PROBLEM SOLVING BEHAVIOUR OF ADULTS AND ADOLESCENTS:
A COMPARATIVE STUDY

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ABSTRACT

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The following thesis studies the way adults solve non-routine mathematical problems, and compares their behaviour to that of school children. To explore the differences between these two groups, we have chosen a set of problems, a style of interviewing our subjects while they are solving these problems, and a questionaire. We are assuming that there will be differences and we have set up some research questions to that effect. We will explore the results in relation to these research questions.

The adults and school children are chosen so as to be at the same (grade 9) level mathematically. We are interested in exploring the ways in which the richer and wider experience of the adults will effect their problem-solving behaviour as compared to that of the school children. We will also look at the effects of school mathematics on the school children as compared to the adults who have been out of school-mathematics classes for some time.

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CHAPTER 1

LITERATURE REVIEW

This study involves two areas of concern. The first is problem solving in general, with it's theories and philosophies, and the second is the teaching of adults, specifically the teaching of mathematics to adults.

General Theories of Problem Solving

Recently there has been an increased interest in the study of problem solving, and literature on this area of exploration is accumulating rapidly. The importance of problem solving to mathematical functioning was always recognized, but the study of problem solving, until recently, did not concern teaching directly. Most of the research and literature concentrates on isolating, categorizing, and analysing problem-solving techniques and abilities. Most of the different theories on this subject attempt to divide the problem solving process into categories or stages in order to simplify this complex topic. There are several theories which try to explain what happens when we are involved in a problem-solving process.

1. Gestalt Theory

During a problem-solving situation, the solver attempts to relate one aspect of the problem-solving situation to another. According to Gestalt theorists, this involves reorganizing the elements of the problem-solving situation in a new way so as to solve the problem.[1] This search is a process in which the mind tries to impose order on the incoming stimuli, resulting in " 'structural understanding', the ability to comprehend how all the parts of a problem fit together to satisfy the requirements of the goal."[2]

Looking at a problem in a new way involves a flash of insight, when the subject suddenly is able to reorganize the given data in a new way. Gestaltists are concerned with creating novel solutions to new situations.[3] They contributed the idea that people get stuck in a problem-solving set, because they can not see a new way to fit the problem elements together.[4] According to Gestalt theory, there are two kinds of thinking: productive thinking (using new organization and insight), and reproductive thinking (using old habits and behaviours.)[5]

Wallas, in 1926, published his analysis of four different

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[1] R. E. Mayer, Thinking and Problem Solving (Illinois, U.S.A.: Scott, Foresman and Co., 1977) p.58

[2] Ibid. p.58

[3] Ibid. p.58

[4] Ibid. p.56

[5] Ibid. p.59

behaviours required in the problem solving process: (1) preparation, which is the gathering of information and preliminary attempts at a solution; (2) incubation, which is putting the problem aside for a while; (3) illumination, the point at which the key to the solution appears; and (4) verification, the checking of the solution.[1] Wallas' research was not based on rigid psychological experimentation, but on introspection by him and by others of what they thought they were doing while solving the problems.[2]

Gestalt problems give a solver the task of arranging the pieces of information or concrete objects in a certain way to solve the problem. These problems involve creative or novel situations.[3] They tend to be non-routine, involving the solver in a new and "different" situation.

Birch, in 1945, claimed that without appropriate past experiences such problems can not be solved. Gestaltists also argue that certain previous experiences can be detrimental to problem solving. Broad and general experience and certain past learning represent an "essential repertoire of behaviour which must be available for restructuring when the new situation demands". However, "productive thinking is impossible if the individual is chained to the past", relying on very specific

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[1] R. E. Mayer, Thinking and Problem Solving, p.65

[2] Ibid. p.65

[3] Ibid. p.58

and limited habits.[1]

A well known Gestalt problem is Luchins' water jar problem. This problem was designed to test the inhibiting effect of functional fixedness (einstellung) - how prior experience can limit an individual's ability to develop a solution.[2] This problem involves presenting subjects with the hypothetical situation of three jars of varying (given) sizes and an unlimited water supply, and asking them to figure out how to obtain a required amount of water (goal). The control group was given four problems (same water jar problem with four different sets of data) requiring the same problem solving set (first jar - second jar -2x third jar). Consequently, when presented with more problems (requiring a change in their problem set) they chose a long method (same as used in previous four problems) rather than a shorter and more productive method (first jar + second jar).[3] This provided the basis for the Gestalt claim that reproductive application of past habits could become an hinderance.

Duncker, in 1945, attempted to study the stages in solving a problem. His studies were empirical. He gave problems to his subjects and asked them to report their thought processes aloud as they were thinking. He concluded that the

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[1] R. E. Mayer, Thinking and Problem Solving, p.80

[2] Ibid. p.75

[3] Ibid. p.76

problem-solving process proceeds by stages from general solutions to functional ones and finally to specific ones. The original problem is continually being reformulated.[1] Duncker describes these stages: Functional solution or value, in which elements of the problem must be seen in terms of their general or functional usefulness, and then the general or functional solution precedes specific solutions; reformulating or recentering, the successive stages of reformulating or restructuring the problem the problem with each new practical solution creating a new, more specific solution; suggestion from above - working backwards, reformulating the goal to make it closer to what is given; suggestion from below - working forward, reformulating what is given to bring it closer to the goal.[2]

The problem Duncker used is the tumor problem: "Given a human being with an inoperable stomach tumor, and rays which destroy organic tissue at sufficient intensity, by what procedures can one free the person of the tumor by these rays and at the same time avoid destroying the healthy tissue which surrounds it?" A general solution would be to lower the intensity of the rays through the healthy tissue. A functional solution would be to give a weak intensity in periphery and concentrate in the place of the tumor. A specific solution would be to use a lens.[3]

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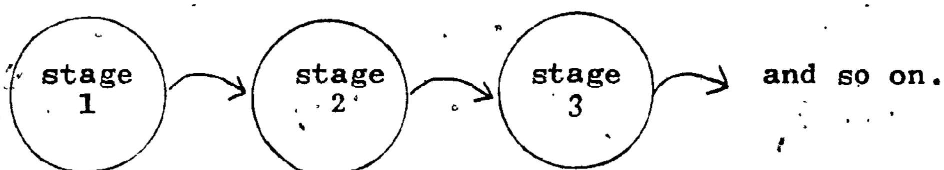
[1] R.E. Mayers, Thinking and Problem Solving, p.69

[2] Ibid. p.70

[3] Ibid. p.69

Here, the general solution of lowering the intensity of the rays while radiating on healthy tissues is a reformulation of the original goal. This is a process of recentering, where the solver restructures the problem to create new, more specific problems. Recentering is the essence of Gestalt theory. The problem is reformulated into smaller problems or subgoals.

Restle and Davis, in 1962, continued with this idea of subgoals. They assumed that problem solving involves completing a sequence of stages. When one stage is completed, the solver continues to the next stage. Each stage is independent, and each is equally difficult. The average time to move from any one stage to the next is constant. These ideas are represented in a diagram:



[1]

This diagram resembles a flow chart, where one progresses from one situation to another. These Gestalt ideas of stages and subgoals, together with the rapid development of computers, led to the formulation of another more recent theory of problem solving called Computer Simulation or Information Processing.

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[1] R.E.Mayer, Thinking and Problem Solving, p.71

2. Information-processing Theory

The information-processing theory approaches human thinking as the processing of information. Thinking is represented as a sequence of mental operations performed on the information in the subject's memory. It is also seen as a sequence of internal states that progress towards a goal.[1] According to this theory, human cognitive processes can be generated and expressed as computer programs and tested by running the programs to see if they produce an accurate representation of the human thought processes. This field of computer simulation is generally referred to as artificial intelligence.

Information-processing theorists are interested in defining precisely the processes and the states that a solver uses in solving a particular problem. They attempt to list the exact sequence of operations used, as in the form of a computer program.[2]

Problem solving, according to the theorists, involves breaking a problem down into subgoals and then achieving each subgoal by applying various problem-solving techniques, each of which changes the problem state in the direction of the subgoal.[3]

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[1] R. E. Mayer, Thinking and Problem Solving, p.133

[2] Ibid. p.133

[3] Ibid. p.142

This resembles the work of Restle and Davis who also looked at problem-solving as passing through consecutive stages, each independent and clear-cut.

In 1969, Ernst and Newell suggested that there are certain general problem solving techniques that are involved in solving a wide variety of problems. They constructed a general computer program called the General Problem Solver (GPS) that was able to solve different types of problems. The computer program tests a problem solving technique, determining whether it changes the problem solving state to one that is closer to the subgoal. If it succeeds, the technique is used and the program starts over. If it fails, then the program tries another technique. This program consists of four major components. The 'initial state' represents the given or initial conditions. The 'goal state' represents the final or goal situation. 'Problem states' are intermediate states which result from an application of an operator to a state. 'Operators' are all the moves or manipulations which may be performed on one state to change it into another.[1]

There are here some similarities to Wallas' four types of behaviours in problem solving. The 'initial state' is similar to the computer determining the success of a technique. The incubation and illumination periods described by Wallas

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[1] R.E. Mayers, Thinking and Problem Solving, p.137

can not be compared to a computer action. They are purely 'human' types of behaviour.

Simon claims that the human information-processor operates "almost entirely serially, one process at a time".[1] Simon and Newell describe a problem solving process, given a task environment (a problem) and an IPS (the solver): (1) An initial process (the input translator) produces inside the problem solver an internal representation of the external development. (2) After the internal representation, the solver responds by selecting a particular problem solving method. This method is formulated in terms of the internal representation. (3) The selected method is applied. It controls both the internal and external behaviour of the solver. The execution of the method may be halted depending on the outcome of the processes that monitor its application. (4) When a method is terminated there are three options: (a) another method may be attempted, (b) a different internal representation is selected and the problem is reformulated, or (c) the attempt to solve the problem is abandoned. (5) The method may have produced new problems (subgoals) and the solver may attempt to solve one of these. The solver may also set aside a new subgoal and continue with another branch of the original method.[2] This process, according to Simon, applies to a human problem-solver as well as to a GPS (computer problem-solver).

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[1] Herbert A. Simon, 'Information-processing Theory of Human Problem Solving', in W.K. Estes (Ed.) <u>Handbook of Learning and Cognitive Process</u>, Volume 5 (Hillsdale, N.J.: Lawrence Evelbaum Associates, 1978) p.273

[2] A. Newell and H.A. Simm, <u>Human Problem Solving</u>, (Englewood Cliffs, N.J.: Prentice-Hall Inc., 1972) p.88

The kind of problems that computers can solve are called 'move' problems. They have three basic characteristics: a definite initial state, a definite goal state, and a well defined set of allowable operators.[1] Simon admits that information processing theory does not deal with tasks that are not laboratory move problems.[2] Therefore, this theory does not work for all types of problems (i.e. it can not be generalized).

The experimental method used in computer simulation involves asking subjects to solve problems aloud while describing their thought processes and behaviours.[3] (This differs from the Gestaltists who tend to use their own introspection while solving a problem.) This method is criticized by Kendler. He cites Watt and Ach, according to whom a person's thinking occurs before s/he knows what s/he is thinking. Therefore, he argues, this verbal reporting might conflict with the thinking and thus change it.[4]

After the verbalizing of the subjects, the experimenter analyses their protocols, deriving a description of each subject's mental processes. S/he then specifies these in a computer program and tests it by running the program and comparing the computer's protocol to the subject's.[5]

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[1] R. E. Mayer, Thinking and Problem Solving, p.131

[2] H. A. Simon, Information-processing Theory of Human Problem Solving, p.279

[3] Mayer, p.136

[4] Howard H. Kendler, Concept of the Concept In: A. W. Melton (Ed.) Categories of Human Learning (New York: Academic Press, 1964) p.224

[5] Mayer, p.136

The information-processing approach reduces a problem solving process to a computer program. This is a simplification which may overlook additional essential elements existing in the process, like the affective aspects involved. The uncomplicated, smooth and straight forward programs that, according to information-processing theorists, represent human problem solving behaviours, hardly describe the complex and rich behaviour of a human problem-solver. The description of problem solving as a serial process (one process at a time) ignores that the human solver appears to be able to process more than one item at a time (parallel processing).

Although the information-processing approach simplifies problem-solving behaviour, the associationist view of thinking attempts to further simplify the activity of the solver.

3. Associationist Theory

The associationist psychologists or behaviourists describe problem solving as a trial and error application of pre-existing responses and tendencies which are called 'habits'.[1] There is a stimulus (a problem solving situation) a response (problem solving behaviour) and the associations between them.[2] Associationists deal with reproductive thinking, with past experiences, as opposed to Gestaltists who are interested in productive or novel ways of thinking.

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[1] H.A. Simon, Information-processing Theory of Human Problem Solving, p.18

[2] Ibid. p.18

E. L. Thorndike is known for his experiments with cats (which led to his formulating the associationist theory). He found that by practice, the cat performed the responses that did not work less often than the responses that did work.[1] Skinner, a well known behaviourist, claimed that behaviour is a response to stimuli.

Associationists see thinking as the selecting of correct responses from a collection of responses. This is similar to the information-processing theory, where the subject applies a series of moves or operators until s/he has changed the situation to solve the problem.[2] But this is where the similarity ends. Information-processing theorists concentrate on the activity of the solver and on the mediating process used to solve a problem. The associationists dismiss the idea of internal processes. They believe that theories of thinking must deal with directly observable (empirical) evidence.[3] They emphasize simplicity, clarity, and noncomplicated events. This is different also from the Gestalists, who are necessarily vague, since they deal with complex experimental situations.

4. Soviet Studies

Research on problem solving was also done by a group of Russian researchers, among whom Krutetskii is the best known.

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[1] R. E. Mayer, Thinking and Problem Solving, p.17

[2] Ibid. p.133

[3] Ibid. p.6

[4] V. A. Krutetskii, The Psychology of Mathematical Abilities in School Children, (Chicago: University Press, 1976) p.4

The emphasis here is on mathematical abilities studied through the solutions of routine high-school problems. Krutetskii claims that abilities are not inate, but are developed through practice and instruction.[1] He suggests three stages of mental activity in the process of solving mathematical problems: receiving information, processing or transforming the obtained information, and retaining the information.[2] The stage of receiving the information is similar to the gathering of information (preparation) stage of Wallas, where the solver prepares him/herself by putting together the given information. Processing or transforming the obtained information is similar to the incubation and illumination stages where the solver is internally processing the data and has a flash of insight when the transformation succeeds. This stage is also similar to Ernst and Newell's 'problem states' and 'operators' where the solver performs manipulations and moves which transform a state into another, nearer to the 'goal state'.

In order to test mathematical abilities, these Russian researchers devised a systematic, classified and organized collection of experiental problems for investigating individual differences in specific areas.

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[1] V.A. Krutetskii, The Psychology of Mathematical Abilities in School Children, (Chicago: University Press, 1976) p.4

[2] Ibid. p.184

In his research, using these systematic techniques, Krutetskii found that mathematically capable pupils, in their initial orientation to the problem, curtailed their processes of reasoning. They tended to single out the features essential for the problem.[1] They grapsed the structure of the problem while perceiving it.[2]

This is similar to the view held by Gestaltists who claim that the solver attempts to understand how the different parts of the problem fit together (structural understanding). At the same time s/he is looking at the specific data, the solver also looks at the problem as a whole.

Krutetskii distinguished mathematically able pupils by great flexibility in their mental processes. They were able to switch easily from one mental operation to another, breaking established solution patterns and replacing them with new ones.[3] This is similar to the Gestalt idea of a 'mental-block' and how successful solvers must break out of the established habits and pre-existing behaviours. Krutetskii also found that the capable students generalized the material during the solution more rapidly and broadly than the other students.[4]

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[1] V.A. Krutetskii, The Psychology of Mathematical Abilities in School Children, p.230

[2] Ibid. p.231

[3] Ibid. p.282

[4] Ibid. p.240

In our research we have used many of Krutetskii's techniques and also many of his problems, some of which have been revised according to our own needs. Krutetskii has classified his problems according to their types. We chose many of our problems from these categories.

5. Theories Related to the Use of Heuristics

More recent research and theories tend to deal with more specific mathematical abilities and problem-solving techniques called heuristics. Emphasis is placed on the importance of problem solving in mathematical learning and, as a consequence, on the use of heuristics.

Polya, one of the founders of the "heuristics movement", defines heuristics as strategies of problem solving.[1] He suggested that problem solving is based on cognitive processes that result in "finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately attainable".[2] Polya's aim is to teach a method (heuristics) which can be applied to solving a variety of problems. He influenced current research with his classification of the problem-solving activity into four stages: understanding the problem, realizing the connections and devising a plan, carrying out the plan, and looking back. Each of his four strategies

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[1] G. Polya, How to Solve It, (New Jersey: Princeton University Press, 1945) p.vii

[2] R. E. Mayer, Thinking and Problem Solving, p.6

consists of more specific heuristics in which Polya asks questions or gives hints to direct the solver.[1] As an example, in a section entitled "Understanding the Problem", Polya asks the following: What is the unknown? What are the data? What is the condition?[2] These questions provide instructions or directions towards systematic thinking.

These stages are similar to Wallas'. The 'understanding' stage is similar to Wallas' 'preparation' stage and both Wallas' 'incubation' and 'illumination' phases. Wallas' verification includes 'carrying out the plan' and 'looking back'.[3]

The emphasis in Polya's theories is the teaching of specific heuristics to students in order to facilitate mathematical problem solving.

Kantowski says that in order to solve a nonroutine problem a subject needs two tools: mathematical knowledge required to solve the problem, and heuristics, which she describes as 'rules of thumb' that result in a strategy that can be used to put together the given information leading to an attempt at a solution.[4]

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[1] G. Polya, How to Solve It, p.5

[2] Ibid. p.8

[3] R. E. Mayer, Thinking and Problem Solving, p.67

[4] M. G, Kantowski, The Use of the Microcomputer in Instruction for Problem Solving. (Paper presented at the N.C.T.M. Annual General Meeting, Toronto, 1982) p.60

In this area, affective aspects are again ignored rendering these theories incomplete. Although the teaching of heuristics will certainly facilitate problem solving, aspects like negative self-image of the solver will hinder his/her problem solving ability no matter how many heuristics s/he has learned.

Schoenfeld claims that it is not enough to know many heuristics. He says that one should be able to organize these heuristics and to make decisions about how and when to use them. He calls such decisions 'managerial' decisions. He claims that the success of solving a problem depends also on these managerial decisions, not only on the simple heuristics.[1]

Some examples of managerial decisions from his protocols are: (1) consideration of the utility of potential alternatives which arise during the problem solving process. (2) progress monitored or (re)assessed, including finding reliable means of terminating wild-goose chases. (3) assessment of potential utility of planned actions.[2] He lists other interesting heuristics such as: (1) constructing an altered version of the problem and working on that. (2) dismissing previous efforts easily.[3] These heuristics can be applied not only to mathematical learning but to general learning as well.

Schoenfeld has composed a list of three frequently used heuristics, each of which consists of sub-heuristics. The three

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[1] A. H. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving. (Paper presented at the A.E.R.A. Annual Meeting, Los Angles, 1981) p.16

[2] Ibid. p.11

[3] Ibid. p.13

are: analysis, exploration, and verifying the solution.[1] These are similar to Polya's stages. The analysis heuristic is similar to 'understanding the problem'. The exploration heuristic includes 'realizing the connections', 'devising a plan' and 'carrying out the plan', verification is 'looking back'.

Theories Related to Adult Learning

General Theories of Adult Learning

As well as problem solving, this thesis deals with adult learning. Today, more and more adults are going back to school and education programs are being re-evaluated and redesigned to satisfy adults' needs. Literature on the subject of adult learning has proliferated. Questions have appeared as to the learning capabilities of adults such as: How does experience and maturity influence the learning of adults? Does age affect learning positively or negatively?

Krutetskii says that abilities are developed by living, working and practice.[2] Although he is not specifically referring to adults, there is a definite connection. Kidd claims that because of richer experience, adults have abilities to secure a 'clear grasp and understanding of the ideas

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[1] A. H. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving. (Paper presented at the A.E.R.A. Annual Meeting, Los Angles, 1981) p.21

[2] V. A. Krutetskii, The Psychology of Mathematical Abilities in School Children, p.4

presented'.[1] They have the capacity and habit of making use of all that they know or can find out in interpreting meanings of ideas read.[2] They have the ability to think critically and to perceive strengths or weaknesses in what they read. They also have a tendency to fuse new ideas with previous experience.[3] Although Kidd does not mention specific ages when he talks about adults, he does mention in the beginning of his book How Adults Learn that he considers the summit before the decline of mental capacities as the age of seventy-five or 'whatever age signals the termination of a healthy organism'.[4] We can assume that the word 'adult' means anyone past childhood, who is between early adulthood and old age.

A. T. Welford has found that older adults' performance tends to be slower and more deliberate than that of younger subjects but subsequently tends to be more accurate.[5] Lorge has found that when he put a time limit on a test, adults of forty years and over scored less than the younger adults. But when time was unlimited their scores increased dramatically.[6]

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[1] J. R. Kidd, How Adults Learn, (New York: Association Press, 1973) p.45

[2] Ibid. P.45

[3] Ibid. p.45

[4] Ibid. p.22

[5] Ibid. p.83

[6] Ibid. p.82

Kidd concludes from these studies that a test conducted with a time limit supports the conclusion that there is a decline in mental capacity with age but when a time limit is not a factor, there is no significant decline associated with age.[1] He says that of all human abilities, judgement and reasoning ability seem to reach a peak late in adult life.[2]

Kesler, Denney and Whiteley, in 1976, found that age differences between younger and older people are negligible in regard to non-verbal intelligence.[3] As for verbal abilities, Lorge has found that vocabulary improves with age.[4] Living longer is conducive to vocabulary. Mayer says that adults tend to classify words both by meaning and grammatical function, and in this they differ from children. Children store words exactly as used in ordinary speech.[5] An interesting observation, as cited by Mayer, is made by Ivan M. Schenou, who claims that during adulthood, thinking can and does become free of language.[6]

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[1] J. R. Kidd, <u>How Adults Learn</u>, (New York: Association Press, 1973) p.82

[2] Ibid. p.83

[3] Agruso, <u>Learning in Later Years</u>, p.108

[4] Kidd, p.85

[5] Mayer, <u>Thinking and Problem Solving</u>, p.170

[6] Kidd, p.88

Age seems to affect memory. Kidd says that there is a decline in the ability to remember isolated facts.[1] Agruso also says that memory declines with increasing age. He found restricted short-term memory in older adults, particularly in unfamiliar material.[2]

How does 'adulthood' affect general learning ability? Kidd cites the Thorndike studies as the "foundation stone" for adult education.[3] E. L. Thorndike has explored the effects of aging. He has found that the decline in capacity for learning from a person's mid-twenties to early forties is approximately 1% per year, a negligible decline.[4] He says that the influence of age on ability to learn is slight, although the most advantageous period for learning (according to him) is between the ages of twenty and twenty-five.[5]

Theories Pertaining to Adults' Mathematics Learning

Research of adult learning of mathematics has been conducted mainly through the study of problem solving by adults. How do adults solve problems? Literature on this subject and related aspects is abundant. For many researchers, the test

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1 Kidd, How Adults Learn, p.91

2 V. M. Agruso, Learning in the Later Years, Principles of Educational Gerontology, (New York: Academic Press, 1978) p.47

3 Kidd, p.79

4 Ibid. p.79

5 Ibid. p.79

of intellectual capacities is in performance, in the intricacies of the mind as it solves problems.[1] Kidd gives four activities that one goes through in the process of problem solving: (1) Awareness, or knowing a problem exists. (2) Preparing to find a solution. (3) Attempting to produce a solution. (4) Evaluating the adequacy of the attempted solution.[2] He says that processes such as these do not seem to be eroded by age; actually the richer experience of the adults seems to enhance these processes.[3] Leone Burton gives a list of procedures for solving problems. He draws a connection between these procedures and their applications to real life problems which adults encounter every day. He gives nine procedures: (1) Collecting and analysing information. (2) Searching for relationships. (3) Making and testing of hypotheses. (4) Distinguishing between objective and subjective information. (5) Working systematically. (6) Using examples to 'feel' a problem. (7) Finding methods of communicating results. (8) Verifying tentative solutions. (9) Deriving patterns (generalization).[4]

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1 Kidd, <u>How Adults Learn</u>, p.90

2 Ibid. p.90

3 Ibid. p.90

4 Leone Burton, "Problems and Puzzles", <u>For the Learning of Mathematics 1, 2</u>, p.21

Many of these have been mentioned by previous researchers like Polya and Schoenfeld, but not in regard to adults or to real-life problems. It is clear that all the above abilities are used by adults in every day life, and therefore, ideally they should be used while solving mathematical problems.

In spite of the positive implications of these every day abilities on the problem solving of adults (and as a consequence, on their mathematical abilities) the prevailing view regarding adults is that they are not efficient mathematics learners.

The phenomena of "math anxiety" has been associated with adults. "Math anxiety" is a fear of mathematics rendering the learner unable to function at his/her best while learning mathematics. There are many educational programs and books that take into account this anxiety and try to aleviate it. S. Tobias, has simplified mathematical ideas in an encouraging and interesting manner in order to minimize anxiety.[1] Darwin Hendel has developed the Math Anxiety Program in the University of Minneapolis to facilitate mathematical learning.[2] The program is based on the use of scales and questionnaires which evaluate the extent of anxiety in individuals. Some of these scales are: the Math Anxiety Rating Scale, the Facilit-

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1 Shelia Tobias, Overcoming Math Anxiety, (New York: Norton, 1978)

2 Darwin D. Hendel, The Math Anxiety Program: Its Genesis and Evaluation in Continuing Education for Women, (Minnesota U., Minneapolis, 1977).

ating Anxiety Scale, the Debilitating Anxiety Scale and the Sentence Completion Questionnaire.[1]

Why do adults, who are equipped with problem-solving skills from their every day lives, exhibit this math anxiety that negatively influences their mathematical problem solving?

Emotions have been found to play a crucial role in the problem-solving process. Kidd says that the presence of apprehension, fear, or a lack of confidence have a strong negative impact on problem solving.[2]

In relation to mathematics, he says that older people have a resigned acceptance that they can not do mathematics since they associate failure and unpleasantness with the learning of mathematics during childhood.[3] Gardner Murphy has found that adults have more emotional associations with the factual material in math problems than do children, and they have more elaborate devices of controlling these emotions.[4] Kidd gives a list of feelings that have a constant influence on learning, and on mathematical problem solving:

(1) Love and associated feelings such as respect and encouragement.
(2) Rage, anger, frustration and similar emotions connected with anxiety about competence.

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[1] Darwin D. Hendel, The Math Anxiety Program: Its Genesis and Evaluation in Continuing Education for Women, (Minnesota U., Minneapolis, 1977).

[2] J. Kidd, How Adults Learn, p.90

[3] Ibid. p.90

[4] Ibid. p.95

(3) Fear and suspicion, which are conducive to indifference.[1] Kidd says that the self-image of the adult (how one perceives one's own behaviour and how one thinks one is perceived by others) affects one's learning.[2] M. Knowls says that adults have a deep need to be self-directing.[3] C. Rogers claims that a learning situation that is threatening to the learner impedes learning significantly.[4] J. Apps claims that almost every returning adult student has a poor academic self-concept.[5]

William More, in Emotions and Adult Learning, explores the affective aspects of learning. ~~He~~ says that a conflict arises when new learning presents itself as a threat to the stability of established habits.[6] He gives a list of aspects that have an affect on learning:

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[1] J. Kidd, How Adults Learn, p.97

[2] Ibid. p.126

[3] M. Knowls, The Adult Learner, a Neglected Species, (Houston, Texas: Gulf Publishing Co., 1978) p.31

[4] Ibid. p.41

[5] J. W. Apps, The Returning Student: Characteristics and Problems. (U. of Wisconsin-Madison)

[6] William S. More, Emotions and Adult Learning, (Lexington, Mass.: Lexington Books, 1974) p.35

(1) Attitudes which have to be unlearned.

(2) Values, which are being challenged.

(3) Motivation which has to be recognized.

(4) Personal involvement in the subject.[1]

R. Wertime says that the feelings before solving a problem also influence the solvers. Their performance will be affected by their perception of the problem as a leisure activity or as a chore.[2] These feelings and emotions depend on 'the importance of the context to our sense of worth' and on the degree of anticipated difficulty.[3] Wertime stresses the relationship between problem solving and self-esteem.[4] Adults seem to have more self-esteem to lose when they are solving problems.

In this study we are going to observe adults in a problem-solving situation and compare their behaviour to that of school children.

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[1] William S. More, Emotions and Adult Learning (Lexington, Mass.: Lexington Books, 1974) p.6

[2] Richard Wertime, Students, Problems and 'Courage Spans', In:Cognitive Process Instruction, J. Lochhead and J. Clement (Eds.) (Philadelphia: Philadelphia Institute Press, 1979) p.193

[3] Ibid. p.193

[4] Ibid. p.194

CHAPTER II

NATURE OF THE RESEARCH

The general aim of this thesis is to investigate differences between the ways in which adults and school children attack mathematical problems. Much of the research on mathematical performance of adults has negative connotations. Math anxiety is considered a trait that many adults have. R. Wertime says that there is a connection between problem solving ability and self-esteem. He looks at solving a problem as a burden that is important to the adult's sense of worth and self-esteem.[1] This aspect of problem-solving motivation shows that adults have a self-image to live up to, and as a result are threatened by failing in mathematics and problem solving. There are other negative aspects related to adult learning. Welford and Peers discuss the lack of speed and logic of older adults.[2] Kidd talks about the resigned acceptance that adults have that they can not do mathematics.[3] Kidd says that

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[1] R. Wertime, "Students, Problems and 'Courage Spans'", p.192

[2] Kidd, How Adults Learn, p.83

[3] Ibid. p.91

it is believed that there is a decline in intellectual capacity associated with getting older but, he claims, this is a myth.[1]

Research shows that adults have certain characteristics that should make them proficient in problem solving. W. More says that adults bring a wide and varied experience of life with them.[2] Lorge has found that there is no significant decline of intellect with age.[3] Kidd talks about the many positive aspects of adult learning. Adults have a wide variety of nonmathematical skills that can help them in problem solving. These abilities or general skills stem from the rich experience of the adults.[4] Solving real-life problems must surely help in solving mathematical ones.

The experience, the awareness of that experience, and the intellectual maturity of the adult solvers in our research may substantially influence their problem solving. We are interested in these positive aspects of adults' abilities in solving problems. We want to minimize the negative aspects, and look for the positive ones.

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[1] Kidd, <u>How Adults Learn</u>, p.70

[2] W. Moore, <u>Emotions and Adult Learning</u>, p.13

[3] Kidd, p.83

[4] Ibid. p.45

An effort has been made to conduct the interviews in a nonthreatening situation, using non-standard problems. By giving non-standard problems, we have attempted to minimize the frustration, distrust and feelings of anxiety felt by adults when confronted by mathematical problems. We will look for the positive effects of adulthood on problem solving by comparing the abilities of adults and school children. (The methodology used in this research is discussed in the next chapter.)

We will now present the questions for research. These have been divided into four categories:

(a) Perception of the problem.

(b) Strategies of solution.

(c) Affective aspects.

(d) Use of language.

(a) Perception of the problem

How one perceives a problem is influenced not only by the solver's mathematical knowledge but also by his/her maturity and experience. The adults who were tested have been out of school for some time. The school children were in grade 9. School mathematics has been criticized for being monotonous, for failing to stimulate the imagination, and for repressing the student's creativity. Being in school or being out of school may influence the subject's understanding of the problem. The school children may ignore nonstandard interpretations of the problem because of the conformist nature of school mathematics.

(1) Will the adults interpret stated problems in nonstandard ways?

The experience of adults might cause them to notice contradictory or unrealistic elements in a problem.

(2) Are adults more likely to recognize problems whose specific data renders them unrealistic?

(b) Strategies of Solution

Solving mathematical problems requires more than just mathematical skills. The learning of mathematics, and the use of that knowledge requires certain general skills as well. Schoenfeld makes a distinction between tactical and strategic or 'managerial' decisions. He says that tactical decisions, which include algorithms and heuristics, are important, but they do not guarantee success.[1]

The managerial decisions, which he calls 'metaheuristics', may 'make or break' a problem solving attempt.[2] Some examples of managerial decisions that he gives are:

(1) Choices of what kind of heuristics to employ.

(2) Deciding what to pursue, for how long.

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[1] A. H. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving, p.2

[2] Ibid. p.16

(3) Salvaging valuable elements of ultimately flawed approaches.[1]

The adult subjects' experience, awareness and maturity may influence their use of these general strategies.

(3) Are adults more likely to be aware of, and articulate about the need to make managerial decisions about strategies?

Stephen Brown talks about linear vs. holistic approaches to problem solving. In the linear approach, information is added bit by bit, without regard to the whole picture. In the holistic approach, one looks at the problem as a whold, viewing all the pieces as united.[2] Davis and McKnight claim that successful students have holistic entities which they can contemplate without going through the entire sequence step by step.[3] There is a connection between the holistic approach and managerial decisions. In order to look at a problem holistically, one has to decide when, how, and what to look at in the problem. These decisions are connected to the perception of a problem.

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[1] A. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving, p.11

[2] Stephen Brown, "Ye Shall be Known by Your Generations" For the Learning of Mathematics, 1, 3, p.29

[3] R. Davis and C. McKnight, Towards Eliminating "Black Boxes": A New Look at Good vs. Poor Mathematics Students (San Francisco, Meeting of the American Ed. Research Ass., April, 1979) p.5

(4) Will adults be more likely to use an holistic approach to solve problems?

Kleinmuntz says that good problem solvers tend to construct diagrams and to generally represent the problem in a physical way.[1] Krutetskii also emphasizes the advantage of diagrams.[2] Often, diagrams facilitate understanding of the problem. Diagrams can also be a way to avoid algebraic manipulations. Since the adults have been out of school for some time, they may tend to look for ways to avoid the use of algebra.

(5) Are adults more likely to use diagrams?

The adults might also rely on their intuition or common sense. After all, in real life one does rely on these.

(6) Will the adults be more likely to choose common-sense (non-algorithmic) solutions?

(c) Affective Aspects

Adults have a higher level of self-demand than children. They have more of a self-image to live up to, and generally feel that they have more to lose in a problem-solving situation.

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[1] B. Kleinmuntz (ed.), Problem Solving: Research, Method, and Theory (Wiley, 1966)

[2] V. A. Krutetskii, The Psychology of Mathematical Abilities in School Children p.188

(7) Will the adults be more likely to perceive and be frustrated by ambiguities in problem statements?

Intuition is often helpful in solving a problem. Because of the general anxiety that adults seem to feel towards mathematics, they are likely to distrust themselves.

(8) Will the adults be more likely to express intuitive solutions, but less likely to trust them?

(d) Use of Language (In "Thinking Aloud")

Because of a wider experience, adults have a larger vocabulary than children. Mayer says that there is a difference between adults and children in the way words are stored in memory.[1] We would like to explore these differences in the use of language. The school children may have an easier time articulating because they are still in touch with school mathematics and its language. On the other hand, the adults may be generally more at ease with language and its use.

(9) Are adults better at articulating their thoughts, and does this articulation facilitate their solutions?

In the next chapter we will describe the methodology used in this research. We will describe how we chose the subjects, the problems, the type of interviews and the questionnaire, so as to explore these research questions.

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[1] R. Mayers, Thinking and Problem Solving, p.170

CHAPTER III

METHODOLOGY

1. The Subjects

This study was conducted by interviewing three adults and three school children. The adult subjects (Linda, Lucy and Uri) are at least twenty-five years of age, and have been out of school for sometime. They have all graduated from high school but have not continued in their studies of mathematics. The school children (Robert, Judy and David) were chosen from grade nine, in Centennial Academy in Montreal. Their teachers were asked to recommend students with average abilities in mathematics (average abilities were emphasized).

2. The Sessions

Each subject is interviewed three times, each session lasting approximately forty-five minutes. The first session is conducted in such a way as to "break the ice". It is made as non-threatening as possible. The subject is asked to solve problems. The choice of the problems is conducive to the non-threatening atmosphere. No other demand is made of the subject in that first session. The interviewer merely observes the actions of the subjects. The solver is presented with the problem written on a card. After several minutes, the card is momentarily taken away and the solver is asked to state the problem in his/her own words. This procedure is intended to gather evidence on the first category of problem solving - understanding the problem. Aside from this intervention, the experimenter acts strictly as an observer during the attempts to solve the problems. Any questions pertinent to the hypothesis of the study is deferred until all the problems are declared solved or unsolvable. At the end of the session, the solver is asked to explain his/her answers. As well, s/he is asked general questions by the interviewer to help clarify his/her behaviour. This portion of the session is audiotaped.

During the second session, the subjects are asked to "think out loud" while solving the problems, and the whole session was audio taped. It was felt that by the second session, the solver would become more at ease with the experimental situation, hence rendering the "thinking out loud" technique more effective. At the end of this session, the sujject is asked to complete a brief questionnaire designed to elicit information about aspects of the affective components of problem solving.

In the third session, each subject is interviewed on the basis of the completed questionnaire and the transcripts of the previous two sessions. This is to allow the subject to give any additional information, as well as to illuminate aspects of the subject's problem-solving behaviour not noted at the time they occured. This session is also audiotaped.

3. The Interview Techniques

(a) Background of the Clinical Interview Method

The clinical interview method is a research technique developed and elaborated by Piaget. It is specially designed to uncover evidence of mental operations. Piaget constructed this method in his early investigations of children's thinking in the 1920's.[1] This method allows the subject to verbalize freely, thereby showing his/her covert intellectual processes.

Two other methods are widely used to explore mental processes. One is naturalistic observation, in which the interviewer observes the subject in a natural setting. The other is standard testing, which is considered objective and reliable.[2]

Confrey describes clinical interviews as being "task-oriented, flexible interviews between a student and interviewer, wherein the interviewer is expected to follow and pursue the student's thinking, asking questions until the student's reasons for responses are understandable to the interviewer".[3] Both the subject and the interviewer assume active roles in

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[1] S. Opper, Piaget's Clinical Method p.91

[2] Ibid. p.91

[3] Jere Confrey, Using Clinical Interviews to Explore Students' Mathematical Understandings (Los Angles, paper presented at the American Educational Research Association National Meeting, 1981) p.6

the process, with the subject guiding the questions. The interviewer does two things: s/he

(1) attempts to clarify the meanings of the subject's statements.

(2) actively hypothesizes about the implications of the responses, posing new questions to test these hypotheses.[1]

This testing of hypotheses is an essential part of the clinical interview. Opper describes the clinical interview as 'a hypothesis-testing situation', where there is a dialogue between the interviewer and the subject.[2] The role of the interviewer is not one of a 'neutral instrument' gathering information, but of an active participant in this dialogue.[3] It is here that Confrey sees some of the perils of this method. She says that there is always the possibility of a dominance or interference by the interviewer in this active interaction.[4] She recommends some basic standards and procedures to minimize these perils.[5]

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[1] J. Confrey, Using Clinical Interviews to Explore Students' Mathematical Understanding, p.15

[2] S. Opper, Piaget's Clinical Method, p.92

[3] J. Confrey, p.25

[4] Ibid. p.25

[5] Ibid. p.26

The clinical interview method can be classified into three kinds:

(1) The subject is given specific tasks and asked specific (standardized) questions. After all the questions are asked, the interviewer has the freedom to go in other directions.

This standardized clinical interview is used today by the followers of Piaget,[1]

(2) The subject is given tasks, observed in his/her attempts to solve them, and then asked questions which are intended to clarify the reasoning behind these attempts.

(3) The subject is given tasks and asked to think out loud while attempting to solve them. The interviewer is free to pursue any statements or ideas of the subject.

This kind of clinical interview was used by Krutetskii.[2] He has found some difficulties with this 'think out loud' technique. Often the subject would observe him/her-self, giving a description of his/her mental processes instead of verbalizing his/her thoughts.[3] Some subjects find it unnatural and awkward to think out loud. Other subjects confuse the instruction to think out loud with explaining how the problem was solved, and they go over the solution itself.[4]

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[1] S. Opper, Piaget's Clinical Method, p.93

[2] V. A. Krutetskii, The Psychology of Mathematical Abilities in School Children, p.92

[3] Ibid. p.92

[4] Ibid. p.93

Krutetskii found that he had to instruct his subjects very carefully as to what he wanted when he asked them to think out loud.[1]

The clinical interview is still looked upon as controversial and risky. It is viewed alternately as too informal and lacking in standardization or as a valuable tool, especially because of its informality.[2] J. Kilpatrick says that the subject's feedback as s/he is solving the problem is subjective but no less valuable. "There is no particular virtue in labelling such data as unscientific and ignoring them".[3] Confrey says that this method seems "cumbersome", "controversial", and "fragmented", and that its acceptability depends, at this time, on the "credability of the researchers, and one's sympathy for the results".[4]

Newell and Simon found that their subjects generated the same expressions and problem-solving directions while thinking

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[1] V. A. Krutetskii, The Psychology of Mathematical Abilities in School Children, p.93

[2] J. Confrey, Using the Clinical Interview to Explore Students' Mathematical Understandings, p.2

[3] J. Kilpatrick, Variables and Methodologies in Research on Problem Solving (Athens, GA: paper presented for the Research Workshop on Problem Solving in Mathematics Education, May 1975) p.7

[4] J. Confrey, p.2

aloud as others who were not verbalizing.[1] But, they say, this can not be generalized and further studies in this area are necessary.[2]

Opper gives the following criticisms of the clinical interview offered by researchers:

(1) It relies too heavily on language and sometimes a subject does not verbalize sufficiently.

(2) It is time consuming.

(3) It depends on the skill of the interviewer.

(4) It lacks rigidly standardized techniques.

Opper also gives the advantages of this method:

(1) It reduces anxiety.

(2) It presents a flexibility of approach and an open-endedness that can uncover patterns of thought that are not obvious.[3]

(b) <u>Interview Methods Used in our Study</u>

Our study consists of three interviews with each subject.

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[1] A. Newell and H.A. Simon, <u>Human Problem Solving</u>, p.474

[2] Ibid. p.475

[3] S. Opper, <u>Piaget's Clinical Method</u>, p.104-5

In the first one, we use the second version of the clinical interview, where the interviewer observes the subject, asking questions only when the subject has finished the tasks. The only interuption in this first interview is made at the beginning when the subject is asked to state the problem in his/her words. This is to find out how s/he understands the problem. After the problems are completed, the subject is asked questions about his/her answers and actions to help clarify them and clarify the means of arriving at them. In this first session, the subject is not asked to think out loud, in order to reduce anxiety and pressure. In the second interview, we use the third type of the clinical method. The subject is asked to think out loud while s/he is solving the problems. The interviewer is free to pursue any statements or actions of the subject. The third interview consists of attempting to clarify any remaining questions that the interviewer has as well as going over the answers to the questionnaire given at the end of the second interview.

In the interview techniques we have used, the only standardization is that of the tasks. All the subjects received the same problems. There was no probe standardization. The questions asked by the interviewer were contingent on the individual subject's responses.

4. Choice of Problems (Tasks)

In choosing problems, one must keep in mind what particular areas one wants to explore. Confrey says that in a clinical interview, problems must be carefully selected to elicit the types of responses desired. One must consider how challenging, frustrating, or provocative one wants the problem to be.[1] Krutetskii, in choosing his own problems, has attempted to equalize conditions like available knowledge, skills and habits. He has tried to "weaken" those influences on the problem-solving behaviour of the subjects.[2] He has selected problems that either required no particular knowledge for their solution, or required knowledge that was available to all the pupils.[3] He claims that his experimental problems are oriented not only towards the solution , but mainly towards revealing the solution process itself; the ways of achieving that solution.[4]

In selecting our problems, we have tried to minimize the necessity of mathematical skills in solving them, making them non-standard. A large number of the problems were taken from Krutetskii's collection.

In the first session, six problems were presented with an extra one to be used if extra time was available. The fol-

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[1] J. Confrey, p.16

[2] V.A. Krutetskii, p.94

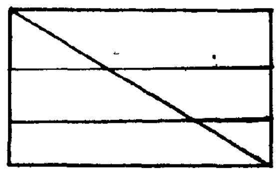
[3] Ibid. p.95

[4] Ibid. p.97

lowing problems were presented in the same order.

(1) Triangles (#4)

How many triangles do you see?



Background

This is a problem with "interpenetrating" elements (Yakimaskaya's term, 1959-61). Yakimanskaya used this problem to study pupils' analyses of geometric drawings. He followed B. Zhuravlev's notion of "mathematical vision as the ability to see in a drawing not only what is striking but also everything that is not in it at all".[1] This problem investigates the subject's skill in discerning and assessing the interpenetrating elements of geometric figures from various points of view, and in isolating figures from a background. Vision and reasoning play a role here.[2]

Our Reason for Choice

One of the reasons this problem is given is "to break

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[1] V. A. Krutetskii, p.112

[2] Ibid. p.112

the ice". This is a non-threatening problem, clear and short, requiring no rigorous mathematical skills or knowledge. It was hoped that this problem would help the subject to relax. This problem also explores geometric visualization and focus.

(2) <u>Circles (#3)</u>

Given 2 circles. The radius of one is 3cm, and the distance between their centres is 10cm. Do the circles intersect?[1]

<u>Background</u>

This is a problem with incomplete information, making it impossible for an exact answer to be given. This problem was given by Krutetskii to see if the subject would notice that data were missing, and to expose certain features of mental perception of a mathematical problem. Krutetskii says that the insufficient facts are noted only when the formal structure of the problem is perceived. The problem can also show how a subject attempts to generate all information possible from the given data when there is a lack of information.[2]

<u>Our Reason for Choice</u>

This problem is given to explore the possibility that

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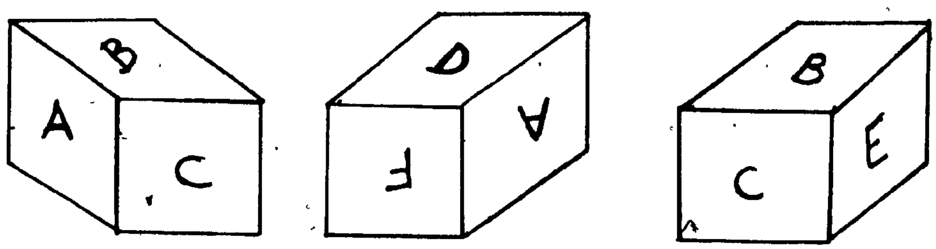
1 V.A. Krutetskii, p.109

2 Ibid. p.107

adults are more likely to perceive ambiguities in problems (research question #7). They might also be more likely to use a diagram here (research question #5). The use of a diagram in this problem will facilitate understanding of it. Who is more likely to notice the lack of information - adults or school children? Who is more likely to trust their observations? (research question #8). Because of their familiarity with the mechanical aspect of school mathematics, where most problems have an answer, school children may tend to assume there is an answer here, ignoring the lack of sufficient information.

(3) Cubes (#1(a))

These are 3 different views of the same cube.



What is the letter on the face opposite A?

opposite B?

opposite C?

Background

Swedish scholar I. Werdelin used this problem to test dynamic spatial concepts. This is considered by Krutetskii as an example of the kind of test used in serious investigations of mathematical ability.[1]

Our Reason for Choice

This problem is given to explore the subject's ability in realizing spatial relationships. This problem requires the subject to work with three views of a cube at the same time. The subject's use of managerial strategies can be explored here (research question #3). The use of a diagram is also helpful here, who will tend to use a diagram - the adults or the school children? (research question #5).

This problem provides a background for a later one (#1(b), given in the second session) that is of a similar type, but which is impossible to solve. When a subject exhibited an understanding of this problem, s/he was given #1(b); otherwise, another problem was given.

(4) Mother-Daughter (#2)

A mother is 3 times as old as her daughter. 3 years from now she will be only twice as old as her daughter. What is the mother's age now?

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[1] V.A. Krutetskii, p.12

Background

This problem is taken from Werdelin's collection and used by Krutetskii. Krutetskii found that even in a direct algebra problem like this one, pupils exhibited differences in approach and thought. One of the subjects (all from Moscow Schools) solved this by using equations, while another drew a diagram using a square for the daughter's age and a circle for the mother's age. Another student solved the problem verbally by articulating it.[1]

The data were changed by us from the original problem, to render this problem technically solvable but unrealistic (the solution is that the mother is 9 years old and the daughter is 3 years old). In his section on unrealistic problems, Krutetskii says that the subject who preferred to look at the problem in a general way, apart from the specific data, might not notice the unrealistic aspect of the solution.[2]

Our Reason for Choice

This problem can be solved by using an equation (an algorithm) or by non-algorithmic ways (like trial and error). We wanted to see if school children, because of school mathematics,

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[1] V.A. Krutetskii, p.15-16

[2] Ibid. p.132

would automatically use equations, while adults who have been out of school for some time would tend to use other, non-algorithmic, ways (research question #6). We also wanted to see if adults would be more likely to recognize problems whose solution is unrealistic (research question #2). The school children might tend to give an answer without regard for the actual meaning of the solution. Krutetskii talks about looking at this problem in a general way, rather than at the specific data. The adults may tend to approach this problem holistically, while the school children approach it linearly (research question #4).

(5) Cryptarithmetic (#8)

In this multiplication, each letter stands for a digit, and different letters represent different digits. Can you find out what digit each letter represents?

$$\begin{array}{r} A\ B \\ \underline{\times\ 4} \\ C\ A \end{array}$$

Background

Cryptarithmetic problems are used by Krutetskii.[1] He gave them to study the "curtailed quality" of the reasoning

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[1] V.A. Krutetskii, p.30

process. He compared the number of links in the actual process of reasoning during the solution of the problem (established before hand) to the number of links in the process of reasoning of the subject while solving the problem.[1]

This type of problem was also used in research by the Gestalt theorists. Bartlet noted that subjects had difficulties with this type of problem because of their past habits of solving arithmetic problems.[2]

Cryptarithmetic problems were also used by the information processing theorists. By carefully observing the general procedures used by subjects, Newell and Simon developed a computer program which solved similar problems in a way apparently similar to human methods.[3]

Our Reason for Choice

This problem explores the relation of numbers and operations. It can be done holistically, by looking at the relationship between A, B, and C, and by putting restrictions on them. The adults may tend to look at it holistically (research question #4). It can also be solved by using trial

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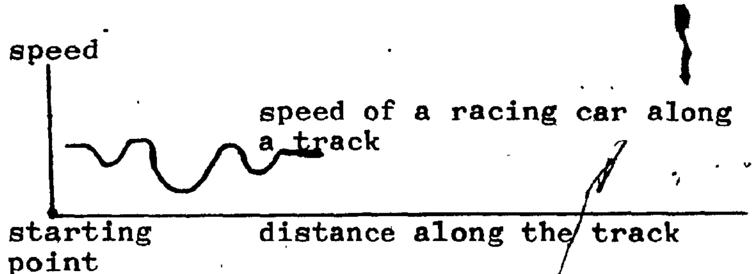
[1] V.A. Krutetskii, p.127

[2] R.E. Mayer, Thinking and Problem Solving, p.76

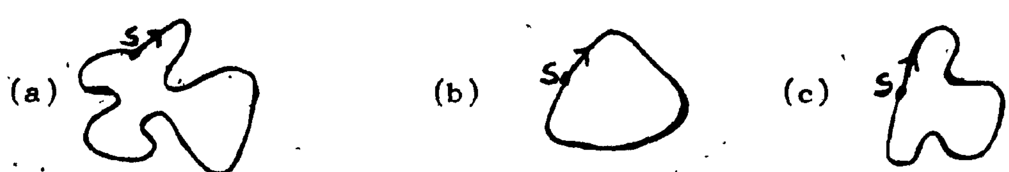
[3] Ibid. p.132

and error, using the errors to improve the trials (a non-algorithmetic strategy). The adults may be more likely to choose a non-algorithmetic strategy (research question #6). This problem requires the making of decisions as to how to approach it and where to start. The adults may be more aware of managerial decisions such as re-evaluating one's thoughts and changing strategies as a result (research question #3).

(6) Racing-Car Graph (#14)



Which of the following corresponds to the shape of the track?



S = starting point

Background

C. Janvier gave different versions of this problem to his subjects to assess formal reasoning and to teach fundamental skills associated with abstract thinking. He says that this task does not require extra scientific knowledge, and does not involve complex symbolism.[1] In his versions, he asked the subjects to either sketch the shape of the track themselves, or even to sketch the graph.[2] He found that individuals used a variety of strategies.[3] He claims that graphical representation is a subtle mixture of meaning and graphical features. This problem has semantic content and the graph has symbolic content.[4] He found that pure visual simplification proved insufficient for many pupils. He says that "minimal use of the situation can sometimes yield reasonably good answers , while an increase of concern for it brings about a noticeable regression".[5]

Our Reason for Choice

This problem explores the ability to interpret and under-

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1 Claude Janvier, The Interpretation of Complex Cartesian Graphs Representing Situations - Studies and Teaching Experiments Shell Centre for Mathematical Education and Universite du Quebec a Montreal (Thesis submitted to University of Nottingham for Doctor of Philosophy, Oct, 1978) p.9.44

2 Ibid. p.9.42

3 Ibid. p.9.43

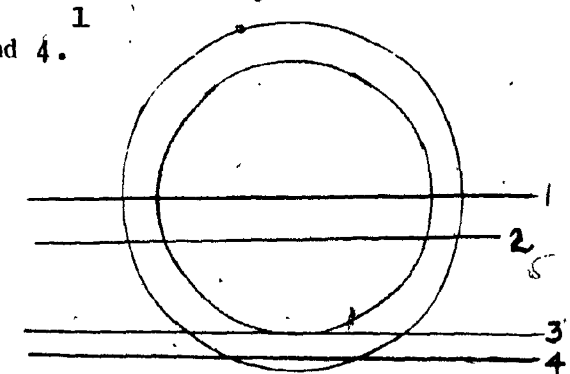
4 Ibid. p.9.42

5 Ibid. p.9.43

stand a graph, using the graph to interpret the situation. One must reverse the usual pattern and go from a graphical situation to a diagram representation. It is a problem where the richer experience of the adults might enable them to better interpret it than the school children.

(7) Torus (#11)

A torus (ring doughnut) is viewed from above. Sketch the cross-sections made by vertical cuts through the lines 1, 2, 3, and 4.[1]



Background

This is a problem related to spatial concepts, requiring 'seeing' of figures mentally. Krutetskii is interested in observing if the subject solves this problem in his/her head, or requires an actual solid to see it more easily showing weakness in spatial concepts. Krutetskii says that rarely does a subject solve this problem by reasoning.[2]

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[1] V. A. Krutetskii, p.171

[2] Ibid. p.169

Our Reason for Choice

This problem explores the use of visualization in relation to spatial concepts. The richer experience of adults may help them in approaching this problem.

Note

This problem was kept as an extra problem to be used when there was extra time available.

The second session includes six problems with an extra seventh problem.

(1) Square Cutting (#6)

Can you cut a square into 7 square pieces?

Background

This problem was given in the I.R.E.M. de Montpellier project in 1975 to analyse elementary sequences of behaviour and heuristics. The work of four subjects was filmed and audio taped and sequences of ideas, sentences and actions were analysed and classified.[1]

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[1] Montpellier project, Heuristique: une technique d'experimentaion et d'analyse en heuristique I.R.E.M. de Montpellier, France

Our Reason for Choice

This problem requires an effective use of managerial decisions (research question #3). It also involves using trial and error systematically. It requires an ability to break out of a particular way of thinking. An example of a mental block here is drawing equal subdivisions or squares over and over again, taking for granted that this is the only possibility. Recentering plays a role here. One must move one's attention from a big square to a small one. The adults may use diagrams more effectively than the children. (research question #5). Although it is not one of our research questions, it will be interesting to see if the adults attempt to generalize and search for other solutions.

(2) Pond (#5)

A large pond is becoming overgrown with vegetation. Every day the overgrown area doubles. On the 8th day it has covered half the pond. On what day will it cover the pond completely?[1]

Background

This is a problem that involves comprehension and logical reasoning. It requires no special knowledge, but a skill in

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[1] V.A. Krutetskii, p.150

logical reasoning is necessary, together with a certain resourcefulness. Krutetskii says that this kind of problem is a good indicator of the presence of mathematical abilities, precisely because it requires only elementary mathematical knowledge and skills. V. I. Zyablovskii says that problems where only quick wittedness is required present the greatest difficulties for students and that such problems appear rarely in mathematics lessons. Intuition also plays a role here. Krutetskii is also interested in the curtailment of reasoning and the number and kinds of links in the reasoning process present while solving this problem.[1]

Our Reason for Choice

This problem is given to explore the possibility that adults may tend to approach a problem holistically (research question #4), since a linear approach (looking at the growth day by day) would not help in solving this problem. A diagram can also be used here (research question #5). Visualizing the pond or drawing a diagram may help in understanding this problem, while using numerical trials can hinder the solving process.

The adults may tend to make managerial decisions (research question #3). The school children may start writing equations and using algorithms, while the adults may use their intuition

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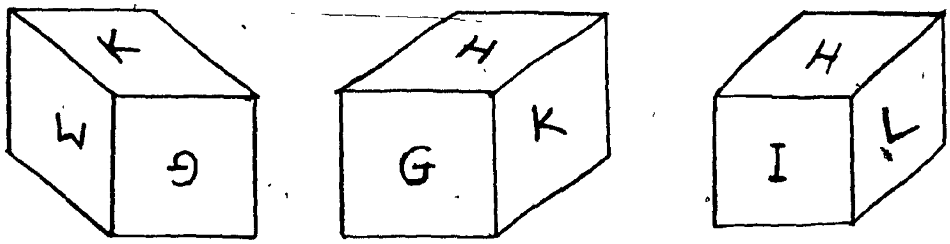
[1] V.A.Krutetskii, p.148

here and then distrust it (research question #8).

(3) Cubes (#1(b))

These are 3 different views of the same cube.



What is the letter on the face opposite G?

opposite H?

opposite I?

Our Reason for Choice

This problem was given to the subjects who exhibited an understanding of problem #1(a) (some subjects who did not succeed in #1(a) were still given this problem). This problem was devised by us to be contradictory and unrealistic. It is an impossible situation. It is given to test the possibility (research question #2) that adults are more likely to recognize that the data render this problem unrealistic. The adults might be frustrated by the contradiction (research question #7). The adults may use their intuition, and they might distrust it (research question #8).

(4) Counters (#9)

Three counters A, B and C are coloured red, white or blue (but not necessarily in this order). Of the following statements, only one is true:

A is red.

B is not blue.

C is not red.

What colour is each counter?

Background

This problem was given as a part of work-session material for the 1978 N.C.T.M. Annual Meeting, on solving nonroutine problems. Three different strategies were explored. One strategy is listing all the possibilities and checking to determine which ones satisfy the conditions of the problem. A second strategy is writing "logical sentences" using symbols like A_r (A is red) to analyse the given data. Every given statement is written as a symbol. The third strategy, is that of using hypothetical deductions. The subject is hypothesizing the colour of one particular counter in order to deduce assumptions. Here, one is seeking contradictions in order to eliminate possibilities.[1]

[1] E. McClintock (with M.G. Kantowski), Solving Non-Routine Problems, (NCTM 56th Annual Meeting, San Diego, California, April 1978)

Our Reason for Choice

This problem explores logical reasoning and the ability to follow logical steps systematically. It also explores the subject's attention span and his/her use of managerial decisions.

(5) Parking Lot (#7)

There are 28 vehicles in a parking lot, cars and motorcycles. All together they have 82 wheels. How many of each kind of vehicle are in the lot?

Background

A version of this problem, with extra information (number of steering wheels), is given by Krutetskii as a problem with surplus information.[1]

Our Reason for Choice

We have changed the number of vehicles and wheels (from 40 vehicles and 100 wheels) to make the solution less obvious. This is a problem where algorithms (equations, etc.) can be used (research question #6). In order to use trial and error

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[1] V.A. Kritetskii, p.110

systematically, the subject must have a knowledge of how the data interact. S/he must be able to coordinate two different bits of information and to realize how they influence each other. This is also a good problem to exhibit the use of managerial decisions since it is rich with possibilities of ways to approach it, and the subject may make decisions such as to change strategies and what information to keep from the old strategy.

(6) S-P (#12)

Write an equation using the variables S and P to represent the following statement:
"There are six times as many students as professors at this university."
Use S for the number of students and P for the number of professors.

Background

This problem was given to one hundred and fifty first-year students at the University of Massachusetts. The researchers (J. Clement, J. Lockhead and E. Soloway) wanted to test the ability to perform complicated translations from English statements into algebraic statements. 63% of the students had the correct answer.[1]

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1 J. Lockhead, J. Clement and E. Soloway, Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems, Cognitive Development Project, University of Massachusetts, Department of Physics and Astronomy, (Amherst, Mass., February 1979).

Our Reason for Choice

This problem requires the use of symbols. Word relationships must be converted into algebraic ones. There is a relationship here between semantic content and algebra manipulations. The adults might use common sense and intuition here, because of their distance from school algebra (research question #6). The school children may automatically write an equation, and might have difficulties with the meaning of this statement.

5. The Questionnaire

At the end of the second session, the subject was presented with a questionnaire that s/he then completed. In the third session, the subject was interviewed on the basis of this completed questionnaire as well as the protocols of the previous two sessions.

This questionnaire is designed to explore the affective components of problem solving. We want to explore how the subject felt while solving the problems. How confident, relaxed, interested or threatened was s/he? How did s/he look at the problems? What expectations did s/he have? What are his/her general feelings about mathematics? We also wanted to know if the subject had seen any of the problems before, since this factor of familiarity can influence the problem-solving process.

Many of the ideas of what to look for and what to ask the subjects were taken from a math anxiety program designed in the

University of Minnesota. The University of Minnesota's math anxiety program consists of a series of of diagnostic tests designed to evaluate the participants' level of math anxiety and ability. The participant is then offered a course designed for his/her level complemented with a support group for discussions of attitudes and fears towards mathematics as well as the help of outside tutors.[1] The diagnostic tests given to the participants are well developed and explore a wide variety of feelings and attitudes towards mathematics. It is from the participants' comments about mathematics and their attitudes towards it that most of the questions in this questionnaire were taken. The questionnaire is reproduced here in the form given to the subjects:

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[1] Darwin D. Hendel, The Math Anxiety Program: its Genesis and Evaluation in Continuing Education for Women, Minnesota University, Minneapolis, Measurement Services Center, May 19, 1977

Questionnaire

(1) Did you ever see any of the problems before?

Which ones?

(2) What were your expectations when starting?

(3) Did you look at the problems as:

(check as many as you wish)

an obligation ________________

an opportunity to learn ________

an unpleasant taks ____________

an opportunity to be creative ________

a mystery __________

a test of intelligence __________

(4) How relaxed were you?

1 2 3 4 5

Very relaxed Very tense

(5) How interested were you?

1 2 3 4 5

Very interested Very bored

(6) How competent did you feel?

1 2 3 4 5

Very competent — Not at all competent

(7) How confident were you?

1 2 3 4 5

Very confident — frustrated

(8) When thinking about the problems, did you usually:

Talk to yourself about the problem ____________

Visualize the problem ____________

Use numbers ____________

(9) These problems were generally:

(check as many as you wish)

interesting ____________

fun ____________

confusing and difficult ____________

a challenge ____________

intimidating ____________

(10) I find math

1 2 3 4 5

Easy Common-sense — Hard Complicated

6. Protocol Analysis

(a) Some Background on Protocol Analysis

The flexible non-standardized and informal nature of the clinical interview renders attempts to analyse the protocols especially difficult. J. Confrey says that the analysis of clinical interview data is the greatest deterrent to the use of clinical interviews. Analysis is always time consuming, costly and usually uncovers missed opportunities and general mistakes of the interviewer. One must then communicate the results to the public in a clear and organized manner.[1]

At the same time, the clinical interview offers an environment for rich, variable and non-obvious responses and, when analysed effectively, can provide invaluable information about problem-solving processes. J. Confrey lists and comments on six ways of analysing protocols:

(1) Develop pre-determined coding categories and use them.

(2) Develop coding schemes from data and have independent raters code the materials.

(3) Construct semantic networks.

(4) Develop coding schemes from half the sample, and try them out on the other half.

(5) Describe themes.

(6) Write case studies.[2]

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[1] J. Confrey, Using the Clinical Interview to Explore Students' Mathematical Understandings P.34

[2] Ibid. p.35

Coding schemes have been used by several researchers in analysing protocols. The codes vary according to individual philosophies and priorities of the researchers.

In her own research, Confrey is currently using codes to classify each ability. She then lists each problem, and under it writes the detailed responses of each subject, using the codes. This is an example of a study of each student across the items, according to pre-determined abilities. By analysing most of the protocols, categories were made clear and used to code the rest of the data.[1]

Alan H. Schoenfeld objects to a detailed analysis. He claims that analysis of tactical responses (like algorithms and most heuristics) are irrelevant if the wrong strategic decisions are made in the problem solving attempts.[2] He also objects to analyses like those of Lucas et al. who have given each event in the protocol a symbol, so that a string of symbols corresponds to a chronological order of events. Only the explicit behaviour was coded.[3] Schoenfeld says that one is 'missing the forest for the trees' when one concentrates on detailed processes and overt behaviour. He says that the artificial-intelligence coding themes have made great progress in analysing at the tactical level only.[4]

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[1] J. Confrey, p.37

[2] A. H. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving p.5

[3] Ibid. p.6

[4] Ibid. p.5

His emphasis is on analysing the use of managerial decisions in problem solving. He warns that when a brief part of the interview suggests a key structure, it is only an intelligent guess, and one must show that other parts of the interview can be accounted by that same structure.[1]

Easley says that analysis of protocols (on the structuralist paradigm) is necessarily a slow and nonmechanical procedure. It begins with subjective but - one hopes- educated judgements, and moves towards objectivity as it attains completeness in accounting for the total protocol.[2] This statement can be applied to the other types of protocol analysis.

Confrey says that a publically agreed upon standard for analysis is needed if researchers want to articulate their results to the public. She says that it is not enough to have one's own standards.[3]

It is clear that there is a great diversity in the ways of conducting protocol analysis. One must combine objectivity and subjectivity to make a rational attempt at integrating the data.

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[1] J.A. Easly Jr., <u>The Structural Paradigm in Protocol Analysis</u>, Journal of Research in Science Teaching, Vol 11, no.3 p.283

[2] J. A. Easley, p.289

[3] J. Confrey, P.39

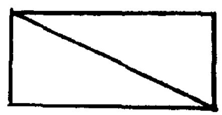
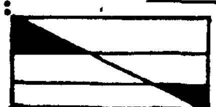
(b) Our Procedures in Analysing Protocols

We have studied each problem across the subjects, comparing the adults' responses to the school children. Under each problem we have listed the research questions we have made pertaining to that problem, and proceeded to analyse the responses of the adults and the school children according to these research questions. When looking at one group (adults or school children) we followed each subject individually but, finally, have made statements about their behaviours as a group to facilitate the comparison between them.

CHAPTER IV

ANALYSIS OF INTERVIEWS

(1) Triangles Problem (#4)

This problem was given as a warming up for the subjects. Therefore the protocols of this problem were not analysed. Most of the subjects found four triangles in the picture, the two large ones:  , and the two smaller ones: 

(2) Circles Problem (#3)

(a) Noticing Incomplete Information

This problem involves incomplete information. Some of the data are missing (the radius of the second circle), rendering it impossible to solve.

Adults

None of the adults noticed the lack of information. They all assumed that both circles have equal radii.

Linda claims that the circles do not intersect because of the distance between them.

L: "Well, if their centres are 10 cm between, then - then the circles don't intersect."

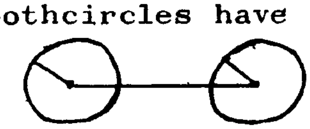
Later, in the third session, Linda concedes that they could intersect, but that she does not think so.

L: "They could intersect. They could have, but I didn't think that way."

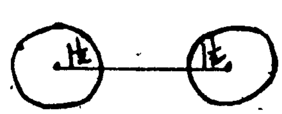
R: "You imagined them not intersecting?"

L: "That's right".

Linda seems to think that there is a possibility that the circles do intersect, but she has chosen the other possibility. It is unclear why.

Lucy assumes that bothcircles have a radius of 3cm each. This is her diagram: 

L: "Well, from what I understood you have two circles - O.K.? (draws) - and if the radius is this here, O.K. let's say this is 3 cm, if their centres is 10 cm, the distance between their centres, so let's say this is 10 cm. How could they intersect if they are apart like this. For me, they don't intersect".

Uri assumes that there are equal radii, but he changes the problem. This is his diagram: 

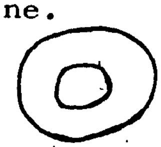
Uri's answer to "do the circles intersect?" is "certainly not!" This is his explanation.

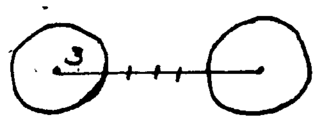
> U: "Those two circles - two circles- each one 3 cm long. That means on each side there is $1\frac{1}{2}$ left. Between the two circles there is 10 cm. That means, from this centre to one circle is $1\frac{1}{2}$, up to the other circle is $11\ \frac{1}{2}$, up to the next centre is 13 cm".

Uri assumes that the diameter is 3 cm in each circle.

School Children

Two school children, David and Robert, notice the missing data. But it is interesting that Robert assumes that there is an answer and therefore gives one.

Initially, Robert Sketches  . He goes on to

draw: 

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where he divides the distance between the circles into ten units. In this drawing, the circles seem equal in size. But Robert says that he doesn't know the second radius, and that it could be bigger.

Ro: "I don't know. It just didn't look right (the sketch). Because it's probably not the same. It's most likely, this is going to be bigger."

He gives as the answer, "Yes, they intersect."

R: "Why did you say yes?"

Ro: "Because, in here they don't give the radius of the second circle, which means it could be anything, and I presume that it will be a bigger circle than 3 cm. And, I didn't think it will be no because I was just trying to figure it out."

Robert assumes that there is a greater probability for the second circle to be bigger, but he acknowledges that there is a possibility for it to be smaller.

Ro: "Because a 3-cm circle is pretty small, so I was just guessing. It's just a guess."

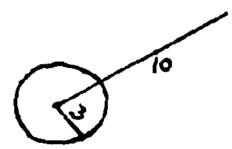
And later:

Ro: "The chance will be that it will be bigger."

Robert feels a need to have an answer, so he rationalizes to get one.

David, on the other hand, notices immediately the missing data, and the nonexistence of a solution.

He draws:



D: "It matters what the other radius is, and there is not enough information."

And later:

D: "But they don't say what the radius of the other circle is, so you can't say if they intersect."

Judy has difficulties in understanding the problem. She answers "no", but when asked to explain her answer, Judy says that she can not.

(b) <u>Diagram Usage</u> (research question #5)

Two adults, Lucy and Uri, have drawn diagrams. Two school children, Robert and David, also used diagrams.

(c) <u>Visualization</u>

One adult, Linda, made a reference to visualization of the situation. She did not use a diagram.

L: "I am picturing them in my mind."

One school child, Judy, asked:

J: "Where are the circles spaced?"

This shows an attempt to visualize. Judy also did not use a diagram.

(3) Cubes Problem (#1(a))

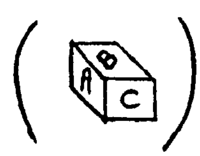
(a) Non Standard Interpretation (research question #1)

Two of the adults, Linda and Uri, interpreted this problem in nonstandard ways. They have both been unsuccessful in solving this problem.

Linda interprets the word "face" as facing her, and the word "opposite" as next to.

L: "...and opposite A on the face is C."

Here Linda means that in the first view, A is next to C and C is a face because it is facing her.



L: "Then you say opposite B. And I don't see a face here."

Linda indicates here that in all three views B is not a face (it is not facing her).

L: "So opposite B would be, well, C or E - And opposite C which is B or E or E or A."

Linda is consistent with her interpretation.

Uri's comments in this session are unclear. During the taped conversation the interviewer, in order to clarify Uri's process of thinking, asks Uri to label the sides of a drawn cube . Uri proceeds to give two different drawings. It is only in the second session, when given problem #1(b), that his interpretation of this problem is finally clear.

Uri claims that the faces of the cubes are labelled according to the order of the alphabet, either in a clockwise or counter-clockwise order. He is working with each view separately. (See #1(b)).

In the third session Uri says that if he had real cubes to work with, it would be easier. He exhibits a need for concrete materials.

The following excerpts show how Uri is following the alphabet in his attempts.

U: "Yes. A, B, C, D, F. Let me see. A, A, B, C, wait, yes. A, B, C, A, B, C, B, C, D, E, F, G, A."

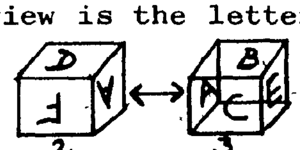
U: "...It depends how you, how you turn the cube."

Uri is saying that the order of the letters depend on the direction one goes (clockwise or counterclockwise). He is trying to find out what is the letter opposite B.

<u>School Children</u>

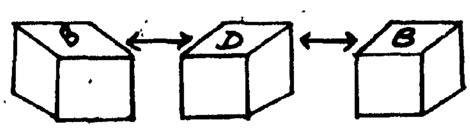
One of the school children, Judy, interprets this problem in a nonstandard way. She has not succeeded in solving this problem, although she shows some understanding of the situation.

Judy's interpretation of the word "opposite" is one side of one cube opposite another side of another cube. Other than this interpretation, Judy does exhibit an awareness of the relationships of the letters and the views. She integrates all three views in one cube. Judy notices that on the left side of the third view is the letter A.

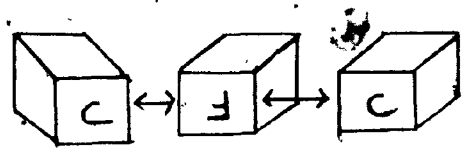


But her answer is that this letter opposite A is <u>A</u>, since A from the second view is opposite A on the third view. She then claims that it can not be done for B or C. She goes on to develop a different interpretation for these. Her answers are:

B is opposite D.



and C is opposite F.



Here she interprets "opposite" as next to. This shows an in-inconsistency in her interpretation. The following is a quotation from the third session.

> J: "A is here and A is here (first and second views). Then A will be there (points to unseen side in third view). Then B I put opposite D, either way. And then C could be opposite F either way."

(b) <u>Holistic vs. Linear Approach</u> (<u>research question #4</u>)

An holistic approach to this problem is indicated by a subject working with all three views at the same time, integrating them into one cube.

On the other hand, a linear approach is indicated by a subject working with each view individually or with a pair of views at a time, without realizing the relationship of all three views to each other.

<u>Adults</u>

Two of the adults, Linda and Uri, exhibit a linear approach. The third adult, Lucy, shows an holistic approach and succeeds in solving this problem.

Linda works in a linear manner. She does not realize that these are views of the same cube. She takes the given diagrams literally, as a group of letters arranged in a certain way. With her nonstandard interpretation of "face" and "opposite", she proceeds to give her solution by looking at the relationships of the letters as she sees them on the problem card. She makes no effort in integrating them.

Uri is also linear in his approach. He is working with each view individually. In the second session, while solving #1(b) it becomes clear that he is associating each question with a corresponding view (first question "What is the letter on the face opposite A?" with the first view). (see problem #1 (b)). After being asked by the interviewer to label the sides of a cube, Uri gives two different cubes as the answers. Was he going to give a third answer, corresponding to the third view?

Lucy looks at the problem in an holistic manner. Although her final answers are wrong, it seems to be so because of disorganization on her part. She does give the right answers previous to her final ones.

Initially, Lucy thinks that there are three different cubes. Here is her explanation from the taped conversation in the first session.

> L: "Well, at first, I did not make, I did not realize
> it was the same cube, but afterwards I did realize

it was the same cube, O.K., but its three different views."

She then proceeds to give her solution (the right one).

L: "So, A it would be E, B would be D, because the D would go here - and the opposite face of C is F."

She then repeats her answers, as the final ones, and here she confuses the letters.

L: "So what is the letter on the face opposite A. So would be A, would be E, opposite B would be F, and opposite C would be D."

School Children

All three children exhibit an holistic approach to this problem and one of them, David, succeeds in solving it.

David starts by drawing a cube and labels the top C, he then changes his strategy and looks at the problem again. He says that he has seen this type of problem before, which may account for his ease with it. He then proceeds to give the correct solution and goes on to explain why this is the solution.

In the following excerpt he refers to the first and third views.

D: "...and C is still on the front in these two. So if you rotate it around, if you rotated it, I was thinking B is still on top so it will be E here, because C is still facing the front."

Robert says he thinks that each question goes with its corresponding view, but he later explains that he is working with the third view, using the first and second views as references. He seems to be integrating all three views into one cube.

Ro: "I was working always with the same triangle."
R: "Ah, you were always working with the third cube."
Ro: "Right. I was just using these (other two views) as references, the way it was set up."

Even though Robert is integrating all three views at once, he does not succeed in solving this problem. It remains unclear as to how he arrives at his solution

A–C
B–F
C–A

Judy, in the beginning, claims that A is on the left of C in the third view, and therefore A is opposite A according to her nonstandard interpretation of "opposite". This shows that she is integrating all views at once, at least initially. She goes on to change her interpretation of "opposite".

(c) Diagram Usage

The three adults do not draw diagrams. Similarly, all the school children do not draw diagrams, although one of them, David, starts by drawing a cube and does not pursue it.

(4) Mother-Daughter Problem (#2)

(a) Perception of the problem as an "algebra" or "equation-type" problem.

This problem is a standard one, typical in mathematics textbooks. It is generally solved by using equations.

Adults

Two of the adults, Linda and Lucy, exhibit a recognition that this problem involves algebraic manipulations or equations. None of the adults succeeded in solving this problem.

Linda, after reading the problem, says that she hopes none of the other problems will be like this one. She says that she does not know algebra and that this is algebra. The following are some excerpts from the third session:

R: "What did you mean when you said algebra, especially in the mother-daughter one, you said that was an algebra question. What did you mean by algebra?

L: "Because there is an unknown factor there, and, uh, I felt it would have been easier to figure out if I'd known algebra."

and later:

R: "In what other way could you have done it?"

L: "Uh, probably by elimination, working backwards perhaps, I don't know."

Here Linda says that she could have done it by using trial and error, trying a possible solution and eliminating it if it does not work out. By 'backwards' Linda means starting with the possible solution and checking if it works by using the given data.

Lucy's initial trial is giving the mother's age as 60. She abandons this, and writes 'x x 3'. This exhibits a realization thatthis is an 'equation-type' problem. It is obvious that she has forgotten how to use equations, but she wants to try anyway.

The third adult, Uri, does not mention the word 'equation' or its concept at all. This is an excerpt from the third session:

R: "O.K. Now; here, in the question with mother and daughter - do you remember? You were trying numbers like 60 and 20. Now, can you think of another way to solve this problem, other than trying numbers?"

U: "Other than numbers? No. If not numbers then age."

Here, Uri still means trying numbers representing the age.

School Children

All three school children exhibit a realization that this is an "equation-type" problem. They all attempt to solve this problem by using equations. None of them succeeded in solving this problem.

Judy begins by asking if she can write it down as an equation. She tries a few equations and then gives up. In the third session, Judy explains the advantages of using equations.

R: "Why did you use an equation?"

J: "Uh, well, because equations are easy to solve, I guess. I would have done it with an equation. I wouldn't have tried to look for another way to do it, because."

J: "When I saw that. When they said a mother is 3 times as old as her daughter, and they didn't give the daughter's age, so I figured that you could write an equation."

David begins by using an equation. When it does not work, he goes on to try different numbers for the ages. After trying a few he stops his efforts.

Robert, after reading the problem, says that he hates this type of problem and asks to pass over it. He claims that he is "bad at this". After being asked to try anyway, he writes down an equation. When the equation does not work out, he asks to leave this problem.

In the third session, Robert claims that he expected all the problems to be "this type". He says that this was the only one of this type.

Ro: "I don't like these. I hate equations."

R: "Why?"

Ro: "Uh, I don't know, I just don't. They are very - You have to figure out her age, the mother's age and then times it by 3, and let X equal something, and it just mix one."

(b) Recognizing Unrealism of Problem (research question #2)

In this problem there are realistic restrictions on the ages of the mother and the daughter. The solution is an unrealistic one in which the mother is nine years old and the daughter is three years old.

Adults

One adult, Uri, realizes that there are some realistic restrictions on the mother's age. After reading the problem, Uri says that the mother can not be younger than 15. In the conversation of the first session he explains:

> U: "...The mother can not be normally, normally a mother is not younger than 15."

He goes on to try numbers larger than 15. Uri also talks about the problem in a real-vs.-theoretical sense. He says that this problem is not real, it is theoretical. In the conversation of the first session, he gives an example in which the mother is 6 and the daughter is 3, and he specifies that this is only in theory.

> U: "As an example, an example, as an example the mother ... later would be 6, and, uh, the daughter would be only 3 years old - in theory."
>
> R: "Why did you pick these numbers now?"
>
> U: "Because this would be reality. Not reality. This is reality (points to example with mother 60 and

daughter 20), but this would be in theory.

School Children

One school child, Judy, when given the solution, claims that she did get the answer before, but because it was unrealistic, she did not pursue it. In one of her equations, Judy did get as an answer X=3. At the time she just continued with another equation. Both of the equations were not accurate representations of the situation.

J: "Nine. That's what I got before, but I thought it is kind of impossible for a nine-years-old mother."

(c) Nonstandard Reaction

One of the adults, Lucy, claims that if the mother is 3 times as old as her daughter, three years from now she will still be 3 times as old. Lucy confuses the differences between the ages with the ratio between them. She assumes that there is a constant ratio between the ages of the mother and the daughter. When trying some numbers, she realizes that her interpretation is inaccurate but she does not change her mind.

L: "I am 21 years difference from my mother. It always stays 21 years, never changes and why would it change for her? She is ten; I am thirty. (L. takes as an example that she is thirty and her daughter is 10). So I'm 3 times as old. 3 years later she is 13, and I am 33. It's not, 13. 3 years older is 16. Oh, it makes no sense to me. I mean, how could a mother be 3 times as old, and then 3 years later be twice as old. It's impossible.

Lucy writes:

10	11	12	13
x3			
30	31	32	33

L: "Now 13x3 gives me 39, doesn't give me 33."

Here she realizes that her claim is mistaken. But her final conclusion is:

L: "The age difference will always be the same. So there is no way I could be twice as old as her, if the age difference is 40, or 30, or 20. I stay that way."

(d) <u>Holistic Approach</u>

An holistic approach to this problem is the coordinating of the two given conditions and working with both at the same time. A subject may realize that in three years the mother's age will increase by a third of it. This realization is a result of working with the two conditions at once.

None of the subjects exhibits an holistic approach. They all, in one way or another, work with one condition at a time switching back and forth between the two conditions. The subjects that use trial and error base their trial on one condition and test if the second condition is satisified. If not, they give another trial.

Uri is an example of a subject disregarding one of the two conditions. He chooses 60 and 20 as the ages (satisfying "3 times as old"). He adds that the daughter will be $31\frac{1}{2}$ years old in three years. He does not realize that in three years she will be twenty-three.

> U: "Three years from now - oh - the mother would be sixty-three, and -uh- and the daughter, she could be at that time, $31\frac{1}{2}$. More or less ($63\div2=31\frac{1}{2}$)."

Uri later claims that this is not the solution, but just an example. Although this example satisifies one condition, it does not satisfy the other. Uri gives one more "theoretical" example, in which the mother is six and the daughter is three. These are the only two possibilities that he tries.

The subjects that have used equations failed to represent these two conditions in an equation.

(e) <u>Algorithm Usage</u> (<u>research question #6</u>)

An algorithm can be defined as a special known-to-succeed method of solving a problem. Examples of algorithms are: the use of equations or a complete systematic search trying all possibilities (starting with daughter is one year old, two years old, three years old, etc.).

The use of an equation is an effective strategy in solving this problem.

Adults

One adult, Lucy, makes an attempt at an equation. It is a weak attempt, in that she writes 'X x 3' and does not follow up on it. She abandons it immediately.

Linda, athough she recognizes this as an algebra problem, does not make any attempt at equations or at using trial and error.

School Children

All the school children attempt to use equations. Judy starts by writing: 3X+3=2X. She realizes the impossibility of a negative solution (X=-3) and changes the equation to 3X-3= 2X(x=3). She does not try any other equations, but stops here. Judy does not translate the problem accurately into an equation. She has difficulties in interpreting 'from now'.

J:
"Because the mother is 3 times as old as her daughter. So you don't give the daughter's age, so you take it ...

> as, I took it as X. So I put 3X. Then I said 3 years from now, so, uh, minus- first of all I put plus 3, from now is O.K., and she, from now she will be only twice as old, so I put 2X. And then to do the equation. (works out equation and gets X=-3, laughs.) So she is -3."

David starts by writing:

X daughter
3X mother
3X+3=2X

At this point, David abandons the use of equations (because of the solution X=-3) and goes on to try different numbers for the ages. He writes them in an equation - like form, labelling years as y.

(i) daughter 1y
mother 3y
3y+3y=3y
6y=3y

David explains:

> D: "I wrote the mother is 3y and the daughter is y. I was like trying to make an equations where you have like, uh, let's say a small thing like this (he writes: X+2=X+3) and then I would subtract the two here and bring the X over there, and then end up with a number."

David does not seem to notice the unrealistic ages. His next trial is:

(ii) daughter is 3y. He does not pursue this.

(iii) daughter 6y
now mother 18y
$18y+3y \neq 6y+3y$

> D: "...Say the daughter was 6 and the mother is 3 times that age, the mother is 18 and the daughter is 6. And then 3 years from now - I added 3 years to 18 to make 21 and 3 years to 6 and that made it 9. But so, that didn't work out because 9 isn't half of that."

At this point, David ignores the two conditions of the problem and gives an answer based on (iii).

> D: "...And it just asks what is the mother's age now, so I say the mother's age now is 24 and the daughter's age is 12."

He adds three more years to give the ages "now". He seems to make a distinction between the "now" in the final question of the problem and the "now" in the earlier (second) sentence of the problem. David later realizes his mistaken distinction between those two.

> D: "The 'now' they are talking about here is the same

> as this part (second sentence of problem)."

He does not continue in his efforts.

Robert, after reading the problem, asks to skip this. He then agrees to spend a few minutes on it.

He writes: Let X be the mot
$3X+3=X$
$2X=$

He stops here and asks to leave this problem.

(5) Cryptarithmetic Problem (#8)

(a) Perception of the Problem as an "Algebra" Problem

This problem consists of numbers and operations, and one must be familiar with them to succeed.

One of the adults, Linda, labels this as an "algebra" problem and consequently claims that she does not know how to do it.

> L: "Well, again this is algebra, I think, and I don't know how. I understand the question, but I don't know how to do it.

Another adult, Lucy, in the midst of her trials says that she remembers from school that they used to write "4xb=A, 4xA=C." It is interesting that she makes no attempt to pursue it.

(b) Holistic Approach

An holistic approach to this problem is exhibited when the subject looks at the relationship between A, B and C, and puts restrictions on them, thereby narrowing down the possibilities.

Adults

Two adults, Lucy and Uri, succeed in solving this problem. They both approach the problem holistically.

Lucy shows a thorough understanding of this problem from the beginning. She realizes immediately that A has to be the same number above and below, and that A and B have to be different. She starts by giving A the value of 3, and then trying different values for B to get the A below to equal to 3. Her trials are:

$$\begin{array}{r}35\\ \underline{\times 4}\\ 140\end{array}\qquad \begin{array}{r}35\\ \underline{\times 4}\\ 3\end{array}\qquad \begin{array}{r}A5\\ \underline{\times 4}\\ \end{array}\qquad \begin{array}{r}A6\\ \underline{\times 4}\\ \end{array}\qquad \left(\times\begin{array}{r}AB\\ \underline{4}\\ CA\end{array}\right)$$

It is after this that Lucy realizes there are some restrictions on A because of C. She starts to approach the problem holistically.

> L: "I was stuck, because C had to be one digit, so if I used any other number higher than 3, I would have a two-digit number for C. So it had to be a small number, and below 10."

She proceeds to solve the problem.

Uri also approaches this problem holistically. He works with the values of 1 and 2 for A. He gives A the value of 3 but rejects it immediately. In the following excerpt he explains.

> U: "...then I found out it couldn't be because if A could be 1, then you could not have 1 as a second letter (the second A), being 4x1, couldn't be 1, 4x1. So it must be another letter - not 1. It could be in the best case 2."

After making this restriction, Uri succeeds in solving this problem. He solves this problem during the taped conversation in the first session. Earlier, during the observation period, he asks to leave this problem after trying a few possibilities with A=1, A=2, and A=3.

School Children

One school child, Robert, succeeds in solving this problem, although he does not, outwardly, exhibit an holistic approach. Another school child, Judy, does exhibit an holistic approach but does not solve the problem.

Robert is at ease with numbers. Initially he tries different possibilities without articulating or writing them down. After thinking for a while, he writes: $\begin{array}{r} 41 \\ \underline{x4} \\ 164 \end{array}$, and claims that "it's too much". He realizes here that C must be a one digit number. His next trial is the solution. Because he does not articulate his thoughts, it is unclear if he approaches the problem holistically or not.

Judy, although she does not succeed in finding the solution, does exhibit an holistic approach to this problem. After reading the problem, she claims that A has to be very small, either 1 or 2, and so does B. Here, she is probably forgetting the possibility of a 'carry'. By restricting B in this way, Judy fails to progress.

> J: "If A is here and A is here - uh - I figured A will have to be a low number, because it is only one digit. It is only two digits here (CA)."

And later:

> J: "But I wasn't sure about B because B has to equal 1 or 2 - so it's - is it impossible?

These excerpts are taken from the taped conversation in the first session. Judy does not continue. At the end of the session, Judy is given the solution by the interviewer. A few days later, during the third session, she is given a different problem which she solves easily.

$$\begin{array}{r} AB \\ \underline{x7} \\ CA \end{array}$$

David assumes immediately that since A is the first letter in the alphabet, it represents 1 and similarly B represents 2. After realizing thatthese values do not work, David claims that there is not enough information given about A, B, and C.

> D: "So that didn't work. They don't give you enough information about the numbers to be able to work it out."

David says that he does not understand how it is possible for AB to equal CA, since "4(ab)=4a4b", and this can not equal
x4
to CA.

> D: "I couldn't figure out how, because if I multiplied that, I find the answer 4B and 4A so, I don't know how they got C in."

<u>Note</u>: Three of the subjects, two adults and one school child, liked this problem and found it interesting. Those were the subjects who succeeded in this problem.

Lucy:

> R: "And why did you like #8 let's say?"
> L: "Because it took me a while to understand it, but it was fun looking for the numbers. That's why I liked it."

Uri

> U: "I know its interesting. I've never encountered this

> kind of problems, but one has to find, once you -."

Robert:

> Ro; "Like this one I found very, very good."

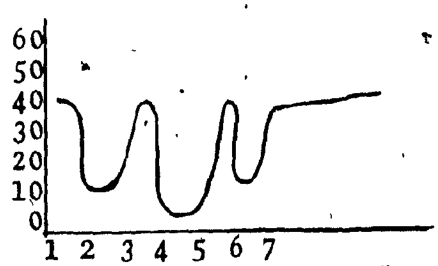
Linda and David both disliked this problem. Judy had no special feelings towards it.

(6) Racing-Car Problem (#14)

This problem requires the ability to interpret a graphical representation of a situation and to convert it into a shape of a track.

Adults

Each adult exhibits a different reaction to this problem. Luch interprets this graph accurately, but she can not make a connection between the graph and the track. She does not give an answer. Lucy's comments exhibit her understanding of the meaning of the graph. She says that the graph does not represent the shape, "it represents how fast the car drives through the distance." After the end of the taped conversation in the first session, Lucy spontaneously starts to discuss this problem. She draws this graph:



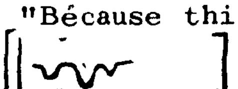
She explains that the car goes 40 mph and then slows down after two hours and again speeds up, but Lucy does not see any connection between the graph and the shape of the track. In the third session she explains.

> R: "Do you remember what was your answer here?"
>
> L: "Yes. It was, this (the graph) could not represent the, the shape of the track, because it was telling how fast the guy was going for the distance he had to go."

Linda does not understand this problem. At one point she says that "figures" are necessary for her to be able to solve it.

> L: "I don't understand the question and I don't know what these three shapes have to do with the question."

Uri sees the graph as the track itself and because of the waves in the graph he says that the track must be very long and choice (a) is the longest. So his answer is (a).

> U: "Because this here, the distance here is straight
>
> 
>
> distance
>
> . And the track is differently shaped, so it has to be the longer, the longest one."

In the third session, Uri compares the track to a wire:

> U: "Just as you take a piece of wire or something, and you bend it, and you have a straight one. The bent one, if you straighten it out will be longer than the straight one, according to the sketch."

School Children

Two school children, Judy and David, are able to interpret the graph accurately and to connect it to the shape of the track.

Judy's first reaction, after reading the problem, is to ask if the graph represents a slowing down and then speeding up. She immediately gives as the answer, choice (b). She explains.

> J: "...I looked at the differences of the turns, and it starts going this way and its a small turn and then this one is bigger and then this one is smaller (points to (b)). And that's why I chose that."

David is the only subject who tries to figure out the shape of the track by drawing it himself while looking at the graph, without paying attention to the three given choices. He draws:

He then compares it with the three choices and gives (b) as the answer. He explains:

> D: "I found that pretty easy. It was fun to work out by figuring out that, how it slowed down when the graph went down, here, so it must be going into a turn, or something. I like working on things, like, I like racing cars and stuff."

Robert says that the curves in the graph correspond to the curves in the track. His answer is (c).

> Ro: "I figured, if you take, O.K., you have the three curves here, three curves eh? (points to graph). This should be your starting point, over there, you should have a space on either side. This (b) is totally different and this (a) has too many."

(7) <u>Torus Problem</u> (<u>#11</u>)

This problem was not used in our study. It was presented to one subject, Judy, who did not understand it and did not attempt to work on it.

(8) Square-Cutting Problem (#6)

(a) <u>Understanding of the Word "Cut"</u>

In this problem, the work "cut" must be given close attention. It is essential to the problem.

One adult, Uri, has an unclear concept of this requirement. He shows this in the third session.

U: "A square into 7 square pieces. You can cut but you have to avoid one piece."

R: "To avoid one piece?"

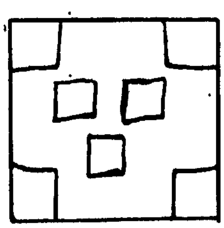
U: "Sure. You have to throw away one piece. Then you can cut it into 7 pieces."

He means that there is bound to be one piece left and it will not be a square. Uri seems to think that one can cut the square into 7 pieces no matter if other pieces are left.

One school child, David, exhibits a similar perception of the problem.

D_6: "Oh yea. I just make like 7 small squares inside and leave a lot of extra space, that would give you 7 squares, and that would cut that big square into 7 smaller ones."

This is David's Solution:



In the third session, David laughs at this problem. He doesn't see why it even should be asked.

D: "I found this a little strange. This question."

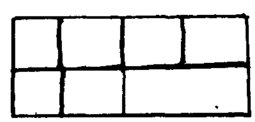
(b) Concept of a Square

The minimum requirement (necessary but not sufficient) in working with this problem, is having a clear concept of a square and being able to consistently work with it.

Adults

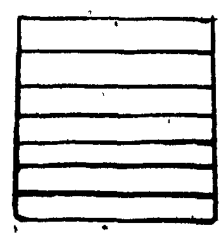
Two adults, Lucy and Uri, do not have a clear concept of a square. None of the adults succeed in solving this problem.

Lucy gives as her solution a rectangle divided (one by one) into 7 pieces.

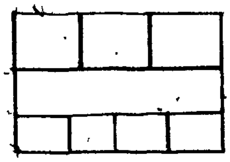


Uri draws a square but then proceeds to divide it into 7 rectangles:

U_2: "A square into 7 pieces. That's it."



In the third session he realizes that the question asks for 7 square pieces. He has ignored the work "square" in his solution above. But even with this realization, he still does not have a clear concept of the word 'cut'.

This is his solution: 

The third adult, Linda, exhibits an understanding of a square:

> L_6: "...A square has four equal sides."

But precisely because of what a square is, she claims that there is no solution to this problem.

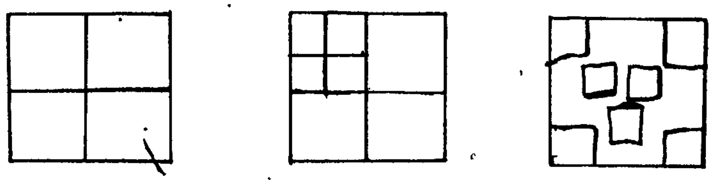
> L_5: "Why do I think it can't be done? Because the squares can't be squares. They will no longer be square evenly, squares."

Later it becomes clear that Linda means the squares will not have equal sides as well as not be equal in size. By "even" she means equal. (See section (f))

<u>School Children</u>

Two school children, David and Judy, have a clear concept of a square.

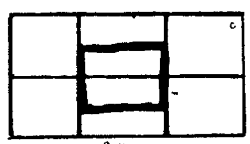
David actually draws the solution but he counts it as 8 squares and as a result continues in his search. In all his trials he exhibits a clear concept of a square. These are his trials:

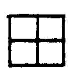


1 2 3

Judy makes the distinction between a square and a rectangle, and claims that she can cut a rectangle into 7 squares.

J_{15}: "I could do it with a rectangle."
R_{15}: "You mean 7 rectangles?"
J_{16}: "No. 7 squares out of a rectangle."

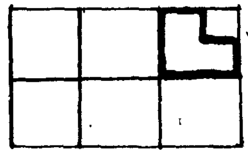
This is her trial: 

Initially, she also makes the same error as David. She is talking about dividing one of the 4 squares in the big square (in ) into 4, but she also assumes it will give her 8 squares.

Robert has an unclear concept of a square. He thinks that any four-sided figure is a square.

Ro_2: "A square is with 4 sides right? So I was trying to make a shape with 4 sides."

Here he is referring to the small square on the right side of his drawing:



He does not consider the darkened shape as a square.

Ro_3: "I want to make a square there but then you have 6 (sides)."

(c) Need for Measurement

Some of the subjects mention the lack of specific measurement in this problem.

Adults

Two of the adults, Lucy and Uri, talk about measurement in this problem.

Lucy concludes that measurement is not important although she does bring the subject up. This is her initial response to the problem:

L_2 : "...Yes I can, because you don't mention anything about the size of it."

Later she comes back to this subject.

L_5:	"Well, I am not using, if I was to use like a 2x2 or 4x4, I would simply measure my squares."
R_6:	"So, the only thing is if you had a measurement for - "
L_6:	"Well, not really. It's just a square. No."

Uri, in the third session, says that to know if something is a square one must measure it.

U: "I would have to take a measurement and make it really square. It's the only thing we would do."

School Children

One of the school children, David, also talks about measurement.

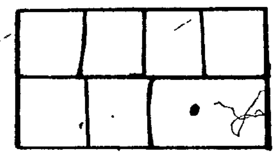
D_5:	"...I would cut a square into 7 square pieces but it will take a little measuring, you know."
R_6:	"Oh. By measuring."
D_6:	"Well. I'll have to - sure -."

(d) Linear Approach

A linear approach to this problem can be exhibited in the action of constructing the squares (inside the big square) one by one until the subject has seven of them.

Adults

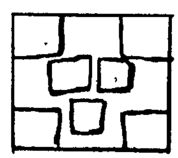
One of the adults, Lucy, exhibits a linear approach. She constructs her pieces one by one. This is her solution:



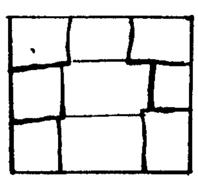
School Children

All the school children exhibit a linear approach during their attempts.

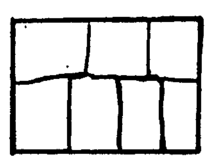
David, who starts out by drawing the actual solution counts it as 8 pieces and goes on to change his strategy. In his next attempt, which he presents as his solution, he draws the pieces individually.



Judy, on her third trial, draws individual squares.

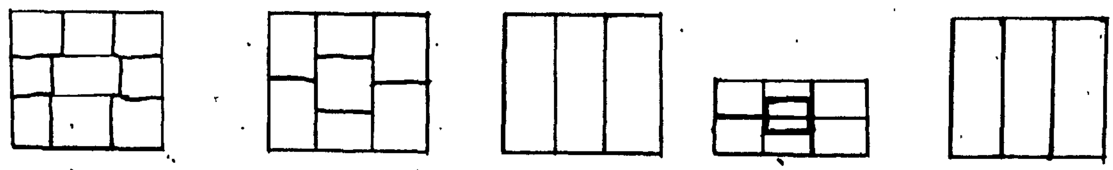


Robert presents as his final solution this drawing, drawn square by square:



(e) Mental Block

One school child, Judy, enters into a mental block, during her fourth trial, and she does not get out of it. She divides the square into three vertical rectangles and proceeds to manipulate, on paper and in her mind, in order to get 7 squares.



3 4 5 6 7

(f) <u>Re-centering</u>

This problem requires the subject to be able to shift his/her attention from a big square to a small square, constructing squares of different sizes.

<u>Adults</u>

Two adults, Linda and Lucy, exhibit difficulties in this area.

Linda claims that when cutting a square into 7 pieces, one is bound to be left with "unequal squares". She seems to associate squares of unequal sizes with squares of unequal sides.

> L_6: "...If you cut it into 7 pieces, it doesn't go. You are going to have one unequal square or several unequal squares-."

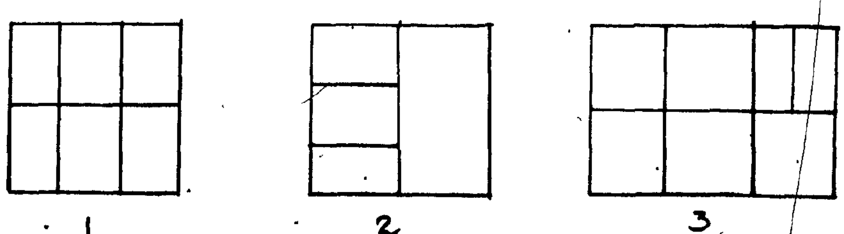
Linda later (in the third session) claims that a square can be cut into any even number of pieces, but she eventually changes her mind and claims that it can only be cut into 4 square pieces.

> L: "Well, I would have 8 squares, but you only want 7."
>
> R: "Oh, you could do it into 8?"
>
> L: "Yea."

But when asked to cut a square into 8 square pieces, Linda does not try.

> L: "...Because you have given me a square, so it has to be 6 pieces. No it can't even be 6 pieces. I don't know. A square can only be divided into 4 square pieces."

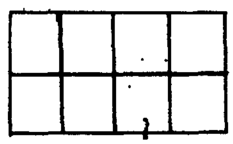
Lucy, in all her trials, does not attempt to cut any of the smaller squares again into squares. These are her three attempts:



In the thrid session she says that the answer is no because 7 is an uneven number:

> L: "Its not true because its uneven."
>
> R: "What's uneven?"
>
> L: "The amount of pieces. Because it's, it always goes even whenever you cut a square."
>
> L: "I could make 6, I could make 8, I could make more. But 7 I can't."

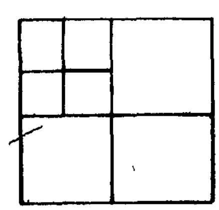
The following is her attempt to divide into 8 square pieces.



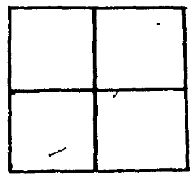
School Children

Two school children exhibit a clear concept of recentering. These are the only subjects who either sketched or were going to sketch the solution (but counted it as 8 square pieces).

David clearly shows how he shifts his attention from big to small squares. His first attempt exhibits it.



Judy, in the third session, succeeds in finding the solution. During the second session, Judy divides a square into four. She then proceeds to ask if she divides the lower left square into 4, will the whole left square be also counted as a square piece. This question shows that Judy is recentering.



> J_8: "...O.K. If I divide this (points to lower left square) like I did this (the whole square) is it counted, is like this part (whole lower left square) counted also?"

The interviewer's answer provides a hint that Judy does not take.

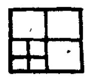
> R_8: "It says here, can you cut, so if you cut it, you have to have 7 squares."

Judy does not continue with this attempt but goes on to use another strategy (dividing square into three rectangles). In the third session she explains:

> J: "At first, cause I counted, this is 4 all to-

J: gether, and then this is 4+4, so it will be 8."

Earlier, Judy realized there were 7 pieces.

J: "Because if you divide (lower left square) into 4 then it's more than 7. Or is it? (draws ) Yea that's (realizes she has 7). Oh. (laughs)."

(g) Diagram Usage

Only one subject (an adult), Linda, has not drawn any diagrams (except her only square).

(9) Pond Problem (#5)

(a) Need for Measurement

Adults

One adult, Linda, expresses a need for measurement. Linda claims that she needs more information about the amount of growth on the first day, as well as about the size of the pond. Throughout this session, Linda attempts to explain what could be done if she had that missing information.

L_6: "It doesn't say, uh, at what stage the pond is at the beginning."

And later:

L_{10}:	"It will depend on how big the pond is. That is what is missing."
R_{11}:	"The - "
L_{12}:	"The size of the pond."

School Children

One school child, Robert, also expresses a need for a measurement describing the situation on the first day.

Ro_{11}:	"I can do it, like. First I have to know, is it the first day is at 1, the second day 2 and the -"

(b) Holistic vs. Linear Approach

An holistic approach to this problem is an initial ability, when perceiving the problem, to coordinate the given amount of growth (half the pond) with the rate of growth (doubling). The subject must take these two factors into consideration, and must work with them both.

A linear approach to this problem is looking at the problem on a day to day basis, trying to solve it by starting with the first day, following it to the eighth day. The subject tends to figure out a daily amount of growth which s/he follows up until the eighth day. Here the subject is ignoring one of the given conditions.

Adults

All the adults exhibit a linear approach to this problem. None of them succeed in solving this problem.

Linda attempts to explain what she could do if she had some measurement. Her proposed plan is to work out the amount of growth on a daily basis.

> R_{12}: "And if you have the size of the pond, what can you do then?"
>
> L_{12}: "Well, then you can calculate how much vegetation grows per day, double it, and decide what happens on the eighth day."

Lucy disregards one of the conditions (rate of growth). She assumes that there is a constant growth each day. The following is her solution given immediately after reading the problem.

> L: "So, 8 days, if it took 8 days to cover the half of the pond, it would take another 8 days for it to be completely covered. So I would say it would take another 8 days. It would be on the sixteenth day. On the sixteenth day the pond would be completely covered."

Uri attempts to follow the amount of growth from the first day to the eighth day. He gives his solution (on the thirty-second day) at the start of the session, and he spends the rest of his time explaining his solution.

U_{10} :	"The overgrown part starts from 0. And on the, on the 0, and every day it doubles."

And later:

U_{11} :	"No. From nothing, doubled is 1 again. Overgrown area is doubled, its 2. On the eighth day its 8x2 is 16. 16 its. 16 is half the pond. On the day everything would be covered would be 32 again. Comes again to 32."

Uri claims that the pond grows by two units each day, up until the eighth day, when there are 16 units of overgrown area. So, 16 units is half the pond. Now, Uri uses the fact that the area doubles every day. He doubles 16, and gets as his answer 32. It is unclear what kind of units he is working with. He seems to shift from square units to time units.

U_{19} :	"Half the covered pond is 8. Twice 8 is 16. Twice the time also."

Although he works throughout with the amount of growth each day, Uri's answer is given by the amount of days.

U_{20}: "On the thirty-second day it will be covered. That's right."

School Children

All the school children exhibit an initial linear approach. Two of them, David and Robert, succeed in solving this problem.

David starts out by explaining the situation in a linear way and suddenly succeeds in changing his perception to an holistic one (has a sudden insight). He initially writes these sequences of numbers: 1/8, 1/4 1/16, 1/4. He later attempts to explain:

D_3: "Let's say the first day its 1/8 full, the next day its 1/4 full, the third day 1/2 - that would work out."

It is clear that David attempts to follow the growth on a daily basis.

R_8: "So the first time you read the question, what was the first thing that you thought?"

D_8: "How much it gets bigger each day. But that wasn't really what I needed."

Immediately after writing the sequences of numbers, while he is trying to explain his thoughts, David realizes the situation the pond is in. The following excerpt shows this sudden change in his perception.

D_2: "I am trying, like, to say, the vegetation squares every day, so in eight days it fills half of it, so I am trying to see - so on the ninth day the pond will be covered completely."

Robert also attempts to work on a day-to-day basis. He works with the pattern of 1, 2, 4, 6, 8.

Ro_6: "...So in the first day, if one area doubled, next day it's going to be 2, next day it's going to be 4, next day it's going to be 6, next day it's 8."

Like David, while verbalizing to himself, Robert realizes the situation.

Ro_{14}: "(reads) on the eighth day half of the

> area has been covered - so, 16, - on the ninth day you will have 16. It would - oh, so on the ninth day."

Judy assumes that there is a constant daily growth. She disregards one of the conditions (the rate of growth). This is her explanation of her solution (sixteenth day).

> J_9: "Everyday the overgrown area has doubled. And on the eighth day it has covered half the pond, so, so then all you do is take it back to the beginning. Everyday the overgrown area has doubled. And so, you take it from 8. Like you start at 8, and it becomes, I count to 16."

(c) Algorithm Usage

One of the school children, Robert, attempts to develop an "equation" that would give him the answer. This attempt is his initial reaction to the problem.

> Ro_2: "...At the eighth, at the eighth day it was half covered, and it it's going to double everyday, like 8, half of 8 is 4. So I figure the twelfth day."

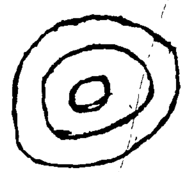
This is an attempt to find a short cut to the problem

$(8+\frac{1}{2}\cdot 8=12)$. Robert immediately claims that the above attempt is wrong. He goes on to generate sequences of numbers.

(d) Use of a Diagram

Adults

One adult, Linda, uses a diagram in her attempts at solving this problem. Linda relies heavily on her diagram, to the point where she claims that her own interpretation is right and the given information is wrong. Her diagram:



This is an attempt to represent a doubling of the over-grown area. She is here violating the condition that the pond is half covered by the eighth day.

L_7: "Let's say it's like this the first day (draws a circle). Every other day it becomes like this. On the eighth day it has covered only half! It doesn't make sense. It's too much time -"

R_8: "It's too much time to be covered only half?"

L_8: "Yeah. Yeah. Yeah. You are not telling me the truth (laughs). You lied! On what day will it cover the pond completely? It would have covered the pond completely on the fourth day or so."

According to her diagram, the pond will be covered completely by the fourth day.

School Children

None of the children use a diagram in this problem.

(e) Visualizing the Situation

Adults

One adult, Uri, says he visualizes the pond.

R_{14}: "O.K. Now. When you were looking at this problem, how were you thinking? Were you visualizing the pond, or were you -"

U_{14}: "Yes, I was visualizing the pond."

R_{15}: "You were seeing it."

U_{15}: "And I didn't take the pond into account, only something that grows on top of the pond."

School Children

One school child, Judy, says that she visualizes the pond.

> J: "I was visualizing it as a, just a circle on a paper, or even a rectangle or just a large shape. And I was picturing it , of half of it. Yeah, I was visualizing it. And then I tried to solve it by the way of numbers."

(f) Display of Frustration (hypothesis #7)

One of the adults, Linda, exhibits frustration. She claims that the problem is not worded properly, that the given information is false, and that there is insufficient information. These claims are made in the beginning of the session and show her initial perceptions of this problem.

> L_3: "(After reading the problem) That's irrelevant, I think."
>
> R_3: "What is irrelevant?"
>
> L_4: "On what day will it cover the pond completely."
>
> R_4: "What do you mean by - ?"
>
> L_5: "Well, it hasn't really got anything to do with the question, I don't think."

Linda then says that the initial state of the pond is missing (see (a)). The whole session consists of her attempts to clarify the problem for herself.

(g) Intuition (hypothesis #8)

One of the school children, Robert, seems to follow and trust his intuition. The following is a situation where he feels that his answer is wrong and he goes on to solve the problem.

Ro_{13}:	"...12 is wrong."
R_{14}:	"Why are you so sure it's wrong?"
Ro_{14}:	"Because it doesn't seem right."

Here he goes on to find the solution (see(b)).

(10) Cubes Problem (#(6))

Note: This problem was not given to Linda, who had difficulties with #1(a).

(a) Recognition of Unrealistic Data (hypothesis #2, #7)

This problem represents an unrealistic situation. The three views given can not be combined to represent one cube.

Adults

One adult, Lucy, exhibits a realization that something is wrong. Although Lucy is perplexed, she does not at any time say that it is the problem that is impossible. Lucy assumes that she herself is at fault. She has no doubts that there exists a solution.

L_{12}: "...Can I come back to that one? Because I just can't figure it out now."

Lucy looks at the first and second views, and she decides that letters can appear twice. This is her sketch:

```
  K
  H
M G K
  G
```

She has not made this assumption (that letters can appear twice) while working on #1(a). She is here attempting to explain away the contradiction.

L_6: "Oui, if I try to match these two (second and third views) they don't fit."

Finally, Lucy claims that with the given information she will get a cube with nine faces.

L_{12}: "...Well, you see, these are three views of the same cube. So it means that cube has got - How could it have nine -? Now, how could I have a cube with nine? I can't -"

Lucy assumes personal responsibility for her difficulties.

L_{12}: "Can I come back to that one? Because I just can't figure it out now. I'll come back."

In the third session, Lucy again talks about the nine faces. She realizes that there are difficulties. She explains here.

R: "...What is the problem that you are encountering here?"

L: "Well, as I was saying, I cannot take one of those cubes, and put it beside the other and say, it goes together, it adds together. I can not do it."

R: "How come?"

L: "Well, if I compare it to the other one (to #1(a)), there is no difference. Well, this is it. The difference from this one is that square #1 you've got upside-down letters."

R: "You have them here too, in 1(a) (points to upside-down letters in #1(a)).

L: "That's right. O.K. - But I am not able to do this here (to combine views in #1(b), like she has done in #1(a)). O.K. Because if I put those two here (first and second views in # 1 (b)), I've got the K in the middle; it makes no sense."

Lucy never says that this is an impossible problem. She just claims that it makes no sense to her.

Uri, who had difficulties in problem # 1(a), is given this problem in the hope of clarifying his approach to #1(a). Here, he continues to use the same interpretation as before, but it becomes clearer to the interviewer.

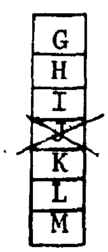
Uri claims that the letters go according to the alphabet, in either a clockwise or a counter-clockwise direction. He answers each of the three questions using it's corresponding view.

U_1: "...There is something now. I have to - to reach into the A, B, C now."

R_3: "Why?"

U_4: "Because it follows the alphabet here. Of course - according to the alphabet."

In this sketch, Uri crosses out the J since it does not appear in the problem.

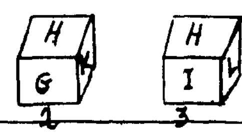


U_{20}: "...It depends if it goes this way (clockwise) or this way (counter-clockwise)."

School Children

None of the school children realizes

the unrealistic nature of this problem. David, in the third session, after being asked to label the sides of a cube, realizes the contradiction. He claims that he initially noticed it in the second session, although he had not mentioned it then.

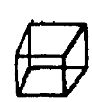
David answers each question by using only two views for each. As an example, while looking for the letter opposite G, David works with the second and third views only. He disregards the first view, and therefore seems unaware of the unrealistic situation. (

)

R_1: "Why is I opposite G?"

D_2: "...Here they show G here and H on top (second view). I guess what they did was turn it around in this picture and H is still on top but I is in the front (third view). If they turned it only a quarter of a turn, the H here will be sideways. But it is still kind of straight. So I guess they turned it 180°."

This explanation is a valid one, but only if the first view did not exist. David answers the other two questions using the same approach. His answers: G - I
H - K
I - G

It is in the third session, where he is asked to label the faces of a given blank cube (), that David realizes the contradiction. He claims that he had noticed it in the second session. He had made no comment at all to that effect in the second session. In that session he possibly was trying to explain away the contradiction by working with two views at a time.

> D: "Something is wrong here....I figured that other thing last time. I noticed that."

In the following excerpt, David shows complete understanding of the situation.

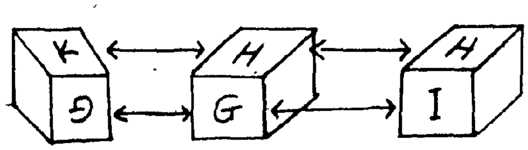
> R: "So what do you think about this problem?"
>
> D: "I think, they have, it's not really, it's either three views of a different cube, or they have not three true views. Like, they are changed so they can't really exist, like if this was a real cube, that wouldn't be there, that would be underneath (K, in second view)."

Judy is given this problem to clarify her nonstandard interpretation in problem # 1(a). She continues with this same interpretation. Her solutions are:

G is opp I and G

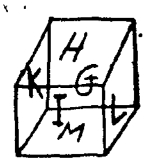
H is opp H and K

I is opp G.



Robert visualizes the cube in the following way:

His solutions: G - I
H - M
I - G



(This is the interviewer's sketch according to his comments.) This sketch agrees with the second and third views. Robert is ignoring the data given in the first view.

(b) <u>Linear Approach</u>

Two school children, David and Robert, exhibit a linear approach. They both work with two views at a time, ignoring the whole picture (all three views at once). They separate the given information into chunks.

David does understand what the problem is asking him. His approach will work in a problem like # 1 (a) where the given information is realistic and there is a soltuion.

Robert seems capable of integrating two views together, although his solutions for #1 (a) were not the right ones. It is unclear how he got them.

(c) <u>Frustration Because of Contradiction</u>

One adult, Lucy, exhibits frustration. She finally asks to continue to the next problem. That is the only sign she makes of frustration.

(d) <u>Distrust of Intuition</u>

<u>Adults</u>

One adult, Lucy, exhibits a distrust of her feelings that something is wrong. She claims personal responsibility for it (see (a)).

School Children

David attempts to explain away the unrealism. This shows a distrust of his intuition if, as he claims, he has noticed the contradiction in the second session.

(11) Counters Problem (#9)

(a) Understanding the Task

This is a problem involving straight logic, therefore it must be solved by a step-by-step linear approach. A subject must take into consideration **all** the information given, especially the statement "only one is true". Then s/he must be able to use logical steps to find the solution.

Adults

All the adults have changed this problem and have answered a "new" problem. They ignore the essential statement "only one is true". They all assume that all three statements are true, and have answered accordingly.

Linda, at first, demands to know what exactly she is asked to find. After reading the problem again she proceeds to give her answer.

L_3: "...B is not blue, so it's white. C is not red,

> so it's blue. There. A is red, B is white, and C is blue."

Lucy, at first, claims that each counter could be any colour. She seems to ignore the three statements. It is unclear what she means here:

> L_1: "...Only one statement, only one is true. I would say A is red. If you say one is true, it could be any colour. Then what colour is each counter? It's up to you. It could be any colour."

After reading the problem again, she changes her answer.

> L_3: "...A is red. If B is not blue, B has got to be white. And if C is not red, it's got to be blue. So A is red, B is white, and C is blue. - O.K.?"

Uri also assumes that these statements are true. This is his answer:

> U_3: "A is red. B is not blue. Then it's - let's say - white. Then it's blue, A is red, and B is white, and C is blue."

Each of the three adults gives his/her solution almost immediately.

School Children

Judy gives her solution immediately.

J_2: "Well, it tells you that A is red. So you know that A is red. And, if B is not blue, then - if B is not blue, then it has to be white, and - the colour left is - is - the only colour left is blue."

Robert, initially, is unsure what the question is.

Ro_1: "Oh, you want me to say which one is not true? Which one is true? A, B, or C?"

After reading the problem again, he proceeds to give his answer.

Ro_3: "What colour - ? Oh, gee. A is red. B is not blue, so it has to be white, and C is blue."

(b) Nonstandard Interpretation

One school child, David, interprets this problem in a nonstandard way. His claim is that each counter has all three colours - red, white and blue. Therefore, only one of the given statements is true, namely "A is red". The other two are false. David claims that the statement does not say that A is completely red, therefore it is true. He arrives at this solution immediately after reading the problem.

D_1: "...So I guess what I understood is counter A is red, white, and blue; counter B is also red, white, and blue; and C is also red, white, and blue. And there they say B is not blue. But if they are each coloured those three red, white and blue, that's false. And that's false (third statement). But that's A is red. They don't say completely red, so I guess A can be red, white and blue, and they say A is red. I guess that's it. 'A is red' is the one that is true, and each counter is red, white, and blue."

David is the only subject who is working with the statement "only one is true".

Notes: (1) After the session with David, the wording in the statement "red, white, and blue" was changed to "red, white, or blue".

(2) All the subjects answered this problem quickly.

(3) Two adults, Lucy and Uri, mentioned that this problem is tricky or especially requires thinking.

L: "But it's just, trick with words. And if you

catch it, all goes well."

U: "Yeah. It's an exercise - a brain exercise I would say."

(12) Parking-Lot Problem (#7)

(a) Holistic Approach Vs. Linear Approach

To exhibit an holistic approach in this problem, a subject must show an initial awareness of the two different conditions, the number of vehicles and the number of wheels, and of the relationship between them. A method that can be used while approaching the problem holistically is that of "trial and adjust", where there is an adjustment and coordination according to trials made, in the right direction. An example would be the realization that more cars give more wheels and more motorcycles give less wheels (when the total number of vehicles is fixed). This realization would direct a subject during his/her trials.

In a linear approach, a subject ignores one of the two conditions and works with the other. S/he may give an answer that satisifies one of the conditions, but not both. The subject may use "trial and error" to arrive at the solution, but it is an attempt to find the number of cars and motorcycles according to the given number of wheels, or according to the given number of vehicles, not both.

Adults

Two adults, Linda and Lucy, exhibit an holistic approach. They also succeed in solving this problem. The third adult, Uri, exhibits a linear approach.

Linda starts out by working with the number of vehicles and the number of wheels, trying to find some connection between them. She wants to find a starting point. In the first few minutes she is constantly dividing and multiplying numbers in an experimental manner. She is attempting to find out if the given information is compatible and realistic.

R_2 : "Why were you dividing into 66?"

(66 ÷ 4 = 16 cars $\begin{array}{r} 82 \\ -66 \\ \hline 16 \end{array}$ x 2 = 32 (motorcycles)

66 + 32 = 98)

L_3 : "I'm just working out, just to see approximately where I stand."

R_3 : "What are you trying to do here with the 68?"

(68 ÷ 4 = 27)

L_4 : "I'm trying to figure out how many wheels equals how many cars."

L_5 : "28 vehicles in the parking lot. 28 vehicles represent how many wheels. First I'll find out if they are right. So how am I going to do that? - 82 wheels. Maybe you are wrong. Maybe there is more than 82 wheels."

Here Linda suspects that there is a possibility of the two conditions being incompatible. In spite of these unnecessary suspicions, Linda is obviously working with these two conditions.

After this episode, Linda re-evaluates her thoughts and gives as her answer 20 cars and 1 motorcycle. She realizes her error immediately.

> L_{11}: "...28 vehicles in the parking lot. Let's say 20 vehicles. That gives me 80. O.K. So there are 20, 20 cars and 1 motorcycle. That's it. That's my answer."
>
> R_{12}: "20 cars and 1 motorcycle?"
>
> L_{12}: "No it isn't. Because there is 28."

From this point on, Linda conducts a "trial and adjust" starting from 15 cars, 13 motorcycles, and then 14 cars and 14 motorcycles, and finally 13 cars and 15 motorcycles (the solution). She shows an awareness that decreasing the number of cars produces less wheels.

Lucy, initially claims that there is no solution. Her reason is unclear. When asked by the interviewer to explain, she repeats the same words.

> R_L: "Can you tell me the reason again?
>
> L_2: "Because cars have 4 wheels and motorcycles

have 2 wheels. And you specify 28 vehicles O.K.? It amounts to 82 wheels. I mean if I was to say, 28, divided by 2 it's 14. 4 times 28, no, it's 28 divided by 4 is 7. No, I can not solve that."

The above explanation is the third time Lucy has given it. When asked to elaborate on that, she changes her mind.

L_5: "Well, that's right. I can not do that (divide 28 by 2 and by 4). I should have taken a number instead, because that's all you have, 28."

She goes on to conduct a "trial and adjust". She starts with 14 cars and 14 motorcycles and decreases to 13 cars and 15 motorcycles, all in the same breath.

L_{15}: "...I choose 14 cars, 64, 5, 14 cars and I am left with 14 and motorcylces have got 2 wheels, gives me 28. Gives me 84. Gives me 2 extra wheels. mm. Let me see. 13 cars, 12, 4, 5 and 15 motorcycles."

Uri immediately ignores the condition of 28 vehicles.

He works with the number of wheels only and does not take all the given information into consideration. His approach is not holistic, it is a linear one. He is looking at parts of the problem only. He goes on to give three solutions.

U_1: "Ah. 82. Well, this is quite simple. I would say 20 cars plus 1 motorcycle. You can do it other ways too."

U_2: "...It could be 15 cars and, uh, 11 motorcycles."

His third solution is 1 car and 39 motorcycles.

<u>School Children</u>

None of the children exhibit an holistic approach, although two of them, David and Judy, show an awareness of the two conditions. None of the school children solve the problem.

David is immediately aware of the two conditions and makes an attempt with 14 cars and 14 motorcycles. It is interesting that, like Lucy, he starts with these numbers. This is his only attempt. He refuses to continue, claiming that he will be guessing and it will take too long.

D_2: "...Let's say it was half cars and half motor-

> cycles, - 14 each. In all, there's 82 wheels and it only adds up to 80."

Here, David makes an arithimetical error when multiplying 14 by 4 and getting as an answer 52. He then claims that this attempt is the only one he can think of.

> D_5: "The only way I could think of is 14 cars and 14 motorcycles, but that doesn't work out. It's the only way I could think of."
>
> D_6: "...Like by. I could pick a few numbers here and there, like 10 cars and 18 motorcycles, but I would just be guessing."
>
> R_7: "And you feel that you don't want to guess?"
>
> D_7: "Well, it probably will take too long."

Because David does not try to adjust his attempt, he is not working on this problem by using an holistic approach. In the third session, he shows how well he understands the relationship between the two conditions.

> D: "<u>It's like a two-combination lock. You got to look for both ones</u>. So you'd take a long time."

This shows an instance of an holistic approach taken by him.

Judy also shows an immediate understanding of the two conditions in her rewording of the problem.

> J_1: "There is 28 vehicles, and there is a mixture of cars and motorcycles, and all together there is 82."

Judy then experiences difficulties in finding a starting point through out her attempts at this problem.

> J_1: "I was thinking - if I could divide 28 into 82. I don't know where to start."

She goes on to divide 82 by 4 and by 2, feeling her way. All her attempts show a "random trial and error" where she tries numbers and gives possible solutions which she rejects. Although her trials are random, Judy shows an awareness of what the solution should not be. She immediately discards her random guesses. Finally, she prefers not to give an answer. One of her guesses is 40 cars and 1 motorcycle.

> J_6: "No, 40 cars and 1 motorcycle - that's it. Yeah, 40 cars. No, wait a minute."

41 motorcycles (no cars) was also given as a possibility, and discarded immediately.

J_{15}:	"Well, if there were, there will be 41 motorcycles. Because 2 times 41 is 82. That won't ... Can we come back to that one?"
J_{16}:	"No, I don't have an answer yet."

Robert does not exhibit an awareness of the two conditions. He works with the number of wheels only, giving two different solutions and claiming that there are many more. This is a linear approach.

Ro_2:	"You can have one motorcycle, for the two wheels and then you could have 20 cars."
Ro_5:	"I can write down a thousand different ways."

His second solution is 11 motorcycles and 15 cars.

(b) Algorithm Usage

Examples of algorithms that can be used in this problem are: equations, or, a complete systematic search starting with one car then 2 cars, three cars, etc.

Adults

None of the adults attempt or even mention the use of an algorithm to solve this problem.

School Children

Two school children, David and Judy, talk about using algorithms.

David searches for a definite algorithm. He refuses to start a "trial and error" which he calls "guessing".

> D_4: "...There could be, a, some way, to do it. I just can't think of it now. I don't know."

Judy, in the third session, says that she could have done this problem by using two equations. She does not try though.

> J: "I could have done it with two equations."
> R: "What do you mean?"
> J: "Well, I could have, like, because they want to find two different kinds of vehicles, and there's 4 wheels on one and 2 on the other."

(c) Managerial Strategies (hypothesis #3)

Re-evaluating an approach to a problem and changing the approach as a result is a managerial strategy if done consciously.

Adults

Two adults, Linda and Lucy, change strategies midway and as a result solve the problem. They both change to a "trial and adjust".

Linda, after a few minutes of working with the number of wheels and vehicles and trying to find a compatibility between them, re-evaluates her train of thought and changes into the strategy of "trial and adjust".

Here she explains her old strategy and why she changed it.

> L_8: "I just picked it out of a hat (66). Just to see whether I was too far, picking too many or to less, then I would."

She claims that this number gave her too many (it is unclear if it is too many wheels or too many cars).

> R_{10}: "Why is it too many?"
> L_{10}: "Because I was on a certain train of thought here, and now I am re-evaluating my thoughts. There are 28 vehicles in the parking-lot, cars and motorcycles. All together, there are 82 wheels."

Linda rereads the problem. After a brief episode of ignoring the number of vehicles, Linda conducts a successful "trial and adjust".

Lucy, initially claims that she can not solve the problem. Although she does not verbalize that she is re-evaluating (Linda does verbalize it), she criticizes her initial approach and claims that she should have tried using trials of specific numbers.

She then goes on to a "trial and adjust".

> L_5: "...I should have taken a number instead, because that's all you have - 28."

(d) Distrust of the Given Information

One adult, Linda, exhibits a tendency in many of the problems, including this one, to distrust the given information. She initially attempts to find out if it is true.

> L_5: "...First I'll find out if they are right... Am I to assume that this is, that this is correct?"

She decides that the information is right and proceeds to solve the problem.

> L_{13}: "Makes sense. O.K. So I got, I have the right number of wheels here (given information) .. You need 28 altogether."

(13) Students-Professors Problem (#12)

This problem involves the converting of semantic content to algebraic content. The subject must use algebraic symbols and relationships to represent the given statement.

<u>Adults</u>

Two of the adults, Lucy and Uri, exhibit an understanding that for each professor there are six students. But, they can not write this in an equation form ($6P=S$).

Lucy is aware of the relationship between students and professors.

L_1: "There are 6 times as many students. Sx6, so it would be 6S for one professor."

Lucy gives the equation $\frac{6S}{P}$. She realizes that ratio has something to do with this problem.

L_2: "What it means is for 6 students you have one teacher. That's what it means to me. That's right. 6 students is equivalent to 1 professor."

This explanation shows that Lucy understands the meaning although she is not able to translate it algebraically.

Uri does not attach any importance to the word "equation". He gives four examples to show that he understands the meaning of the problem.

His examples:

$12S + 2P$
$6S + 1P$
$18S + 3P$
$24S + 4P$

He understands that for every professor there are six students, but he has no concept of "equation".

Linda says that the number of professors, which she calls the "unknown", equals 6 times the students. She writes it: $X = \frac{6xS}{P}$

She reads it as:

L_4: "Unknown equals 6 times students totals professors."

R_5: "What does the unknown mean to you?"

L_6: "How many professors. The number of professors."

The word "equation" might have caused Linda to assume that there is an "unknown". In the third session, Linda asks if she must accurately give the number of professors. She then says that she is only asked for the equation. Her statement that the number of professors equals 6 times the number of students is equivalent to the algebraic expression P=6S.

School Children

All the school children have difficulties in writing an equation in this problem.

One school child, David, exhibits an understanding of the meaning. He writes: 6S:P

R_1: "What is this in the middle?"

D_1: "That - I - that means for a ratio - the thing

that sort of - six times as many - like 6 students to one teacher."

He gives as an example, 12S to 2P.

Judy has difficulties in translating "as many as" algebraically. She finally decides to denote it as "+". She writes: P + 6S.

Robert exhibits a dislike of this problem as soon as he reads "write an equation". He finally writes: Let S be 3 x many P. (He writes a 3 instead of a 6.) He reads it as:

Ro: "Let S be 3 times as many professors."

If he would have continued, could he have written S=3P?

CHAPTER V

CRITICAL EVALUATION

This research involved choosing problems to test certain research questions, presenting these problems to subjects in such a way as not to cause any undue pressure, and observing the subjects as well as questionning them about their attempts at solving these problems. While carrying out all of the above, we were faced with unexpected situations, overlooked information and errors on the part of the interviewer. This chapter is devoted to these.

1. Problems

Some unexpected situations occured as a result of the problems chosen for this thesis. Each problem was chosen to test specific research questions. Sometimes certain difficulties arose for the subject, and consequtently the research question could not be tested. At other times, the wording of the problem presented unexpected interpretations of it.

Problem #1 (a) (Cubes) was given to test spatial understanding. It was a solvable problem, provided as a background for a later one, #1(b). Problem #1(b) presented an impossible situation, and was given to see who would recognize the unrealistic situation. It is not possible for a subject to recognize the unrealistic information given in #1(b) if s/he had not initially understood the spatial situation of #1(a). Therefore, the research question that #1(b) was supposed to test (will adults be more likely to recognize the unrealistic data?) was not tested effectively since some subjects did not understand the problem to begin with. But, in some instances, these subjects clarified their interpretation of #1(a) while working on #1(b).

Uri was given #1(b) (after exhibiting difficulties with #1(a)). His attempt to solve it shed light on his interpretation of #1(a), which was a nonstandard one, and helped us to analyse his protocol of #1(a) more effectively.

Problem #2 (mother daughter) was solvable but the solution itself was unrealistic. This problem was given to see who would notice that the solution is impossible in the real world, as well as to see how the subject would arrive at this solution (by using an equation, guessing, et.). Some of the subjects could not solve this problem at all, and therefore could not be tested as to their recognition of the unrealistic nature of the solution.

Problem #9 (Counters) was initially worded (not intentionally) in such a way as to allow wider interpretation of it. "Three counters are coloured red, white and blue" can be interpreted as saying that each counter is coloured with any combination of these three colours. The intended meaning here is that each counter is coloured by one colour only. One school child, David, interpreted the problem in the non-intended way. After the session with him, the wording was changed to "red, white, or blue". This wording is also somewhat ambiguous, although the rest of the subjects interpreted the problem in the intended way.

Some of the problems required certain skills that some of the subjects lacked. Therefore these subjects had difficulties in working on or understanding these problems. The racing car problem, the mother-daughter problem and the students-professors problem required such skills as interpreting a graph and constructing an equation. As a result, in some instances the research questions pertaining to these problems could not be tested.

2. Presenting the Problems

During a few of the initial interviews, the interviewer had all the problem cards in her hand and she presented them one by one to the subject. As a result, some of the subjects felt hurried and seemed to expect another problem as soon as they had finished one. Therefore, during the rest of the interviews, the problem cards were kept hidden from the subject so that s/he would not watch to see how many were to be used during the interview.

Robert, a school child, was interviewed during his lunch hour, and consequently he seemed to want to get it all over with as fast as possible. He raced through the problems quickly because of the time limit.

Sometimes a subject asked to go ahead to the next problem when having difficulties. The interviewer did not press the subject to continue, and presented the next problem as asked. Perhaps some pushing for more efforts from these subjects would have given us more information, but we did not want the subjects to feel pressured.

For some subjects, six problems per session were too many, while others seemed to feel comfortable with this number of problems.

3. Interview Technique

Opper describes the clinical interview as an hypothesis testing situation where there is a constant dialogue between the interviewer and the subject.[1] The interviewer attempts to clarify the subject's actions and statements by asking questions or by asking for explanations. The interviewer must also keep from giving hints and leading the subject. There must be minimal outside influence on the subject so as to test the hypotheses (or research questions) effectively. It is difficult not to give a hint, especially when the subject is either very close to an effective path or when s/he has made a technical error that can throw him/her off the path completely.

There were some instances in this research when the

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1 S.Opper, Piaget's Clinical Method p.92

interviewer kept from giving hints or pointing out obvious errors at some critical points. In the parking-lot problem, David made an arithmetical mistake (14x4=52). Since the numbers did not work out, David refused to continue. He did not want to try other possibilities. This arithmetical error did not really change anything because David disliked trial and error throughout his interviews. There was one instance when the error made by the subject influenced the rest of the interview. In the square-cutting problem, David drew the solution, but proceeded to count the squares as eight. The interviewer did not comment on this, and David proceeded to try many interesting possibilities, never realizing that he had the solution all along.

On the other hand, there were some instances where the interviewer missed opportunities to follow up on some comment or action of the subject. The interviewer sometimes failed to ask relevant questions or had worded the question in an unclear way. It was while analysing the protocols that these instances of missed opportunities on the part of the interviewer came to light. It is easy to say "I should have" after the event. The following is an example where the interviewer did not ask for a clarification of a statement made by the subject, Linda.

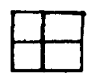
Only after Linda finished with the problem, the interviewer referred back to the statement and asked for an explanation from her, but by that time, Linda's mind was on the next problem. Here Linda claims that one can not cut a square into 7 pieces.

L_7 : "Because a square has four equal sides. If you cut it into 7 pieces it doesn't go. You are going to have an unequal square or several unequal squares. O.K.?"

R_7 : "O.K. Here is the next one."

The interviewer did not ask the subject to elaborate on the meaning of "unequal squares".

In the following instance, the interviewer gave the subject an important hint. This is an example of interference on the part of the interviewer.

J_8 : "O.K. If I divide this (in  , lower left square) like I did this (points to whole of square) is it counted, is like this part (the whole lower left square) counted also? Like..."

R_8 : "It says here, can you cut. So if you cut it, you have to have 7 squares."

The word "cut" is important in this problem.

It was very useful to have the third session, during which the interviewer could ask any remaining questions or could request clarifications. The subjects seemed to remember the previous interviews and what had happened during them. After the protocol analysis of these third sessions it was evident that the interviewer had missed further opportunities.

Note: In Linda's third session the problem cards were not available, so the interviewer reminded her of the specific problems. It did not create any difficulties.

In the beginning of the first session, subjects were asked to repeat the problem statement in their own words, so some data could be gathered about their perception of the problem. The subjects either repeated the problem word for word as they remembered it, or got nervous and could not repeat the problem at all. As a result, this requirement that we have made of the subjects did not yield any information. On the contrary, it seemed to have caused some subjects to be anxious since they felt that they were being put on the spot.

4. Research Questions

We have chosen the problems so as to test our research questions and explore them. Some of these questions were not explored effectively because of their vagueness.

Research question #3 involved managerial decisions. Managerial decisions are difficult to look for because of their complex nature. Schoenfeld used complicated mathematical problems to explore managerial decisions.[1] Our problems

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[1] A.H. Schoenfeld, Episodes and Executive Decisions in Mathematical Problem Solving

were not intricate enough, and as a result there were not many instances of managerial decisions noticed by us in this study.

Some research questions were not precise enough and their vagueness caused them to have several interpretations. Research question #7 mentioned frustrations due to ambiguities in the problem statements. The word 'ambiguities' is a general one, describing several things. It can mean contradictions, unrealism, insufficient information, etc. It can also mean that the subject has not understood the problem and has therefore found it ambiguous. One of the subjects, Linda, has found most of the problems ambiguous, at least initially. This was because of her general distrust of the situation rather than the fact that she is an adult. Therefore, when the question asks if adults will be more likely to be frustrated by ambiguities in the problem, in Linda's case, her being an adult is irrelevant. The question should be more precise.

Research question #8 mentioned two factors, using intuition and distrusting it. It was possible for subjects to use their intuition but not to distrust it. Two separate research questions can be made from this one, exploring intuition and separately, the distrust of one's intuition.

Research question #2 mentioned unrealistic problems. It failed to differentiate between problems that were unrealistic and therefore unsolvable, and problems that were solvable, but the solution itself was unrealistic.

Although, this did not harm the effectiveness of this question, more precision would have made protocol analysis of this question easier.

5. The Questionnaire

The questionnaire was given to explore the subject's feelings during the interviews, their state of relaxation and interest, and their attitudes towards mathematics. The subjects were left alone while answering it and were told to take their time.

One disadvantage of such a questionnaire is that the subjects might answer the questions in the way they think the interviewer would like them to. When a subject answers that s/he found the problems interesting, s/he might do so for approval or as not to insult anyone.

Most of the subjects wrote that they felt relaxed (see summary). It is possible that since the questionnaire was presented to them at the end of the sessions, they felt relaxed now that it was over.

CHAPTER VI

SUMMARY

1. Subjects

The subjects in this research were selected in the following way: The school children were selected by their teacher, who was asked to suggest average math students. The adults were selected on the basis of having no mathematical background beyond high school. All three adults are quite successful in their occupations. Linda is the manager of a sports boutique, Lucy is a medical secretary, and Uri is a well known tourist guide.

2. Evidence Related to Questions for Research

While formulating our research questions, we looked at the characteristics of adults vs. those of school children, as well as the nature of school mathematics and its influence (or lack of influence) on the school children and on the adults. Our questions were explored in this research by observing the problem-solving behaviour of our subjects. Because of our small sample, our results are not conclusive.

We have tabulated the data, organizing the questions for research across subjects in one table and the problems across subjects in another table. We have also commented on the data and the evidence, and the reader can refer to the tables at his or her convience. The following is a summary of our results.

(1) Non-Standard Interpretation

We tested to see who would tend to interpret the problems in a non-standard way - the adults or the school children. We asked if the adults would have this tendency.

There were five instances of non-standard interpretation in this research, three of those exhibited by adults. No noticeable difference between the adults and the school children was observed. The problem where three non-standard interpretations occured (two by adults, one by a child) was #1(a) (the cubes problem).

In some instances, this non-standard interpretation was a result of the subject's own individual problem-solving style. An example is Uri, who interpreted the cubes problem #1(a) in a non-standard way because he was always looking for patterns.

(2) Recognition of Unrealistic Data

We asked if the adults might notice unrealistic elements in the problems more often than the school children would. These elements included data rendering an unrealistic solution, as in the mother-daughter problem, or contradicting data, as in the cubes problem.

There were four instances of recognition of unrealistic data in a problem in this research. Two of them were exhibited by adults. The other two (by school children) were not expressed clearly. Both school children (Judy in the mother-daughter problem, and David in the cubes problem) claimed to have initially realized the unrealistic elements, although they did not exhibit this realization. David realized the situation by himself during the third session. Judy was told that the solution was unrealistic, and she then claimed to have known it.

One adult, Uri, put realistic restrictions on the mother-daughter problem. Before starting to solve the problem, Uri claimed that the mother can not be younger than fifteen years old, and conducted his trials with this in mind. He did not realize that the solution was unrealistic because he never got to this solution.

The other adult, Lucy, had a vague realization that something was wrong (in the cubes problem) but she assumed personal responsibility for it.

In the mother-daughter problem, no subject came close to the actual solution, and as a result they did not realize that the data led to an unrealistic solution.

Two subjects (Uri and Judy) interpreted #1(a) in a non-standard way, and continued with these interpretations in #1(b). Therefore, for them, #1(b) held no contradictions.

The adults seemed to be more aware of realistic restrictions on the given information than the school children were, since the two instances of recognition of unrealistic details by adults were more definite and obvious than those of the school children

(3) Managerial Decisions

Managerial decisions are consciously made strategic decisions and choices that often change the nature or direction of solution paths or assess the progress in a local as well as global way. Because the subjects do not always articulate about what they are thinking or doing, it is difficult to detect managerial decisions. There can be instances of such decisions made that are not clear or articulated and therefore are missed by the interviewer. Another difficulty with detecting managerial decisions is that our problems were not complex enough for such behaviour to surface.

We thought that the adults would tend to make more managerial decisions than the school children. There was not much evidence of managerial decisions during our sessions because of the difficulties in detecting such evidence. There were only two detected instances of managerial decisions taken by subjects, both of them by adults, and both in the same problem (the parking-lot problem). In one instance, Lucy articulated her intent to re-evaluate her thoughts, and went on to try another strategy that turned out to be a successful one. In the other instance, Lucy articulated some criticisms of the current strategy and also changed to a successful one. It is worth noting that both of these managerial decisions led to a successful solution of the problem, which supports Schoenfeld's contention that managerial decisions "make or break" a solution effort. In this research, the instances of managerial decisions discovered by the researchers were taken by adults only.

(4) Holistic Approach

We asked if the adults would be more likely to approach a problem holistically. This question is also a difficult one to test because the subjects do not always articulate their thoughts and actions, and it is not always clear how they initially look at the problem. Some subjects (David in the pond problem) initially approached the problem linearly and then changed their interpretation to an holistic one. Some approaches to a problem were neither holistic nor linear.

There were nine instances of an overt and initial holistic approach to a problem. Five of these were exhibited by adults. In four of these the adults went on to solve the problem. In all the four instances involving the school children, although their approach was an holistic one, they did not succeed in solving the problem. One school child, David, exhibited an holistic approach by comparing the parking lot problem to a "two-combination lock". But in David's case, although his comment exhibited an holistic approach, he did not work on the problem using this approach. He made only one try at the problem and did not continue.

There were six instances of an overt linear approach to a problem, and only one of these by an adult. Four of these linear approaches (the one by an adult included) were all in the same problem (the square-cutting problem). The remaining two instances were in the cube problem. It is worth noting that the school children approached the problems in a linear way more often than the adults did.

(5) Diagram Usage.

We asked whether the adults would be more likely to use diagrams when solving a problem. There were eleven instances of diagram usage in this research, five of them were done by adults. This shows no differences in diagram usage between the adults and the school children.

The problems that seemed to elicit the most diagrams were the two circles and the square-cutting problem. One child, David, used a diagram to solve the racing-car problem. One adult, Linda, used a diagram to solve the pond problem (between the second and third sessions without the presence of the interviewer).

(6) Algorithm Usage

We asked whether the school children would be more likely to use algorithms. There were seven instances of algorithm usage, six of them attempted by school children.

This shows a noticeable difference in algorithm usage between the adults and the school children. The school children used equations or mentioned equations more often. In the mother-daughter problem, all three school children used equations, while only one adult, Lucy, made an attempt at an equation (and a very weak attempt at that). Another adult, Linda, recognized this problem as an "algebraic type", but made no attempt at equations.

In the parking-lot problem, two school children mentioned the use of algorithms but no adults did. One of the school children, Judy, claimed that one could use equations in this problem. Most of the subjects exhibited a dislike for equations, especially the school children.

The school children often attempted or were aware of algorithms, while the adults exhibited more "common-sense" behaviour and were aware of realistic restricitions (Uri in mother-daughter problem) rather than algebraic ones.

Some adults, (Lucy, Uri in S-P problem) exhibited an understanding of the situation but were unable to translate it into an equation form. One of the school children (in the S-P problem) also exhibited an understanding of the problem but was not able to write an equation.

(7) Frustration

We asked if the adults would be more likely to exhibit frustration with any ambiguities or generally in their attempts to solve the problems.

There were two instances of an overt exhibition of frustration in this research, both by adults. These took the form of claims of false data, suspicions about improper wording, and a general distrust of themselves or of the problem. One instance of frustration, by Lucy, had a valid reason. Lucy felt that something was wrong in the cubes problem (#1(b)) where there really was a contradictory situation. Therefore her frustration was well founded. She exhibited a distrust of herself and eventually asked to go to the next problem.

(8) Intuition

We asked whether the adults would be more likely to use their intuition and less likely to trust it. This is a difficult question to explore, because subjects do not want to admit to intuitive feelings (because of the negative connotations attached to intuition) so they do not mention them.

There were three observed instances where the subject made intuitive statements and decisions, but there are bound to be more instances that escaped our notice.

In one of these instances, the subject (a school child, Robert) exhibited a trust in his intuition. After giving a solution, Robert felt intuitively that it was wrong, and he continued in his efforts. He could not explain why it was a wrong solution, saying that he just knew that it was wrong. In the other two instances, the subjects (Lucy, an adult, and David, a school child) exhibited a distrust of their intuition.

(9) Articulation

We thought that the adults might be better at articulating about the problems. No attempt has been made by us to analyse the subjects' articulations systematically, and therefore any of our statements concerning this subject are made from general observations only.

We have looked at articulation as a verbalization of the subjects about their thinking processes while they were solving the problems. We were interested in their external and verbal expressions about what they were doing or thinking.

All the subjects had some difficulty in articulation because of a lack of knowledge in mathematical language and generally because of inexperience in articulating about a mathematical situation.

Some subjects came up with clear and interesting comparisons which affected their articulation by enhancing and clarifying it. Uri, in the racing-car problem, compared the track, to a wire that can be stretched or bent as in the diagram. David compared the situation in the parking-lot problem to a two-combination lock, which communicates his understanding of the problem effectively.

Table 1 Research Questions

	(1) Non-Standard Interpretations	(2) Recognition of Unrealistic Data	(3) Managerial Decisions	(4) Holistic Approach	(5) Diagrams	(6) Algorithm Usage	(7) Frustration	(8) Distrust or Trust of Intuition
Adults								
Uri	#1(a) Faces of cube labelled in order of alphabet in a clock-wise or counter clockwise order.	#2 puts realistic restrictions on ages before he works with them.		#8 puts restrictions on A.	#3 #6			
Linda	#1(a) "face" as facing her, "opposite" as next to.		#7 evaluates her efforst and changes strategy	#7 works with both conditions	#5		#5 Claims insufficient and false information, improper wording.	
Lucy	#2 assumes a constant ratio between the ages.	#1(b) vague realization that something is wrong. Assumes it is her fault.	#7 criticizes her attempts and changes her strategy.	#1(a) works with all 3 views #8 realizes relationship between letters #7 works with both conditions	#3 #6	#2 Makes a weak attempt at an equation "x x 3". Does not continue.	#1(b) Weak evidence of frustration. After many trys, she asks to go on to the next problem.	#1(b) Distrusts her intuitive feelings that something is wrong.
Adults	3 instances	2 instances	2 instances	5 instances	5 instances	1 instance	2 instances	1 instance
School Children								
David	#9 Interprets that each counter has all three colours.	#1(b) In third session realizes contradiction in data. Claims he has noticed it before.		#7 Exhibits an instance of holistic approach but does not work in an holistic way.	#3 #14 draws track himself before choosing the answer. #6	#2 Starts by using an equation. #7 Claims he needs a definite algorithm, that he can not just guess.		#1(b) Distrusts his intuition, he attempts to explain away the contradiction.
Judy	#1(a) "opposite", one side of a cube opposite another. "Opposite" as next to.	#2 Claims she realized that solution is unrealistic in third session.		#1(a) Realizes position of "A" on the cube. #8 Puts restrictions on A.	#6	#2 Starts by using equations. #7 Mentions that this can be done by using 2 equations.		
Robert				#1(a) Works with one view and uses other two as references.	#3 #6	#2 Starts by using an equation. #5 Attempts to develop an "equation-like" short-cut.		#5 After giving an answer, feels that it is wrong and continues in his efforts. He trusts his intuition.
	2 instances	2 instances	none	4 instances	6 instances	6 instances	none	2 instances

Table 2 Problems

	#4 Triangles	#3 Circles	#1(a) Cubes	#2 Mother-daughter	#8 Crypt	#14 Graph	#6 Square-Cut	#5 Pond	#1(b) Cubes	#9 Counters	#7 Parking-lot	#12 Student-Professor
Uri	Partial Sees 4 triangles	Nil Changes problem. Assumes equal radii of 1½ each.	Nil Non-standard interpretaion.	Nil Disregards one of 2 conditions.	Complete Solves this problem.	Nil Does not interpret graph accurately.	Nil Does not have clear concept of a square and of "cut".	Nil Linear approach	Nil Non-standard interpretation	Nil Changes problem. Assumes all statements are true.	Nil Disregards one of two conditions.	Nil Understands meaning. Can not write an equation.
Linda	Partial Sees 2 triangles.	Nil Assumes equal radii.	Nil Non-standard interpretation.	Nil Claims she does not know the solution and goes on to the next problem.	Nil Claims she does not know how to solve this.	Nil Does not understand problem.	Nil Difficulties in recentering Does not try any diagrams.	Nil Claims missing data. (measurement) Linear Approach	Was not given this problem.	Nil Changes problem. Assumes all statements are true.	Complete Solves problem.	Nil Assumes there is an unknown.
Lucy	Partial Sees 2 triangles.	Nil Assumes equal radii.	Complete Solves the problem.	Nil Non-standard interpretation.	Complete Solves this problem.	Partial Interprets graph accurately but does not know shape of track.	Nil Does not have clear concept of a square. Linear approach.	Nil Disregards one condition, linear approach	Partial Realizes something is wrong but does not see contradictions.	Nil Changes problem. Assumes all statements are true.	Complete Solves problem.	Nil Understands meaning but can not write an equation to represent it.
David	Partial Sees 4 triangles.	Complete Notices missing data.	Complete	Nil Disregards one of 2 conditions.	Nil Assumes A=1, B=2, Claims insufficient information.	Complete Solves problem.	Partial Gives solution but goes on to a linear approach.	Complete Initial linear approach. Success in solving problem.	Partial In third session realizes contradiction. Claims he noticed it before.	Nil Non-standard interpretation.	Nil Understands problem but refuses to go on with his trials.	Nil Understands meaning, can not write an equation.
Judy	Partial Sees 4 triangles.	Nil Does not understand.	Partial Non-standard interpretation. Shows understanding of views. Knows where "A" is.	Nil Uses equations. Later claims she saw the contradiction.	Partial Puts restrictions on A, but fails to progress.	Complete Solves problem.	Partial Finds solution in third session. Shows a linear approach previously.	Nil Assumes a constant daily growth. Disregards one condition.	Nil Non-standard interpretation.	Nil Changes problem. Assumes all statements are true.	Nil Makes random guesses and discards them all.	Nil Can not write an equation.
Robert	Partial Sees 4 triangles.	Partial Notices missing data but assumes there is an answer.	Partial Works with all views at once. Gives incorrect answer.	Nil After trying one equation, asks to go on to next problem.	Complete Solves problem.	Nil Does not interpret graph accurately.	Nil Unclear concept of a square. Linear approach.	Complete Initially claims missing measurement. Succeeds in solving problem.	Nil Is not working with all three views.	Nil Changes problem. Assumes all statements are true.	Nil Disregards one of two conditions.	Nil Can not write an equation.

3. Other Observations

In addition to the questions described above, other factors surfaced from the protocols, and as a result other questions can be formulated and other topics can be explored.

(a) Individual Styles

While analysing the protocols, it became obvious that the individual problem-solving styles of the subjects created more of a pattern than the style of solving one problem across the subjects. Three subjects had certain individual techniques and characteristics that they exhibited throughout their interviews. Linda exhibited a distrust and disbelief in the data throughout her sessions. She often felt that she had not been given all the information, and even suspected that she had been lied to. Uri tried to impose patterns upon some problems (even when the problems did not contain any patterns), arranging the given data in a pattern when patterns were irrelevant to the problem. David exhibited a reluctance to use trial and error, saying that it would be a waste of time. When he did not find the solution immediately on his first trial, he did not try any other possibilities, but simply stopped in his attempts. An interesting project would be to analyse these individual styles and their effectiveness in problem solving.

(b) Need for Measurement

Some subjects expressed a need for measurement in problems where measurement was irrelevant. There were five instances of an expression of a need for measurement, three of them by adults. One of the adults (Lucy, in the square-cutting problem) brought the subject of exact measurement up, but she admitted that it was not necessary for that problem. The two problems where the need for measurement was exhibited were the pond problem (one adult and one school child) and the square-cutting problem (two adults, one school child).

There are three related topics that can be explored, surfacing from our observations. An interesting project would be to explore the need for measurement of adults vs. that of school children. In our research, the adults and the school children exhibited a relatively equal need for measurement. It would be interesting to observe a larger sample as to who needs measurement more, adults or school children. It may be that there is no difference between them in this respect.

It is interesting to note that the adults and the school children exhibited this need for measurement while working on two particular problems. Another project would be to explore if adults and school children generally need exact measurement in the same types of problems (where measurement is irrelevant).

One can also research the types of problems that lead to a general need for measurement without comparing adults and school children. Is there a particular type of problen that renders a subject to ask for exact measurements even though measurements are irrelevant to its solution?

(c) Visualization

During the attempts at solving certain problems, the subject of visualization came up, sometimes as a response to the interviewer's questions relating to visualization and at other times brought up by the subject him/herself.

There were four instances (two by adults) of subjects saying that they were visualizing. Of these, three were a direct response to inquiry by the interviewer. In all four instances, the subjects did not draw a diagram, rendering the interviewer to ask them if they were visualizing. Because the interviewer inquired about visualization only in these instances where diagrams were expected by the interviewer but not drawn, we do not know whether visualization occured in other instances.

In the questionnaire given at the end of the second session (question #8) five subjects (three adults) answered that they visualized while thinking about the problems. Only one subject, David, said that he did not visualize. It would be interesting to explore the visualization during problem solving of adults vs. that of school children.

4. The Questionnaire

We have tabulated the responses to the questionnaire given at the end of the second session. These responses show us how the subjects see themselves in relation to the interviews, to mathematics, and to their general competence. The questionnaire has given us some feedback as to the feelings of the subjects, as well as having given them a chance to voice certain opinions. There were some differences between the adults and the school children, although for the most part their responses were alike.

Two subjects, David and Robert, claimed to have seen some of the problems previously, but this did not seem to have affected their performance.

In response to the question about their expectations before the interviews, four subjects (2 adults) claimed to have had no expectations. The other two subjects (1 adult) had expected to be able to solve "math problems" or to do "more problem solving". They had expected to be given standard math problems.

It is interesting to note that none of the subjects looked at the problems as an obligation. Only one subject (a school child) looked at the problems (sometimes) as an unpleasant task. Only one subject (an adult) looked at them as an opportunity to be creative. Five of the six subjects looked at the problems as a mystery (2 adults) and four as a test of intelligence. One of the four was not too sure, writing "I wish ~~it~~ would" (be a test of intelligence). More school children than adults looked at the problems as a test of intelligence.

Adults and school children both had the same level of relaxation. They were all relatively relaxed, claiming that they had felt more relaxed as they got used to the problems. One adult said that she was very relaxed. As this questionnaire was given at the end of the sessions, by that time the subjects might have felt more relaxed.

All the subjects felt that the problems were interesting and a challenge. Only one subject (a school child) did not find the problems fun. Two subjects found the problems intimidating (one adult and a school child). Only two subjects (both school children) did not find the problems confusing and difficult. More adults than school children found the problems confusing and difficult. It is possible that the adults were more willing to admit confusion than the school children. This seems to contradict the idea that adults have more to lose in admitting weakness and therefore are afraid to admit to feelings of confusion.

All the subjects showed a high level of interest (three adults and two school children rated their level of interest at 1 or $1\frac{1}{2}$). This high level of interest relates to the subjects finding the problems interesting or challenging, and not looking at the problems as an obligation or an unpleasant task.

Most of the subjects did not see themselves as competent. Only one subject (a school child) rated himself at level 2. Two adults rated themselves as not at all competent (level 5). The adults seemed to feel less competent than the school children.

Most of the subjects rated themselves in the middle of the scale from very confident to frustrated. One subject (an adult, Linda) rated herself as frustrated (level 5), and she did exhibit frustration throughout her sessions.

Of four subjects who claimed to use numbers when thinking about the problems, three were school children. The evidence did not show that the school children were using more numbers; but it seems possible that they related the use of numbers to the use of equations and algorithms. The analysis of our protocols showed that the school children used more equations and algorithms than the adults. (See Research Questions Table).

Slightly more adults claimed to visualize (three adults, two school children) and to verbalize (two adults, one school child) while working on problems. Only half of the sample claimed to verbalize while thinking about the problems.

It is interesting to note that the adults found math hard and complicated (two adults rated math at level 5, and the third wrote that math is complicated as well as full of common sense), while all the school children rated math as relatively easy and full of common sense (levels $1\frac{1}{2}$ and 2).

There is here a noticeable difference between the views of the adults and of the school children towards mathematics.

Questionnaire

Looking at the problems as:

Adults	An Obligation	An opportunity to learn	An unpleasant task	An opportunity to be creative	A mystery	A test of intelligence	Level of relaxation 1...5	Level of interest 1...5	Level of competency 1...5	Level of confidence 1...5
Uri	-	Yes	-	so, so	Yes	"I wish it were."	3 "Very, after my second reply"	1	5 "Not at all at the start of questions"	3
Linda	-	-	-	-	Yes	-	1	1	5	5
Lucy	-	Yes	-	-	-	Yes	3 felt giving wrong answer	1	$3\frac{1}{2}$	4
School Children										
David	-	-	-	-	Yes	Yes	2	1	3	3
Judy	-	Yes	sometimes	-	some	"in a way"	3	2	4	3
Robert	-	Yes	-	-	Yes	-	3	$1\frac{1}{2}$	2	$3\frac{1}{2}$

Questionnaire

Looking at the problems as:

Adults	Interesting	Fun	Confusing & difficult	Challenge	Intimidating	Math is: Easy-hard 1...5	Seen problems before	Verbalize	Visualize	Use Numbers
Uri	Yes	Yes	Yes	Yes	-	"Common sense but also complicated"	No	-	Yes	-
Linda	Yes	-	Yes	Yes	Yes	5	No	Yes	Yes	Yes
Lucy	Yes	Yes	Circled word confusing	Yes	-	5	No	Yes	Yes	-
School Children										
David	Yes	Yes	-	Yes	-	2	Yes, #1a, 2, 4	Yes	-	Yes
Judy	Yes	Yes some were	-	Yes some were	-	2	No	-	Yes	Yes
Robert	Yes	-	Yes	Yes	Yes	$1\frac{1}{2}$	Yes, #7 animal people #12 equation	-	Yes	Yes

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