complex coefficients obtained from the expansion of $P(s)/Q(s)$
$z_{11}(s)$	Driving point impedance at the input terminal of the two-port network
$z_{22}(s)$	Driving point impedance at the output terminal of the two-port netowork
$z_{12}(s)$	Transfer impedance at the output to input terminal of the two-port network
$x_{11}(s)$	Driving point reactance at the input terminal of the two-port network
$x_{22}(s)$	Driving point reactance at the output terminal of the two-port network
$x_{12}(s)$	Transfer reactance at the output to input terminal of the two-port network

Chapter 1

INTRODUCTION

What is network synthesis? A generally accepted definition of network analysis and synthesis, includes three key words.

- Excitation
- Network
- Response

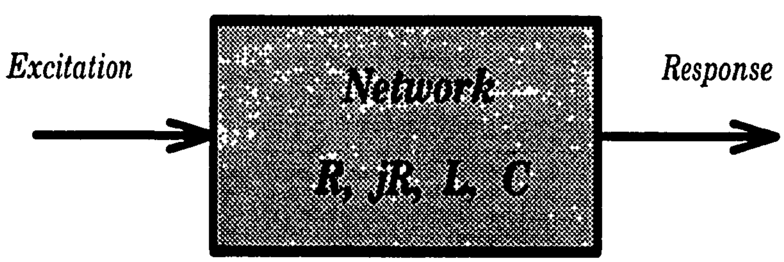

Figure 1.1: The objects of our concern

As depicted in Figure 1.1, network analysis is concerned with determining the response, given the excitation and the network. In network synthesis, the problem is to design the network, given the excitation and the desired response. Since the transfer function of any system or network is completely described by input and output signals of that system, then a mathematical synthesis of this function leads to a network synthesis.

1.1 Motivation

The methods for testing the stability of a polynomial with real coefficients are well known. A multivariable polynomial with real coefficients can be reduced to one variable polynomial with complex coefficients. To test the stability of a multivariable polynomial we can therefore test the stability of a complex polynomial with one variable which is the first motivation for studying complex polynomials.

To test stability of complex polynomials we have to extend the Routh array and the Hurwitz determinant of real polynomials to include complex polynomials as well. This extension will be more general for testing the stability of any polynomial with real or complex coefficients. We can prove that these methods of testing the stability of a real polynomial are a special case of that extension.

The second motivation behind our interest in studying the stability of complex polynomials is that it will open a window in circuit theory to allow us to construct complex rational functions. Synthesis of these functions will enable us to include a new type of element, namely a pure imaginary resistor (jR).

The interpretation of this imaginary resistance is understood by studying harmonic states with complex amplitudes. Superconductor materials (i.e zero-loss material), which are represented by imaginary resistances, have found many practical applications in microwave, millimeter-wave devices and circuits [1, 2]. Barely five years after the discovery of high-temperature superconductors, researchers succeeded in developing a variety of useful circuits and devices using thin film technology based on new materials. There have been many experimental research attempts to obtain a superconductor material with high T_c, near room temperatures, which can represent (jR).

The first motivation behind our interest in imaginary resistances is that they will enable us to study such complex networks as free mathematical creations. In addition, the theory of complex networks is of practical interest in modulation problems, and this is a second motivation for studying the imaginary resistances.

1.2 Outline of The Thesis

This thesis is concerned with network synthesis of complex rational functions, which leads to a study of the stability of the complex polynomial denominators of these functions.

The thesis is organized as follows: In Chapter 2, we present different methods of studying the stability conditions of any polynomial $P(s)$ with complex coefficients. Chapter 3 contains a synthesis of certain complex functions e.g complex impedance $Z(s)$ or complex reactance $X(s)$ functions in the s-plane. Chapter 4 is mainly concerned with transferring the whole idea of network synthesis from the s-domain to the z-domain. Chapter 5 addresses an extension of network synthesis of Chapter 3 to include two-port networks. Finally Chapter 6 gives some concluding remarks concerning the research described in this thesis and suggestions for future work.

In the remainder of this chapter we outline specific contents of each chapter.

1.2.1 Chapter 2: Stability conditions for a complex polynomial and its applications

One of the main problems in system stability analysis is the location of the roots of polynomials. This chapter introduces the problem of the stability of complex polynomials, which implies the stability of complex rational functions for complex networks.

Three methods are used to derive the necessary and sufficient conditions for the stability of a given complex polynomial. In the first one, a powerful mathematical method referred to as the J-fraction [6] expansion is explored. In the second method a generalized Routh array of a complex polynomial is derived using the division process and two cross multiplying processes. In the third method, the generalized Hurwitz determinant and the modified generalized Hurwitz determinant of a complex polynomial are derived. The second and third methods for testing the stability of complex polynomials are the first contribution of the thesis. The relationship among the coefficients in the J-fraction expansion, the terms in the first column in the generalized Routh array and the successive principal minor determinants in the modified generalized Hurwitz determinant are found.

One of the most important aspects of studying the stability of a complex polynomial $P(s)$ is its application to the stability of multidimensional systems.

1.2.2 Chapter 3: Synthesis of complex impedance and complex reactance functions in the s-plane

The importance of synthesis procedures with real components is well known in network theory. Chapter 3 addresses an extension of the present methods of circuit theory to include a new type of element, namely a pure imaginary resistor (jR). Initially, this chapter deals with finding a network realization of a complex impedance or admittance and a complex reactance or susceptance in the form of a rational function of the two polynomials with complex coefficients. The method of J-fraction expansion has opened a new research area in circuit theory for the synthesis of complex functions. These functions are rational and have non-negative real parts in the closed right-half of the frequency s-plane.

A complex impedance is synthesized by using four kinds of elements; a real resistor (R), imaginary resistors (jR), real inductors (L) and real capacitors (C). Complex reactance is synthesized either by using three kinds of elements (jR, L, C) or by using two kinds of elements (jR, L *or* jR, C). The relationship between a complex impedance $Z(s)$ and its associated complex reactance $X(s)$ is also derived in this chapter. This is the second contribution of the thesis.

1.2.3 Chapter 4: Synthesis of a complex reactance function $X(z)$ in the z-plane

In this chapter, we extend the ideas of Chapter 3 (in s-domain) to the z-domain synthesis and this is the third contribution of the thesis. A complex reactance function X(s) is analogue of a discrete complex reactance function X(z). A mathematical transformation method is used to transfer an analog complex reactance X(s) from the s-plane to a complex reactance function X(z) in the z-plane. X(z) can be obtained from any complex polynomial which has all roots inside the unit circle in the z-plane. Discrete complex reactance function can be implemented or synthesized by the directly obtained algebraic equation from X(z). The algebraic equations can be implemented by computer program, digital circuitry, or programmable integrated circuits. Direct evaluation of algebraic equations is one of the many possible realizations of the discrete complex reactance function.

The purpose of this chapter is to provide a realization of a discrete complex reactance function X(z), e.g algebraic realization, partial fraction expansion realization (parallel realization), cascade realization, J-fraction expansion realization, continued fraction expansion realization, ... etc. The implementation of a discrete complex reactance function can be achieved by using delay elements which are equivalent to inductors, capacitors or imaginary resistors (energy storage elements) in an analog complex reactance function X(s).

1.2.4 Chapter 5: Synthesis of complex reactance two-port networks in the s-plane

Chapter 5 addresses extensions of the complex reactance function of Chapter 3 to the two-port networks in the s-domain and this is the fourth contribution of the thesis. The extensions are related to the construction and synthesis of a complex impedance matrix $[Z(s)]$ or a complex reactance matrix $[X(s)]$ of two-port networks from any stable complex polynomial. Realizability conditions for complex impedance and complex reactance two-port parameters are given. A complex impedance two-port matrix is synthesized by using four kinds of elements (R, jR, L, C). Complex reactance matrix is synthesized by either three kinds of elements (jR, L, C) or by using two kinds of elements $(jR, L$ *or* $jR, C)$.

1.2.5 Chapter 6: Conclusions and future work

Network synthesis presented in this thesis is summarized and discussed in this chapter. Suggestions for future work in this area are also made in this chapter.

Chapter 2

STABILITY CONDITIONS FOR A COMPLEX POLYNOMIAL AND ITS APPLICATIONS

2.1 Introduction

One of the main problems in system analysis is the location of the roots of polynomials in the complex s-plane. A mathematical method referred to as the J-fraction is explored in this chapter to find the necessary and sufficient conditions for the stability of a polynomial with complex coefficients. Some theorems are presented to illustrate the stability conditions. Also three criteria are found to determine the number of roots that lie in the left half and the right half of the complex s-plane, respectively. A generalized Routh array for a complex polynomial is considered. Two methods are used to construct that array. The first method involves a division process, and the second method involves two cross multiplying processes. The generalized Hurwitz determinant and the modified generalized Hurwitz determinant for the stability of a complex polynomial are derived. Relationships between coefficients in the J-fraction expansion, the terms in the first column of the generalized Routh array, and the successive odd principal minor determinants in the modified generalized Hurwitz determinant are established. The analysis is supported by numerical examples.

2.2 J-Fraction Expansion

Let $P(s)$ be a polynomial of degree n with complex coefficients,

$$P(s) = s^n + (a_1 + jb_1)s^{n-1} + (a_2 + jb_2)s^{n-2} + \ldots + (a_{n-1} + jb_{n-1})s + (a_n + jb_n) \quad (2.1)$$

The main problem considered in this chapter is to determine the necessary and sufficient condition for the polynomial given by equation (2.1) to be stable, namely condition when all the roots lie in the open left half of the complex s-plane. In (2.1) the condition $a_1 > 0$ is necessary but obviously not sufficient for stability. To establish the sufficients condition for $P(s)$ to be stable, we derive some theorems.

From $P(s)$ we seek another polynomial $Q(s)$ [6] of degree $n - 1$ such that properties of $P(s)$ could be determined from the J-fraction expansion of the quotient $Q(s)/P(s)$. There is one choice for $Q(s)$ which is particularly convenient, called the alternant of $P(s)$. The polynomial $Q(s)$ has the following form,

$$Q(s) = a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + a_5 s^{n-5} + jb_6 s^{n-6} + \ldots \quad (2.2)$$

The quotient $Q(s)/P(s)$ has in general a J-fraction expansion of the form [6, 7, 8],

$$\frac{Q(s)}{P(s)} = \cfrac{1}{F_1 s + 1 + E_1 + \cfrac{1}{F_2 s + E_2 + \cfrac{1}{F_3 s + E_3 + \cfrac{1}{F_4 s + E_4 + \cfrac{1}{F_5 s + E_5 + \cfrac{1}{\ddots + \cfrac{1}{F_n s + E_n}}}}}}} \quad (2.3)$$

called the test-fraction of P(s)

Theorem 2.1 *[6] Let $P(s)$ in equation (2.1) be a polynomial of degree $n > 0$ with complex coefficients. Let $Q(s)$ in equation (2.2) be the alternant of $P(s)$. All the roots of $P(s)$ have negative real parts if and only if, $P(s)$ has a test-fraction of the form (2.3) and all $F_1, F_2, F_3, ..., F_n$ are real and positive and $E_1, E_2, E_3, ..., E_n$ are purely imaginary or zero.*

As an illustration of Theorem 2.1, consider the following example:

Example 2.1 *Given a polynomial $P(s)$ with complex coefficients*

$$P(s) = s^4 + (5 - j5)s^3 + (0 - j19.75)s^2 + (-18.375 - j17.875)s - (13.125 - j0.625) \tag{2.4}$$

the alternant polynomial of $P(s)$ can be written as

$$Q(s) = 5s^3 - j19.75s^2 - 18.375s + j0.625 \tag{2.5}$$

then the J-fraction expansion of $Q(s)/P(s)$ is obtained as follows:

$$\frac{Q(s)}{P(s)} = \cfrac{1}{0.2s + 1 - j0.21 + \cfrac{1}{0.6392s - j0.7387 + \cfrac{1}{1.2527s - j1.6615 + \cfrac{1}{3.162s - j4.6418}}}} \tag{2.6}$$

This is the test-fraction of the polynomial $P(s)$, and all the coefficients of s and the constant terms in (2.6) are positive real and purely imaginary, respectively. This means that the polynomial $P(s)$ with complex coefficients is stable, and all the roots lie in the left half of the complex s-plane. However, they need not appear in conjugate pairs, as shown in Figure 2.1. The roots are given by (2.7).

$$\lambda_1 = -2 + j0.5, \qquad \lambda_2 = -1.5 + j1, \qquad \lambda_3 = -1 + j2, \qquad \lambda_4 = -0.5 + j1.5 \tag{2.7}$$

For a stable polynomial with real coefficients $p(s)$, all the roots lie in the left half of the complex s-plane, and they occur in conjugate pairs.

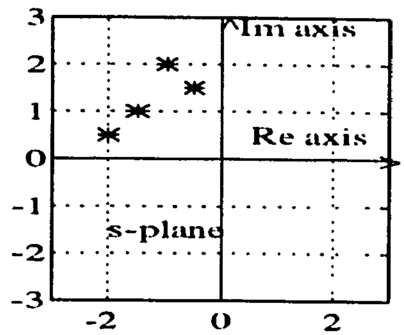

Figure 2.1: Roots of stable $P(s)$

2.2.1 Determination of the number of roots of $P(s)$ in each of the half-planes {Re s > 0 and Re s < 0 } using J-fraction expansion

We have seen from Theorem 2.1 and Example 2.1, that all the roots of $P(s)$ are in the left half of the s-plane {Re s < 0 } if and only if the coefficients F_i's in equation (2.3) are all positive. Also, we see in (2.3) that if all the F_i's have negative signs, it implies that all the roots of $P(s)$ have positive real parts. The criterion for determination of the number of roots of $P(s)$ that lie in the left half and right half of the complex s-plane respectively, is given:

Criterion 2.1 *The number of roots with positive real parts of any complex polynomial is equal to the number of negative signs of F_i in the J-fraction (2.3).*

To illustrate this criterion consider the following example:

Example 2.2 *Consider a polynomial $P(s)$ with complex coefficients and its $Q(s)$*

$$P(s) = s^4-(5+j5)s^3+(0+j19.75)s^2+(18.375-j17.875)s-(13.125+j0.625) \quad (2.8)$$

$$Q(s) = -5s^3 + j19.75s_2 + 18.375s - j0.625 \quad (2.9)$$

Then the J-fraction of $Q(s)/P(s)$ takes the following form

$$\frac{Q(s}{P(s)} = \frac{1}{-0.2s + 1 + j0.21 + \frac{1}{-0.6392s+j0.7387+\frac{1}{-1.2527s+j1.6615+\frac{1}{-3.162s+j4.6418}}}} \quad (2.10)$$

All coefficients of s and the constant terms in (2.10) having negative real and purely imaginary values respectively, means that $P(s)$ is unstable and all the roots lie in the right half of the complex s-plane. The roots are given by (2.11) and shown in Figure 2.2

$$\lambda_1 = 2 + j0.5, \qquad \lambda_2 = 1.5 + j1, \qquad \lambda_3 = 1 + j2, \qquad \lambda_4 = 0.5 + j1.5 \quad (2.11)$$

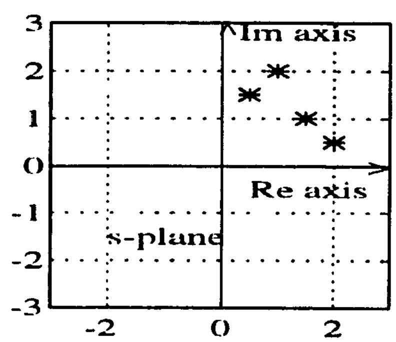

Figure 2.2: Roots of unstable $P(s)$

The case corresponding to m roots of $P(s)$ lie in the left half of the s-plane, and n-m roots in the right half of the s-plane is illustrated in Example 2.3

Example 2.3 *consider a polynomials P(s), the corresponding Q(s), and the quotient Q(s)/P(s)*

$$P(s) = s^4 + (3 - j5)s^3 - (8 + j13.75)s^2 - (22.375 + j0.375)s + (-7.375 + j10.875) \quad (2.12)$$

$$Q(s) \;=\; 3s^3 - j13.75s^2 - 22.375s + j10.875 \tag{2.13}$$

$$\frac{Q(s)}{P(s)} = \cfrac{1}{0.3333s + 1 - j0.1389 + \cfrac{1}{2.1928s + j1.3418 + \cfrac{1}{-0.1101s + j0.37 + \cfrac{1}{27.5561s + j50.5483}}}} \tag{2.14}$$

In this expansion three coefficients of s are positive real and one coefficient is negative real. This means that $P(s)$ has three roots in the $Re\ s < 0$ plane and one root in the $Re\ s > 0$ plane. The roots of $P(s)$ are shown in Figure 2.3

$$\lambda_1 = 1 + j2, \qquad \lambda_2 = -0.5 + j1.5, \qquad \lambda_3 = -2 + j0.5, \qquad \lambda_4 = -1.5 + j1 \tag{2.15}$$

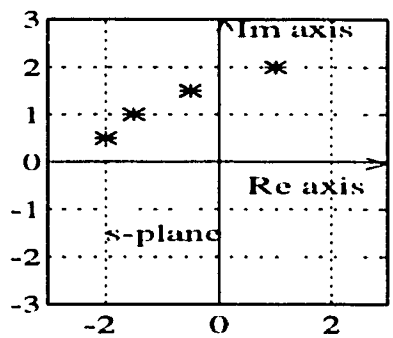

Figure 2.3: Roots of $P(s)$

2.3 Generalized Routh Array

We use two methods, to construct the Routh array for a general complex polynomial. In the first one, a long division process is used, and in the second, two cross multiplying processes are used. Detailed analysis is given in the next two subsections.

2.3.1 Generalized Routh array using division process.

In the previous section, the stability condition for a general complex polynomial was illustrated on the basis of Theorem 2.1 . Now we present another method for determining whether a given complex polynomial is stable or not. We start from $P(s)$ as in Equation (2.1) and the corresponding $Q(s)$ as in Equation (2.2). $P(s)$ can always be written as

$$P(s) \; = \; r_1(s)\, Q(s) \; + \; R_1(s) \tag{2.16}$$

where, $r_1(s)$ is a polynomial of degree one and $R_1(s)$ is the remainder polynomial of degree $n-2$. Both $r_1(s)$ and $R_1(s)$ could be found by formally dividing $P(s)$ by $Q(s)$, since this gives

$$\frac{P(s)}{Q(s)} \; = \; r_1(s) \; + \; \frac{R_1(s)}{Q(s)} \tag{2.17}$$

where $R_1(s)$ is

$$R_1(s) = c_1 s^{n-2} + c_2 s^{n-3} + c_3 s^{n-4} + c_4 s^{n-5} + \ldots + c_{n-3} s^2 + c_{n-2} s^1 + c_{n-1} \tag{2.18}$$

We can also write $Q(s)$ in the last term of (2.17) in terms of $R_1(s)$ as

$$Q(s) \; = \; r_2(s)\, R_1(s) \; + \; R_2(s) \tag{2.19}$$

where, $r_2(s)$ is a first order polynomial and $R_2(s)$ is the remainder polynomial of degree $n-3$. Both $r_2(s)$ and $R_2(s)$ could be found by formally dividing $Q(s)$ by $R_1(s)$, since this gives

$$\frac{Q(s)}{R_1(s)} \; = \; r_2(s) \; + \; \frac{R_2(s)}{R_1(s)} \tag{2.20}$$

where the remainder polynomial $R_2(s)$ is

$$R_2(s) = d_1 s^{n-3} \; + \; d_2 s^{n-4} \; + \; d_3 s_{n-5} \; + \; d_4 s^{n-6} \; + \; \ldots + \; + \; d_{n-3} s^1 \; + \; d_{n-2} \tag{2.21}$$

We continue this process of division of two polynomials until the remainder polynomial equals zero.

The coefficients of polynomials $P(s)$, $Q(s)$, $R_1(s)$, $R_2(s)$,..., etc. can be arranged in rows and columns according to the following pattern:

s^n	1	$a_1 + jb_1$	$a_2 + jb_2$	....	$a_n + jb_n$	$\longleftarrow$ *1st row* $P(s)$
s^{n-1}	a_1	jb_2	a_3	....	a_n *or* jb_n	$\longleftarrow$ *2nd row* $Q(s)$
s^{n-2}	c_1	c_2	c_3	....	0	$\longleftarrow$ *3rd row* $R_1(s)$
s^{n-3}	d_1	d_2	d_3	....	0	$\longleftarrow$ *4th row* $R_2(s)$
....	....	....	....	....	....	....
s^2	g_1	g_2	0	0	0	$\longleftarrow$ $(n)th$ *row* $R_{n-2}(s)$
s^1	h_1	0	0	0	0	$\leftarrow$ $(n+1)th$ *row* $R_{n-1}(s)$
0	0	0	0	0	0	$\longleftarrow$ $(n+2)th$ *row* $R_n(s)$

(2.22)

- In this pattern the first row corresponds to the coefficients of the given polynomial $P(s)$.
- The second row corresponds to the coefficients of the alternant polynomial $Q(s)$.
- The constants $c_1, c_2, c_3, ..., etc$, in the third row are coefficients of the remainder polynomial obtained from the quotient of $P(s)/Q(s)$.
- The coefficients of the remainder polynomial obtained from the division of the two previous rows give the constant coefficients $d's, ..., g's, h's$.
- Evaluation of the coefficients in this pattern is continued until the $(n+2)th$ row is completed, in other words, until the remainder polynomial obtained from the two previous rows equals zero.

This pattern is the generalized Routh array for a complex polynomial $P(s)$. It is similar to the Routh array for the real polynomial $p(s)$.

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-2} + a_4 s^{n-2} + a_5 s^{n-2} + ... + a_{n-1} s + a_n \quad (2.23)$$

In other words, (2.22) is a general array constructed from a complex polynomial, of which, the commonly used Routh array in control theory and system analysis for real polynomials is a particular case. To this end, to obtain the Routh array for the real polynomial from the pattern in (2.22), we follow the procedure given below:

- Let all imaginary elements in (2.22) equal zero.
- Delete all the even columns.
- The resulting array is the Routh array for a real polynomial [9, 10] as in (2.24)

$$\begin{array}{lllllllll} s^n & 1 & a_2 & a_4 & a_6 & a_8 & a_{10} & \dots & \leftarrow 1st\ row\ even\ of\ P(s) \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & a_9 & a_{11} & \dots & \leftarrow 2nd\ row\ odd\ of\ P(s) \\ s^{n-2} & c_1 & c_3 & c_5 & c_7 & c_9 & c_{11} & \dots & \leftarrow -3nd\ row\ R_1(s) \\ s^{n-3} & d_1 & d_3 & d_5 & d_7 & d_9 & d_{11} & \dots & \leftarrow -4nd\ row\ R_2(s) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s^2 & g_1 & g_3 & 0 & 0 & 0 & 0 & 0 & \leftarrow (n-1)th\ row\ R_{n-1}(s) \\ s^1 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & \leftarrow (n-1)th\ row\ R_{n-1}(s) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \leftarrow (n-2)th\ row\ R_{n-1}(s) \end{array} \tag{2.24}$$

Note that in developing the array in pattern (2.22), the entire row may be divided or multiplied by a positive number in order to simplify the numerical calculation without altering the results of stability. The next theorem states the stability condition.

Theorem 2.2 *A necessary and sufficient condition for all the roots of any complex polynomial $P(s)$ to have negative real parts is that all the coefficients in the first column of the array in (2.22) have positive sign.*

Proof:

The J-fraction of $P(s)$ with respect to $Q(s)$ takes the form

$$\frac{P(s)}{Q(s)} = \frac{1}{a_1}s + 1 + E_1 + \cfrac{1}{\frac{a_1}{c_1}s + E_2 + \cfrac{1}{\frac{c_1}{d_1}s + E_3 + \cfrac{1}{\frac{d_1}{e_1}s + E_4 + \cfrac{1}{+\cdots}}}} \quad (2.25)$$

From Theorem 2.1, $P(s)$ is stable if and only if all the coefficients of s in (2.25) are real and positive. This means that all a_1, c_1, d_1, ... should have positive signs. These a_1, c_1, d_1, ... are the same as the coefficients in the first column of the generalized Routh array (2.22), and the stability condition is established. Q. E. D.

2.3.2 Generalized Routh array using two cross multiplying processes.

In the previous subsection 2.3.1, the creation of the generalized Routh array depended on a division process which is very tedious, even using a computer. We shall give a simple way to evaluate the generalized Routh array. The procedure for the construction of the generalized Routh array is given below:

- Write the given polynomial $P(s)$ of degree n as in (2.1)
- Find the alternant polynomial of $P(s)$ of degree n-1 as in (2.2)
- Arrange the coefficients of the polynomial $P(s)$ and $Q(s)$ in the first two rows of the following pattern:

$$
\begin{array}{lllllll}
s^{n} & 1 & a_1+jb_1 & a_2+jb_2 & \ldots & a_n+jb_n & \longleftarrow 1st\ row\ P(s) \\
s^{n-1} & a_1 & jb_2 & a_3 & \ldots & a_n\ or\ jb_n & \longleftarrow 2nd\ row\ Q(s) \\
s^{n-2} & c_1 & c_2 & c_3 & \ldots & 0 & \longleftarrow 3rd\ row\ c_i \\
s^{n-3} & d_1 & d_2 & d_3 & \ldots & 0 & \longleftarrow 4th\ row\ d_i \\
,\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ \ldots\ \ldots\ \ldots \\
s^{2} & g_1 & g_2 & 0 & 0 & 0 & \longleftarrow (n-1)th\ row\ g_i \\
s^{1} & h_1 & 0 & 0 & 0 & 0 & \longleftarrow (n+1)th\ row\ h_i \\
0 & 0 & 0 & 0 & 0 & 0 & \longleftarrow (n+2)th\ row\ zeros
\end{array}
\tag{2.26}
$$

The coefficients $c_1, c_2, c_3, ..., etc.$, are evaluated using the two cross multiplying processes.

$$
\begin{aligned}
c_1 &= \frac{a_1\ (a_2+jb_2) - a_3}{a_1} - \frac{jb_2}{a_1}\ \frac{a_1(a_1+jb_1) - jb_2}{a_1} \\
c_2 &= \frac{a_1\ (a_3+jb_3) - jb_4}{a_1} - \frac{a_3}{a_1}\ \frac{a_1(a_1+jb_1) - jb_2}{a_1} \\
c_3 &= \frac{a_1\ (a_4+jb_4) - a_5}{a_1} - \frac{jb_4}{a_1}\ \frac{a_1(a_1+jb_1) - jb_2}{a_1} \\
c_4 &= \frac{a_1\ (a_5+jb_5) - jb_6}{a_1} - \frac{a_5}{a_1}\ \frac{a_1(a_1+jb_1) - jb_2}{a_1} \\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots
\end{aligned}
\tag{2.27}
$$

The evaluation of the $c's$ is continued until the remaining ones are all zeros. The same pattern of the two cross multiplying the coefficients of the two previous rows is followed in evaluating the $d's, e's, g's, ...etc.$

$$
\begin{aligned}
d_1 &= \frac{c_1\ a_3 - a_1 c_3}{c_1} - \frac{c_2}{c_1}\ \frac{c_1(jb_2) - a_1\ c_2}{c_1} \\
d_2 &= \frac{c_1\ jb_4 - a_1 c_4}{c_1} - \frac{c_3}{c_1}\ \frac{c_1(jb_2) - a_1\ c_2}{c_1} \\
d_3 &= \frac{c_1\ a_5 - a_1 c_5}{c_1} - \frac{c_4}{c_1}\ \frac{c_1(jb_2) - a_1\ c_2}{c_1} \\
d_4 &= \frac{c_1\ a_6 - a_1 c_6}{c_1} - \frac{c_5}{c_1}\ \frac{c_1(jb_2) - a_1\ c_2}{c_1} \\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
e_1 &= \frac{d_1 c_3 - c_1 d_3}{d_1} - \frac{d_2}{d_1}\frac{d_1 c_2 - c_1 d_2}{d_1} \\
e_2 &= \frac{d_1 c_4 - c_1 d_4}{d_1} - \frac{d_3}{d_1}\frac{d_1 c_2 - c_1 d_2}{d_1} \\
e_3 &= \frac{d_1 c_5 - c_1 d_5}{d_1} - \frac{d_4}{d_1}\frac{d_1 c_2 - c_1 d_2}{d_1} \\
e_4 &= \frac{d_1 c_6 - c_1 d_6}{d_1} - \frac{d_5}{d_1}\frac{d_1 c_2 - c_1 d_2}{d_1} \\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots
\end{aligned} \tag{2.29}
$$

This process is continued until the $(n+2)th$ row has been completed. The complete array of the coefficients is triangular as in (2.26).

2.3.3 Determination of the number of roots of $P(s)$ in each of the half-planes $\{\text{Re } s > 0 \text{ and Re } s < 0\}$ using Routh array

The number of roots in the left half and the right half of the s-plane can be determined from Criterion 2.2.

Criterion 2.2 *The number of roots with positive real parts of any complex polynomial is equal to the number of changes in the sign of the coefficients of the first column of the generalized Routh array pattern (2.26).*

Note that the values of the terms in the first column are not important for us instead only signs of the terms are needed. Consider the following example to illustrate Theorem 2.2.

Example 2.4 *Consider a complex polynomial $P(s)$ and its $Q(s)$ as in Example 2.1.*

The Routh array of the coefficients is constructed, following the analysis of the generalized Routh array just presented. The first two rows can be obtained directly from $P(s)$, and $Q(s)$. The remaining terms are obtained from equations (2.27). (2.28) and (2.29). The generalized Routh array now is

$$
\begin{array}{lllllll}
s^4 & 1 & 5-j5 & 0-j19.75 & -18.375-j17.875 & -13.125+j0.625 & \leftarrow 1st\ row\ P(s) \\
s^3 & 5 & -j19.75 & -18.375 & j0.625 & 0 & \leftarrow 2nd\ row\ Q(s) \\
s^2 & 7.8225 & -j21.8588 & 13.2562 & 0 & 0 & \leftarrow 3rd\ row\ c_i \\
s^1 & 6.2446 & -j9.167 & 0 & 0 & 0 & \leftarrow 4th\ row\ d_i \\
s^0 & 1.9749 & 0 & 0 & 0 & 0 & \leftarrow 5th\ row\ e_i \\
0 & 0 & 0 & 0 & 0 & 0 & \leftarrow 6th\ row\ f_i
\end{array}
\tag{2.30}
$$

The number of changes in the sign of the coefficients of the first column is zero. This implies that there are no roots with positive real parts, and the given polynomial $P(s)$ is stable.

To illustrate Criterion 2.2, consider the following example:

Example 2.5 *Consider a complex polynomial P(s) and its Q(s) of Example 2.2,*

The corresponding generalized Routh array is

$$
\begin{array}{lllllll}
s^4 & 1 & -5-j5 & 0+j19.75 & 18.375-j17.875 & -13.125+j0.625 & \leftarrow 1st\ row\ P(s) \\
s^3 & -5 & j19.75 & 18.375 & -j0.625 & 0 & \leftarrow 2nd\ row\ Q(s) \\
s^2 & 7.8225 & -j21.8588 & -13.2562 & 0 & 0 & \leftarrow 3rd\ row\ c_i \\
s^1 & -6.2446 & j9.167 & 0 & 0 & 0 & \leftarrow 4th\ row\ d_i \\
s^0 & 1.9749 & 0 & 0 & 0 & 0 & \leftarrow 5th\ row\ e_i \\
0 & 0 & 0 & 0 & 0 & 0 & \leftarrow 6th\ row\ d_i
\end{array}
\tag{2.31}
$$

The number of changes in the sign of the coefficients in the first column is four, which implies that there are four roots with positive real parts, and $P(s)$ is unstable.

Example 2.6 *Consider a polynomial P(s) with complex coefficients of Example 2.3. The generalized Routh array is*

$$
\begin{array}{lllllll}
s^4 & 1 & 3-j5 & -8-j13.75 & -22.375-j0.375 & -7.375+j10.875 & \leftarrow 1st\ row\ P(s) \\
s^3 & 3 & -j13.75 & -22.375 & +j10.875 & 0 & \leftarrow 2nd\ row\ Q(s) \\
s^2 & 1.3681 & -j7.1076 & -8.8854 & 0 & 0 & \leftarrow 3rd\ row\ c_i \\
s^1 & -12.4278 & j22.7973 & 0 & 0 & 0 & \leftarrow 4th\ row\ d_i \\
s^0 & -0.451 & 0 & 0 & 0 & 0 & \leftarrow 5th\ row\ e_i \\
0 & 0 & 0 & 0 & 0 & 0 & \leftarrow 6th\ row\ f_i
\end{array}
\tag{2.32}
$$

$P(s)$ is unstable with one root lying on the right half of the s-plane because there is one change of sign in the first column in (2.32).

2.3.4 J-fraction expansion and the generalized Routh array relationship.

We first note that the construction of the generalized Routh array using a division process or two cross multiplying processes gives us the same pattern as in (2.22) and (2.26). The stability conditions for any complex polynomial can be summarized as follows:

- From the J-fraction expansion, as in (2.3) all F_1, F_2, F_3 , ... are real positive.
- From the generalized Routh array, all the terms in the first column of the array in (2.22) or (2.26) have positive signs.

Now we try to find a relation between the coefficients, $\{F_1, F_2, F_3, ...\}$, in the J-fraction expansion and the coefficients in the first column of the generalized Routh array, $\{1, a_1, c_1, d_1, e_1, ...\}$. The division of any two successive terms in the first column of (2.26) gives us one value of the coefficients $F's$ in the J-fraction expansion (2.3).

$$F_1 = \frac{1}{a_1}, \quad F_2 = \frac{a_1}{c_1}, \quad F_3 = \frac{c_1}{d_1}, \quad F_4 = \frac{d_1}{e_1}, \quad ...., \quad .... \quad (2.33)$$

Table 2.1 shows the relation between the constants $F_1, F_2, ...$ in the J-fraction and the coefficients $a_1, c_1, d_1, e_1, ...$ .in the generalized Routh array. Also from Table 2.1 the stability criteria 2.1 and 2.2 should be observed

2.4 Generalized Hurwitz Determinant

In the previous two sections, two methods were found to test stability of a complex polynomial on the basis of the J-fraction expansion, and of the generalized Routh

J-Fraction	Routh	J-F & Routh
	1	
F_1	a_1	$F_1 = \frac{1}{a_1}$
F_2	c_1	$F_2 = \frac{a_1}{c_1}$
F_3	d_1	$F_3 = \frac{c_1}{d_1}$
F_4	e_1	$F_4 = \frac{d_1}{e_1}$
...	...	...

Table 2.1: J-fraction, Routh array relationship

array. Now a third method which involves the generalized Hurwitz determinant for a complex polynomial will be presented. Starting from a polynomial $P(s)$, and its $Q(s)$ as in (2.1) and (2.2), the procedure is outlined below:

- Write the given polynomial $P(s)$ of degree n
- Arrange the coefficients of $Q(s)$ and $P(s)$ in the following matrix:

$$A = \begin{pmatrix} a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & \dots & \dots \\ 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & a_4+jb_4 & a_5+jb_5 & \dots & \dots \\ 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & \dots & \dots \\ 0 & 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & a_4+jb_4 & \dots & \dots \\ 0 & 0 & a_1 & jb_2 & a_3 & jb_4 & \dots & \dots \\ 0 & 0 & 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & \dots & \dots \\ 0 & 0 & 0 & a_1 & jb_2 & a_3 & \dots & \dots \\ 0 & 0 & 0 & 1 & a_1+jb_1 & a_2+jb_2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (2.34)$$

The determinant of the matrix A of rank $(2n-1)$ is the generalized Hurwitz determinant, whose successive principal minors are denoted by D_q.

The commonly used Hurwitz determinant for a real polynomial p(s) is a particular case of the Hurwitz determinant of the matrix A in (2.34). To obtain the

Hurwitz determinant for a real polynomial from the matrix A in (2.34), the steps shown below are to be followed.

- Set all the imaginary elements in the matrix A in (2.34) equal zero.
- Delete all the even columns in the matrix A.
- Delete the zero rows of A created by the above two operations.
- If the first two operations create some rows in (2.34) having the same elements, retain one of them and delete the rest.
- The resulting determinant is the Hurwitz determinant for a real polynomial.

$$D = |A| = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & a_{11} & \cdots & \cdots \\ 1 & a_2 & a_4 & a_6 & a_8 & a_{10} & \cdots & \cdots \\ 0 & a_1 & a_3 & a_5 & a_7 & a_9 & \cdots & \cdots \\ 0 & 1 & a_2 & a_4 & a_4 & a_6 & \cdots & \cdots \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 & \cdots & \cdots \\ 0 & 0 & 1 & a_2 & a_4 & a_6 & \cdots & \cdots \\ 0 & 0 & 0 & a_1 & a_3 & a_5 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & a_2 & a_4 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{2.35}$$

The stability condition is stated in Theorem 2.3.

Theorem 2.3 *All the roots of a complex polynomial P(s) have negative real parts if and only if, the modified generalized Hurwitz determinant*

$$\Delta_0 = 1 \; , \; \Delta_q = (-1)^{\frac{q^2-1}{8}} \; D_q > 0 \; , \; q = 1,3,5,7,9,11,....,(2n-1) \quad (2.36)$$

Proof:

From Theorem 2.2, $P(s)$ is stable if, and only if, the elements in the first column of the array in (2.26) have positive signs. We want to indicate the dependence of the determinant Δ_q upon the elements in the first column in the Routh array (2.26). After some matrix operations, the matrix A in (2.34) can be transformed into an upper triangular matrix as in (2.37),

$$B = (-1)^{\frac{q^2-1}{8}} \begin{pmatrix} a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & jb_8 & a_9 & jb_{10} & \dots & \dots \\ 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & jb_8 & a_9 & \dots & \dots \\ 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & \dots & \dots \\ 0 & 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \dots & \dots \\ 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & d_5 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & e_4 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (2.37)$$

The successive principal minors of the generalized Hurwitz determinant of the matrix A in (2.34) are equal to the successive principal minors of the determinant of the matrix B given by (2.37).

In Equation (2.37) it should be observed that the coefficients of the polynomial $Q(s)$ appear in the first two rows. The rows containing the $c's, d's, e's, f's, ...$ in (2.37) are the same as those in the Routh array (2.26). Further, the elements of the first column in the Routh array are the same as the diagonal elements of the matrix B (2.37). From Equation (2.37), the successive principal minors of the Hurwitz determinant will be:

$$D_q = (-1)^{\frac{q^2-1}{8}} \begin{vmatrix} a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & jb_8 & a_9 & jb_{10} & \ldots & \ldots \\ 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & a_7 & jb_8 & a_9 & \ldots & \ldots \\ 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & \ldots & \ldots \\ 0 & 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & d_4 & d_5 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & e_4 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & e_3 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \tag{2.38}$$

$$D_q = (-1)^{\frac{q^2-1}{8}} \; a_1^2 \; c_1^2 \; d_1^2 \; e_1^2 \; \ldots\ldots\ldots \; g_1^2 \; h_1 \tag{2.39}$$

$$\Delta_q = (-1)^{\frac{q^2-1}{8}} \; D_q = (-1)^{\frac{q^2-1}{8}} \; (-1)^{\frac{q^2-1}{8}} a_1^2 \; c_1^2 \; d_1^2 \; e_1^2 \; \ldots\ldots\ldots \; g_1^2 \; h_1 \tag{2.40}$$

where, q is equal to the sum of the numbers $a_1, c_1, d_1, e_1, \ldots$ etc. Since all the coefficients $a_1, c_1, d_1, e_1, f_1, \ldots.$ etc. are greater than zero for stability (Theorem 2.2), this implies that all $\Delta_q > 0$ for stability. Q. E. D.

2.4.1 Determination of the number of roots of $P(s)$ in each of the half-planes {Re s > 0 and Re s < 0 } using the Hurwitz determinant

The number of roots in the left and right halve of the s-plane can be determined from the following criterion:

Criterion 2.3 *The number of roots with positive real parts of any complex polynomial, is equal to the number of changes in the sign of the odd determinants Δ_q in (2.36).*

The general form of the determinant Δ_q is

$$\Delta_q = (-1)^{\frac{q^2-1}{8}} \begin{vmatrix} a_1 & jb_2 & a_3 & jb_4 & a_5 & jb_6 & \cdots & \cdots \\ 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & a_4+jb_4 & a_5+jb_5 & \cdots & \cdots \\ 0 & a_1 & jb_2 & a_3 & jb_4 & a_5 & \cdots & \cdots \\ 0 & 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & a_4+jb_4 & \cdots & \cdots \\ 0 & 0 & a_1 & jb_2 & a_3 & jb_4 & \cdots & \cdots \\ 0 & 0 & 1 & a_1+jb_1 & a_2+jb_2 & a_3+jb_3 & \cdots & \cdots \\ 0 & 0 & 0 & a_1 & jb_2 & a_3 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & a_1+jb_1 & a_2+jb_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \quad (2.41)$$

Note that the exact values of the determinants Δ_q are not needed. Instead, only signs of these determinants are sufficient for the stability criterion.

Example 2.7 *Given a complex polynomial as in Example 2.4.*

Following the procedures in the generalized Hurwitz determinant just presented, and constructing the generalized Hurwitz determinant of the coefficients gives:

$$D_q = \begin{vmatrix} 5 & -j19.75 & -18.375 & j0.625 & 0 & \dots \\ 1 & 5-j5 & -j19.75 & -18.375-j17.875 & -13.125+j0.625 & \dots \\ 0 & 5 & -j19.75 & -18.375 & j0.625 & \dots \\ 0 & 1 & 5-j5 & -j19.75 & -18.375-j17.875 & \dots \\ 0 & 0 & 5 & -j19.75 & -18.375 & \dots \\ 0 & 0 & 1 & 5-j5 & -j19.75 & \dots \\ 0 & 0 & 0 & 5 & -j19.75 & \dots \\ 0 & 0 & 0 & 1 & 5-j5 & \dots \end{vmatrix} \tag{2.42}$$

The modified generalized Hurwitz determinant is

$$\Delta_q = (-1)^{\frac{q^2-1}{8}} \begin{vmatrix} 5 & -j19.75 & -18.375 & j0.625 & 0 & \dots \\ 1 & 5-j5 & -j19.75 & -18.375-j17.875 & -13.125+j0.625 & \dots \\ 0 & 5 & -j19.75 & -18.375 & j0.625 & \dots \\ 0 & 1 & 5-j5 & -j19.75 & -18.375-j17.875 & \dots \\ 0 & 0 & 5 & -j19.75 & -18.375 & \dots \\ 0 & 0 & 1 & 5-j5 & -j19.75 & \dots \\ 0 & 0 & 0 & 5 & -j19.75 & \dots \\ 0 & 0 & 0 & 1 & 5-j5 & \dots \end{vmatrix} \tag{2.43}$$

In this example, the odd successive principal minors of the generalized Hurwitz determinants D_q and the modified generalized Hurwitz determinant Δ_q are

$$\begin{array}{llll} D_1 = +5 & D_3 = -195.56 & D_5 = -9553 & D_7 = +117800 \\ \Delta_1 = +5 & \Delta_3 = +195.56 & \Delta_5 = +9553 & \Delta_7 = +117800 \end{array} \tag{2.44}$$

Since all the principal minors of the modified Hurwitz determinant Δ_q are positive real, this means that all the roots have negative real parts, which implies that the given complex polynomial $P(s)$ is stable.

2.4.2 The stability relationship between the generalized Routh array and the modified generalized Hurwitz determinant

Necessary and sufficient conditions for $P(s)$ to be stable are:

- From the generalized Routh array, all the terms in the first column of that array in (2.26) have positive signs.
- From the modified generalized Hurwitz determinant (2.41), all $\Delta_1, \Delta_3, \Delta_5$, ... are positive real.

The relation between these odd principal minor determinants Δ_i and $\{1, a_1, c_1, d_1, e_1, ...\}$ in the pattern (2.26) are derived as

$$a_1 = \Delta_1, \quad c_1 = \frac{\Delta_3}{\Delta_1^2}, \quad d_1 = \frac{\Delta_5 \, \Delta_1^2}{\Delta_3^2}, \quad e_1 = \frac{\Delta_7 \, \Delta_3^2}{\Delta_5^2 \Delta_1^2} \quad , ... \tag{2.45}$$

Table 2.2 shows the relation between the odd determinants $\Delta_1, \Delta_3, \Delta_5$... of the modified Hurwitz determinant and the coefficients a_1, c_1, d_1, e_1, ...in the generalized Routh array. It should also be observed that the stability Criteria 2.2 and 2.3 can be obtained from Table 2.2.

Routh	Hurwitz	Routh & Hurwitz
1		1
a_1	Δ_1	$a_1 = \Delta_1$
c_1	Δ_3	$c_1 = \Delta_3/\Delta_1^2$
d_1	Δ_5	$d_1 = \Delta_5/\Delta_3^2/\Delta_1^2$
e_1	Δ_7	$e_1 = \Delta_7/\Delta_5^2/\Delta_3^2/\Delta_1^2$
....	....	....
....	....	....

Table 2.2: Routh array, Hurwitz determinant relationship

2.5 Relationships Between J-Fraction Expansion, the Generalized Routh Array, and the Modified Generalized Hurwitz Determinant

The relations between $F_1, F_2, F_3, \ldots$.. in the J-fraction expansion (2.3), $a_1, c_1, d_1, e_1, \ldots$.. in the Routh array (2.26), and $\Delta_1, \Delta_3, \Delta_5, \ldots$.. in the modified Hurwitz determinant (2.36) may be presented as in Table 2.3.

J-Fraction	Routh	Hurwitz	J-F & Routh	Routh & Hurwitz	J-F & Hurwitz
	1				
F_1	a_1	Δ_1	$F_1 = \frac{1}{a_1}$	$a_1 = \Delta_1$	$F_1 = \frac{1}{\Delta_1}$
F_2	c_1	Δ_3	$F_2 = \frac{a_1}{c_1}$	$c_1 = \frac{\Delta_3}{\Delta_1^2}$	$F_2 = \frac{\Delta_1^3}{\Delta_3}$
F_3	d_1	Δ_5	$F_3 = \frac{c_1}{d_1}$	$d_1 = \frac{\Delta_5 \Delta_1^2}{\Delta_3^2}$	$F_3 = \frac{\Delta_3^3}{\Delta_5 \Delta_1^4}$
F_4	e_1	Δ_7	$F_4 = \frac{d_1}{e_1}$	$e_1 = \frac{\Delta_7 \Delta_3^2}{\Delta_5^2 \Delta_1^2}$	$F_4 = \frac{\Delta_5^3 \Delta_1^4}{\Delta_7 \Delta_3^4}$
...	...	...	... ...	....	....
...	...	...	... ...	....	....

Table 2.3: J-fraction, Routh array, modified Hurwitz determinant relationships

For the stability of a complex polynomial, the interrelations among the constant coefficients in the J-fraction expansion, the terms in the first column of the generalized Routh array and the odd successive principal minors of the modified generalized Hurwitz determinant can be obtained by directly using Table 2.4. Also from Table 2.4 the stability Criteria 2.1, 2.2 and 2.3 can be obtained.

From \ To	J-Fraction	Routh	Hurwitz
J-Fraction	$F_0 = 1$ F_1 F_2 F_3 F_4	$a_1 = 1/F_1$ $c_1 = 1/F_1F_2$ $d_1 = 1/F_1F_2F_3$ $e_1 = 1/F_1F_2F_3F_4$	$\Delta_1 = 1/F_1$ $\Delta_3 = 1/F_1^3F_2$ $\Delta_5 = 1/F_1^5F_2^3F_3$ $\Delta_7 = 1/F_1^7F_2^5F_3^3F_4$ ....
Routh	$F_1 = 1/a_1$ $F_2 = a_1/c_1$ $F_3 = c_1/d_1$ $F_4 = d_1/e_1$	1 a_1 c_1 d_1 e_1	$\Delta_1 = a_1$ $\Delta_3 = a_1^2c_1$ $\Delta_5 = a_1^2c_1^2d_1$ $\Delta_7 = a_1^2c_1^2d_1^2e_1$ ....
Hurwitz	$F_1 = 1/\Delta_1$ $F_2 = 1/\Delta_3/\Delta_1^3$ $F_3 = 1/\Delta_5/\Delta_3^3/\Delta_1^4$ $F_4 = 1/\Delta_7/\Delta_5^3/\Delta_3^4/\Delta_1^4$	$a_1 = 1/\Delta_1$ $c_1 = \Delta_3/\Delta_1^2$ $d_1 = \Delta_5/\Delta_3^2/\Delta_1^2$ $e_1 = \Delta_7/\Delta_5^2/\Delta_3^2/\Delta_1^2$	$\Delta_0 = 1$ Δ_1 Δ_3 Δ_5 Δ_7 ....

Table 2.4: The parameter interrelations among J-fraction, Routh array and modified Hurwitz determinant

2.6 Application of the Stability Conditions for a Complex Polynomial to a Multivariable System.

The stability of multidimensional systems and multidimensional polynomials have been discussed by Bose, Jury and others [12, 13, 14, 15]. From our analysis in this chapter, we observe that our study of the stability of a complex polynomial associated with one dimensional digital or analog systems, forms the kernel for the study of the stability of multidimensional systems. In the previous sections we have used three methods, namely:

- The J-fraction expansion
- The generalized Routh array
- The modified generalized Hurwitz determinant

to test the stability of a complex polynomial $P(s)$ describing a one dimensional system. We shall use these methods to test the stability of multidimensional polynomials with real coefficients. The stability of $P(s)$ with one variable opens a window for a practical test of the stability of two-variable polynomials $P(s_1, s_2)$. Also another practical application is to test the stability for two-variable discrete functions $P(z_1, z_2)$ e.g 2-D digital filter by using a double bilinear transformation to transfer $P(z_1, z_2)$ to $P(s_1, s_2)$

$$z_1 = \frac{s_1 + 1}{s_1 - 1} \quad , \quad z_2 = \frac{s_2 + 1}{s_2 - 1} \tag{2.46}$$

In general an important part of testing the stability of a polynomial with multivariables s_i, $i = 1, 2, ..., k$, for an analog system described by

$$P\,(\,s_1\,,\ s_2\,,\ \ldots,\ s_m\ ,\ldots,\ s_k\,) \tag{2.47}$$

or a polynomial with multivariables z_i, $i = 1, 2, ..., k$, for a discrete system by using multi-bilinear transformation

$$z_1 = \frac{s_1+1}{s_1-1}, \quad z_2 = \frac{s_2+1}{s_2-1}, \quad \ldots., \quad z_m = \frac{s_m+1}{s_m-1}, \quad \ldots., \quad z_k = \frac{s_k+1}{s_k-1} \tag{2.48}$$

to transfer $P(z_1, z_2, ..., z_m, ..., z_k)$ to $p(s_1, s_2, ..., s_m, ..., s_k)$, leads to test the stability of a complex polynomial with one variable

$$P(j\omega_1,\ j\omega_2, ...,\ s_m,\ ...,\ j\omega_k),\ where\ 0\ \leq\ \omega_i\ \leq\ \infty\ i, m\ =\ 1, 2, ..., k,\ i\ \neq\ m \tag{2.49}$$

Definition 2.1 *A multivariable polynomial* $P(s_1,\ s_2,\ ..., s_m,\ ..., s_k)$ *of degree* n *in* s_i, $i = 1, 2, ..., k$ *with real coefficients is said to be separable if it can be written as follows:*

$$\begin{aligned}
P(s_1, ..., s_m, ..., s_k) &= \textstyle\sum_{n1=0}^{n} \cdots \sum_{nm=0}^{n} \cdots \sum_{nk=0}^{n} A_{n1,...,nm...nk}\ \ s_1^{n1} ... s_m^{nm} ... s_k^{nk} \\
&= \textstyle\sum_{n1=0}^{n} a_{n1} s_1^{n1} \ldots. \sum_{nm=0}^{n} b_{nm} s_m^{nm} \ldots. \sum_{nm=0}^{n} c_{nk} s_k^{nk} \\
&= (a_0 s_1^0 + a_1 s_1^1 + a_2 s_1^2 + a_3 s_1^3 + ... + a_n s_1^n) \ldots. \\
&\quad (b_0 s_m^0 + b_1 s_m^1 + b_2 s_m^2 + b_3 s_m^3 + ... + b_n s_m^n) \ldots. \\
&\quad (c_0 s_k^0 + c_1 s_k^1 + c_2 s_k^2 + c_3 s_k^3 \ldots. + c_n s_k^n) \\
&= P_1(s_1)\ \ldots.\ P_m(s_m)\ \ldots.\ P_k(s_k)
\end{aligned} \tag{2.50}$$

Otherwise $P(s_1,\ s_2,\ ..., s_m,\ ..., s_k)$ is unseparable

Once we get a complex polynomial with one-variable from separable or unseparable real polynomial with multivariables, then the stability methods of sections 2.2, 2.3, and 2.4 can be applied

2.6.1 Stability test for multivariable polynomials using the J-fraction expansion

Consider a multivariable real polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$ of degree n in s_i, $i = 1, 2, ..., k$. This polynomial will be transferred to one-variable polynomial in s_m with complex coefficients by putting $s_i = j\omega_i$, $i = 1, 2, 3, ..., k$, and $i \neq m$.

$$\begin{aligned} P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k) &= s_m^n + (a_{1m} + jb_{1m})s_m^{n-1} + (a_{2m} + jb_{2m})s_m^{n-2} + \\ &+ \ ... + (a_{n-1,m} + jb_{n-1,m})s_m + (a_{nm} + jb_{nm}) \end{aligned} \tag{2.51}$$

where the coefficients a_{im} and b_{im} are functions of $\{j\omega_1, j\omega_2, ..., j\omega_k\}$.
From $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$, the alternant polynomial will be

$$Q(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k) = a_{1m}s_m^{n-1} + jb_{2m}s_m^{n-2} + a_{3m}s_m^{n-3} + jb_{4m}s_m^{n-4} + ... \tag{2.52}$$

The J-fraction expansion of $Q(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)/P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ takes the form

$$\frac{Q(j\omega_1, .., s_m, .., j\omega_k)}{P(j\omega_1, .., s_m, .., j\omega_k)} = \cfrac{1}{F_{1m}s_m + 1 + E_{1m} + \cfrac{1}{F_{2m}s_m + E_{2m} + \cfrac{1}{F_{3m}s_m + E_{3m} + \cfrac{1}{\ddots + \cfrac{1}{F_{nm}s_m + E_{nm}}}}}} \tag{2.53}$$

where F_{im} and E_{im} are functions of $\{j\omega_1, j\omega_2, ..., j\omega_k\}$. The stability condition for multivariable polynomial is stated in the following theorem:

Theorem 2.4 *If a multivariable polynomial $P(s_1,..,s_m,..,s_k)$ is stable, then for each variable s_m, the complex polynomial $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ has a J-fraction of the form (2.53) and all $F_{1m}, F_{2m}, ..., F_{nm}$ are real and positive and $E_{1m}, E_{2m}, ..., E_{nm}$ are pure imaginary or zero. This is valid for all m=1,2,3,...,k.*

Proof

When $P(s_1, s_2, ..., s_m, ..., s_k)$ is a stable polynomial, then for $s_i = j\omega_i$, $i = 1, 2, ..., k$, $i \neq m$, the polynomial $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$, is stable with the J-fraction as in Equation (2.53) and all $F_{1m}, F_{2m}, F_{3m}, ..., F_{nm}$ are real and positive, otherwise, Theorem 2.1 would be violated. **Q. E. D.**

Now for each variable s_m in $P(s_1, s_2, ..., s_m, ..., s_k)$, a criterion for determining the number of roots of $P(j\omega_1, .., s_m, .., j\omega_k)$ in the left half and right half of the complex s_m-plane is given below.

Criterion 2.4 *For each variable s_m of any multivariable real polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$, the number of roots lying on the right half-plane Re $s_m > 0$ is equal to the number of negative signs of that F_{im} in the J-fraction (2.53).*

To illustrate the analysis, consider the example of Bose's paper [13]. His analysis deals with a real multivariable polynomial using a resultant matrix to test stability for a two dimensional filter. But in this analysis we start with a multivariable polynomial with real coefficients, and the stability test is based on complex polynomials.

Example 2.8 *It is required to determine whether $P(s_1, s_2)$ is stable or not*

$$\begin{aligned} P(s_1, s_2) &= \sum_{n1=0}^{2} \sum_{n2=0}^{2} A_{n1,n2}\, s_2^{n1}\, s_1^{n2} \\ &= 2s_2^2 s_1^2 + 2s_2^2 s_1^1 + s_2^2 s_1^0 + 2s_2^1 s_1^2 + 4s_2^1 s_1^1 + 2s_2^1 s_1^0 + s_2^0 s_1^2 + 2s_2^0 s_1^1 + 2s_2^0 s_1^0 \end{aligned} \tag{2.54}$$

by putting $s_1 = j\omega_1$ in the two-variable real polynomial in (2.54) we get a single variable polynomial with complex coefficients as

$$P(j\omega_1, s_2) = (-2\omega_1^2 + j2\omega_1^1 + 1)s_2^2 + (-2\omega_1^2 + j4\omega_1^1 + 2)s_2^1 + (-\omega_1^2 + j2\omega_1^1 + 2) \quad (2.55)$$

where $0 \leq \omega_1 \leq \infty$. We can write (2.55) in this form

$$P(j\omega_1, s_2)/\kappa_1 = s_2^2 + \left[\frac{(-2\omega_1^2 + j4\omega_1^1 + 2)}{(-2\omega_1^2 + j2\omega_1^1 + 1)}\right] s_2^1 + \left[\frac{(-\omega_1^2 + j2\omega_1^1 + 2)}{(-2\omega_1^2 + j2\omega_1^1 + 1)}\right] \quad (2.56)$$

where

$$\kappa_1 = (-2\omega_1^2 + j2\omega_1^1 + 1) \quad (2.57)$$

or in the form of equation (2.51):

$$P(j\omega_1, s_2)/\kappa_1 = s_2^2 + \left[\frac{(4\omega_1^4 + 2\omega_1^2 + 2)}{(4\omega_1^4 + 1)} + j\frac{(-4\omega_1^3)}{(4\omega_1^4 + 1)}\right] s_2^1 + \left[\frac{(2\omega_1^4 - 2\omega_1^2 + 2)}{(4\omega_1^4 + 1)} + j\frac{(-2\omega_1^3 - 2\omega_1^1)}{(4\omega_1^4 + 1)}\right] \quad (2.58)$$

The alternant $Q(j\omega_1, s_2)/\kappa_1$ polynomial of (2.58) is

$$Q(j\omega_1, s_2)/\kappa_1 = \left[\frac{(4\omega_1^4 + 2\omega_1^2 + 2)}{(4\omega_1^4 + 1)}\right] s_2^1 + \left[j\frac{(-2\omega_1^3 - 2\omega_1^1)}{(4\omega_1^4 + 1)}\right] \quad (2.59)$$

For the variable s_2, the J-fraction of $Q(j\omega_1, s_2)/P(j\omega_1, s_2)$ as in (2.53) becomes

$$\frac{Q(j\omega_1, s_2)}{P(j\omega_1, s_2)} = \frac{1}{F_{11}s_2 + 1 + E_{11} + \frac{1}{F_{21}s_2 + E_{11}}} \quad (2.60)$$

where F_{11}, F_{21}, and E_{11}, E_{21} are functions of $\{j\omega_1\}$. For any value of ω_1, $0 \leq \omega_1 \leq \infty$ we found expressions of F_{11} and F_{21}. The values of these expressions have positive signs for all values of ω_1. This means that $P(j\omega_1, s_2)$ is a stable polynomial. As a numerical calculation for $\omega_1 = 4$, equation (2.56) becomes

$$P(j4, s_2)/\kappa_1 = s_2^2 + (0.399 - j0.6712)s_2^1 + (0.4859 - j0.1327) \quad (2.61)$$

and the alternant $Q(j4, s_2)/\kappa_1$ polynomial of (2.61) is

$$Q(j4, s_2)/\kappa_1 = 0.399s_2^1 - j0.1327 \quad (2.62)$$

For the variable s_2, the J-fraction of $Q(j4,s_2)/P(j4,s_2)$ as in (2.53) becomes

$$\frac{Q(j4,s_2)}{P(j4,s_2)} = \frac{1}{2.50633s_2 + 1 - j0.8487 + \frac{1}{0.6667s_2 - j0.2217}} \tag{2.63}$$

From equation (2.63) it is clear that all the coefficients of s_2 are real positive which means that $P(j4,s_2)$ is stable, the same is true for any value of ω_1.

Also by putting $s_2 = j\omega_2$ in the two-variable real polynomial in (2.54) we get another single variable polynomial with complex coefficients:

$$P(s_1,j\omega_2) = (-2\omega_2^2 + j2\omega_2^1 + 1)s_2^2 + (-2\omega_2^2 + j4\omega_2^1 + 2)s_2^1 + (-\omega_2^2 + j2\omega_2^1 + 2) \tag{2.64}$$

where $0 \leq \omega_2 \leq \infty$. We can write (2.64) in the form:

$$P(s_1,j\omega_2)/\kappa_2 = s_1^2 + \left[\frac{(-2\omega_2^2 + j4\omega_2^1 + 2)}{(-2\omega_2^2 + j2\omega_2^1 + 1)}\right] s_1^1 + \left[\frac{(-\omega_2^2 + j2\omega_2^1 + 2)}{(-2\omega_2^2 + j2\omega_2^1 + 1)}\right] \tag{2.65}$$

where

$$\kappa_2 = (-2\omega_2^2 + j2\omega_2^1 + 1) \tag{2.66}$$

or in the form of equation (2.51)

$$P(s_1,j\omega_2)/\kappa_2 = s_1^2 + \left[\frac{(4\omega_2^4+2\omega_2^2+2)}{(4\omega_2^4+1)} + j\frac{(-4\omega_2^3)}{(4\omega_2^4+1)}\right] s_1^1 + \left[\frac{(2\omega_2^4-2\omega_2^2+2)}{(4\omega_2^4+1)} + j\frac{(-2\omega_2^3-2\omega_2^1)}{(4\omega_2^4+1)}\right] \tag{2.67}$$

The alternant $Q(s_1,j\omega_2)/\kappa_2$ polynomial of (2.67) is

$$Q(s_1,j\omega_2)/\kappa_2 = \left[\frac{(4\omega_2^4 + 2\omega_2^2 + 2)}{(4\omega_2^4 + 1)}\right] s_1^1 + \left[j\frac{(-2\omega_2^3 - 2\omega_2^1)}{(4\omega_2^4 + 1)}\right] \tag{2.68}$$

and for the variable s_1, the J-fraction of $Q(s_1,j\omega_2)/P(s_1,j\omega_2)$ as in (2.53) becomes

$$\frac{Q(s_1,j\omega_2)}{P(s_1,j\omega_2)} = \frac{1}{F_{12}s_1 + 1 + E_{12} + \frac{1}{F_{22}s_1 + E_{22}}} \tag{2.69}$$

where F_{12}, F_{22} and E_{12}, E_{22} are functions of $j\omega_2$. For any value of ω_2, $0 \leq \omega_2 \leq \infty$, we also found expressions of F_{12} and F_{22}. The values of these expressions have positive signs for all values of ω_2. This means that $P(s_1,j\omega_2)$ is a stable polynomial. Since $P(s_1,j\omega_2)$ and $P(j\omega_1,s_2)$ are stable polynomials this leads to the stability of $P(s_1,s_2)$. As a numerical calculation for $\omega_2 = 4$, equation (2.65) becomes

$$P(s_1,j4)/\kappa_2 = s_1^2 + (0.399 - j0.6712)s_1^1 + (0.4859 - j0.1327) \tag{2.70}$$

and the alternant $Q(s_1, j4)/\kappa_2$ polynomial of (2.70) is

$$Q(s_1, j4)/\kappa_2 = 0.399s_1^1 - j0.1327 \qquad (2.71)$$

For the variable s_1, the J-fraction expansion of $Q(s_1, j4)/P(s_1, j4)$ as in (2.53) becomes

$$\frac{Q(s_1, j4)}{P(s_1, j4)} = \frac{1}{2.5063s_1 + 1 - j0.8487 + \frac{1}{0.6667s_1 - j0.2217}} \qquad (2.72)$$

From (2.72) it is clear that all the coefficients of s_1 are real positive, which means that $P(s_1, j4)$ is stable.

The previous example was given in Bose's paper [13] and it is a second-order two-variables real polynomial. The next example is to test the stability of a fourth-order two-variables real polynomial

Example 2.9 *Consider a fourth order real polynomial with two variables, it is required to study the stability of $P(s_1, s_2)$*

$$\begin{aligned} P(s_1, s_2) &= \textstyle\sum_{n1=0}^{4} \sum_{n2=0}^{4} A_{n1,n2}\, s_2^{n1}\, s_1^{n2} \\ &= 2s_2^4 s_1^4 + 6s_2^4 s_1^3 + 9s_2^4 s_1^2 + 6s_2^4 s_1^1 + 2s_2^4 s_1^0 + \\ &+ 5s_2^3 s_1^4 + 20s_2^3 s_1^3 + 37s_2^3 s_1^2 + 38s_2^3 s_1^1 + 20s_2^3 s_1^0 + \\ &+ 7s_2^2 s_1^4 + 28s_2^2 s_1^3 + 62s_2^2 s_1^2 + 60s_2^2 s_1^1 + 23s_2^2 s_1^0 + \\ &+ 4s_2^1 s_1^4 + 22s_2^1 s_1^3 + 45s_2^1 s_1^2 + 36s_2^1 s_1^1 + 13s_2^1 s_1^0 + \\ &+ 2s_2^0 s_1^4 + 6s_2^0 s_1^3 + 9s_2^0 s_1^2 + 6s_2^0 s_1^1 + 2s_2^0 s_1^0 \end{aligned} \qquad (2.73)$$

The two variable polynomial in (2.73) will transform to a one variable polynomial in s_2 with complex coefficients by setting $s_1 = j\omega_1$

$$\begin{aligned} P(j\omega_1, s_2) &= (2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)s_2^4 + \\ &+ (5\omega_1^4 - j20\omega_1^3 - 37\omega_1^2 + j38\omega_1 + 20)s_2^3 + \\ &+ (7\omega_1^4 - j28\omega_1^3 - 62\omega_1^2 + j60\omega_1 + 23)s_2^2 + \\ &+ (4\omega_1^4 - j22\omega_1^3 - 45\omega_1^2 + j36\omega_1 + 13)s_2^1 + \\ &+ (2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)s_2^0 \end{aligned} \qquad (2.74)$$

where $0 \leq \omega_1 \leq \infty$. We can write (2.74) in this form

$$P(j\omega_1, s_2)/\kappa_1 = s_2^4 + \left[\frac{(5\omega_1^4 - j20\omega_1^3 - 37\omega_1^2 + j38\omega_1 + 20)}{(2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)}\right] s_2^3 + \left[\frac{(7\omega_1^4 - j28\omega_1^3 - 62\omega_1^2 + j60\omega_1 + 23)}{(2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)}\right] s_2^2 +$$

$$+ \left[\frac{(4\omega_1^4 - j22\omega_1^3 - 45\omega_1^2 + j36\omega_1 + 13)}{(2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)}\right] s_2^1 + \left[\frac{(2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)}{(2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2)}\right] s_2^0 \tag{2.75}$$

where

$$\kappa_1 = (2\omega_1^4 - j6\omega_1^3 - 9\omega_1^2 + j6\omega_1 + 2) \tag{2.76}$$

or in the form of equation (2.51)

$$P(j\omega_1, s_2)/\kappa_1 = s_2^4 + \left[\frac{(10\omega_1^8 + \omega_1^6 + 35\omega_1^4 - 26\omega_1^2 + 40)}{(4\omega_1^8 + 17\omega_1^4 + 4)} + j\frac{(-10\omega_1^7 + 4\omega_1^5 - 40\omega_1^3 - 44\omega_1^1)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^3$$

$$+ \left[\frac{(14\omega_1^8 - 19\omega_1^6 + 90\omega_1^4 + 29\omega_1^2 + 46)}{(4\omega_1^8 + 17\omega_1^4 + 4)} + j\frac{(-14\omega_1^7 - 42\omega_1^5 - 86\omega_1^3 - 18\omega_1^1)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^2$$

$$+ \left[\frac{(8\omega_1^8 + 6\omega_1^6 + 91\omega_1^4 + 9\omega_1^2 + 26)}{(4\omega_1^8 + 17\omega_1^4 + 4)} + j\frac{(-20\omega_1^7 - 24\omega_1^5 - 20\omega_1^3 - 6\omega_1^1)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^1 + 1 \tag{2.77}$$

The alternant polynomial $Q(j\omega_1, s_2)/\kappa_1$ of (2.77) is

$$Q(j\omega_1, s_2)/\kappa_1 = + \left[\frac{(10\omega_1^8 + \omega_1^6 + 35\omega_1^4 - 26\omega_1^2 + 40)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^3 + \left[+j\frac{(-14\omega_1^7 - 42\omega_1^5 - 86\omega_1^3 - 18\omega_1^1)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^2$$

$$+ \left[\frac{(8\omega_1^8 + 6\omega_1^6 + 91\omega_1^4 + 9\omega_1^2 + 26)}{(4\omega_1^8 + 17\omega_1^4 + 4)}\right] s_2^1 \tag{2.78}$$

For the variable s_2, the J-fraction of $Q(j\omega_1, s_2)/P(j\omega_1, s_2)$ as in (2.53) becomes

$$\frac{Q(j\omega_1, s_2)}{P(j\omega_1, s_2)} = \cfrac{1}{F_{11}s_2 + 1 + E_{11} + \cfrac{1}{F_{21}s_2 + E_{21} + \cfrac{1}{F_{31}s_2 + E_{31} + \cfrac{1}{F_{41}s_2 + E_{41}}}}} \tag{2.79}$$

where F_{11}, F_{21}, F_{31}, F_{41}, and E_{11}, E_{21}, E_{41}, E_{41}, are functions of $j\omega_1$. For any value of ω_1, $0 \leq \omega_1 \leq \infty$ we found expressions of F_{11}, F_{21}, F_{31} and F_{41}. The values of these expressions have positive sign for all values of ω_1. This means that $P(j\omega_1, s_2)$ is stable polynomial. Also for $s_2 = j\omega_2$ we have another complex polynomial in s_1

$$
\begin{aligned}
P(s_1, j\omega_2) \;=\; & (2\omega_2^4 - j5\omega_2^3 - 7\omega_2^2 + j4\omega_1 + 2)s_1^4 + \\
+ \; & (6\omega_2^4 - j20\omega_2^3 - 28\omega_2^2 + j22\omega_2 + 6)s_1^3 + \\
+ \; & (9\omega_2^4 - j37\omega_2^3 - 62\omega_2^2 + j45\omega_2 + 9)s_1^2 + \\
+ \; & (6\omega_2^4 - j38\omega_2^3 - 60\omega_2^2 + j36\omega_2 + 6)s_1^1 + \\
+ \; & (2\omega_2^4 - j20\omega_2^3 - 23\omega_2^2 + j13\omega_2 + 2)s_1^0
\end{aligned}
\tag{2.80}
$$

where $0 \leq \omega_2 \leq \infty$. We can write (2.80) in the form:

$$
\begin{aligned}
P(s_1, j\omega_2)/\kappa_1 \;=\; & s_1^4 + \left[\frac{(6\omega_2^4 - j20\omega_2^3 - 28\omega_2^2 + j22\omega_2 + 6)}{(2\omega_2^4 - j5\omega_2^3 - 7\omega_2^2 + j4\omega_2 + 2)}\right] s_1^3 + \left[\frac{(9\omega_2^4 - j37\omega_2^3 - 62\omega_2^2 + j45\omega_2 + 9)}{(2\omega_2^4 - j5\omega_2^3 - 7\omega_2^2 + j4\omega_2 + 9)}\right] s_1^2 + \\
+ \; & \left[\frac{(6\omega_2^4 - j38\omega_2^3 - 60\omega_2^2 + j36\omega_2 + 6)}{(2\omega_2^4 - j5\omega_1^3 - 7\omega_1^2 + j4\omega_1 + 2)}\right] s_1^1 + \left[\frac{(2\omega_2^4 - j20\omega_1^3 - 23\omega_2^2 + j13\omega_2 + 2)}{(2\omega_2^4 - j5\omega_1^3 - 7\omega_1^2 + j4\omega_2 + 2)}\right] s_1^0
\end{aligned}
\tag{2.81}
$$

where

$$
\kappa_2 \;=\; (2\omega_2^4 - j5\omega_2^3 - 7\omega_2^2 + j4\omega_2 + 2) \tag{2.82}
$$

or in the form of equation (2.51)

$$
\begin{aligned}
P(s_2, j\omega_2)/\kappa_2 = s_1^4 \; & + \left[\frac{(12\omega_2^8 + 2\omega_2^6 + 3\omega_2^4 - 10\omega_2^2 + 12)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)} + j\,\frac{(-10\omega_2^7 + 20\omega_2^5 - 52\omega_2^3 - 20\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^3 \\
& + \left[\frac{(18\omega_2^8 - 2\omega_2^6 + 97\omega_2^4 - 7\omega_2^2 + 18)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)} + j\,\frac{(-29\omega_2^7 + 3\omega_2^5 - 96\omega_2^3 + 54\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^2 \\
& + \left[\frac{(12\omega_2^8 + 28\omega_2^6 + 112\omega_2^4 - 18\omega_2^2 + 12)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)} + j\,\frac{(-46\omega_2^7 + 14\omega_2^5 - 58\omega_2^3 + 48\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^1 \\
& + \left[\frac{(4\omega_2^8 + 40\omega_2^6 + 24\omega_2^4 - 8\omega_2^2 + 4)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)} + j\,\frac{(-30\omega_2^7 + 43\omega_2^5 - 29\omega_2^3 + 18\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^0
\end{aligned}
\tag{2.83}
$$

The alternant $Q(s_1, j\omega_2)/\kappa_2$ polynomial of (2.83) is

$$
\begin{aligned}
Q(s_1, j\omega_2)/\kappa_2 = \; & \left[\frac{(12\omega_2^8 + 2\omega_2^6 + 3\omega_2^4 - 10\omega_2^2 + 12)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^3 + \left[j\,\frac{(-29\omega_2^7 + 3\omega_2^5 - 96\omega_2^3 + 54\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^2 \\
& + \left[\frac{(12\omega_2^8 + 28\omega_2^6 + 112\omega_2^4 - 18\omega_2^2 + 12)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^1 + \left[j\,\frac{(-30\omega_2^7 + 43\omega_2^5 - 29\omega_2^3 + 18\omega_2^1)}{(4\omega_2^8 - 3\omega_2^6 + 17\omega_2^4 - 12\omega_2^2 + 4)}\right] s_1^0
\end{aligned}
\tag{2.84}
$$

For the variable s_1, the J-fraction of $Q(s_1, j\omega_2)/P(s_1, j\omega_2)$ as in (2.53) becomes

$$\frac{Q(s_1, j\omega_2)}{P(s_1, j\omega_2)} = \frac{1}{F_{12}s_1 + 1 + E_{12} + \frac{1}{F_{22}s_1 + E_{22} + \frac{1}{F_{32}s_1 + E_{32} + \frac{1}{F_{42}s_1 + E_{42}}}}} \tag{2.85}$$

where F_{12}, F_{22}, F_{32}, F_{42}, and E_{12}, E_{22}, E_{32}, E_{42}, are functions of $j\omega_2$. For any value of ω_2, $0 \leq \omega_2 \leq \infty$, we found expressions of F_{12}, F_{22}, F_{32}, F_{42}. The values of these expressions have positive signs for all values of ω_2. This means that $P(s_1, j\omega_2)$ is a stable polynomial. Since $P(s_1, j\omega_2)$ and $P(j\omega_1, s_2)$ are stable polynomials, this leads to the stability of $P(s_1, s_2)$. As a numerical calculation for $\omega_1 = 4$, equation (2.74) becomes

$$P(j4, s_2)/\kappa_1 = s_2^4 + (2.5067 - j0.6097)s_2^3 + (3.2836 - j0.9998)s_2^2 + (2.1476 - j1.3267)s_2 + 1 \tag{2.86}$$

and the $Q(j4, s_2)/\kappa_1$ polynomial of (2.86) is

$$Q(j4, s_2)/\kappa_1 = 2.5067s_2^3 - j0.9998s_2^2 + 2.1476s_2 \tag{2.87}$$

The J-fraction of $Q(j4, s_2)/P(j4, s_2)$ for the variable s_2 becomes

$$\frac{Q(j4, s_2)}{P(j4, s_2)} = \frac{1}{0.3989s_2 + 1 - j0.0841 + \frac{1}{0.9983s_2 + j0.0575 + \frac{1}{2.3177s - j0.9394 + \frac{1}{1.0271s - j0.0546}}}} \tag{2.88}$$

Since all the coefficients of s_2 in (2.88) are real positive, $P(j4, s_2)$ is stable. Also for $\omega_2 = 4$, equation (2.80) becomes

$$\begin{aligned} P(s_1, j4)/\kappa_2 &= s_1^4 + (3.1578 - j0.5772)s_1^3 + (4.7597 - j1.8434)s_1^2 + \\ &+ (3.6592 - j2.9244)s_1 + (1.7007 - j1.7686) \end{aligned} \tag{2.89}$$

and the $Q(s_1, j4)/\kappa_2$ polynomial of (2.89) is

$$Q(s_1, j4)/\kappa_2 = 3.1578s_1^3 - j1.8434s_1^2 + 3.6592s_1 - j1.7686 \tag{2.90}$$

For the variable s_1, the J-fraction of $Q(s_1, j4)/P(s_1, j4)$ becomes

$$\frac{Q(s_1,j4)}{P(s_1,j4)} = \cfrac{1}{0.3167s_1 + 1 + j0.0021 + \cfrac{1}{0.8779s_1 + j0.0664 + \cfrac{1}{1.7878s_1 + j0.4928 + \cfrac{1}{2.6131s_1 - j2.4436}}}} \tag{2.91}$$

Since all the coefficients of s_1 in (2.91) are real and positive, $P(s_1, j\omega_2)$ is stable.

2.6.2 Stability test for multivariable polynomials using the generalized Routh array

- Obtain $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ and its $Q(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ for each variable s_m, from multivariable real polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$.

- Arrange the coefficients of the polynomials $P(j\omega_1, .., s_m, .., j\omega_k)$ and $Q(j\omega_1, ..., s_m, .., j\omega_k)$ in rows and columns according to the following pattern:

s_m^n	1	$a_{1m} + jb_{1m}$	$a_{2m} + jb_{2m}$	....	$a_{nm} + jb_{nm}$	$\leftarrow P(j\omega_1, .., s_m, ., j\omega_k)$
s_m^{n-1}	a_{1m}	jb_{2m}	a_{3m}	....	a_{nm} or jb_{nm}	$\leftarrow Q(j\omega_1, .., s_m, .., j\omega_k)$
s_m^{n-2}	c_{1m}	c_{2m}	c_{3m}	....	0	$\leftarrow$ *3nd row* c_{im}
s_m^{n-3}	d_{1m}	d_{2m}	d_{3m}	....	0	$\leftarrow$ *4nd row* d_{im}
....	....	....	....	. ..	...	....
s_m^1	g_{1m}	g_{2m}	0	0	0	$\leftarrow (n)th$ *row* g_{im}
s_m^0	h_{1m}	0	0	0	0	$\leftarrow (n+1)th$ *row* h_{im}
0	0	0	0	0	0	$\leftarrow -(n+2)th$ *row*

(2.92)

The coefficients $c_{im}, d_{im}, e_{im}, ...$,etc., are evaluated using two cross multiplying processes as illustrated in Section 2.3.

The stability conditions for a multivariable polynomial are stated in the following theorem:

Theorem 2.5 *If a multivariable real polynomial is stable, then for each variable s_m in $P(s_1, s_2, ..., s_m, ..., s_k)$, all the coefficients in the first column of the Routh array (2.92), constructed from $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ and $Q(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ have positive signs.*

Proof

When $P(s_1, s_2, ..., s_m, ..., s_k)$ is a stable polynomial, then for $s_i = j\omega_i$, $i = 1, 2, ..., k$, $i \neq m$, the polynomial $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$, is stable with the Routh array as in (2.92) and all the elements in the first column $a_{1m}, c_{1m}, d_{1m}, ...$ are real and positive, or otherwise Theorem 2.2 would be violated. **Q. E. D.**

The number of roots in the left half and right half of s_m-plane can be determined from Criterion 2.5

Criterion 2.5 *In any multivariable polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$, for each variable s_m, the number of roots lying on the right half-plane, Re $s_m > 0$ is equal to the number of changes in the sign of the coefficients in the first column of the generalized Routh array pattern (2.92).*

To illustrate how we could use the generalized Routh array for a complex polynomial to test stability of a two variable polynomial, consider the following example

Example 2.10 *Consider a fourth order real polynomial with two variables, $P(s_1, s_2)$ as in Example 2.9. It is required to use the generalized Routh array to test stability of $P(s_1, s_2)$.*

For the variables s_1 and s_2 in $P(s_1, s_2)$, obtain the two complex polynomials $P(j\omega_1, s_2)$ and $P(s_1, j\omega_2)$. Now the pattern of the Routh array for $P(s_1, j\omega_2)$ is given by equation (2.93)

$$
\begin{array}{|c|cccccc|}
\hline
s_1^4 & 1 & a_{11}+jb_{11} & a_{21}+jb_{21} & a_{31}+jb_{31} & a_{41}+jb_{41} \\
s_1^3 & a_{11} & jb_{21} & a_{31} & jb_{41} & 0 \\
s_1^2 & c_{11} & c_{21} & c_{31} & 0 & 0 \\
s_1^1 & d_{11} & d_{21} & 0 & 0 & 0 \\
s_1^0 & e_{11} & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad (2.93)
$$

and the pattern of the Routh array for $P(j\omega_1, s_2)$ is given by equation (2.94)

$$
\begin{array}{|c|cccccc|}
\hline
s_1^4 & 1 & a_{12}+jb_{12} & a_{22}+jb_{22} & a_{32}+jb_{32} & a_{42}+jb_{42} \\
s_1^3 & a_{12} & jb_{22} & a_{32} & jb_{42} & 0 \\
s_1^2 & c_{12} & c_{22} & c_{32} & 0 & 0 \\
s_1^1 & d_{12} & d_{22} & 0 & 0 & 0 \\
s_1^0 & e_{12} & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad (2.94)
$$

The first column in the two generalized Routh arrays are:

$$
\begin{array}{cccc}
 & \begin{array}{c} The\ first\ column\ in\ the \\ generalized\ Routh\ array \\ for\ P(s_1, j\omega_2) \end{array} & & \begin{array}{c} The\ first\ column\ in\ the \\ generalized\ Routh\ array \\ for\ P(j\omega_1, s_2) \end{array} \\
s_1^4 & 1 & s_2^4 & 1 \\
s_1^3 & a_{11} & s_2^3 & a_{12} \\
s_1^2 & c_{11} & s_2^2 & c_{12} \\
s_1^1 & d_{11} & s_2^1 & d_{12} \\
s_1^0 & e_{11} & s_2^0 & e_{12}
\end{array}
\qquad (2.95)
$$

The coefficients $a_{11}, c_{11}, d_{11}, e_{11}$, are functions of $j\omega_2$. For any value of ω_2, $0 \leq \omega_2 \leq \infty$, we found expressions of $a_{11}, c_{11}, d_{11}, e_{11}$. The values of these expressions have positive signs for all values of ω_2. This means that $P(s_1, j\omega_2)$ is a stable polynomial. Also the coefficients $a_{12}, c_{12}, d_{12}, e_{12}$, are functions of $j\omega_1$. For any value of ω_1, $0 \leq \omega_1 \leq \infty$, we found expressions of $a_{12}, c_{12}, d_{12}, e_{12}$. The values of these expressions have positive signs for all values of ω_1. This means that $P(j\omega_1, s_2)$ is stable polynomial.

Stability of $P(s_1, j\omega_2)$ and $P(j\omega_1, s_2)$ leads to the stability of $P(s_1, s_2)$. Consider a numerical calculation for $\omega_1 = 4$ and $\omega_2 = 4$. The first column in the two generalized Routh array now are

	The first column in the generalized Routh array for $P(s_1, j4)$		*The first column in the generalized Routh array for $P(j4, s_2)$*
s_1^4	1	s_2^4	1
s_1^3	3.1578	s_2^3	2.5067
s_1^2	3.5971	s_2^2	2.511
s_1^1	2.012	s_2^1	1.0834
s_1^0	0.7699	s_2^0	1.0538

(2.96)

2.6.3 Stability test for multivariable polynomials using the generalized Hurwitz determinant.

- For each variable s_m in any multivariable real polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$, arrange the coefficients of $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ and $Q(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$ in the form of the following matrix:

$$A_m = \begin{pmatrix} a_{1m} & jb_{2m} & a_{3m} & jb_{4m} & a_{5m} & \cdots & \cdots \\ 1 & a_{1m}+jb_{1m} & a_{2m}+jb_{2m} & a_{3m}+jb_{3m} & a_{4m}+jb_{4m} & \cdots & \cdots \\ 0 & a_{1m} & jb_{2m} & a_{3m} & jb_{4m} & \cdots & \cdots \\ 0 & 1 & a_{1m}+jb_{1m} & a_{2m}+jb_{2m} & a_{3m}+jb_{3m} & \cdots & \cdots \\ 0 & 0 & a_{1m} & jb_{2m} & a_{3m} & \cdots & \cdots \\ 0 & 0 & 1 & a_{1m}+jb_{1m} & a_{2m}+jb_{2m} & \cdots & \cdots \\ 0 & 0 & 0 & a_{1m} & jb_{2m} & \cdots & \cdots \\ 0 & 0 & 0 & 1 & a_{1m}+jb_{1m} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{matrix} \leftarrow Q(j\omega_1, ..., s_m, ..., j\omega_k) \\ \leftarrow P(j\omega_1, ..., s_m, ..., j\omega_k) \\ \leftarrow Q(j\omega_1, ..., s_m, ..., j\omega_k) \\ \leftarrow P(j\omega_1, ..., s_m, ..., j\omega_k) \\ \cdots\ \cdots \\ \cdots\ \cdots \\ \cdots\ \cdots \\ \cdots\ \cdots \\ \cdots\ \cdots \\ \cdots\ \cdots \end{matrix} \quad (2.97)$$

The determinant of the matrix A_m of rank $(2n-1)$, is the generalized Hurwitz determinant for the variable s_m. The stability conditions for a multivariable polynomial is stated in the following theorem:

Theorem 2.6 ***If a multivariable real polynomial is stable, then for each variable s_m in $P(s_1, s_2, ..., s_m, ..., s_k)$***

$$\Delta_0 = 1 \quad , \quad \Delta_{qm} = (-1)^{\frac{q^2-1}{8}} D_{qm} > 0 \ , \qquad q = 1,3,5,....., (2n-1) \quad (2.98)$$

where D_{qm} are the principal minors of the generalized Hurwitz determinant of the matrix A_m.

Proof

If $P(s_1,\ s_2,\ ..., s_m,\ ..., s_k)$ is a stable polynomial, then, for $s_i = j\omega_i$, $i = 1, 2, ..., k$, $i \neq m$, the polynomial $P(j\omega_1, j\omega_2, ..., s_m, ..., j\omega_k)$, is stable with the Hurwitz matrix as in equation (2.97) and all the modified generalized Hurwitz determinant $\Delta_{1m}, \Delta_{1m}, \Delta_{1m}, ...$ are real and positive, otherwise Theorem 2.3 would be violated. **Q. E. D.**

The number of roots in each of the half-planes $Re\ s_m > 0$, and $Re\ s_m < 0$, can be determined from the criterion:

Criterion 2.6 ***For any multivariable polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$, the number of roots that lie on the Re $s_m > 0$ plane is equal to the number of changes in the sign of the odd determinants Δ_{qm} in (2.98).***

To illustrate how one can use the generalized Hurwitz determinant for a complex polynomial to test the stability of a two variable polynomial, consider the following example.

Example 2.11 *Consider the fourth order real polynomial with two variables, as in Example 2.9. It is required to use the generalized Hurwitz determinant to test the stability of* $P(s_1, s_2)$

For the variables s_1 and s_2 in $p(s_1, s_2)$, obtain the two complex polynomials $P(j\omega_1, s_2)$ and $P(s_1, j\omega_2)$. Now the Hurwitz matrix for $P(s_1, j\omega_2)$ is given by equation (2.99)

$$\Lambda_1 = \begin{pmatrix} a_{11} & jb_{21} & a_{31} & jb_{41} & 0 & 0 & 0 \\ 1 & a_{11}+jb_{11} & a_{21}+jb_{21} & a_{31}+jb_{31} & a_{41}+jb_{41} & \cdots & \cdots \\ 0 & a_{11} & jb_{21} & a_{31} & jb_{41} & 0 & 0 \\ 0 & 1 & a_{11}+jb_{11} & a_{21}+jb_{21} & a_{31}+jb_{31} & a_{41}+jb_{41} & 0 \\ 0 & 0 & a_{11} & jb_{21} & a_{31} & jb_{41} & 0 \\ 0 & 0 & 1 & a_{11}+jb_{11} & a_{21}+jb_{21} & a_{31}+jb_{31} & a_{41}+jb_{41} \\ 0 & 0 & 0 & a_{11} & jb_{21} & a_{31} & jb_{41} \end{pmatrix}. \tag{2.99}$$

and the Hurwitz matrix for $P(j\omega_1, s_2)$ is given by equation (2.100)

$$\Lambda_2 = \begin{pmatrix} a_{12} & jb_{22} & a_{32} & jb_{42} & 0 & 0 & 0 \\ 1 & a_{12}+jb_{12} & a_{22}+jb_{22} & a_{32}+jb_{32} & a_{42}+jb_{42} & \cdots & \cdots \\ 0 & a_{12} & jb_{22} & a_{32} & jb_{42} & 0 & 0 \\ 0 & 1 & a_{12}+jb_{12} & a_{22}+jb_{22} & a_{32}+jb_{32} & a_{42}+jb_{42} & 0 \\ 0 & 0 & a_{12} & jb_{22} & a_{32} & jb_{42} & 0 \\ 0 & 0 & 1 & a_{12}+jb_{12} & a_{22}+jb_{22} & a_{32}+jb_{32} & a_{42}+jb_{42} \\ 0 & 0 & 0 & a_{12} & jb_{22} & a_{32} & jb_{42} \end{pmatrix}. \tag{2.100}$$

The odd successive principal minors of the generalized Hurwitz determinant D_{11}, D_{31}, D_{51}, D_{71} and the odd successive principal minors of the modified generalized Hurwitz determinant Δ_{11}, Δ_{31}, Δ_{51}, Δ_{71} from Equation 2.98, for the polynomial $P(s_1, j\omega_2)$ are given by:

$$\begin{aligned} D_{11} &= a_{11} & \Delta_{11} &= a_{11} \\ D_{31} &= -a_{11}^2 \ c_{11} & \Delta_{31} &= a_{11}^2 \ c_{11} \\ D_{51} &= -a_{11}^2 \ c_{11}^2 \ d_{11} & \Delta_{51} &= a_{11}^2 \ c_{11}^2 \ d_{11} \\ D_{71} &= a_{11}^2 \ c_{11}^2 \ d_{11}^2 \ e_{11} & \Delta_{71} &= a_{11}^2 \ c_{11}^2 \ d_{11}^2 \ e_{11} \end{aligned} \quad (2.101)$$

Also the odd successive principal minors of the generalized Hurwitz determinants D_{12}, D_{32}, D_{52}, D_{72} and the odd successive principal minors of the modified generalized Hurwitz determinants Δ_{11}, Δ_{32}, Δ_{52}, Δ_{72} for the polynomial $P(j\omega_1, s_2)$ are given by:

$$\begin{aligned} D_{12} &= a_{12} & \Delta_{12} &= a_{12} \\ D_{32} &= -a_{12}^2 \ c_{12} & \Delta_{32} &= a_{12}^2 \ c_{12} \\ D_{51} &= -a_{12}^2 \ c_{12}^2 \ d_{12} & \Delta_{52} &= a_{12}^2 \ c_{12}^2 \ d_{12} \\ D_{72} &= a_{12}^2 \ c_{12}^2 \ d_{12}^2 \ e_{12} & \Delta_{72} &= a_{12}^2 \ c_{12}^2 \ d_{12}^2 \ e_{12} \end{aligned} \quad (2.102)$$

All the Hurwitz determinants $D_{11}, D_{31}, D_{51}, D_{71}$, and the modified Hurwitz determinants $\Delta_{11}, \Delta_{31}, \Delta_{51}, \Delta_{71}$, are functions of $j\omega_2$. For any value of ω_2, $0 \leq \omega_2 \leq \infty$, we found expressions of $\Delta_{11}, \Delta_{31}, \Delta_{51} and \Delta_{71}$. The values of these expressions have positive signs for all values of ω_2. This means that $P(s_1, j\omega_2)$ is stable polynomial. Also all the Hurwitz determinants $D_{12}, D_{32}, D_{52}, D_{72}$, and the modified Hurwitz determinants $\Delta_{12}, \Delta_{32}, \Delta_{52}, \Delta_{72}$, are functions of $j\omega_1$. For any value of ω_1, $0 \leq \omega_1 \leq \infty$, we found expressions of $\Delta_{12}, \Delta_{32}, \Delta_{52} and \Delta_{72}$. The values of these expressions have positive signs for all values of ω_1. This means that $P(j\omega_1, s_2)$ is a stable polynomial. Stability of $P(s_1, j\omega_2)$ and $P(j\omega_1, s_2)$ implies the stability of $P(s_1, s_2)$. Consider a numerical calculation for $\omega_1 = 4$ and $\omega_2 = 4$. The odd successive principal minors of the generalized Hurwitz determinants D_{q1}, D_{q2} and the modified generalized Hurwitz determinant Δ_{q1}, Δ_{q2} for $P(j4, s_2)$ and $P(s_1, j4)$are

$$\begin{array}{llll} D_{11} = +3.1578 & D_{31} = -35.8691 & D_{51} = -259.59 & D_{71} = +402.04 \\ \Delta_{11} = +3.1578 & \Delta_{31} = +35.8691 & \Delta_{51} = +259.59 & \Delta_{71} = +402.04 \end{array} \tag{2.103}$$

$$\begin{array}{llll} D_{12} = +2.5067 & D_{32} = -15.7777 & D_{52} = -42.9221 & D_{72} = +49.0015 \\ \Delta_{12} = +2.5067 & \Delta_{32} = +15.7777 & \Delta_{52} = +42.9221 & \Delta_{72} = +49.0015 \end{array}$$

2.6.4 Relationships between J-fraction, the generalized Routh array, and the modified generalized Hurwitz determinant of multivariable real polynomials

In any multivariable real polynomial $P(s_1, s_2, ..., s_m, ..., s_k)$, for each variable s_m, the parameter interrelations among the coefficients F_{im} in J-fraction expansion, the elements $a_{1m}, b_{1m}, c_{1m}, d_{1m}, ...$ in the generalized Routh array and the odd successive principal minors of the modified generalized Hurwitz determinant Δ_{qm} are given in Table 2.5.

To / From	J-Fraction	Routh	Hurwitz
J-Fraction	$F_{0m} = 1$ F_{1m} F_{2m} F_{3m} F_{4m}	$a_{1m} = 1/F_{1m}$ $c_{1m} = 1/F_{1m}F_{2m}$ $d_{1m} = 1/F_{1m}F_{2m}F_{3m}$ $e_{1m} = 1/F_{1m}F_{2m}F_{3m}F_{4m}$	$\Delta_{1m} = 1/F_{1m}$ $\Delta_{3m} = 1/F^3_{1m}F_{2m}$ $\Delta_{5m} = 1/F^5_{1m}F^3_{2m}F_{3m}$ $\Delta_{7m} = 1/F^7_{1m}F^5_{2m}F^3_{3m}F_{4m}$ ....
Routh	$F_{1m} = 1/a_{1m}$ $F_{2m} = a_{1m}/c_{1m}$ $F_{3m} = c_{1m}/d_{1m}$ $F_{4m} = d_{1m}/e_{1m}$	1 a_{1m} c_{1m} d_{1m} e_{1m}	$\Delta_{0m} = 1$ $\Delta_{1m} = a_{1m}$ $\Delta_{3m} = a^2_{1m}c_{1m}$ $\Delta_{5m} = a^2_{1m}c^2_{1m}d_{1m}$ $\Delta_{7m} = a^2_{1m}c^2_{1m}d^2_{1m}e_{1m}$ ... .
Hurwitz	$F_{1m} = 1/\Delta_{1m}$ $F_{2m} = 1/_{3m}\Delta^3_{1m}$ $F_{3m} = 1/\Delta_{5m}/\Delta^3_{3m}/\Delta^4_{1m}$ $F_{4m} = 1/\Delta_{7m}/\Delta^3_{5m}/\Delta^4_{3m}/\Delta^4_{1m}$	$a_{1m} = 1/\Delta_{1m}$ $c_{1m} = \Delta_{3m}/\Delta^2_{1m}$ $d_{1m} = \Delta_{5m}/\Delta^2_{3m}/\Delta^2_{1m}$ $e_{1m} = \Delta_{7m}/\Delta^2_{5m}/\Delta^2_{3m}/\Delta^2_{1m}$	$\Delta_{0m} = 1$ Δ_{1m} Δ_{3m} Δ_{5m} Δ_{7m} ... .

Table 2.5: Parameter interrelations among the J-fraction, Routh array and Hurwitz determinant. for each variable s_m in $P(s_1, s_2, ..., s_m, ..., s_k)$. when $s_i = j\omega_i$, $i = 1, 2, ..., k$, $i \neq m$

A summary of the main contributions in chapter 2.

- Generalization of the Routh array of a complex polynomial $P(s)$ using division processes.
- Generalization of the Routh array using two cross multiplying processes.
- Derivation of stability conditions using the generalized Routh array.
- Stability relationship between J-fraction expansion and the generalized Routh array.
- Generalization of the Hurwitz matrix of a complex polynomial $P(s)$ and the modified generalized Hurwitz determinant.
- Condition for the stability of $P(s)$ using a modified generalized Hurwitz determinant.
- Stability relationship between the generalized Routh array and the modified generalized Hurwitz determinant.
- Relationship between J-fraction expansion, the generalized Routh array, and the modified generalized Hurwitz determinant.
- The interrelations among the constant coefficients in the J-fraction expansion, the terms in the first column of the generalized Routh array and the odd successive principal minors of the modified generalized Hurwitz determinant.
- Stability test for multivariable polynomials using the J-fraction expansion.
- Stability test for multivariable polynomials using the generalized Routh array.
- Stability test for multivariable polynomials using the generalized Hurwitz determinant.

Chapter 3

SYNTHESIS OF COMPLEX IMPEDANCE AND COMPLEX REACTANCE FUNCTIONS IN s-plane

3.1 Introduction

The importance of network synthesis procedures with real components is well known in circuit theory. This chapter addresses an extension to the present methods in circuit theory to include a new element, namely a pure imaginary resistor (jR). Some authors have already visualized an analytic investigation of circuits containing imaginary resistors [16, 17, 18].

In Chapter 2 we illustrated the use of J-fraction expansion to test the stability of any complex polynomial. In this chapter we shall illustrate how one can use this expansion to synthesize certain complex functions. These functions are rational and have non-negative real parts in the right-half of the frequency s-plane. The stability study of a complex polynomial in Chapter 2, allows us to construct a complex impedance function $Z(s)$, and a complex reactance function $X(s)$. Realizable complex impedance and complex reactance networks are obtained by synthesizing $Z(s)$ and $X(s)$ using a J-fraction expansion.

We use this particular expansion to synthesize $Z(s)$ with four kinds of elements (real resistors, imaginary resistors, real inductors, and real capacitors). Also $X(s)$ is synthesized using three kinds of elements (jR, L, C) and two kinds of elements (jR, L) or (jR, C). The relationship between $X(s)$ and its associative $Z(s)$ is found. The analysis is supported by numerical examples.

3.1.1 General Remarks about synthesis procedures

In network theory we are concerned with the following three aspects: an excitation, a response, and a network. If the network itself is one of the two given aspects, then the problem becomes a network analysis problem. When the network is the unknown quantity, the process of its identification is commonly known as network synthesis.

The problem that we are initially concerned with is to find a network realization for a complex impedance $Z(s)$ or admittance $Y(s)$, also for a complex reactance $X(s)$ or susceptance $B(s)$ in the form of a rational function which is a ratio of two polynomials with complex coefficients.

3.1.2 What is imaginary resistance (jR)

A current in the harmonic state with complex amplitude can be written as [16]

$$i\, e^{j\omega t} = I\, e^{j\phi}\, e^{j\omega t} \tag{3.1}$$

where the real part of equation (3.1) is

$$Re\ i\ e^{j\omega t} = Re\ I\ e^{j\phi}\ e^{j\omega t} = I\ \cos(\omega t + \phi) \tag{3.2}$$

If in equation (3.2) either I or ϕ are varying functions of time, this is of practical interest in the case of amplitude or phase modulation. The complex amplitude i becomes a function of t. Since the equation of resistance is linear and does not involve the operator d/dt, it remain valid for such varying complex amplitudes i and v. On the other hand, the equations of an inductance and capacitance are linear and involve the operator d/dt, then the variation of complex amplitudes i and v leads to the inductance and capacitance equations as

$$v\ e^{j\omega t} = L\ \frac{d}{dt}\ (\ i\ e^{j\omega t}\ ) = (\ L\ \frac{di}{dt} + j\omega\ L\ i\ )\ e^{j\omega t} \tag{3.3}$$

$$i\ e^{j\omega t} = C\ \frac{d}{dt}\ (\ v\ e^{j\omega t}\ ) = (\ C\ \frac{dv}{dt} + j\omega\ C\ v\ )\ e^{j\omega t} \tag{3.4}$$

Since the term $j\omega Li$ in Equation 3.3 does not involve the operator d/dt, it represents the voltage drop across a time-independent element $j\omega L$. Also the term $j\omega Cv$ in Equation 3.4 does not involve the operator d/dt, it represents the current through a time-independent element $j\omega C$. By writing $R = L\omega$, in 3.3 and $G = C\omega$, in (3.4) we get,

$$v\ e^{j\omega t} = (\ L\ \frac{di}{dt} + j\ R\ i\ )\ e^{j\omega t} \tag{3.5}$$

$$i\ e^{j\omega t} = (\ C\ \frac{dv}{dt} + j\ G\ v\ )\ e^{j\omega t} \tag{3.6}$$

Equation (3.5) can be interpreted as representing an inductance L in series with an imaginary constant resistance jR which is a new element defined by the instantaneous relation

$$v_{jR}\ (\ t\ ) = j\ R\ i(t) \tag{3.7}$$

Also equation (3.6) can be interpreted as having a capacitance C in parallel with imaginary constant conductance jG which is also a new element defined by the instantaneous relation

$$i_{jG}\ (\ t\ ) = j\ G\ v(t) \tag{3.8}$$

The only new elements generated by this approach [16], are (jR) and (jG). Now the real networks with complex states in the form $I\ e^{j\phi}\ e^{j\omega t}$ and $V\ e^{j\phi}\ e^{j\omega t}$ are identical to the complex networks with real state $I\ e^{j\omega t}$ and $V\ e^{j\omega t}$.

In this chapter we shall construct complex rational functions from complex polynomials. Once we construct these functions, then their synthesis will reveal complex imaginary resistance or complex imaginary conductance, as we shall see from the synthesis of $Z(s)$ and $X(s)$.

The first motivation behind our interest in imaginary resistances is that they will enable us study such complex networks as free mathematical creations. In addition, the theory of complex networks is of practical interest in modulation problems, and this was a second motivation for studying the imaginary resistances.

3.2 Construction of Complex Impedance Functions From Complex Polynomials

In the previous chapter we illustrated the stability test of any complex polynomial $P(s)$. Now consider that stable polynomial of degree n, as in (2.1)

$$P(s) = s^n + (a_1 + jb_1)s^{n-1} + (a_2 + jb_2)s^{n-2} + \ldots + (a_{n-1} + jb_{n-1})s + (a_n + jb_n) \quad (3.9)$$

The problem now is to see how we can find a complex impedance from a given complex polynomial (3.9). To do so we seek to find another polynomial $Q(s)$ such that properties of $Z(s)$ can be determined from the J-fraction of $Z(s)$. Here $Q(s)$ is the alternant of $P(s)$, defined by:

$$Q(s) = a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + a_5 s^{n-5} + jb_6 s^{n-6} + \ldots \quad (3.10)$$

Theorem 3.1 *The alternant polynomial $Q(s)$ of any stable complex polynomial $P(s)$ has all its roots on the $j\omega$-axis.*

Proof:

By assumption $P(s)$ is a polynomial whose roots are in the left half of the s-plane, $Re(s) < 0$. Denote by $P^*(s)$ the polynomial obtained from $P(s)$ by replacing each coefficient by its complex conjugate:

$$P^*(s) = s^n + (a_1 - jb_1)s^{n-1} + (a_2 - jb_2)s^{n-2} + \ldots + (a_{n-1} - jb_{n-1})s + (a_n + jb_n) \quad (3.11)$$

Then the roots of $P(s)$ are symmetrical to the roots of $P^*(-s)$ with respect to the imaginary axis. If we regard the modulus of the polynomial as the product of the lengths of the vectors from s to its roots, then

$$|P(s)| > |P^*(-s)| \qquad if\ Re(s) > 0 \quad (3.12)$$

$$|P(s)| < |P^*(-s)| \qquad if\ Re(s) < 0 \quad (3.13)$$

Consequently,

$$|P(s) \pm P^*(-s)| > 0 \qquad for\ Re(s) \neq 0 \quad (3.14)$$

The alternant of $P(s)$ is $Q(s)$ which admits one of the following two forms

$$Q(s) = \frac{P(s) + P^*(-s)}{2}, \qquad or \qquad Q(s) = \frac{P(s) - P^*(-s)}{2} \quad (3.15)$$

Therefore, by (3.14), the roots of $Q(s)$ are all on the $j\omega$-axis, **Q. E. D.**

Now we have two polynomials $P(s)$ and $Q(s)$. From these polynomials one can find a complex impedance $Z(s)$ or admittance $Y(s)$. $Z(s)$ can be represented as:

$$Z(s) = \frac{P(s)}{Q(s)} \qquad or \qquad Z(s) = \frac{Q(s)}{P(s)} \quad (3.16)$$

We shall study first the case $Z(s) = P(s)/Q(s)$

$$Z(s) = \frac{s^n + (a_1 + jb_1)s^{n-1} + (a_2 + jb_2)s^{n-2} + (a_3 + jb_3)s^{n-3} + ... + (a_n + jb_n)}{a_1 s^{n-1} \;+\; jb_2 s^{n-2} \;+\; a_3 s^{n-3} \;+\; jb_4 s^{n-4} \;+\; a_5 s^{n-5} \;+\; ...} \tag{3.17}$$

$Z(s)$ has zeros that lie in the left-half of the complex s-plane, and poles that lie on the $j\omega$-axis as shown in Figure 3.1

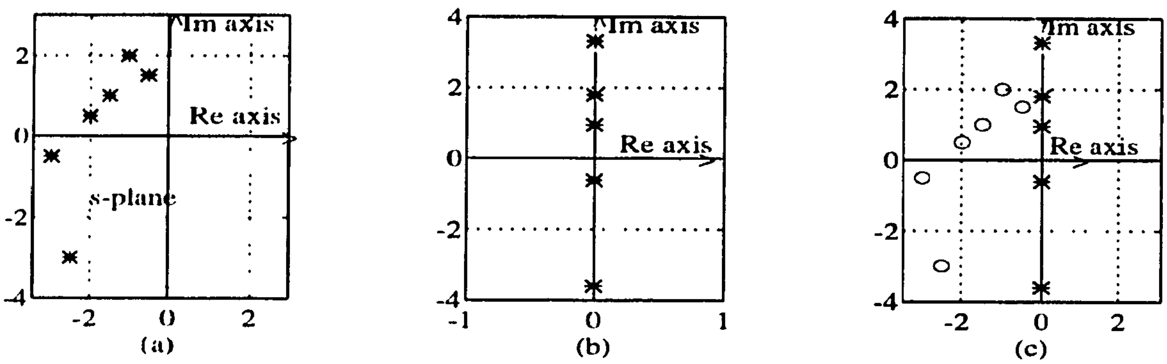

Figure 3.1: (a) Roots of $P(s)$, (b) Roots of $Q(s)$, (c) Pole-zero diagram of $Z(s) = P(s)/Q(s)$.

If we divide $Q(s)$ by $P(s)$ we then obtain a quotient $\alpha_1 s + 1 + j\beta_1$, and a remainder term $R_1(s)/Q(s)$

$$\frac{P(s)}{Q(s)} \;=\; \alpha_1 s + 1 + j\beta_1 \;+\; \frac{R_1(s)}{Q(s)} \tag{3.18}$$

where the degree of the remainder polynomial, $R_1(s)$, is one lower than the degree of $Q(s)$:

$$R_1(s) = c_1 s^{n-2} \;+\; c_2 s^{n-3} \;+\; c_3 s^{n-4} \;+\; c_4 s^{n-5} \;+\; ...+c_{n-3}s^2 \;+\; c_{n-2}s^1 \;+\; c_{n-1} \tag{3.19}$$

Therefore, if we invert the remainder term, we have

$$\frac{Q(s)}{R_1(s)} \;=\; \alpha_2 s + j\beta_2 \;+\; \frac{R_2(s)}{R_1(s)} \tag{3.20}$$

where the degree of $R_2(s)$ is one lower than the degree of $R_1(s)$

$$R_2(s) = d_1 s^{n-3} \;+\; d_2 s^{n-4} \;+\; d_3 s_{n-5} \;+\; d_4 s^{n-6} \;+\; ...+\; d_{n-3}s^1 \;+\; d_{n-2} \tag{3.21}$$

Inverting and dividing again, we obtain

$$\frac{R_1(s)}{R_2(s)} = \alpha_3(s) + j\beta_3 + \frac{R_3(s)}{R_2(s)} \tag{3.22}$$

We continue this division process of two polynomials until the remainder term of this division equal zero. The quotient $P(s)/Q(s)$ has in general a J-fraction expansion of the form

$$Z(s) = \frac{P(s)}{Q(s)} = \alpha_1 s + 1 + j\beta_1 + \cfrac{1}{\alpha_2 s + j\beta_2 + \cfrac{1}{\alpha_3 s + j\beta_3 + \cfrac{1}{\alpha_4 s + j\beta_4 + \cfrac{1}{\alpha_5 s + j\beta_5 + \cfrac{1}{+ \cfrac{1}{\alpha_n s + j\beta_n}}}}}} \tag{3.23}$$

We have seen that the process of obtaining $Z(s)$ simply involves two divisions and an inversion, and we shall call it a 'two divisions one inversion' process or a 'division-division-inversion' process. This terminology is consistent with that used in the literature, in the context of the s fraction expansion of real reactance or impedance functions, which involves one division and one inversion, process or a 'division - inversion' process to construct ladder networks (R, L), (R, C) and (L, C) [19, 20, 21].

It is known from the theory of continued fractions that, if the continued fraction expansion of the complex polynomial $P(s)$ to its alternant polynomial $Q(s)$ yields positive quotient terms, then the complex polynomial must have all roots in the left half of the complex s-plane. See Theorem 2.1.

3.2.1 Synthesis of $Z(s)$ by four kinds of elements (R, jR, L, C).

In the previous section we discussed how to perform a continued fraction expansion of $Z(s)$. Now we want to synthesise $Z(s)$ to find a realizable network representing $Z(s)$ as given in (3.17). By dividing the denominator polynomial $Q(s)$ by the numerator polynomial $P(s)$ we obtain a quotient first-order polynomial. This is equivalent to the removal of a series branch consisting of the series connection of a real resistor, an imaginary resistor, and a real inductor from $Z(s)$, i.e. $(L_1 s + 1 + jR_1)$ and a remainder function which is the ratio of an $(n-2)$ to an $(n-1)$ degree polynomial, as indicated below.

$$Z(s) = L_1 s + 1 + jR_1 + \frac{c_1 s^{n-2} + c_2 s^{n-3} + c_3 s^{n-4} + \ldots + c_{n-3} s^1 + c_{n-1}}{a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + \ldots} \quad (3.24)$$

Inverting the remainder and repeating the process is equivalent to removing a shunt branch consisting of an imaginary resistor in parallel with a real capacitor from $Z(s)$ i-e $1/(C_2 s + jG_2)$. The second remainder function is the ratio of an $(n-3)$ to an $(n-2)$ degree polynomial.

$$Z(s) = L_1 s + 1 + jR_1 + \frac{1}{C_2 s + jG_2 + \frac{d_1 s^{n-3} + d_2 s^{n-4} + d_3 s^{n-5} + d_4 s^{n-6} + \ldots + d_{n-2} s^1 + d_{n-1}}{c_1 s^{n-2} + c_2 s^{n-3} + c_3 s^{n-4} + c_4 s^{n-5} + c_5 s^{n-6} + \ldots}} \quad (3.25)$$

Continuation of this process evidently yields the following expansion development of $Z(s)$:

$$Z(s) = L_1 s + 1 + jR_1 + \cfrac{1}{C_2 s + jG_2 + \cfrac{1}{L_3 s + jR_3 + \cfrac{1}{C_4 s + jG_4 + \cfrac{1}{L_5 s + jR_5 + \cfrac{1}{+ \cfrac{1}{C_n s + jG_n}}}}}} \quad (3.26)$$

It should be observed that if n is even in equation (3.26) then the final term in the J-fraction is $(C_n s + jG_n)$ and the last branch in the network realization is a shunt branch consisting of the parallel combination of a real capacitor with an imaginary resistor see Figure 3.2.a. Alternatively if n is odd then the final term is $(L_n s + jR_n)$ and the last branch in the network realization is a series branch, consisting of a real inductor in series with an imaginary resistor. The network realization of $Z(s)$ in (3.17), corresponding to the J-fraction development (3.26) is shown in Figure 3.2.a and 3.2.b for the cases when n is even and odd respectively.

Next consider the case of $Z(s) = Q(s)/P(s)$ where

$$Z(s) = \frac{a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + a_5 s^{n-5} + \ldots}{s^n + (a_1 + jb_1)s^{n-1} + (a_2 + jb_2)s^{n-2} + \ldots + (a_n + jb_n)} \quad (3.27)$$

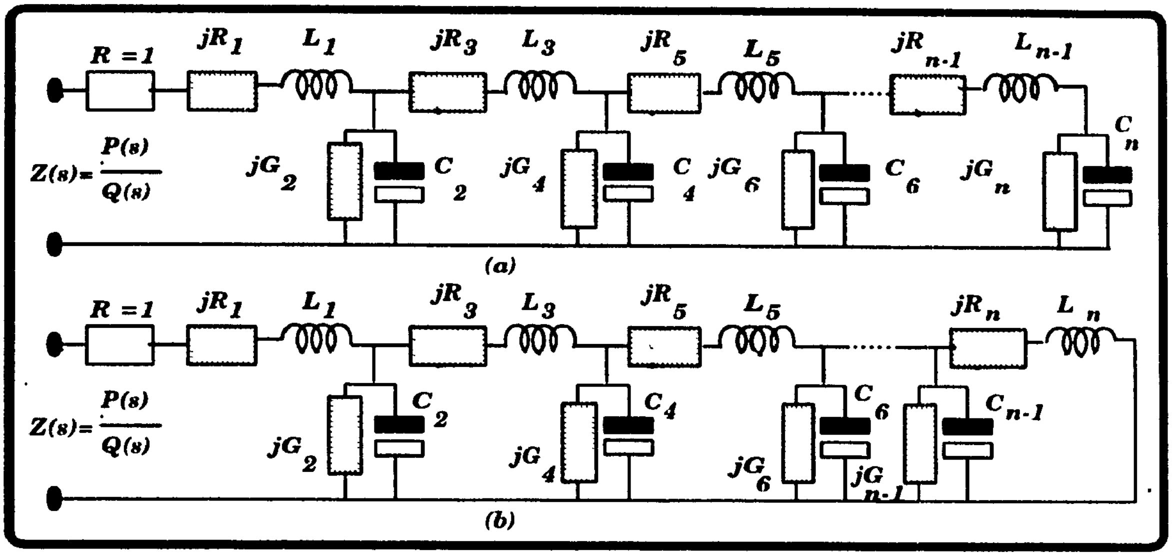

Figure 3.2: Realization of a complex impedance $Z(s) = P(s)/Q(s)$ in (3.17) (a) n is even (b) n is odd.

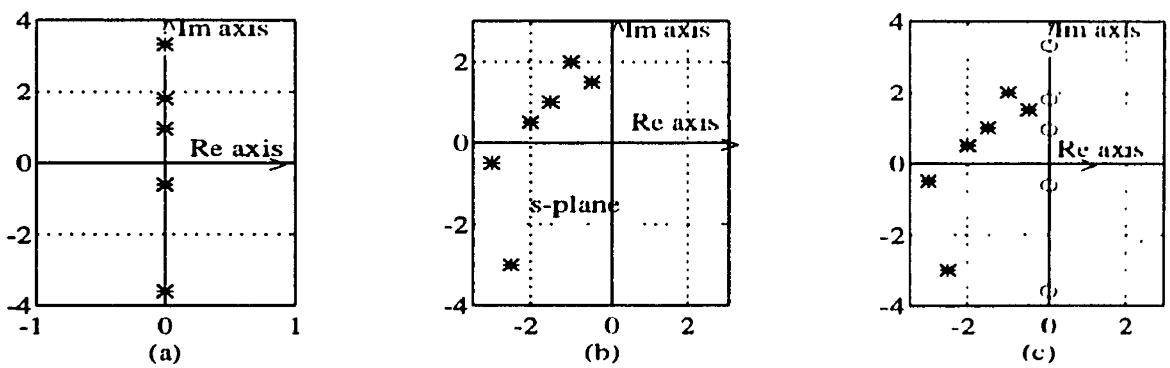

Figure 3.3: (a) Roots of $Q(s)$, (b) Roots of $P(s)$, (c) Pole-zero diagram of $Z(s) = Q(s)/P(s)$.

In this case $Z(s)$ has poles lying in the left-half of the complex s-plane, and zeros lying on the $j\omega$-axis as shown in Figure 3.3.

The J-fraction expansion of $Z(s) = Q(s)/P(s)$ is

$$Z(s) = \cfrac{1}{C_1 s + 1 + jG_1 + \cfrac{1}{L_2 s + jR_2 + \cfrac{1}{C_3 s + jG_3 + \cfrac{1}{L_4 s + jR_4 + \cfrac{1}{C_5 s + jG_5 + \cdots + \cfrac{1}{L_n s + jR_n}}}}}} \tag{3.28}$$

It should be observed that if n is even in (3.28), then the final term in the continued fraction expansion is $(L_n s + jR_n)$ and the last branch in the network realization is series branch consisting of a real inductor in series with an imaginary resistor. When n is odd the final term in the continued fraction expansion is $(C_n s + jG_n)$ and the last branch in the network realization is a shunt branch consisting of a real capacitor in parallel with an imaginary resistor. The network realization of $Z(s)$, corresponding to the continued fraction development (3.28), is shown in Figure 3.4 when the degree n of the complex polynomial $P(s)$ is even or odd.

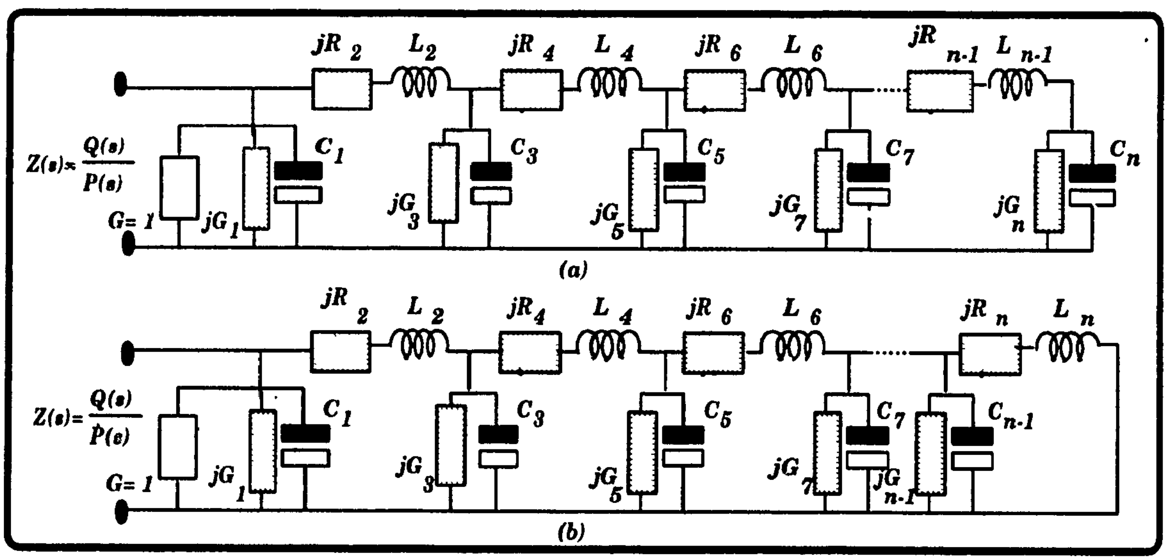

Figure 3.4: Realization of a complex impedance $Z(s) = Q(s)/P(s)$ in (3.27) (a) n is odd (b) n is even.

To illustrate the synthesis of $Z(s)$ consider the following example.

Example 3.1 *Given a polynomial $P(s)$ with complex coefficients*

$$P(s) = s^4 + (5-j5)s^3 + (0-j19.75)s^2 + (-18.375-j17.875)s - (13.125-j0.625) \tag{3.29}$$

A complex impedance which can be created from $P(s)$ and its $Q(s)$ is

$$Z(s) = \frac{s^4 + (5-j5)s^3 + (0-j19.75)s^2 + (-18.375-j17.875)s - (13.125-j0.625)}{5s^3 - j19.75s^2 - 18.375s} \tag{3.30}$$

with the roots of $P(s)$ and $Q(s)$ given by:

$$\begin{aligned} &\textit{Roots of } P(s) \textit{ are} \quad \lambda_1 = -2+j0.5 \quad \lambda_2 = -1.5+j1 \quad \lambda_3 = -1+j2 \\ &\qquad\qquad\qquad\quad \lambda_4 = -0.5+j1.5 \\ &\textit{Roots of } Q(s) \textit{ are} \quad \eta_1 = +j2.5 \quad \eta_2 = +j1.4147 \quad \eta_3 = +j0.0353 \end{aligned} \tag{3.31}$$

The J-fraction of $Z(s) = P(s)/Q(s)$ is

$$Z(s) = \frac{P(s)}{Q(s)} = 0.2s + 1 - j0.21 + \cfrac{1}{0.6392s - j0.7387 + \cfrac{1}{1.2527s - j1.6615 + \cfrac{1}{3.162s - j4.6418}}} \tag{3.32}$$

All the coefficients of s and the constant terms in this expansion are either positive real or pure imaginary numbers respectively. This means that we can synthesise a complex impedance $Z(s)$ by using a real resistor, imaginary resistors, real inductors and real capacitors. The corresponding network is shown in Figure 3.5

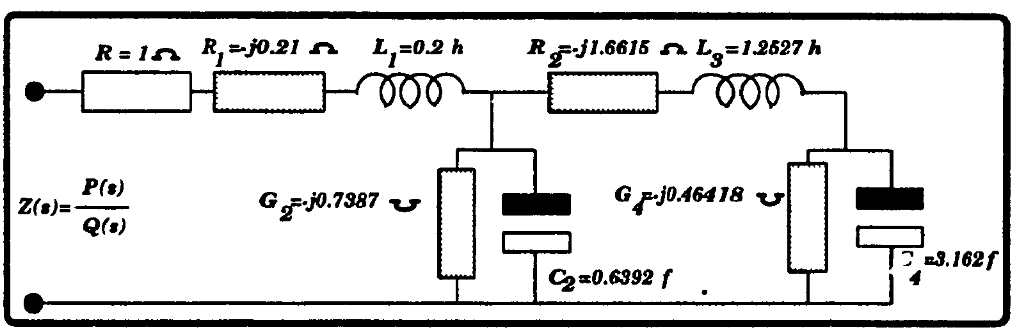

Figure 3.5: Realization of $Z(s)$ in equation (3.30)

In the previous example, the degree of the numerator of $Z(s)$ is higher than that of the denominator by one and the corresponding network started by a series branch consisting of the series of a real resistor, an imaginary resistor, and a real inductor. In addition the last branch is a shunt branch consisting of an imaginary resistor in parallel with a real capacitor. Now the synthesis of a complex impedance $Z(s) = Q(s)/P(s)$ is given in the next example.

Example 3.2 *Consider a complex $P(s)$ as in Example 3.1, then construct a complex impedance $Z(s) = Q(s)/P(s)$*

$$Z(s) = \frac{5s^3 - j19.75s^2 - 18.375s}{s^4 + (5 - j5)s^3 + (0 - j19.75)s^2 + (-18.375 - j17.875)s - (13.125 - j0.625)} \tag{3.33}$$

The J-fraction expansion of $Z(s) = Q(s)/P(s)$ is given by the following.

$$Z(s) = \frac{Q(s)}{P(s)} = \frac{1}{0.2s + 1 - j0.21 + \frac{1}{0.6392s - j0.7387 + \frac{1}{1.2527s - j1.6615 + \frac{1}{3.162s - j4.6418}}}} \tag{3.34}$$

$Z(s)$ can be synthesized by using G, jR, L, and C and the corresponding network is shown in Figure 3.6.

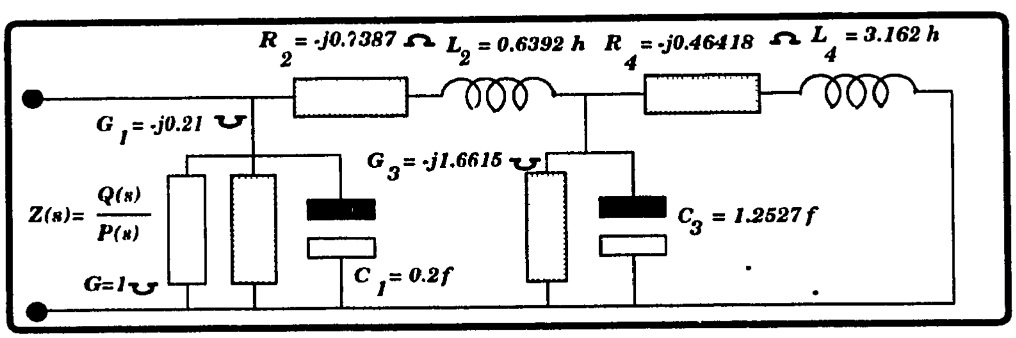

Figure 3.6: Realization of $Z(s)$ in equation (3.33)

In this example the degree of the numerator of $Z(s)$ is lower than the degree of the denominator by one, and the corresponding network started by a shunt branch consisting of the parallel combination of a real resistor, an imaginary resistor, and a real capacitor.

3.3 Construction of Complex Reactance Functions From Complex Polynomials

Before we discuss complex reactance functions, and show how to synthesise $X(s)$ using the J-fraction expansion, we first discuss, in brief, the real reactance function [19, 20, 21].

In general a real reactance function $x(s)$ is the ratio of even to odd or odd to even parts of a real stable polynomial $p(s)$. The poles and zeros of $x(s)$ lie on the $j\omega$-axis and alternate. The words even and odd are sufficient to describe the two parts of the real polynomial. However when we started to study complex polynomials and complex reactances we found that we can not extend the commonly understood definition of real reactance to complex reactance. We therefore need to generalize the definition of real reactance functions to include complex reactance functions as well.

It will be general if we describe the two parts of a real polynomial by a real and an imaginary or {quasi real and quasi imaginary} parts when $s = j\omega$. It is true that the real, {quasi} real part of the real polynomial $p(s)$ at $s = j\omega$ is the even part of $p(s)$ and that the imaginary, {quasi} imaginary part of $p(s)$ at $s = j\omega$ is the odd part of $p(s)$.

Definition 3.1 *[22]: A polynomial $P(s)$ with complex coefficients is said to be quasi-real (quasi-imaginary) if its value for $s = j\omega$ is purely real (purely imaginary).*

Now we can say that a real reactance function $x(s)$, in general, is the ratio of quasi real to quasi imaginary or quasi imaginary to quasi real parts of the real stable polynomial when $s = j\omega$.

The reason for this comes from a study of a complex reactance function $X(s)$ and a complex polynomial $P(s)$. We can state here that:

- A real polynomial $p(s)$ is a special case of a complex polynomial $P(s)$.
- A real reactance function $x(s)$ is a special case of a complex reactance function $X(s)$.
- For a real polynomial $p(s)$, the quasi-real part of $p(s)$ equals the even part of $p(s)$.
- For a real polynomial $p(s)$, the quasi-imaginary part of $p(s)$ equals the odd part of $p(s)$.
- If a complex polynomial $P(s) \overset{becomes}{\longrightarrow} p(s)$ a real polynomial, then a complex reactance $X(s) \overset{becomes}{\longrightarrow} x(s)$ a real reactance.

Here we can say that a complex reactance function is the ratio of a quasi real ($qRe\,P(s)$) to a quasi imaginary ($qIm\,P(s)$) or vice-versa. With this introduction of real reactance and complex reactance functions we shall now study in detail how to find a complex reactance function, and how to synthesise this function using the J-fraction expansion.

We start from a stable complex polynomial $P(s)$ of degree n as in (2.1). The problem now is how we can find $X(s)$ from $P(s)$. To do so we have to divide $P(s)$ into its quasi-real ($qRe\,P(s)$) and quasi-imaginary ($qIm\,P(s)$) parts, respectively.

$$P(s) \quad = \quad qRe\,P(s) \quad + \quad qImP(s) \tag{3.35}$$

where for even n the quasi-real and quasi-imaginary parts of $P(s)$ are given by:

$$qReP(s) = s^n + jb_1s^{n-1} + a_2s^{n-2} + jb_2s^{n-3} + \ldots + jb_{n-1}s + a_n \tag{3.36}$$

$$qImP(s) = a_1s^{n-1} + jb_2s^{n-2} + a_3s^{n-3} + jb_4s^{n-4} + \ldots + a_{n-1}s + jb_n \tag{3.37}$$

It should be observed that if n is odd, then the quasi-imaginary and the quasi-real parts of $P(s)$ are given by:

$$qImP(s) = s^n + jb_1s^{n-1} + a_2s^{n-2} + jb_2s^{n-3} + \ldots + a_{n-1}s + jb_n \tag{3.38}$$

$$qReP(s) = a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_3 s^{n-3} + ... + jb_{n-1} s + a_n \qquad (3.39)$$

If we examine very carefully the $qImP(s)$ when n is even in equation (3.37) or the $qReP(s)$ in equation (3.39) when n is odd we can conclude that these equations are the same as the alternant polynomial $Q(s)$ of $P(s)$ in equation (2.2).

$$Q(s) \;=\; qImP(s) \qquad if \;\; n \;\; even \qquad (3.40)$$

$$Q(s) \;=\; qReP(s) \qquad if \;\; n \;\; odd \qquad (3.41)$$

From the $qReP(s)$ and $qImP(s)$ polynomials we can find complex reactance or susceptance functions. The complex reactance function $X(s)$ can be represented by

$$X(s) \;=\; \frac{qReP(s)}{qImP(s)} \qquad or \qquad X(s) \;=\; \frac{qImP(s)}{qReP(s)} \qquad (3.42)$$

There are some papers [23, 24] which discuss complex reactance functions and each of them has a different way to deal with the complex reactance function For example we have defined here a complex reactance function from $qReP(s)$ and $qImP(s)$. In Bose's paper [23] he defines a complex reactance function from $P(s)$ as

$$X(s) \;=\; \frac{P(s) \;-\; P^*(-s)}{P(s) \;+\; P^*(-s)} \qquad (3.43)$$

In Reza's paper [24] he defines the numerator and the denominator of $X(s)$ from a set of arbitrary points within the right-half of the frequency s-plane.

$$X(s) \;=\; \frac{1 \;-\; \frac{(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)\ldots(s-\alpha_n)}{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)(s+\bar{\alpha}_3)\ldots(s+\bar{\alpha}_n)}}{1 \;+\; \frac{(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)\ldots(s-\alpha_n)}{(s+\bar{\alpha}_1)(s+\alpha_2)(s+\alpha_3)\ldots(s+\alpha_n)}} \qquad (3.44)$$

3.3.1 Synthesis of X(s) by three kinds of elements (jR, L, C).

Once we construct $X(s)$ from a given $P(s)$, consider the case $X(s) = qReP(s)/qImP(s)$, when the degree n of $P(s)$ is even:

$$X(s) = \frac{s_n + jb_1 s_{n-1} + a_2 s_{n-2} + jb_3 s_{n-3} + a_4 s_{n-4} + ... + jb_{n-1} s + a_n}{a_1 s_{n-1} + jb_2 s_{n-2} + a_3 s_{n-3} + jb_4 s_{n-4} + ... + a_{n-1} s + jb_n} \qquad (3.45)$$

To synthesise $X(s)$ we have to expand it using J-fraction expansion as

$$X(s) = L_1 s + jR_1 + \cfrac{1}{C_2 s + jG_2 + \cfrac{1}{L_3 s + jR_3 + \cfrac{1}{C_4 s + jG_4 + \cfrac{1}{L_5 s + jR_5 + \cfrac{1}{+ \cfrac{1}{C_n s + jG_n}}}}}} \tag{3.46}$$

The network realization for a complex reactance function $X(s)$ in (3.45) which is synthesised by the expansion in (3.46) is shown in Figure 3.7.

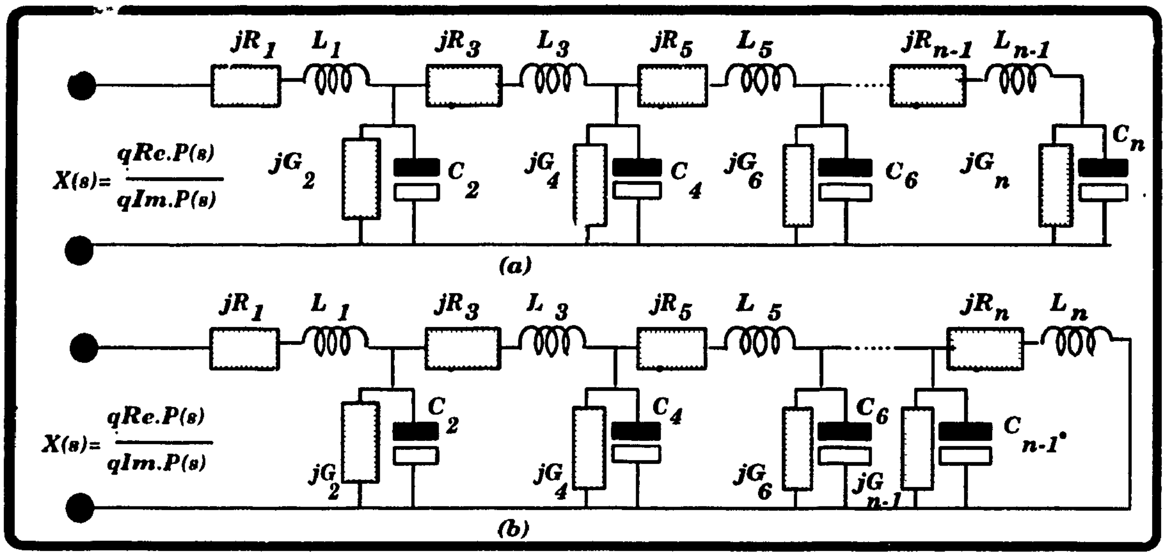

Figure 3.7: Realization of a complex reactance $X(s) = qReP(s)/qImP(s)$ in (3.45) (a) n is even (b) n is odd.

Consider the case when $X(s) = qImP(s)/qReP(s)$

$$X(s) = \frac{a_1 s_{n-1} + jb_2 s_{n-2} + a_3 s_{n-3} + jb_4 s_{n-4} + \ldots + a_{n-1} s + jb_n}{s_n + jb_1 s_{n-1} + a_2 s_{n-2} + jb_3 s_{n-3} + \ldots + jb_{n-1} s + a_n} \tag{3.47}$$

The J-fraction expansion in this case becomes

$$X(s) = \cfrac{1}{C_1 s + jG_1 + \cfrac{1}{L_2 s + jR_2 + \cfrac{1}{C_3 s + jG_3 + \cfrac{1}{L_4 s + jR_4 + \cfrac{1}{C_5 s + jG_5 + \cfrac{1}{+ \cfrac{1}{L_n s + jR_n}}}}}}} \tag{3.48}$$

The network realization for $X(s)$ in 3.47 which is synthesised by the J-fraction in 3.48 is shown in Figure 3.8.

To illustrate this synthesis consider the following example.

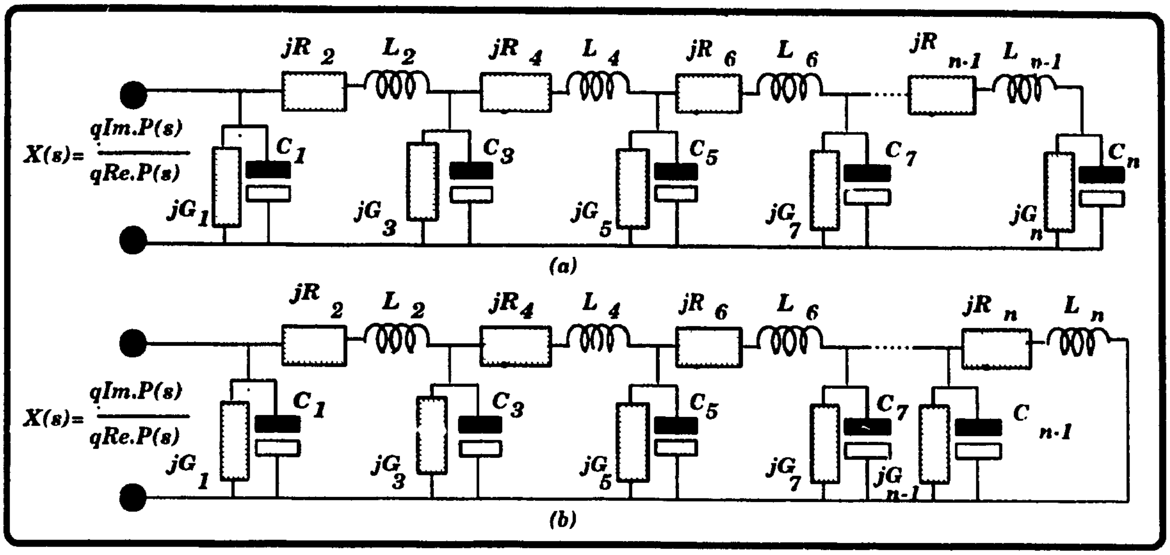

Figure 3.8: Realization of a complex reactance $X(s) = qImP(s)/qReP(s)$ in (3.47) (a) n is even (b) n is odd.

Example 3.3 *Consider the stable complex polynomial $P(s)$ as in Example 3.1,*

The alternant polynomial of $P(s)$ is given by

$$Q(s) = 5s^3 - j19.75s^2 - 18.375s + j0.625 \tag{3.49}$$

the quasi real part of $P(s)$ is given by

$$qReP(s) = s^4 - j5s^3 + 0s^2 - j17.875s - 13.125 \tag{3.50}$$

and the quasi imaginary part of $P(s)$ is given as

$$qImP(s) = 5s^3 - j19.75s^2 - 18.375s + j0.625 \tag{3.51}$$

As we mentioned before, when the degree of $P(s)$ is even, the alternant polynomial $Q(s)$ of the complex polynomial $P(s)$ is the quasi imaginary part of $P(s)$, see equations (3.49) and (3.51). From these polynomials one can construct $X(s)$ as

$$X(s) = \frac{qReP(s)}{qImP(s)} = \frac{s^4 - j5s^3 + 0s^2 - j17.875)s - 13.125}{5s^3 - j19.75s^2 - 18.375s + j0.625} \tag{3.52}$$

The pole-zero diagram of $X(s)$ is shown in Figure 3.9

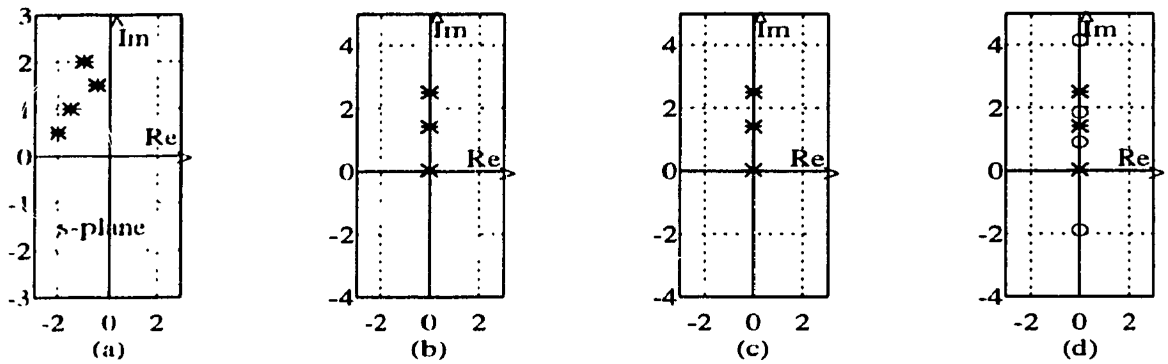

Figure 3.9: (a) Roots of $P(s)$, (b) Roots of $Q(s)$, (c) Roots of $qImP(s)$ (d) Pole-zero diagram of $X(s) = qReP(s)/qImP(s)$.

The J-fraction expansion of $X(s)$ is

$$X(s) = \frac{qReP(s)}{qImP(s)} = 0.2s - j0.21 + \cfrac{1}{0.6392s - j0.7387 + \cfrac{1}{1.2527s - j1.6615 + \cfrac{1}{3.162s - j4.6418}}}. \tag{3.53}$$

and the corresponding network is shown in Figure 3.10.

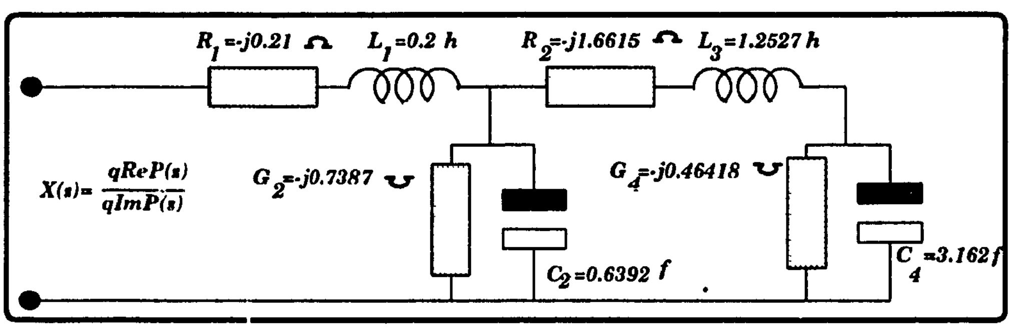

Figure 3.10: Realization of $Z(s)$ in equation (3.52)

Now we want to synthesize a complex reactance $X(s) = qImP(s)/qReP(s)$ for the following example

Example 3.4 *Consider the stable complex polynomial P(s) as in Example 3.1.*

In this case,

$$X(s) = \frac{qImP(s)}{qReP(s)} = \frac{5s^3 - j19.75s^2 - 18.375s + j0.625}{s^4 - j5s^3 + 0s^2 - j17.875)s - 13.125} \tag{3.54}$$

The roots of $P(s)$ and $Q(s)$, and the poles and zeros of $X(s)$ are given by (3.55)

$$
\begin{array}{llll}
Zeros\ of\ X(s)\ are & z_1 = +j2.5 & z_2 = +j1.4147 & z_3 = +j0.0353 \\
Poles\ of\ X(s)\ are & p_1 = +j4.1433 & p_2 = +j1.8499 & p_3 = +j0.9031 \\
 & P_4 = -j1.8962 & & \\
Roots\ of\ Q(s)\ are & \eta_1 = +j2.5 & \eta_2 = +j1.4147 & \eta_3 = +j0.0353 \\
Roots\ of\ P(s)\ are & \lambda_1 = -2 + j0.5 & \lambda_2 = -1.5 + j1 & \lambda_3 = -1 + j2 \\
 & \lambda_4 = -0.5 + j1.5 & &
\end{array}
\qquad (3.55)
$$

The J-fraction expansion of $X(s)$ is

$$X(s) = \cfrac{1}{0.2s - j0.21 + \cfrac{1}{0.6392s - j0.7387 + \cfrac{1}{1.2527s - j1.6615 + \cfrac{1}{3.162s - j4.6418}}}} \qquad (3.56)$$

The network realization is shown in Figure 3.11.

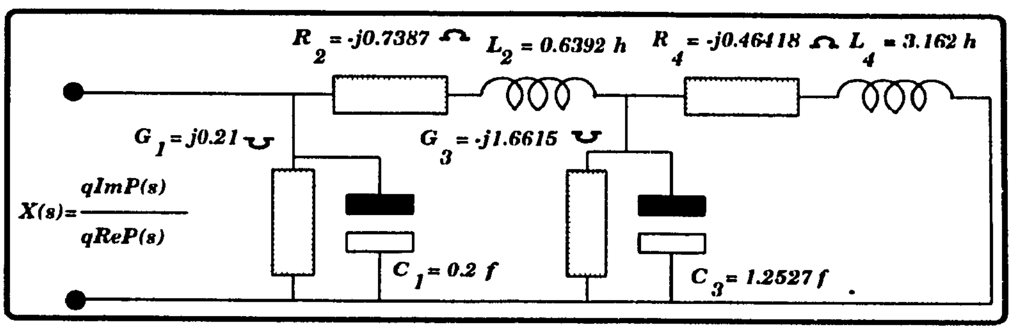

Figure 3.11: Realization of $Z(s)$ in equation (3.54)

3.3.2 Synthesis of $X(s)$ by two kinds of elements (jR, L or jR, C).

In the previous subsection we synthesized $X(s)$ by using a J-fraction expansion which depends on the division and inversion process and we named it a (division division-inversion) process or a (two divisions one inversion) process. These division processes are equivalent to the removal of some elements like imaginary resistors.

inductors and capacitors from a complex reactance $X(s)$. The quotient from the division process is a first order polynomial equivalent to a removal of two elements at a time such as imaginary resistor-inductor or imaginary conductor-capacitor from a complex reactance or susceptance function.

Now we shall use the S-fraction expansion to synthesise the same complex reactance function $X(s)$ in 3.45 but in this case we shall restrict the division process to be equivalent to the removal of only one element from $X(s)$. Here we can say that the S-fraction expansion depends on the (division-inversion) process.

Consider a complex reactance function $X(s)$ as in (3.45). By using the (division-inversion) process, the S-fraction expansion of $X(s) = qReP(s)/qImP(s)$ takes the form:

$$X(s) = \frac{qReP(s)}{qImP(s)} = L_1 s + \cfrac{1}{jG_2 + \cfrac{1}{L_3 s + \cfrac{1}{jG_4 + \cfrac{1}{L_5 s + \cfrac{1}{\ddots + \cfrac{1}{L_{2n-1} + \cfrac{1}{jG_{2n}}}}}}}} \qquad (3.57)$$

The realization for $X(s)$ in (3.57) is shown in Figure 3.12.

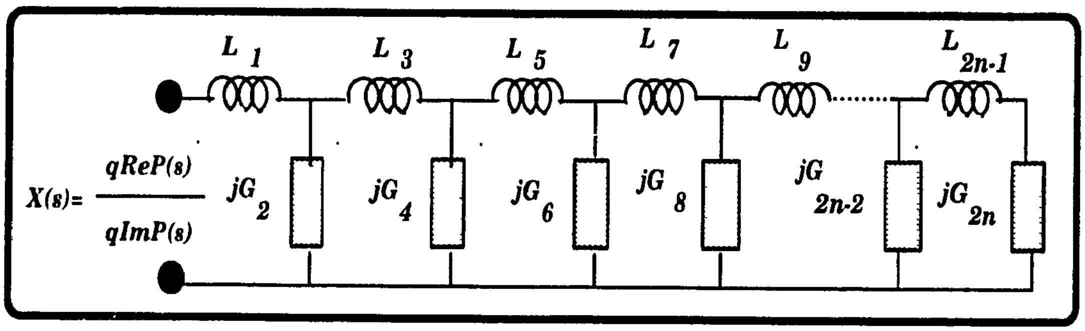

Figure 3.12: Realization of $X(s)$ in equation (3.57)

Consider now a complex reactance function equal to $qImP(s)/qReP(s)$, as in (3.47) then the S-fraction expansion in this case becomes

$$X(s) = \frac{qImP(s)}{qReP(s)} = \cfrac{1}{C_1 s + \cfrac{1}{jR_2 + \cfrac{1}{C_3 s + \cfrac{1}{jR_4 + \cfrac{1}{C_5 s + \cfrac{1}{\ddots + \cfrac{1}{C_{2n-1} + \cfrac{1}{jR_{2n}}}}}}}}} \qquad (3.58)$$

and the network realization of $X(s)$ is shown in Figure 3.13.

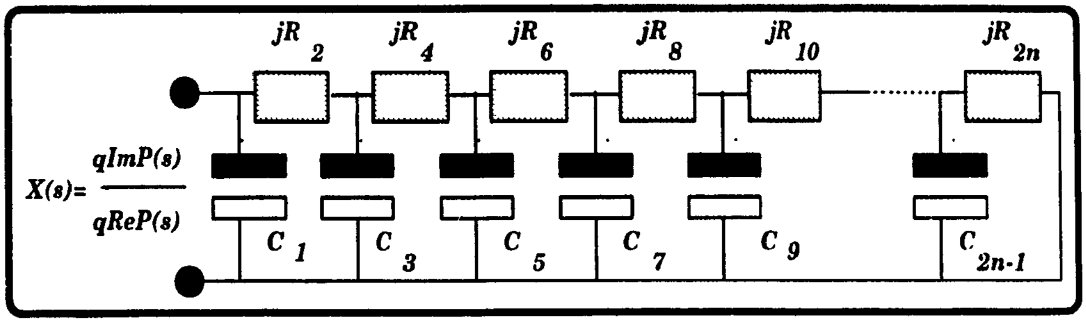

Figure 3.13: Realization of $X(s)$ in equation (3.58)

3.3.3 An alternative synthesis of $X(s)$ by two kinds of elements (jR, L or jR, C).

To have an alternative realization of a complex reactance function in (3.45), using S-fractions expansion, arrange both the numerator and the denominator of $X(s)$ in ascending powers of s as in (3.59).

$$X(s) = \frac{a_n + jb_{n-1} + \ldots + jb_3 s^{n-3} + a_2 s^{n-2} + jb_1 s^{n-1} + s^n}{jb_{n-1} + a_{n-1} + \ldots + a_3 s_{n-3} + jb_2 s^{n-2} + a_1 s^{n-1}} \tag{3.59}$$

then divide the lowest power of the denominator by the lowest power of the numerator, invert the remainder and divide again. We defined this as a (division-inversion) process.

$$X(s) = \frac{qReP(s)}{qImP(s)} = jR_1 + \cfrac{1}{C_2 s + \cfrac{1}{jR_3 + \cfrac{1}{C_4 s + \cfrac{1}{jR_5 + \cfrac{1}{C_6 s + \cfrac{1}{+ \cfrac{1}{jR_{2n-1} + \cfrac{1}{C_{2n} s}}}}}}}} \tag{3.60}$$

The realization of $X(s)$ in (3.59) which is synthesised by the S-fraction in (3.60) is shown in Figure 3.14.

Also to obtain an alternative realization of a complex reactance $X(s)$ in (3.47), arrange both the numerator and the denominator of $X(s)$ in ascending power of s

$$X(s) = \frac{jb_n + a_{n-1} + \ldots + a_3 s_{n-3} + jb_2 s^{n-2} + a_1 s^{n-1}}{a_n + jb_{n-1} + \ldots + jb_3 s^{n-3} + a_2 s^{n-2} + jb_1 s^{n-1} + s^n} \tag{3.61}$$

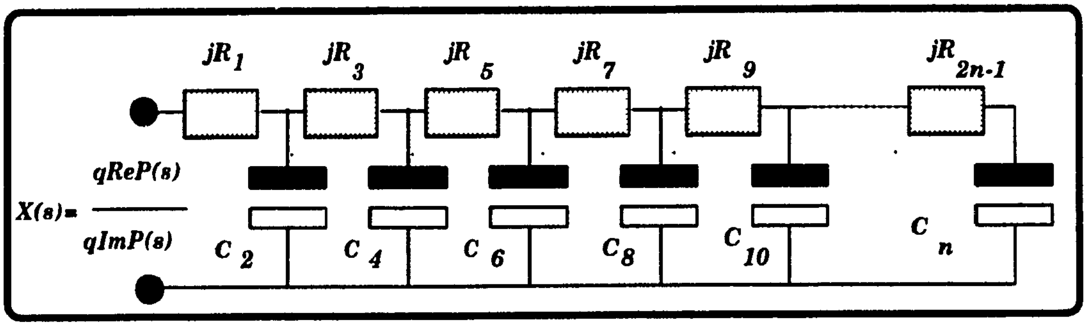

Figure 3.14: Realization of $X(s)$ in equation (3.59)

then use the (division-inversion) process to find the S-fraction expansion.

$$X(s) = \frac{qImP(s)}{qReP(s)} = \cfrac{1}{jG_1 + \cfrac{1}{L_2 s + \cfrac{1}{jG_3 + \cfrac{1}{L_4 s + \cfrac{1}{jG_5 + \cfrac{1}{L_6 s + \cfrac{1}{\cdots + \cfrac{1}{jG_{2n-1} + \cfrac{1}{L_{2n} s}}}}}}}}} \qquad (3.62)$$

The network realization for $X(s)$ in (3.62) is shown in Figure 3.15.

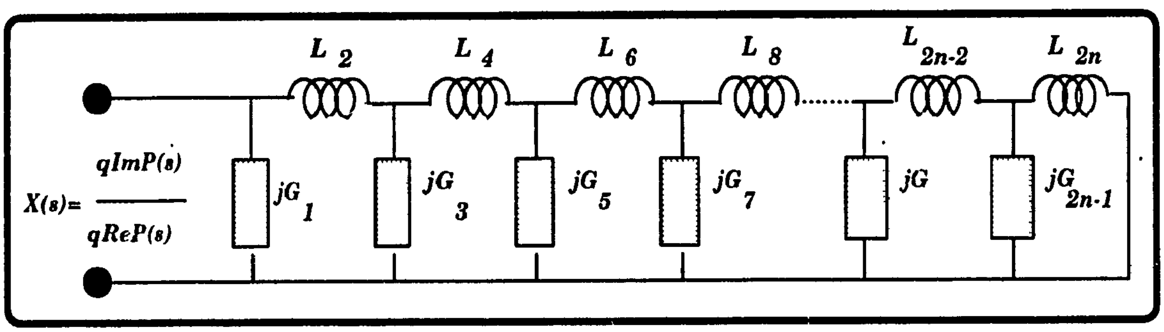

Figure 3.15: Realization of $X(s)$ in equation (3.61)

To illustrate the synthesis of complex reactance functions using the (division-inversion) process, consider the following examples.

Example 3.5 *Consider a complex reactance X(s) as in Example 3.3.*

The S-fraction expansion of $X(s) = qReP(s)/qImP(s)$ is

$$X(s) = \frac{qReP(s)}{qImP(s)} = 0.2s + \cfrac{1}{j4.7619 + \cfrac{1}{0.0282s + \cfrac{1}{-j5.6362 + \cfrac{1}{0.8945s + \cfrac{1}{j0.7337 + \cfrac{1}{8.2252s + \cfrac{1}{j0.0933}}}}}}} \qquad (3.63)$$

$X(s)$ can be synthesised by using imaginary conductances and real inductors, connected as a ladder network. The corresponding network is shown in Figure 3.16.

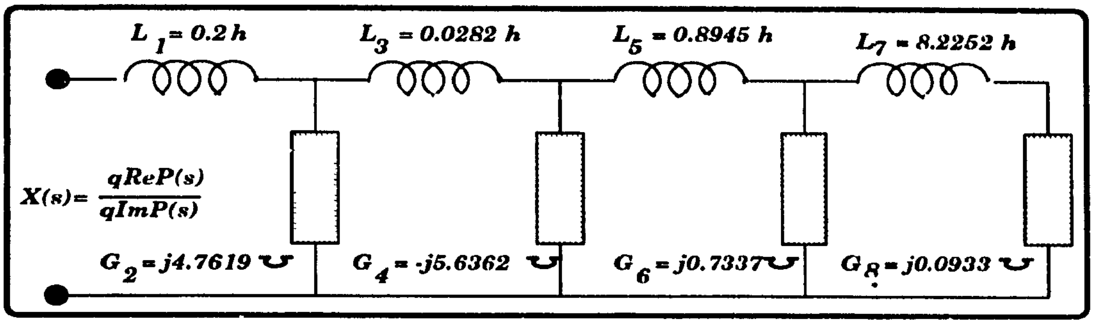

Figure 3.16: Realization of $X(s)$ in equation (3.63)

Example 3.6 *Consider a complex reactance $X(s)$ as in Example 3.4.*

The S-fraction expansion of $X(s) = qImP(s)/qReP(s)$ is

$$X(s) = \frac{qImP(s)}{qReP(s)} = \cfrac{1}{0.2s + \cfrac{1}{j4.7619 + \cfrac{1}{0.0282s + \cfrac{1}{-j5.6362 + \cfrac{1}{0.8945s + \cfrac{1}{j0.7337 + \cfrac{1}{8.2252s + \cfrac{1}{j0.0933}}}}}}}} \qquad (3.64)$$

In this example, $X(s)$ can be synthesised by using imaginary resistances and real capacitors which connected as a ladder network. The corresponding network is shown in Figure 3.17.

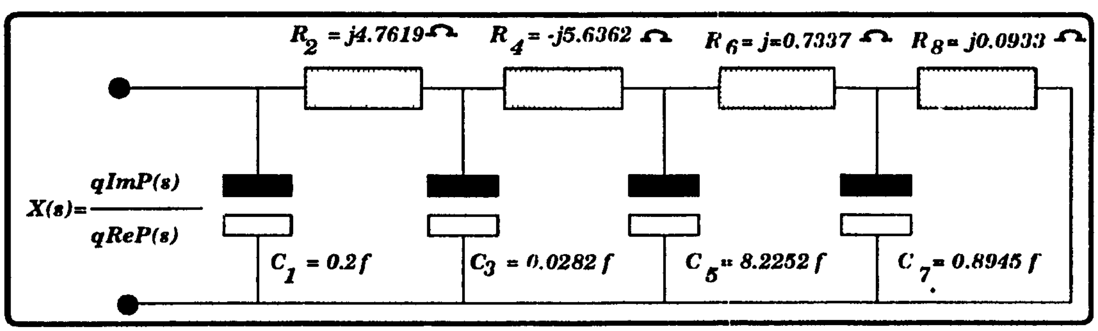

Figure 3.17: Realization of $X(s)$ in equation (3.64)

Now the alternative ladder network of a complex reactance function in Example 3.5 can be obtained by writing the polynomials of $X(s)$ in (3.52), in ascending power of s as in the following example:

Example 3.7

$$X(s) = \frac{qReP(s)}{qImP(s)} = \frac{-13.125 - j17.875s + 0s^2 - j5s^3 + s_4}{j0.625 - 18.375s - j19.75s^2 + 5s^3} \qquad (3.65)$$

The S-fraction expansion of $X(s)$ is given by

$$X(s) = \frac{qReP(s)}{qImP(s)} = j21 + \cfrac{1}{0.0017s + \cfrac{1}{-j20.8256 + \cfrac{1}{2.4089s + \cfrac{1}{-j1.3756 + \cfrac{1}{2.2735s + \cfrac{1}{-j7.4248 + \cfrac{1}{0.3159s}}}}}}} \qquad (3.66)$$

The corresponding network is shown in Figure 3.18

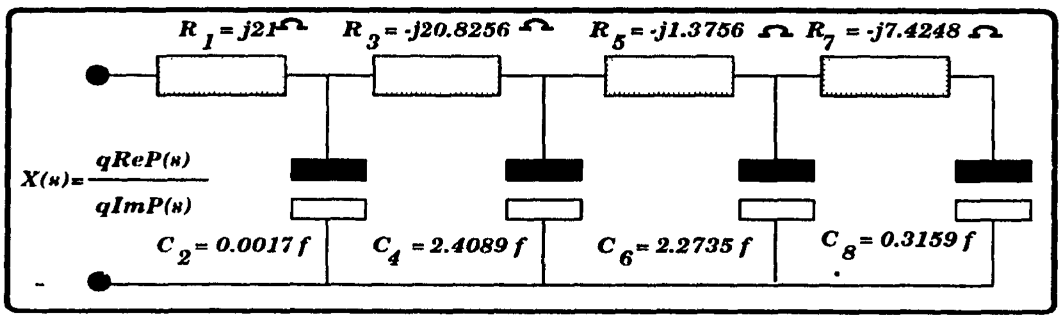

Figure 3.18: Realization of $X(s)$ in equation (3.65)

Example 3.8 *The alternative ladder network of the complex reactance function in Example 3.6 can be obtained by writing the polynomials of $X(s)$ (3.54) in ascending power of s as follows:*

$$X(s) = \frac{qReP(s)}{qImP(s)} = \frac{j0.625 - 18.375s - j19.75s^2 + 5s^3}{-13.125 - j17.875s + 0s^2 - j5s^3 + s_4} \qquad (3.67)$$

$$X(s) = \frac{qImP(s)}{qReP(s)} = \cfrac{1}{j21 + \cfrac{1}{0.0017s + \cfrac{1}{-j20.8256 + \cfrac{1}{2.4089s + \cfrac{1}{-j1.3756 + \cfrac{1}{2.2735s + \cfrac{1}{-j7.4248 + \cfrac{1}{0.3159s}}}}}}}} \qquad (3.68)$$

We can synthesize the complex reactance $X(s)$ by using imaginary conductances and real inductors connected as a ladder network. The corresponding network is shown in Figure 3.19.

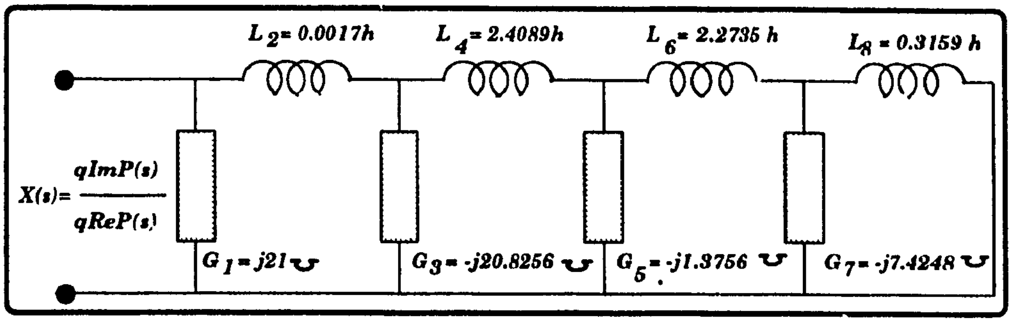

Figure 3.19: Realization of $X(s)$ in equation (3.67)

3.4 Relationship Between Complex Reactance and its Associated Complex Impedance Function

A complex polynomial $P(s)$ can be separated into its quasi parts as in (3.36) and (3.37) for n even or as in (3.38) and (3.39) for n odd. A complex reactance and a complex impedance functions are defined by equations (3.16) and (3.42). Also the alternant polynomial $Q(s)$ is defined in (2.2).

From (3.35) and (3.40), we can rewrite $Z(s)$ for n even as

$$Z(s) = \frac{P(s)}{Q(s)} = \frac{qReP(s) + qImP(s)}{qImP(s)} \tag{3.69}$$

or

$$Z(s) = \frac{Q(s)}{P(s)} = \frac{qImP(s)}{qReP(s) + qImP(s)} \tag{3.70}$$

Also for n odd we can write $Z(s)$ as

$$Z(s) = \frac{P(s)}{Q(s)} = \frac{qReP(s) + qImP(s)}{qReP(s)} \tag{3.71}$$

or

$$Z(s) = \frac{Q(s)}{P(s)} = \frac{qReP(s)}{qReP(s) + qImP(s)} \tag{3.72}$$

Next we discuss the case when the degree n of a complex polynomial $P(s)$ is even and $Z(s) = P(s)/Q(s)$. The same analysis follows when n is odd. From (3.71) we have

$$Z(s) = \frac{P(s)}{Q(s)} = \frac{qReP(s) + qImP(s)}{qImP(s)} = \frac{qReP(s)}{qImP(s)} + 1 = X(s) + 1 \quad (3.73)$$

We can write this relation in term of rational function as

$$\frac{qReP(s)}{qIm.P(s)} = \frac{P(s)}{Q(s)} - 1 = \frac{P(s) - Q(s)}{Q(s)} \quad (3.74)$$

It should be observed that if we know either $Z(s)$ or $X(s)$ then the other can be obtained, (see equations (3.17) and (3.45)). Also if we know the J-fraction of $Z(s)$ or $X(s)$ then the J-fraction of the second can be obtained (see equations (3.26) and (3.46)). See also Figures (3.2) and (3.7).

The relation between complex polynomials $P(s)$, $Q(s)$, $qReP(s)$, and $qImP(s)$ and complex functions $Z(s)$ and $X(s)$ and the corresponding networks is shown in Figure 3.20.

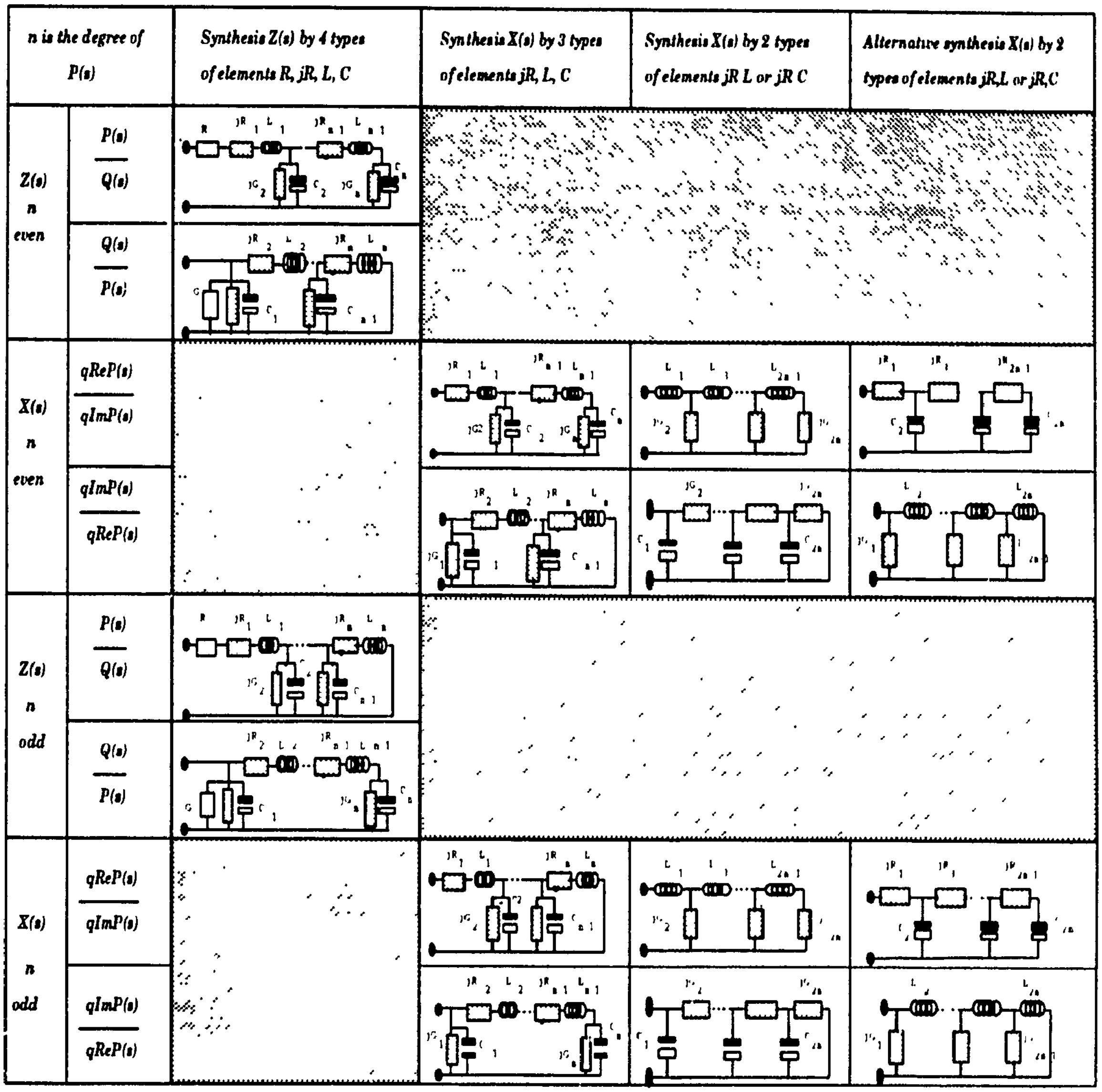

Figure 3.20: Relationships between $P(s)$, $Q(s)$ $qReP(s)$, $qImP(s)$, $Z(s)$, and $X(s)$, and the corresponding networks

Chapter 4

SYNTHESIS OF A COMPLEX REACTANCE FUNCTION X(z) in the z-plane

4.1 Introduction

In Chapter 3 we illustrated how an analog complex reactance function $X(s)$ can be synthesised in the s-domain. In this chapter we shall illustrate the synthesis of a discrete complex reactance function $X(z)$ in the z-domain. $X(z)$ can be obtained directly from $X(s)$ by using certain transformation. Further, it can be obtained from any stable complex polynomial which has all the roots inside the unit circle in the z-plane. Discrete complex reactance function could be implemented or synthesized by an algebraic equation obtained directly from $X(z)$. The algebraic equation could be implemented by a computer program, a digital circuitry, or a programmable integrated circuit. Direct evaluation of the algebraic equation is one of the many possible ways of realizing a discrete complex reactance function. The purpose of this chapter is to illustrate the realization of a discrete complex reactance function

$X(z)$, e.g algebraic realization, partial fraction expansion realization (parallel realization), cascade realization, J-fraction expansion realization, S-fraction expansion realization, ... etc. A discrete complex reactance function can be implemented by using delay elements, which are equivalent to inductors, capacitors or imaginary resistors {jR} (energy storage elements) in the s-domain, as illustrated in Chapter 3. In the present chapter, a mathematical transformation method known as bilinear transformation [25] is used to transfer $X(s)$ to a discrete complex reactance function $X(z)$ in the z-plane. The bilinear transformation is given by

$$z = \frac{s+1}{s-1} \tag{4.1}$$

Using this transformation, the left hand side of the s-plane in Figure 4.1.a, can be mapped into the unit circle in the z-plane as shown in Figure 4.1.b.

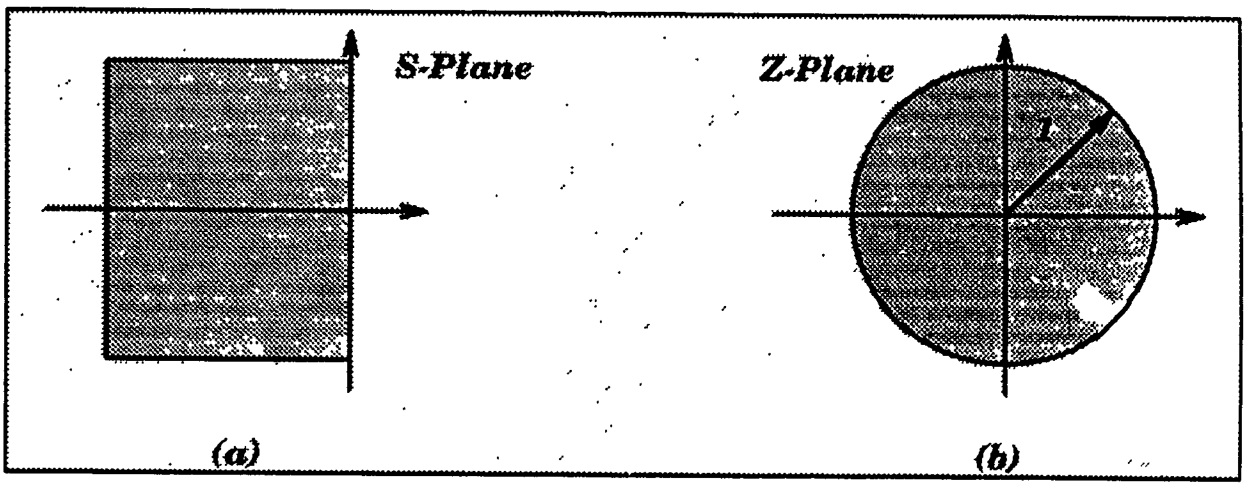

Figure 4.1: (a) Left side of the s-plane. (b) The left side of the s-plane mapped into unit circle in the z-plane by a bilinear transformation.

A complex reactance function $X(s)$ in the continuous domain, given in equation (3.43) can be represented by

$$X(s) = \frac{P(s) - P^{*}(-s)}{P(s) + P^{*}(-s)} \tag{4.2}$$

where $P(s)$ is a complex polynomial with all its roots in the left hand side of the s plane, and $P^*(-s)$ is the polynomial obtained from $P(s)$ by replacing the coefficients by the respective complex conjugates, and s by $-s$. The roots of $P(s)$ and $P^*(-s)$ are symmetric with respect to the imaginary axis as shown in Figure 4.2.a.

An analog complex reactance function $X(s)$ in (4.2) can be transformed to a discrete complex reactance function $X(z)$ using the bilinear transformation (4.1), as follows

$$X(z) = \frac{P(z) - z^N P^*(z^{-1})}{P(z) + z^N P^*(z^{-1})} \tag{4.3}$$

where $P(z)$ is a complex polynomial with all its roots inside the unit circle in the z-plane, and $P^*(z^{-1})$ denotes the polynomial obtained from $P(z)$ by replacing each of its coefficient by its complex conjugate and z by z^{-1}. The roots of $P(z)$ are reciprocals of those of $z^N P^*(z^{-1})$. See Figure 4.2.b.

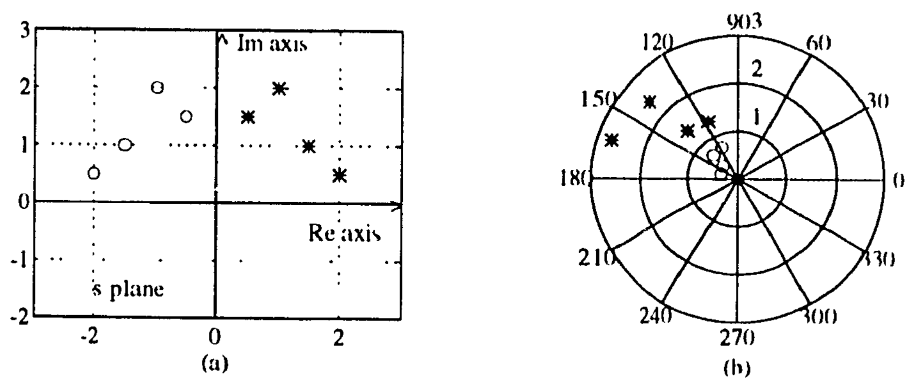

Figure 4.2: (a) Roots of $P(s)$ and $P^*(-s)$ in the s-plane. (b) Roots of $P(z)$ and $z^N P^*(z^{-1})$in the z-plane.

The poles and zeros of a complex reactance function $X(s)$ in (4.2) alternate on the $j\omega$-axis in the s-plane, but they need not appear in conjugate pairs as shown in Figure 4.3.a. The poles and zeros of a discrete complex reactance function $X(z)$ depicted in equation (4.3) alternate on the unit circle in the z-plane, however they need not appear in conjugate pairs, as shown in Figure 4.3.b.

Different methods of realizing a discrete complex reactance function $X(z)$ are

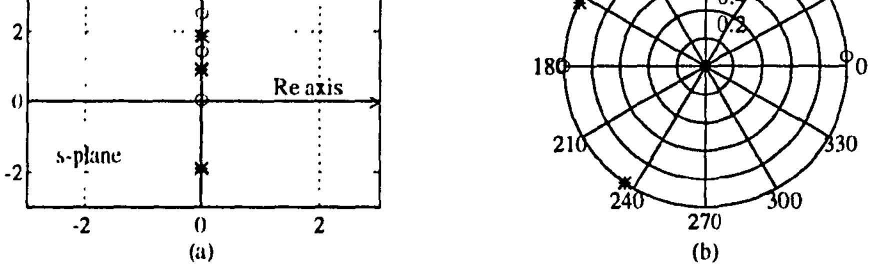

Figure 4.3: (a) Pole-zero diagram of $X(s)$ in the s-plane. (b) Pole-zero diagram of X (z) in the z-plane.

illustrated in the following sections. Classical methods for realizing a system function can be found in [26, 27, 28], and those for realizing complex functions, in [29].

4.2 Algebraic Realization of a Discrete Complex Reactance Function X(z)

A discrete complex reactance function $X(z)$ can be characterized by the following rational function:

$$X(z) = \frac{P(z) - z^N P^*(z^{-1})}{P(z) + z^N P^*(z^{-1})} = \frac{\sum_{k=0}^{N}(a_k + jb_k)z^{-k}}{\sum_{k=0}^{N}(c_k + jd_k)z^{-k}} = \frac{V(z)}{I(z)} \qquad (4.4)$$

The algebraic equation of $X(z)$ is given as

$$(c_0 + d_0)V(z) = -\sum_{k=1}^{N}(c_k + jd_k)z^{-k}V(z) + \sum_{k=0}^{N}(a_k + jb_k)z^{-k}I(z) \qquad (4.5)$$

A realization of a discrete complex reactance function $X(z)$ using (4.5) as shown in Figure 4.4, will be called the algebraic realization. The delay blocks represent a form of storage and delay, $'\times'$ represents a multiplication, and $'\sum'$ represents a summing operation. The number of delay blocks is equal to $2N$. The delay blocks (sample storage elements) are equivalent to the energy storage elements: inductors, capacitors or imaginary resistors in lumped circuit complex reactance function $X(s)$ as we illustrated in Chapter 3.

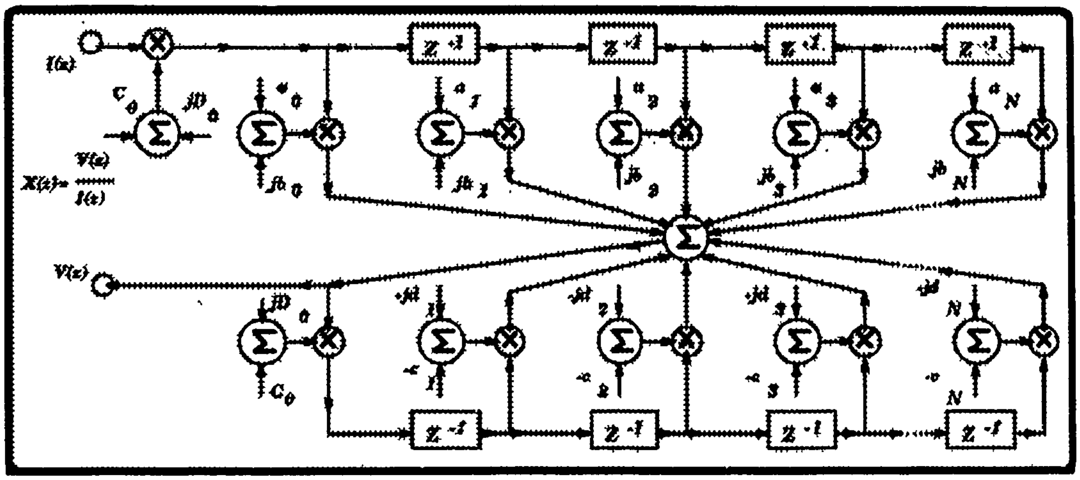

Figure 4.4: Algebraic realization of a discrete complex reactance function X(z).

4.3 Intermediate Result Realization of a Discrete Complex Reactance Function X(z)

Another realization of Equation (4.4) can be obtained by breaking $X(z)$ into a product of two functions $X_1(z)$ and $X_2(z)$, where $X_1(z)$ contains only the denominator or the poles of $X(z)$ and $X_2(z)$ contains only the numerator or the zeros of $X(z)$ as shown below:

$$X(z) = X_1(z) \, . \, X_2(z) = V(z)/I(z) \tag{4.6}$$

where

$$X_1(z) = \frac{1}{\sum_{k=0}^{N}(c_k + jd_k)z^{-k}} \qquad X_2(z) = \sum_{k=0}^{N}(a_k + jb_k)z^{-k} \tag{4.7}$$

The intermediate result $W(z)$, is the output of $X_1(z)$ and the input of $X_2(z)$ as shown in Figure 4.5.

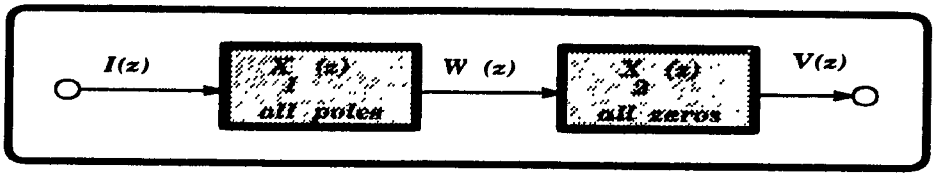

Figure 4.5: Decomposition of a discrete complex reactance function X(z).

$X_1(z)$ and $X_2(z)$ in Equation (4.7) can be written in the form

$$X_1(z) = \frac{W(z)}{I(z)} = \frac{1}{\sum_{k=0}^{N}(c_k + jd_k)z^{-k}} \tag{4.8}$$

$$X_2(z) = \frac{V(z)}{W(z)} = \sum_{k=0}^{N}(a_k + jb_k)z^{-k} \tag{4.9}$$

The algebraic equations involving $X_1(z)$ and $X_2(z)$ are

$$(c_0 + d_0)W(z) = -\sum_{k=1}^{N}(c_k + jd_k)z^{-k}W(z) + I(z) \tag{4.10}$$

$$V(z) = -\sum_{k=0}^{N}(a_k + jb_k)z^{-k}W(z) \tag{4.11}$$

A realization of $X(z)$ using equations (4.10) and (4.11), shown in Figure 4.6, will be called the intermediate result realization.

For simplicity, the analysis of $X(z)$ can be started in (4.4) by making the coefficient $(c_0 + jd_0)$ equal to unity. Then, the two branches of delay elements in Figure 4.6, can be combined into one branch with the delay elements as shown in Figure 4.7. This will be called the intermediate result simple realization of a discrete complex reactance function $X(z)$. The number of delay blocks is equal to N, which is the order of the algebraic equation. It can be shown that N is the minimum number of delay elements.

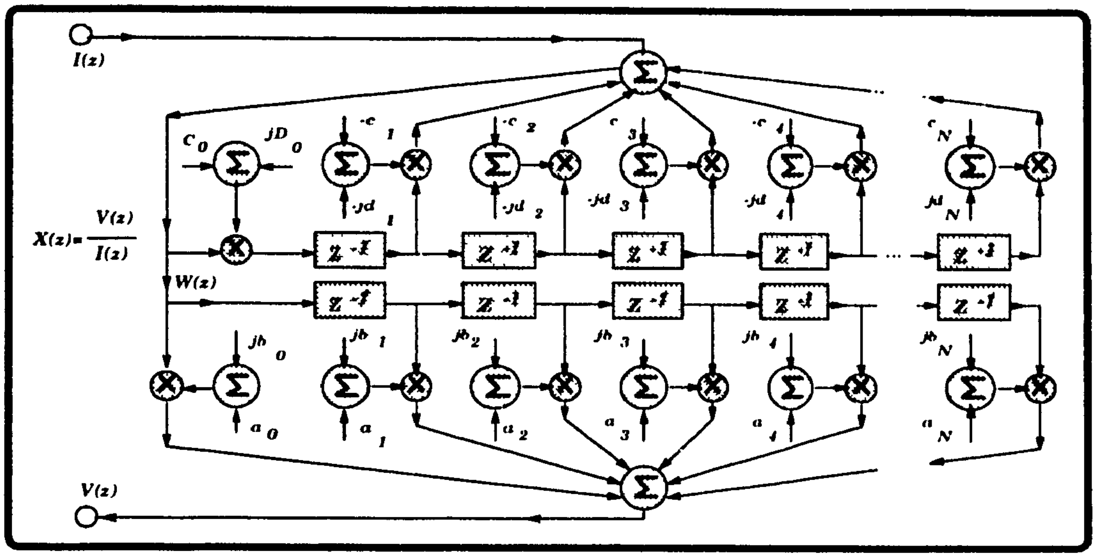

Figure 4.6: Intermediate result realization of a discrete complex reactance function X(z).

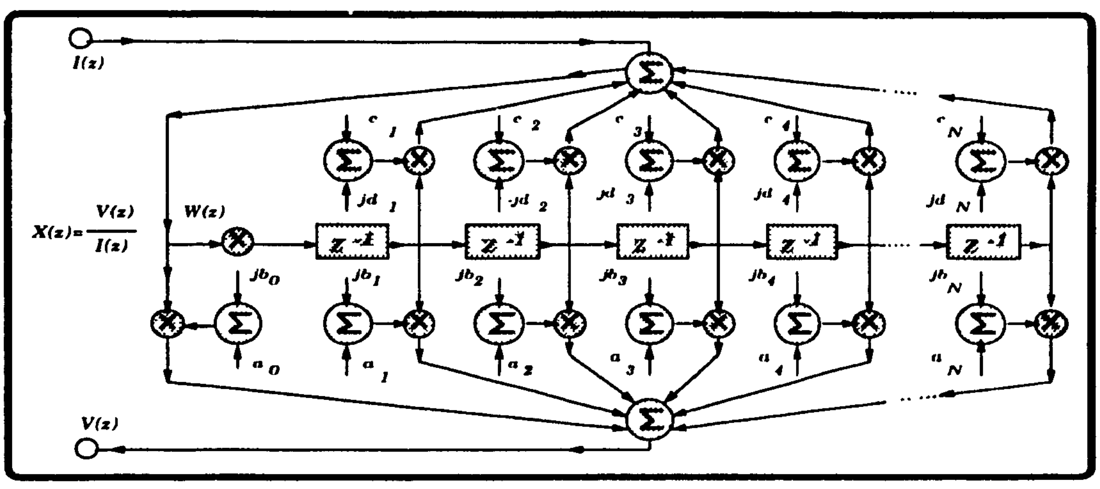

Figure 4.7: Intermediate result simple realization of a discrete complex reactance function X(z).

4.4 Cascade Realization of a Discrete Complex Reactance Function X(z)

In the cascade realization, a discrete complex reactance function $X(z)$ is broken into a product of complex reactance functions $X_1(z), X_2(z), ..., X_k(z)$,as follows

$$X_1(z) \,.\, X_2(z) \,.\, X_3(z) \,.\, ... \, X_k(z) \tag{4.12}$$

and is shown in Figure 4.8

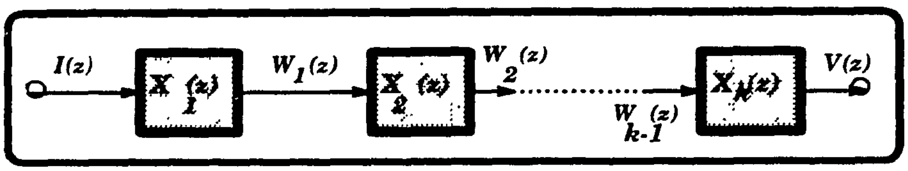

Figure 4.8: Cascade representation of a discrete complex reactance function X(z).

Each X_k is a rational expression in z^{-1}. $X(z)$ could be broken up in many ways. However, the most common cascade realization requires each of the X_k's to be of the form:

$$X_k(z) = \frac{(a_{0k} + jb_{0k}) + (a_{1k} + jb_{1k})z^{-1}}{(c_{0k} + jd_{0k}) + (c_{1k} + jd_{1k})z^{-1}} \tag{4.13}$$

By letting a_{1k}, b_{1k} equal to zero, $X_k(z)$ will contains only poles. Letting c_{1k} and d_{1k} equal to zero, $X_k(z)$ will contains only zeros. Each of the $X_k(z)$ could then realized by using either the algebraic realization as in Section 4.2 or the intermediate result simple realization as in Section 4.3. For simplicity, the coefficient $(c_{0k} + jd_{0k})$ could be made equal to unity. The general cascade realization using the algebraic realization is shown in Figure 4.9, and the intermediate simple realization is shown in Figure 4.10.

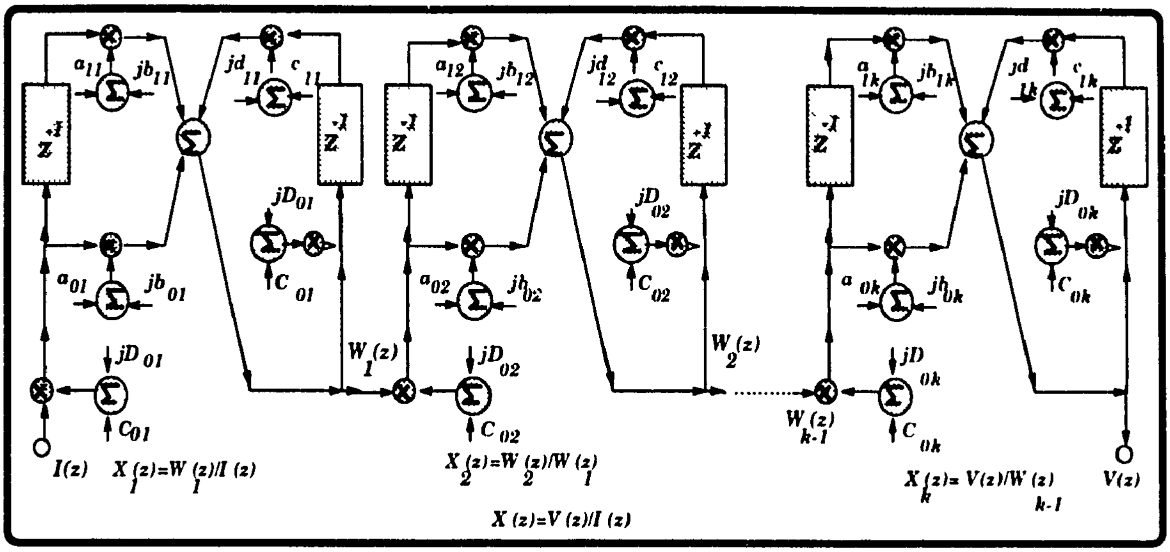

Figure 4.9: A cascade algebraic realization of a discrete complex reactance function X(z).

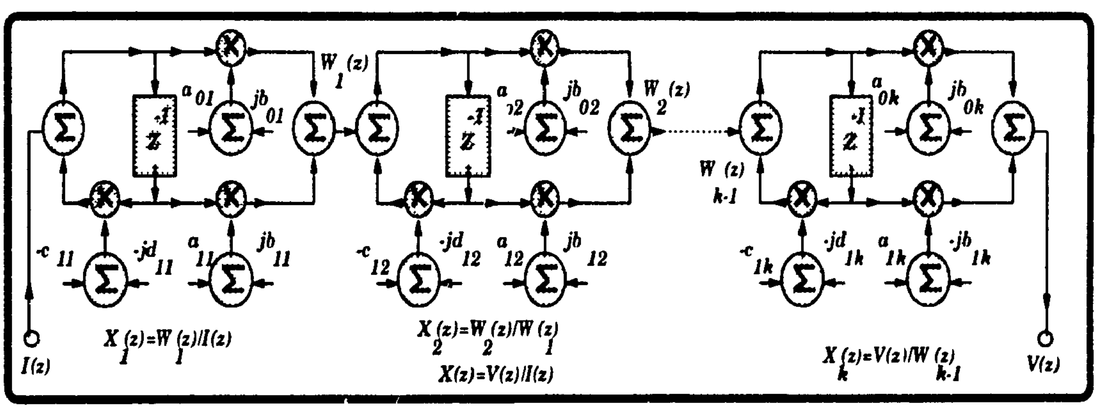

Figure 4.10: A cascade intermediate result simple realization of a discrete complex reactance function X(z).

4.5 Partial Fraction Expansion (Parallel) Realization of a Discrete Complex Reactance Function X(z)

In the parallel realization, a discrete complex reactance function $X(z)$ can be written as a sum of complex reactance functions $X_1(z)$, $X_2(z)$, $X_3(z)$, ... , $X_k(z)$ as follows:

$$X(z) = X_1(z) + X_2(z) + X_3(z) + ... + X_k(z) \tag{4.14}$$

This is shown in Figure 4.11.

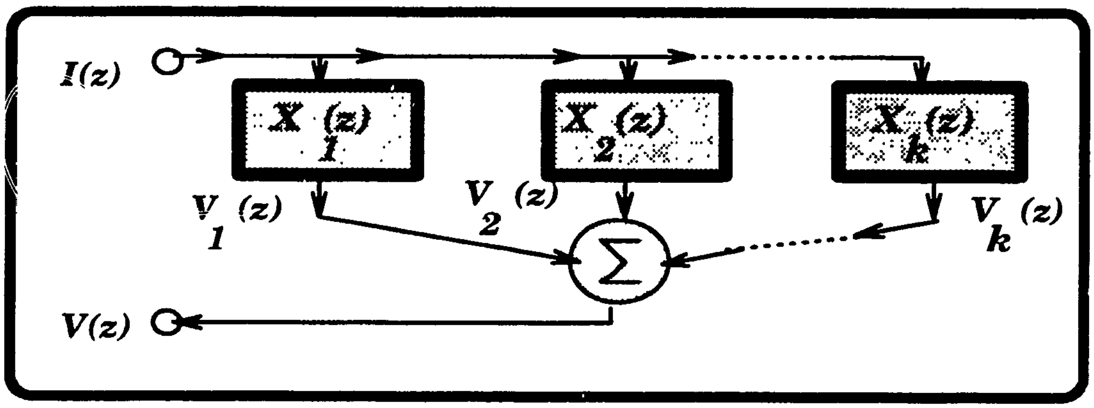

Figure 4.11: Parallel representation of a discrete complex reactance function X(z).

Each X_k is a rational function in z^{-1}. $X(z)$ could be broken up in many ways; however, the most common parallel realization requires each of the X_k's to take the same form as in (4.13). Note that $X_k(z)$ includes a special case. For example, by letting a_{1k} , b_{1k} , c_{1k} , and d_{1k} equal zero, $X_k(z)$ contains neither poles nor zeros. And letting $c_{1k} + jd_{1k}$ equal to zero, $X_k(z)$ will contain only zeros; and letting a_{1k} , b_{1k} equal to zero, $X_k(z)$ will contain only poles. Each of the $X_k(z)$ can then be realized using either the algebraic realization or the intermediate simple realization. The general parallel realization is shown in Figures 4.12 and 4.13 .

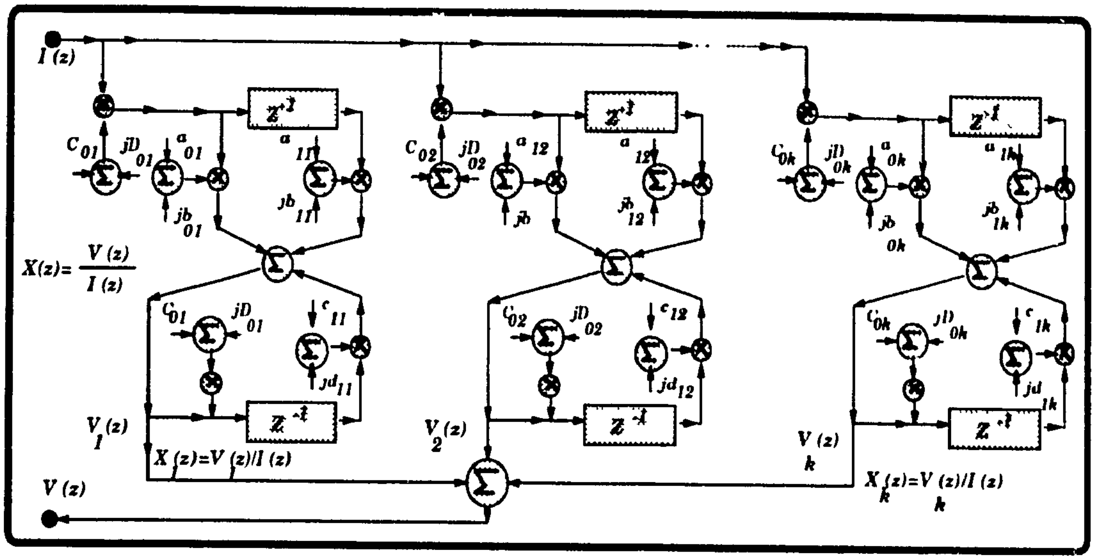

Figure 4.12: A parallel algebraic realization of a discrete complex reactance function X(z).

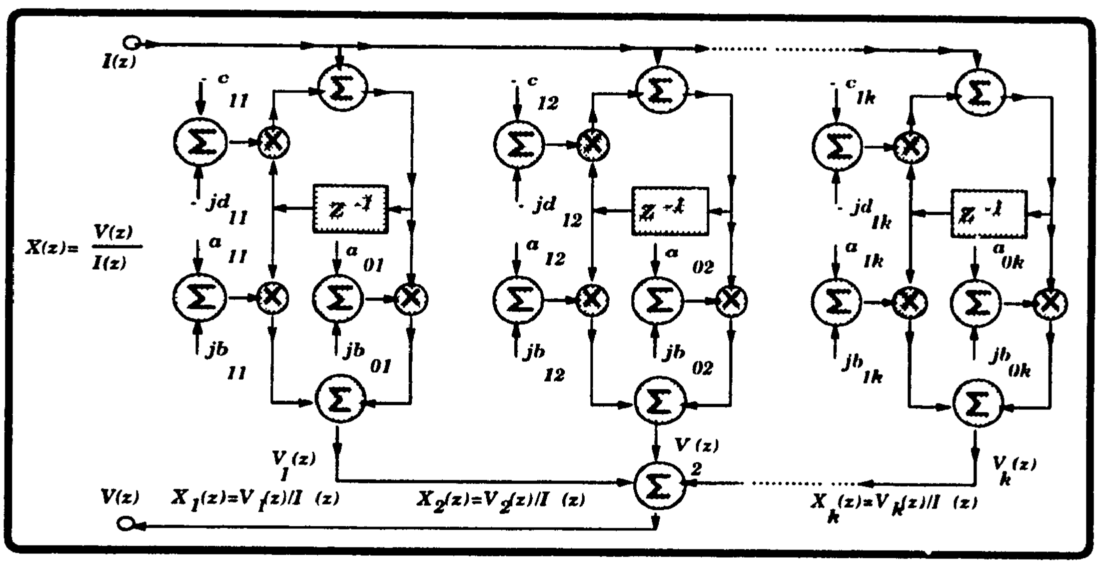

Figure 4.13: A parallel intermediate result simple realization of a discrete complex reactance function X(z).

4.6 J-Fraction Expansion Realization of a Discrete Complex Reactance Function X(z)

In the J-fraction expansion illustrated in Chapter 3, the realization of a discrete complex reactance function $X(z)$, given by

$$X(z) = \frac{(a_0 + jb_0)z^N + (a_1 + jb_1)z^{N-1} + (a_2 + jb_2)z^{N-2} + \ldots + (a_N + jb_N)z^0}{(c_0 + jd_0)z^N + (c_1 + jd_1)z^{N-1} + (c_2 + jd_2)z^{N-2} + \ldots + (c_N + jd_N)z^0} \tag{4.15}$$

can be written as:

$$X(s) = (d_0 + jb_0) + \frac{1}{(q_1 + jp_1)z + (d_1 + jb_1) + \frac{1}{(q_2+jp_2)z+(d_2+jb_2)+\frac{1}{(q_3+jp_3)z+(d_3+jb_3)+\frac{1}{+}}}} \tag{4.16}$$

$X(z)$ in equation (4.16) can be realized through a series of synthesis cycles involving the extraction of the first and the zeroth order complex functions as follows

$$X(z) \;=\; (d_0 + jb_0) \;+\; X_1(z) \tag{4.17}$$

This yields a parallel continuation of $d_0 + jb_0$ with $X_1(z)$. The complex function $X_1(z)$ is now given by

$$X_1(z) = \frac{1}{(q_1 + jp_1)z + (d_1 + jb_1) + X_2(z)} = \frac{\frac{z^{-1}}{(q_1+jp_1)+(d_1+jb_1)z^{-1}}}{1 + \frac{z^{-1}}{(q_1+jp_1)+(d_1+jb_1)z^{-1}} X_2(z)} \tag{4.18}$$

$$X_1(z) = \frac{\frac{(q_1+jp_1)^{-1}z^{-1}}{1+(d_1+jb_1)(q_1+jp_1)^{-1}z^{-1}}}{1 + \frac{(q_1+jp_1)^{1}z^{-1}}{1+(d_1+jb_1)(q_1+jp_1)^{-1}z^{-1}}.X_2(z)} = \frac{\frac{(Q_1+jP_1)z^{-1}}{1+(d_1+jb_1)(Q_1+jP_1)z^{-1}}}{1 + \frac{(Q_1+jP_1)z^{-1}}{1+(d_1+jb_1)(Q_1+jP_1)z^{-1}}.X_2(z)} \tag{4.19}$$

Thus $X_1(z)$ can be realized as a closed loop with a feed-forward term and a feed-back term $X_2(z)$. The feed-forward term itself can be realised as a closed loop with a feed-forward term $((q_1 + jp_1)\ z)^{-1}$ and a feed-back term $(d_1 + jb_1)$. The feed-back term $X_2(z)$ has an expression given by

$$X_2(z) = \frac{1}{(q_2 + jp_2)z + (d_2 + jb_2) + X_3(z)} \tag{4.20}$$

and

$$X_3(z) = \frac{1}{(q_3 + jp_3)z + (d_3 + jb_3) + X_4(z)} \tag{4.21}$$

Continuation of the realization of X_i , $i = 1, 2, 3, \ldots$ yields the realization shown in Figure 4.14

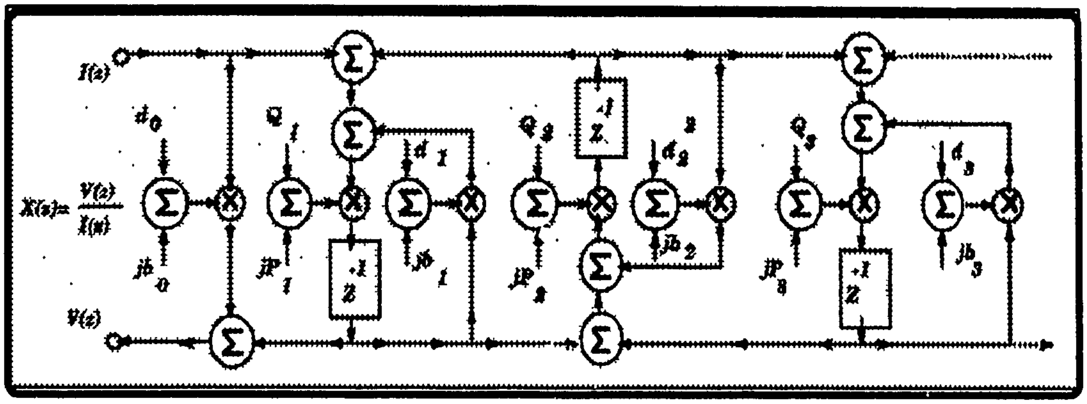

Figure 4.14: J-fraction expansion realization of a discrete complex reactance function X(z).

4.7 S-Fraction Expansion Realization of a Discrete Complex Reactance Function X(z)

In the S-fraction expansion realization, a discrete complex reactance function $X(z)$ can be written as follows;

$$X(s) = (e_0 + jf_0) + \cfrac{1}{(g_1 + jh_1)z + \cfrac{1}{(e_2+jf_2) + \cfrac{1}{(g_3+jh_3)z + \cfrac{1}{(e_4+jf_4) + \cfrac{1}{\cdots}}}}} \tag{4.22}$$

X(z) in (4.22) can be realised through a series of synthesis cycles involving the extraction of a constant, and the delay elements as follows

$$X(z) = (e_0 + jf_0) + X_1(z) \tag{4.23}$$

which yield a parallel continuation of $e_0 + jf_0$ with $X_1(z)$. The complex function $X_1(z)$ is now given by

$$X_1(z) = \frac{1}{(g_1 + jh_1)z + X_2(z)} = \frac{(g_1 + jh_1)^{-1}z^{-1}}{1 + (g_1 + jh_1)^{-1}z^{-1}X_2(z)} = \frac{(G_1 + jH_1)z^{-1}}{1 + (G_1 + jH_1)z^{-1}X_2(z)} \tag{4.24}$$

which implies the removal of a closed loop with a feed-forward term $z^{-1}(g_1 + jh_1)^{-1}$ and a feed-back term of $X_2(z)$. This means the removal of a multiplier of value $(G_1 + jH_1)$ in series with a delay z^{-1}. The discrete complex reactance $X_2(z)$ is given by

$$X_2(z) = \frac{1}{(e_2 + jf_2)z + X_3(z)} = \frac{(e_2 + jf_2)^{-1}z^{-1}}{1 + (e_2 + jf_2)^{-1}z^{-1}X_3(z)} = \frac{(E_2 + jF_2)z^{-1}}{1 + (E_1 + jF_1)z^{-1}X_3(z)} \tag{4.25}$$

$X_2(z)$ is realised as a closed loop having a feed-forward term of $(e_2 + jf_2)^{-1}z^{-1}$ and a feed-back term of $X_3(z)$ which is given by

$$X_3(z) = \frac{1}{(g_2 + jh_2)z + X_4(z)} \tag{4.26}$$

Continuation of the realization of X_i , $i = 1, 2, 3, \ldots$ yields the realization shown in Figure 4.15.

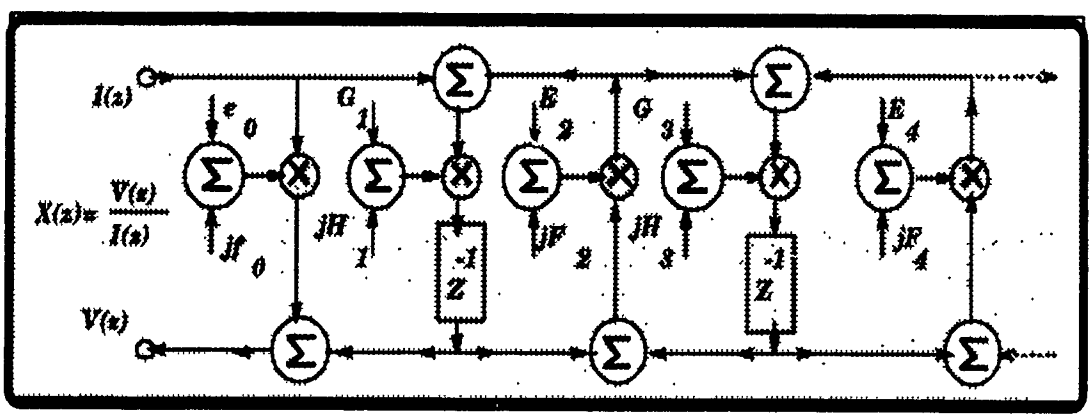

Figure 4.15: S-fraction expansion realization of a discrete complex reactance function X(z).

4.8 An Alternative S-Fraction Expansion Realization of a Discrete Complex Reactance Function X(z)

By arranging both the numerator and the denominator of a discrete complex reactance function $X(z)$ in ascending powers of z, we get:

$$X(z) = \frac{(a_N + jb_N)z^0 + ... + (a_2 + jb_2)z^{N-2} + (a_1 + jb_1)z^{N-1} + (a_0 + jb_0)z^N}{(c_N + jd_N)z^0 + ... + (c_2 + jd_2)z^{N-2} + (c_1 + jd_1)z^{N-1} + (c_0 + jd_0)z^N} \tag{4.27}$$

The S-fraction expansion realization of $X(z)$ can be written as follows;

$$X(s) = (r_0 + ju_0) + \cfrac{1}{(k_1 + jl_1)z^{-1} + \cfrac{1}{(r_2 + ju_2) + \cfrac{1}{(k_3 + jl_3)z^{-1} + \cfrac{1}{(r_4 + ju_4) + \frac{1}{\cdot}}}}} \tag{4.28}$$

$X(z)$ in (4.28) can be realised through a series of synthesis cycles involving the extraction of the first and the zeroth order complex functions as follows

$$X(z) \; = \; (r_0 + ju_0) \; + \; X_1(z) \tag{4.29}$$

which yield a parallel continuation of $r_0 + ju_0$ with $X_1(z)$. The complex function $X_1(z)$ is now given by

$$X_1(z) = \frac{1}{(k_1 + jl_1)z^{-1} + \frac{1}{X_2(z)}} = \frac{1}{(k_1 + jl_1)z^{-1} + X_2^{-1}(z)} = \frac{X_2(z)}{1 + (k_1 + jl_1)z^{-1}X_2(z)} \tag{4.30}$$

The term $X_2(z)$ and $(k_1 + jl_1)z^{-1}$ can be extracted from $X_1(z)$ as a feed-forward and feed-back in the closed loop of $X_1(z)$.

$$X_2(z) \; = \; (r_2 + ju_2) \; + \; X_3(z) \tag{4.31}$$

Continuation of the realization of X_i , $i = 1, 2, 3, ...$ yields the realization shown in Figure 4.16.

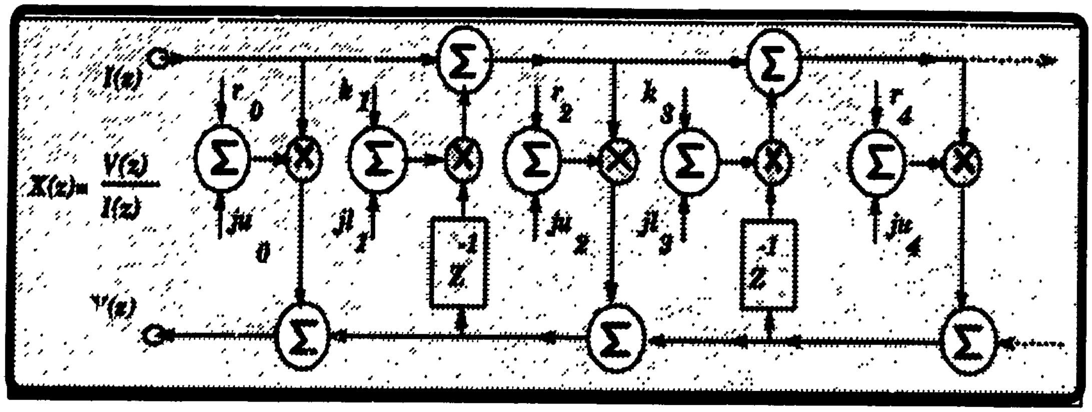

Figure 4.16: An alternative S-fraction expansion realization of a discrete complex reactance function X(z).

The following example illustrates the synthesis of $X(z)$.

Example 4.1 *Consider the complex polynomial $P(s)$ of Example 2.1,*

$$P(s) = s^4 + (5 - j5)s^3 + (0 - j19.75)s^2 + (-18.375 - j17.875)s - (13.125 - j0.625) \tag{4.32}$$

The roots of $P(s)$ are given by

$$\lambda_1 = -2 + j0.5, \quad \lambda_2 = -1.5 + j1, \quad \lambda_3 = -1 + j2, \quad \lambda_4 = -0.5 + j1.5 \tag{4.33}$$

which lie in the left half of the s-plane. The polynomial $P^*(-s)$ is given by

$$P^*(-s) = s^4 - (5 + j5)s^3 + (0 + j19.75)s^2 + (18.375 - j17.875)s - (13.125 + j0.625) \tag{4.34}$$

The roots of $P^*(-s)$ are seen to lie in the right half of the s-plane:

$$\lambda_1 = 2 + j0.5, \quad \lambda_2 = 1.5 + j1, \quad \lambda_3 = 1 + j2, \quad \lambda_4 = 0.5 + j1.5 \tag{4.35}$$

It should be observed that the roots of $P(s)$ and those of $P^*(-s)$ are symmetrically located with respect to the imaginary axis. The bilinear transformation given in (4.1) maps the complex polynomials $P(s)$ and $P^*(-s)$ onto the z-plane as follows

$$\begin{aligned} P(z) &= [(-25.5 - j42)z^4 + (-103.25 - j23.25)z^3 + (-72.75 + j43.25)z^2 \\ &+ (-9.75 + 28.25)z + (1.25 + j3.75)](z - 1)^{-4} \end{aligned} \tag{4.36}$$

The roots, which lie inside the unit circle in the z-plane, are given by

$$\zeta_1 = -0.3333 + j0.6667 \qquad \zeta_2 = -.0.5 + j0.5$$
$$\zeta_3 = -.0.3103 + j0.2759 \qquad \zeta_4 = -0.3514 + j10.81 \tag{4.37}$$

In the polar form, the roots are:

$$\zeta_1 = 0.7454e^{j2.0344} \qquad \zeta_2 = 0.7071e^{j2.3562}$$
$$\zeta_3 = 0.4152e^{j2.415} \qquad \zeta_4 = 0.3676e^{j2.8431} \tag{4.38}$$

The polynomial $z^N P^*(z^{-1})$ can be obtained from $P^*(-s)$ using equation (4.1) or directly from $P(z)$ in (4.36)

$$\begin{aligned} z^N P(z^{-1}) &= [(1.25 - j3.75)z^4 + (-9.75 - j28.25)z^3 + (-72.75 + j43.25)z^2 \\ &+ (-103.25 + j23.25)z + (-25.5 + j42)](z-1)^{-4} \end{aligned} \tag{4.39}$$

whose roots lie outside the unit circle in the z-plane, and are given by

$$\eta_1 = -2.6 + j0.8 \qquad \eta_2 = -1.8 + j1.6$$
$$\eta_3 = -1 + j \qquad \eta_4 = -0.6 + j1.2 \tag{4.40}$$

In the polar form, the roots are:

$$\eta_1 = 2.7203e^{j2.8431} \qquad \eta_2 = 2.4083e^{j2.415}$$
$$\eta_3 = 1.4141e^{j2.3562} \qquad \eta_4 = 1.3416e^{j2.0344} \tag{4.41}$$

From equations (4.32), (4.34) and (4.2), a complex reactance $X(s)$ in the s-plane can be written as

$$\begin{aligned} X(s) &= \frac{\mathrm{P(s)} - \mathrm{P^*(-s)}}{\mathrm{P(s)} + \mathrm{P^*(-s)}} \\ &= \frac{5s^3 - j19.75s^2 - 18.375s - j0.625}{s^4 - j5s^3 + 0s^2 - j17.875s - 13.125} \end{aligned} \tag{4.42}$$

where the poles and zeros of $X(s)$ alternate on the $j\omega$-axis i.e.,

$$\begin{aligned} &\textit{Zeros of } X(s) \textit{ are} \quad z_1 = +j2.5 \qquad z_2 = +j1.4147 \qquad z_3 = +j0.0353 \\ &\textit{Poles of } X(s) \textit{ are} \quad p_1 = +j4.1433 \qquad p_2 = +j1.8499 \qquad p_3 = +j0.9031 \\ &\qquad\qquad\qquad\quad P_4 = -j1.8962 \end{aligned} \tag{4.43}$$

Also from equations (4.36), (4.39) and (4.3) a complex reactance $X(z)$ in the z-plane can be written as

$$X(z) = \frac{P(z) + z^N P(z^{-1})}{P(z) + z^N P(z^{-1})}$$

$$= \frac{(-26.75-j38.25)z^4+(-93.5+j5)z^3(0+j86.5)z^2+(93.5+j5)z+(26.75-j38.75)}{(-24.25-j45.75)z^4+(-113-j51.5)z^3+(-145.5)z^2+(-113+j51.5)z^1+(-24.25+j45.75)} \quad (4.44)$$

where poles and zeros of $X(z)$ alternate on the unit circle in the z-plane i.e.,

$$\begin{aligned} Zeros\ of\ X(z)\ are\quad & z_1 = 0.9975 + j0.0706 \quad & z_2 = -0.3336 + j0.9427 \\ & z_3 = -0.7241 + j0.6897 \quad & z_4 = -1 \\ Poles\ of\ X(z)\ are\quad & p_1 = -0.5648 - j0.8252 \quad & p_2 = -0.8899 + j0.4561 \\ & p_3 = -0.5477 + j0.8367 \quad & P_4 = 0.1016 + j0.9948 \end{aligned} \quad (4.45)$$

In the polar form,

$$\begin{aligned} Zeros\ of\ X(z)\ are\quad & z_1 = 1e^{0.0707} \quad & z_2 = 1e^{1.9109} \\ & z_3 = 1e^{2.3806} \quad & z_4 = 1e^{3.1416} \\ Poles\ of\ X(z)\ are\quad & p_1 = 1e^{-2.1710} \quad & p_2 = 1e^{2.6679} \\ & p_3 = 1e^{2.1504} \quad & P_4 = 1e^{1.469} \end{aligned} \quad (4.46)$$

$X(z)$ can be broken into a product of complex reactance functions as follows

$X(z) =$

$$\left(\frac{(-0.2235-j16.4632)-(0.9393-j16.4378)z^{-1}}{(0.4015-j13.4487)-(-11.3246+j7.2645)z^{-1}}\right) \cdot \left(\frac{(0.7379+j0.6415)-(-0.8509+j0.4816)z^{-1}}{(0.6892+j0.6122)-(5.7767+j12.1511)z^{-1}}\right) \cdot$$

$$\left(\frac{(0.4433-j0.0341)-(-0\ 2975+j0.3304)z^{-1}}{(0.8023-j0.6978)-(0.1444+j1.0535)z^{-1}}\right) \cdot \left(\frac{(2.1450-j6.1580)-(-2.1450+j6.1580)z^{-1}}{(3.3920-j1.9770)-(2.3113+j3.1735)z^{-1}}\right) \quad (4.47)$$

We can now use algebraic or intermediate simple realization methods to get a cascade realization as in Figures 4.9 and 4.10.

Also the complex reactance in (4.44) can be synthesized using the J-fraction as in (4.16) or the S-fraction as in (4.22). Another alternative synthesis of $X(z)$

can be obtained by rearranging both the denominator and the numerator of $X(z)$ in ascending powers of z as follows:

$$X(z) =$$

$$\frac{(26.75-j38.25)z^0+(93.5+j5)z^1(0+j86.5)z^2+(-93.5+j5)z^3+(-26.75-j38.75)z^4}{(-24.25+j45.75)z^0+(-113+j51.5)z^1+(-145.5)z^2+(-113-j51.5)z^3+(-24.25-j45.75)z^4} \tag{4.48}$$

and the S-fraction of (4.48) will take the form as in (4.28). Further, if we rewrite $X(z)$ in (4.44) in decreasing powers of z^{-1}, we get

$$X(z) =$$

$$\frac{(-26.75-j38.25)z^0+(-93.5+j5)z^{-1}(0+j86.5)z^{-2}+(93.5+j5)z^{-3}+(26.75-j38.25)z^{-4}}{(-24.25-j45.75)z^0+(-113-j51.5)z^{-1}+(-145.5)z^{-2}+(-113+j51.5)z^{-3}+(-24.25+j45.75)z^{-4}} \tag{4.49}$$

We can now use the algebraic realization method as in (4.5) or the intermediate realization method as in (4.7) to synthesise $X(z)$. In both methods, the number of delay elements is $2N$. For simplicity, if we divide both the denominator and the numerator of $X(z)$ in (4.49) by the constant term of the denominator, this will reduce the number of delay elements to N

$$X(z) =$$

$$\frac{(0.8946-j0.1105)z^0+(0.7604-j1.6407)z^{-1}(-1.4760-j0.7824)z^{-2}+(-0.9310+j1.5502)z^{-3}+(0.4107+j0.8024)z^{-4}}{(1z^0+(1.9008-j1.4624)z^{-1}+(1.3160-j2.4828)z^{-2}+(0.1433-j2.3940)z^{-3}+(-0.5613-j0.8276)z^{-4}} \tag{4.50}$$

The synthesis of $X(z)$ in 4.50 is called intermediate result simple realization, shown in Figure 4.7.

Chapter 5

SYNTHESIS OF COMPLEX REACTANCE TWO-PORT NETWORKS in s-plane

5.1 Introduction

In electronics and communication, two-port networks are the most commonly used circuits. This chapter is an extension to Chapter 3 and addresses the complex reactance of a two-port networks in the s-plane. An open circuit impedance parameters matrix of a two-port network is obtained from any stable complex polynomial $P(s)$. J-fraction expansion, S-fraction expansion, and partial fraction expansion [19, 21, 30, 32] are used to synthesise any complex impedance matrix of two-port network by R, jR, L, and C elements; and any complex reactance matrix $[X(s)]$ by jR, L, and C elements, and jR, L or jR, C elements. In this chapter, realizability conditions for complex impedance and complex reactance two-port parameters are given.

5.2 Realizability Conditions for Impedance or Reactance Two-Port Parameters

Consider a linear passive (R, jR, L, C) two-port circuit shown in Figure 5.1.

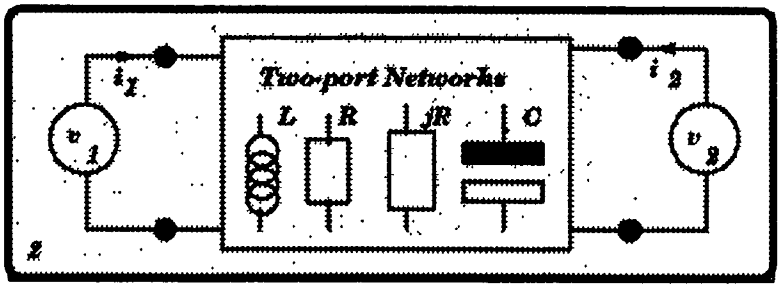

Figure 5.1: Two-port linear circuit with the notations used in the analysis.

From Figure 5.1, the V-I relationship is given by

$$[V(s)] = [Z(s)]\,[I(s)] \tag{5.1}$$

where $[Z(s)]$ is the open circuit impedance matrix

$$[Z(s)] = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \tag{5.2}$$

The necessary conditions for an open circuit impedance matrix $[Z(s)]$ of any complex impedance two port network are

- All elements $z_{11}(s)$, $z_{12}(s)$ and $z_{22}(s)$ of $[Z(s)]$ must be complex rational function of s, with $z_{12} = z_{21}$.
- All elements $z_{11}(s)$ and $z_{22}(s)$ of $[Z(s)]$ must have a J-fraction expansion with its stated properties (see Theorem 2.1).

- Each element $z_{11}(s)$, $z_{12}(s)$ and $z_{22}(s)$ of $[Z(s)]$ must have a partial-fraction expansion with real residues $k_{11}(s)$, $k_{12}(s)$ and $k_{22}(s)$ respectively and at each pole it should satisfy the residue conditions, i.e.

$$k_{11} \geq 0 \qquad k_{22} \geq 0 \qquad k_{11}\, k_{22} - k_{12}^2 \geq 0 \tag{5.3}$$

- The zeros of $z_{12}(s)$ can be anywhere in the s-plane but the poles must also be present in $z_{11}(s)$ or $z_{22}(s)$ or both. In the reverse situation, a pole for either or both of $z_{11}(s)$ and $z_{22}(s)$ but not for $z_{12}(s)$ is possible.

5.3 Construction and Synthesis of a Complex Impedance Matrix of Two-Port Networks by (R, jR, L, C)

5.3.1 Using J-fraction expansion

Consider a polynomial $P(s)$ of degree n with complex coefficients whose roots are all in the left-half of the s-plane as in Equation (2.1). The alternant of $P(s)$ is $Q(s)$ as in Equation (2.2). Since $z_{11}(s)$ and $z_{22}(s)$ are driving point impedances of the two-port network, then

$$z_{11}(s) = \frac{P(s)}{Q(s)} \tag{5.4}$$

and

$$z_{22}(s) = \frac{P_1(s)}{Q(s)} \tag{5.5}$$

Now we shall show that the the polynomial $P_1\ (s)$ in equation (5.5), depends on the polynomial $P(s)$. Using equation (5.4), a rational function complex impedance $z_{11}(s)$ can be written as

$$Z_{11}(s) = \frac{P(s)}{Q(s)} = \frac{s^n + (a_1 + jb_1)s^{n-1} + (a_2 + jb_2)s^{n-2} + \ldots + (a_n + jb_n)}{a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + a_5 s^{n-5} + \ldots} \tag{5.6}$$

then the J-fraction expansion of $z_{11}(s)$ in (5.6) becomes

$$Z_{11}(s) = L_{11}s + 1 + jR_{11} + \cfrac{1}{C_2s + jG_2 + \cfrac{1}{L_3s+jR_3+\cfrac{1}{C_4s+jG_4+\cfrac{1}{+\cfrac{1}{C_ns+jG_n}}}}} \qquad (5.7)$$

Now, if the arrangement of the last $(n-1)$ terms in the J-fraction (5.7) are reversed and the value of L_{11} changed to another value L_{22} (arbitrary), applying the same for R_{11}, then the right side of equation (5.7) can be written as

$$L_{22}s + 1 + jR_{22} + \cfrac{1}{C_ns + jG_n + \cfrac{1}{L_{n-1}s+jR_{n-1}+\cfrac{1}{+\cfrac{1}{L_3s+jR_3+\cfrac{1}{C_2s+jG_2}}}}} = \frac{P_1(s)}{Q(s)} \qquad (5.8)$$

where $P_1(s)/Q(s)$ is a rational function obtained from the J-fraction in (5.8) and it represents the driving point impedance $z_{22}(s)$ given by

$$Z_{22} = L_{22}s + 1 + jR_{22} + \cfrac{1}{C_ns + jG_n + \cfrac{1}{L_{n-1}s+jR_{n-1}+\cfrac{1}{+\cfrac{1}{C_2s+jG_2}}}} = \frac{P_1(s)}{Q(s)} \qquad (5.9)$$

At this point, we have defined and found two elements $z_{11}(s)$ and $z_{22}(s)$ in the open circuit impedance parameters matrix $[Z(s)]$ of a complex impedance two-port network. The only element left now in the matrix $[Z(s)]$ is $z_{12}(s)$ which can be defined as

$$z_{12}(s) = \frac{N(s)}{Q(s)} \qquad (5.10)$$

and the polynomial $N(s)$ is equal to the multiplication of the inverse of the admittance terms in the J-fraction expansion (5.7) or (5.9). These terms are the shunt elements in the ladder two-port network, therefore $N(s)$ is given by

$$N(s) = (C_2s+jG_2)^{-1}(C_4s+jG_4)^{-1}(C_6s+jG_6)^{-1}....(C_{n-1}s+jG_{n-1})^{-1}(C_ns+jG_n)^{-1} \qquad (5.11)$$

and $z_{12}(s)$ in (5.10) becomes

$$Z_{12}(s) = \frac{(C_2s + jG_2)^{-1}(C_4s + jG_4)^{-1}(C_6s + jG_6)^{-1}....(C_{n-1}s + jG_{n-1})^{-1}(C_ns + jG_n)^{-1}}{a_1s^{n-1} + jb_2s^{n-2} + a_3s^{n-3} + jb_4s^{n-4} + a_5s^{n-5} + ...} \qquad (5.12)$$

After the derivation of $z_{11}(s)$, $z_{12}(s)$, and $z_{22}(s)$, the open circuit impedance parameters of the two-port complex impedance network in (5.2) can be put in this form

$$[Z(s)] = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} = \begin{bmatrix} \frac{P(s)}{Q(s)} & \frac{N(s)}{Q(s)} \\ \frac{N(s)}{Q(s)} & \frac{P_1(s)}{Q(s)} \end{bmatrix}. \tag{5.13}$$

$$[Z(s)] = \begin{bmatrix} L_{11}s + 1 + jR_{11} + \frac{1}{C_2s+jG_2+\frac{1}{+\frac{1}{C_ns+jG_n}}} & \frac{N(s)}{Q(s)} \\ \frac{N(s)}{Q(s)} & L_{22}s + 1 + jR_{22} + \frac{1}{C_ns+jG_n+\frac{1}{\cdot+\frac{1}{C_2s+jG_2}}} \end{bmatrix} \tag{5.14}$$

The two-port network realization of the impedance matrix $[Z(s)]$ which is synthesized by J-fraction expansion is shown in Figure 5.2.

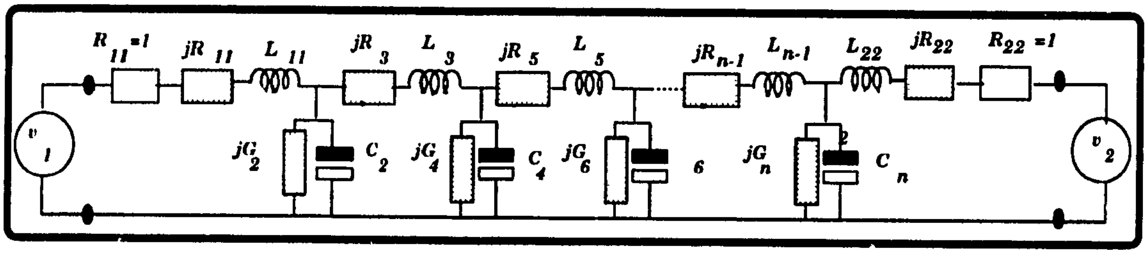

Figure 5.2: Two port realization of $Z(s)$ using J-Fraction expansion.

5.3.2 Using Partial Fraction Expansion

From a given complex $P(s)$ and its alternant $Q(s)$, the open circuit impedance parameters z_{11}, z_{12} and z_{22} of the matrix $[Z(s)]$ are constructed as in (5.14). Once the matrix $[Z(s)]$ is constructed with each element being a rational function as in equation (5.13), meeting the realizability conditions described in Section 5.2, then we can always design an impedance two-port network using a partial-fraction expansion.

Since $z_{11}(s)$ and $z_{22}(s)$ are driving-point impedances measured at the input and the output ports, respectively, they can be expanded into partial fractions as

$$z_{11}(s) = R_{11} + jR_{11} + L_{11}s + \sum_{k=1}^{n} \frac{A_{11k}}{s + js_k} \tag{5.15}$$

$$z_{22}(s) \;=\; R_{22} + jR_{22} \;+\; L_{22}s \;+\; \sum_{k=1}^{n} \frac{A_{22k}}{s + js_k} \qquad (5.16)$$

and the transfer impedance $z_{12}(s)$ can be expanded as

$$z_{12}(s) \;=\; \sum_{k=1}^{n} \frac{A_{12k}}{s + js_k} \qquad (5.17)$$

where all residues are real positive, satisfying equation (5.3). We start by collecting those terms which correspond to the poles of $z_{11}(s)$ and are not the poles of $z_{12}(s)$ into an impedance $Z_a(s)$, and similarly for $z_{22}(s)$ as shown in Figure 5.3.

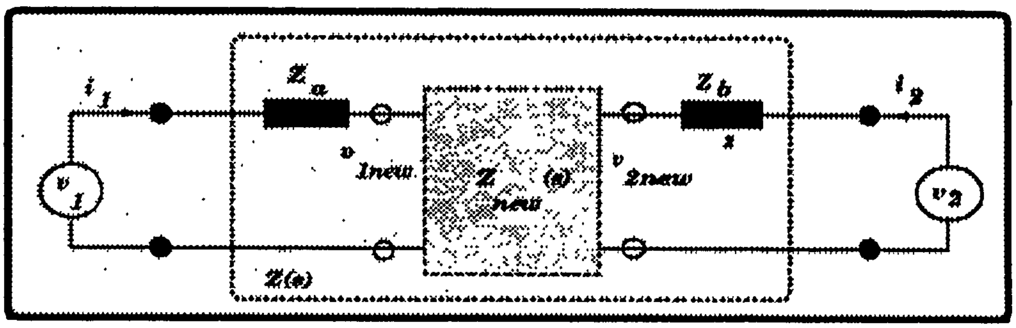

Figure 5.3: The realization of the poles of z_{11} and z_{22} that are not in z_{12}.

Hence, we write

$$z_{11}(s) \;=\; Z_a(s) \;+\; \sum_{k=1}^{n} \frac{A_{11k}}{s + js_k} \;=\; Z_a(s) \;+\; z_{11new} \qquad (5.18)$$

$$z_{22}(s) \;=\; Z_b(s) \;+\; \sum_{k=1}^{n} \frac{A_{22k}}{s + js_k} \;=\; Z_b(s) \;+\; z_{22new} \qquad (5.19)$$

Clearly, $Z_a(s)$ and $Z_b(s)$ are realizable complex impedance functions.

The subtraction of $Z_a(s)$ from the driving point complex impedance $z_{11}(s)$, as shown in Figure 5.3 and in equation (5.18), produces a driving point complex impedance z_{11new}. Similarly, subtraction of $Z_b(s)$ from the driving point complex impedance $z_{22}(s)$ produces a driving point complex impedance z_{22new}.

It is clear that the series complex impedances $Z_a(s)$ and $Z_b(s)$ contribute only to $z_{11}(s)$ and $z_{22}(s)$, respectively, which contain poles of the impedances z_{11} and z_{22}

that are not in $z_{12}(s)$. Now the rest of the new network as shown in Figure 5.3 can be realized by z_{12} as in (5.17), and $z_{11new}(s)$ and $z_{22new}(s)$ as in (5.18) and (5.19) respectively which have common poles in all the four elements of Z_{new} as,

$$[Z_{new}(s)] = \begin{bmatrix} z_{11new} & z_{12new} \\ z_{12new} & z_{22new} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} \frac{A_{11k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} \\ \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{22k}}{s+js_k} \end{bmatrix} \tag{5.20}$$

$$[Z_{new}(s)] = \sum_{k=1}^{n} [Z_k(s)] \tag{5.21}$$

Equation (5.21) is the synthesis of the $[Z_{new}(s)]$ as a sum of simple terms. The network of $Z_{new}(s)$ can be realized as in Figure 5.4 where each elementary two-port network corresponds to an elementary impedance matrix on the right-hand side of (5.21). A dual synthesis, which results in a different realization for the two-port, can be carried out from Y(s).

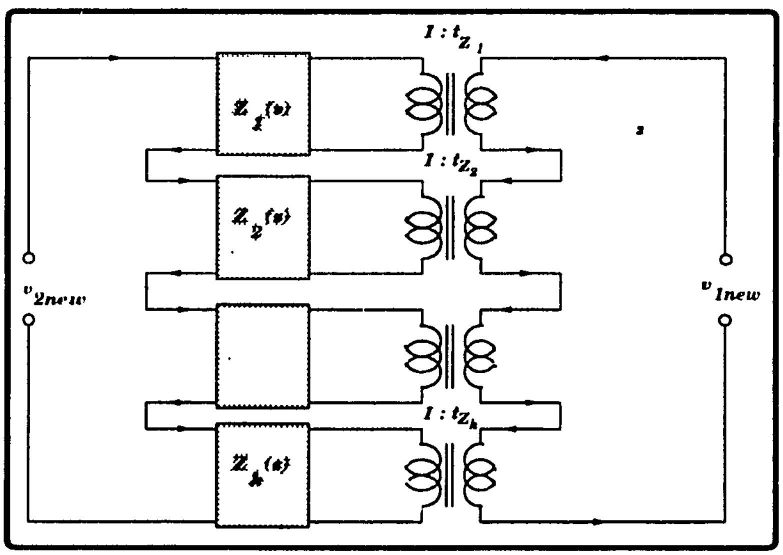

Figure 5.4: The realization of z_{new}.

From the partial fraction expansions (5.15), (5.16) and (5.17), the open circuit impedance matrix $[Z(s)]$ in (5.13) can be written as

$$[Z(s)] = \begin{bmatrix} R_{11} + jR_{11} + L_{11}s + \sum_{k=1}^{n} \frac{A_{11k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} \\ \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} & R_{22} + jR_{22} + L_{22}s + \sum_{k=1}^{n} \frac{A_{22k}}{s+js_k} \end{bmatrix} \tag{5.22}$$

The realizable network for the partial fraction expansion of the complex impedance matrix elements of the two-port in (5.22) is shown in Figure 5.5.

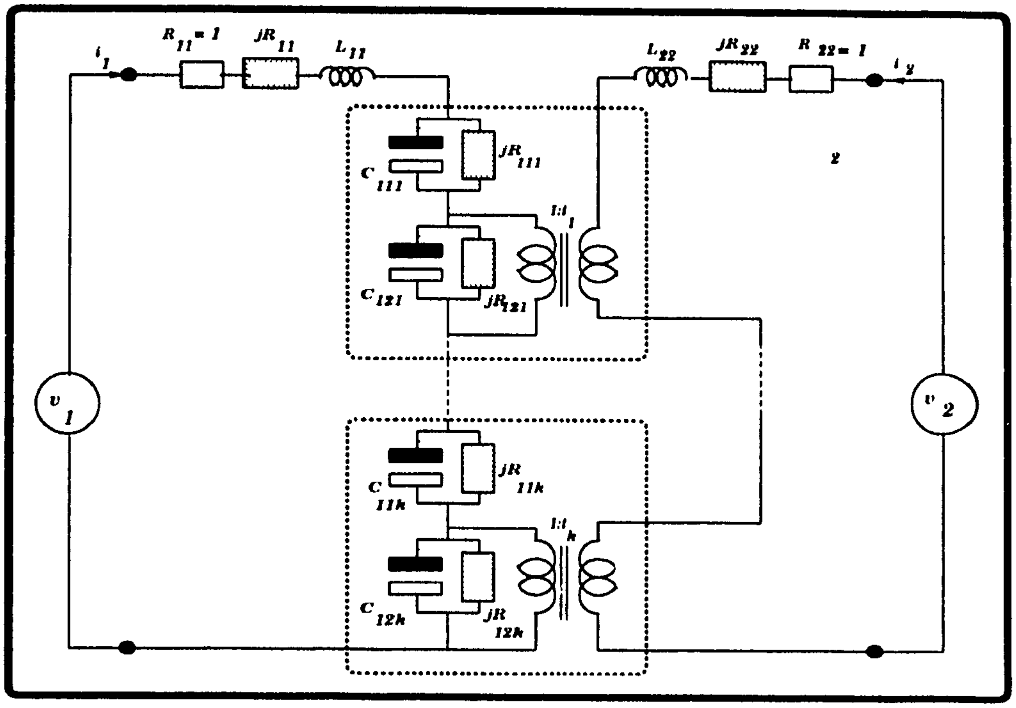

Figure 5.5: Two-port realization of $Z(s)$ using partial fraction expansion.

When the parallel branches $(jR_{111} - C_{111})$,...,$(jR_{11k} - C_{11k})$ are moved into the series input branch, the circuit of Figure 5.6 results.

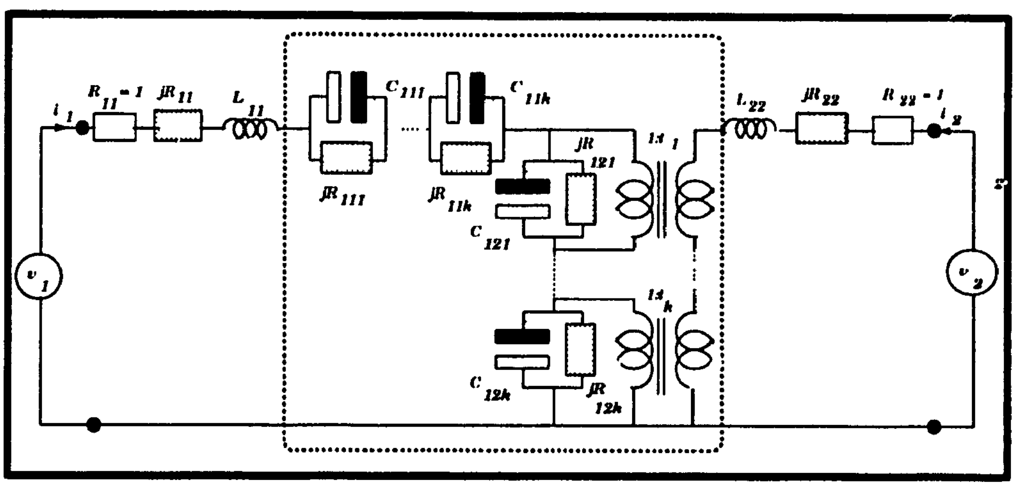

Figure 5.6: Final two-port realization of $Z(s)$ using partial fraction expansion.

5.4 Construction and Synthesis of a Complex Reactance Matrix of Two-Port Networks by (jR, L, C)

5.4.1 Using J-fraction expansion

Any complex polynomial $P(s)$ can be separated into its quasi-real and quasi-imaginary parts, respectively as given in Chapter 3 by

$$P(s) = qRe11.P(s) + qIm11.P(s). \quad (5.23)$$

If the degree of P(s) is even, then

$$qRe11P(S) = s^n + jb_1 s^{n-1} + a_2 s^{n-2} + jb_2 s^{n-3} + ... + jb_{n-1}s + a_n \quad (5.24)$$

$$qIm11P(S) = a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + ... + a_{n-1}s + jb_n \quad (5.25)$$

From these two polynomials, one may define the driving point reactance $x_{11}(s)$ and $x_{22}(s)$ of the two-port as

$$x_{11}(s) = \frac{qRe11P(s)}{qIm11P(s)} \quad (5.26)$$

$$x_{22}(s) = \frac{qRe22P(s)}{qIm11P(s)} \quad (5.27)$$

The value of $qRe22P(s)$ is a function of $qRe11P(s)$ as we shall see next. The poles and zeros of $x_{11}(s)$ alternate on the $j\omega$-axis, but they do not need to appear in conjugate pairs, the same is true for $x_{22}(s)$.

The matrix form of the open circuit reactance parameters of the two-port complex reactance $[X(s)]$ is given as

$$[X(s)] = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \tag{5.28}$$

The rational function $x_{11}(s)$ in (5.28) is given by

$$x_{11}(s) = \frac{qRe11P(s)}{qIm11P(s)} = \frac{s_n + jb_1 s_{n-1} + a_2 s_{n-2} + jb_3 s_{n-3} + \ldots + jb_{n-1}s + a_n}{a_1 s_{n-1} + jb_2 s_{n-2} + a_3 s_{n-3} + jb_4 s_{n-4} + \ldots + a_{n-1}s + jb_n} \tag{5.29}$$

applying a J-fraction expansion, on $x_{11}(s)$ we get

$$x_{11}(s) = L_{11}s + jR_{11} + \cfrac{1}{C_2 s + jG_2 + \cfrac{1}{L_3 s + jR_3 + \cfrac{1}{C_4 s + jG_4 + \cfrac{1}{+ \cfrac{1}{C_n s + jG_n}}}}} \tag{5.30}$$

Now if the arrangement of the last $(n-1)$ terms in the J-fraction expansion (5.30) is reversed and the value of L_{11} changed to another value L_{22}, (which may or may not equal), applying the similarly for R_{11}, then the right side of (5.30) can be written as

$$L_{22}s + jR_{22} + \cfrac{1}{C_n s + jG_n + \cfrac{1}{L_{n-1}s + jR_{n-1} + \cfrac{1}{+ \cfrac{1}{C_2 s + jG_2}}}} = \frac{qRe22P(s)}{qIm11(s)} \tag{5.31}$$

where $qRe22P(s)/qIm11P(s)$ is a rational function obtained from the J-fraction in (5.31) and it represents the driving point reactance $x_{22}(s)$ which is given as

$$x_{22} = L_{22}s + R_{22} + \cfrac{1}{C_n s + G_n + \cfrac{1}{L_{n-1}s + R_{n-1} + \cfrac{1}{+ \cfrac{1}{C_2 s + G_2}}}} = \frac{qRe22P(s)}{qIm11(s)} \tag{5.32}$$

$x_{12}(s)$ now is the only element left in the matrix $[X(s)]$, which can be defined as

$$x_{12}(s) = \frac{N(s)}{qIm11(s)} \tag{5.33}$$

where the polynomial $N(s)$ is equal to the inverse multiplication of the admittance terms in the J-fraction expansions (5.32) or (5.30) which are the shunt elements in the ladder two-port network. Now the $x_{12}(s)$ in (5.33) becomes

$$x_{12}(s) = \frac{(C_2 s + jG_2)^{-1}(C_4 s + jG_4)^{-1}(C_6 s + jG_6)^{-1} \ldots (C_{n-1}s + jG_{n-1})^{-1}(C_n s + jG_n)^{-1}}{a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + a_5 s^{n-5} + \ldots} \tag{5.34}$$

The open circuit reactance parameters of the two-port complex reactance network in (5.28) can be written in the following form

$$[X(s)] = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} = \begin{bmatrix} \frac{qRe11P(s)}{qIm11P(s)} & \frac{N(s)}{qIm11P(s)} \\ \frac{N(s)}{qIm11P(s)} & \frac{qRe22P(s)}{qIm11P(s)} \end{bmatrix} \tag{5.35}$$

$$[X(s)] = \begin{bmatrix} L_{11}s + R_{11} + \frac{1}{C_2s+G_2+\frac{1}{+\frac{1}{C_ns+G_n}}} & \frac{N(s)}{Q(s)} \\ \frac{N(s)}{Q(s)} & L_{22}s + 1 + R_{22} + \frac{1}{C_ns+G_n+\frac{1}{+\frac{1}{C_2s+G_2}}} \end{bmatrix} \tag{5.36}$$

The two-port network realization of a reactance matrix $[X(s)]$ which is synthesized by a J-fraction expansion is shown in Figure 5.7

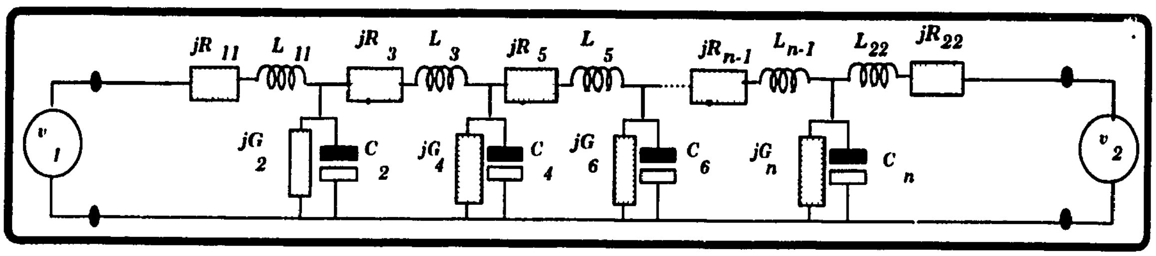

Figure 5.7: Two port realization of $X(s)$ using J-Fraction expansion.

5.4.2 Using partial fraction expansion

From the quasi-real and th quasi-imaginary parts of a given complex polynomial, the open circuit reactance parameters of $[X(s)]$ can be constructed from (5.36) to be in the form of rational functions as in (5.35). Once the matrix $[X(s)]$ is constructed satisfying the realizability conditions, then we can design a complex reactance two-port using partial-fraction expansion.

Since $x_{11}(s)$ and $x_{22}(s)$ are driving-point impedances measured at the input and the output ports, respectively, then we can expand them into partial fraction as

$$x_{11}(s) \;=\; jR_{11} \;+\; L_{11}s \;+\; \sum_{k=1}^{n} \frac{A_{11k}}{s + js_k} \tag{5.37}$$

$$x_{22}(s) \;=\; jR_{22} \;+\; L_{22}s \;+\; \sum_{k=1}^{n} \frac{A_{22k}}{s + js_k} \tag{5.38}$$

and the transfer reactance $x_{12}(s)$ can be expanded as

$$x_{12}(s) \;=\; \sum_{k=1}^{n} \frac{A_{12k}}{s + js_k}, \tag{5.39}$$

where all residues are real, positive, satisfying (5.3). We start by collecting those terms which correspond to the poles of $x_{11}(s)$ and are not the poles of $x_{12}(s)$ into a reactance $X_a(s)$, and similarly for $x_{22}(s)$. This is shown in Figure 5.8.

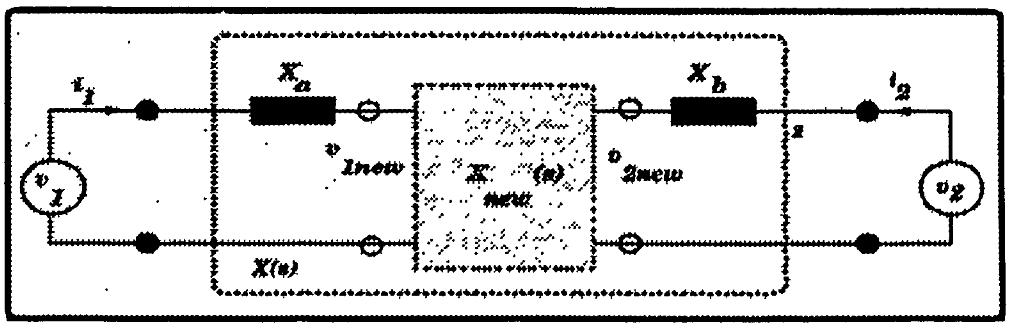

Figure 5.8: The realization of the poles of x_{11} and x_{22} and are not in x_{12}.

Hence we can write $x_{11}(s)$ and $x_{22}(s)$ as

$$x_{11}(s) \;=\; X_a(s) \;+\; \sum_{k=1}^{n} \frac{A_{11k}}{s + js_k} \;=\; X_a(s) \;+\; x_{11new} \tag{5.40}$$

$$x_{22}(s) \;=\; X_b(s) \;+\; \sum_{k=1}^{n} \frac{A_{22k}}{s + js_k} \;=\; X_b(s) \;+\; x_{22new} \tag{5.41}$$

Clearly, $X_a(s)$ and $X_b(s)$ are realizable reactance functions.

The subtraction of $X_a(s)$ from the driving point reactance $x_{11}(s)$ as shown in Figure 5.8 and Equation (5.40) produces a driving point reactance x_{11new}. Similarly subtraction of $X_b(s)$ from the driving point reactance $x_{22}(s)$ produces a driving point reactance x_{22new}.

It is clear that the series reactances $X_a(s)$ and $X_b(s)$ contribute only to $x_{11}(s)$ and $x_{22}(s)$, respectively, which contain poles of these reactances x_{ii}, i = 1, 2, which are not in $x_{12}(s)$. Now the rest of the new network as shown in Figure 5.8 can be realized by (5.39), and $x_{11\ new}(s)$ and $x_{22\ new}(s)$ in (5.40) and 5.41 respectively which have common poles in all the four elements of X_{new} as,

$$[X_{new}(s)] = \begin{bmatrix} x_{11new} & x_{12new} \\ x_{12new} & x_{22new} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} \frac{A_{11k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} \\ \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{22k}}{s+js_k} \end{bmatrix} \tag{5.42}$$

$$[X_{new}(s)] = \sum_{k=1}^{n} [X_k(s)] \tag{5.43}$$

Equation (5.43) is the synthesis of the $X_{new}(s)$ by expanding it as a sum of simple terms. The network of $X_{new}(s)$ can be realized as in Figure 5.9, where each elementary two-port network corresponds to an elementary reactance matrix on the right-hand side of (5.42).

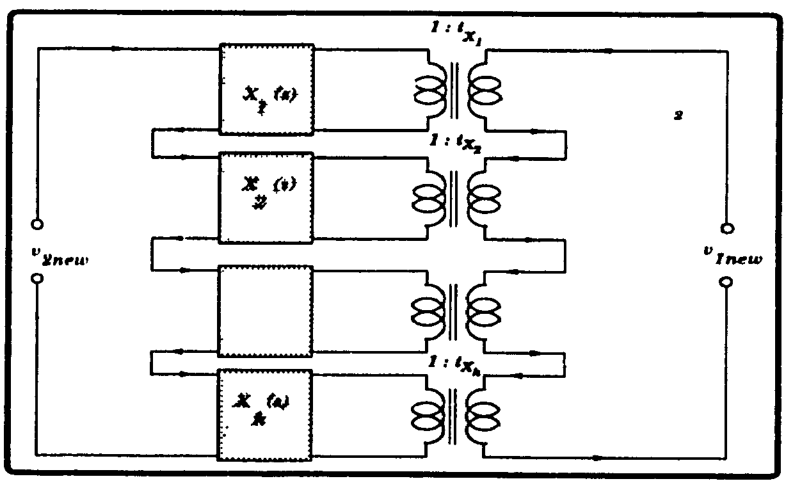

Figure 5.9: The realization of x_{new}.

From the partial fraction expansions (5.37), (5.38) and (5.39), the open circuit reactance $[X(s)]$ in (5.34) can be written as

$$[X(s)] = \begin{bmatrix} jR_{11} + L_{11}s + \sum_{k=1}^{n} \frac{A_{11k}}{s+js_k} & \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} \\ \sum_{k=1}^{n} \frac{A_{12k}}{s+js_k} & jR_{22} + L_{22}s + \sum_{k=1}^{n} \frac{A_{22k}}{s+js_k} \end{bmatrix} \tag{5.44}$$

The realizable network for the partial fraction expansion of the complex reactance matrix elements of the two-port in 5.44 is shown in Figure 5.10.

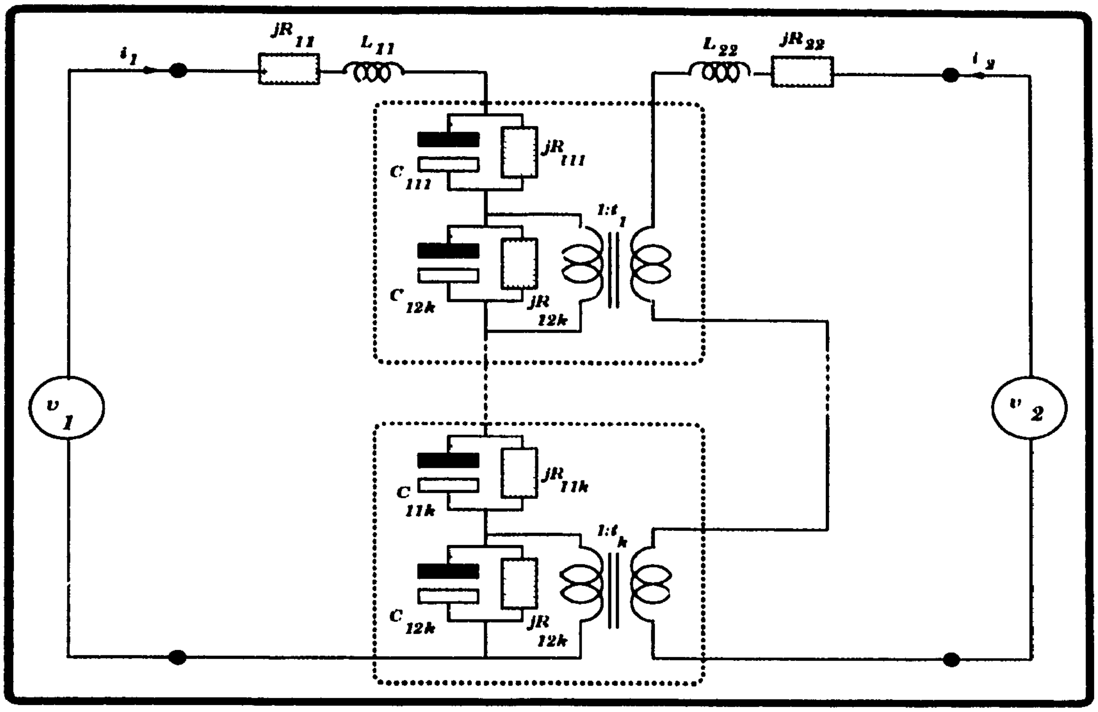

Figure 5.10: Two-port realization of $X(s)$ using partial fraction expansion.

When the parallel branches $(jR_{111} - C_{111})$,...,$(jR_{11k}$ - $C_{11k})$ are moved into the series input branch, the circuit of Figure 5.11 results.

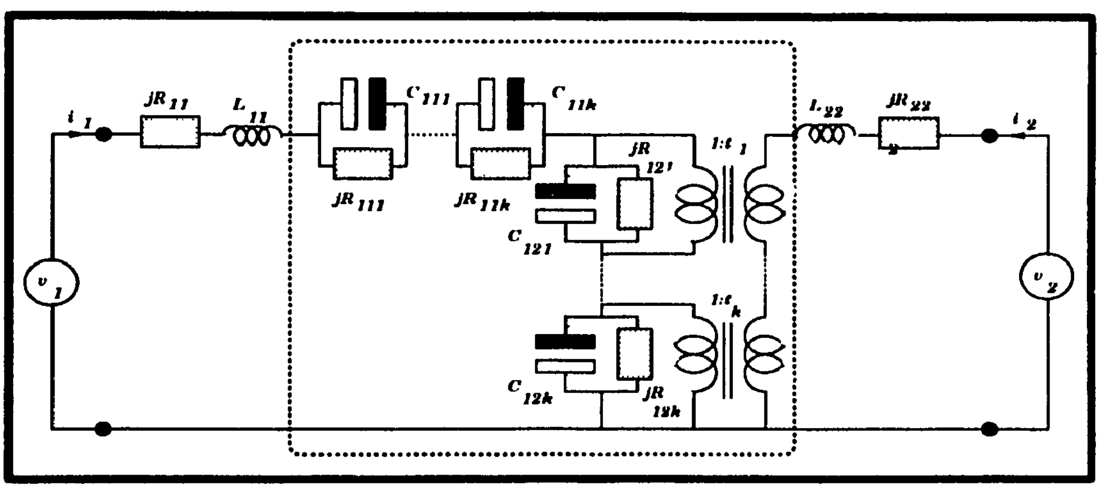

Figure 5.11: Final two-port realization of $X(s)$ using partial fraction expansion.

5.5 Construction and Synthesis of a Complex Reactance Matrix of Two-Port Networks by $(jR, L$ or $jR, C)$

In Section 5.4, a complex reactance matrix of a two-port network was synthesized by three kinds of elements (jR, L, C) using a J-fraction expansion. In this section we shall use the S-fraction expansion to synthesize a complex reactance matrix $[X(s)]$ by two kinds of element $(jR, L$ or $jR, C)$. The S-fraction of the driving point reactance $x_{11}(s)$ in (5.26) is given by

$$x_{11}(s) = \frac{qRe11P(s)}{qIm11P(s)} = L_{11}s + \cfrac{1}{jG_2 + \cfrac{1}{L_3 s + \cfrac{1}{jG_4 + \cfrac{1}{+\frac{1}{jG_n}}}}} \tag{5.45}$$

The driving point reactance $x_{22}(s)$ can be found by the same procedure as used in Section 5.4 except that in this expansion only one element L_{11} is changed to L_{22}.

$$x_{22} = L_{22}s + \cfrac{1}{jG_n + \cfrac{1}{L_{n-1}s + \cfrac{1}{+\cfrac{1}{jG_4 + \cfrac{1}{L_3 s + \frac{1}{jG_2}}}}}} = \frac{qRe22P(s)}{qIm11(s)} \tag{5.46}$$

where $qRe22P(s)/qIm11P(s)$ is a rational function obtained from the S-fraction in (5.46).

The only element left now in the matrix $[X(s)]$ is $x_{12}(s)$, where the numerator is equal to the multiplication of the inverse of the admittance terms in the S-fraction in (5.46) or (5.45) which are the shunt elements in the ladder two-port network see Figure 5.12.

$$x_{12}(s) = \frac{N(s)}{qIm11P(s)} = \frac{(jG_2)^{-1}(jG_4)^{-1}(jG_6)^{-1}(jG_8)^{-1}....(jG_{n-1})^{-1}(jG_n)^{-1}}{a_1s^{n-1} + jb_2s^{n-2} + a_3s^{n-3} + jb_4s^{n-4} + a_5s^{n-5} + ...} \tag{5.47}$$

The open circuit reactance parameters of the two-port complex reactance network in (5.28) can be written as

$$[X(s)] = \begin{bmatrix} L_{11}s + \cfrac{1}{jG_2 + \cfrac{1}{L_3 s + \cfrac{1}{\ddots + \cfrac{1}{L_{n-1}s + \frac{1}{jG_n}}}}} & \dfrac{N(s)}{qIm11P(s)} \\ \dfrac{N(s)}{qIm11P(s)} & L_{22}s + \cfrac{1}{jG_n + \cfrac{1}{L_{n-1}s + \cfrac{1}{\ddots + \cfrac{1}{L_3 s + \frac{1}{jG_2}}}}} \end{bmatrix} \tag{5.48}$$

The two-port network realization of the reactance matrix $[X(s)]$ of 5.48, which is synthesized by the S-fraction expansion, is shown in Figure 5.12.

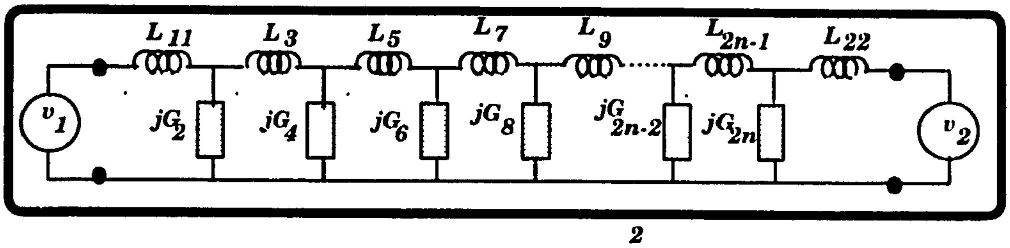

Figure 5.12: Twos port realization of $X(s)$ using S-Fraction expansion.

5.6 An Alternative Synthesis of Complex Reactance Matrix of Two-Port Networks by $(jR, L$ or $jR, C)$.

One can find an alternative realization of a complex reactance matrix $[X(s)]$, by arranging both the numerator and the denominator of the driving point impedance $x_{11}(s)$ in ascending powers of s as follows:

$$x_{11}(s) = \frac{qRe11P(s)}{qIm11P(s)} = \frac{jb_n + a_{n-1}s + \ldots + a_3 s^{n-3} + jb_2 s^{n-2} + \ldots + a_1 s^{n-1}}{a_n + jb_{n-1} + \ldots + jb_3 s^{n-3} + a_2 s^{n-2} + \ldots + jb_1 s^{n-1} + s^n} \tag{5.49}$$

The S-fraction of the driving point reactance $x_{11}(s)$ in (5.49) is given by,

$$x_{11}(s) = jR_{11} + \cfrac{1}{C_2 s + \cfrac{1}{jR_3 + \cfrac{1}{C_4 s + \cfrac{1}{jR_5 + \cfrac{1}{\ddots + \frac{1}{C_n s}}}}}} \tag{5.50}$$

and following the procedure of earlier sections, the driving point reactance $x_{22}(s)$ can be written as

$$x_{22}(s) = jR_{22} + \cfrac{1}{C_n s + \cfrac{1}{jR_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{C_4 s + \cfrac{1}{jR_3 + \cfrac{1}{C_2 s}}}}}} = \frac{qRe22P(s)}{qIm11(s)} \tag{5.51}$$

where $(qRe22P(s)/qIm11P(s))$ is a rational function obtained from the S-fraction in (5.51).

Using the same procedure as in the previous section, the transfer reactance $x_{12}(s)$ is given by

$$x_{12} = \frac{N(s)}{qIm11P(s)} = \frac{(jG_2)^{-1}(jG_4)^{-1}(jG_6)^{-1}(jG_8)^{-1}...(jG_{n-1})^{-1}(jG_n)^{-1}}{a_1 s^{n-1} + jb_2 s^{n-2} + a_3 s^{n-3} + jb_4 s^{n-4} + ... + a_{n-1} s + jb_n} \tag{5.52}$$

The open circuit reactance parameters of the two-port complex reactance network in (5.35) can be expressed as

$$[X(s)] = \begin{bmatrix} jR_{11} + \cfrac{1}{C_2 s + \cfrac{1}{jR_3 + \cfrac{1}{\ddots + \cfrac{1}{jR_{n-1} s + \cfrac{1}{C_n s}}}}} & \dfrac{N(s)}{qIm11P(s)} \\ \dfrac{N(s)}{qIm11P(s)} & jR_{22} + \cfrac{1}{C_n s + \cfrac{1}{R_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{jR_3 + \cfrac{1}{C_2 s}}}}} \end{bmatrix} \tag{5.53}$$

The two-port network realization of a reactance matrix $[X(s)]$ which is synthesized by the S-fraction expansion, is shown in Figure 5.13.

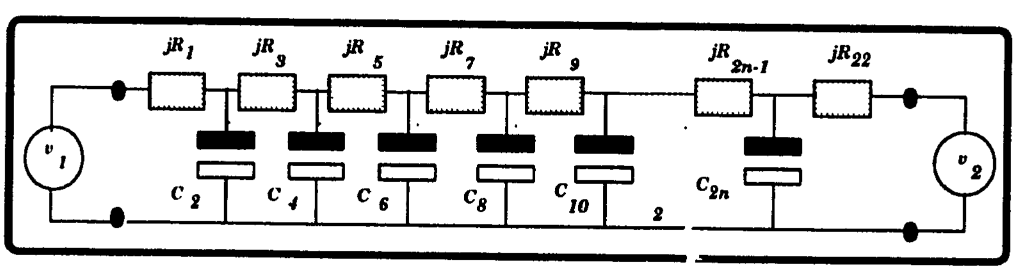

Figure 5.13: An alternative two port realization of $X(s)$ using S-Fraction expansion.

Chapter 6

CONCLUSIONS AND FUTURE WORK

An important problem in system analysis and circuit theory is system stability and synthesis of a realizable network. In this thesis, we have proposed methods for testing the stability of a system specified by complex rational functions. In Chapter 2, we derived the necessary and sufficient conditions for the existence of stability of a given polynomial with complex coefficients using J-fraction expansion. The generalized Routh array of a complex polynomial was developed and the stability conditions using this array were found. Two methods were used to construct the Routh array on the basis of division process and two cross multiplying processes. The generalized Hurwitz determinant and the modified generalized Hurwitz determinant were also derived. The modified Hurwitz determinant yields stability condition of a complex polynomial. Application of these methods of complex polynomials to test stability of real multivariable polynomials in continuous systems is discussed. Another application is stability testing of real polynomials with multivariables in discrete system by transferring the polynomials from the z-domain to the s-domain. Methods of testing stability using the generalized Routh array and the modified generalized Hurwitz determinant is the first contribution of this thesis.

Given a stable complex polynomial, the second contribution of this thesis provides answers to two questions: first, how can we construct complex impedance and complex reactance functions? second, how can we synthesise these complex functions? Complex impedance function is constructed from the stable complex polynomial $P(s)$ and its alternant polynomial $Q(s)$. Complex reactance function is constructed from the quasi real and the quasi imaginary parts of $P(s)$. Both $Z(s)$ and $X(s)$ are synthesized using J-fraction expansion. The division process in that expansion is equivalent to the removal of two elements from $Z(s)$ or $X(s)$. Also $X(s)$ is synthesized using S-fraction expansion with division process equivalent to the removal of one element from $X(s)$. The relationship between $X(s)$ and its associative $Z(s)$ is determined. The networks realization of the complex impedance function is found by using four kinds of elements: R, jR, L, C. We have synthesized the complex reactance function with three kinds of elements: jR, L, C; and with two kinds of elements jR, L or jR, C.

It should be pointed out that this thesis is concerned with network synthesis. This represents one part of the network analysis and synthesis area. The other important part of circuit theory is analysis of networks containing imaginary resistance (jR), which we plan to undertake as future work. For example,

- Investigate the analysis of regular systems with real polynomials as being synthesized in terms of complex polynomial considerations.
- Study the state space representation of a complex network.
- Evaluate the transient and steady state response of linear lumped-constant systems containing imaginary resistance (jR).
- Study a high temperature T_c superconductor to get a material with high T_c near room temperature, representing the imaginary resistance (jR).

The third contribution of this thesis is the z-domain synthesis. The main idea of network synthesis of $X(s)$ is transferred to synthesise discrete complex reactance

$X(z)$ in the z-domain. A discrete complex reactance function $X(z)$ was constructed from the equivalent analog complex reactance function $X(s)$ by using a bilinear transformation, or from any complex polynomial $P(z)$ with its roots inside the unit circle in the z-plane. $X(z)$ is synthesized using J-fraction expansion, S-fraction expansion and partial fraction expansion. The network realizations of discrete complex reactance functions are found by using delay blocks, called sample storage elements, which are equivalent to inductors, capacitors or imaginary resistors in the complex reactance function $X(s)$.

It is evident that once the synthesis of complex function in the s-domain is accomplished, we can extend it to construct and synthesise the complex matrix of a two-port network. This is the fourth contribution of the thesis. We have derived a complex impedance matrix $[Z(s)]$ of two-port network from $P(s)$ and its alternant. A complex reactance matrix was constructed from the quasi-real and the quasi-imaginary parts of $P(s)$. Realizability conditions of the complex impedance matrix $[Z(s)]$ and the complex reactance $[X(s)]$ of two-port networks have been discussed. The matrices are synthesized by J-fraction, S-fraction, and partial fraction expansions. The network realizations of complex impedance of two port networks were found by using four kinds of elements (R, jR, L, C). Realizable complex reactance two-port networks with three kinds of elements (jR, L, C) and with two kinds of elements $(jR, L$ or $jR, C)$ were also derived.

Bibliography

[1] S.A. Reible,"Wideband Analog Signal Processing With Superconductive Circuits",in 1982 IEEE Ultrasonics Symp, Proc, (San Diego, CA), Oct. 1982, pp. 190-201.

[2] J.M. Pond, C.M. Krowne, W. Carter,"On the Application of complex resistive Boundary Conditions to Model Transmission Lines Consisting of Very Thin Superconductors", IEEE Transaction on Microwave Theory and Techniques, Vol. 37, No. 1, January 1989.

[3] R. Simon,"High-T_c Thin Film and Electronic Devices", Physics Today, pp. 64-70, June 1991.

[4] B. Batlogg,"Physical Properties of High-T_c Superconductors", Physics Today, pp. 44-50, June 1991.

[5] J. Rowell,"High-Temperature Superconductivity", Physics Today, pp. 22-23, June 1991.

[6] H. S. Wall, "Analytic Theory of Continued Fractions," , pp. 174, 1967.

[7] E. Frank,"On the Zeros of the Polynomials With Complex Coefficients," *Bull. Amer. Math. Soc.*, vol. 52, pp. 144-172, 1946.

[8] H. S. Wall,"Polynomials Whose Zeros Have Negative Real Parts," *Amer. Math. Monthly.*, vol. 52, pp. 308-322, 1945.

[9] Katsuhiko Ogata, "State Space Analysis of Control Systems", Prentice Hall, INC. 1967.

[10] Katsuhiko Ogata, "Modern Control Engineering", Prentice Hall, INC. 1967.

[11] M.G. Strintzis,"Test of Stability of Multidimensional Filters IEEE Trans. on Circuit and Systems, Vol. CAS-24, No. 8, pp. 432-437, August 1977.

[12] E.I. Jury,"Stability of Mulyidimensional Scalar and Matrix Polynomials," Proc. IEEE, vol. 66, pp. 1018-1047, Sept. 1978.

[13] N.K. Bose,"Implementation of a New Stability Test for two Dimensional Fil ters," IEEE Trans. Acoust., Speech, Signal Proc., vol. ASSP-25, pp. 117-120, Apr. 1977.

[14] N.K. Bose,"Problems and Progress in Multidimensional Systems Theory," Proc. IEEE, vol. 65, no. 6, pp. 824-840, June 1977.

[15] E.I. Jury,"Inners and Stability of Dynamic Systems," New York: Wiley, 1974.

[16] Belevitch, V.,"Classical Network Theory", Holden-Day, 1968.

[17] Baum, R.F.,"A Modification of Brune's Method for Narrow-Band Filters", IRE Trans actions on Circuit Theory, Vol. 5, pp. 264-267, December 1958.

[18] Belevitch, V.,"An Alternative Derivation of Brune's Cycle", IRE Transactions on Circuit Theory, Vol. 6, pp. 389-390, December 1959.

[19] Franklin F.Kuo "Network Analysis And Synthesis", John wiley and Sons, Inc. NewYork 1962.

[20] Ernst A. Guillemin "Synthesis of Passive Networks", John wiley sons, Inc. NewYork 1957

[21] Van Valkenburg ."Introduction to Modern Network Synthesis". John wiley Sons, In c. NewYork 1960.

[22] Agashe, S. D."A New General Routh-Like Algorithm to Determine the Number of RHP Roots of A Real or Complex Polynomial" IEEE Transaction on Automatic Cont rol, Vol. AC-30, No. 4, April 1985, pp. 406-409.

[23] Bose N.K., and Y.Q. Shi."Network realizability theory approach to stability of complex polynomials",T-CAS pp. 216-218Feb. 1987

[24] Reza "A Note On Complex Reactance Function" Computers Elect. Engng Vol, 18, No. 2, pp, 183-188, 1992.

[25] Someshwar C. Gupta,"Transformation and State Variable Methods in Linear Systems" , John Wiley Sons. Inc. 1966.

[26] Reger M. Golden,"Digital Filter Synthesis by Sampled-Data Transformation", Transactions on Audio and Electroacoustics Vol. AU-16, No. 3 september 1968.

[27] Alan V. Oppenheim and Roland W. Schafer,"Discrete-Time Signal Processing", Prentice Hall 1989.

[28] Lonnie C. Ludeman."Fundamentals of Digital Signal Processing". Harper Row. Publishers, 1986.

[29] Thomas H. Crystal and Leonard Ehrman,"The Design and Applications of Digital Filter With Complex Coefficients",IEEE transactions on Audio and Electroacoustics Vol. AU-16, No. 3 september 1968.

[30] Balabanian, N.,"Network Synthesis", Prentice Hall, Englewood Cliffs, N.J., 1960

[31] Robert W. Newcomb," Linear Multiport Synthesis", McGraw-Hill book company 1966.

[32] G.C. Temes, and J.W. La Patra, "Introduction to Circuit Synthesis and design", McGraw-Hill 1977.