PROOFS OF EQUIVALENCE

AND

EQUIVALENCE OF PROOFS

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A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirement
for the degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

January, 1976

ABSTRACT

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In this thesis we study the notion of equivalence of proofs restricted to "proofs of equivalence".

We present empirical evidence for the existence of three distinct levels of equivalence:

- Strong
- Tauberian
- Weak

and give a syntactic characterisation of these notions. We develop an example from each level and show that it satisfies the criteria.

Finally we discuss the problems associated with the cut elimination algorithm for systems with equality and mathematical axioms.

ACKNOWLEDGEMENTS

I am endebted to Dr. M.E. Szabo for all the inspiration, assistance, moral support, constructive criticism and overall guidance that he gave me from the inception to the finish of this work. He also generously allowed me access to his manuscript and many unpublished works in his possession.

Thanks are also due to Fran Rousseau who spent many long evenings "cultivating trees" so that this thesis could be read by others.

Special thanks to Maggi, Muffin and Jason, my human and animal room-mates for their non-mathematical contribution to this paper.

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CHAPTER I

INTRODUCTION

1: *Historical Background.* Studies in the foundations of mathematics were concerned at first with the syntax of the subject. A syntactic approach seemed more plausible and amenable to a mathematical investigation. Frege, Russell, Hilbert and others formulated some deductive systems which were intended to formalize mathematical proofs. Soon, many results were known about these systems but the question as to whether or not they encompassed the proofs of "actual mathematics" was still unsettled. One way to answer this question was to obtain completeness and consistency results for these systems, since these kinds of theorems can be understood as efforts to determine the extent to which provability measures truth. Gödel's 1931 incompleteness theorem put a stop to this approach but did not settle the question.

A different, more semantic approach was developed. Logicians sought representation theorems for semantics, such as the Stone representation theorem, and more recently categorical interpretation of proofs (*see* [SI-VII]). Other concepts, such as the notion of "validity of derivations" developed by Prawitz in [PII], join the general effort to justify the deductive systems used. We believe that another way of helping to solve the question is to investigate actual mathematical proofs as they are produced by the "working mathematicians".

In the writings of Russell and Frege, a *number* is an equivalence class of sets (*see* [HI]). There is no ambiguity about the notion of equivalence used, and these classes are studied to gather information about numbers. In the same sense, we can think of a *proof* as an equivalence class of derivations. This idea is implicit in Kreisel [K,I] and Prawitz [PII]. However there is no accepted notion of equivalence of proofs.

One of the basic questions of interest in proof theory is to decide when two proofs are equivalent. This problem was approached by many logicians and several criteria have been developed. These criteria are usually motivated philosophically, like the criterion that two proofs should be equivalent only if they have the same spirit, namely if they use the same kind of ideas; and mathematically, like the criterion that demands that the rules of inference of the system preserve the equivalence.

But, although these criteria have merit, they have evolved from reflection *upon* proofs rather than from direct empirical studies.

The desirability of an empirical type of investigation was suggested by the late Professor A. Robinson during a discussion about the future of proof theory at the Orleans Logic Congress in 1972.

In this thesis we propose to start such an investigation.

2: *The Equivalence of Proofs.* Many approaches have been used to attempt to partition the class of mathematical proofs.

One of the first was the notion of the "scope of a proof" introduced by J. Lambek in [LI] and soon abandoned for the notion of "generality of a proof" in [LII]. This concept was revived by

M.E. Szabo in [SI] but had to be discarded later in [SII] for it failed to have the desired universality and invariance properties necessary for the analysis of some classes of proofs, even elementary ones.

The concepts of "graph and co-graph" of a proof were introduced by Szabo in [SII] as partial substitutes for generality, and although these concepts alleviate the problem of keeping track of particular subformulas throughout the proof and distinguished proofs considered non-equivalent, it did not produce the general theorem that one hopes to get.

At present the state of things seems to be that one uses techniques largely developed by Gentzen, like the cut elimination theorem, and through a manipulation of the class of derivations of the particular system, gets a Church-Rosser property, so that each equivalence class has a unique representative of a special kind. This is the approach of Szabo in [$SIII$] and [SVI].

The problem of equivalence of proofs is related, and in some instances equivalent, to the problem of coherence in categories. (For a definition of coherence, see [SIV].)

The problem is essentially that of deciding when a diagram commutes, or whether it is possible to partition the class of arrows of a category sensibly into classes closed under composition, and with the commutativity property, i.e. every diagram made up of arrows from the same class commutes.

Questions of coherence were first raised by S. MacLane who, together with Kelly and others, attempted to answer them using the idea of graphs and extended graphs. In the difficult cases, however, their efforts have led only to partial results. It seems that Szabo's method of representing arrows by derivations and using the cut elimination process together with a Church-Rosser type of reduction (familiar from combinatory logic and the theory of λ-definability), is in many instances, the only viable approach.

3: *Proofs of Equivalence*. Since our study is based on empirical investigations of unbounded scope, we found it necessary, in order to make the data manageable, to restrict our considerations to a special class of mathematical proofs.

We have chosen "proofs of equivalence", i.e., cycles, proofs of the type (A iff B), (A iff B iff C), etc. . ., for several reasons:

- they are an easily defined class.
- they occur in all branches of mathematics with a fairly high frequency.
- they are related to the notion of equivalence (for (A iff B) really says that A and B are equivalent as statements; and further, that a proof of A can be converted via the implication $(A \to B)$ into a proof of B and vice versa).

And so, throughout this paper, we shall deal only with proofs of equivalence (i.e., proofs of the kind: from A we may deduce B and from B we may deduce A).

The notion of a proof, and more particularly of a proof of equivalence, will be defined formally in *Chapter 11*.

4: *Proofs of Equivalence by Levels*. We shall distinguish between three levels of proofs of equivalence:

- *STRONG EQUIVALENCE*
- *TAUBERIAN EQUIVALENCE*
- *WEAK EQUIVALENCE*

Here we shall give only an intuitive sketch of these levels. A formal definition will be given in *Chapter 11*.

In looking at mathematical statements of equivalence, certain concepts seem so "strongly equivalent" that they are almost interdefinable. For a typical representative of the strong equivalence level, recall a proposition of elementary group theory:

Let $f: G \to H$ a group homomorphism

f is monic iff $\ker f = \{e_G\}$ where e_G is the identity element of G.

On the other hand, some concepts are not equivalent, but become so when an additional condition is assumed. This sort of equivalence we call "Tauberian Equivalence" after the mathematician A. Tauber and in the sense of Rudin in [R1]. A Tauberian theorem is one in which $(A \to B)$ is provable, $(B \to A)$ is false, but $((B \text{ and } C) \to A)$ is true for some "minimal" C, so that although (A iff B) is false, ((A and C) iff (B and C)) is a theorem. The following theorem from topology is a typical representative of the "Tauberian Equivalence" level:

A metric space is compact iff it is countably compact.

As we know, compactness implies countable compactness, but the converse is false for arbitrary topological spaces.

Finally, hard problems in mathematics give rise to a third level of equivalence, the "weak equivalence". Very often, in order to comprehend a situation, one has to translate it into an equivalent form which is more familiar. We shall consider the cycle of equivalent forms of the axiom of choice as the representative of this level:

Axiom of choice iff Zermelo's principle iff counting principle iff multiplicative principle iff Zorn's Lemma etc.

We note here that the majority of these equivalent forms evolved in the days when attemps were made to deduce the axiom of choice from the other axioms of set theory, which we now know to be impossible.

5: *Strong Equivalence.* From the search we have carried out in various branches of mathematics, a few observations about the strong equivalence level can be made:

Proofs of strong equivalence tend to occur at the beginning of the development of a theory. Indeed, when we start working we usually try to define the concepts involved and it is only natural to try and relate these new concepts to ones already known.

This level of proofs is found in all branches of mathematics, with the exception perhaps of Analysis (here we mean Analysis in the narrower sense, not including topology etc. . .). It appears that, because of the eclectic nature of analysis as a theory, the richness of each individual concept, and the lack of a good axiomatisation, the concepts stand remote from each other, and interdefinability is out. The two examples in the Appendix were, in fact, the only ones found.

A case however should be made for Analysis or, more generally for subjects that are so rich and fertile that some of their topics break away and become branches of mathematics in their own right. M.E. Szabo conjectured during a discussion about the three levels of equivalence that,

"The canonicity of a theory is directly related to the paucity of equivalence, and the creation of new subjects from the theory to the richness of equivalence".

If one considers the number of proofs in each level, proofs of strong equivalence are by far the most abundant.

The notion of truth has a fairly natural relation to the various levels. In the case of strong equivalence, the truth value of each statement carries practically no weight. Here one merely proves that statements A and B are equivalent each of which may be true or false. In no way does a proof of strong equivalence attempt to establish the truth of one of the statements, while, as we shall see, a proof of weak equivalence is often a device used to overcome the difficulty of proving the truth of an assertion.

A last fact noticed about the strong equivalence level is the simplicity of the proofs involved. This is related to the place in which one finds these proofs in a theory, which as we saw is the beginning of the theory. The concepts in each of the statements are fairly elementary and the proofs are therefore simple, almost mechanical.

6: *Tauberian Equivalence*. In the history of a theory we encounter proofs of Tauberian equivalence towards the middle of the development, for obvious reasons. We have to first establish certain concepts, build some "machinery" which helps prove, for example, that $(A \to B)$ is true and $(B \to A)$ is not, and that therefore A and B are not equivalent, then we search for the conditions under which they are the same. Also, sometimes we define structures with particular properties and as it happens notions that are in general different, coincide in these particular structures.

It is interesting to compare this with some situations that arise in mathematics and particularly in logic. Some properties hold in certain weak structures, not so much because of features the structures have, but more because of features they are lacking. This is the case of the completeness theorem for predicate calculus. In Tauberian equivalence it is not so: we need an additional assumption to prove a theorem, i.e., a property the structure has.

Although proofs of Tauberian equivalence are fairly common to all branches of mathematics, it is in analysis that they are most often found. The reasons are not very clear, but one can try to understand them by realizing that proofs of strong equivalence are quasi-inexistent in analysis, hence that concepts that stand apart come together eventually.

Proofs of the Tauberian equivalence level are less frequent than proofs of the strong equivalence level. They occur only as a complement to an already established fact.

Here, truth plays a more important role than in the case of strong equivalence. By adjoining an assumption and changing a false implication into a true one, we are in fact establishing that the truth of the final statement is dependant upon the adjunction of the of the additional assumption.

If we consider that proofs of Tauberian equivalence are of the form ((A and C) iff (B and C)), where (A → B) is true and (B → A) is false, then it appears that ((B and C) → (A and C)) is a more complex proof than (A → B). Since the concepts here are more involved and at least half of each proof is somewhat "unnatural", the proofs investigated turned out to be considerably more complex than proofs of the strong equivalence level.

7: *Weak Equivalence*. Proofs of this level are found solely at the sophisticated levels of a theory. Usually a certain assertion either forces itself upon the mathematician through the need for tools, or is formulated as an interesting conjecture. The task of proving such a statement is not always easy and, one way to confront the difficulty may be to transfer the problem to an area which is more manageable, given the expertise of the mathematician or the state of development of the area to which the problem has been transfered.

Decidability and undecidability of a theory are good examples. The best tool to prove the undecidability of a theory is to reduce it to one known to be undecidable, by means of an equivalence. In every branch of mathematics, there are "prototypes" which are used as references in trying to establish statements about an object in the theory. For example, when testing a series for convergence we may compare it to $\Sigma\, ^1/n$ or $\Sigma\, ^1/_{n^2}$, known divergent and convergent series respectively.

Proofs of weak equivalence are sometimes used to make certain statements plausible, add credibility and give respectability. For example, the proofs that λ-definability iff Turing computable iff recursive, have made Church's thesis more acceptable.

Sometimes, because they transfer the problem to an area in which we have a stronger intuition, proofs of weak equivalence make the way to the solution easier. In fact, in the conclusion of his "Set theory and the continuum hypothesis", Paul Cohen writes:

"*A point of view which the author feels may eventually come to be accepted is that* [*the*] CH [*continnum hypothesis*] *is* *obviously false*."

And this because we will have enough equivalent forms of the continuum hypothesis, one of them in an area where our intuition is very good.

We found proofs of weak equivalence in all the branches of mathematics with a little higher frequency in the branches that are not very well axiomatised. In fact, these proofs establish almost invariably the equivalence of a statement from a not so well axiomatised and developed field to a statement from a better axiomatised and usually older branch of mathematics.

Overall, they are less frequent than proofs of the other two levels.

Truth is an important motivation for these propositions. It is in the search to establish an assertion that one arrives at formulating equivalent problems, since establishing the truth of one will imply the validity of the others.

Coming from a sophisticated level of the theory, the proofs are much more complex. They very often involve ad hoc constructions and techniques that are unconventional. These equivalences therefore tend not to be obtained mechanically.

§: *Procedures of Chapter II and III*. In the following chapters we attempt to define formally each of the levels intuitively introduced above, and proceed to work out one example from each level. For reasons that we shall discuss in the next section we had to restrict ourselves to first order predicate logic. Therefore the examples are simple. However, we discuss ways to extend the basic language and the changes necessary to permit work within natural mathematical theories.

After establishing the language, we describe the deductive system. It will be a Gentzen classical first order sequent calculus known in Gentzen's writings as *LK* (see [S VII]). Note that there is no difficulty in dealing with an intuitionist calculus - the only difference is that the theorems involved must be intuitionistically true. Therefore, our study could have been carried out in intuitionist logic.

In the Gentzen calculus LK we represent each proof by a tree in the usual way and give a cut elimination procedure. Here we insist in writing out all steps explicitly, mostly because we have not seen it done anywhere.

We show that this procedure yields complete cut elimination by the usual induction methods, and we obtain a Church-Rosser property for the calculus.

For the purposes of the Church-Rosser property our cut elimination procedure is presented as a reducibility relation (>) on trees.

We then look at proofs of equivalence. As we define them later, proofs of equivalence on cycles of length 2 (A iff B, for example) turn out to be pairs (δ_1, δ_2) where δ_1 is a proof of $A \to B$ and δ_2 is a proof of $B \to A$. Using the cut rule we obtain two new derivations:

$$1/\quad \dfrac{\overset{\delta_1}{A \to B} \qquad \overset{\delta_2}{B \to A}}{A \to A}$$

$$2/\quad \dfrac{\overset{\delta_2}{B \to A} \qquad \overset{\delta_1}{A \to B}}{B \to B}$$

By applying the reduction procedure to these two derivations, they may or may not collapse to the axioms $A \to A$ and $B \to B$ (or to identities if one thinks of the categorical interpretation of these systems).

If both derivations (1/ and 2/) reduce to axioms we say that the proof of equivalence is of the *strong level.*

If only one derivation reduces to an axiom, we say that the proof of equivalence is of the *Tauberian level.*

If neither of the derivations reduce to an axiom we say that the proof of equivalence is of the *weak level.*

We pause for a moment to see what is going on. The strong equivalence level involves interdefinability, simple concepts and simple proofs and therefore the trees will collapse to two identities. The Tauberian equivalence involves one general and fairly straight forward proof which collapses to an identity and another proof which needs an additional assumption, which as we show in our example in *Chapter III* cannot collapse to an identity. Proofs of weak equivalence always have some ad hoc constructions and there is little chance that they reduce to identities (in fact, the trees that formalize these proofs are so huge and "scary" that one could quite honestly spend a ~~week or~~ so trying to reduce a single branch).

9: *The Difficulty of > with Natural Mathematical Theories.* If one stays with first order predicate logic, there are no problems and the work described above can be carried out nicely. The key is that the only axioms allowed in the deductive system are of the form $A \to A$ where A is an arbitrary formula of the language. Unfortunately, to do mathematics we need also some axioms of a different form. Usually, they are of the form $(\forall x)\varphi(x)$ or $(\exists x)\varphi(x)$, where $\varphi(x)$ is a formula of the language involving x as a variable, which in a Gentzen type system are interpreted as: $\rightarrow (\forall x)\varphi(x)$ and $\rightarrow (\exists x)\varphi(x)$. Naturally enough, we would like to allow these at the top of any branch of a tree. However, in this case, the cut is no longer eliminable and this of course destroys the procedure we proposed to follow.

There are various problems we encounter. One of them is that in a proof, we are really interested in a substitution instance of an axiom rather than its quantified form (eg. in group theory, we have that $(\exists e)(\forall x)(xe = x)$ is an axiom, but in a proof we actually only use the fact that $ae = a$ for some particular element a of the group). In general the problem is that getting from $\rightarrow (\forall x)\varphi(x)$ to $\rightarrow \varphi(a)$, involves a non-eliminable cut, since the simplest derivation for this is the following:

$$\frac{\rightarrow (\forall x)(\varphi(x)) \qquad \dfrac{\varphi(a) \rightarrow \varphi(a)}{(\forall x)(\varphi(x)) \rightarrow \varphi(a)}}{\rightarrow \varphi(a)}\ (\text{cut})$$

and as we shall see the reduction algorithm does not provide for such occurences of the cut. One way of aleviating this problem is to take all substitution instances of the axioms for axioms, thus $\rightarrow \varphi(a)$ becomes an axiom and we may provide in our algorithm a reduction of the above tree to $\rightarrow \varphi(a)$.

A second problem caused by mathematical axioms is the following:

Consider $\dfrac{\rightarrow A \quad A \rightarrow B}{\rightarrow B}$ (cut) where $\rightarrow A$ is an axiom. This constitutes a loss of information, for $\rightarrow B$ really states $\rightarrow B$ is an axiom, while it is only a theorem directly obtainable from the axioms. To avoid this situation we "sign" all our axioms with an antecedent so as not to allow an axiom to have an empty antecedent. We may, for example, for every group axiom write in the antecedent "G is a group", or a predicate letter P(G); so that a group axiom will be written like:

$$P(G) \rightarrow (\forall x)(x \in G \Rightarrow xe_G = x)$$

or a substitution instance thereof.

Thus the situation under discussion is not a problem anymore, now $P(G) \rightarrow B$ is at the bottom of the tree. But we cannot distinguish anymore between the axioms and the theorems directly obtainable from the axioms.

The third kind of problem relates to the cut elimination.

Consider $\dfrac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$ (cut) where $A \rightarrow B$, $B \rightarrow C$ are axioms.

This cut is not eliminable because the above is its derivation. To suppress the problem we can do one of two things.

Close the class of axioms under simple transitivity, that is if $A \to B$ and $B \to C$ are axioms then so is $A \to C$, and collapse the tree mentioned to $A \to C$.

OR

allow for mathematical axioms only sequents with a "signature" for antecedent in the sense mentioned, and a meaningful mathematical statement (not a "signature") for succedent. Then the problem does not exist anymore, because the situation does not occur.

In general when introducing new mathematical axioms in a deductive system, we should worry not only about their content but also their form, because this influences the eliminability of the cut. Consider the axioms for equality as in Gentzen's "Investigations into Logical Deductions":

$$(\forall x)\,(x = x)$$

$$(\forall x)\,(\forall y)\,(x = y \Rightarrow y = x)$$

$$(\forall x)\,(\forall y)\,(\forall z)\,((x = y \wedge y = z) \Rightarrow x = z)$$

These axioms do not perturb the cut elimination for they are not sequents but merely formulas; and, he calls a formula B "derivable" iff there is derivation for

$A_1, \ldots, A_n, \to B$ where $A_1, \ldots, A_n$ are axiom formulas.

So that the branches of all trees start with axioms of the form $A \to A$. These same axioms considered as sequents give rise to some serious problems some of which are the ones we discussed, and others like the fact that immediate consequences of the axioms using the cut are not eliminable. This particular difficulty can be treated in two ways:

> by adjoining to the axioms all immediate consequences of the cut (as done by S. Feferman in [*FI*]).
>
> OR
>
> by introducing new rules of inference (as does Lopez-Escobar in [*L-EI*]).

Gentzen's way of solving the mathematical axioms problem, is in the spirit of the "keine Umwege" (not roundabout) proofs. He does not carry any more information than is absolutely necessary to prove a statement. In the case of the "signature" method for example, we have to carry all the axioms to the bottom of the tree, regardless of the fact that we might have used only some in the proof.

To summarize, the difficulties of $>$ with natural mathematical theories can be overcome by various methods, the most elegant one due, of course, to Gentzen himself.

10: *Other Methods of Investigation*: The method described in 8 for the classification of proofs of equivalence appears to us as quite natural, and we have tried throughout this chapter to motivate it by showing how it relates to real proofs in the various branches of mathematics.

Several other approaches were suggested by M.E. Szabo, one of which, the one we believe to be the most interesting, is related to model theory and we shall discuss it here.

Given a formula A of predicate calculus, it determines a class of Heyting algebras $H(A)$, in which it is valid.

Therefore given formulas A and B and a proof δ_1 for $A \rightarrow B$, interpreting the arrow as an implication and the implication as the relative pseudo-complement, $A \rightarrow B$ determines a class of Heyting algebras $H(A,B)$.

Using similar methods to the ones for constructing counter-examples developed by Kleene and others, we may be able to build a canonical Heyting algebra for any proof δ_1 of $A \rightarrow B$, call it $H_{\delta_1}(A,B)$.

Clearly $H_{\delta_1}(A,B) \in H(A,B)$, but it is a distinguished member of this class. If we take the class of Heyting algebras modulo isomorphisms (ie. identify any two isomorphic Heyting algebras), $H_{\delta_1}(A,B)$ is unique.

We then call two proofs δ_1 and δ_2 of $A \to B$ equivalent iff $H_{\delta_1}(A,B) = H_{\delta_2}(A,B)$. Of course we have an equivalence relation with infinitely many equivalence classes, but this may be closer to reality.

If we look at $\mathcal{H} = \{H(A,B) : A,B \text{ are formulas}\}$, we can order it by inclusion, and it is not empty since $A \to A$ is always valid (i.e., the class of all Heyting algebras belongs to $\mathcal{H}$).

For any (A,B) the cardinality of $H(A,B)$ is informative as to the number of non-equivalent proofs of $A \to B$.

Some interesting theorems to look for are:

T_1: $A \overset{\delta_1}{\to} B > A \overset{\delta}{\nrightarrow} B$ and $A \overset{\delta_2}{\to} B > A \overset{\delta}{\nrightarrow} B$

iff $H_{\delta_1}(A,B) = H_{\delta_2}(A,B)$.

(Where $A \overset{\delta_1}{\to} B > A \overset{\delta}{\nrightarrow} B$ is according to the description in the previous sections).

T_2: $(A \overset{\delta_1}{\to} B, B \overset{\delta_2}{\to} A)$ is a proof of strong equivalence

iff $H_{\delta_1}(A,B) = H_{\delta_2}(B,A)$

T_3: $((A \text{ and } C) \overset{\delta_1}{\to} (B \text{ and } C), (B \text{ and } C) \overset{\delta_2}{\to} (A \text{ and } C))$

(where there is a proof δ_3 of $A \to B$ and there is no proof of $B \to A$).

is a proof of Tauberian equivalence

iff $H(A \text{ and } C, B \text{ and } C) \cap H(B \text{ and } C, A \text{ and } C) = H(C)$

T_4: $(A \xrightarrow{\delta_1} B, B \xrightarrow{\delta_2} A)$ is a proof of weak equivalence

iff $H(A,B) \cap H(B,A) = \phi$

T_1 states that the reduction algorithm and this method leads to the same classification.

T_2, T_3 and T_4 are the interpretations of our notion of equivalence under this classification.

This method would hopefully help to explain the nature of proofs through the study of classes of Heyting algebras which seem more amenable to investigation.

CHAPTER II

CUT ELIMINATION AND NORMALISATION

In this chapter, we define a formal language L adequate for first order logic. Then, we describe a deductive system LK for L, modeled after Gentzen's calculus of sequents for classical logic. For the LK derivations, we give a cut-elimination algorithm in the form of the reducibility relation " $>$ " on trees. On cut-free trees, we induce a relation " $\geqslant$ " given by a normalisation algorithm. Finally, we define rigorously our notion of equivalence described in *Chapter I*.

1. *The Language L.*

a. *Alphabet*

(i) Variables designated by:

$a, a_1, \ldots, b, b_1, \ldots, x, x_1, \ldots$

(ii) Predicate symbols designated by:

$P^0_0, P^0_1, \ldots, P^0_n, \ldots, P^1_0, \ldots P^n_0, \ldots$

where the upper index is called the *degree* of the predicate.

(iii) Logical Symbols:

$\wedge$ for conjunction

$\vee$ for disjunction

$\rightarrow$ for implication

$\neg$ for negation

$\forall$ for the universal quantifier

$\exists$ for the existential quantifier

(iv) Auxiliary symbols:

(;) ; , ; $\rightarrow$.

b. *Formulas.*

(i) Let P_k^n be a predicate symbol of degree n and $x_1, \ldots, x_n$ be n variables then $P_k^n(x_1, \ldots, x_n)$ is an *atomic formula*.

(ii) Every atomic formula is a *formula*.

(iii) If A and B are formulas then $\neg A, A \wedge B$ $A \vee B$ and $A \Rightarrow B$ are formulas.

(iv) By "A(a) is a formula" we mean that A is a formula in which a occurs as a variable.

If $A(a)$ is a formula and x a variable then $(\exists x)A(x)$ and $(\forall x)A(x)$ are formulas, where $A(x)$ is obtained by substituting x for all occurrences of a in A.

(v) Nothing else is a formula.

c. *Sequents.*

A *sequent* is an expression of the form

$A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m$ where $A_1, \ldots, A_n, B_1, \ldots B_m$ are formulas. $A_1, \ldots, A_n$ form the *antecedent* and $B_1, \ldots, B_m$ form the *succedent*. We shall call any of $A_1, \ldots, A_n$ an *antecedent formula* and any of $B_1 \ldots B_m$ a *succedent formula*.

2. *The deductive system LK*.

a. *Axioms of LK*

The only axioms of *LK* are sequents of the form

$$A \to A$$

where A is a formula of *L*.

b. *Rules of inference of LK*.

We represent the rules of inference as *inference figures*. By an inference figure we mean:

$$\text{(i)}\quad \frac{\vec{A}_1}{\vec{A}_2} \qquad \text{or} \qquad \text{(ii)}\quad \frac{\vec{A}_1 \quad \vec{A}_2}{\vec{A}_3}$$

where $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$ are sequents, (i) means from $\vec{A}_1$ we may infer $\vec{A}_2$, and (ii) means from $\vec{A}_1$ and $\vec{A}_2$ we may infer $\vec{A}_3$.

In (i), $\vec{A}_1$ is called the *upper sequent* and $\vec{A}_2$ the *lower sequent*.

In (ii), $\vec{A}_1$ and $\vec{A}_2$ are the *upper sequents* and $\vec{A}_3$ the *lower sequent*.

INFERENCE FIGURE SCHEMATA

Capital Greek letters stand for arbitrary (possibly empty) sequences of formulas separated by commas.

Lower case Greek letters stand for arbitrary formulas of L.

Lower case Roman letters stand for arbitrary variables of L.

STRUCTURAL RULES.

CUT

$$R_1 \quad \frac{\Gamma \rightarrow \Phi, \gamma, \Psi \qquad \Delta, \gamma, \Lambda \rightarrow \Theta}{\Delta, \Gamma, \Lambda \rightarrow \Phi, \Theta, \Psi}$$

THINNING

$$R_2 \quad \frac{\Gamma, \Delta \rightarrow \Phi}{\Gamma, \alpha, \Delta \rightarrow \Phi} \qquad\qquad R_3 \quad \frac{\Gamma \rightarrow \Delta, \Phi}{\Gamma \rightarrow \Delta, \alpha, \Phi}$$

CONTRACTION

$$R_4 \quad \frac{\Gamma, \alpha, \alpha, \Delta \rightarrow \Phi}{\Gamma, \alpha, \Delta \rightarrow \Phi} \qquad\qquad R_5 \quad \frac{\Gamma \rightarrow \Delta, \alpha, \alpha, \Phi}{\Gamma \rightarrow \Delta, \alpha, \Phi}$$

INTERCHANGE

$$R_6 \quad \frac{\Gamma, \alpha, \beta, \Delta \rightarrow \Phi}{\Gamma, \beta, \alpha, \Delta \rightarrow \Phi} \qquad\qquad R_7 \quad \frac{\Gamma \rightarrow \Delta, \alpha, \beta, \Phi}{\Gamma \rightarrow \Delta, \beta, \alpha, \Phi}$$

OPERATIONAL RULES

CONJUNCTION

$$R_8 \quad \frac{\Gamma,\alpha,\Delta \rightarrow \Phi}{\Gamma,\alpha\wedge\beta,\Delta \rightarrow \Phi} \qquad R_9 \quad \frac{\Gamma \rightarrow \Phi,\alpha,\Psi \qquad \Gamma \rightarrow \Phi,\beta,\Psi}{\Gamma \rightarrow \Phi,\alpha\wedge\beta,\Psi}$$

$$R_8' \quad \frac{\Gamma,\alpha,\Delta \rightarrow \Phi}{\Gamma,\beta\wedge\alpha,\Delta \rightarrow \Phi}$$

DISJUNCTION

$$R_{10} \quad \frac{\Gamma,\alpha,\Delta \rightarrow \Phi \qquad \Gamma,\beta,\Delta \rightarrow \Phi}{\Gamma,\alpha\vee\beta,\Delta \rightarrow \Phi} \qquad R_{11} \quad \frac{\Gamma \rightarrow \Delta,\alpha,\Psi}{\Gamma \rightarrow \Delta,\alpha\vee\beta,\Psi}$$

$$R_{11}' \quad \frac{\Gamma \rightarrow \Delta,\alpha,\Psi}{\Gamma \rightarrow \Delta,\beta\vee\alpha,\Psi}$$

NEGATION

$$R_{12} \quad \frac{\Gamma \rightarrow \Phi,\alpha}{\neg\alpha,\Gamma \rightarrow \Phi} \qquad R_{13} \quad \frac{\alpha,\Gamma \rightarrow \Phi}{\Gamma \rightarrow \Phi,\neg\alpha}$$

IMPLICATION

$$R_{14} \quad \frac{\Gamma \rightarrow \Phi,\alpha \qquad \beta,\Delta \rightarrow \Lambda}{\alpha \Rightarrow \beta,\Gamma,\Delta \rightarrow \Phi,\Lambda} \qquad R_{15} \quad \frac{\alpha,\Gamma \rightarrow \Phi,\beta}{\Gamma \rightarrow \Phi,\alpha \Rightarrow \beta}$$

UNIVERSAL QUANTIFIER

$$R_{16} \quad \frac{\alpha(a),\Gamma \rightarrow \Phi}{(\forall x)\alpha(x),\Gamma \rightarrow \Phi} \qquad R_{17} \quad \frac{\Gamma \rightarrow \Phi,\alpha(a)}{\Gamma \rightarrow \Phi,(\forall x)\alpha(x)}$$

EXISTENTIAL QUANTIFIER

$$R_{18}\ \frac{\alpha(a), \Gamma \rightarrow \Phi}{(\exists x)\alpha(x), \Gamma \rightarrow \Phi} \qquad R_{19}\ \frac{\Gamma \rightarrow \Phi, \alpha(a)}{\Gamma \rightarrow \Phi, (\exists x)\alpha(x)}$$

Restrictions on the rules.

In R_{17} a must not occur in Γ, Φ or $(\forall x)\alpha(x)$.

In R_{18} a must not occur in Γ, Φ or $(\exists x)\alpha(x)$.

Remark.

Except for R_1, all rules of inference R_n with n odd, affect the succedent of the sequent only. In R_m with m even, the rule only affects the antecedent of the sequent.

In view of this remark we make the following definitions:

A rule of inference R_n is called an *antecedent rule* if n is even and a *succedent rule* if n is odd and greater than 1.

c. *Derivations*

A *derivation* is a non-empty, finite list of sequents (possibly with repetition) combined via the inference figures in the following way:

- Each sequent except for one (the end-sequent) is an upper sequent of exactly one inference figure.

- Each sequent is a lower sequent of at most one inference figure.

Under these restrictions, each derivation is representable by a tree, and in fact we shall use the words tree and derivation, interchangeably.

A *proof* of a sequent $\Gamma \rightarrow \Delta$ is a tree in which the end-sequent is $\Gamma \rightarrow \Delta$, and all the sequents that are not lower sequents of an inference figure are axioms. Example of a proof: the law of "the excluded middle" is provable in *LK*, i.e., there is a proof for $\rightarrow A \vee \neg A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\rightarrow A, \neg A}\,R_{13}}{\rightarrow A, A\vee\neg A}\,R_{11}}{\rightarrow A\vee\neg A, A\vee\neg A}\,R_{11}}{\rightarrow A\vee\neg A}\,R_{5}$$

We shall designate derivations by the lower case Greek letters δ and λ in the following ways: $\delta, \delta_1, \delta_2, \ldots \delta_n, \ldots$

$\lambda, \lambda_1, \lambda_2, \ldots \lambda_m, \ldots$

We write $\Gamma \xrightarrow{\delta_1} \Delta$ to mean that δ_1 is a proof for $\Gamma \rightarrow \Delta$.

d. *Proofs of equivalence.*

A *proof of equivalence* for the ordered pair (Γ, Δ) is an ordered pair (δ_1, δ_2), where δ_1 is a derivation for $\Gamma \rightarrow \Delta$ and δ_2 a derivation for $\Delta \rightarrow \Gamma$.

3. *The relation $\check{>}$.*

The relation $>$ is defined on trees, and it will be used as an algorithm which, when applied to a given tree, will in a finite number of steps eliminate all occurences of the rule R_1 from the derivation.

Some definitions are necessary:

- A *cut* is an application of the rule R_1.
- A *cut-out formula* is a formula which disappears as a result of a cut.

For example, in

$$\frac{\Gamma \rightarrow \Delta,\gamma,\Phi \qquad \Psi,\gamma,\Lambda \rightarrow \Theta}{\Psi,\Gamma,\Lambda \rightarrow \Delta,\Theta,\Phi}$$

γ is the cut-out formula.

- An application of a rule of inference is *passive with respect to the cut* if: a/ it does not occur immediately before the cut;

or b/ it immediately preceeds the cut, but does not manufacture or affect the cut-out formula.

For example, in

$$R_2 \frac{\dfrac{\alpha \rightarrow \beta}{\gamma, \alpha \rightarrow \beta} \quad \beta \rightarrow \phi}{\gamma, \alpha \rightarrow \phi},$$

R_2 is passive with respect to the cut.

- By (R_n, R_m) we mean a cut where the rule R_n was used to produce the left premise and R_m the right one.
- By $(R_n, -)$ we mean (R_n, R_m) where $m = 2$, or 3, or . . ., or 19; or a cut where the left premise is obtained by R_n and the right premise is an axiom.
- $(-, R_m)$ is defined similarly.
- $R_{(even)}$ will stand for any antecedent rule.
- $R_{(odd)}$ will stand for any succedent rule.

In view of the above definitions we have some immediate consequences:

a/ In $(R_{(even)}, -)$, $R_{(even)}$ is passive with respect to the cut. By investigating our rules of inference, we see that $R_{(even)}$ rules affect only the antecedent.

In $(R_{(even)}, -)$, $R_{(even)}$ has produced the left premise, and since the cut-out formula belongs to its succedent, it could not have been affected by $R_{(even)}$.

b/ In $(-, R_{(odd)})$, $R_{(odd)}$ is passive with respect to the cut. (The argument is similar to the one in a/).

c/ In (R_{11}, R_8), either R_{11} or R_8 is passive with respect to the cut. If R_{11} is not passive then the cut-out formula is $\alpha \vee \beta$, which could not have been affected by R_8 since all R_8 does is change some formula say γ to $\gamma \vee \varphi$ for some φ.

Similarly if R_8 is not passive, then R_{11} is.

d/ A similar argument to the one in c/ yields a list of pairs in which at least one of the members of every pair is passive with respect to the cut. By adjoining to this list the results in a/ and b/ we obtain a new list to which we shall refer as the *passive list.*

$(R_{(even)}, -)$, $(-, R_{(odd)})$

(R_{11}, R_8) , (R'_{11}, R_8) , (R_{13}, R_8) , (R_{15}, R_8) ,

(R_{17}, R_8) , (R_{19}, R_8) , (R_{11}, R'_8) , (R'_{11}, R'_8) ,

(R_{13}, R'_8) , (R_{15}, R'_8) , (R_{17}, R'_8) , (R_{19}, R'_8) ,

(R_9, R_{10}) , (R_9, R_{12}) , (R_9, R_{18}) , (R_{13}, R_{10}) ,

(R_{15}, R_{10}) , (R_{17}, R_{10}) , (R_{19}, R_{10}) , (R_{11}, R_{12}) ,

(R_{11}, R_{14}) , (R_{11}, R_{16}) , (R_{11}, R_{18}) , (R'_{11}, R_{12}) ,

(R'_{11}, R_{14}) , (R'_{11}, R_{16}) , (R'_{11}, R_{18}) , (R_{15}, R_{12}) ,

(R_{17}, R_{12}) , (R_{19}, R_{12}) , (R_{13}, R_{14}) , (R_{13}, R_{16}) ,

(R_{13}, R_{18}) , (R_{17}, R_{14}) , (R_{19}, R_{14}) , (R_{15}, R_{16}) ,

(R_{15}, R_{18}) , (R_{19}, R_{16}) , (R_{17}, R_{18}) , (R_9, R_{14}) ;

(R_9, R_{16}).

The cut elimination algorithm.

(1) The cut permutes with all the rules of inference when their applications are passive with respect to the cut, as in the following two typical instances:

$>_1$

$$\mathrm{R}_1\ \dfrac{\mathrm{R}_2\ \dfrac{\overset{\delta_1}{\Gamma,\Delta \to \Phi,\alpha}}{\Gamma,\beta,\Delta \to \Phi,\alpha} \quad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\beta,\Delta,\Psi \to \Phi,\Theta} \quad > \quad \dfrac{\dfrac{\overset{\delta_1}{\Gamma,\Delta \to \Phi,\alpha} \quad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Delta,\Psi \to \Phi,\Theta}\ \mathrm{R}_1}{\Lambda,\Gamma,\beta,\Delta,\Psi \to \Phi,\Theta}\ \mathrm{R}_2$$

$>_2$

$$\mathrm{R}_1\ \dfrac{\mathrm{R}_{10}\ \dfrac{\overset{\delta_1}{A \to B} \quad \overset{\delta_2}{C \to B}}{A \vee C \to B} \quad \overset{\delta_3}{B \to D}}{A \vee C \to D} \quad > \quad \dfrac{\mathrm{R}_1\ \dfrac{\overset{\delta_1}{A \to B} \quad \overset{\delta_3}{B \to D}}{A \to D} \quad \dfrac{\overset{\delta_2}{C \to B} \quad \overset{\delta_3}{B \to D}}{C \to D}\ \mathrm{R}_1}{A \vee C \to D}\ \mathrm{R}_{10}$$

(2) All trivial cuts are eliminated. (By a trivial cut we mean a cut involving an axiom.)

$>_3$

$$\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda} \quad \alpha \to \alpha}{\Gamma \to \Delta,\alpha,\Lambda} \quad > \quad \overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda}$$

$>_4$

$$\dfrac{\alpha \to \alpha \quad \overset{\delta_1}{\Delta,\alpha,\Lambda \to \Gamma}}{\Delta,\alpha,\Lambda \to \Gamma} \quad > \quad \overset{\delta_1}{\Delta,\alpha,\Lambda \to \Gamma}$$

(3) $(-, R_2)$

$>_5$

$$\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda} \qquad \dfrac{\overset{\delta_2}{\Psi,\Phi \to \Theta}}{\Psi,\alpha,\Phi \to \Theta}}{\Psi,\Gamma,\Phi \to \Delta,\Theta,\Lambda} \;>\; \begin{array}{c} \delta_2 \\ \Psi,\Phi \to \Theta \\ \hline\hline \Psi,\Gamma,\Phi \to \Theta \\ \hline\hline \Psi,\Gamma,\Phi \to \Delta,\Theta \\ \hline\hline \Psi,\Gamma,\Phi \to \Delta,\Theta,\Lambda \end{array}$$

(Double lines indicate as many thinnings as there are formulas in the sequences introduced.)

This reduction is justified, since the position taken by applying the cut makes δ_1 alienated from the result of the cut. Therefore our reduction codes as much information as the original tree. What is of importance here is that the result was obtained by a thinning.

(4) $(R_3, -)$

$>_6$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\Phi}}{\Gamma \to \Delta,\alpha,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Psi \to \Delta,\Theta,\Phi} \;>\; \begin{array}{c} \delta_1 \\ \Gamma \to \Delta,\Phi \\ \hline\hline \Lambda,\Gamma \to \Delta,\Phi \\ \hline\hline \Lambda,\Gamma,\Psi \to \Delta,\Phi \\ \hline\hline \Lambda,\Gamma,\Psi \to \Delta,\Theta,\Phi \end{array}$$

(Double lines are as in (3).)

The justification for $>_6$ is the same as the one in (3).

(5) $(-, R_4)$

$>_7$

$$\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda} \qquad \dfrac{\overset{\delta_2}{\Psi,\alpha,\alpha,\Phi \to \Theta}}{\Psi,\alpha,\Phi \to \Theta}}{\Psi,\Gamma,\Phi \to \Delta,\Theta,\Lambda} \quad >$$

$$\begin{array}{c} \dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda} \qquad \dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Lambda} \qquad \overset{\delta_2}{\Psi,\alpha,\alpha,\Phi \to \Theta}}{\Psi,\Gamma,\alpha,\Phi \to \Delta,\Theta,\Lambda}}{\Psi,\Gamma,\Gamma,\Phi \to \Delta,\Delta,\Theta,\Lambda,\Lambda} \\ \hline\hline \Psi,\Gamma,\Phi \to \Delta,\Delta,\Theta,\Lambda,\Lambda \\ \hline\hline \Psi,\Gamma,\Phi \to \Delta,\Theta,\Lambda,\Lambda \\ \hline\hline \Psi,\Gamma,\Phi \to \Delta,\Theta,\Lambda \end{array}$$

(Here, double lines indicate several interchanges and thinnings, depending on the number of formulas in the sequences contracted.)

(6) $(R_5, -)$

$>_8$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\alpha,\Phi}}{\Gamma \to \Delta,\alpha,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Psi \to \Delta,\Theta,\Phi} \quad >$$

$$\begin{array}{c}\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\alpha,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Psi \to \Delta,\alpha,\Theta,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Lambda,\Gamma,\Psi,\Psi \to \Delta,\Theta,\Theta,\Phi} \\ \hline\hline \Lambda,\Gamma,\Psi,\Psi \to \Delta,\Theta,\Theta,\Phi \\ \hline\hline \Lambda,\Gamma,\Psi \to \Delta,\Theta,\Theta,\Phi \\ \hline\hline \Lambda,\Gamma,\Psi \to \Delta,\Theta,\Phi \end{array}$$

(Double lines are as in (5).)

(7) $(\neg, R_6)$

$>_9$

$$\dfrac{\overset{\delta_1}{\Psi \to \Lambda,\alpha,\Theta} \qquad \dfrac{\overset{\delta_2}{\Delta,\beta,\alpha,\Gamma \to \Phi}}{\Delta,\alpha,\beta,\Gamma \to \Phi}}{\Delta,\Psi,\beta,\Gamma \to \Lambda,\Phi,\Theta} \quad > \quad \begin{array}{c}\dfrac{\overset{\delta_1}{\Psi \to \Lambda,\alpha,\Theta} \qquad \overset{\delta_2}{\Delta,\beta,\alpha,\Gamma \to \Phi}}{\Delta,\beta,\Psi,\Gamma \to \Lambda,\Phi,\Theta} \\ \hline\hline \Delta,\Psi,\beta,\Gamma \to \Lambda,\Phi,\Theta \end{array}$$

(Double line indicates as many interchanges as there are formulas in Ψ.)

(8) $(R_7, \neg)$

$>_{10}$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\beta,\Phi}}{\Gamma \to \Delta,\beta,\alpha,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Psi \to \Delta,\beta,\Theta,\Phi} \quad > \quad \begin{array}{c}\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\beta,\Phi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Psi \to \Theta}}{\Lambda,\Gamma,\Psi \to \Delta,\Theta,\beta,\Phi} \\ \hline\hline \Lambda,\Gamma,\Psi \to \Delta,\beta,\Theta,\Phi \end{array}$$

(Double line indicates as many interchanges as there are formulas in Θ.)

(9) (R_9, R_8)

$>_{11}$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Phi,\alpha,\Psi} \qquad \overset{\delta_2}{\Gamma \to \Phi,\beta,\Psi}}{\Gamma \to \Phi,\alpha\wedge\beta,\Psi} \qquad \dfrac{\overset{\delta_3}{\Delta,\alpha,\Lambda \to \Theta}}{\Delta,\alpha\wedge\beta,\Lambda \to \Theta}}{\Delta,\Gamma,\Lambda \to \Phi,\Theta,\Psi} \quad >$$

$$\dfrac{\overset{\delta_1}{\Gamma \to \Phi,\alpha,\Psi} \qquad \overset{\delta_3}{\Delta,\alpha,\Lambda \to \Theta}}{\Delta,\Gamma,\Lambda \to \Phi,\Theta,\Psi}$$

Since we are dropping δ_2 in our reduction, a justification is mandatory. In the original tree, δ_2 was used as a way of producing $\alpha\wedge\beta$, which ended up being the cut-out formula. Also, δ_3 produces $\alpha\wedge\beta$ by R_8 which is a similar rule to R_2. So that $\alpha\wedge\beta$ has been introduced for the purpose of the cut only and no information is lost by cutting before introducing $\alpha\wedge\beta$.

(10) (R_9, R_8')

$>_{12}$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Phi,\alpha,\Psi} \qquad \overset{\delta_2}{\Gamma \to \Phi,\beta,\Psi}}{\Gamma \to \Phi,\alpha\wedge\beta,\Psi} \qquad \dfrac{\overset{\delta_3}{\Delta,\beta,\Lambda \to \Theta}}{\Delta,\alpha\wedge\beta,\Lambda \to \Theta}}{\Delta,\Gamma,\Lambda \to \Phi,\Theta,\Psi} \quad >$$

$$\dfrac{\overset{\delta_2}{\Gamma \to \Phi,\beta,\Psi} \qquad \overset{\delta_3}{\Delta,\beta,\Lambda \to \Theta}}{\Delta,\Gamma,\Lambda \to \Phi,\Theta,\Psi}$$

The justification is the same as in (9).

(11) (R'_{11}, R_{10})

$>_{13}$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Psi}}{\Gamma \to \Delta,\alpha\vee\beta,\Psi} \qquad \dfrac{\overset{\delta_2}{\Lambda,\alpha,\Phi \to \Theta} \quad \overset{\delta_3}{\Lambda,\beta,\Phi \to \Theta}}{\Lambda,\alpha\vee\beta,\Phi \to \Theta}}{\Lambda,\Gamma,\Phi \to \Delta,\Theta,\Psi} \quad >$$

$$\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\alpha,\Psi} \qquad \overset{\delta_2}{\Lambda,\alpha,\Phi \to \Theta}}{\Lambda,\Gamma,\Phi \to \Delta,\Theta,\Psi}$$

A similar argument to the one in (9) would justify dropping δ_3.

(12) (R_{11}, R'_{10})

$>_{14}$

$$\dfrac{\dfrac{\overset{\delta_1}{\Gamma \to \Delta,\beta,\Psi}}{\Gamma \to \Delta,\alpha\vee\beta,\Psi} \qquad \dfrac{\overset{\delta_2}{\Lambda,\alpha,\Phi \to \Theta} \quad \overset{\delta_3}{\Lambda,\beta,\Phi \to \Theta}}{\Lambda,\alpha\vee\beta,\Phi \to \Theta}}{\Lambda,\Gamma,\Phi \to \Delta,\Theta,\Psi} \quad >$$

$$\dfrac{\Gamma \xrightarrow{\delta_1} \Delta,\beta,\Psi \qquad \Lambda,\beta,\Phi \xrightarrow{\delta_3} \Theta}{\Lambda,\Gamma,\Phi \to \Delta,\Theta,\Psi}$$

Same argument as in (9).

(13) (R_{13}, R_{12})

$>_{15}$

$$\dfrac{\dfrac{\alpha,\Gamma \xrightarrow{\delta_1} \Phi}{\Gamma \to \Phi,\neg\alpha} \qquad \dfrac{\Lambda \xrightarrow{\delta_2} \Delta,\alpha}{\neg\alpha,\Lambda \to \Delta}}{\Gamma,\Lambda \to \Phi,\Delta} \quad > \quad \dfrac{\dfrac{\dfrac{\Lambda \xrightarrow{\delta_2} \Delta,\alpha \qquad \alpha,\Gamma \xrightarrow{\delta_1} \Phi}{\Lambda,\Gamma \to \Delta,\Phi}}{\overline{\Gamma,\Lambda \to \Delta,\Phi}}}{\overline{\Gamma,\Lambda \to \Phi,\Delta}}$$

(Double lines indicate several interchanges depending on the number of formulas in the sequences interchanged.)

(14) (R_{15}, R_{14})

$>_{16}$

$$\dfrac{\dfrac{\alpha,\Gamma \xrightarrow{\delta_1} \Phi,\beta}{\Gamma \to \Phi,\alpha \Rightarrow \beta} \qquad \dfrac{\Delta \xrightarrow{\delta_2} \Lambda,\alpha \qquad \beta,\Psi \xrightarrow{\delta_3} \Theta}{\alpha \Rightarrow \beta,\Delta,\Psi \to \Lambda,\Theta}}{\Gamma,\Delta,\Psi \to \Phi,\Lambda,\Theta} \quad >$$

$$\cfrac{\cfrac{\cfrac{\overset{\delta_2}{\Delta \to \Lambda,\alpha} \qquad \overset{\delta_1}{\alpha,\Gamma \to \Phi,\beta}}{\Delta,\Gamma \to \Lambda,\Phi,\beta} \qquad \overset{\delta_3}{\beta,\Psi \to \Theta}}{\Delta,\Gamma,\Psi \to \Lambda,\Phi,\Theta}}{\cfrac{\overline{\Gamma,\Delta,\Psi \to \Lambda,\Phi,\Theta}}{\overline{\Gamma,\Delta,\Psi \to \Phi,\Lambda,\Theta}}}$$

(Double lines as in (13).)

We note the similarity between $>_{16}$ and $>_{15}$. Indeed, negation is a special case of implication, as $\neg A$ can be expressed by $A \Rightarrow \perp$ where $\perp$ is a fixed object in the language standing for falsehood. Gentzen uses $1=2$ for $\perp$, so that $\neg A$ is then $A \Rightarrow (1=2)$.

(15) $(R_{17},\ R_{16})$

$>_{17}$

$$\cfrac{\cfrac{\overset{\delta_1}{\Gamma \to \Phi,\alpha(a)}}{\Gamma \to \Phi,(\forall x)\alpha(x)} \qquad \cfrac{\overset{\delta_2}{\alpha(b),\Lambda \to \Theta}}{(\forall x)\alpha(x),\Lambda \to \Theta}}{\Gamma,\Lambda \to \Phi,\Theta} \quad > \quad \cfrac{\overset{\delta_1'}{\Gamma \to \Phi,\alpha(c)} \qquad \overset{\delta_2'}{\alpha(c),\Lambda \to \Theta}}{\Gamma,\Lambda \to \Phi,\Theta}$$

Since the number of variables in a formula is finite, the number of formulas in a sequent is finite, and the number of sequents in a derivation is finite; it follows that the number of variables in a tree is finite.

The number of variables in our language being infinite, it is always possible to find a variable which does not occur in a given tree.

In $>_{17}$ c is a variable which does not occur in δ_1 or in δ_2, and, δ_1' is obtained by substituting c for all occurrence of a in δ_1, δ_2' is obtained by substituting c for all occurrences of b in δ_2.

We must remark that this procedure does not violate the restrictions on R_{17} and R_{18} wherever they occur in δ_1 or δ_2 because of our choice for c.

It is also clear that this reduction is justified since the choice of the initial variables a and b was arbitrary and substituting a different variable for them does not change the spirit of the proof. As to cutting one step earlier, it simply avoids an unecessary detour, and no information is lost by doing so.

(16) (R_{19}, R_{18})

$>_{18}$

$$\frac{\dfrac{\overset{\delta_1}{\Gamma \rightarrow \Phi, \alpha(a)}}{\Gamma \rightarrow \Phi, (\exists x)\alpha(x)} \qquad \dfrac{\overset{\delta_2}{\alpha(b), \Lambda \rightarrow \Delta}}{(\exists x)\alpha(x), \Lambda \rightarrow \Delta}}{\Gamma, \Lambda \rightarrow \Phi, \Delta} \quad > \quad \frac{\overset{\delta_1'}{\Gamma \rightarrow \Phi, \alpha(c)} \qquad \overset{\delta_2'}{\alpha(c), \Lambda \rightarrow \Delta}}{\Gamma, \Lambda \rightarrow \Phi, \Delta}$$

δ_1', and δ_2' are obtained as in (15) and the justification for this procedure and this reduction is given in (15).

This completes our cut-elimination algorithm.

There are in all 400 possible distinct cases.

- Our passive list takes care of 341 cases.
- $>_5$ deals with 10 additional cases.
- $>_6$ deals with 9 additional cases.
- $>_7$ deals with 9 additional cases.
- $>_8$ deals with 8 additional cases.
- $>_9$ deals with 8 additional cases.
- $>_{10}$ deals with 7 additional cases.
- $>_{11}$ to $>_{18}$ deal with 1 additional case each, i.e., 8 in all.

Therefore, all distinct cases are considered.

It suffices to show now that the reduction prescribed will totally eliminate the cut from any given tree. The proof that this is the case, is by double induction and is modeled on a proof of M.E. Szabo as presented in his "Algebra of Proofs".

Let us first define some terms.

Let δ be the derivation.

$$\frac{\overset{\delta_1}{\Gamma \rightarrow \Phi,\gamma,\Psi} \quad \overset{\delta_2}{\Delta,\gamma,\Lambda \rightarrow \Theta}}{\Delta,\Gamma,\Lambda \rightarrow \Phi,\Theta,\Psi}$$

where δ_1 and δ_2 are cut free.

(i) The degree of δ_1 = degree of $\delta_2 = 0$

(ii) The *degree of* δ = 1 + number of occurrences of $\wedge, \vee, \neg$ and $\Rightarrow$ in γ.

(iii) The *rank of* δ_1 = rank of δ_2 = 1.

(iv) The *left rank of* δ is the number of rules of inference in the longest branch of δ_1 terminating with $\Gamma \rightarrow \Phi, \gamma, \Psi$, which have as a premise a sequent containing γ as a succedent formula.

(v) The *right rank of* δ is the number of rules of inference in the longest branch of δ_2 terminating with $\Delta, \gamma, \Lambda \rightarrow \Theta$, which have as a premise a sequent containing γ as an antecedent formula.

(vi) Let m be the number of consecutive applications of R_5 with which δ_1 terminates.

(vii) Let n be the number of consecutive applications of R_4 with which δ_2 terminates.

(viii) *Rank of* δ = $(m+1) \cdot$ left rank of δ.
$+ (n+1) \cdot$ right rank of δ.

Now it is routine to check that each of the reductions prescribed, either lowers the rank or the degree of the cut, therefore an induction on rank and degree yields the desired result.

It is important to mention that the algorithm must be applied to each cut separately, starting from the uppermost cut and finishing with the lowest one.

4. *The Relation "$\geqslant$".*

The relation $\geqslant$ is defined on cut-free derivations and is given in the form of an algorithm. It establishes an order of priority for the rules of inference, eliminates steps which do not alter the spirit of the proof, specifies certain "shortest routes" to obtain a result, and identifies particular derivations, always with many specific real mathematical proofs in mind.

In the description of the algorithm we shall omit sequences of formulas (Γ, Δ, etc. . .) unless they play a role in the reduction.

a/ *Reduction to axioms.*

$\geqslant_1$

$$\dfrac{\dfrac{A \to A}{A \wedge B \to A} \qquad \dfrac{B \to B}{A \wedge B \to B}}{A \wedge B \to A \wedge B} \geqslant A \wedge B \to A \wedge B$$

$\geqslant_2$

$$\dfrac{\dfrac{A \to A}{A \to A \vee B} \qquad \dfrac{B \to B}{B \to A \vee B}}{A \vee B \to A \vee B} \geqslant A \vee B \to A \vee B$$

$\geqslant_3$

$$\cfrac{\cfrac{A(a) \to A(a)}{(\forall x)A(x) \to A(a)}}{(\forall x)A(x) \to (\forall x)A(x)} \geqslant \quad (\forall x)A(x) \to (\forall x)A(x)$$

$\geqslant_4$

$$\cfrac{\cfrac{A(a) \to A(a)}{A(a) \to (\exists x)A(x)}}{(\exists x)A(x) \to (\exists x)A(x)} \geqslant \quad (\exists x)A(x) \to (\exists x)A(x)$$

$\geqslant_5$

$$\cfrac{\cfrac{\cfrac{A \to A \qquad B \to B}{A \Rightarrow B, A \to B}}{A, A \Rightarrow B \to B}}{A \Rightarrow B \to A \Rightarrow B} \geqslant \quad A \Rightarrow B \to A \Rightarrow B$$

$\geqslant_6$

$$\cfrac{\cfrac{\cfrac{A \to A}{\neg A, A \to}}{A, \neg A \to}}{\neg A \to \neg A} \geqslant \quad \neg A \to \neg A$$

$\geqslant_7$

$$\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to \neg A, A}}{\neg A \to \neg A} \geqslant \neg A \to \neg A$$

$\geqslant_1$ to $\geqslant_7$ identify to axioms the simplest ways of obtaining these axioms from axioms of the form $A \to A$ where A is atomic. These steps are necessary since we have allowed our axioms to be of the form $B \to B$ where B is an arbitrary formula not necessarily atomic. The justification for this identification is: When starting a tree we have a *choice* and the most economical choice (the more elaborate axiom) should be made since it eliminates unecessary steps.

This will be our attitude throughout the algorithm, we shall always try to introduce the most elaborate form needed as soon as possible.

b/ *Reduction of thinnings and interchanges.*

$\geqslant_8$

$$\dfrac{\dfrac{\overset{\delta_1}{A \to B}}{A, A \to B}}{A \to B} \geqslant \overset{\delta_1}{A \to B}$$

$\geqslant_9$

$$\dfrac{\dfrac{\overset{\delta_1}{A \to B}}{A \to B, B}}{A \to B} \geqslant \overset{\delta_1}{A \to B}$$

$\geqslant_{10}$

$$\begin{array}{c} A,A \overset{\delta_1}{\to} B \\ \hline A \to B \\ \hline A,A \to B \end{array} \geqslant A,A \overset{\delta_1}{\to} B$$

$\geqslant_{11}$

$$\begin{array}{c} A \overset{\delta_1}{\to} B,B \\ \hline A \to B \\ \hline A \to B,B \end{array} \geqslant A \overset{\delta_1}{\to} B,B$$

$\geqslant_{12}$

$$\begin{array}{c} A,B \overset{\delta_1}{\to} C \\ \hline B,A \to C \\ \hline A,B \to C \end{array} \geqslant A,B \overset{\delta_1}{\to} C$$

$\geqslant_{13}$

$$\begin{array}{c} A \overset{\delta_1}{\to} B,C \\ \hline A \to C,B \\ \hline A \to B,C \end{array} > A \overset{\delta_1}{\to} B,C$$

$\geqslant_{14}$

$$\begin{array}{c} A,A \overset{\delta_1}{\to} B \\ \hline A,A \to B \end{array} \geqslant A,A \overset{\delta_1}{\to} B$$

$\geqslant_{15}$

$$\begin{array}{c} A \overset{\delta_1}{\to} B,B \\ \hline A \to B,B \end{array} \geqslant A \overset{\delta_1}{\to} B,B$$

In $\geqslant_{14}$ and $\geqslant_{15}$ the rules of inference are interchanges.

In $\geqslant_8$ to $\geqslant_{15}$ the reductions made are not all valid under the categorical interpretation of Szabo and others. However, from a proof theoretical point of view, no information is lost and all we are doing is eliminating unecessary steps, allowing therefore the proof to proceed faster.

$\geqslant_{16}$

$$\cfrac{\cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A,B \to C}}{B,A \to C} \geqslant \cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{B,A \to C}$$

$\geqslant_{17}$

$$\cfrac{\cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A \to B,C}}{A \to C,B} \geqslant \cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A \to C,B}$$

$\geqslant_{18}$

$$\cfrac{\cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{B,A \to C}}{A,B \to C} \geqslant \cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A,B \to C}$$

$\geqslant_{19}$

$$\cfrac{\cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A \to C,B}}{A \to B,C} \geqslant \cfrac{\begin{matrix}\delta_1\\ A \to C\end{matrix}}{A \to B,C}$$

$\geqslant_{16}$ to $\geqslant_{19}$ avoid interchanges made unecessary by the fact that our thinning rules are general enough.

$\geqslant_{19}$

$$a\,\frac{\begin{matrix}\delta_1\\ \Gamma, A_1, \ldots, A_n, \Delta \to \Lambda\end{matrix}}{\Gamma, A_{i_1}, \ldots, A_{i_n}, \Delta \to \Lambda} \geqslant \frac{\begin{matrix}\delta_1\\ \Gamma, A_1, \ldots, A_n, \Delta \to \Lambda\end{matrix}}{\Gamma, A_{i_1}, \ldots, A_{i_n}, \Delta \to \Lambda}\,b$$

Where a is an arbitrary number of interchanges, and b is the canonical number of interchanges necessary to pass from $A_1, \ldots, A_n$ to $A_{i_1}, \ldots, A_{i_n}$.

Here canonical means, by successive interchanges, bring A_{i_1} to the first position, then bring A_{i_2} to the second position.

etc. . . as illustrated in the following example:

$$\begin{array}{c}\Gamma, A_1, A_2, A_3, A_4, \Delta \to \Lambda \\ \hline \Gamma, A_3, A_4, A_2, A_1, \Delta \to \Lambda\end{array} \geqslant \begin{array}{c}\Gamma, A_1, A_2, A_3, A_4, \Delta \to \Lambda \\ \hline \Gamma, A_1, A_3, A_2, A_4, \Delta \to \Lambda \\ \hline \Gamma, A_3, A_1, A_2, A_4, \Delta \to \Lambda \\ \hline \Gamma, A_3, A_1, A_4, A_2, \Delta \to \Lambda \\ \hline \Gamma, A_3, A_4, A_1, A_2, \Delta \to \Lambda \\ \hline \Gamma, A_3, A_4, A_2, A_1, \Delta \to \Lambda\end{array}$$

$\geqslant_{19}$ is not always the shortest way of doing things but it is a uniform one. Other methods are available, in general the problem of finding a method for the above is equivalent to the so called "motel on a rainy day" problem.

c) *Succedent before antecedent principle.*

Even rules precede odd rules whenever possible, as in the following example:

$\geqslant_{20}$.

$$\begin{array}{c}\delta_1 \\ \Gamma \to \Lambda \\ \hline \Gamma \to \Lambda, \alpha \\ \hline \beta, \Gamma \to \Lambda, \alpha\end{array} \geqslant \begin{array}{c}\delta_1 \\ \Gamma \to \Lambda \\ \hline \beta, \Gamma \to \Lambda \\ \hline \beta, \Gamma \to \Lambda, \alpha\end{array}$$

d/ *Left precedes right principle.*

Left precedes right whenever possible as in:

$\geqslant_{21}$

$$\dfrac{\dfrac{\dfrac{\delta_1}{\Phi,\Delta,\Gamma \rightarrow \Lambda}}{\Phi,\Delta,\beta,\Gamma \rightarrow \Lambda}}{\Phi,\alpha,\Delta,\beta,\Gamma \rightarrow \Lambda} \geqslant \dfrac{\dfrac{\dfrac{\delta_1}{\Phi,\Delta,\Gamma \rightarrow \Lambda}}{\Phi,\alpha,\Delta,\Gamma \rightarrow \Lambda}}{\Phi,\alpha,\Delta,\beta,\Gamma \rightarrow \Lambda}$$

$\geqslant_{20}$ and $\geqslant_{21}$ are only examples of the substantial numbers of instances in which the antecedent before succedent and left before right principles is used. The words *whenever possible* are not in the spirit of an algorithm, but there is no ambiguity as to how these two principles are applied in practice.

The antecedent before succedent principle can be justified by the naïve interpretation of the meaning of a sequent. We can understand $\Delta \rightarrow \Gamma$ to mean from Δ we can deduce Γ. Therefore it is sensible to collect and prepare all the data we need before setting out to prove an assertation which depends on this data. This is in fact how it happens in actual mathematical proofs. Another way of justifying it is through the categorical interpreation of deductive systems, where $\Delta \rightarrow \Gamma$ is interpreted as an arrow with domain Δ and codomain Γ. Here again it seems more sensible to built the domain first.

The left before right principle cannot be justified with many convincing arguments but, one has a choice and a canonnical one should be made.

e/ Reduction of introductions by pieces.

If a formula is introduced by pieces, the final form should be introduced first, if possible, as in the following example:

$$\geqslant_{22}\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\delta_1\\ A \rightarrow C\end{array}}{A, B \rightarrow C}}{A, B\wedge D \rightarrow C}}{A, (B\wedge D)\wedge E \rightarrow C} \;\geqslant\; \dfrac{\begin{array}{c}\delta_1\\ A \rightarrow C\end{array}}{A, (B\wedge D)\wedge E}$$

This principle simply shortens the proofs without changing them in any way.

f/ Operational rules are more important than structural ones.

This principle is based on the fact that structural rules are mostly used to prepare the sequents for the application of operational rules. It is in the spirit of this normalisation that an application of a structural rule can be used, eliminated or displaced to accommodate the application of an operational rule. For examples of this principle, see example 1 and 2 at the end of paragraph g/.

g/ *Priority for the rules of inference.*

We establish an order of priority for the rules of inference in general. This order of priority should be used *whenever possible*, two examples will be given at the end of the description of the order.

(1) R_4, R_5

(2) R_2, R_3

(3) R_8', R_8, R_9

(4) R_{10}, R_{11}', R_{11}

(5) R_6, R_7

(6) R_{12}, R_{13}

(7) R_{14}, R_{15}

(8) R_{18}, R_{19}

(10) R_{16}, R_{17}

The numbering is to be interpreted as follows: if a rule assigned number (4) precedes a rule assigned number (2), for example, and it is possible to permute them there, it should be done.

There are two reasons for requiring R_4 and R_5 to precede all other rules. They are problematic and should be kept as high in the tree as possible, and, it is useless to carry a formula which will be contracted away eventually, for several steps if it is not used during these steps.

Next, R_2 and R_3 should be applied as soon as possible since they provide the "bricks and mortar" for the proof.

R_6 and R_7 should be applied after the conjunction and disjunction rule, since all the rules in (1) to (4) are symmetric and therefore do not require interchanges. It is preferable to use R_6 and R_7 after the symmetric rules, so that the proof is "cleaned up" from interchanges before the non-symmetric rules of negation and implication are applied. It is not reasonable to put the interchanges lower than the non-symmetric rules, since in practice they will always precede them.

R_{18} and R_{19} precede R_{16} and R_{17} because in a mathematical proof, we often try to establish an "AE sentence" (a sentence of the form $(\forall x)(\exists y)\varphi(x,y)$). In such sentences we can only introduce the quantifiers in the prescribed order.

We present two examples of reductions involving the order of priority of the rules of inference.

Example 1.

$$\geqslant_{23}\quad \dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\to A\vee B}}{\neg(A\vee B),A\to}}{A,\neg(A\vee B)\to}\qquad \dfrac{\dfrac{\dfrac{B\to B}{B\to A\vee B}}{\neg(A\vee B),B\to}}{B,\neg(A\vee B)\to}}{A\vee B,\neg(A\vee B)\to} \quad\geqslant\quad \dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\to A\vee B}\qquad\dfrac{B\to B}{B\to A\vee B}}{A\vee B\to A\vee B}}{\neg(A\vee B),A\vee B\to}}{A\vee B,\neg(A\vee B)\to}$$

In the original tree R_{12} precedes R_{10}, in the reduced tree R_{10} precedes R_{12}.

Example 2.

$$\geq_{24} \quad \dfrac{\dfrac{\overset{\delta_1}{A \to B} \qquad \overset{\delta_2}{C \to B}}{B \Rightarrow C, A \to B} \qquad \dfrac{\overset{\delta_2}{C \to B}}{B \Rightarrow C, C \to B}}{B \Rightarrow C, A\vee C \to B} \quad > \quad \dfrac{\dfrac{\overset{\delta_1}{A \to B} \qquad \overset{\delta_2}{C \to B}}{A\vee C \to B} \qquad \overset{\delta_2}{C \to B}}{B \Rightarrow C, A\vee C \to B}$$

In the original tree R_{14} precedes R_{10}, in the reduced tree R_{10} precedes R_{14}.

We have deliberately chosen border line examples, in most cases it is much clearer whether the permutation is possible or not. It may appear from those examples that we are liberal in interpreting our order, but it can be shown that our interpretation is in harmony with the spirit of the reduction. In this particular case, since it is in the spirit of the reduction to consider operational rules more important than structural ones, our interpretation of the order is justified.

We can now summarize the spirit of our normalisation procedure by the following slogans:

- Operational rules are more important than structural ones.

- Structural rules, except for interchanges, are higher in the tree than operational rules.
- Interchanges are as low as possible in the tree.
- Thinning is done in the right place and with the most elaborate form needed right away.
- Nothing is done which gets undone in the following steps.
- Left precedes right.

5. *Equivalence of proofs of equivalence.*

We are now in a position to define rigourously our notion of equivalence for proofs of equivalence.

a/ *Strong equivalence level.*

Given a proof of equivalence (δ_1, δ_2) for a pair (A,B), if δ_1' and δ_2' are the cut-free forms of δ_1 and δ_2 then (δ_1, δ_2) is of the *strong equivalence level* iff

$$\frac{\overset{\delta_1'}{A \to B} \qquad \overset{\delta_2'}{B \to A}}{A \to A} \geqslant A \to A$$

and

$$\frac{\overset{\delta_2'}{B \to A} \qquad \overset{\delta_1'}{A \to B}}{B \to B} \geqslant B \to B$$

b/ *Tauberian equivalence level.*

Given a proof of equivalence (δ_1,δ_2) for a pair (A,B), if δ_1' and δ_2' are the cut-free forms for δ_1 and δ_2, then (δ_1,δ_2) is of the *Tauberian equivalence level* iff exactly one of

$$\frac{\overset{\delta_1'}{A \to B} \quad \overset{\delta_2'}{B \to A}}{A \to A}$$

or $$\frac{\overset{\delta_2'}{B \to A} \quad \overset{\delta_1'}{A \to B}}{B \to B}$$

reduces to an axiom.

c/ *Weak equivalence level.*

Given a proof of equivalence (δ_1,δ_2) for a pair (A,B), if δ_1' and δ_2' are the cut-free forms for δ_1 and δ_2, then (δ_1,δ_2) is of the *weak equivalence level* iff neither of

$$\frac{\overset{\delta_1'}{A \to B} \quad \overset{\delta_2'}{B \to A}}{A \to A}$$

and $$\frac{\overset{\delta_2'}{B \to A} \quad \overset{\delta_1'}{A \to B}}{B \to B}$$

reduces to an axiom.

CHAPTER III

EXAMPLES

In this chapter we present one example of each of the three levels of equivalence.

1: *Strong Equivalence.*

Let (δ_1,δ_2) be a proof of equivalence given below for $(\neg(A\vee B), \neg A\wedge\neg B)$, then (δ_1,δ_2) is a proof of strong equivalence.

Proof:

δ_1 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A,\neg A}}{\to \neg A, A}}{\to \neg A, A\vee B}}{\neg(A\vee B) \to \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\to B,\neg B}}{\to \neg B, B}}{\to \neg B, A\vee B}}{\neg(A\vee B) \to \neg B}}{\neg(A\vee B) \to \neg A\wedge\neg B}$$

δ_2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}}{A,\neg A \to}}{A,\neg A\wedge\neg B \to} \qquad \dfrac{\dfrac{\dfrac{B \to B}{\neg B, B \to}}{B,\neg B \to}}{B,\neg A\wedge\neg B \to}}{A\vee B,\neg A\wedge\neg B \to}}{\neg A\wedge\neg B \to \neg(A\vee B)}$$

a/ we verify that

$$\dfrac{\overset{\delta_1}{\neg(A\vee B)\to\neg A\wedge\neg B} \qquad \overset{\delta_2}{\neg A\wedge\neg B\to\neg(A\vee B)}}{\neg(A\vee B)\to\neg(A\vee B)} \;>\; \neg(A\vee B)\to\neg(A\vee B)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\to A,\neg A}}{\to\neg A,A}}{\to\neg A,A\vee B}}{\neg(A\vee B)\to\neg A} \quad \dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\to B,\neg B}}{\to\neg B,B}}{\to\neg B,A\vee B}}{\neg(A\vee B)\to\neg B}}{\neg(A\vee B)\to\neg A\wedge\neg B} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg A,A\to}}{A,\neg A\to}}{A,\neg A\wedge\neg B\to} \quad \dfrac{\dfrac{\dfrac{B\to B}{\neg B,B\to}}{B,\neg B\to}}{B,\neg A\wedge\neg B\to}}{A\vee B,\neg A\wedge\neg B\to}}{\neg A\wedge\neg B\to\neg(A\vee B)}}{\neg B(A\vee B)\to\neg(A\vee B)}$$

$>$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\to A,\neg A}}{\to\neg A,A}}{\to\neg A,A\vee B}}{(A\vee B)\to\neg A} \quad \dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\to B,\neg B}}{\to\neg B,B}}{\to\neg B,A\vee B}}{\neg(A\vee B)\to\neg B}}{\neg(A\vee B)\to\neg A\wedge\neg B} \quad \dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg A,A\to}}{A,\neg A\to}}{A,\neg A\wedge\neg B\to} \quad \dfrac{\dfrac{\dfrac{B\to B}{\neg B,B\to}}{B,\neg B\to}}{B,\neg A\wedge\neg B\to}}{A\vee B,\neg A\wedge\neg B\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)}$$

$>$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\to A,\neg A}}{\to\neg A,A}}{\to\neg A,A\vee B}}{\neg(A\vee B)\to\neg A} \quad \dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\to B,\neg B}}{\to\neg B,B}}{\to\neg B,A\vee B}}{\neg(A\vee B)\to\neg B}}{\neg(A\vee B)\to\neg A\wedge\neg B} \quad \dfrac{\dfrac{\dfrac{A\to A}{\neg A,A\to}}{A,\neg A\to}}{A,\neg A\wedge\neg B\to}}{A,\neg(A\vee B)\to} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\to A,\neg A}}{\to\neg A,A}}{\to\neg A,A\vee B}}{\neg(A\vee B)\to\neg A} \quad \dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\to B,\neg B}}{\to\neg B,B}}{\to\neg B,A\vee B}}{\neg(A\vee B)\to\neg B}}{\neg(A\vee B)\to\neg A\wedge\neg B} \quad \dfrac{\dfrac{\dfrac{B\to B}{\neg B,B\to}}{B,\neg B\to}}{B,\neg A\wedge\neg B\to}}{B,\neg(A\vee B)\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)}$$

$>$

Here, we shall apply the algorithm simultaneously to both branches of the tree.

```
A → A                                   B → B
→ A, ¬A                                 →B, ¬B
→ ¬A,A            A → A                 → ¬B,B              B → B
→ ¬A,AvB          ¬A,A →                → ¬B,AvB            ¬B,B →
¬(AvB) → ¬A       A, ¬A →               ¬(AvB) → ¬B         B, ¬B →
      A, ¬(AvB) →                                B, ¬(AvB) →
                          AvB, ¬(AvB) →
                        ¬(AvB) → ¬(AvB)
```

>

```
A → A                                 B → B
→ A, ¬A                               → B, ¬B
→ ¬A,A                                → ¬B,B
→ ¬A,AvB           A → A              → ¬B,AvB          B → B
¬(AvB) → ¬A        ¬A,A →             ¬(AvB) → ¬B       ¬B,B →
   ¬(AvB) ,A →                        ¬(AvB) ,B →
   A, ¬(AvB) →                        B, ¬(AvB) →
                     AvB, ¬(AvB) →
                    ¬(AvB) → ¬(AvB)
```

>

```
A → A                                   B → B
→ A, ¬A                                 → B, ¬B
→ ¬A,A            A → A                 → ¬B,B             B → B
→ ¬A,AvB          ¬A,A →                → ¬B,AvB           ¬B,B →
   A → AvB                                    B → AvB
   ¬(AvB) ,A →                                ¬(AvB) ,B →
   A, ¬(AvB) →                                B, ¬(AvB) →
                     AvB, ¬(AvB) →
                    ¬(AvB) → ¬(AvB)
```

>

```
A → A                                 B → B
→ A, ¬A            A → A              → B, ¬B            B → B
→ ¬A,A             ¬A,A →             → ¬B,B             ¬B,B →
        A → A                            B → B
        A → AvB                          B → AvB
        ¬(AvB) ,A →                    ¬(AvB) ,B →
        A, ¬(AvB) →                      B, ¬(AvB) →
                     AvB, ¬(AvB) →
                    ¬(AvB) → ¬(AvB)
```

>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\to A,\neg A}\quad\dfrac{A\to A}{\neg A,A\to}}{A\to A}}{A\to A\vee B}}{\neg(A\vee B),A\to}}{A,\neg(A\vee B)\to}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\to B,\neg B}\quad\dfrac{B\to B}{\neg B,B\to}}{B\to B}}{B\to A\vee B}}{\neg(A\vee B),B\to}}{B,\neg(A\vee B)\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)} > \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A\quad A\to A}{A\to A}}{A\to A\vee B}}{\neg(A\vee B),A\to}}{A,\neg(A\vee B)\to}\qquad\dfrac{\dfrac{\dfrac{\dfrac{B\to B\quad B\to B}{B\to B}}{B\to A\vee B}}{\neg(A\vee B),B\to}}{B,\neg(A\vee B)\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)}$$

$$> \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\to A\vee B}}{\neg(A\vee B),A\to}}{A,\neg(A\vee B)\to}\qquad\dfrac{\dfrac{\dfrac{B\to B}{B\to A\vee B}}{\neg(A\vee B),B\to}}{B,\neg(A\vee B)\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)} > \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\to A\vee B}\quad\dfrac{B\to B}{B\to A\vee B}}{A\vee B\to A\vee B}}{\neg(A\vee B),A\vee B\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)}$$

$$> \dfrac{\dfrac{\dfrac{A\vee B\to A\vee B}{\neg(A\vee B),A\vee B\to}}{A\vee B,\neg(A\vee B)\to}}{\neg(A\vee B)\to\neg(A\vee B)} > \neg(A\vee B\to\neg(A\vee B).$$

b/ Next, we verify that

$$\dfrac{\overset{\delta_2}{\neg A\wedge\neg B\to\neg(A\vee B)}\qquad\overset{\delta_1}{\neg(A\vee B)\to\neg A\wedge\neg B}}{\neg A\wedge\neg B\to\neg A\wedge\neg B} > A\wedge\neg B\to A\wedge\neg B$$

\begin{prooftree}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$A \vee B, \neg A \wedge \neg B \to$}
\UnaryInfC{$\neg A \wedge \neg B \to \neg(A \vee B)$}
\AxiomC{$A \to A$}
\UnaryInfC{$\to A, \neg A$}
\UnaryInfC{$\to \neg A, A$}
\UnaryInfC{$\to \neg A, A \vee B$}
\UnaryInfC{$\neg(A \vee B) \to \neg A$}
\AxiomC{$B \to B$}
\UnaryInfC{$\to B, \neg B$}
\UnaryInfC{$\to \neg B, B$}
\UnaryInfC{$\to \neg B, A \vee B$}
\UnaryInfC{$\neg(A \vee B) \to \neg B$}
\BinaryInfC{$\neg(A \vee B) \to \neg A \wedge \neg B$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A \wedge \neg B$}
\end{prooftree}

>

\begin{prooftree}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$A \vee B, \neg A \wedge \neg B \to$}
\UnaryInfC{$\neg A \wedge \neg B \to \neg(A \vee B)$}
\AxiomC{$A \to A$}
\UnaryInfC{$\to A, \neg A$}
\UnaryInfC{$\to \neg A, A$}
\UnaryInfC{$\to \neg A, A \vee B$}
\UnaryInfC{$\neg(A \vee B) \to \neg A$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A$}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$A \vee B, \neg A \wedge \neg B \to$}
\UnaryInfC{$\neg A \wedge \neg B \to \neg(A \vee B)$}
\AxiomC{$B \to B$}
\UnaryInfC{$\to B, \neg B$}
\UnaryInfC{$\to B, B$}
\UnaryInfC{$\to \neg B, A \vee B$}
\UnaryInfC{$\neg(A \vee B) \to \neg B$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg B$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A \wedge \neg B$}
\end{prooftree}

>

Here we shall apply the algorithm, simultaneously to both branches of the tree.

\begin{prooftree}
\AxiomC{$A \to A$}
\UnaryInfC{$\to A, \neg A$}
\UnaryInfC{$\to \neg A, A$}
\UnaryInfC{$\to \neg A, A \vee B$}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$A \vee B, \neg A \wedge \neg B \to$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A$}
\AxiomC{$B \to B$}
\UnaryInfC{$\to B, \neg B$}
\UnaryInfC{$\to \neg B, B$}
\UnaryInfC{$\to \neg B, A \vee B$}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$A \vee B, \neg A \wedge \neg B \to$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg B$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A \wedge \neg B$}
\end{prooftree}

>

\begin{prooftree}
\AxiomC{$A \to A$}
\UnaryInfC{$\to A, \neg A$}
\UnaryInfC{$\to \neg A, A$}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \wedge \neg B \to$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A$}
\AxiomC{$B \to B$}
\UnaryInfC{$\to B, \neg B$}
\UnaryInfC{$\to \neg B, B$}
\AxiomC{$B \to B$}
\UnaryInfC{$\neg B, B \to$}
\UnaryInfC{$B, \neg B \to$}
\UnaryInfC{$B, \neg A \wedge \neg B \to$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg B$}
\BinaryInfC{$\neg A \wedge \neg B \to \neg A \wedge \neg B$}
\end{prooftree}

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to \neg A, A} \quad \dfrac{\dfrac{A \to A}{\neg A, A \to}}{A, \neg A \to}}{\neg A \to \neg A}}{\neg A \wedge \neg B \to \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}}{\to \neg B, B} \quad \dfrac{\dfrac{B \to B}{\neg B, B \to}}{B, \neg B \to}}{\neg B \to \neg B}}{\neg A \wedge \neg B \to \neg B}}{\neg A \wedge \neg B \to \neg A \wedge \neg B}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to \neg A, A} \quad \dfrac{A \to A}{\neg A, A \to}}{\neg A \to \neg A}}{\neg A \wedge \neg B \to \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}}{\to \neg B, B} \quad \dfrac{B \to B}{\neg B, B \to}}{\neg B \to \neg B}}{\neg A \wedge \neg B \to \neg B}}{\neg A \wedge \neg B \to \neg A \wedge \neg B}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to \neg A, A} \quad A \to A}{\to \neg A, A}}{\neg A \to \neg A}}{\neg A \wedge \neg B \to \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}}{\to \neg B, B} \quad B \to B}{\to \neg B, B}}{\neg B \to \neg B}}{\neg A \wedge \neg B \to \neg B}}{\neg A \wedge \neg B \to \neg A \wedge \neg B} \quad > \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to \neg A, A}}{\neg A \to \neg A}}{\neg A \wedge \neg B \to \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{\to B, \neg B}}{\to \neg B, B}}{\neg B \to \neg B}}{\neg A \wedge \neg B \to \neg B}}{\neg A \wedge \neg B \to \neg A \wedge \neg B}$$

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$$\dfrac{\dfrac{\neg A \to \neg A}{\neg A \wedge \neg B \to \neg A} \qquad \dfrac{\neg B \to \neg B}{\neg A \wedge \neg B \to \neg B}}{\neg A \wedge \neg B \to \neg A \wedge \neg B} \quad > \quad \neg A \wedge \neg B \to \neg A \wedge \neg B$$

This completes the proof.

2: *Tauberian Equivalence.*

Let (δ_1, δ_2) be a proof of equivalence given below for $(A \land \neg C, (A \lor C) \land \neg C)$, then (δ_1, δ_2) is a proof of Tauberian equivalence.

(Remark: $A \rightarrow A \lor C$ is always a theorem, while $A \lor C \rightarrow A$ is not, and $\neg C$ is the "minimal" formula necessary to make it a theorem.)

Proof:

δ_1 is

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{A \land \neg C \rightarrow A}}{A \land \neg C \rightarrow A \lor C} \qquad \dfrac{\neg C \rightarrow \neg C}{A \land \neg C \rightarrow \neg C}}{A \land \neg C \rightarrow (A \lor C) \land \neg C}$$

δ_2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \qquad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A \lor C \rightarrow A}}{(A \lor C) \land \neg C, (A \lor C) \rightarrow A}}{(A \lor C) \land \neg C, (A \lor C) \land \neg C \rightarrow A}}{(A \lor C) \land \neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{(A \lor C) \land \neg C \rightarrow \neg C}}{(A \lor C) \land \neg C \rightarrow A \land \neg C}$$

a/ We first show that

$$\dfrac{\overset{\delta_1}{A\wedge\neg C \to (A\vee C)\wedge\neg C} \qquad \overset{\delta_2}{(A\vee C)\wedge\neg C \to A\wedge\neg C}}{A\wedge\neg C\to A\wedge\neg C} > A\wedge\neg C \to A\wedge\neg C$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg C,A\to A}\quad\dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C,C\to A}}{\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{(A\vee C)\wedge\neg C\to A} \quad \dfrac{\neg C\to\neg C}{(A\vee C)\wedge\neg C\to\neg C}}{(A\vee C)\wedge\neg C\to A\wedge\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg C,A\to A}\quad\dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C,C\to A}}{\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C\to A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C} \quad \dfrac{\neg C\to\neg C}{(A\vee C)\wedge\neg C\to\neg C}}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

We call this tree λ_1 and first reduce its right branch.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A \wedge \neg C \rightarrow A}}{A \wedge \neg C \rightarrow A \vee C} \qquad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}}{A \wedge \neg C \rightarrow (A \vee C) \wedge \neg C} \qquad \dfrac{\neg C \rightarrow \neg C}{(A \vee C) \wedge \neg C \rightarrow \neg C}}{A \wedge \neg C \rightarrow \neg C}$$

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$$\dfrac{\dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C} \qquad \neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C} \quad > \quad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}$$

Thus λ_1 $>$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A \wedge \neg C \rightarrow A}}{A \wedge \neg C \rightarrow A \vee C} \qquad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}}{A \wedge \neg C \rightarrow (A \vee C) \wedge \neg C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \qquad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, (A \vee C) \wedge \neg C \rightarrow A}}{(A \vee C) \wedge \neg C \rightarrow A}}{A \wedge \neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}}{A \wedge \neg C \rightarrow A \wedge \neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C}\quad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C}\quad\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg C,A\to A}\quad\dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C,C\to A}}{\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C,A\wedge\neg C\to A}}{A\wedge\neg C\to A}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C}\quad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A}{A\wedge\neg C\to A}}{A\wedge\neg C\to A\vee C}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to(A\vee C)\wedge\neg C}\quad\dfrac{\dfrac{\dfrac{A\to A}{\neg C,A\to A}\quad\dfrac{\dfrac{C\to C}{C,\to A,C}}{\neg C,C\to A}}{\neg C,A\vee C\to A}}{(A\vee C)\wedge\neg C,A\vee C\to A}}{A\wedge\neg C,A\vee C\to A}}{A\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C,A\wedge\neg C\to A}}{A\wedge\neg C\to A}\quad\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A\wedge\neg C \to A}}{A\wedge\neg C\to A\vee C} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to (A\vee C)\wedge\neg C} \quad \dfrac{\dfrac{\dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C} \quad \dfrac{\dfrac{A\to A}{\neg C, A\to A} \quad \dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C, C\to A}}{\neg C, A\vee C\to A}}{A\wedge\neg C, A\vee C\to A}}{A\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C, A\wedge\neg C\to A}}{A\wedge\neg C\to A} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A\wedge\neg C \to A}}{A\wedge\neg C\to A\vee C} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to (A\vee C)\wedge\neg C} \quad \dfrac{\dfrac{\dfrac{\neg C\to\neg C \quad \dfrac{\dfrac{A\to A}{\neg C, A\to A} \quad \dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C, C\to A}}{\neg C, A\vee C\to A}}{\neg C, A\vee C\to A}}{A\wedge\neg C, A\vee C\to A}}{A\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C, A\wedge\neg C\to A}}{A\wedge\neg C\to A} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A\wedge\neg C \to A}}{A\wedge\neg C\to A\vee C} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to (A\vee C)\wedge\neg C} \quad \dfrac{\dfrac{\dfrac{\dfrac{A\to A}{\neg C, A\to A} \quad \dfrac{\dfrac{C\to C}{C\to A,C}}{\neg C, C\to A}}{\neg C, A\vee C\to A}}{A\wedge\neg C, A\vee C\to A}}{A\wedge\neg C,(A\vee C)\wedge\neg C\to A}}{A\wedge\neg C, A\wedge\neg C\to A}}{A\wedge\neg C\to A} \quad \dfrac{\neg C\to\neg C}{A\wedge\neg C\to\neg C}}{A\wedge\neg C\to A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A\vee C} \qquad \dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A\vee C \rightarrow A}}{A\wedge\neg C, A\vee C \rightarrow A}}{\dfrac{A\wedge\neg C, A\wedge\neg C \rightarrow A}{A\wedge\neg C \rightarrow A}} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A\vee C} \qquad \dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A\vee C \rightarrow A}}{\neg C, A\wedge\neg C \rightarrow A}}{\dfrac{A\wedge\neg C, A\wedge\neg C \rightarrow A}{A\wedge\neg C \rightarrow A}} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A} \quad \dfrac{A \rightarrow A}{\neg C, A \rightarrow A}}{\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C} \quad > \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A} \quad A \rightarrow A}{A\wedge\neg C \rightarrow A}}{\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C}$$

>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A}}{\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C} \;\geqslant\; \dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A}}{A\wedge\neg C, A\wedge\neg C \rightarrow A}}{A\wedge\neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C}$$

$$\geqslant$$

$$\dfrac{\dfrac{A \rightarrow A}{A\wedge\neg C \rightarrow A} \qquad \dfrac{\neg C \rightarrow \neg C}{A\wedge\neg C \rightarrow \neg C}}{A\wedge\neg C \rightarrow A\wedge\neg C} \;\geqslant\; A\wedge\neg C \rightarrow A\wedge\neg C.$$

b/ Next, we show that

$$\dfrac{\overset{\delta_2}{(A\vee C)\wedge\neg C \rightarrow A\wedge\neg C} \qquad \overset{\delta_1}{A\wedge\neg C \rightarrow (A\vee C)\wedge\neg C}}{(A\vee C)\wedge\neg C \rightarrow (A\vee C)\wedge\neg C}$$

does not reduce to $(A\vee C)\wedge\neg C \rightarrow (A\vee C)\wedge\neg C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, (A \vee C) \wedge \neg C \rightarrow A}}{(A \vee C) \wedge \neg C \rightarrow A} \quad \dfrac{\neg C \rightarrow \neg C}{(A \vee C) \wedge \neg C \rightarrow \neg C}}{(A \vee C) \wedge \neg C \rightarrow A \wedge \neg C} \quad \dfrac{\dfrac{\dfrac{A \rightarrow A}{A \wedge \neg C \rightarrow A}}{A \wedge \neg C \rightarrow A \vee C} \quad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}}{A \wedge \neg C \rightarrow (A \vee C) \wedge \neg C}}{(A \vee C) \wedge \neg C \rightarrow (A \vee C) \wedge \neg C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, (A \vee C) \wedge \neg C \rightarrow A}}{(A \vee C) \wedge \neg C \rightarrow A} \quad \dfrac{\neg C \rightarrow \neg C}{(A \vee C) \wedge \neg C \rightarrow \neg C}}{(A \vee C) \wedge \neg C \rightarrow A \wedge \neg C} \quad \dfrac{\dfrac{A \rightarrow A}{A \wedge \neg C \rightarrow A}}{A \wedge \neg C \rightarrow A \vee C}}{(A \vee C) \wedge \neg C \rightarrow A \vee C} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\neg C, A \rightarrow A} \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow A, C}}{\neg C, C \rightarrow A}}{\neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, A \vee C \rightarrow A}}{(A \vee C) \wedge \neg C, (A \vee C) \wedge \neg C \rightarrow A}}{(A \vee C) \wedge \neg C \rightarrow A} \quad \dfrac{\neg C \rightarrow \neg C}{(A \vee C) \wedge \neg C \rightarrow \neg C}}{(A \vee C) \wedge \neg C \rightarrow A \wedge \neg C} \quad \dfrac{\neg C \rightarrow \neg C}{A \wedge \neg C \rightarrow \neg C}}{(A \vee C) \wedge \neg C \rightarrow \neg C}}{(A \vee C) \wedge \neg C \rightarrow (A \vee C) \wedge \neg C}$$

We apply the algorithm to both branches of the tree simultaneously.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg C, A \to A} \quad \dfrac{\dfrac{C \to C}{C \to A, C}}{\neg C, C \to A}}{\neg C, A \lor C \to A}}{(A \lor C) \land \neg C, A \lor C \to A}}{(A \lor C) \land \neg C, (A \lor C) \land \neg C \to A}}{(A \lor C) \land \neg C \to A} \quad \dfrac{\neg C \to \neg C}{(A \lor C) \land \neg C \to \neg C}}{(A \lor C) \land \neg C \to A \land \neg C} \quad \dfrac{A \to A}{A \land \neg C \to A}}{(A \lor C) \land \neg C \to A}}{(A \lor C) \land \neg C \to A \lor C} \quad \dfrac{\dfrac{\neg C \to \neg C}{(A \lor C) \land \neg C \to \neg C} \quad \neg C \to \neg C}{(A \lor C) \land \neg C \to \neg C}}{(A \lor C) \land \neg C \to (A \lor C) \land \neg C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg C, A \to A} \quad \dfrac{\dfrac{C \to C}{C \to A, C}}{\neg C, C \to A}}{\neg C, A \lor C \to A}}{(A \lor C) \land \neg C, A \lor C \to A}}{(A \lor C) \land \neg C, (A \lor C) \land \neg C \to A}}{(A \lor C) \land \neg C \to A} \quad A \to A}{(A \lor C) \land \neg C \to A}}{(A \lor C) \land \neg C \to A \lor C} \quad \dfrac{\neg C \to \neg C}{(A \lor C) \land \neg C \to \neg C}}{(A \lor C) \land \neg C \to (A \lor C) \land \neg C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg C, A \to A} \qquad \dfrac{\dfrac{C \to C}{C \to A, C}}{\neg C, C \to A}}{\neg C, A\vee C \to A}}{(A\vee C)\wedge\neg C, (A\vee C) \to A}}{(A\vee C)\wedge\neg C, (A\vee C)\wedge\neg C \to A}}{(A\vee C)\wedge\neg C \to A}}{(A\vee C)\wedge\neg C \to (A\vee C)} \qquad \dfrac{\neg C \to \neg C}{(A\vee C)\wedge\neg C \to \neg C}}{(A\vee C)\wedge\neg C \to (A\vee C)\wedge\neg C}$$

which does not reduce any further.

This completes the proof.

We note that it is a/ that reduced to an axiom. Indeed, $A \to A\vee C$ is provable without the adjunction of the additional condition $\neg C$. This will happen in all cases.

Examples abound.

Take for instance:

$$((A \Rightarrow B)\wedge\neg C) \text{ iff } (((A\vee C) \Rightarrow (B\vee C))\wedge\neg C),$$

as in the case treated:

$$\dfrac{\overset{\delta_1}{(A \Rightarrow B)\wedge\neg C \to ((A\vee C) \Rightarrow (B\vee C))\wedge\neg C} \qquad \overset{\delta_2}{((A\vee C) \Rightarrow (B\vee C))\wedge\neg C \to (A \Rightarrow B)\wedge\neg C}}{(A \Rightarrow B)\wedge\neg C \to (A \Rightarrow B)\wedge\neg C}$$

reduces to an axiom since $A \Rightarrow B \rightarrow (A \vee C) \Rightarrow (B \vee C)$ is a theorem, while the other cut does not reduce to an axiom, since $(A \vee C) \Rightarrow (B \vee C) \rightarrow A \Rightarrow B$ is not a theorem.

3: *Weak equivalence.*

Let (δ_1, δ_2) be a proof of equivalence given below for $(\neg B \Rightarrow B, A \vee \neg A)$, then (δ_1, δ_2) is a proof of weak equivalence.

Proof:

δ_1 is
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A, \neg B \Rightarrow B \rightarrow A}}{\neg B \Rightarrow B \rightarrow A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, A \vee \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A}$$

δ_2 is
$$\dfrac{\dfrac{\dfrac{\dfrac{B \rightarrow B}{B, A \vee \neg A \rightarrow B}}{A \vee \neg A \rightarrow B, \neg B}}{\neg B, A \vee \neg A \rightarrow B}}{A \vee \neg A \rightarrow \neg B \Rightarrow B}$$

a/ First we show that

$$\dfrac{\overset{\delta_1}{\neg B \Rightarrow B \to A \vee \neg A} \qquad \overset{\delta_2}{A \vee \neg A \to \neg B \Rightarrow B}}{\neg B \Rightarrow B \to \neg B \Rightarrow B}$$

does not reduce to $\neg B \Rightarrow B \to \neg B \Rightarrow B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{\neg B \Rightarrow B \to A \vee \neg A, \neg A}}{\neg B \Rightarrow B \to A \vee \neg A, A \vee \neg A}}{\neg B \Rightarrow B \to A \vee \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A \vee \neg A \to B}}{A \vee \neg A \to B, \neg B}}{\neg B, A \vee \neg A \to B}}{A \vee \neg A \to \neg B \Rightarrow B}}{\neg B \Rightarrow B \to \neg B \Rightarrow B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{\neg B \Rightarrow B \to A \vee \neg A, \neg A}}{\neg B \Rightarrow B \to A \vee \neg A, A \vee \neg A}}{\neg B \Rightarrow B \to A \vee \neg A} \qquad \dfrac{\dfrac{\dfrac{B \to B}{B, A \vee \neg A \to B}}{A \vee \neg A \to B, \neg B}}{\neg B, A \vee \neg A \to B}}{\neg B, \neg B \Rightarrow B \to B}}{\neg B \Rightarrow B \to \neg B \Rightarrow B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A, \neg B \Rightarrow B \rightarrow A}}{\neg B \Rightarrow B \rightarrow A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, A \vee \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A} \qquad \dfrac{\dfrac{B \rightarrow B}{B, A \vee \neg A \rightarrow B}}{A \vee \neg A \rightarrow B, \neg B}}{\neg B \Rightarrow B \rightarrow B, \neg B}}{\neg B, \neg B \Rightarrow B \rightarrow B}}{\neg B \Rightarrow B \rightarrow \neg B \Rightarrow B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A, \neg B \Rightarrow B \rightarrow A}}{\neg B \Rightarrow B \rightarrow A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A, A \vee \neg A}}{\neg B \Rightarrow B \rightarrow A \vee \neg A} \qquad \dfrac{B \rightarrow B}{B, A \vee \neg A \rightarrow B}}{B, \neg B \Rightarrow B \rightarrow B}}{\neg B \Rightarrow B \rightarrow B, \neg B}}{\neg B, \neg B \Rightarrow B \rightarrow B}}{\neg B \Rightarrow B \rightarrow \neg B \Rightarrow B}$$

$$A \to A$$
$$A, \neg B \Rightarrow B \to A$$
$$\neg B \Rightarrow B \to A, \neg A$$
$$\neg B \Rightarrow B \to A \vee \neg A, \neg A$$
$$\neg B \Rightarrow B \to A \vee \neg A, A \vee \neg A \qquad \dfrac{B \to B}{B, A \vee \neg A \to B}$$
$$B, \neg B \Rightarrow B \to A \vee \neg A, B \qquad \dfrac{B \to B}{B, A \vee \neg A \to B}$$
$$B, B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to B, \neg B$$
$$\neg B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to \neg B \Rightarrow B$$

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$$B \to B$$
$$B, \neg B \Rightarrow B \to B$$
$$B, \neg B \Rightarrow B \to A \vee \neg A, B \qquad \dfrac{B \to B}{B, A \vee \neg A \to B}$$
$$B, B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to B, \neg B$$
$$\neg B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to \neg B \Rightarrow B$$

$>$

$$B \to B$$
$$B, B \to B$$
$$B, B, \neg B \Rightarrow B \to B$$
$$B, B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to B, \neg B$$
$$\neg B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to \neg B \Rightarrow B$$

$>$

$$B \to B$$
$$B, \neg B \Rightarrow B \to B$$
$$B, \neg B \Rightarrow B \to B, B$$
$$B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to B, \neg B$$
$$\neg B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to \neg B \Rightarrow B$$

$>$

$$B \to B$$
$$B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to B, \neg B$$
$$\neg B, \neg B \Rightarrow B \to B$$
$$\neg B \Rightarrow B \to \neg B \Rightarrow B$$

which does not reduce any further.

b/ Next we show that

$$\dfrac{\overset{\delta_2}{A\vee\neg A \to \neg B \Rightarrow B} \qquad \overset{\delta_1}{\neg B \Rightarrow B \to A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

does not reduce to $A\vee\neg A \to A\vee\neg A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A\vee\neg A \to B}}{A\vee\neg A \to B, \neg B}}{\neg B, A\vee\neg A \to B}}{A\vee\neg A \to \neg B \Rightarrow B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{\neg B \Rightarrow B \to A\vee\neg A, \neg A}}{\neg B \Rightarrow B \to A\vee\neg A, A\vee\neg A}}{\neg B \Rightarrow B \to A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A\vee A \to B}}{A\vee\neg A \to B, \neg B}}{\neg B, A\vee\neg A \to B}}{A\vee A \to \neg B \Rightarrow B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{\neg B \Rightarrow B \to A\vee\neg A, \neg A}}{\neg B \Rightarrow B \to A\vee\neg A, A\vee\neg A}}{A\vee\neg A \to A\vee\neg A, A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A\vee\neg A \to B}}{A\vee\neg A \to B, \neg B}}{\neg B, A\vee\neg A \to B}}{A\vee\neg A \to \neg B \Rightarrow B} \qquad \dfrac{\dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{\neg B \Rightarrow B \to A\vee\neg A, \neg A}}{A\vee\neg A \to A\vee\neg A, \neg A}}{A\vee\neg A \to A\vee\neg A, A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A\vee\neg A \to B}}{A\vee\neg A \to B, \neg B}}{\neg B, A\vee\neg A \to B}}{A\vee\neg A \to \neg B \Rightarrow B} \qquad \dfrac{\dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{\neg B \Rightarrow B \to A, \neg A}}{A\vee\neg A \to A, \neg A}}{A\vee\neg A \to A\vee\neg A, \neg A}}{\dfrac{A\vee\neg A \to A\vee\neg A, A\vee\neg A}{A\vee\neg A \to A\vee\neg A}}$$

>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{B, A\vee\neg A \to B}}{A\vee\neg A \to B, \neg B}}{\neg B, A\vee\neg A \to B}}{A\vee\neg A \to \neg B \Rightarrow B} \qquad \dfrac{A \to A}{A, \neg B \Rightarrow B \to A}}{A, A\vee\neg A \to A}}{A\vee\neg A \to A, \neg A}}{A\vee\neg A \to A\vee\neg A, \neg A}}{A\vee\neg A \to A\vee\neg A, A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

$>$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, A\vee\neg A \to A}}{A\vee\neg A \to A, \neg A}}{A\vee\neg A \to A\vee\neg A, \neg A}}{A\vee\neg A \to A\vee\neg A, A\vee\neg A}}{A\vee\neg A \to A\vee\neg A}$$

which does not reduce any further.

This completes the proof.

We remark on the meaning of this proof of equivalence. It can be regarded as an effort to show that the validity of $A \vee \neg A$ (the excluded middle) is equivalent to that of $\neg\neg B \Rightarrow B$. Indeed neither $\rightarrow A \vee \neg A$, nor $\rightarrow \neg\neg B \Rightarrow B$ is intuitionistically derivable.

In attempting to prove that the removal of the law of the excluded middle from a system, will also prevent one from being able to prove that $\neg\neg B \Rightarrow B$, this proof of equivalence can be invoked. Thus, it belongs to the weak equivalence level according to the criteria developed in *Chapter I*.

CHAPTER IV

AN EXTENSION OF L AND LK

In this chapter we extend L and LK to accomodate certain proofs from group theory. Following the description of the new language and deductive system, we give a cut-free formalisation of a proof of equivalence.

1: *The language L^* and the deductive system LK^*.*

a/ *Alphabet of L^*.*

(i) Variables: In addition to the variables of L we have $a', a_1', \ldots, b', b_1', \ldots, x', x_1', \ldots$

(ii) Logical symbols: Same as in L.

(iii) Auxiliary symbols: Same as in L.

(iv) Equality symbol:

$=$

(v) Functions symbols:

$f_0^0, f_1^0, \ldots, f_h^0, \ldots, f_0^1, \ldots, f_0^n, \ldots$

where the upper index is called the *degree* of the function.

(We shall write P_0^2 for f_0^2 and it will stand for product.)

(vii) Relational symbols:

$R, R_1, R_2, \ldots$

b/ *Formulas.*

(i) If x is a variable of L^* then x is a *term*.

(ii) If t_1 and t_2 are terms of L^* then $P_0^2(t_1,t_2)$ is a *term*.

(In practice we shall write x_1x_2 for $P_0^2(x_1,x_2)$).

(iii) If f_k^n is a function symbol of degree n and $t_1,\ldots,t_n$ are terms then $f_k^n(t_1,\ldots,t_n)$ is a *term*.

(In practice, we write $f_k^n t_1t_2\ldots t_n$ for $f_k^n(t_1,\ldots,t_n)$; in our proof, we shall use f for f_0^1).

(iv) If t_1 and t_2 are terms then $t_1 = t_2$ is an *atomic formula*.

(v) Every atomic formula is a *formula*.

(vi) If A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$ and $A \to B$ are *formulas*.

(vii) If $A(a)$ is a formula and t a term then $(\exists t)A(t)$ and $(\forall t)A(t)$ are *formulas*. ("$A(a)$ is a formula" means that A is a formula and a is a term occuring in A. In $(\exists t)A(t)$, $A(t)$ is obtained from $A(a)$ by substituting all occurences of a by t in A. Similarly for $(\forall t)A(t)$.)

(viii) Nothing else is a formula.

c/ *Sequents.*

The sequents of L^* are described exactly like the sequents of L except that the formulas in the sequents are from L^*.

d/ *The deductive system LK^*.*

LK^* is exactly as LK, except for the fact that the formulas are the ones in L^* and that the word term should replace the word variable wherever it occurs in the description of LK.

2: *Mathematical axioms and distinguished formulas.*

For the purpose of carrying out some mathematical proofs, we need certain mathematical axioms which regulate the behaviour of $=, P_0^2$ and f_0^1 .

a/ For equality.

E_1: $(\forall t)(t = t)$

E_2: $(\forall t)(\forall v)(t = v \rightarrow v = t)$

E_3: $(\forall t)(\forall v)(\forall w)((t = v \wedge v = w) \rightarrow t = w)$

Let E be $(E_1 \wedge E_2) \wedge E_3$

b/ For groups.

G_1: $(\forall t)(te = t)$

G_1': $(\forall t)(et = t)$

G_1'': $(\forall t)(t = et)$

G_1''': $(\forall t)(t = te)$

G_2: $(\forall t)(\forall v)(\forall w)((tv)w = t(vw))$

G_2': $(\forall t)(\forall v)(\forall w)(t(vw) = (tv)w)$

G_3: $(\forall t)(tt' = e)$

G_3': $(\forall t)(t't = e)$

Let G be $((((((G_1 \wedge G_1') \wedge G_1'') \wedge G_1''') \wedge G_2) \wedge G_2') \wedge G_3) \wedge G_3'$

c/ *For product.*

P_1: $(\forall t)(\forall v)(\forall u)(\forall w)((t = uv \wedge u = w) \Rightarrow t = wv)$

P_1': $(\forall t)(\forall v)(\forall u)(\forall w)((uv = t \wedge u = w) \Rightarrow wv = t)$

P_2: $(\forall t)(\forall v)(\forall u)(\forall w)((t = uv \wedge v = w) \Rightarrow t = uw)$

P_2': $(\forall t)(\forall v)(\forall u)(\forall w)((uv = t \wedge v = w) \Rightarrow uw = t)$

Let P be $((P_1 \wedge P_1') \wedge P_2) \wedge P_2'$

d/ *For functions.*

F_1: $(\forall t)(\exists v)(ft = v)$

F_2: $(\forall t)(\forall v)(t = v \Rightarrow ft = fv)$

e/ *For group homomorphisms.*

F_3: $(\forall t)(\forall v)(ftv = ftfv)$

F_3': $(\forall t)(\forall v)(ftfv = ftv)$

Let F be $((F_1 \wedge F_2) \wedge F_3) \wedge F_3'$

f/ *For substitution:*

S_1: $(\forall t)(\forall u)(\forall v)(t = u \Rightarrow tv = uv)$

S_2: $(\forall t)(\forall u)(\forall v)(t = u \Rightarrow vt = vu)$

Let S be $S_1 \wedge S_2$.

Let Axioms be $(((E\wedge G)\wedge P)\wedge F)\wedge S$

g/ *Distinguished formulas.*

In addition to the axiom formulas, we have two distinguished formulas of L^* :

$$F_4: \quad (\forall t)\,(\forall v)\,(ft = fv \to t = v)$$

and $\quad K: \quad (\forall t)\,(ft = e \to (\forall v)\,(tv = v \wedge vt = v))$

F_4 means that f is a group monomorphism and K means that $\mathrm{Ker}\, f = \{e\}$ where $f\colon G \to H$. Really, K only says $\mathrm{Ker}\, f \subseteq \{e\}$ but we prove in our derivation that $\{e\} \subseteq \mathrm{Ker}\, f$ so that there is no need to carry this information.

3: *A proof of equivalence.*

In order to illustrate the fact that the cut elimination problems discussed in *Chapter* I are solved completely by Gentzen's method of axiom formulas, we formalize a theorem of elementary group theory.

We *do not*, however, carry out on this proof the procedure of *Chapter* III for reasons that will become obvious after looking at the trees.

Before giving the theorem, we would like to comment on the fact that the derivation is cut-free and intuitionistically valid.

Theorem: Let $f: G \to H$ a group homomorphism then:

$$\text{Ker } f = \{e\} \text{ iff } f \text{ is a monomorphism.}$$

Proof:

We first give a proof in plain English arranged in such a way to make its formalisation simple.

Let f be a monomorphism

and let $c \in \text{Ker } f$, i.e., let $fc = e$.

For every a we have

$fca = fcfa$ (since f is a homomorphism)

$fca = efa$ (since $fc = e$)

$fca = fa$ (since $efa = fa$)

but f is a monomorphism, therefore $ca = a$. Similarly $ac = a$ for every a. Therefore if f is a monomorphism, $\text{Ker } f = \{e\}$.

Conversely: Let $\text{Ker } f = \{e\}$

i.e., $fc = e$ implies $c = e$

we have that

$fe = fee$ (since $e = ee$)

$fe = fefe$ (since f is a homomorphism)

this implies that

$(fe)(fe)' = (fefe)(fe)'$

$(fe)(fe)' = (fe)(fe(fe)')$ (associativity)

$(fe)(fe)' = (fe)e$ $((fe(fe)') = e)$

$(fe)(fe)' = fe$ $((fe)e = fe)$

but $(fe)(fe)' = e$

therefore $e = fe$.

Now if $fb = fa$

we have $fbfa' = fafa'$

$fbfa' = faa'$ (f is a homomorphism)

$fbfa' = fe$ $(aa' = e)$

$fbfa' = e$ $(e = fe)$

but $fbfa' = fba'$ (f is a homomorphism)

therefore $fba' = e$

which implies that $ba' = e$ $(\mathrm{Ker}\ f = \{e\})$

$$
\begin{aligned}
\text{i.e.,}\quad & (ba')a = ea & \\
& (ba')a = a & (ea = a) \\
\text{or}\quad & b(a'a) = a & (\text{associativity}) \\
& be = a & (a'a = e) \\
& b = a & (be = b)
\end{aligned}
$$

thus f is a monomorphism.

We now formalize this proof.

Throughout the proof, whenever we recognize a formula to be an axiom formula or any distinguished formula of L^*, we shall replace it by its namesake. For example, $(\forall t)(t = t)$ will be replaced by E_1 in the derivation.

We first prove that if f is a monomorphism, then $\text{Ker } f = \{e\}$.

[redacted]

$$fca = fcfa \to fca = fcfa \qquad fc = e \to fc = e$$

$$fca = fcfa, fc = e \to fca = fcfa \qquad fca = fcfa, fc = e \to fc = e$$

$$fca = fcfa, fc = e \to fca = fcfa \wedge fc = e$$

$$(\forall v)\,(fcv = fcfv), fc = e \to fca = fcfa \wedge fc = e$$

$$F_3, fc = e \to fca = fcfa \wedge fc = e \qquad fca = efa \to fca = efa$$

$$(fca = fcfa \wedge fc = e) \Rightarrow fca = efa, F_3, fc = e \to fca = efa$$

$$(\forall w)\,(fca = fca \wedge fc = w) \Rightarrow fca = wfa), F_3, fc = e \to fca = efa$$

$$(\forall u\,(\forall w)\,((fca = ufa \wedge u = w) \Rightarrow fca = wfa), F_3, fc = e \to fca = efa \qquad efa = f$$

$$(\forall v)\,(\forall u)\,(\forall w)\,((fca = uv \wedge u = w) \Rightarrow fca = wv), F_3, fc = e \to fca = efa \qquad efa = fa,$$

$$P_1, F_3, fc = e \to fca = efa \qquad efa = fa, P$$

$$efa = fa, P_1, F_3, fc = e \to fca = efa \qquad efa = fa, P_1, F$$

$$efa = fa, P_1, F_3, fc = e \to fca = efa \wedge efa = fa$$

$$G_1', P_1, F_3, fc = e \to fca = efa \wedge efa = fa$$

$$(fca = efa \wedge efa = fa) \Rightarrow fca = fa, G_1', P_1, F_3, f$$

$$(\forall w)\,((fca = ew \wedge ew = w) \Rightarrow fca = w), G_1', P_1, F_3, f$$

$$(\forall v)\,(\forall w)\,((fca = v \wedge v = w) \Rightarrow fca = w), G_1', P_1, F_3, f$$

$$E_3, G_1', P_1, F_3, fc = e \to fca = fa$$

We call this tree δ_1.

$c = e$

$a, fc = e \rightarrow fc = e$

$fc = e$

$cfa \wedge fc = e$

$e \qquad fca = efa \rightarrow fca = efa$

$, fc = e \rightarrow fca = efa$

$fa), F_3, fc = e \rightarrow fca = efa$

$wfa), F_3, fc = e \rightarrow fca = efa \qquad efa = fa \rightarrow efa = fa$

$a = wv), F_3, fc = e \rightarrow fca = efa \qquad efa = fa, P_1 \rightarrow efa = fa$

$= e \rightarrow fca = efa \qquad efa = fa, P_1, fc = e \rightarrow efa = fa$

$_3, fc = e \rightarrow fca = efa \qquad efa = fa, P_1, F_3, fc = e \rightarrow efa = fa$

$efa = fa, P_1, F_3, fc = e \rightarrow fca = efa \wedge efa = fa$

$G_1', P_1, F_3, fc = e \rightarrow fca = efa \wedge efa = fa \qquad fca = fa \rightarrow fca = fa$

$(fca = efa \wedge efa = fa) \Rightarrow fca = fa, G_1', P_1, F_3, fc = e \rightarrow fca = fa$

$(\forall w)((fca = ew \wedge ew = w) \Rightarrow fca = w), G_1', P_1, F_3, fc = e \rightarrow fca = fa$

$(\forall v)(\forall w)((fca = v \wedge v = w) \Rightarrow fca = w), G_1', P_1, F_3, fc = e \rightarrow fca = fa$

$E_3, G_1', P_1, F_3, fc = e \rightarrow fca = fa$

We call this tree δ_1.

$$\overset{\delta_1}{E_3, G_1', P_1, fc = e \rightarrow fca = fa} \qquad fa = fca \rightarrow$$

$$\overline{fca = fa \rightarrow fa = fca, E_3, G_1', P_1, fc = e \rightarrow fa}$$

$$\overline{(\forall v)\,(fa = v \rightarrow v = fa), E_3, G_1', P_1, fc =}$$

$$\overline{E_2, E_3, G_1', P_1, fc = e \rightarrow fa = fc}$$

$$\overline{fa = fca \rightarrow a = ca, E_2, E_3, G_1', P_1,}$$

$$\overline{(\forall v)\,(fa = fv \rightarrow a = v), E_2, E_3, G_1',}$$

$$\overline{F_4, E_2, E_3, G_1', P_1, fc = e \rightarrow ac}$$

$$\overset{\delta_1}{E_3, G_1', P_1, F_3, fc = e \rightarrow fca = fa} \qquad ca = a \rightarrow ca = a$$

$$\overline{fca = fa \rightarrow ca = a, E_3, G_1', P_1, fc = e \rightarrow ca = a}$$

$$\overline{(\forall v)\,(fca = fv \rightarrow ca = v), E_3, G_1', P_1, fc = e \rightarrow ca = a}$$

$$\overline{F_4, E_3, G_1', P_1, fc = e \rightarrow ca = a}$$

$$\overline{F_4, E_2, E_3, G_1', P_1, fc = e \rightarrow ca = a}$$

$$\overline{F_4, E_2, E_3, G_1', P_1, fc = e \rightarrow ca = a \wedge ac = a}$$

$$\overline{F_4, E_2, E_3, G_1', fc = e, P_1 \rightarrow ca = a \wedge ac = a}$$

$$\overline{F_4, E_2, E_3, fc = e, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\overline{F_4, E_2, fc = e, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\overline{F_4, fc = e, E_2, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\overline{fc = e, F_4, E_2, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\overline{fc = e, F_4, E_2, E_3, G_1', P_1 \rightarrow (\forall v)\,(cv = v \wedge vc = v)}$$

$$\overline{F_4, E_2, E_3, G_1', P_1 \rightarrow fc = e \rightarrow (\forall v)\,(cv = v \wedge vc = v)}$$

$$F_4, E_2, E_3, G_1', P_1 \rightarrow K$$

We call this tree δ_2.

$$\delta_1$$

$$\frac{E_3, G_1', P_1, fc = e \rightarrow fca = fa \qquad fa = fca \rightarrow fa = fca}{fca = fa \Rightarrow fa = fca, E_3, G_1', P_1, fc = e \rightarrow fa = fca}$$

$$\frac{fca = fa \Rightarrow fa = fca, E_3, G_1', P_1, fc = e \rightarrow fa = fca}{(\forall v)\,(fa = v \Rightarrow v = fa), E_3, G_1', P_1, fc = e \rightarrow fa = fca}$$

$$\frac{(\forall v)\,(fa = v \Rightarrow v = fa), E_3, G_1', P_1, fc = e \rightarrow fa = fca}{E_2, E_3, G_1', P_1, fc = e \rightarrow fa = fca \qquad ac = a \rightarrow ac = a}$$

$$\frac{E_2, E_3, G_1', P_1, fc = e \rightarrow fa = fca \qquad ac = a \rightarrow ac = a}{fa = fca \Rightarrow a = ca, E_2, E_3, G_1', P_1, fc = e \rightarrow ac = a}$$

$$\frac{fa = fca \Rightarrow a = ca, E_2, E_3, G_1', P_1, fc = e \rightarrow ac = a}{(\forall v)\,(fa = fv \Rightarrow a = v), E_2, E_3, G_1', P_1, fc = e \rightarrow ac = a}$$

$$\frac{(\forall v)\,(fa = fv \Rightarrow a = v), E_2, E_3, G_1', P_1, fc = e \rightarrow ac = a}{F_4, E_2, E_3, G_1', P_1, fc = e \rightarrow ac = a}$$

$$\ldots a = a \rightarrow ca = a$$

$$\ldots a$$

$$\ldots e \rightarrow ca = a$$

$$\ldots \rightarrow ca = a$$

$$\ldots \rightarrow ca = a$$

$$\frac{}{F_4, E_2, E_3, G_1', P_1, fc = e \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{F_4, E_2, E_3, G_1', fc = e, P_1 \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{F_4, E_2, E_3, fc = e, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{F_4, E_2, fc = e, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{F_4, fc = e, E_2, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{fc = e, F_4, E_2, E_3, G_1', P_1 \rightarrow ca = a \wedge ac = a}$$

$$\frac{}{fc = e, F_4, E_2, E_3, G_1', P_1 \rightarrow (\forall v)\,(cv = v \wedge vc = v)}$$

$$\frac{}{F_4, E_2, E_3, G_1', P_1 \rightarrow fc = e \Rightarrow (\forall v)\,(cv = v \wedge vc = v)}$$

$$F_4, E_2, E_3, G_1', P_1 \rightarrow K$$

We call this tree δ_2.

By successive thinnings and interchanges on δ_2 we obtain

δ_3:

$$\dfrac{\overset{\delta_2}{F_4, E_2, E_3, G_1', P_1 \to K}}{E_1, E_2, E_3, G_1, G_1', G_1'', G_1''', G_2, G_2', G_3, G_3', P_1, P_1', P_2, P_2', F_1, F_2', F_3, F_3', S, F_4 \to K}$$

On δ_3 we operate as in the following example:

$$\dfrac{\dfrac{\dfrac{A, B \to C}{A \wedge B, B \to C}}{A \wedge B, A \wedge B \to C}}{A \wedge B \to C}$$

and obtain δ_4 : $((((E \wedge G) \wedge P) \wedge F) \wedge S) \wedge F_4 \to K$

i.e., $\overset{\delta_4}{\text{Axioms} \wedge F_4 \to K}$

Now from δ_4 we obtain δ by:

$$\dfrac{\dfrac{\text{Axioms} \to \text{Axioms}}{\text{Axioms} \wedge F_4 \to \text{Axioms}} \qquad \overset{\delta_4}{\text{Axioms} \wedge F_4 \to K}}{\text{Axioms} \wedge F_4 \to \text{Axioms} \wedge K}$$

Remark: Our δ_4 states that from the axioms and the fact that f is a monomorphism we can deduce that $\operatorname{Ker} f = \{e\}$.

We now tackle the converse.

[redacted]

$$e = ee \rightarrow e = ee \qquad fe = fee \rightarrow fe = fee$$

$$e = ee \Rightarrow fe = fee, e = ee \rightarrow fe = fee$$

$$(\forall v)(e = v \Rightarrow fe = fv), e = ee \rightarrow fe = fee$$

$$F_2, e = ee \rightarrow fe = fee$$

$$e = ee, F_2 \rightarrow fe = fee$$

$$G_1'', F_2 \rightarrow fe = fee$$

$$G_1'', F_2, F_3 \rightarrow fe = fee$$

$$fee = fefe \rightarrow fee = fefe$$

$$G_1'', fee = fefe \rightarrow fee = fefe$$

$$G_1'', F_2, fee = fefe \rightarrow fee = fefe$$

$$G_1'', F_2, (\forall v)(fev = fefv) \rightarrow fe = fefe$$

$$G_1'', F_2, F_3 \rightarrow fee = fefe$$

$$G_1'', F_2, F_3 \rightarrow fe = fee \wedge fee = fefe \qquad fe = fefe \rightarrow fe = fe$$

$$(fe = fee \wedge fee = fefe) \Rightarrow fe = fee, G_1'', F_2, F_3 \rightarrow fe = fefe$$

$$(\forall w)((fe = fee \wedge fee = w) \Rightarrow fe = w), G_1'', F_2, F_3 \rightarrow fe = fefe$$

$$(\forall v)(\forall w)((fe = v \wedge v = w) \Rightarrow fe = w), G_1'', F_2, F_3 \rightarrow fe = fefe$$

$$E_3, G_1'', F_2, F_3 \rightarrow fe = fefe \qquad (fe)(fe)' = (fefe)$$

$$fe = fefe \Rightarrow (fe)(fe)' = (fefe)(fe)', E_3 G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fe$$

$$(\forall v)(fe = fefe \Rightarrow (fe)v = (fefe)v), E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (f$$

$$(\forall u)(\forall v)(fe = u \Rightarrow (fe)v = uv) E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)$$

$$S_1, E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)'$$

We call this tree λ_1.

$$\underline{fee = fefe \rightarrow fee = fefe}$$

$$\underline{G_1'', fee = fefe \rightarrow fee = fefe}$$

$$\underline{G_1'', F_2, fee = fefe \rightarrow fee = fefe}$$

$$\underline{G_1'', F_2, (\forall v)(fev = fefv) \rightarrow fe = fefe}$$

$$\underline{G_1'', F_2, F_3 \rightarrow fee = fefe}$$

$$\underline{fee = fefe} \qquad \underline{fe = fefe \rightarrow fe = fefe}$$

$$\underline{e) \Rightarrow fe = fee, G_1'', F_2, F_3 \rightarrow fe = fefe}$$

$$\underline{e = w) \Rightarrow fe = w), G_1'', F_2, F_3 \rightarrow fe = fefe}$$

$$\underline{= w) \Rightarrow fe = w), G_1'', F_2, F_3 \rightarrow fe = fefe}$$

$$\underline{E_3, G_1'', F_2, F_3 \rightarrow fe = fefe \qquad (fe)(fe)' = (fefe)(fe)' \rightarrow (fe)(fe)' = (fefe)(fe)'}$$

$$\underline{fe = fefe \Rightarrow (fe)(fe)' = (fefe)(fe)', E_3 G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)'}$$

$$\underline{(\forall v)(fe = fefe \Rightarrow (fe)v = (fefe)v), E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)'}$$

$$\underline{(\forall u)(\forall v)(fe = u \Rightarrow (fe)v = uv) E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)'}$$

$$S_1, E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)'$$

We call this tree λ_1.

$$b(a'a) = (ba')a \to b(a'a) = (ba')a$$

$$(\forall w)(b(a'w) = (ba')w) \to b(a'a) = (ba')a$$

$$(\forall v)(\forall w)(b(vw) = (bv)w) \to b(a'a) = (ba')a$$

$$G_2' \to b(a'a) = (ba')a$$

$$K, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_2' \to b(a'a) = (ba')a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_2' \to b(a'a) = (ba')a \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb =$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = (ba')a \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa,$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = (ba')a \land (ba')a$$

$$(b(a'a) = (ba')a \land (ba')a = a) \Rightarrow b(a'a) = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G$$

$$(\forall w)((b(a'a) = (ba')a \land (ba')a = w) \Rightarrow b(a'a) = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1'''$$

$$(\forall v)(\forall w)((b(a'a) = v \land v = w) \Rightarrow b(a'a) = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b($$

$$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$$

$$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$$

We call this tree. λ_{12}

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$$

[redacted]

2.

$a')a \to b(a'a) = (ba')a$

$) = (ba')w) \to b(a'a) = (ba')a$

$= (bv)w) \to b(a'a) = (ba')a$

$G_2^1 \to b(a'a) = (ba')a$

$K, G_2^1 \to b(a'a) = (ba')a$

$K, E_3, G_2^1 \to b(a'a) = (ba')a$

$_3, E_2, G_2^1 \to b(a'a) = (ba')a$

$_2, P_2', G_2' \to b(a'a) = (ba')a$

$_2', S_1, G_2' \to b(a'a) = (ba')a$

$_1, G_1'', G_2' \to b(a'a) = (ba')a$

$_1'', F_2, G_2' \to b(a'a) = (ba')a$

$_2, F_3, G_2' \to b(a'a) = (ba')a$

$_3, G_2, G_2' \to b(a'a) = (ba')a$

$_2, G_3, G_2' \to b(a'a) = (ba')a$

$_3, G_1, G_2' \to b(a'a) = (ba')a$

$_1, fb = fa, G_2' \to b(a'a) = (ba')a$

$= fa, G_1''', G_2' \to b(a'a) = (ba')a$ $\qquad$ λ_{11} $K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \to (ba')a = a$

$, G_1''', G_1', G_2' \to b(a'a) = (ba')a$ $\qquad$ $K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to (ba')a = a$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = (ba')a \wedge (ba')a = a$ $\qquad$ $b(a'a) = a \to b(a'a) = a$

$(ba')a \wedge (ba')a = a) \Rightarrow b(a'a) = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

$= (ba')a \wedge (ba')a = w) \Rightarrow b(a'a) = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

$= v \wedge v = w) \Rightarrow b(a'a) = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2' \to b(a'a) = a$

[redacted]

$$(fefe)(fe)' = (fe)(fe(fe)') \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$(\forall w)((fefe)w = fe(few)) \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$(\forall v)(\forall w)((fev)w = fe(vw)) \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$S_1, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$S_1, E_3, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$S_1, E_3, G_1'', G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$\lambda_1$$

$$S_1, E_3, G_1'', F_2, F_3 \rightarrow (fe)(fe)' = (fefe)(fe)' \qquad S_1, E_3, G_1'', F_2, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fefe)(fe)' \qquad S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$$

$$S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fefe)(fe)' \wedge (fefe)(fe)' = (fe)(fe(fe)') \qquad (fe)(fe)' = ($$

$$((fe)(fe)' = (fefe)(fe)' \wedge (fefe)(fe)' = (fe)(fe(fe)') \Rightarrow (fe)(fe)' = (fe)(fe(fe)'), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe$$

$$(\forall w)(((fe)(fe)' = (fefe)(fe)' \wedge (fefe)(fe)' = w) \Rightarrow (fe)(fe)' = w), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$$

$$(\forall v)(\forall w)(((fe)(fe)' = v \wedge v = w) \Rightarrow (fe)(fe)' = w), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$$

$$E_3, S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$$

$$E_3, E_3, S_1, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$$

$$E_3, S_1, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$$

We call this tree λ_2

$(fefe)(fe)' = (fe)(fe(fe)') \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$(\forall w)((fefe)w = fe(few)) \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$(\forall v)(\forall w)((fev)\ w = fe(vw)) \rightarrow (fefe)(fe)' = (fe)(fe)fe)')$

$G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$S_1, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$S_1, E_3, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$S_1, E_3, G_1'', G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$fe)' \qquad S_1, E_3, G_1'', F_2, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$fe)' \qquad S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fefe)(fe)' = (fe)(fe(fe)')$

$F_3, G_2 \rightarrow (fe)(fe)' = (fefe)(fe)' \wedge (fefe)(fe)' = (fe)(fe(fe)') \qquad (fe)(fe)' = (fe)(fe(fe)') \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$)(fe(fe)') \Rightarrow (fe)(fe)' = (fe)(fe(fe)'), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$e)' = w) \Rightarrow (fe)(fe)' = w), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$= w), S_1, E_3, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

$, G_1'', F_2, F_3, G_2 \rightarrow (fe)(fe)' = (fe)(fe(fe)')$

We call this tree λ_2

[redacted]

$$\underline{fe(fe)'=e \to fe(fe)'=e}$$
$$\underline{G_3 \to fe(fe)'=e}$$
$$\underline{E_3, G_3 \to fe(fe)'=e}$$
$$\underline{E_3, S_1, G_3 \to fe(fe)'=e}$$
$$\underline{E_3, S_1, G_1'', G_3 \to fe(fe)'=e}$$
$$\underline{E_3, S_1, G_1'', F_2, G_3 \to fe(fe)'=e}$$

$$\lambda_2$$
$$\underline{E_3, S_1, G_1'', F_2, F_3, G_2 \to (fe)(fe)'=(fe)(fe(fe)')} \qquad \underline{E_3, S_1, G_1'', F_2, F_3, G_3 \to fe(fe)'=e}$$
$$\underline{E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)(fe(fe)')} \qquad \underline{E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to fe(fe)'=e}$$

$$\underline{E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)(fe(fe)') \wedge fe(fe)'=e} \qquad (fe)(fe$$
$$\underline{((fe)(fe)'=(fe)(fe(fe)') \wedge fe(fe)'=e) \Rightarrow (fe)(fe)'=(fe)e, E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe}$$
$$\underline{(\forall w)(((fe)(fe)'=(fe)(fe(fe)') \wedge fe(fe)'=w) \Rightarrow (fe)(fe)'=few), E_3, S_1, G_1'', F_2, F_3, G}$$
$$\underline{(\forall u)(\forall w)(((fe)(fe)'=u(fe(fe')) \wedge (fe(fe)')=w) \Rightarrow (fe)(fe)'=uw), E_3, S_1, G_1'', F_2, F_3,}$$
$$\underline{(\forall v)(\forall u)(\forall w)(((fe)(fe)'=uv \wedge v=w \Rightarrow (fe)(fe)'=uw), E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(}$$
$$P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e$$

We call this tree λ_3

$$\underline{fe(fe)'=e \to fe(fe)'=e}$$

$$\underline{G_3 \to fe(fe)'=e}$$

$$\underline{E_3, G_3 \to fe(fe)'=e}$$

$$\underline{E_3, S_1, G_3 \to fe(fe)'=e}$$

$$\underline{E_3, S_1, G_1'', G_3 \to fe(fe)'=e}$$

$$\underline{E_3, S_1, G_1'', F_2, G_3 \to fe(fe)'=e}$$

$$e)') \qquad \underline{E_3, S_1, G_1'', F_2, F_3, G_3 \to fe(fe)'=e}$$

$$e(fe)') \qquad \underline{E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to fe(fe)'=e}$$

$$\underline{{}_3, G_2, G_3 \to (fe)(fe)'=(fe)(fe(fe)') \wedge fe(fe)'=e} \qquad \underline{(fe)(fe)'=(fe)e \to (fe)(fe)'=(fe)e}$$

$$\underline{e)(fe(fe)') \wedge fe(fe)'=e) \Rightarrow (fe)(fe)'=(fe)e, E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e}$$

$$\underline{(fe)'=(fe)(fe(fe)') \wedge fe(fe)'=w) \Rightarrow (fe)(fe)'=few), E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e}$$

$$\underline{fe)(fe)'=u(fe(fe)')) \wedge (fe(fe)')=w) \Rightarrow (fe)(fe)'=uw), E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e}$$

$$\underline{w)(((fe)(fe)'=uv \wedge v=w \Rightarrow (fe)(fe)'=uw), E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e}$$

$$P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \to (fe)(fe)'=(fe)e$$

We call this tree λ_3^*

[redacted]

$$\frac{(fe)e = fe \rightarrow (fe)e = fe}{G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', E_3, G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', E_3, S_1, G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', E_3, S_1, G_1'', G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', E_3, S_1, G_1'', F_2, G_1 \rightarrow (fe)e = fe}$$

$$\frac{}{P_2', E_3, S_1, G_1'', F_2, F_3, G_1 \rightarrow (fe)e = fe}$$

λ_3

$$\frac{P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3 \rightarrow (fe)(fe)' = (fe)e}{P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = (fe)e} \qquad \frac{P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_1 \rightarrow (fe)e = fe}{P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)e = fe}$$

$$\frac{P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = (fe)e \wedge (fe)e = fe \qquad (fe)(fe)' = fe \rightarrow (}{((fe)(fe)' = (fe)e \wedge (fe)e = fe) \Rightarrow (fe)(fe)' = fe, P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe}$$

$$\frac{}{(\forall w)(((fe)(fe)' = (fe)e \wedge (fe)e = w) \Rightarrow (fe)(fe)' = w), P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe}$$

$$\frac{}{(\forall v)(\forall w)(((fe)(fe)' = v \wedge v = w) \Rightarrow (fe)(fe)' = w), P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe}$$

$$\frac{}{E_3, P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe}$$

$$\frac{}{E_3, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe}$$

$$\frac{E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe \qquad fe = (fe)(fe)' \rightarrow}{(fe)(fe)' = fe \Rightarrow fe = (fe)(fe)', E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (}$$

$$\frac{}{(\forall v)((fe)(fe)' = v \Rightarrow v = (fe)(fe)'), E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow f}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)(fe)'}$$

We call this tree λ_4.

$$(fe)e = fe \rightarrow (fe)e = fe$$

$$G_1 \rightarrow (fe)e = fe$$

$$P_2', G_1 \rightarrow (fe)e = fe$$

$$P_2', E_3, G_1 \rightarrow (fe)e = fe$$

$$P_2', E_3, S_1, G_1 \rightarrow (fe)e = fe$$

$$P_2', E_3, S_1, G_1'', G_1 \rightarrow (fe)e = fe$$

$$P_2', E_3, S_1, G_1'', F_2, G_1 \rightarrow (fe)e = fe$$

$$P_2', E_3, S_1, G_1'', F_2, F_3, G_1 \rightarrow (fe)e = fe$$

$$)e \qquad P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_1 \rightarrow (fe)e = fe$$

$$(fe)e \qquad P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)e = fe$$

$$, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = (fe)e \wedge (fe)e = fe \qquad (fe)(fe)' = fe \rightarrow (fe)(fe)' = fe$$

$$)e \wedge (fe)e = fe) \Rightarrow (fe)(fe)' = fe, P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe$$

$$)' = (fe)e \wedge (fe)e = w) \Rightarrow (fe)(fe)' = w), P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe$$

$$(fe)' = v \wedge v = w) \Rightarrow (fe)(fe)' = w), P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe$$

$$E_3, P_2', E_3, S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe$$

$$E_3, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe$$

$$E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = fe \qquad fe = (fe)(fe)' \rightarrow fe = (fe)(fe)'$$

$$(fe)(fe)' = fe \Rightarrow fe = (fe)(fe)', E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)(fe)'$$

$$(\forall v)((fe)(fe)' = v \Rightarrow v = (fe)(fe)'), E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)(fe)'$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)(fe)'$$

We call this tree λ_4

$$\underline{(fe)\,(fe)' = e \rightarrow (fe)\,(fe)' = e}$$

$$\underline{G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3 \rightarrow (fe)\,(fe)' = e}$$

$$\lambda_4$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)\,(fe)' \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)\,(fe)' = e}$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)\,(fe)' \wedge (fe)\,(fe)' = e}$$

$$\underline{(fe = (fe)\,(fe)' \wedge (fe)\,(fe)' = e) \Rightarrow fe = e, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe}$$

$$\underline{(\forall w)\,((fe = (fe)\,(fe)' \wedge (fe)\,(fe)' = w) \Rightarrow fe = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3}$$

$$\underline{(\forall v)\,(\forall w)\,((fe = v \wedge v = w) \Rightarrow fe = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe =}$$

$$\underline{E_3, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$\underline{E_2, E_3, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e$$

We call this tree λ_5

$$\frac{(fe)(fe)' = e \rightarrow (fe)(fe)' = e}{G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_1'', G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_1'', F_2, G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_3 \rightarrow (fe)(fe)' = e}$$

$$\frac{}{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3 \rightarrow (fe)(fe)' = e}$$

$$fe = (fe)(fe)' \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow (fe)(fe)' = e$$

$$\frac{, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = (fe)(fe)' \wedge (fe)(fe)' = e \qquad fe = e \rightarrow fe = e}{fe = (fe)(fe)' \wedge (fe)(fe)' = e) \Rightarrow fe = e, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$\frac{}{\forall w)((fe = (fe)(fe)' \wedge (fe)(fe)' = w) \Rightarrow fe = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$\frac{}{(\forall v)(\forall w)((fe = v \wedge v = w) \Rightarrow fe = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$\frac{}{E_3, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$\frac{}{E_2, E_3, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e}$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow fe = e$$

We call this tree λ_5

$$fb = fa \to fb = fa \qquad fbfa' = fafa' \to fbfa' = fafa'$$

$$fb = fa \Rightarrow fbfa' = fafa', fb = fa \to fbfa' = fafa' \qquad fafa' = faa' \to fafa' = faa'$$

$$(\forall v)(fb = fa \Rightarrow fbv = fav), fb = fa \to fbfa' = fafa' \qquad (\forall v)(fafv = fav) \to fafa' = faa' \qquad aa' = e \to aa' = e$$

$$(\forall u)(\forall v)(fb = u \Rightarrow fbv = uv), fb = fa \to fbfa' = fafa' \qquad F'_3 \to fafa' = faa' \qquad aa' = e$$

$$S_1, fb = fa \to fbfa' = fafa' \qquad F'_3, S_1 \to fafa' = faa' \qquad (\forall v)(aa' = v$$

$$F'_3, S_1, fb = fa \to fbfa' = fafa' \qquad F'_3, S_1, fb = fa \to fafa' = faa'$$

$$F'_3, S_1, fb = fa \to fbfa' = fafa' \wedge fafa' = faa' \qquad fbfa' = faa' \to fbfa' = f$$

$$(fbfa' = fafa' \wedge fafa' = faa') \Rightarrow fbfa' = faa', F'_3, S_1, fb = fa \to fbfa' = faa'$$

$$(\forall w)((fbfa' = fafa' \wedge fafa' = w) \Rightarrow fbfa' = w), F'_3, S_1, fb = fa \to fbfa' = faa'$$

$$(\forall v)(\forall w)((fbfa' = v \wedge v = w) \Rightarrow fbfa' = w), F'_3, S_1, fb = fa \to fbfa' = faa'$$

$$E_3, F'_3, S_1, fb = fa \to fbfa' = faa'$$

$$E_3, F'_3, S_1, F_2, fb = fa \to fbfa' = faa'$$

$$E_3, F'_3, S_1, F_2, G_3, fb = fa \to fbfa' = faa' \qquad E_3$$

$$E_3, F'_3, S_1, F_2, G_3, fb = fa \to fbfa' = faa' \wedge fa$$

We call this tree λ_6.

$fa' = fafa'$

$= fafa'$ $\qquad$ $fafa' = faa' \rightarrow fafa' = faa'$

$= fafa'$ $\qquad$ $(\forall v)(fafv = fav) \rightarrow fafa' = faa'$ $\qquad$ $aa' = e \rightarrow aa' = e$ $\qquad$ $faa' = e \rightarrow faa' = e$

$= fafa'$ $\qquad$ $F_3' \rightarrow fafa' = faa'$ $\qquad$ $aa' = e \Rightarrow faa' = e, aa' = e \rightarrow faa' = e$

afa' $\qquad$ $F_3', S_1 \rightarrow fafa' = faa'$ $\qquad$ $(\forall v)(aa' = v \Rightarrow faa' = fv), aa' = e \rightarrow faa' = fe$

afa' $\qquad$ $F_3', S_1, fb = fa \rightarrow fafa' = faa'$ $\qquad$ $F_2, aa' = e \rightarrow faa' = fe$

$b = fa \rightarrow fbfa' = fafa' \wedge fafa' = faa'$ $\qquad$ $fbfa' = faa' \rightarrow fbfa' = faa'$ $\qquad$ $aa' = e, F_2 \rightarrow faa' = fe$

$fa' = fafa' \wedge fafa' = faa') \Rightarrow fbfa' = faa', F_3', S_1, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $G_3, F_2 \rightarrow faa' = fe$

$bfa' = fafa' \wedge fafa' = w) \Rightarrow fbfa' = w), F_3', S_1, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $F_2, G_3 \rightarrow faa' = fe$

$((fbfa' = v \wedge v = w) \Rightarrow fbfa' = w), F_3', S_1, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $E_3, F_2, G_3 \rightarrow faa' = fe$

$E_3, F_3', S_1, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $E_3, F_3', F_2, G_3 \rightarrow faa' = fe$

$E_3, F_3', S_1, F_2, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $E_3, F_3', S_1, F_2, G_3 \rightarrow faa' = fe$

$E_3, F_3', S_1, F_2, G_3, fb = fa \rightarrow fbfa' = faa'$ $\qquad$ $E_3, F_3', S_1, F_2, G_3, fb = fa \rightarrow faa' = fe$

$$E_3, F_3', S_1, F_2, G_3, fb = fa \rightarrow fbfa' = faa' \wedge faa' = fe$$

We call this tree λ_6

$$\lambda_6$$

$$E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = faa' \wedge faa' = fe \qquad fbfa' = fe \to fbfa' = fe$$

$$(fbfa' = faa' \wedge faa' = fe) \Rightarrow fbfa' = fe, E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$(\forall w)((fbfa' = faa' \wedge faa' = w) \Rightarrow fbfa' = w), E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$(\forall v)(\forall w)((fbfa' = v \wedge v = w) \Rightarrow fbfa' = w, E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_3, E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, P_2', F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, P_2', S_1, F_3', F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, P_2', S_1, G_1'', F_3', F_2, G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3', G_3, fb = fa \to fbfa' = fe$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, E_3, G_2, G_3, fb = fa \to fbfa' = fe \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, E_3, G_2, G_3, G_1, fb = fa \to fbfa' = fe \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to fbfa' = fe \wedge fe = e$$

We call this tree λ_7.

$\wedge faa' = fe \qquad fbfa' = fe \to fbfa' = fe$

$E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$w), E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$E_3, F_3', S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$S_1, F_2, G_3, fb = fa \to fbfa' = fe$

$F_3', F_2, G_3, fb = fa \to fbfa' = fe$

$F_3', F_2, G_3, fb = fa \to fbfa' = fe$

$F_2, F_3', G_3, fb = fa \to fbfa' = fe \qquad\qquad \lambda_5$

$E_3, G_2, G_3, fb = fa \to fbfa' = fe \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \to fe = e$

$G_2, G_3, G_1, fb = fa \to fbfa' = fe \qquad E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to fe = e$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to fbfa' = fe \wedge fe = e$$

We call this tree λ_7

$$\text{fba}' = \text{fbfa}' \rightarrow \text{fba}' = \text{fbfa}'$$

$$(\forall v)(\text{fbv} = \text{fbfv}) \rightarrow \text{fba}' = \text{fbfa}'$$

$$F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1 \rightarrow \text{fba}' = \text{fbfa}'$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = \text{fbfa}'$$

$$\lambda_7$$

$$E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fbfa}' = \text{fe} \wedge \text{f}$$

$$(\text{fbfa}' = \text{fe} \wedge \text{fe} = e) \Rightarrow \text{fbfa}' = e, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2$$

$$(\forall w)((\text{fbfa}' = \text{fe} \wedge \text{fe} = w) \Rightarrow \text{fbfa}' = w), E_2, E_3, P_2', S_1, G_1'', F$$

$$(\forall v)(\forall w)((\text{fbfa}' = v \wedge v = w) \Rightarrow \text{fbfa}' = w), E_2, E_3, P_2', S_1, G_1''$$

$$E_3, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G$$

$$E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = \text{fbfa}' \wedge \text{fbfa}' = e$$

$$(\text{fba}' = \text{fbfa}' \wedge \text{fbfa}' = e) \Rightarrow \text{fba}' = e, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = e$$

$$(\forall w)((\text{fba}' = \text{fbfa}' \wedge \text{fbfa}' = w) \Rightarrow \text{fba}' = w), E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}$$

$$(\forall v)(\forall w)((\text{fba}' = v \wedge v = w) \Rightarrow \text{fba}' = w), E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = e$$

$$E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, \text{fb} = \text{fa} \rightarrow \text{fba}' = e$$

We call this tree λ_8

$$\lambda_7$$

$$\underline{E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = fe \wedge fe = e \qquad fbfa' = e \rightarrow fbfa' = e}$$

$$\underline{(fbfa' = fe \wedge fe = e) \Rightarrow fbfa' = e, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{(\forall w)((fbfa' = fe \wedge fe = w) \Rightarrow fbfa' = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{(\forall v)(\forall w)((fbfa' = v \wedge v = w) \Rightarrow fbfa' = w), E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{E_3, E_2, E_3, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{fa'}$$

$$\underline{ba' = fbfa'} \qquad \underline{E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fbfa' = e}$$

$$\underline{F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = fbfa' \wedge fbfa' = e \qquad fba' = e \rightarrow fba' = e}$$

$$\underline{fbfa' = e) \Rightarrow fba' = e, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e}$$

$$\underline{a' \wedge fbfa' = w) \Rightarrow fba' = w), E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e}$$

$$\underline{= w) \Rightarrow fba' = w), E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e}$$

$$\underline{E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e}$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e$$

We call this tree λ_8

$$ba' = (ba)'e \rightarrow ba' = (ba')e$$

$$G_1''' \rightarrow ba' = (ba')e$$

$$E_3, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, G_1''' \rightarrow ba' = (ba')e$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = (ba')e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = (ba')e$$

$$\lambda_8$$

$$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow fba' = e$$

$$fba' = e \Rightarrow (\forall v)((ba')v = v \wedge v(ba)' = v), E_3, E_2, P_2', S$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \rightarrow (b$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1'''$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = (ba')e \wedge (ba')e = e$$

$$(ba' = (ba')e \wedge (ba')e = e) \Rightarrow ba' = e, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow$$

$$(\forall w)((ba' = (ba')e \wedge (ba')e = w) \Rightarrow ba' = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb$$

$$(\forall v)(\forall w)((ba' = v \wedge v = w) \Rightarrow ba' = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G$$

$$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = e$$

$$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = e$$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$ba')e$

$= (ba')e$

$\to ba' = (ba')e$

$G_1''' \to ba' = (ba')e$

$(ba')e = e \to (ba')e = e$

$(ba')e = e \wedge e(ba') = e \to (ba')e = e$

λ_8

$E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to fba' = e$ $\qquad (\forall v)((ba')v = v \wedge v(ba') = v) \to (ba')e = e$

$fba' = e \Rightarrow (\forall v)((ba')v = v \wedge v(ba)' = v), E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to (ba')e = e$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa \to (ba')e = e$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to (ba')e = e$

$, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = (ba')e \wedge (ba')e = e$ $\qquad ba' = e \to ba' = e$

$e \wedge (ba')e = e) \Rightarrow ba' = e, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

$' = (ba')e \wedge (ba')e = w) \Rightarrow ba' = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

$((ba' = v \wedge v = w) \Rightarrow ba' = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \to ba' = e$

We call this tree λ_9.

$$\overset{\lambda_9}{K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow ba' = e} \qquad (ba')a = ea \rightarrow (ba')a = ea$$

$$ba' = e \Rightarrow (ba')a = ea, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$(\forall v)\ (ba' = e \Rightarrow (ba')v = ev), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$(\forall u)\ (\forall v)\ (ba' = u \Rightarrow (ba')v = uv), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$S_1, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$K, S_1, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$K, E_3, S_1, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$K, E_3, E_2, S_1, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$K, E_3, E_2, P_2', S_1, S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea$$

We call this tree λ_{10}

$$\lambda_9$$

$$\underline{G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to ba') = e \qquad (ba')a = ea \to (ba')a = ea}$$

$$\underline{(ba')a = ea, K, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{e \Rightarrow (ba')v = ev), K, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{u \Rightarrow (ba')v = uv), K, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{S_1, K, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{K, S_1, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{K, E_3, S_1, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{K, E_3, E_2, S_1, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$\underline{K, E_3, E_2, P'_2, S_1, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea}$$

$$K, E_3, E_2, P'_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1 \to (ba')a = ea$$

We call this tree λ_{10}

$$ea = a \rightarrow ea =$$

$$G_1' \rightarrow ea$$

$$K, G_1' \rightarrow ea$$

$$K, E_3, G_1' \rightarrow ea$$

$$K, E_3, E_2, G_1' \rightarrow ea$$

$$K, E_3, E_2, P_2', G_1' \rightarrow ea$$

$$K, E_3, E_2, P_2', S_1, G_1' \rightarrow ea$$

$$K, E_3, E_2, P_2', S_1, G_1'', G_1' \rightarrow ea$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, G_1' \rightarrow ea$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_1' \rightarrow ea =$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1' \rightarrow ea =$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1' \rightarrow e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, G_1' \rightarrow$$

$$\lambda_{10}$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = ea \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = ea \wedge ea = a$$

$$((ba')a = ea \wedge ea = a) \Rightarrow (ba')a = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''',$$

$$(\forall w)(((ba')a = ea \wedge ea = w) \Rightarrow (ba')a = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb =$$

$$(\forall v)(\forall w)(((ba')a = v \wedge v = w) \Rightarrow (ba')a = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa,$$

$$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a =$$

$$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a =$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a =$$

We call this tree λ_{11}.

[redacted]

$$ea = a \rightarrow ea = a$$

$$G_1' \rightarrow ea = a$$

$$K, G_1' \rightarrow ea = a$$

$$K, E_3, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1' \rightarrow ea = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, G_1' \rightarrow ea = a$$

$$\lambda_{10}$$

$$= fa, G_1''' \rightarrow (ba')a = ea \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1' \rightarrow ea = a$$

$$= fa, G_1''', G_1' \rightarrow (ba')a = ea \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow ea = a$$

$$E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = ea \wedge ea = a \qquad (ba')a = a \rightarrow (ba')a = a$$

$$)a = ea \wedge ea = a) \Rightarrow (ba')a = a, K, E_3, E_2, P_2', S_1, G_1''', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

$$(((ba')a = ea \wedge ea = w) \Rightarrow (ba')a = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

$$(((ba')a = v \wedge v = w) \Rightarrow (ba')a = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

$$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

$$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a$$

K,E

K,E_3,E

K,E_3,E_2,P

K,E_3,E_2,P_2',S

K,E_3,E_2,P_2',S_1,G

$K,E_3,E_2,P_2',S_1,G_1'',F$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_3,G_1$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_3,G_1,fb$

λ_{12}

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2' \to b(a'a)=a$ $\quad$ $K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_3,G_1,fb=f$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to b(a'a)=a$ $\quad$ $K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_3,G_1,fb=fa,G$

$K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to b(a'a)=a \wedge a'a=e$

$(b(a'a)=a \wedge a'a=e) \Rightarrow be=a,K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3$

$(\forall w)((b(a'a)=a \wedge a'a=w) \Rightarrow bw=a),K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1$

$(\forall u)(\forall w)((u(a'a)=a \wedge a'a=w) \Rightarrow uw=a),K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G$

$(\forall v)(\forall u)(\forall w)((uv=a \wedge v=w) \Rightarrow uw=a),K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',$

$P_2',K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to bc$

$K,P_2',E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to be$

$K,E_3,P_2',E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to bc$

$K,E_3,E_2,P_2',P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to be$

We call this tree λ_{13} $\quad$ $K,E_3,E_2,P_2',S_1,G_1'',F_2,F_3,G_2,G_1,fb=fa,G_1''',G_1',G_2',G_3' \to be$

$$a'a = e \to a'a = e$$

$$G_3' \to a'a = e$$

$$K, G_3' \to a'a = e$$

$$K, E_3, G_3' \to a'a = e$$

$$K, E_3, E_2, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_3' \to a'a = e$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_3' \to a'a = e$$

λ_{12}

$$\ldots, G_2' \to b(a'a) = a \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_3' \to a'a = e$$

$$\ldots, G_2', G_3' \to b(a'a) = a \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to a'a = e$$

$$\ldots, S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b(a'a) = a \wedge a'a = e \qquad be = a \to be = a$$

$$\ldots \wedge a'a = e) \Rightarrow be = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$\ldots a) = a \wedge a'a = w) \Rightarrow bw = a), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$\ldots u(a'a) = a \wedge a'a = w) \Rightarrow uw = a), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$\ldots w)\,((uv = a \wedge v = w) \Rightarrow uw = a), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$P_2', K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$K, P_2', E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$K, E_3, P_2', E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$K, E_3, E_2, P_2', P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to be = a$$

[redacted]

$$b = be \rightarrow b = be$$

$$G_1''' \rightarrow b = be$$

$$K, G_1''' \rightarrow b = be$$

$$K, E_3, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2' \rightarrow b = be$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow b = be \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow b = be \wedge be = a$$

$$(b = be \wedge be = a) \Rightarrow b = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow b =$$

$$(\forall w)\,((b = be \wedge be = w) \Rightarrow b = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G$$

$$(\forall v)\,(\forall w)\,((be = v \wedge u = w) \Rightarrow b = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2$$

$$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow$$

$$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow$$

we call this tree λ_{14}

be

be

be

be

be

be

be

be

be

e

e

$\to b = be$

$\to b = be$

$, G_2' \to b = be$

$G_2', G_3' \to b = be \qquad K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \xrightarrow{\lambda_{13}} be = a$

$_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = be \wedge be = a \qquad\qquad b = a \to b = a$

$= a) \Rightarrow b = a, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$be \wedge be = w) \Rightarrow b = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$(be = w \wedge u = w) \Rightarrow b = w), K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$E_3, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$K, E_3, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \to b = a$

$$\lambda_{14}$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, fb = fa, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, fb = fa, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, fb = fa, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, fb = fa, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', fb = fa, F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, fb = fa, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', fb = fa, S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, fb = fa, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, fb = fa, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, fb = fa, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$fb = fa, K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_1', G_2', G_3' \rightarrow b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_2', G_3' \rightarrow fb = fa \supset b = a$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_2', G_3' \rightarrow (\forall v)(fb = fv \supset b = v)$$

$$K, E_3, E_2, P_2', S_1, G_1'', F_2, F_3, G_2, G_1, G_1''', G_2', G_3' \rightarrow F_4$$

We call this tree λ_{15}

By successive thinnings and interchanges on λ_{15} we obtain

λ_{16}:

$$E_1, E_2, E_3, G_1, G_1', G_1'', G_1''', G_2, G_2', G_3, G_3', P_1, P_1', P_2, P_2', F_1, F_2, F_3, F_3', S_1, S_2, K \to F_4 .$$

On λ_{16} we operate as in the following example

$$\begin{array}{c} A, B \to C \\ \hline A\wedge B, B \to C \\ \hline A\wedge B, A\wedge B \to C \\ \hline A\wedge B \to C \end{array} \quad \text{and obtain } \lambda_{17}: (((E\wedge G)\wedge P)\wedge F)\wedge S)\wedge K \to F_4$$

$$\text{i.e. , } \overset{\lambda_{17}}{\text{Axioms} \wedge K \to F_4}$$

Now from λ_{17} we obtain λ by:

$$\dfrac{\dfrac{\text{Axioms} \to \text{Axioms}}{\text{Axioms}\wedge K \to \text{Axioms}} \qquad \overset{\lambda_{17}}{\text{Axioms}\wedge K \to F_4}}{\text{Axioms}\wedge K \to \text{Axioms}\wedge F_4}$$

Remark: λ_{17} says that from the axioms and the fact that $\ker f = \{e\}$ we can deduce that f is a monomorphison.

This establishes the theorem and the sympathy of the reader.

4: *Remarks on the proof.*

The problems of cut elimination created by equality do not appear in our proof. The reason is that our substitution axioms are not as general as the usual one namely $(\forall x)(\forall y)((x=y \wedge \varphi(x)) \Rightarrow \varphi(y))$. We deliberately restricted ourselves to the atomic level, and, as shown, it alleviated the problems.

The number of axiom formulas of our language can be reduced considerably. The advantage of having G_2 and G_2', for example, is that one can avoid using E_2 which is an axiom involving conjunction. There is no problem with conjunction. However, as portrayed very nicely in our derivation, it lengthens the trees considerably.

Our derivation is an illustration of the fact that a cut-free proof is in general much longer than a proof using the cut. Consider for example the tree λ_{11}, by using a cut, the top 17 lines of the tree become 6 lines, namely:

$$\dfrac{(ba')a = ea \rightarrow (ba')a = ea}{(ba')a = ea, G_1' \rightarrow (ba')a = ea} \qquad \dfrac{\dfrac{ea = a \rightarrow ea = a}{G_1' \rightarrow ea = a}}{(ba')a = ea, G_1' \rightarrow ea = a}$$

$$(ba')a = ea, G_1' \rightarrow (ba')a = ea \wedge ea = a$$

$$\dfrac{K, E_3, E_2, P_2^1, S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''' \rightarrow (ba')a = ea \quad \lambda_{10} \quad ((ba')a = ea \wedge ea = a) \rightarrow (ba')a = a, (ba')a = ea, G_1'}{((ba')a = ea \wedge ea = a) \rightarrow (ba')a = a, K, E_3, E_2, P_2^1, S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')}$$

Similarly in λ_{12}, the top 20 lines become 8 lines:

$$\dfrac{b(a'a) = (ba')a \rightarrow b(a'a) = (ba')a}{(\forall w)(b(a'w) = (ba')w) \rightarrow b(a'a) = (ba')a}$$

$$\dfrac{(\forall v)(\forall w)(b(vw) = (bv)w) \rightarrow b(a'a) = (ba')a}{G_2^1 \rightarrow b(a'a) = (ba')a}$$

$$\dfrac{G_2^1 \rightarrow b(a'a) = (ba')a}{(ba')a = a, G_2^1 \rightarrow b(a'a) = (ba')a} \qquad \dfrac{(ba')a =}{(ba')a = a, G}$$

$$(ba')a = a, G_2^1 \rightarrow b(a'a) = (ba')a \wedge (ba')a =$$

$$\dfrac{K, E_3, E_2, P_2^1, S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1' \rightarrow (ba')a = a \quad \lambda_{11} \quad (b(a'a) = (ba')a \wedge (ba')a = a) \rightarrow b(a'a) = a, (b}{(b(a'a) = (ba')a \wedge (ba')a = a) \rightarrow b(a'a) = a, K, E_3, E_2, P_2^1, S_1, G_1'', F_2, F_3, G_2, G_3, G_1, fb = fa, G_1''', G_1',}$$

$$ea = a \rightarrow ea = a$$

$$(ba')a = ea \rightarrow (ba')a = ea \qquad G'_1 \rightarrow ea = a$$

$$(ba')a = ea, G'_1 \rightarrow (ba')a = ea \qquad (ba')a = ea, G'_1 \rightarrow ea = a$$

$$\lambda_{10} \qquad (ba')a = ea, G'_1 \rightarrow (ba')a = ea \wedge ea = a \qquad (ba')a = a \rightarrow (ba')a = a$$

$$, G'''_1 \rightarrow (ba')a = ea \qquad ((ba')a = ea \wedge ea = a) \Rightarrow (ba')a = a, (ba')a = ea, G' \rightarrow (ba')a = a$$

$$(ba')a = a, K, E_3, E_2, P^1_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1, G'_1 \rightarrow (ba')a = a$$

0 lines become 8 lines:

$$b(a'a) = (ba')a \rightarrow b(a'a) = (ba')a$$

$$(\forall w)(b(a'w) = (ba')w) \rightarrow b(a'a) = (ba')a$$

$$(\forall v)(\forall w)(b(vw) = (bv)w) \rightarrow b(a'a) = (ba')a$$

$$G^1_2 \rightarrow b(a'a) = (ba')a \qquad (ba')a = a \rightarrow (ba')a = a$$

$$(ba')a = a, G^1_2 \rightarrow b(a'a) = (ba')a \qquad (ba')a = a, G^1_2 \rightarrow (ba')a = a$$

$$\lambda_{11} \qquad (ba')a = a, G^1_2 \rightarrow b(a'a) = (ba')a \wedge (ba')a = a \qquad b(a'a) = a \rightarrow b(a'a) = a$$

$$, G'''_1, G'_1 \rightarrow (ba')a = a \qquad (b(a'a) = (ba')a \wedge (ba')a = a) \Rightarrow b(a'a) = a, (ba')a = a, G^1_2 \rightarrow b(a'a) = a$$

$$a = a) \Rightarrow b(a'a) = a, K, E_3, E_2, P^2_2, S_1, G''_1, F_2, F_3, G_2, G_3, G_1, fb = fa, G'''_1, G'_1, G^1_2 \rightarrow b(a'a) = a$$

The rest of the proof would provide many more examples, since practically every tree could be reduced by a judicious use of a cut.

There are some cases, however, where a cut-free derivation is shorter than a derivation using the cut. In fact for systems that do not involve equality or mathematical axioms, it is generally true that cut-free proofs are shorter.

It seems that the cut-elimination algorithm, since it is such a powerful idea, has captured almost exclusively the attention of the people involved with equivalence of proofs. It is only recently with the advances made in the fields of automata and machine proving, that mathematicians have started to realize the fact that allowing certain forms of the cut in the systems would simplify tremendously the data to be analysed. In the conclusion of his paper on "Resolution, Paramodulation and Gentzen-Systems", Michael Richter writes:

"Now an important point concerning the efficiency is that in Gentzen-systems the elimination of the cut rule increases the length of the proofs (in general exponentially). In particular there are formulas where the shortest proof without the cut rule is longer than some proof using the cut rule.' . . . Although this does not mean that for every derivable formula the shortest proof uses a cut it indicates that the re-introduction of the cut in some controlled manner can lead to more efficient proof procedures and test systems."

In view of these remarks it seems that the approach should be that normal proofs in the sense of Szabo are not necessarily the the objects to study, and that more attention should be given to such concepts as "minimal cuts", "complexity of non-eliminable cuts", "equivalence of cuts", etc . . . .

APPENDIX

In this appendix, we list a number of proofs of equivalence, by levels. For each proof we give a reference, and all the concepts involved in the theorems are defined in the references given.

Strong equivalence level.

1. Let $f: G \to H$ be a group homomorphism, then f is a monomorphism

iff $\ker f = \{e_G\}$

(see [MLI]).

2. Let N be a subgroup of G, then N is normal in G

iff $gN = Ng, \forall g \in G$

iff $g^{-1}Ng \subseteq N, \forall g \in G$

iff $g^{-1}Ng = N, \forall g \in G$

(see [MLI])

3. A ring R is Noetherian

iff every non-empty family of ideals of R has a maximal element

iff every ideal of R is finitely generated.

(see [BI])

4. A ring R with unity is local

iff $\{r \in R: r \text{ is non-invertible}\}$ is an ideal of R

iff $(\forall a)(\forall b)$ (a and b non-invertible $\Rightarrow$ (a+b) is non-invertible)

iff $\forall r \in R$, r or 1-r is invertible.

(see [BI])

5. One can define a lattice in two different ways, and the proof of the equivalence of these two definitions is of the strong level.

(see [KuI]).

6. Let B be a boolean algebra and F a filter on B, then F is an ultrafilter

iff $B/_F \simeq 2$

iff F is prime

iff $\forall x \in B \Rightarrow x \in F$ or $x^* \in F$.

(see [BSI]).

7. Let $f: X \to Y$ be a map of topological spaces, then f is continuous

iff the inverse image of every closed set is closed

iff for every $x \in X$ and each neighborhood W of $f(x)$ in Y, there is a neighborhood $V(x)$ in x, such that $f(V(x)) \subseteq W$

iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$

iff $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$,

(see [DI]).

8. A space Y is connected

iff ϕ and Y are the only open and closed sets of Y

iff $f: Y \to 2$ continuous $\Rightarrow f$ is not surjective.

(see [DI]).

9. Let $f: E \to R^*$ a function then for all $r \in R$ $f^{-1}(r,\infty)$ is measurable

iff for all $r \in R$ $f^{-1}[r,\infty)$ is measurable

iff for all $r \in R$ $f^{-1}(-\infty,r)$ is measurable iff for all $r \in R$ $f^{-1}(-\infty, r]$ is measurable

(see [*HuI*]).

10. A function $f: I \to R$ is absolutely continuous

iff there exists a Lebesgue integrable function $g: I \to R^*$ such that $f(x) = f(a) + \int_a^x g(t)\,dt$ for every $x \in I$.

(see [*RoI*]).

11. Let $\leq$ be a partial ordering, then $\leq$ is a well ordering

iff $\leq$ is a simple ordering and every non-empty class A contained in $\mathrm{Fld}(\leq)$ has a $\leq$-least element

iff every non-empty class A contained in $\mathrm{Fld}(\leq)$ has a $\leq$-least element.

(see [*MI*]).

Tauberian equivalence level.

1. An abelian group is locally finite

iff it is periodic.

(see [*KuI*]).

2. In a Noetherian ring, if M is a maximal ideal and Q any ideal, then Q is M-primary.

iff $\{x \in Q: x^n \in Q$ for some $n > o\} = M$

iff $M^n \subseteq Q \subseteq M$ for $n > o$.

(see [AMI]).

3. A metric space is Lindelöf

iff it is separable

(see [SSI]).

4. A metric space is compact

iff it is sequentially compact

iff it is countably compact.

(see [SSI]).

5. If X and Y are topological vector spaces, then $f: X \to Y$ is linear

iff f is continuous

iff f is bounded

iff $x_n \to 0 \Rightarrow \{fx_n : n = 1,2,3,\ldots\}$ is bounded.

(see [RI]).

Weak equivalence level.

1. Every finitely generated periodic group if finite

iff every periodic group is locally finite.

(see

2. Each of the following is equivalent to the four colour conjecture.

- Every planar graph is 4 - colourable
- If G is a planar graph then $f(G,4)>0$ ($f(G,t)$ is the chromatic polynomial of Birkhoff and Lewis.)
- Every connected 5- chromatic diagram is contractible to K_5.

(see [HaI]).

3. Let F: $\mathbb{N}^k \to \mathbb{N}$ a function, then f is computable

iff as a relation, f is a decidable

iff as a relation, f is effectively ennumerable.

(see [EI]).

4. The following concepts are equivalent:

- Gödel - Herbrand recursive functions.
- λ - definable functions (Church - Kleene).
- Turing computable functions.
- The normal algorithms of Markov.
- The canonical systems of Post.
- The register machines programs (Shepherdson-Sturgis).

(see [BLI]).

5. The following are equivalent:

- The axiom of choice.
- The multiplicative principle.
- Zermelo 's principle.

(see [MI]).

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