ON STANDBY REDUNDANCY WITH ERLANG REPAIR DISTRIBUTION

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A THESIS

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Science at

Concordia University

Montreal, Canada

September 1974

ABSTRACT

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This thesis deals with a problem in the theory of standby redundancy. A system consisting of $n + 1$ $(n = 1,2,\ldots.)$ identical units is considered. One unit, the basic, is in operation at any given time while the other n are in "cold" standby, under repair, or waiting for repair. The failure time of the basic unit obeys an arbitrary distribution $F(t)$.

Two cases are considered:

1. The repair time of a failed unit has an Erlang-distribution $G(t) = 1 - \sum_{i=0}^{\mu-1} \frac{(\mu\lambda t)^i}{i!} e^{-\mu\lambda t}$ where λ and μ are the scale and shape parameters of the Erlang distribution, respectively, with $\lambda > 0$ a constant and $\mu = 1,2,3\ldots.$ are integers. There are n repair facilities in the system.

2. The repair-time obeys an exponential distribution $G(t) = 1 - e^{-vt}$, $v > 0$ is a constant. There is one repair facility in the system, available to work only after the accumulation of k failed units.

The system breaks down if the basic unit fails while all the others are under repair or waiting for repair. We identify the states where the system can be found during time t. One of these states is its down state and then we set up integral equations to characterize the probabilities of the system being in the down state, given that it is in state i at $t = 0$, where i can be any of its up states.

We use Laplace-Stiltjes transforms technique to solve the system of integral equations.

Finally we obtain the LS transform of the distribution of the first time to system failure, and the mean-time of faultless operation in the two cases.

ACKNOWLEDGEMENTS

My great thanks go to Dr. Zohel Khalil, who proposed the problem and directed this thesis. I would say that this study would not have been possible without his suggestions and encouragement.

My sincere thanks are due to my wife Dimitra for her cooperation and patience during the period of this work.

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CHAPTER I

STANDBY REDUNDANCY

1.1 Introduction

We define reliability as the ability of a system to operate for a specified period of time under given operational conditions. Criteria of a reliable system are its readiness to perform at the desired instant and satisfactory performance while in operation. By system operation we mean the set of all phases of its existence.

A partial or total loss of the ability of a system to operate is called a "system-failure". System-failure during operation is costly and may probably be dangerous. An attempt to prevent such a failure, really is an attempt to increase reliability.

The problem of increasing system's reliability is important in connection with the fact that many automatic systems must operate for a long time without failure. In order to increase reliability, we have to choose either units of highest quality which give high reliability, or ease the operational conditions of the system, or employ standby units. Every one of the above methods is a case of redundancy and is an extension on structure, size or cost of a system, in order to ensure that it has a required reliability. In the present work we shall consider only standby redundancy. The term "unit" will be used to mean an element, a part of a system, a whole system, or the like.

Standby redundancy is one of the basic methods of increasing reliability of technical systems, and of constructing highly reliable systems from less reliable units. In general a system consists of a number of homogeneous units or subsystems. Standby redundancy consists mainly in joining one or several units to the system in which, as failures occur, the standby units assume the function of the failed units, allowing the continuation of system's operation. The number of standby units can be equal, less or greater than the number of acting units in the system. Standby redundancy increases the time of failure-free operation of a system and enables us, at least in principle, to obtain a system reliability arbitrarily close to one.

To take out a unit that has failed and bring in its place a standby unit, we need a device, which we shall call a "switch". The time required to disconnect the unit which has failed, and put in its place the standby unit, is called switchover time. In general this time is randomly distributed according to some distribution, but in some cases, for simplicity, we can assume it to be instantaneous.

As nothing in real life is a hundred per cent reliable, the switch itself can fail. A switch failure can occur at the instant of switching, in case of instantaneous switchover time. In case of random switchover time a switch failure can occur during that time. In the first case we can consider a constant probability of a switch failure, while in the second one we can assume that the time up to this failure is distributed according to some law.

In what follows we shall consider first the case of instantaneous switchover time, and second a randomly distributed switchover time.

We can have three types of standby redundancy depending on the state of the redundant unit before it is put into action:

1. Active standby redundancy. The redundant units can fail before they are put into operation, according to the same failure-time distribution as the basic units, and their reliability is independent of the instant at which they take the place of the basic units.

2. Completely inactive standby redundancy. The redundant units cannot fail until they are put in the place of the basic units.

3. Partially active standby redundancy. The redundant units can fail during the period they are in standby but with probability less than the basic units.

Standby redundancy is also differentiated as follows:

1. Standby redundancy with renewal. In such a case each unit which has failed can be repaired and after repair recovers its function perfectly, then it can be put into standby. The renewal time is randomly distributed according to some law.

In most of the systems which have been studied up till now this law has been assumed exponential.

2. Standby redundancy without renewal. In standbys without renewal, the unit which has failed is eliminated and it no longer takes part in the system's operation. However our work will deal mainly with standby redundancy with renewal.

1.2 The Two-unit Standby Redundant System

Consider the following simple problem:

A system consists of two identical units, one of which is the basic unit and the other is a standby. This is the so-called two-unit standby redundant system which has been examined by many authors under different conditions.

Gnedenko [1] has analyzed the most generalized model of a two-unit system. The failure-time distributions of the two units, one operating and one in warm standby, are exponential with failure rates λ and λ_1 respectively where $\lambda_1 < \lambda$. For $\lambda_1 = 0$ we obtain the case of a cold standby and for $\lambda_1 = \lambda$ we obtain the case of hot standby. The renewal time has a general distribution $G(t)$ and the switchover time is instantaneous.

When the basic unit fails the standby takes its place and the failed unit starts to be repaired. When the repair is completed, the two units reverse rules. The length of time between two successive renewals is defined as a cycle.

A system failure occurs when one unit fails, in some cycle, while the other is still under repair.

Let $R(t)$ be the probability of the event that the time of faultless operation of the standby system will be at least t. It is easy to show that this event can be decomposed into the following disjoint events:

1. The first failure occurs after the instant t. The probability of this event is $e^{-(\lambda+\lambda_1)t}$.

2. The first failure occurs prior to t but the first cycle ends after the instant t. The standby unit that is put into operation at the instant of failure operates without failure up to the instant t. The probability of this event is

$$\int_0^t (\lambda + \lambda_1) e^{-(\lambda+\lambda_1)x} [1 - G(t - x)] e^{-\lambda(t-x)} dx .$$

3. The first cycle ends prior to the instant t, the second unit operates without failure during the first renewal period, and the pair operates without failure for the remaining time up to the instant t. The probability of such an event is

$$\int_0^t R(t-x)\left[\int_0^x (\lambda+\lambda_1)e^{-(\lambda+\lambda_1)z-\lambda(x-z)}g(x-z)dz\right]dx$$

where $g(x) = G'(x)$. Adding these probabilities we obtain the integral equation

$$R(t) = e^{-(\lambda+\lambda_1)t} + e^{-\lambda t}(\lambda+\lambda_1)\int_0^t e^{-\lambda_1 x}[1 - G(t-z)]\,dx$$

$$+ \int_0^t R(t-x)e^{-\lambda x}(\lambda+\lambda_1)dx\int_0^x e^{-\lambda_1 z}g(x-z)dz .$$

From this integral equation we obtain

$$R(t) = A(t) + \int_0^t R(t-x)B(x)dx ,$$

which is an equation of the renewal type with:

$$A(t) = e^{-(\lambda+\lambda_1)t} + e^{-\lambda t}(\lambda+\lambda_1)\int_0^t e^{-\lambda_1 x}[1 - G(t-x)]\,dx ,$$

$$B(t) = e^{-\lambda t}(\lambda+\lambda_1)\int_0^t e^{-\lambda_1 z}g(t-z)dz .$$

We define the Laplace transforms:

$$a(s) = \int_0^\infty e^{-st}A(t)dt , \qquad c(s) = \int_0^\infty e^{-st}dG(t) ,$$

$$b(s) = \int_0^\infty e^{-st} B(t)dt, \qquad \phi(s) = \int_0^\infty e^{-st} R(t)dt ,$$

So that ; $$\phi(s) = \frac{a(s)}{1 - b(s)}$$

Calculating a(s) and b(s) we obtain:

$$\phi(s) = \frac{s + \lambda + (\lambda + \lambda_1)[1 - c(\lambda + s)]}{(s + \lambda)[\lambda + \lambda_1 + s - (\lambda_1 + \lambda)c(\lambda + s)]}$$

The mean time of faultless operation of the system is:

$$T_0 = \int_0^\infty R(t)dt = \phi(0) = \frac{\lambda + (\lambda + \lambda_1)[1 - c(\lambda)]}{\lambda(\lambda + \lambda_1)[1 - c(\lambda)]}$$

Let us now consider the same system in a more general case of general failure-time distribution for the acting unit and general distribution for renewal time. We also assume that we have cold standby. The behavior of the system is that the basic unit, having operated for a random time τ_o, fails and is then renewed during the course of a random time ξ_1. At the instant of its failure, the standby unit is immediately put into operation and it functions for a random time τ_1. After time τ_1 the unit fails, and is renewed during random time ξ_2 etc. (Figure 1)

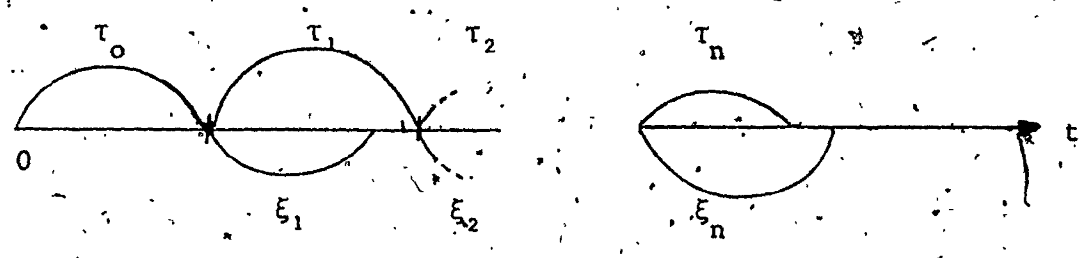

Figure 1

The quantities τ_i and ξ_i are independent and we have

$$P[\tau_i < t] = F(t), \qquad P[\xi_j < t] = G(t) .$$

The system fails when the renewal of one unit has not been completed at the time the other fails. We denote by τ the random time of the faultless operation of the system.

Obviously $\tau = \tau_0 + \tau_1 + \ldots + \tau_n$.

We consider now the quantity $\tau' = \tau_1 + \tau_2 + \ldots + \tau_n$, and let $Q_1(t)$ be the distribution of τ', i.e., $Q_1(t) = P[\tau' < t]$.

Then we can easily obtain the probability of faultless operation of the system for time at least t, and is given by:

$$1 - Q_1(t) = 1 - F(t) + \int_0^t [1 - Q_1(t - x)] G(x) dF(x) \text{, and}$$

If $Q(t)$ is the distribution law of $\tau = \tau_0 + \tau'$, we have

$$Q(t) = \int_0^t Q_1(t - x) dF(x) .$$

We define the Laplace transforms:

$$a(s) = \int_0^\infty e^{-st} dF(t) , \qquad b(s) = \int_0^\infty e^{-st} G(t) dF(t) ,$$

$$A_1(s) = \int_0^\infty e^{-st} dQ_1(t) , \qquad A(s) = \int_0^\infty e^{-st} dQ(t) ,$$

and from the above equation we obtain

$$A(s) = a(s) \cdot A_1(s) = a(s) \cdot \frac{a(s) - b(s)}{1 - b(s)}$$

which obviously is the $\mathcal{L}$ transform of the time of failure-free operation of the system.

The two-unit standby redundant system has been examined also in [4]. The failure time distribution of the basic unit is an arbitrary $F(t)$, and the repair-time distribution $G(t)$ is also arbitrary.

Furthermore it is assumed that the standby unit can fail according to an arbitrary distribution $H(t)$ different from $F(t)$. The assumption is that the failure time of a unit obeys $F(t)$, regardless of how long it has been in standby since the instant of the last repair completion. The renewal time always obeys $G(t)$, whether a unit fails in the operative interval or in the standby interval.

In order to derive the LS transform of the distribution of the first time to system-failure we define three states of the system:

State S_0: One unit begins to operate and the other unit begins a standby status.

State S_1: One unit begins to operate and the other unit begins to be repaired.

State S_2: Two units are under repair or failure simultaneously. This is the system's down-state. Figure 2 gives the state-transition diagram of the system.

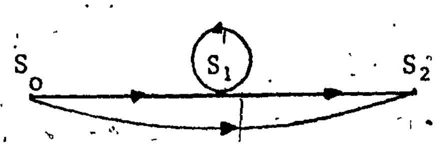

Figure 2

At the instant $t = 0$ the system is in state S_0 and during the time interval $[0,t]$ two transitions can occur: one is to state S_1, and the other is to state S_2.

A transition to state S_1 will occur if at the instant of basic unit's failure the other unit is in standby.

The probability that the operating unit fails in the time interval $(t, t+dt)$ is $dF(t)$, and the probability that the other unit is in standby up to time t are $\overline{H}(t)$, $H(t) * G(t) * \overline{H}(t)$, $H(t) * G(t) * H(t) * G(t) * \overline{H}(t)$, and so on. The transition probability $P_{01}(t)$ is

$$P_{01}(t) = \int_0^t [\overline{H}(t) + H(t) * G(t) * \overline{H}(t) + \ldots] \, dF(t)$$

$$= \int_0^t [\overline{H}(t) * (1 - H(t) * G(t))^{(-1)}] \, dF(t) ,$$

where we note: $(1 - M(t))^{(-1)} = 1 + M(t) + M(t) * M(t) + \ldots$,

and $*$ stands for convolution.

Then $$P_{01}(s) = \int_0^\infty e^{-st} [\overline{H}(t) * (1 - H(t) * G(t))^{(-1)}] \, dF(t) .$$

Similarly we can get the LS transforms $P_{12}(s)$, $P_{02}(s)$, $P_{11}(s)$. Defining $\phi_i(s)$ $(i = 0,1)$ the LS transform of the distribution of the first time to system-down, starting from state S_i at $t = 0$ we obtain

$$\phi_0(s) = P_{02}(s) + \frac{P_{01}(s) P_{12}(s)}{1 - P_{11}(s)} ,$$

which is the LS transform of the first time to system down starting from state S_0 at $t = 0$.

The same system has been examined in [5]. The failure times of the units are exponentially distributed with failure rates λ and

λ_1 respectively. An integral equation for the probability of a system failure has been obtained by identifying a semi-Markov process for the system. The LS transform $\phi(s)$ of the distribution of the first time to system failure and the mean life-time of the system are derived. These are given by:

$$\phi(s) = \frac{\lambda + s + (\lambda + \lambda_1)[1 - g(\lambda + s)]}{(\lambda + s)[\lambda + \lambda_1 + s - (\lambda + \lambda_1)g(\lambda + s)]},$$

$$T_0 = \frac{\lambda + (\lambda + \lambda_1)[1 - g(\lambda)]}{\lambda(\lambda + \lambda_1)[1 - g(\lambda)]},$$

where $g(s)$ is the LS transform of $G(t)$.
The above results are similar to those given in [1] for exponential failure-time distribution of the two units.

In [3] the two-unit warm standby system has been considered and the failure times of the two units are distributed according to Erlang-law with different scale parameters λ_1, λ_2 but the same shape parameter k. The distribution of the renewal time has been considered general or Erlang with parameters λ and v. The Laplace transform of the distribution of time to system failure and the mean time to system failure have been derived, for the above two cases of the renewal time distribution, using integral equations to calculate all possible events of the system during time interval $[0,t]$.

An extended standby redundant system has been considered in [2]. The system consists of $n + 1$ $(n = 1,2,\ldots.)$ identical units in which only one is in operation at any given time while the other units are in cold standby, or are under repair. There are r $(1 \leq r \leq n)$ repair facilities. The time to failure of each operating unit has an arbitrary distribution $F(x)$. The renewal time has a distribution $G(x) = 1 - e^{-vx}$.

Denote by $R_{n+1}(t)$ the probability of reliable operation for a time period t of this system. Assume that all the units are in working order and that the system is switched on at time $t = 0$. Denote by $h_{n+1,k}(t)$ the conditional probability of reliable operation for a time t of the system consisting of $n + 1$ units, assuming that at the instant $t = 0$ a unit fails, and that there are k failed units in the system.

In order to solve this problem the idea is that the system of k standbys and one basic unit can survive in two disjoint cases.

1. The subsystem of $k - 1$ standbys and one basic survive for time t.

2. The subsystem of $k - 1$ standbys and one basic fails at some instant u $(u < t)$ and the complete system of $k + 1$ units survive for the remaining period $t - u$.

If we consider this for any $k = 1,2,\ldots.n$ we obtain the following system of equations of the renewal type.

$$R_2(t) = R_1(t) + \int_0^t h_{2,1}(t - u)d(-R_1(u)),$$

$$R_3(t) = R_2(t) + \int_0^t h_{3,2}(t - u)d(-R_2(u)), \qquad (1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_{r+1}(t) = R_r(t) + \int_0^t h_{r+1,r}(t - u)d(-R_r(u)),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_{n+1}(t) = R_n(t) + \int_0^t h_{n+1,n}(t - u)d(-R_n(u)).$$

The functions $h_{n+1,k}(t)$ are given by:

$$h_{n+1,k}(t) = \overline{F}(t) + \sum_{i=1}^{k} \int_0^t h_{n+1,k-i+1}(t - u)C_{k,i}(u)dF(u)$$

$$+ \int_0^t h_{n+1,k+1}(t - u)C_{k,o}(u)dF(u) , \quad 1 \leqslant k \leqslant n - 1 .$$

The functions $C_{n,k}(u)$ are the following:

1. For $n \leqslant r$

$$C_{n,k}(u) = \binom{n}{k}\overline{G}^{n-k}(u)G^{k}(u) .$$

2. For $n > r$

$$C_{n,o}(u) = \overline{G}^{r}(u) ,$$

$$C_{n,1}(u) = \binom{r}{1}\overline{G}^{r-1}(u)\int_0^u G(u - v)dG(v) ,$$

$$C_{n,2}(u) = \binom{r}{2}\overline{G}^{r-1}\int_0^u G(u - v)dG^2(v) + \binom{r}{2}\overline{G}^{r-2}(u)\int_0^u\int_0^u G(u - v_1) \cdot$$

$$\cdot \, dG(v_1)G(u - v_2)dG(v_2) ,$$

etc. We shall not continue writing these functions because their form becomes increasingly unwieldy.

Writing the above system (1) in terms of LS transforms and solving it, using Cramer's rule, we obtain:

$$a_{n+1}(s) = f(s)\frac{f^{n-r}(s + rv) \prod_{k=1}^{r} f(s + kv)}{\Delta_n(s)} ,$$

where $$a_{n+1}(s) = \int_0^\infty e^{-st} d(-R_{n+1}(t)) ,$$

$$g_n(s) = \int_0^\infty e^{-st} G_n(t) dF(t) ,$$

$$f(s) = \int_0^\infty e^{-st} dF(t) ,$$

and $\Delta_n(s)$ is the determinant:

$$\Delta_n(s) = \begin{vmatrix} 1-\gamma_{11}(s) & -\gamma_{10}(s) & 0 & 0 & \cdot & 0 \\ -\gamma_{22}(s) & 1-\gamma_{22}(s) & -\gamma_{20}(s) & & \vdots & \cdot \\ \cdot & \cdot & & & & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\gamma_{rr}(s) & -\gamma_{r,r-1}(s) & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\gamma_{n-1,0}(s) \\ -\gamma_{nn}(s) & -\gamma_{n,n-1}(s) & \cdot & \cdot & \cdot & 1-\gamma_{n1}(s) \end{vmatrix} ,$$

and $$\gamma_{ij}(s) = \int_0^\infty e^{-st} C_{ij}(t) dF(t) .$$

The mean-time of system's faultless operation is

$$T_o = -f'(0) - \frac{f'(0) \sum_{i=1}^{n} (-1)^{i-1} \delta_i(0)}{f^{n-r}(rv) \prod_{k=1}^{r} f(kv)} ,$$

$\delta_i(s)$ are the subdeterminants of $\Delta_n(s)$ obtained by crossing out the first row and the i-th column.

CHAPTER II

ON STANDBY REDUNDANCY WITH ERLANG'S REPAIR DISTRIBUTION

2.1 Statement of the Problem. Notations

We consider an $(n + 1)$-identical units redundant system, as in [2], with the following assumptions:

The failure-time distribution of the basic unit is assumed to be arbitrary while the repair time distribution is assumed to be Erlang with shape parameter μ and scale parameter λ. That is, when a unit is under repair it passes through μ phases and the time, spent in each phase, is distributed according to an exponential law, with parameter λ.

The system is supplied with n repair facilities, the switch is perfect and the switchover time is instantaneous.

The failure and repair processes are completely independent. The behavior of our system is that at $t = 0$ one unit is put to work and n are kept in cold standby.

Upon failure of the active unit a standby unit is switched on to operation and the failed unit is taken for repair immediately.

When repair has been completed the unit is kept in standby, and further it is assumed that after repair a unit recover its function perfectly. The repair of each unit starts from phase 1 and by a succession of $\mu - 1$ shocks goes through μ phases until repair is completed. Each time a unit has failed there is one facility available for its repair and therefore the repair times of the units are independent. A system failure occurs if the operating unit fails when all the other units are under repair.

Our aim is to derive the LS transform of the distribution of the first time to system-down and the mean time of system's faultless operation.

We introduce the following notations:

$F(t)$: Failure time distribution of the operating unit.

$G(t)$: Repair-time distribution of a unit.

$$= 1 - \sum_{i=0}^{\mu-1} \frac{(\mu\lambda t)^i}{i!} e^{-\mu\lambda t}$$

$g(t)$: Probability density function of repair time of a unit.

$$= \mu\lambda \frac{(\mu\lambda t)^{\mu-1}}{(\mu - 1)!} e^{-\mu\lambda t}$$

$P_{ij}(t)$: Transition probability of the system from state i to state j.

$Q_i(t)$: Probability that the system fails prior to t, given that it is in state i at $t = 0$ $(i = 0,1,2,\ldots\ldots.,n)$

$R(t)$: System's reliability.

$$f(s) = \int_0^\infty e^{-st} dF(t) .$$

$$Q_i(s) = \int_0^\infty e^{-st} dQ_i(t) .$$

$$\bar{G}(t) = 1 - G(t) .$$

$C_{m,p}(t)$: Probability that during time t, p units have been repaired out of m.

$$= \binom{m}{p} \bar{G}^{m-p}(t) G^p(t) , \quad p = 0,1,2\ldots m .$$

$$a_{m,p}(s) = \int_0^\infty e^{-st} C_{m,p}(t) dF(t) ,$$ where in particular

$$a_{m,o}(s) = \int_0^{\infty} e^{-st} C_{m,o}(t) dF(t) = \int_0^{\infty} e^{-st} \left[\sum_{i=0}^{\mu-1} \frac{(\lambda t)^i}{i!} e^{-\mu\lambda t} \right]^m dF(t) .$$

$Q(t) = 1 - R(t)$, system's unreliability.

LS: Laplace-Stieltjes

2.2 Analysis of the System

The behavior of our system is described by a semi-Markov process. The "Markov points" of the process are the instants at which the state transitions take place, that is, the instants at which failures occur. The process can be found in $n+2$ states during the time interval t. At any instant of time between 0 to t we identify the state of the process by the number of units under repair or waiting for repair. The last state is the down-state for the system.

Let us assume that at the instant x $(0 \leq x \leq t)$ the process is in state P, $(0 < P \leq n)$, that is, the system has one operating unit, P under repair, and $n - P$ in standby, then the following transitions of the system can occur. First, transition to state $(P+1)$ if failure of the operating unit occurs before any unit's repair is completed. Second, transition to state $(P-i)$, $i = 0,1,2...P-1$ if $(i+1)$ units have completed their repair before failure of the operating unit.

The distribution function of the time which the process spends in any state i, is $F(t)$, that is, the failure time distribution of the operating unit.

We begin our investigation of this system from the special case of $n = 2$. The case $n = 1$, the two-unit system has been considered in [3], if we put $H(t) = 0$ and $G(t)$ the Erlang distribution with parameters μ and λ.

The system of 3 units can be found in one of the following four states during time interval t :

State 0: One unit begins to operate, two units in standby.

State 1: One unit begins to operate, one unit starts a repair, one unit in standby.

State 2: One unit begins to operate, two units undergoing repair.

State 3: Two units undergoing repair, one waiting for repair; the system's down state.

Figure 3 gives the flow graph of a 3-identical unit redundant system.

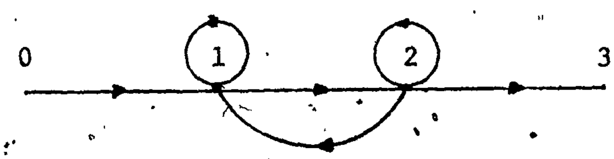

Figure 3

By definition, $Q_i(t)$ $(i = 0,1,2)$ is the probability that the system has been found in state 3 prior to instant t, given that it is in state i at $t = 0$.

Our aim is to derive $Q_o(s)$. It is easy to find the system of equations:

$$Q_o(t) = \int_0^t P_{01}(x)Q_1(t - x)dx ,$$

$$Q_1(t) = \int_0^t P_{11}(x)Q_1(t - x)dx + \int_0^t P_{12}(x)Q_2(t - x)dx ,$$

$$Q_2(t) = \int_0^t P_{21}(x)Q_1(t - x)dx + \int_0^t P_{22}(x)Q_2(t - x)dx + P_{23}(t) .$$

The transition probabilities of the system and their LS transforms are:

$$P_{01}(t) = \int_0^t dF(x)\ , \quad P_{01}(s) = f(s)\ , \qquad (2)$$

$$P_{11}(t) = \int_0^t C_{11}(x)dF(x)\ , \quad P_{11}(s) = a_{11}(s)\ , \qquad (3)$$

$$P_{12}(t) = \int_0^t C_{10}(x)dF(x)\ , \quad P_{12}(s) = a_{10}(s)\ , \qquad (4)$$

$$P_{22}(t) = \int_0^t C_{21}(x)dF(x)\ , \quad P_{22}(s) = a_{21}(s)\ , \qquad (5)$$

$$P_{23}(t) = \int_0^t C_{20}(x)dF(x)\ , \quad P_{23}(s) = a_{20}(s)\ , \qquad (6)$$

$$P_{21}(t) = \int_0^t C_{22}(x)dF(x)\ , \quad P_{21}(s) = a_{22}(s)\ . \qquad (7)$$

Writing the system (1) in terms of LS transforms we get:

$$\begin{aligned} Q_0(s) - P_{01}(s)Q_1(s) &= 0\ , \\ [1 - P_{11}(s)]\,Q_1(s) - P_{12}(s)Q_2(s) &= 0\ , \\ -P_{21}(s)Q_1(s) + [1 - P_{22}(s)]\,Q_2(s) &= P_{23}(s)\ . \end{aligned} \qquad (8)$$

From system (8), by Cramer's rule, we can find

$$Q_0(s) = f(s)\,\frac{a_{10}(s)\cdot a_{20}(s)}{W_2(s)}\ , \qquad (9)$$

where

$$W_2(s) = \begin{vmatrix} 1 - a_{11}(s) & -a_{10}(s) \\ -a_{22}(s) & 1 - a_{21}(s) \end{vmatrix}\ . \qquad (10)$$

We proceed now to the general case of such a system with $n+1$ identical units. Figure 4 gives the flow graph of the system.

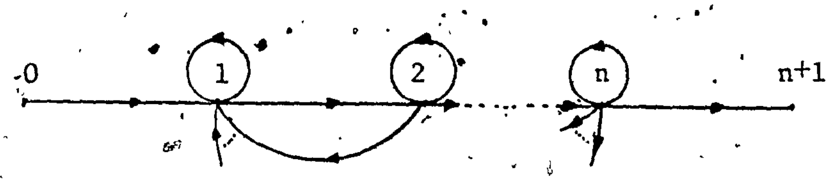

Figure 4 Flow graph of the n+1-identical unit system.

It is easy to see that the following system of renewal equations holds:

$$
\begin{aligned}
Q_0(t) &= \int_0^t P_{01}(x)Q_1(t-x)dx\,, \\
Q_1(t) &= \int_0^t P_{11}(x)Q_1(t-x)dx + \int_0^t P_{12}(x)Q_2(t-x)dx\,, \\
Q_2(t) &= \int_0^t P_{21}(x)Q_1(t-x)dx + \int_0^t P_{22}(x)Q_2(t-x)dx + \int_0^t P_{23}(x)Q_3(t-x)dx \qquad (11) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
Q_n(t) &= \int_0^t P_{n1}(x)Q\ (t-x)dx + \int_0^t P_{n2}(x)Q_2(t-x)dx + \ldots + \int_0^t P_{nn}(x)Q_n(t-x)dx \\
&\quad + P_{n,n+1}(t)\,.
\end{aligned}
$$

Writing system (11) in terms of LS transforms we get:

$$
\begin{aligned}
Q_0(s) \quad - P_{01}(s)Q_1(s) \ldots\ldots\ldots\ldots\ldots &= 0\,, \\
(1 - P_{11}(s))Q_1(s) - P_{12}(s)Q_2(s) \ldots\ldots\ldots\ldots &= 0\,, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \qquad (12)\\
- P_{n1}(s)Q_1(s) - P_{n2}(s)Q_2(s) \ldots\ldots + (1 - P_{nn}(s))Q_n(s) &= \\
&= P_{n,n+1}(s)\,.
\end{aligned}
$$

By Cramer's rule we obtain:

$$Q_0(s) = \frac{P_{01}(s)\, P_{12}(s) \ldots\ldots\ldots P_{n,n+1}(s)}{W_n(s)} \,. \tag{13}$$

The transition probabilities and their LS transforms are given by:

$$P_{m,m+1}(t) = \int_0^t C_{mo}(x)dF(x)\,, \quad P_{m,m+1}(s) = a_{mo}(s)\,, \quad m = 1,2,\ldots,n \tag{14}$$

and

$$P_{01}(t) = \int_0^t dF(x)\,, \qquad P_{01}(s) = f(s)\,; \tag{15}$$

$$P_{m,j}(t) = \int_0^t C_{m,j-m+1}(x)dF(x)\,, \qquad m = 1,\ldots n\,, \quad j \leqslant m \tag{16}$$

$$P_{m,j}(s) = a_{m,j-m+1}(s)\,. \tag{17}$$

Therefore

$$Q_0(s) = f(s)\,\frac{\prod_{v=1}^{n} a_{vo}(s)}{W_n(s)}\,, \tag{18}$$

where $W_n(s)$ is the following determinant:

$$W_n(s) = \begin{vmatrix} 1 - a_{11}(s) & -a_{10}(s) & 0 \cdots\cdots\cdots 0 \\ -a_{22}(s) & 1 - a_{21}(s) & -a_{20}(s) \cdots\cdots 0 \\ \cdots & \cdots & \cdots \\ -a_{n-1,n-1}(s) & \cdots\cdots & -a_{n-1,o}(s) \\ -a_{nn}(s) & -a_{n,n-1}(s) \cdots\cdots & 1 - a_{n1}(s) \end{vmatrix} \quad (19)$$

By definition of the function $C_{m,j}(t)$ we get

$$\sum_{j=0}^{m} C_{m,j}(t) = 1 , \quad (20)$$

and then
$$\int_0^\infty e^{-st} \sum_{j=0}^{m} C_{m,j}(t) dF(t) = \int_0^\infty e^{-st} dF(t) ,$$

or
$$\sum_{j=0}^{m} a_{m,j}(s) = f(s) , \quad m = 1, \ldots, n . \quad (21)$$

We transform now the determinant (19) as in [2], by adding the first column to the second, the resulting column to the third, and so on.

$$W_n(s) = \begin{vmatrix} 1 - a_{11}(s) & 1 - f(s) & 1 - f(s) \cdots\cdots 1 - f(s) \\ -a_{22}(s) & 1 - a_{22} - a_{21}(s) & 1 - f(s) \cdots\cdots \\ -a_{33}(s) & -a_{32}(s) - a_{33}(s) \cdots\cdots & \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 - f(s) \\ -a_{nn}(s) & -a_{n,n-1}(s) - a_{nn}(s) \cdots\cdots & 1 - f(s) + a_{no}(s) \end{vmatrix} \quad (22)$$

In determinant (22) we subtract the first row from all the others. We obtain the determinant:

$$W_n(s) = \begin{vmatrix} 1 - a_{11}(s) & 1 - f(s) \ldots\ldots\ldots\ldots 1 - f(s) \\ -1 - a_{22}(s) + a_{11}(s) & f(s) - a_{22}(s) - a_{21}(s) \ldots\ldots\ldots\ldots 0 \\ -1 - a_{33}(s) + a_{11}(s) & -1 - a_{32}(s) - a_{33}(s) + f(s) \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots 0 \\ -a_{nn}(s) - 1 + a_{11}(s) \ldots\ldots & \ldots\ldots\ldots\ldots a_{n,o}(s) \end{vmatrix} \tag{23}$$

Expanding determinant (23) with respect to the first row we get:

$$W_n(s) = [1 - f(s) + a_{1o}(s)]\, w_1(s) + [1 - f(s)] \sum_{j=2}^{n} (-1)^{j-1} w_j(s), \tag{24}$$

where $w_j(s)$ is the subdeterminant of $W_n(s)$ obtained by crossing out the first row and the j-th column.

But $\quad w_1(s) = \prod_{v=2}^{n} a_{vo}(s)$.

Relation (24) becomes

$$W_n(s) = \prod_{v=1}^{n} a_{vo}(s) + (1 - f(s)) \sum_{j=1}^{n} (-1)^{j-1} w_j(s). \tag{25}$$

Finally we get

$$Q_o(s) = f(s) \frac{\prod_{v=1}^{n} a_{vo}(s)}{\prod_{v=1}^{n} a_{vo}(s) + [1 - f(s)] \sum_{j=1}^{n} (-1)^{j-1} w_j(s)} \tag{26}$$

2.3 System's Mean Time of Faultless Operation

The system's mean time is given directly by:

$$T_o = \frac{-dQ_o(s)}{ds}\bigg/_{s=o} ,$$

$$T_o = -f'(0) - f'(0) \frac{\sum_{i=1}^{n}(-1)^{j-1} w_j(0)}{\prod_{v=1}^{n} a_{vo}(0)} . \tag{27}$$

Results (26), (27) coincide with those in [2] if we put $\mu = 1$, that is, considering exponential repair-time distribution.

2.4 Variance of System's Life Length

The variance of system's life length is given by

$$DT_o = \frac{d^2[Q_o(s)]}{ds}\bigg/_{s=o} - \left[\frac{d}{ds}[Q_o(s)]\right]^2$$

$$= f''(0) - [f'(0)]^2 + 2f'(0) + 1 - [2\ f'(0)]^2 \frac{\sum_{j=1}^{n}(-1)^{j-1} w_j(0) + 2f'(0) + 4}{\prod_{v=1}^{n} a_{vo}(0)}$$

$$- f'(0) \sum_{j=1}^{n} (-1)^{j-1} w_j(0) \frac{f'(0) \sum_{j=1}^{n} (-1)^{j-1} w_j(0) - 2f'(0) - 2}{\left[\prod_{v=1}^{n} a_{vo}(0)\right]^2}$$

$$+ 2f'(0) \sum_{j=1}^{n} (-1)^{j-1} w_j(0) \frac{\left[2 - f'(0) \sum_{j=1}^{n} (-1)^{j-1} w_j(0)\right]}{\left[\prod_{v=1}^{n} a_{vo}(0)\right]^3}$$

$$= f''(0) - [f'(0)]^2 + 2f'(0) + 1 - [2(f'(0))^2\zeta_o + 2f'(0) + 4]z_o^{-1}$$

$$- [(f'(0))^2\zeta_o^2 - 2(f'(0))^2\zeta_o - 2f'(0)\zeta_o]z_o^{-2}$$

$$+ [2f'(0)\zeta_o - 2(f'(0))^2\zeta_o]z_o^{-3},$$

where
$$\zeta_o = \prod_{v=1}^{n} a_{vo}(0),$$

$$z_o = \sum_{j=1}^{n} (-1)^{j-1} w_j(0).$$

2.5 An Application of the Main Problem

Consider a system of (n + 1)-identical units where one unit is operative and n units are in cold standby. Our objective is to derive the LS transform of the distribution of the first time to system down when the switchover time of a unit from standby state to operation is a random variable with an arbitrary distribution $L(t)$.

We introduce the following new notations:

$$H(t) = \int_0^t L(t-u)dF(u), \tag{1}$$

$$h(s) = \int_0^\infty e^{-st}dH(t) = f(s)\ell(s), \tag{2}$$

$$\ell(s) = \int_0^\infty e^{-st}dL(t), \tag{3}$$

$$c_n(s) = \int_0^\infty e^{-st}G_n(t)dH(t), \tag{4}$$

$$b_{m,j}(s) = \int_0^\infty e^{-st} C_{m,j}(t)dH(t) , \tag{5}$$

$$b_{mo}(s) = \int_0^\infty e^{-st} C_{mo}(t)dH(t) . \tag{6}$$

Again the system can be found in the same $n + 2$ states and its flow graph is that of Figure 4.

So

$$Q_o(s) = \frac{P_{o1}(s)P_{12}(s)\ldots P_{n,n+1}(s)}{W_n(s)} . \tag{7}$$

The transition probabilities and their LS transforms are

$$P_{m,m+1}(t) = \int_0^t C_{mo}(x)dH(x) , \quad m = 1,2,\ldots n ; \tag{8}$$

$$P_{m,m+1}(s) = b_{mo}(s) . \tag{9}$$

For $m = 0$ we get

$$P_{o1}(t) = \int_0^t dF(x) , \qquad P_{o1}(s) = f(s) , \tag{10}$$

$$P_{m,j}(t) = \int_0^t C_{m,m-j+1}(x)dH(x) , \quad P_{m,j}(s) = b_{m,m-j+1}(s) , m = 1,\ldots,n, j \leq m . \tag{11}$$

Finally we get

$$Q_o(s) = f(s) \frac{\prod_{\nu=1}^{n} b_{\nu o}(s)}{W_n(s)} , \tag{12}$$

where $W_n(s)$ after the same transformations as before is:

$$W_n(s) = \begin{vmatrix} 1 - b_{11} & 1 - h(s) & 1 - h(s) \ldots\ldots & 1 - h(s) \\ -1 - b_{22} + b_{11} & -b_{20} \ldots\ldots & & \\ \ldots\ldots & & & \\ \ldots\ldots & & & 0 \\ -b_{nn} & -1 + b_{11} \ldots\ldots & & b_{no} \end{vmatrix} \quad (13)$$

Expanding determinant (13) with respect to the first row we get:

$$W_n(s) = [1 - h(s) + b_{10}(s)]w_1(s) + [1 - h(s)]\sum_{j=2}^{n}(-1)^{j-1}w_j(s)$$

$$= \prod_{v=1}^{n} b_{vo}(s) + [1 - h(s)]\sum_{j=1}^{n}(-1)^{j-1}w_j(s) \quad (14)$$

Then, $$Q_o(s) = f(s)\frac{\prod_{v=1}^{n} b_{vo}(s)}{\prod_{v=1}^{n} b_{vo}(s) + [1 - h(s)]\sum_{j=1}^{n}(-1)^{j-1}w_j(s)}; \quad (15)$$

where $w_j(s)$ is the subdeterminant of $W_n(s)$ obtained by crossing out the first row and the j-th column.

The mean faultless operation time T_1 of this system is given by:

$$T_1 = -\frac{dQ_o(s)}{ds}\Big/_{s=o},$$

$$T_1 = -f'(0) - \frac{h'(0)\sum_{j=1}^{n}(-1)^{j-1}w_j(0)}{\prod_{v=1}^{n} b_{vo}(0)}. \quad (16)$$

CHAPTER III

ON STANDBY REDUNDANCY WITH k DELAYS

3.1 Statement of the Problem

We consider a system consisting of $n+1$ $(n = 1,2,\ldots)$ identical units in which only one is operating while the other n units are in cold standby, or waiting for repair, or under repair. The time to failure of the operating unit is distributed according to an arbitrary distribution $F(t)$. The renewal time is a random variable distributed according to an exponential law $G(t) = 1 - e^{-vt}$ and there is only one repair facility in the system.

We assume that the repair facility is available only after the accumulation of k failures $(1 \leq k \leq n)$.

The behavior of our system is that at $t = 0$ we put one unit to operate and we keep n in cold standby. When a failure occurs a standby unit is switched on to operate, instantaneously and the failed unit waits for repair. At the instant of a second failure another standby unit starts to operate, and the failed unit waits for repair.

Upon the accumulation of k failed units the repair facility starts working until the queue is emptied. A system-failure occurs if at the instant of a failure no standby is available. Our objective is to derive the LS transform of the distribution of the time to system down and the mean time of faultless operation.

3.2 Analysis of the System

The system can be found in $n+2$ states during the time interval t and each state can be identified by the number of units waiting for repair. Transitions from state to state occur at the instants of failures of the operating unit.

Viewing the nature of this system, the following system of equations can be obtained:

$$
\begin{aligned}
Q_0(t) &= P_{01}(t) * Q_1(t) , \\
Q_1(t) &= P_{12}(t) * Q_2(t) , \\
&\cdots\cdots\cdots\cdots \\
Q_{k-1}(t) &= P_{k-1,k}(t) * Q_k(t) , \\
Q_k(t) &= P_{k1}(t) * Q_1(t) + P_{k2}(t) * Q_2(t) + \ldots + P_{k,k+1}(t) * Q_{k+1}(t) , \\
&\cdots\cdots\cdots\cdots \\
Q_n(t) &= P_{n1}(t) * Q_1(t) + P_{n2}(t) * Q_2(t) + \ldots + P_{n,n+1}(t) ,
\end{aligned}
\tag{1}
$$

where $Q_i(t)$ is the probability that the system fails prior to t given that it is in state i at $t = 0$ $(i = 0,1,2,\ldots n)$.

We write now the system (1) in terms of LS transforms:

$$
\begin{aligned}
&Q_0(s) - P_{01}(s)Q_1(s) = 0 , \\
&0 \quad Q_1(s) - P_{12}(s)Q_2(s) = 0 , \\
&\cdots\cdots\cdots\cdots \\
&0 \quad - P_{k1}(s)Q_1(s) - P_{k2}(s)Q_2(s) \ldots - P_{k,k+1}(s)Q_{k+1}(s) = 0 , \\
&\cdots\cdots\cdots\cdots \\
&0 \quad - P_{n1}(s)Q_1(s) - P_{n2}(s)Q_2(s) \ldots\ldots [1 - P_{nn}(s)]\, Q_n(s) = \\
&\qquad = P_{n,n+1}(s) .
\end{aligned}
\tag{2}
$$

By Cramer's rule

$$Q_0(s) = \frac{P_{01}(s)P_{12}(s)\ldots\ldots P_{n,n+1}(s)}{W_n(s)}, \tag{3}$$

where $W_n(s)$ is the determinant:

$$W_n(s) = \begin{vmatrix} 1 & -P_{12}(s) & 0 & \cdots & & & & 0 \\ 0 & 1 & -P_{23}(s) & & & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1-P_{k-1,k}(s) & 0 & & 0 \\ -P_{k1}(s) & -P_{k2}(s) & \cdots & -P_{k,k-1}(s) & 1-P_{kk}(s) & -P_{k,k+1}(s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -P_{n1}(s) & -P_{n2}(s) & \cdots & \cdots & \cdots & -P_{n,k+1}(s) & \cdots & [1-P_{nn}(s)] \end{vmatrix}. \tag{4}$$

The transition probabilities and their LS transform are:

$$P_{01}(t) = P_{12}(t) = \ldots = P_{k-1,k}(t) = \int_0^t dF(x), \quad P_{01}(s) = \ldots \; P_{k-1,k}(s) = f(s), \tag{5}$$

$$P_{m,m+1}(t) = \int_0^t C_{mo}(t-x)\,dF(x) \quad \forall\, k \leqslant m \leqslant n.$$

Since $C_{mo}(t) = \overline{G}(t)$ we get

$$P_{m,m+1}(s) = a_{mo}(s) = f(s) - g_1(s), \tag{6}$$

and $P_{m,j}(t) = \int_0^t C_{m,m-j+1}(t-x)\,dF(x)$, where $m \geqslant j$.

Since $C_{m,m-j+1}(t) = \int_0^t \overline{G}(t-u)\,dG_{(m-j+1)}(u)$ we get

$$\underset{m\geqslant j}{P_{m,j}}(s) = g_{m-j+1}(s) - g_{m-j}(s) = a_{m,j}(s). \tag{7}$$

We also have $$\sum_{j=0}^{m} C_{m,j}(x) = 1 ,$$

and $$\sum_{j=0}^{m} a_{m,j}(s) = f(s) , \tag{8}$$

Using relations 4 - 7 we get

$$Q_o(s) = \frac{[f(s)]^k [f(s) - g_1(s)]^{n-k+1}}{W_n(s)} , \tag{9}$$

where

$$W_n(s) = \begin{vmatrix} 1 & -f(s) & 0 & 0 & 0 & 0 \cdots\cdots\cdots & 0 \\ 0 & 1 & -f(s) & 0 & 0 & 0 \cdots\cdots\cdots & 0 \\ 0 & 0 & 1 & -f(s) & 0 & 0 \cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots 1 & -f(s) & 0 \quad 0 \cdots & 0 \\ -g_k & -[g_{k-2}-g_{k-1}] & \cdots\cdots & 1-g_1+g_2 & -[f(s)-g_1] & 0 \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -[f(s)-g_1] \\ -g_n & -[g_{n-2}-g_{n-1}] & \cdots\cdots & \cdots & \cdots & \cdots & 1-g_1+g_2 \end{vmatrix} , \tag{10}$$

where

$$g_n(s) = \int_0^\infty e^{-st} G_n(t) dF(t) .$$

Transforming determinant (9) in the same way as before we obtain finally.

$$W_n(s) = \begin{vmatrix} 1 & 1-f(s) & 1-f(s) & \cdots & & & 1-f(s) \\ -1 & f(s) & 0 & \cdots & & & \cdots \\ -1 & f(s)-1 & f(s) & \cdots & & & \cdots \\ \cdots & \cdots & \cdots & \cdots & & & \cdots \\ -1 & f(s)-1 & \cdots & f(s) & 0 & & \cdots \\ -1-g_k & -1+f(s)-g_k-g_{k-2}+g_{k-1} & \cdots & \cdots & f(s)-g_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1-g_n & \cdots & \cdots & \cdots & \cdots & \cdots & f(s)-g_1 \end{vmatrix} \tag{11}$$

Expanding determinant (10) with respect to the first row we get:

$$\begin{aligned} W_n(s) &= [f(s)]^{k-2}[f(s) - g_1(s)]^{n-k+1} \\ &\quad + [1 - f(s)] \sum_{j=2}^{n} (-1)^{j-1} w_j(s) \\ &= [f(s)]^{k-1}[f(s) - g_1(s)]^{n-k+1} + [1 - f(s)] \sum_{j=1}^{n} (-1)^{j-1} w_j(s). \end{aligned} \tag{12}$$

Note that $w_1(s) = [f(s)]^{k-2}[f(s) - g_1(s)]^{n-k+1}$.

Now relation (9) becomes:

$$\begin{aligned} Q_o(s) &= [f(s)]^k \frac{[f(s) - g_1(s)]^{n-k+1}}{[f(s)]^{k-1}[f(s) - g_1(s)]^{n-k+1} + [1 - f(s)] \sum_{j=1}^{n} (-1)^{j-1} w_j(s)} \\ &= [f(s)]^k \frac{[f(s + v)]^{n-k+1}}{[f(s)]^{k-1}[f(s + v)]^{n-k+1} + [1 - f(s)] \sum_{j=1}^{n} (-1)^{j-1} w_j(s)} \end{aligned} \tag{13}$$

3.3 System's Mean Time of Faultless Operation

The system's mean time is given by

$$T_o = -\frac{d}{ds}\left[Q_o(s)\right]\Big/_{s=o}$$

$$= -f'(0) - f'(0)\,\frac{\sum_{j=1}^{n}(-1)^{j-1}w_j(0)}{[f(v)]^{n-k+1}}$$

. . . .

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