Quantum Stochastic Processes and Dynamical Semi-Groups

Fatemeh Afsharnejad

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ABSTRACT

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Two problems have been studied in this dissertation, quantum stochastic processes and quantum dynamical semi-groups and their generators. For an arbitrary one-parameter semi-group, defining the infinitesimal generators under different continuity conditions, we can construct and determine the structure of the corresponding semi-groups. Two such classes of semi-groups, namely quantum stochastic processes, and dynamical semi-groups have been considered, and the relationship between them is analyzed.

The theory of stochastic processes was developed by Davies [2] around 1969, in an attempt to describe mathematically the time evolution of quantum mechanical systems, which are subject to randomly repeated measurements.

The concept of a dynamical semi-group was developed by Kossakowski [14] in 1972. The theory of dynamical semi-groups starts from certain basic assumptions on the nature of a general time evolution of a physical system, and one can deduce conditions under which these semi-groups describe quantum stochastic processes in the above sense.

We apply the general theory to the time evolution of a quantum mechanical system, to a good approximation by given a certain class of dynamical semi-groups. Specifically we look at a spin-1/2 system evolving in time via a dynamical semi-group and an unstable system decaying in the presence of randomly repeated measurements.

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CHAPTER 1

Introduction

We study in this dissertation, the mathematical description of physical systems, which in the course of their time evolution, undergo measurements at randomly repeated times. The description is quantum mechanical, so that in the background one has a separable Hilbert space $\mathcal{H}$, with the states of the physical system under consideration, being given by the normalized, positive trace class operators on it. Any such state ϱ evolves in time as a result of two processes. Firstly, it evolves as a result of the system dynamics, in between two acts of observation made on the system. This evolution is governed by a Hamiltonian H (which is a self-adjoint operator on $\mathcal{H}$) and is time reversible. Moreover, it preserves pure states, in the sense that if ϱ_t is the state ϱ evolved to time t, as a result of this evolution alone, and if ϱ is a one-dimensional projection operator, then so also is ϱ_t. Secondly, the state ϱ changes, each time a measurement is made on the system. This change is irreversible in time and hence the combined evolution of the system is irreversible and takes "pure" states into "mixed" states — ie., one-dimensional projection operators are not preserved.

Consider, for example, a typical photon experiment. A weak beam of light is directed into a photon counter and is split into two parts which are directed into different counters. Then the numbers of photons arriving at each counter is a random variable and depends strongly on whether the light is coherent or incoherent. There may be a statistical correlation between the arrivals of the photons at the two counters.

We intend to study a mathematical model for such a situation. The framework for this description is a quantum mechanical theory of stocastic processes, which we discuss in Chapter 3.

E.B. Davies and J.T. Lewis had introduced in [1] a model of quantum probability theory, to provide a mathematical framework for studying changes that occur in quantum mechanical states as a result of a measurement performed upon it. Then, in 1969, Davies continued this work and developed a theory of quantum stochastic processes, which concerns successive observations of a system during its evolution in time under certain fixed exterior conditions.

Among the results in Chapter 3, given a stochastic process, we obtain a one-parameter semi-group determined by it. A related concept is that of a dynamical semi-group which we study in Chapter 4. Following the work of Kossakowski [13, 14], we study the generators of these semi-groups and how the generators uniquely fix them.

Chapter 5 illustrates the interconnection between Chapters 3 and 4 by means of an example.

CHAPTER 2

Mathematical Preliminaries

The purpose of this chapter is to introduce some of the fundamental concepts (mathematical background) and definitions which will be used through this work.

Section 1: Measure Theory

A σ-field F on a set X is a non-empty collection of subsets of X which is closed under complements, countable unions and countable intersections. Every σ-field contains the empty set $\emptyset$ and the whole space X as members. If X is any topological space, then C(X) denotes the space of all complex-valued continuous functions on X and $C_{\mathbb{R}}(X)$ the space of all real-valued continuous functions. The Baire σ-field of X is the smallest σ-field with respect to which all continuous functions are measurable, while the Borel σ-field is the smallest σ-field containing all open sets. In general, the Baire σ-field is contained in the Borel σ-field. If X is a compact Hausdorff space, there is a natural one-one correspondence between

(i) the positive linear functionals on $C_{\mathbb{R}}(X)$;

(ii) the Baire measures on X;

(iii) the Borel measures on X which are regular in the sense that for all Borel sets E

$$\mu(E) = \operatorname{Sup}\{\mu(K): K \subseteq E \text{ and } K \text{ is compact}\}.$$

If X is a compact Hausdorff space, the following three conditions are equivalent:

(i) X is second countable (there is a countable base to the topology of X);

(ii) X is metrizable;

(iii) $C_{\mathbb{R}}(X)$ is separable (contains a countable dense set).

If these conditions are satisfied then the Baire and Borel σ-fields coincide, and every Borel measure on X is regular.

In quantum-mechanical measurement theory it is almost always the case that a measurable quantity takes its values in a locally-compact space X which is second countable, actually $X \subset R^n$. The one-point compactification of X is a compct metrizable space $\bar{X}$ obtained by adjoining one point, called ∞, to X. There is a one-one correspondence between the (finite) Borel measures on X and the Borel measures μ on $\bar{X}$ such that $\mu(\infty) = 0$.

From the mathematical point of view almost every theorem can be extended from a compact metrizable space to a compact Hausdorff space by replacing the word "Borel" by "regular Borel" or "Baire".

Section 2: Trace Class Operators

We use the symbol $\mathcal{H}$ to denote a complex Hilbert space with an inner product $\langle\varphi,\psi\rangle$ which is complex linear in φ and conjugate linear in ψ. $L(\mathcal{H})$ shall denote the set of all bounded operators on $\mathcal{H}$, and $\mathcal{T}(\mathcal{H})$ the Banach space of all trace class operators on $\mathcal{H}$ (cf. Definition 2.2.2 below), with respect to the trace-norm. We denote by $\mathcal{T}_s(\mathcal{H})$ the Banach space of self-adjoint trace class operators with the trace norm and by $\mathcal{T}_s(\mathcal{H})^+$ the positive cone of trace class operators $(\mathcal{T}(\mathcal{H}))$.

Definition 2.2.1

An operator A in $L(\mathcal{H})$ is said to be self-adjoint if $A = A^*$, (A^* is the adjoint operator of A in $L(\mathcal{H})$). A is said to be compact if $\{A\varphi_n\}$ has a norm convergent subsequence for every norm bounded sequence $\{\varphi_n\}$ in $\mathcal{H}$.

If A is compact then $|A|$ is also compact, with non-negative eigen values $\{\lambda_n\}_{n=1}^{\infty}$; where we put $|A|$ to be: $|A| = (A^* A)^{\frac{1}{2}}$.

If $A = A^*$ we define the positive and negative parts A^+ and A^- of A to be

$$A^+ = \frac{|A| + A}{2}, \quad A^- = \frac{|A| - A}{2}.$$

Definition 2.2.2

We say that A is of trace class if it has finite trace norm, where

$$\| A \|_{tr} = \sum_{n=1}^{\infty} \lambda_n$$

Now if $\{ \varphi_n \}$ is an orthonormal basis in $\mathcal{H}$ and $A \in \mathcal{T}(\mathcal{H})$ the sum

$$tr [A] = \sum_n < A\varphi_n, \varphi_n >$$

is absolutely convergent and independent of the orthonormal basis chosen.

The linear functional tr: $\mathcal{T}(\mathcal{H}) \to \mathbb{C}$ defined by $A \to tr [A] = \sum_n < A\varphi_n, \varphi_n >$ is norm bounded and satisfies

$$| tr [A] | \leq \| A \|_{tr} \qquad \forall A \in \mathcal{T}(\mathcal{H})$$

If A is compact and self adjoint with eigenvalues $\{ \lambda_n \}$ then

$$\| A \|_{tr} = \sum_n | \lambda_n |.$$

If this sum is finite then

$$\mathrm{tr}\,[\,A\,] = \sum_n \lambda_n$$

If $A \in \mathcal{T}_s(\mathcal{H})^+$ then

$$\mathrm{tr}\,[\,A\,] = \|\,A\,\|_{tr}.$$

For an arbitrary element $A \in \mathcal{T}_s(\mathcal{H})$, both A^+ and A^- lies in $\mathcal{T}_s(\mathcal{H})^+$ and

$$\|\,A\,\|_{tr} = \|\,A^+\,\|_{tr} + \|\,A^-\,\|_{tr}.$$

Definition 2.2.3

Let $\mathcal{T}_s(\mathcal{H})$ be the real Banach space of self-adjoint trace class linear operators on $\mathcal{H}$. Elements of $\mathcal{T}_s(\mathcal{H})$ will be denoted by $\varrho, \partial, A, \ldots$ The norm in $\mathcal{T}_s(\mathcal{H})$ is given by the formula

$$\|\,\varrho\,\|_{tr} = \sup \sum_{n=1}^{\infty} |\,(x_n, \varrho y_n)\,|$$

where the supremum is taken over all orthonormal and complete bases $\{\,x_n\,\}$ and $\{\,y_n\,\}$ in $\mathcal{H}$.

Section 3: State Space and Observable

The material of this section and the next is derived from Davies and Lewis [1] and Davies [2]. The results will be needed in the next chapter.

The set $\mathcal{T}_s(\mathcal{H})$ of self-adjoint trace class operators on the Hilbert space $\mathcal{H}$ forms a real Banach space V under the trace norm, and the states form a positive norm closed cone V^+ in V. The states are defined as the non-negative trace class operators of trace one, elsewhere called density matrices.

Definition 2.3.1

A state space is a triple (V, V^+, tr) consisting of a real Banach space V, a closed cone V^+ which generates V, and a linear functional tr on V such that

$$tr[\varrho] = \|\varrho\| \qquad \text{for all } \varrho \in V.$$

A linear functional φ on V is said to be positive if

$$\varphi[\varrho] \geq 0 \qquad \text{for all } \varrho \in V^+.$$

A linear mapping $T:V \to V$ is said to be positive if $T(V^+) \subseteq V^+$. Every positive linear functional on a state space is necessarily continuous and bounded. The set of all positive linear mappings on V forms a closed cone $L^+(V)$ in the linear space $L(V)$ taken with the norm topology of all bounded linear mappings of V into itself.

Definition 2.3.2

A state ρ is a non-zero element of V^+, satisfying $\mathrm{tr}\,\rho = 1$.

Let V^* be the Banach space dual to V which can be identified with the space of bounded self-adjoint operators on $\mathcal{H}$.

Definition 2.3.3

An observable is a triple (X, F, a) consisting of a set X with a σ-field F of subsets of X, and a mapping $a: F \to V^*$ satisfying:

(i) $0 \leq a(E) \leq a(X)$, for all $E \in F$;

(ii) $a(X) = I$, the identify operator on $\mathcal{H}$;

(iii) for each countable family $\{E_i\}$ of pairwise disjoint sets in F,

$$a\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} a(E_i),$$

where the right hand side converges in the weak operator topology of V^*. The Borel space (X, F) is called the <u>value space</u> of the observable (X, F, a).

<u>Definition 2.3.4</u>

An instrument is a triple $(X, F, \mathcal{E})$ consisting of a Borel space (X,F) and a mapping $\mathcal{E}: F \to L^+(V)$ satisfying

(i) $\operatorname{tr}[\mathcal{E}(X)\rho] = \operatorname{tr}[\rho] \qquad \forall \rho \in V;$

(ii) for each countable family $\{E_i\}$ of pairwise disjoint sets in F,

$$\mathcal{E}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathcal{E}(E_i),$$

where the series converges in the strong operator topology.

If $\mathcal{E}^1$ and $\mathcal{E}^2$ are instruments on X and Y respectively with values in a state space V, then we define the composition $\mathcal{E}$ on $X \times Y$, of $\mathcal{E}^1$ following $\mathcal{E}^2$, denoted by $\mathcal{E}^1 \circ \mathcal{E}^2$, by

$\mathcal{E}(E \times F) = \mathcal{E}^1(E)\, \mathcal{E}^2(F)$, for all subsets of $E \subseteq X$ and $F \subseteq Y$.

Section 4: Sample Space

To define a quantum stochastic process, we need to develop the notion of an event and a sample space.

Suppose we are given an apparatus which records events for example, the space time point at which a photon is absorbed. Each event is represented by a point $(x, t) \in X \times (0, \infty)$ where $t>0$ is the time at which the event occurs and $x \in X$, where X is a separable locally compact Hausdorff space representing the set of possible values of some observable the apparatus is measuring.

Definition 2.4.1

A sample point is defined as a sequence of events $\{(x_i, t_i): i=1,2,\ldots\}$ such that $0<t_1<t_2\ldots$ and either the sequence terminates or $t_n \to \infty$ as $n \to \infty$. The case $n=0$ gives rise to the sample point Z with no events.

Definition 2.4.2

For each time $t>0$, we define the sample space X_t as the set of all finite sequences $\{ (x_i, t_i): i=1,2,\ldots,n \}$ of events of arbitrary finite length n, such that each x_i is the position of an event and $t_i \in (0,t]$ i.e.:

$$X_t = \{ (x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n) \}.$$

The sample space is a Borel space in a natural way; also $\overline{X_\infty}$ is defined as the set of all sample points and is a Borel subset of

$$\bigcup_{m=0}^{\infty} \{ \prod_{n=0}^{\infty} X \times (0,\infty) \}.$$

Given any $s, t>0$, there is a continuous map $\lambda : X_s \times X_t \xrightarrow{\text{onto}} X_{s+t}$ defined by:

$$\lambda\{ (x_i, s_i)_{i=1}^{m}, (y_j, t_j)_{j=1}^{n} \} = (Z_k, U_k)_{k=1}^{m+n} \qquad (2.1)$$

where $(Z_k, U_k) = (y_k, t_k)$ if $1 \leq k < n$

and $(Z_k, U_k) = (x_{k-n}, s_{k-n} + t)$ if $n < k \leq m+n.$

CHAPTER 3

Quantum Stochastic Processes

The material in this chapter consists of an introduction to the theory of quantum stochastic processes (Davies [2] and [3]).

A stochastic process is a family of random variables X_t, where t is a parameter runing over a suitable index T. If the index T is countable set, ie: $T = \{ 0, 1, 2, 3, \dots \}$, then we call X_t a discrete-time stochastic process, and if T is continuous, we call it a continuous-time process. A continuous-time stochastic process $\{ X_t, t \geq 0 \}$, in analogy with the definition of a discrete-time Markov-chain [4, chapter 4 and 5], is a continuous-time Markov chain.

Stochastic processes for which $T = [0, \infty)$ are particularly important here. In classical probability theory [see 1], if the sample space (Ω, μ) is a standard Borel space and the state space V is the space of all bounded Borel measures on Ω, this state space has all the properties laid down in section 3 of chapter 2, and we have

$$\mathrm{tr}\,[\mu] = \mu(\Omega) \qquad \text{for all } \mu \in V.$$

Definition 3.1.1

A random variable is defined as a Borel map $\alpha : \Omega \rightarrow R$ where (R, F) is a Borel space. We define the associated measurable observable (R, F, a) by putting $a(E) = \chi_{\alpha^{-1}E}$ for all $E \in F$.

We shall define a quantum stochastic process in close analogy with the classical case, and eventually obtain a generalization of the latter.

This will be followed by a discussion concerning the classification of a class of stochastic processes, in which we analyze the structure of such processes.

Section 1: The Basic Formulation

In this section we formulate the idea of a stochastic process in the general situation. Throughout this section, we suppose that the state space (V, tr) consists of $V = \mathcal{T}_s(\mathcal{H})$ and tr, the normalized trace on V.

Definition 3.1.2

A quantum stochastic process on X, V is a family of instruments $\mathcal{E}^t$ defined on X_t, V for all $t \geq 0$ satisfying the following.

(i) $\operatorname*{s-lim}_{t \to 0} \mathcal{E}^t (X_t, \varrho) = \varrho, \qquad \forall \varrho \in V$ (3.1)

(ii) $\operatorname{tr} [\mathcal{E}^t (X_t, \varrho)] = \operatorname{tr} [\varrho], \qquad \forall \varrho \in V, t \geq 0$ (3.2)

(iii) for all $\varrho \in V$ and S, t>0

$$\mathcal{E}^t (F, \mathcal{E}^s (E, \varrho)) \equiv \mathcal{E}^t \circ \mathcal{E}^s (F \times E, \varrho) = \mathcal{E}^{s+t}(\lambda(F \times E), \varrho) \qquad (3.3).$$

is defined as in (2.1).

The last condition says that the evolution after time t depends only on the state at time t, and the evolution is homogenous in time. It is a generalization of the Chapman-Kolmogorov equation. The instrument $\mathcal{E}^t(E, \varrho)$ is the state at time t having the Markovian property [4], that the conditional distribution of the future state at time t+s, given the present state at t and all past states, depends only on the present state and is independent of the past, obtaining a sample point in the set $E \subseteq X_t$ at the later time.

The following proposition whose proof may be found in [2] will be important later.

Proposition 3.1.1

Given a stochastic process $\mathcal{E}^t$ we obtain a one-parameter semi-group on V as follows. If we define $T_t : V \rightarrow V$ by

$$T_t(\varrho) = \mathcal{E}^t(X_t, \varrho).$$

Then T_t is bounded positive linear operator mapping V into V satisfying:

$$\mathrm{tr}\,[T_t(\varrho)] = \mathrm{tr}[\varrho] \qquad \text{for all } \varrho \in V \text{ and } t \geq 0$$

since $\lambda(X_s \times X_t) = X_{s+t}$ we obtain $T_s \cdot T_t = T_{s+t}$ from the (generalized) Chapman-Kolmogorov equations. The continuity condition (3.1) shows that T_t is a strongly continuous one-parameter semi-group on V.

There is another semi-group of interest on V that we can define. Writing Z for the sample point in X_t which consists of zero events, we define the bounded positive operator $S_t : V \longrightarrow V$ by

$$S_t(\varrho) = \mathcal{E}^t(Z, \varrho) \qquad \forall t \geq 0 \text{ and } \varrho \in V^+$$

and

$$0 \leq \mathrm{tr}\,[S_t(\varrho)] \leq \mathrm{tr}\,[\varrho]$$

Moreover, as $\lambda(Z, Z) = Z$ we have $S_s\, S_t = S_{s+t}$ for all $s, t \geq 0$.

It is not obvious however, whether this semi-group is strongly continuous. It is necessary now to make a further assumption about the stochastic process, that there exists a constant $k < \infty$ such that

$$\mathrm{tr}\,[\mathcal{E}^t(X_t - Z, \varrho)] \leq k\, t\, \mathrm{tr}\,[\varrho] \qquad \forall \varrho \in V^+ \text{ and } t > 0 \qquad (3.4)$$

This states that the stochastic process has a bounded interaction rate. In order to investigate the implication of this we introduce the Borel sets,

$$A_t^n = \{\text{all sample points in } X_t \text{ containing exactly n events}\},$$

and

$$B_t^n = \bigcup_{m=n}^{\infty} A_t^m = \{\text{all points in } X_t \text{ containing at least n events}\}.$$

observing that

$$A_{s+t}^q = \bigcup_{m+n=q} \lambda(A_s^m \times A_t^n).$$

Let $m \geq n$ and let F_m^n be the family of all subsets a of $(1,\dots,m)$ containing exactly n points. For a F_m^n and $1 \leq r \leq m$ define $C_{a,r} \subseteq X_{m^{-1}t}$ by

$$C_{a,r} = \begin{cases} X_{m^{-1}t} & \text{if } r \notin a \\ B^1_{m^{-1}t} & \text{if } r \in a. \end{cases}$$

Then define $D_{t,m}^n \subseteq X_t$ by

$$D_{t,m}^n = \bigcup_{a \in F_m^n} \lambda(C_{a,1} \dots C_{a,m})$$

clearly $D_{t,m}^n \subseteq B_t^n$; conversely the characteristic functions of these sets satisfy

$$\chi(B_t^n)(x) = \lim_{m \to \infty} \chi(D_{t,m}^n)(x) \qquad \text{for all } x \in X_t.$$

Now for any state $\varrho \in V^+$ the hypothesis of a bounded interaction rate K implies that

$$\operatorname{tr}[\mathcal{E}^t(D^n_{t,m}, \varrho)] \leqq \frac{m!\, k^n\, t^n\, \operatorname{tr}[\varrho]}{(m-n)!\, n!\, m^n} \qquad \forall \varrho \in V^+$$

Therefore, by Fatou's lemma: [see 6]

$$\operatorname{tr}[\mathcal{E}^t(B^n_t, \varrho)] \leqq (n!)^{-1} k^n t^n \operatorname{tr}[\varrho] \qquad (3.5)$$

since $Z = \lambda(Z\ Z \dots Z)$ we have for all $\varrho \in V^+$

$$\operatorname{tr}[\mathcal{E}^+(Z, \varrho) \geqq (1 - Kt/m)^m \operatorname{tr}[\varrho]$$

and letting $m \longrightarrow \infty$ we obtain

$$\operatorname{tr}[S_t(\varrho)] \geqq e^{-kt} \operatorname{tr}[\varrho]$$

as the semi-group T_t is strongly continuous and

$$T_t(\varrho) = S_t(\varrho) + \mathcal{E}^t(B^1_t, \varrho)$$

it follows from equation (3.5) that S_t is also a strongly continuous one-parameter semi-group on V, i.e.:

$$\lim_{t \to 0} S_t(\varrho) = \lim_{t \to 0} T_t(\varrho) = \varrho.$$

Using the projection mapping π from $A^1_t = X \times (0,t)$ onto X then we define

$$\mathcal{J}^t(E, \varrho) = t^{-1} \mathcal{E}^t(\pi^{-1}(E), \varrho), \qquad (3.6)$$

and observe that $\mathcal{J}^t$ satisfies all of the axioms for an instrument (in chapter 2, section 3) except that instead of

$$\operatorname{tr}[\mathcal{J}^t(X, \varrho)] = \operatorname{tr}[\varrho] \qquad \forall \varrho \in V$$

we have

$$0 \leqq \operatorname{tr}[\mathcal{J}^t(X, \varrho)] \leqq k \operatorname{tr}[\varrho] \qquad \forall \varrho \in V^+ \qquad (3.7).$$

Equation (3.5) implies that

$$t^{-1}(T_t \varrho - \varrho) = t^{-1}(S_t\varrho - \varrho) + \mathcal{J}^t(X,\varrho) + O(t) \qquad (3.8)$$

We wish to show that $\mathcal{J}^t$ converges as $t \to 0$, but this is technically difficult at this point. If we assume for the moment that $\mathcal{J}^t$ converges to a limit as $t \to 0$ and the infinitesimal generators $\frac{\partial}{\partial t}$ of T_t and H of S_t have a common dense domain D then from (3.8) we obtain the following evolution equation

$$\frac{\partial}{\partial t}(\varrho) = H(\varrho) + \mathcal{J}(X,\varrho) \qquad (3.9)$$

This equation is precisely the version of the Kolmogorov forward-equation relevant to this generalized stochastic process.

Section 2: Structure of Stochastic Processes

We now start the analysis of stochastic processes introduced in the previous section. Throughout this section we shall suppose that $V = \mathcal{T}_s(\mathcal{H})$ where $\mathcal{H}$ is a given separable Hilbert space, and we shall suppose that the stochastic process has an interation rate bounded by the constant $K < \infty$.

The one-parameter semi-group T_t has the property that for any partition $\{E_n\}_{n=1}^{\infty}$ of X_t, and any $\varrho \in V^+$

$$T_t(\varrho) = \mathcal{E}^t(X_t, \varrho) = \sum_{n=1}^{\infty} \mathcal{E}^t(E_n, \varrho)$$

(E_n are Borel sets of X_t).

The above equation is telling us that T_t will transform pure states into mixed states. As t increases $T_t(\varrho)$ will become more mixed rather than pure. But from the manner in which S_t is defined, such an argument does not apply to that semi-group, even though as t increases we have a transformation $\varrho \to S_t(\varrho)$ which will give us more information about but this information is not enough to figure out whether $S_t(\varrho)$ becomes pure or mixed. Therefore it is reasonable to suppose that the evolution S_t is of the "simplest kind", which we shall interpret in physical terms, that if ϱ is a pure state so also is $S_t(\varrho)$ for all $t \geq 0$.

As a consequence of theorem 2.3.1 in [3] there exists a strongly continuous one-parameter contraction semi-group B_t on $\mathcal{H}$ such that

$$S_t(\varrho) = B_t \varrho B_t^* \tag{3.10}$$

since $e^{-kt}\, tr[\varrho] \leq tr[S_t(\varrho)] \leq tr[\varrho]$ for all $\varrho \in V^+$

we see that

$$e^{-\frac{1}{2}kt}\|\zeta\| \leq \|B_t\zeta\| \leq \|\zeta\| \qquad \forall\, \zeta \in \mathcal{H} \text{ and } t \geq 0$$

We shall say more about the infinitesimal generator Z of B_t later.

The following lemma will be needed in the proof of theorem (3.2.1).

Lemma 3.2.1

Let B_t be a strongly continuous bounded semi-group of operators on $\mathcal{H}$ and let S_t be the corresponding strongly continuous semi-group on $V = \mathcal{T}_s(\mathcal{H})$. If $\rho_1 \in V^+$ is such that

$$t_n^{-1} \; (B_{t_n} \rho_1 B_{t_n}^* - \rho_1) = t_n^{-1} (S_{t_n} (\rho_1) - \rho_1)$$

converges in the weak operator topology to a limit in V for some sequence $t_n \longrightarrow 0$, then ρ_1 is in the domain D_H of the infinitesimal generator H of S_t.

Proof

Given a sequence $t_n \longrightarrow 0$ we define

$D' = \{ \rho \in V: t_n^{-1} \; (S_{t_n}(\rho) - \rho)$ converges in the weak operator topology to a limit in V$\}$

and we define $H'(\rho)$ as the said limit for $\rho \in D'$. Then $\bar{D} \supseteq D_H$ and H' is an extension of H. If we can show that $(I - H')$ is one-one then since $(I - H)$ maps D_H one-one onto V by ((A7) in [3]), it follows that $D_H = D'$. Note that since each S_t: $V \longrightarrow V$ is continuous for the weak operator topology, D' is invariant under the action of S_t.

Suppose there exists some non-zero $\varrho \in D'$ such that

$(I - H')\varrho = 0$ i.e. $H'\varrho = \varrho$

Let φ be a weakly continuous functional on V such that $\varphi(\varrho) = 1$ and let $\zeta \in \mathcal{H}$ be such that $\zeta \neq 0$, if we define $f: \mathbb{R}^+ \to \mathbb{R}$ by

$$f(t) = \varphi(S_t(\varrho)\zeta)/\zeta$$

then f is continuous, $f(0) = 1$ and

$$\lim_{t_n \to 0} t_n^{-1} \{f(t+t_n) - f(t)\}$$

$$= \lim_{t_n \to 0} \varphi\{t_n^{-1}(S_{t+t_n}(\varrho)\zeta - S_t(\varrho)\zeta)/\zeta\}$$

$$= \lim_{t_n \to 0} \varphi\{S_t(t_n^{-1}(S_{t_n}(\varrho) - \varrho)\zeta)/\zeta\}$$

$$= \frac{\varphi(S_t(H'\varrho)\zeta)}{\zeta} = \frac{\varphi(S_t(\varrho)\zeta)}{\zeta}$$

$$= f(t)$$

The set $\{t \geq 0, \; f(t) \geq e^{t/2}\}$ is therefore non-empty, closed and has no right-hand end point. This contradicts the fact that f is a bounded function and leads to the conclusion that $(I - H')$ is one-one.

Theorem 3.2.1

The domains of the infinitesimal generators of T_t and S_t are equal. Moreover for all $\varrho \in V$, $\mathcal{J}^t(X,\varrho)$ converges in trace norm as $t \to 0$.

Proof

If $\varrho \in V$ and $\varrho^{\pm}$ are defined by

$$\varrho^{+} = \frac{|\varrho| + \varrho}{2} \quad , \quad \varrho^{-} = \frac{|\varrho| - \varrho}{2}$$

Then by equation (3.7) the operators $\mathcal{J}^t(X,\varrho^{\pm})$ are uniformly bounded, and there exists a sequence $t_n \longrightarrow 0$ such that $\mathcal{J}^{t_n}(X,\varrho^{\pm})$ converge in the weak operator topology to limits $\sigma_{\pm} \in L_s(\mathcal{H})^+$. ((A1) in [3]) and equation (3.7) imply that $\sigma_{\pm} \in V^+$ and

$$\mathrm{tr}[\sigma_{\pm}] \leq K\,\mathrm{tr}[\varrho^{\pm}] < \infty.$$

If $\sigma = \sigma_+ - \sigma_-$ then $\sigma \in V$, $\|\sigma\|_{tr} \leq K\,\|\varrho\|_{tr}$ and σ is the limit in the weak operator topology of $\mathcal{J}^{t_n}(X,\varrho)$ as $t_n \to 0$.

Now suppose $\varrho \in V^{+} \cap D_H$, by the above argument, lemma 3.2.1 and equation (3.8) we see that ϱ is in the domain of the infinitesimal generator of T_t. From (3.8) it now follows that $\mathcal{J}^t(X, \varrho)$ converges in trace norm as $t \to 0$. But since the domain of the infinitesimal generator of T_t is dense in V so $\mathcal{J}^t(X, \varrho)$ converges in trace norm to a limit in V as $t \to 0$ for all $\varrho \in V$. Again by equation (3.8) we conclude that the domain of the infinitesimal generator of S_t and T_t are equal.

We next want to consider the convergence as $t \to 0$ of $\mathcal{J}^t(E, \varrho)$ for arbitrary $E \subseteq X$ and $\varrho \in V$. In order to use separability arguments, we are forced to change the problem somewhat. The $\mathcal{J}^t$ are not instruments in the sense of [1] because they do not satisfy the normalization conditions so we make the following slight modifications.

If X is a separable locally compact Hausdorff space and (V, tr) is a state space, a bounded stochastic kernel $\mathcal{J}$ on X, V can be defined in three possible ways.

(S1) $\mathcal{J}_1$ is a bounded positive σ-additive measure on the σ-field of Borel sets in X with values in $L(V, V)$

(S2) $\mathcal{J}_2$ is a bilinear map $\mathcal{J}_2$: $B(X) \times V \to V$, where B(X) is the space of bounded Borel functions on X such that

(i) If $f \in B(X)^+$ and $\varrho \in V^+$ then $\mathcal{J}_2(f,\varrho) \in V^+$;

(ii) If $0 \leqq f_n \uparrow f$ in $B(X)$ and $\varrho \in V^+$ then $\mathcal{J}_2(f_n,\varrho)$ converges to $\mathcal{J}_2(f,\varrho)$ in norm.

(S3) $\mathcal{J}_3$ is a bilinear map $\mathcal{J}_3: C_{\mathbb{R}}(X) \times V \to V$ where $C_{\mathbb{R}}(X)$ is the space of continuous functions of compact support on X, such that

(i) If $f \in C_{\mathbb{R}}(X)^+$ and $\varrho \in V^+$ then $\mathcal{J}_3(f,\varrho) \in V^+$,

(ii) For some constant $K < \infty$

$\|\mathcal{J}_3(f,\varrho)\| \leqq K \|\varrho\| \max \{|f(x)|: x \in X\}$.

From now on we shall take $\mathcal{J}^t$ as bounded stochastic kernels in the sense of (S3), the advantage being that $C_{\mathbb{R}}(X)$ has a countable dense subset.

Lemma 3.2.2

If X is compact there exists a sequence $t_n \to 0$ of the form $t_n = 2^{-m_n}$ and a bounded stochastic kernel $\mathcal{J}$ on X such that for all $f \in C_{\mathbb{R}}(X)$ and $\varrho \in V$, $\mathcal{J}^{t_n}(f,\varrho)$ converges in trace norm to $\mathcal{J}(f,\varrho)$.

Proof

Using the separability of $C_{\mathbb{R}}(X)$ and V, the uniform boundedness of $\mathcal{J}^{2^{-n}}$, and the compactness of $\{A \in L_s(\mathcal{H}) : \|A\| \leq 1\}$ in the weak operator topology, there exists a sequence $t_n = 2^{-m_n}$ and a bilinear map

$$\mathcal{J} : C_R(X) \times V \rightarrow L_s(\mathcal{H})$$

such that for all $f \in C_{\mathbb{R}}(X)$ and $\rho \in V$, $\mathcal{J}^{t_n}(f, \rho)$ converges in the weak operator topology to $\mathcal{J}(f, \rho)$.

If $f \in C_{\mathbb{R}}(X)^+$ and $\rho \in V^+$ then it follows that $\mathcal{J}(f, \rho) \geq 0 \in L_s(\mathcal{H})^+$ and

$$\mathrm{tr}[\mathcal{J}(f, \rho)] \leq \lim_{n \to \infty} \inf \mathrm{tr}[\mathcal{J}^{t_n}(f, \rho)] \leq K\, \mathrm{tr}[\rho] < \infty$$

by ((A1), in [3]), so $\mathcal{J}(f, \rho) \in V$, and $\mathcal{J}$ is a bounded stochastic kernel. Moreover, if $\rho \in V^+$ and $0 \leq f \leq C1$ then

$$\mathrm{tr}[\mathcal{J}(C1, \rho)] = \lim \mathrm{tr}[\mathcal{J}^{t_n}(C1, \rho)]$$

$$\geq \lim_{n \to \infty} \inf \mathrm{tr}[\mathcal{J}^{t_n}(f, \rho)] + \lim_{n \to \infty} \inf \mathrm{tr}[\mathcal{J}^{t_n}(1-f, \rho)]$$

$$\geq \mathrm{tr}[\mathcal{J}(f, \rho)] + \mathrm{tr}[\mathcal{J}(1-f, \rho)]$$

$$= \mathrm{tr}[\mathcal{J}(1, \rho)]$$

which implies that

$$\lim_{n\to\infty} \operatorname{tr}[\mathcal{J}^{t_n}(f,\varrho)] = \operatorname{tr}[\mathcal{J}(f,\varrho)]\,.$$

therefore, by ((A1) in [3]), $\mathcal{J}^{t_n}(f,\varrho)$ converges in trace to $\mathcal{J}(f,\varrho)$.

Definition 3.2.1

The total interaction rate R of the quantum stochastic process is the unique operator $R\in L_s(\mathcal{H})^+$ such that for all $\varrho\in V$

$$\operatorname{tr}[\varrho R] = \operatorname{tr}[\mathcal{J}(X,\varrho)] = \lim_{t\to 0} \operatorname{tr}[\mathcal{J}^t(X,\varrho)]$$

Lemma 3.2.3

If $\zeta\in D_Z$ where D_Z is the domain in $\mathcal{H}$ of the infinitesimal generator Z on semi-group B_t then

$$\langle R\zeta,\zeta\rangle = -2\,\mathrm{Re}\,\langle Z\zeta,\zeta\rangle \qquad (3.11)$$

If ϱ is an arbitrary state in V then $\operatorname{tr}[S_t(\varrho)]$ is differentiable and

$$\frac{d}{dt}\operatorname{tr}[S_t(\varrho)] = -\operatorname{tr}[R\,S_t(\varrho)] \qquad (3.12)$$

Proof

If $\zeta \in D_Z$ then $|\zeta\rangle\langle\zeta|$ D_H since

$$\lim_{t \to 0} t^{-1}\{S_t(|\zeta\rangle\langle\zeta|) - |\zeta\rangle\langle\zeta|\}$$

$$= \lim_{t \to} \{t^{-1} |B_t\zeta - \zeta\rangle\langle\zeta| + t^{-1} |B_t\zeta\rangle\langle B_t\zeta - \zeta|\}$$

$$= |Z\zeta\rangle\langle\zeta| + |\zeta\rangle\langle Z\zeta|.$$

Taking the trace of both sides of equation (3.8) it follows that

$$\langle Z\zeta, \zeta\rangle + \langle\zeta, Z\zeta\rangle + \mathrm{tr}[\mathcal{J}(X, |\zeta\rangle\langle\zeta|)] = 0$$

which gives equation (3.11) for all states $\varrho = |\zeta\rangle\langle\zeta|$ with $\zeta \in D_Z$. Since $\zeta \in D_Z$ implies $B_t\zeta \in D_Z$ for all $t \geq 0$; it follows that if ϱ lies in

$$L = \text{linear span } [|\zeta\rangle\langle\zeta| : \zeta \in D_Z] \subseteq V$$

we see that

$$\mathrm{tr}[S_t(\varrho)] = \mathrm{tr}[\varrho] - \int_0^t \mathrm{tr}[S_s(\varrho)R]\, ds, \tag{3.13}$$

this being the integral version of equation (3.12). Since L is dense in V so equation (3.13) depends continuously on ϱ it holds for all $\varrho \in V$. Differentiating this now proves equation (3.12) for all $\varrho \in V$.

Comments

Equation (3.12) gives a clear reason why we call R the total interaction rate, since $tr[S_t(\rho)]$ is defined as the probability of no interaction between the quantum system and the measuring apparatus up to time t, we note that the interaction rate is independent of the state iff R is a scalar multiple of the identity operator. This clearly is an unreasonable condition if the measuring apparatus is localised in some finite region of space.

Having completed the preliminary work we can now prove the main theorem of the section.

Theorem 3.2.2

Let $\mathcal{E}_a^{\,t}$ be a bounded stochastic process on X, V where X is a separable locally compact Hausdorff space and $V = \mathcal{T}_s(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. If the semi-group S_t takes pure states to pure states then $\mathcal{E}_a^{\,t}$ is uniquely determined by the infinitesimal generator Z of the semi-group B_t on $\mathcal{H}$ and a bounded stochastic kernel $\mathcal{J}$ on X, which are related by the equation

$$tr[\mathcal{J}(X, |\zeta\rangle\langle\zeta|] = -2\,\mathrm{Re}\,\langle Z\zeta, \zeta\rangle \qquad \text{for all } \zeta \in D_Z.$$

Proof

If $t>0$, $f \in C_{\mathbb{R}}(X)$ and $\varrho \in V$ then, using that fact that $\mathcal{E}^t$ has a bounded interaction rate, by equations (3.3) and (3.6)

$$2^{-m}[2^m t]\mathcal{J}^{2^{-m}[2^m t]}(f, \varrho) = \sum_{r=1}^{[2^m t]} 2^{-m} S_{2^{-m}\{[2^m t]-r\}}\mathcal{J}^{2^{-m}}(f, S_{2^{-m}(r-1)}\varrho)$$

where $[2^m t]$ denotes the largeset integer less than or equal to $2^m t$. Using lemma 3.2.2 it follows, letting $m_n \longrightarrow \infty$, that

$$\mathcal{J}^t(f, \varrho) = t^{-1}\int_0^t S_{t-s}(f, S_s\varrho)ds \qquad (3.14)$$

for all $t \geq 0$, $f \in C_{\mathbb{R}}(X)$ and $\varrho \in V$, the integer existing as a vector valued norm-convergent Riemann integral. The above integral in fact exists in the same sense for all $f \in B(X)$ and using familiar dominated convergence arguments for the ordinary Lebesgue integral,

$$\int_0^t tr[S_{t-s}\mathcal{J}(f, S_s\varrho)B]\, ds,$$

where $B \in L_s(\mathcal{H})$ is arbitrary, we see that the equation (3.14) holds for all $f \in B(X)$. Therefore, for all $\varrho \in V$ and $f \in B(X)$,

$$\mathcal{J}(f, \varrho) = \lim_{t \to 0}\mathcal{J}^t(f, \varrho)$$

in the trace norm.

This greatly improves lemma 3.2.2 and shows that it is not necessary to assume that X is compact, and proves that $\mathcal{J}$ is uniquely determined by $\mathcal{E}^t$.

Now let $0<r_1<t_1<\ldots<r_m<t_m\leq t$ and $\{E_i\}_{i=1}^m$ are Borel sets in X, and let $E \subseteq X_t$ be the Borel set defined by

$$E=\{(x_i, s_i)_{i=1}^m : r_i<s_i\leq t_i \text{ and } x_i \in E_i\}, \tag{3.15}$$

then by its defining properties

$$\mathcal{E}^t(E,\rho)=\prod_{i=1}^m (t_i-r_i)\{S_{t-t_m}\mathcal{J}_{E_m}^{t_m-r_m} S_{r_m-t_{m-1}}\ldots S_{r_2-t_1}\mathcal{J}_{E_1}^{t_1-r_1} S_{r_1}\rho\}$$

$$=\int_{r_i<s_i\leq t_i} S_{t-t_m}(S_{t_m-s_m}\mathcal{J}_{E_m}S_{s_m-r_m})\ldots S_{r_2-t_1}(S_{t_1-s_1}\mathcal{J}_{E_1}S_{s_1-r_1})S_{r_1}\rho\, ds_1\ldots ds_m$$

or, simplifying

$$\mathcal{E}^t(E,\rho)=\int_{r_i<s_i\leq t_i} S_{s-s_m}\mathcal{J}_{E_m}S_{s_m-s_{m-1}}\mathcal{J}_{E_{m-1}}\cdots\mathcal{J}_{E_1}S_{s_1}\rho\, ds_1\ldots ds_m.$$

This shows that for $\rho \in V$, S_t and $\mathcal{J}$ determine $\mathcal{E}^t(E,\rho)$ for any set E of the form of equation (3.15), and hence for any Borel set E.

Definition 3.2.2

Z and $\mathcal{J}$ are called the infinitesimal generators of the stochastic process $\mathcal{E}^t$. The following result shows that the stated conditions on Z and $\mathcal{J}$ are the only ones necessary for them to be the generators of a stochastic process.

Theorem 3.2.3

Let X be a separable locally compact Hausdorff space and $V = \mathcal{T}_s(\mathcal{H})$ the state space of a separable Hilbert space $\mathcal{H}$. Let Z be the infinitesimal generator of a strongly continuous one-parameter-contraction B_t on $\mathcal{H}$ and $\mathcal{J}$ be a bounded stochastic kernel on X, V such that for all $\zeta \in D_Z$

$$\mathrm{tr}[\mathcal{J}(X,|\zeta\rangle\langle\zeta|)] = -2\,\mathrm{Re}\,\langle Z\zeta,\zeta\rangle \ .$$

Then there exists a unique stochastic process such that Z, $\mathcal{J}$ are its infinitesimal generators.

Proof

We define $\mathcal{E}^t$ independently on each of the sets $A_t^n \subseteq X_t$. If $n=0$ we construct the semi-groups S_t and B_t from Z and define

$$\mathcal{E}^t(Z,\varrho)=S_t(\varrho)=B_t\varrho B_t^*\qquad \forall t\geq 0 \text{ and } \varrho\in V.$$

If $n\geq 1$ and $0<t_1<\ldots<t_n\leq t$ there is a unique bounded stochastic kernel $\mathcal{J}_{t_1,t_2,\ldots,t_n}$ on X^n such that if $E_1,\ldots,E_n\subseteq X$

$$\mathcal{J}_{t_1,\ldots,t_n}(E_1\, E_2\,\ldots\, E_n,\varrho)=S_{t-t_n}\mathcal{J}_{E_n}S_{t_n-t_{n-1}}\cdots\mathcal{J}_{E_1}S_{t_1}.$$

In order to apply theorem 4.2.2 in [3] we regard S_s as a bounded stochastic kernel on a set of one element. If E is a Borel set in A_t^n and

$0<t_1<\ldots<t_n\leq t$ we define $E_{t_1,\ldots,t_n}\subseteq X^n$ by

$$E_{t_1,\ldots,t_n}=\{(x_1,\ldots,x_n):(x_r,t_r)_{r=1}^n\in E\}.$$

We then define the bounded stochastic kernel $\mathcal{E}_n^t$ on A_t^n by

$$\mathcal{E}_n^t(E,\varrho)=\int_{0<t_1<\ldots<t_n\leq t}\mathcal{J}_{t_1,\ldots,t_n}(E_{t_1,\ldots,t_n},\varrho)dt_1\ldots dt_n,\qquad (3.16)$$

if $0\leq \operatorname{tr}[\mathcal{J}(X,\varrho)]\leq K\operatorname{tr}[\varrho]$

for all $\varrho\in V^+$, then it is immediate that

$$0\leq \operatorname{tr}[\mathcal{E}_n^t(A_t^n,\varrho)]\leq\frac{K^n t^n \operatorname{tr}[\varrho]}{n!}\qquad \forall\varrho\in V^+.\qquad (3.17)$$

We now define the bounded stochastic process $\mathcal{E}^t$ on X_t by

$$\mathcal{E}^t(E,\varrho)=\sum_{n=0}^{\infty}\mathcal{E}^t(E\cap A_t^n,\varrho),$$

equation (3.17) implies that this series converges. Indeed

$$0\leq \mathrm{tr}\,[\mathcal{E}^t(X_t,\varrho)]\leq e^{kt}\,\mathrm{tr}[\varrho] \qquad \forall\,\varrho\in V^+.$$

Equation (3.3) is now immediate from equation (3.16).

If we define $T_t: V\longrightarrow V$ by

$$T_t(\varrho)=\mathcal{E}^t(X_t,\varrho)$$

then it follows from equation (3.17) that for all $\varrho\in V$,

$$T_t(\varrho)=S_t(\varrho)+\int_0^t S_{t-s}\mathcal{J}_X S_s\varrho\,ds+0(t^2)$$

$$=S_t(\varrho)+t\,\mathcal{J}_X(\varrho)+0(t^2) \tag{3.18}$$

as $t\longrightarrow 0$, it proves equation (3.1). i.e., it shows that $\mathcal{E}^t$ is strongly continuous in the sense of the definition of stochastic process.

Finally, by taking the trace of both sides of equations (3.18) and differentiating, gives, by lemma 3.2.3

$$\frac{d}{dt}\,\mathrm{tr}\,[T_t \varrho]_{t=0} = -\mathrm{tr}\,[R\varrho] + \mathrm{tr}\,[\mathcal{J}_x(\varrho)]$$

$$=0$$

i.e. $\frac{d}{dt}\,\mathrm{tr}\,[T_t\varrho] = 0$ for all $\varrho \in V$.

So $\mathrm{tr}\,[T_t\varrho] = \mathrm{tr}\,[\varrho]$ for all $\varrho \in V$ and $t \geq 0$

which is equation (3.2).

This concludes the proof that $\mathcal{E}^t$ is a stochastic process.

Among the various results presented in this chapter, the most important one was the derivation of the differential equation (3.9), and relatively we showed how to reconstruct the stochastic process from this differential equation. We then obtain two semi-groups S_t and T_t, given the stochastic process. The semi-groups can be determined by and constructed from their infinitesimal generators too.

In the following chapter, we make a detailed study of these generators.

CHAPTER 4

Semi—Groups and Generators

This chapter consists of an introduction to the theory of semi-groups of bounded linear operators in a Banach space $\mathfrak{X}$. It is concerned with the problem of determining the most general linear bounded operator valued function T_t, $t \geq 0$ which satisifies:

$$T_t T_s = T_{t+s}.$$

Subsequently, we construct the infinitesimal generator L of T_t defined as:

$$L = s - \lim_{t \to 0} t^{-1} (T_t - I),$$

where I is the identity operator on the Banach space $\mathfrak{X}$.

Here we are only concerned with the Banach space $\mathfrak{X} = \mathcal{T}_s(\mathcal{H})$.

Section 1: Semi-Groups

Let $\mathcal{T}_s(\mathcal{H})$ be the Banach space of self-adjoint trace class linear operators on $\mathcal{H}$. A function T_t on $[0,\infty)$ with values in $\mathcal{T}_s(\mathcal{H})$ is defined to be a one-parameter semi-group on $\mathcal{T}_s(\mathcal{H})$.

Definition 4.1.1

Let $\{T_t, t \geq 0\}$ be a one-parameter family of bounded linear operators on $\mathcal{T}_s(\mathcal{H})$ satisfying the conditions:

$$T_t T_s = T_{t+s} \qquad \forall t,s \geq 0, \tag{4.1}$$

$$T_0 = I. \tag{4.2}$$

Then $\{T_t\}$ is called a one-parameter semi-group. T_t is said to be strongly measurable if for each $\zeta \in \mathcal{T}_s(\mathcal{H})$, $T_t(\zeta)$ is the limit almost everywhere of a sequence of step functions. i.e. for any $\zeta \in \mathcal{T}_s(\mathcal{H})$, there exists a sequence $\{t_n\}$ such that $T_{t_n}(\zeta)$ is a sequence of step functions of elements in $\mathcal{H}$. and

$$T_{t_n}(\zeta) \rightarrow T_t(\zeta) \text{ a.e.}$$

Dunford and Hille [9] showed that if T_t is strongly measurable and $\| T_t \|$ is bounded in each interval $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$, then $T_t(\zeta)$ is strongly continuous for $t \geq 0$ and $\zeta \in \mathcal{T}_s(\mathcal{H})$. (For more details and proofs of the above statements, cf, Phillips [7], Hille and Phillips [9].)

Here we show briefly that the first hypothesis of this theorem implies the second.

Proposition 4.1.1

If $\{ T_t \}$ is strongly measurable, then $\| T_t \|$ is bounded in each interval $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$.

Proof

Making use of the uniform boundedness theorm [cf, 10], it is sufficient to show that $\| T_t f \|$ is bounded in $[\alpha, \beta]$ for each $f \in \mathcal{T}_s(\mathcal{H})$. Suppose on the contrary, that $\| T_t f \|$ is not bounded for some f. Then there will exist a $\gamma \in [\alpha, \beta]$ and a sequence $\{t_n\} \subset [\alpha, \beta]$ such that $t_n \rightarrow \gamma$ and

$$\| T_{t_n} f \| \geq n \qquad n=1,2,3,\ldots$$

On the other hand, since $\| T_t f \|$ is measurable, there will exist a constant M and a measurable set $F \subset [0, \gamma]$ with $m(F) > \gamma/2$ such that

$$\sup_{t \in F} \| T_t F \| \leq M.$$

Now set $E_n = \{(t_n - \eta) : \eta \in F \cap [0, t_n]\}$. Then E_n is measurable and for n sufficiently large $m(E_n) \geq \gamma/2$.

For $\eta \in F \cap [0, t_n]$ we have

$$n \leq \| T_{t_n} f \| \leq \| T_{t_n - \eta} \| \, \| T_\eta f \| \leq \| T_{t_n - \eta} \| M .$$

Therefore

$$\| T_t \| \geq \frac{n}{M} \qquad \text{for all } t \in E_n.$$

Hence denoting $\limsup_n E_n = E$, we see that

$$\| T_t \| = \infty \qquad \text{for all } t \in E \text{ and } m(E) \geq \gamma/2.$$

This contradicts the fact that $\| T_t \|$ is finite-valued for $t \in (0, \infty)$.

Thus, $\| T_t \|$ is bounded, i.e. $\| T_t \| \leq M$.

So from the last statement of this proposition, we get the second result that T_t is continuous for $t>0$.

Definition 4.1.2

The semi-group T_t is said to be strongly continuous if

$$\underset{t\to 0^+}{\text{s-lim}}\, T_t f = f \qquad \text{for each } f \in \mathcal{T}_s(\mathcal{H}) \qquad (4.3)$$

From now on, we consider the semi-group T_t to be strongly continuous on $[0,\infty)$ with $T_0 = I$ (I stands for the identity operator on $\mathcal{T}_s(\mathcal{H})$).

Definition 4.1.3

The positive cone $V_s^+(\mathcal{H})$ in $\mathcal{T}_s(\mathcal{H})$ is the set of all semi-positive definite elements of $\mathcal{T}_s(\mathcal{H})$. $V_s^+(\mathcal{H})$ can also be defined as follows:

$$V_s^+(\mathcal{H}) = \{\, \rho \in \mathcal{T}_s(\mathcal{H}) : \|\rho\|_{tr} = \operatorname{tr}\rho \,\}, \qquad (4.4)$$

So $\rho \in V_s^+(\mathcal{H})$ iff $\|\rho\|_{tr} = \operatorname{tr}\rho$ holds, [cf,14].

The Banach space $\mathcal{T}_s(\mathcal{H})$ is the smallest linear space in which the set $P(\mathcal{H})$ of all density operators can be embedded.

More precisely

$$P(\mathcal{H}) = \{\varrho \in V_s^+(\mathcal{H}) : \|\varrho\|_{tr} = 1\}$$

Definition 4.1.4

From the proposition 4.1.1, if in particular $M \leq 1$ that is if,

$$\|T_t\| \leq 1.$$

Then the semi-group $\{T_t\}$ is called a contraction semi-group.

We introduce now the notion of a dynamical semi-group, which is a strongly continuous one-parameter contracting semi-group of trace preserving linear operators on $\mathcal{T}_s(\mathcal{H})$.

Definition 4.1.5

A family $S(\mathcal{H}) = \{T_t, t \geq 0\}$ of linear operators on $\mathcal{T}_s(\mathcal{H})$ is said to be a dynamical semi-group of a quantum system provided the following ocnditions are satisfied:

(i) $T_t : V_s^+(\mathcal{H}) \to V_s^+(\mathcal{H})$

(ii) $\| T_t \varrho \|_{tr} = \| \varrho \|_{tr}$ for all $\varrho \in V_s^+(\mathcal{H})$ and $t \geqq 0$.

(iii) $T_t T_s = T_{t+s}$ $\forall t,s \geqq 0$

(iv) $\underset{t \downarrow 0}{\text{s-lim}}\ T_t = I$.

Lemma 4.1.1

Let S be a linear operator on $\mathcal{T}_s(\mathcal{H})$. The conditions

(i) $S: V_s^+(\mathcal{H}) \to V_s^+(\mathcal{H})$

(ii) $\| S\varrho \|_{tr} = \| \varrho \|_{tr}$ — $\forall\ \varrho \in V_s^+(\mathcal{H})$

and

(a) $\| S\varrho \|_{tr} \leqq \| \varrho \|_{tr}$ $\forall\ \varrho \in \mathcal{T}_s(\mathcal{H})$

(b) $\mathrm{tr}\,(S\varrho) = \mathrm{tr}\,\varrho$ $\forall\ \varrho \in \mathcal{T}_s(\mathcal{H})$

are equivalent.

Proof

The implication (i), (ii) $\to$ (a), (b) has been shown by Kossakowski [13]. Suppose now that the conditions (a) and (b) hold. Taking (4.4) into account it follows that for all $\varrho \in V_s^+(\mathcal{H})$ the inequalities

$$\| \varrho \|_{tr} = tr(S\varrho) \leq \| S\varrho \|_{tr} \leq \| \varrho \|_{tr}$$

holds, i.e., $\|S\varrho\|_{tr} = \|\varrho\|_{tr}$ and $tr(S\varrho) = \|S\varrho\|_{tr}$ for all $\varrho \in V_s^+(\mathcal{H})$. Since $\varrho \in V_s^+(\mathcal{H})$ iff $\|\varrho\|_{tr} = tr\,\varrho$, the last equality implies that $S\varrho \in V_s^+(\mathcal{H})$ whenever $\varrho \in V_s^+(\mathcal{H})$ i.e.

$$S: V_s^+(\mathcal{H}) \to V_s^+(\mathcal{H}).$$

Taking Definition (4.1.5) and Lemma (4.1.1) into account we have

Theorem 4.1.1

A family $S(\mathcal{H}) = \{T_t, t \geq 0\}$ of linear operators on $\mathcal{T}_s(\mathcal{H})$ is a dynamical semi-group iff the following conditions are satisfied

(i)	$tr(T_t\varrho) = tr\,\varrho$	$\forall\, \varrho \in \mathcal{T}_s(\mathcal{H})$ and $t \geq 0$
(ii)	$\| T_t\varrho \|_{tr} \leq \| \varrho \|_{tr}$	for all $\varrho \in \mathcal{T}_s(\mathcal{H})$ and $t \geq 0$
(iii)	$T_t T_s = T_{t+s}$	$t, s \geq 0$
(iv)	$\underset{t \downarrow 0}{\text{s-lim}}\ T_t = I$	

Section 2: The Infinitesimal Generator

We now undertake a further study of the structure of the semi-group $G = \{ T_t \}$. Throughout this section we assume that T_t is strongly continuous for $t>0$. The infinitesimal generator A of T_t is defined as

$$A_t = \frac{1}{t}[T_t - I],$$

$$A \rho = \text{s-}\lim_{t \to 0^+} A_t \rho .$$

whenever the limit exists, i.e., A is the linear operator whose domain is the set

$$D(A) = \{ \rho \in \mathcal{T}_s(\mathcal{H}); \lim_{t \to 0^+} t^{-1} (T_t - I) \rho \ \text{ exists in } \mathcal{T}_s(\mathcal{H}) \}.$$

It is clear that $D(A)$ is a linear subspace of $\mathcal{T}_s(\mathcal{H})$ and for $\rho \in D(A)$, $A\rho = \lim_{t \to 0^+} t^{-1} (T_t - I) \rho$. $D(A)$ is non-empty; it contains at least the vector 0. Actually $D(A)$ is larger. We shall prove that $D(A)$ is dense in $\mathcal{T}_s(\mathcal{H})$.

We start by showing that $D(A)$ never reduces to the zero element. We set

$$\varphi_\beta = \int_0^\beta T_s\, y\, ds \qquad \text{for } y \in \Upsilon_s(\mathcal{H}),\ 0 < \beta < \infty \quad (4.5)$$

then

$$A_t\, \varphi_\beta = \frac{1}{t} \int_0^\beta [\, T_t - I\,]\, T_s\, y\, ds$$

$$= \frac{1}{t} \int_0^\beta [\, T_{t+s} - T_s\,]\, y\, ds$$

$$A_t \varphi_\beta = \frac{1}{t} \int_t^{t+\beta} T_\delta\, y\, d\delta - \frac{1}{t} \int_0^t T_\delta\, y\, d\delta$$

which will tend to $[\, T_\beta - T_I\,]y$ as $t \longrightarrow 0^+$. Hence every element of the type φ_β belongs to D(A). [9]

Now let R_α be the range of the transformation T_t, $t>0$. Clearly

$$R_\alpha \supset R_\beta \qquad \text{if } \alpha < \beta .$$

We define $R = \cup[\, R_\alpha\ ;\ \alpha > 0]$. Thus R is the smallest linear subspace containing the range space of G and R is dense in $\Upsilon_s(\mathcal{H})$.

Theorem 4.2.1

If $[\, T_t;\ t>0\,]$ is strongly continuous, then D(A) is dense in R, the two sets have the same closure, and the range of A is contained in $\bar{R}$.

Proof

If $\varphi \in R$ there exists an $S>0$ and a $y \in \mathfrak{J}_S(\mathcal{H})$ such that $\varphi = T_S y$. The element φ_β defined by (4.5) belongs to $D(A) \cap R$ and

$$\lim_{\beta \to 0} [\frac{1}{\beta}] \varphi_\beta = \varphi ;$$

that is every point of R lies in the closure of $D(A) \cap R$. Conversely, if $y \in D(A)$, then

$$\lim_{t \to 0^+} T_t y = y$$

so that $y \in \bar{R}$. It follows that the closure of R and $D(A)$ coincide. Finally, if $y \in D(A)$, then $A_t y \in \bar{R}$ and so does A .

Theorem 4.2.2

If T_t is strongly continuous for $t>0$, then for $\varrho \in D(A)$

$$\frac{d}{dt}(T_t \varrho) = A(T_t \varrho) = T_t(A \varrho), \qquad t>0 \qquad (4.6)$$

Proof

We have

$$\frac{1}{\zeta}[T_{t+\zeta} - T_t]\varrho = A_\zeta(T_t \varrho) = T_t(A_\zeta \varrho).$$

Since the right-hand side tends to $T_t(A\varrho)$ as $\zeta \to 0^+$, we see that $T_t(\varrho) \in D(A)$. And $A(T_t\varrho) = T_t(A\varrho)$. So that the right-sided derivative of $T_t\varrho$ exists and satisfies (4.6). It is easy to see also that the left-sided derivative exists. For that we make use of the fact that T_t is strongly continuous at $t>0$.

$$\frac{-1}{\zeta}[T_{t-\zeta} - T_t]\varrho = (T_{t-\zeta})A\varrho ,$$

and this tends to $T_t(A\varrho)$ as $\zeta \to 0^+$; which satisfies (4.6).

We next consider the problem: What properties should an operator A possess in order to be the infinitesimal generator of a semi-group G of linear operators?

We consider the semi-group G, in particular the properties of its generating operator A and its resolvent $R(\lambda, A)$. We use the definition of the resolvent of the generator [9] to be the Laplace transformation of the semi-group G, i.e.

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T_s \, ds, \qquad 0<\lambda<\infty.$$

Namely, $R(\lambda, A)$ satisfies the inequality

$$\lambda^{n+1} \| R^{(n)}(\lambda; A) \| \leq Mn! \qquad n=0,1,2,3,\ldots \quad \lambda>0 \tag{4.7}$$

for some constant M. [cf, 9]

Theorem 4.2.3

A necessary and sufficient condition that a closed linear operator A generates a semi-group $G = \{ T_t, t>0\}$ such that $\|T_t\| \leq M$ is that D(A) be dense in $\mathcal{T}_s(\mathcal{H})$ and

$$\| [R(\lambda; A)]^n \| \leq M \lambda^{-n} \tag{4.8}$$

for $\lambda > 0$, $n=1,2,3,\ldots$ and some constant M.

Proof

We note that the two inequalities (4.7) and (4.8) are equivalent as

$$R^{(n)}(\lambda; A) = (-1)^n n! [R(\lambda; A)]^{n+1}$$

the necessity is now immediate. By Theorem (4.2.1), for T_t with infinitesimal generator A, D(A) is dense in $\mathcal{T}_s(\mathcal{H})$. Further for $\lambda>0$

$$R(\lambda ; A)\varrho = \int_0^\infty e^{-\lambda s}\, T_s \varrho \, ds \qquad \forall \varrho \in \mathcal{J}_s(\mathcal{H})$$

and hence

$$R^{(n)}(\lambda ; A)\varrho = \int_0^\infty e^{-\lambda s} (-S)^n \, T_s \varrho \, ds.$$

Since $\| T_t \| \leqq M$ we obtain

$$\| R^{(n)}(\lambda ; A) \| \leqq \int_0^\infty e^{-\lambda s} s^n M \, ds = \frac{M n!}{\lambda^{n+1}}, \quad n=0,1,2,\ldots$$

which is formula (4.7).

This result is very close to the original Hille-Yosida Theorem, which now appears as corollary.

Further, if $M=1$, and if (4.8) holds for $n=1$, then it automatically holds for $n \geqq 1$; this proves the Hille-Yosida Theorem, namely,

Corollary 4.2.1

If A is a closed linear operator with dense domain, if $R(\lambda ; A)$ exists for $\lambda > 0$, and if

$$\| R(\lambda ; A) \| \leqq 1 \qquad \lambda > 0, \qquad (4.9)$$

then A is the infinitesimal generator of a semi-group $[T_t]$ such that $\| T_t \| \leqq 1$ for $t \geqq 0$.

Applying the Hille-Yosida Theorem to a dynamical semi-group $S(\mathcal{H}) = \{ T_t, t \geqq 0 \}$, it follows that there exists a linear operator L on $\mathcal{T}_s(\mathcal{H})$ called the generator of dynamical semi-group $S(\mathcal{H})$, whose domain is dense in $\mathcal{T}_s(\mathcal{H})$, with the property

$$\frac{d}{dt}(T_t \varrho) = L(T_t \varrho) = T_t(L\varrho) \qquad (4.10)$$

for all $\varrho \in D(L)$. Moreover, T_t, $t \geqq 0$ is the solution of (4.10) with the initial condition $T_0 \varrho = \varrho \qquad \forall \varrho \in D(L)$.

Now using the Hille-Yosida Theorem and Theorem (4.1.1) we immediately get:

Theorem 4.2.4

A linear operator L with domain D(L) and range R(L) both in $\mathcal{T}_s(\mathcal{H})$ generates a dynamical semi-group $S(\mathcal{H})$ iff the following conditions are satisfied

(i) The domain D(L) is dense in $\mathcal{T}_s(\mathcal{H})$

(ii) $R(\lambda I - L) = \mathcal{T}_s(\mathcal{H})$

(iii) $\| \lambda \varrho - L\varrho \|_{tr} \geqq \lambda \| \varrho \|_{tr}$ for all $\varrho \in D(L)$ and each $\lambda > 0$

(iv) $tr(L\varrho) = 0 \qquad \forall \varrho \in D(L)$.

<u>Proof</u>

The implication in one direction is trivial, since if L generates the dynamical semi-group $S(\mathscr{H})$, conditions (i) $\rightarrow$ (iv) holds. Conversely, we have to show that if conditions (i) $\rightarrow$ (iv) are satisfied, then L is a generator of a dynamical semi-group $S(\mathscr{H})$. Conditions (i) and (ii) have been proved in Theorem (4.2.1), since T_t is a strongly continuous semi-group. (iii) Using the heuristic correspondence principle (see [9], Chapter 11), since the resolvent is given by $R(\lambda, L) = (\lambda I - L)^{-1}$, then for $\varrho \in D(L)$ we have $R(\lambda, L)\varrho = (\lambda I - L)^{-1}\varrho \quad , \lambda > 0.$

Taking the trace norm of both sides, we get:

$$\frac{\|\varrho\|_{tr}}{\|R(\lambda, L)\|_{tr}} = \|\lambda\varrho - L\varrho\|.$$

But by virtue of the Corollary (4.2.1), $\|R(\lambda, L)\|_{tr} \leq \lambda^{-1}$, this implies

$$\|\lambda\varrho - L\varrho\|_{tr} \geq \lambda\|\varrho\|_{tr} \qquad \lambda > 0$$

which proves (iii).

Using Theorem (4.1.1) and Definition (2.2.2), for $\varrho \in D(L)$, $L\varrho \in D(L)$, we have

$$tr[L(T_t\varrho)] = tr[T_t(L\varrho)] = tr[L\varrho].$$

On the other hand by equation (4.10)

$$\mathrm{tr}[\,L(T_t \varrho)\,] = \mathrm{tr}[\,\frac{d}{dt}(T_t \varrho)\,] = \frac{d}{dt}\mathrm{tr}[T_t \varrho\,] = \frac{d}{dt}\mathrm{tr}[\,\varrho\,] = 0.$$

Hence,

$$\mathrm{tr}[\,L\varrho\,] = 0$$

which completes the proof.

Other necessary and sufficient conditions for the existence of a generator can be found using the result of Lumer and Phillips [11]. To introduce this idea, we need the following definitions.

<u>Definition 4.2.1</u>[cf,12]

To each pair $\{\sigma, \varrho\}$ of elements of $\mathcal{T}_s(\mathcal{H})$ there corresponds a real number $[\sigma, \varrho]$, called semi-inner product in $\mathcal{T}_s(\mathcal{H})$, in such a way that:

(i) $[\sigma + \varrho, \tau] = [\sigma, \tau] + [\varrho, \tau] \quad \forall \sigma, \varrho, \tau \in \mathcal{T}_s(\mathcal{H})$

$[\lambda\sigma, \varrho] = \lambda[\sigma, \varrho]$ and λ real.

(ii) $[\sigma, \sigma] = \|\sigma\|^2 > 0$ for $\sigma \neq 0$

(iii) $|[\sigma, \varrho]| \leq \|\sigma\|_{tr} \|\varrho\|_{tr}$.

Definition 4.2.2[cf, 11]

A linear operator on $\mathcal{T}_s(\mathcal{H})$ is called dissipative (with respect to $[\sigma,\rho]$) if

$$[\rho, L\rho] \leqq 0 \qquad \text{for } \rho \in D(L).$$

Definition 4.2.3[cf, 14]

Let E_ρ be the spectral measure corresponding to $\rho \in \mathcal{T}_s(\mathcal{H})$. To each element $\rho \in \mathcal{T}_s(\mathcal{H})$ we associate a linear operator $\mathrm{sign}\,\rho$ on the Banach space $\mathcal{T}_s(\mathcal{H})^*$ (dual to $\mathcal{T}_s(\mathcal{H})$), defined as follows

$$\mathrm{sign}\,\rho = \int_{-\infty}^{\infty} (\mathrm{sign}\,\lambda)\, E_\rho \, d\lambda \tag{4.11}$$

where

$$\mathrm{sign}\,\lambda = \begin{cases} 1, & \lambda > 0 \\ 0, & \lambda = 0 \\ -1, & \lambda < 0. \end{cases}$$

As an example, a semi-inner product in $\mathcal{T}_s(\mathcal{H})$ is defined by

$$[\sigma, \rho] = \|\sigma\|_{tr}\, \mathrm{tr}((\mathrm{sign}\,\sigma)\,\rho) \tag{4.12}$$

Theorem 4.2.5

A linear operator on $\mathcal{T}_s(\mathcal{H})$ generates a dynamical semi-group $S(\mathcal{H})$ in $\mathcal{T}_s(\mathcal{H})$ iff

(i) The domain $D(L)$ is dense in $\mathcal{T}_s(\mathcal{H})$;

(ii) $R(I - L) = \mathcal{T}_s(\mathcal{H})$ $\qquad \lambda = 1,$

(iii) $[\varrho, L\varrho] \leq 0$ $\qquad \forall \varrho \in D(L),$

(iv) $\mathrm{tr}[L\varrho] = 0$ $\qquad \forall \varrho \in D(L).$

Proof

Since T_t is a strongly continuous semi-group of contraction operators. Then

$$[(T_t\varrho - \varrho), \varrho] = [T_t\varrho, \varrho] - [\varrho, \varrho]$$
$$= [T_t\varrho, \varrho] - \|\varrho\|^2$$

using (4.11) and (4.12), we get

$$[(T_t\varrho - \varrho), \varrho] = [T_t\varrho, \varrho] - \|\varrho\|^2 \leq 0.$$

Hence, for $\varrho \in D(L)$

$$[\varrho, L\varrho] = \lim_{t \to 0^+} t^{-1} [(T_t\varrho - \varrho), \varrho] \leq 0.$$

Thus, the generator L is dissipative. Moreover, it is shown in Theorems (4.2.1) and (4.2.4) that (i), (ii), and (iv) are satisfied. Conversely, if L is dissipative, we see by Lemma 3.1 of [11] that $\| (\lambda I - L)^{-1} \|_{tr} \leq \lambda^{-1}$ for all $\lambda > 0$. By assumption $R(I-L) = \mathcal{T}_s(\mathcal{H})$ so $\lambda = 1$ is in the resolvent set of L. Denoting resolvent of L at λ by $R(\lambda; L)$, it follows that

$$\| R(\lambda ; L) \| \leq \| R(\lambda ; L) \|_{tr} \leq \lambda^{-1} \qquad \lambda > 0$$

which satisfies (4.9). Since D(L) is assumed to be dense in $\mathcal{T}_s(\mathcal{H})$, therefore, it now follows by Hille-Yosida Theorem that L generates a dynamical semi-group $S(\mathcal{H})$.

Now if the linear operator L is bounded on $\mathcal{T}_s(\mathcal{H})$, then only the two following conditions are necessary in order for it to be a generator of $S(\mathcal{H})$.

(i) $[\varrho, L\varrho] \leq 0 \qquad \forall \varrho \in \mathcal{T}_s(\mathcal{H})$,

(ii) $tr[L\varrho] = 0 \qquad \forall \varrho \in \mathcal{T}_s(\mathcal{H})$.

CHAPTER 5

An Example and Comments

In this chapter, we introduce an example on physical grounds. The aim is to describe the time evolution of a state ρ which is subject to random measurements. The time evolution of a quantum mechanical system is given by a dynamical semi-group; so we expect, then, that for any given state ρ at time t=0 will uniquely determine another state ρ_t at time $t\neq0$. As a result of this evolution ρ transforms to ρ_t at time t, and the transformation $\rho \rightarrow \rho_t$ is strongly continuous. The time evolution of a system of quantum mechanics can be described in the form of spin 1/2 system or system of decaying unstable elementary particle.

Let us consider a two-dimensional Hilbert space $\mathcal{H}$, and if we assume ρ to be a state at t=0, entering a bubble chamber and decaying there [15] and [16], then at random times ρ_t will be the state ρ evolved to time t, assuming that a measurement is made on ρ_t. We assume that for all $\rho \in \mathcal{T}(\mathcal{H})$, ρ_t is given by $\sum_t(\rho)$, where $\sum_t$ is a semi-group on $\mathcal{T}(\mathcal{H})$. We now study the nature of $\sum_t$ in a specific example. But before introducing the problem and giving all calculations, we need to introduce some of the concepts to be used here.

Definition 5.1

A stochastic dynamical process is a strongly continuous semi-group $\sum_t : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$, for $t \in R^+$; such that

(i) $\sum_o = I$

(ii) $\sum_{t_1+t_2} = \sum_{t_1} \sum_{t_2}$.

Clearly a stochastic dynamical process $\sum_t$ is a generalization of the usual time evolution of quantum mechanics i.e. for

$$\rho \in \mathcal{T}(\mathcal{H}), \qquad \rho_t = \sum_t (\rho).$$

$\sum_t$ may be written as the sum of two parts, $\sum_t^I$ and $\sum_t^{II}$, and $\sum_t$ can be given by $\exp(Xt)$, where X is an infinitesimal generator.

Let H be the Hamiltonian operator mapping pure states into pure states, and J be a bounded positive linear operator on $\mathcal{T}(\mathcal{H})$ mapping pure states into mixed states. Then, using the result of Davies [2] and also [17], we may write the differential equation

$$\frac{\partial \rho_t}{\partial t} = Z\rho_t + \rho_t Z^* + \lambda J(\rho_t), \tag{5.1}$$

where

$$Z = -iH - \lambda_{/2} R$$

λ denotes the mean frequency of reduction and R is a bounded linear operator on $\mathcal{H}$, (defined in Chapter 3). The integral form of the differential equation (5.1) is then given by:

$$\varrho_t = \sum_t (\varrho) = \exp(tZ)\varrho \exp(tZ^*) +$$

$$\lambda \int_0^t \exp[(t-t')Z]\, J(\varrho_{t'}) \exp[(t-t')Z^*]\, dt' \qquad (5.3)$$

specifically we take in (5.3)

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

then by (5.2)

$$Z = \begin{pmatrix} -2\lambda & -i \\ -i & -\lambda \end{pmatrix}.$$

Consider the formal series

$$\sum_t = S_t + \lambda \int_0^t dt_1\, S_t J_{t_1} + \lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, S_t J_{t_1} J_{t_2} + \ldots$$

$$+ \lambda^n \int_0^t dt_1 \int_0^{t_1} dt_{t_2} + \ldots + \int_0^{t_{n-1}} dt_n\, S_t J_{t_1} J_{t_2} \ldots J_{t_n} + \ldots \qquad (5.4)$$

Where S_t and J_t are positive operators such that:

$$\sum_t^{I}(\rho) = S_t(\rho) = \exp(Zt)\,\rho\,\exp(Z^* t)$$

$$= \exp[-iHt - \lambda/2\ Rt]\,\rho\,\exp[iHt - \lambda/2 Rt] \qquad (5.5)$$

and

$$\sum_t^{II}(\rho) = \lambda \int_1^t dt_1\, S_t\, J_{t_1} + \lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, S_t\, J_{t_1}\, J_{t_2} + \ldots, \qquad (5.6)$$

where

$$J_t = S_{-t}\, J\, S_t \qquad (5.7)$$

We should note that $\sum_t(\rho)$ in (5.4) is the formal series we would have for ρ_t.

Therefore, to complete $\sum_t(\rho)$ for our example, one can find $\sum_t^{I}(\rho)$ and $\sum_t^{II}(\rho)$ separately, then $\sum_t^{I} + \sum_t^{II}$ will give us $\sum_t$. Now to find $\sum_t I(\rho)$, we take (5.5) and, considering the calculations of Appendix, i.e. (A1) into account, so for $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{T}(\mathcal{H})$ we have,

$$\sum_t^{I}(\rho)$$

$$\begin{pmatrix} e^{-3\lambda t}\left(\operatorname{Cosh}\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\ \operatorname{Sinh}\alpha\frac{t}{2}\right)^2 & \frac{e^{-3\lambda t}}{2i\alpha}(-4)\operatorname{Sinh}\alpha\frac{t}{2}\left(\operatorname{Cosh}\alpha t/2 - \lambda/2\ \operatorname{Sinh}\alpha\frac{t}{2}\right) \\ \frac{e^{-3\lambda t}}{2i\alpha}\ 16\ \sinh\alpha\frac{t}{2}\left(\operatorname{Cosh}\alpha\frac{t}{2} - \frac{\lambda}{2}\cdot\operatorname{Sinh}\alpha\frac{t}{2}\right) & \frac{e^{-3\lambda t}}{4\alpha^2}\ 16\ \operatorname{Sinh}^2\alpha\frac{t}{2} \end{pmatrix}$$

Also to find $\sum_t^{II}(\varrho)$, we take (5.6), (5.7) and (A2) into account, for $\varrho \in \Upsilon(\mathcal{H})$, we have

$$\sum_t^{II}(\varrho) = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

$$f_1 = \frac{e^{-3\lambda t}}{4\alpha^2} \sinh\alpha\frac{t}{2}(\cosh\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2})\Big[\frac{1}{\alpha}(1-\frac{\lambda^2}{\alpha^2})$$

$$(\cosh 2\alpha t - \frac{\lambda}{\alpha}\sinh 2\alpha t) + \frac{4}{\alpha}(1+\frac{\lambda^2}{\alpha^2})\cosh\alpha t - \frac{8\lambda^3}{\alpha^3}\sinh\alpha t - \frac{5}{\alpha}$$

$$-\frac{3\lambda^2}{\alpha^3} + 2t\frac{\lambda}{\alpha}\Big]$$

$$f_2 = \frac{e^{-3\lambda t}}{8i\alpha}(\cosh\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2})^2\Big[\frac{1}{\alpha}(1-\frac{\lambda^2}{\alpha^2})$$

$$(\cosh 2\alpha t - \frac{\lambda}{\alpha}\sinh 2\alpha t) + \frac{4}{\alpha}(1+\frac{\lambda^2}{\alpha^2})\cosh\alpha t - \frac{5}{\alpha} + 8\frac{\lambda^3}{\alpha^4}\sinh\alpha t$$

$$-2t\frac{\lambda}{\alpha}(1+3\frac{\lambda^2}{\alpha^2} - 3\frac{\lambda^2}{\alpha^3})\Big]$$

$$f_3 = \frac{e^{-3\lambda t}}{2i\alpha^3}\sinh\alpha\frac{t}{2}\Big[\frac{1}{\alpha}(1-\frac{\lambda^2}{\alpha^2})(\cosh 2\alpha t - \frac{\lambda}{\alpha}\sinh 2\alpha t) + \frac{4}{\alpha}(1+\frac{\lambda^2}{\alpha^2})\cosh\alpha t$$

$$-\frac{8\lambda^3}{\alpha^4}\sinh\alpha t + 2t + \frac{3\lambda^2 t}{\alpha^2}\Big]$$

$$f_4 = \frac{e^{-3\lambda t}}{\alpha^4}\sinh^2\alpha\frac{t}{2}\Big[\frac{1}{\alpha}(1-\frac{\lambda^2}{\alpha^2})\sinh 2\alpha t + 8\frac{\lambda^2}{\alpha^2}\sinh\alpha t$$

$$-2t(1+\frac{\lambda^2}{\alpha^2})\Big]$$

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APPENDIX

(A1) Since $Z = \begin{pmatrix} -2\lambda & -i \\ -i & -\lambda \end{pmatrix}$, then the characteristic equation φ (r) is as:

$$\varphi(r) = 2\lambda^2 + r^2 + 3\lambda r + 1 = 0$$

$$r = \frac{-3\lambda \pm \sqrt{\lambda^2 - 4}}{2}$$

so the diagonal metrix is as:

$$\Lambda = \begin{bmatrix} \frac{-3\lambda}{2} + \frac{\sqrt{\lambda^2 - 4}}{2} & 0 \\ 0 & \frac{-3\lambda}{2} - \frac{\sqrt{\lambda^2 - 4}}{2} \end{bmatrix}$$

and,

$$m = \begin{bmatrix} -i & i \\ \frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4}}{2} & \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4}}{2} \end{bmatrix}$$

$$m^{-1} = \begin{bmatrix} \frac{\lambda - \sqrt{\lambda^2 - 4}}{2i(\sqrt{\lambda^2 - 4})} & \frac{1}{\sqrt{\lambda^2 - 4}} \\ \frac{\lambda + \sqrt{\lambda^2 - 4}}{2i\sqrt{\lambda^2 - 4}} & \frac{1}{\sqrt{\lambda^2 - 4}} \end{bmatrix}$$

Therefore

$$e^{Zt} = m \cdot e^{\Lambda t} m^{-1} = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$$

$$r_1 = e^{-3\lambda t/2} \left(\operatorname{Cosh}\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\operatorname{Sinh}\alpha\frac{t}{2}\right)$$

$$r_2 = \frac{-2i}{\alpha} e^{-3\lambda t/2} \sinh\alpha\frac{t}{2}$$

$$r_3 = \frac{e^{-3\lambda t/2}}{2i\alpha} \sinh\alpha\frac{t}{2} (\lambda^2 - \alpha^2)$$

$$r_4 = e^{-3\lambda t/2} \left(\operatorname{Cosh}\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2}\right)$$

where $\alpha = \sqrt{\lambda^2 - 4}$

and $e^{Z^* t} = (m^*)^{-1} e^{\Lambda t} m^*$

$$m^* = \begin{bmatrix} i & \frac{\lambda + \alpha}{2} \\ -i & \frac{-\lambda + \alpha}{2} \end{bmatrix} \qquad (m^*)^{-1} = \begin{bmatrix} \frac{-\lambda + \alpha}{2i\alpha} & \frac{-\lambda - \alpha}{2i\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix}$$

Thus,

$$e^{Z^* t} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$$

$$Z_1 = e^{-3\lambda t/2}\left(\mathrm{Cosh}\,\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\mathrm{Sinh}\,\alpha\frac{t}{2}\right)$$

$$Z_2 = e^{-3\lambda t/2}\frac{\alpha^2-\lambda^2}{2i\alpha}\mathrm{Sinh}\,\alpha\frac{t}{2}$$

$$Z_3 = e^{-3\lambda t/2}\frac{2i}{\alpha}\mathrm{Sinh}\,\alpha\frac{t}{2}$$

$$Z_4 = e^{-3\lambda t/2}\left(\mathrm{Cosh}\,\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\mathrm{Sinh}\,\alpha\frac{t}{2}\right)$$

So for $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$\hat{S}_t(\rho) = e^{Zt}\rho\, e^{Z^* t}.$$

(A2) To figure out the second sum, $\sum_t^{II}$, we have first to find $J = S_{-t}\, J\, S_t$, where

$$S_{-t}(\rho) = \sum_{-t}^{I}(\rho) = e^{-tZ}\rho\, e^{-tZ^*} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

$$g_1 = e^{+3\lambda t}\left(\mathrm{Cosh}\,\alpha\frac{t}{2} + \frac{\lambda}{2}\mathrm{Sinh}\,\alpha\frac{t}{2}\right)^2$$

$$g_2 = \frac{e^{3\lambda t}}{2i\alpha}(\lambda^2-\alpha^2)\mathrm{Sinh}\,\alpha\frac{t}{2}\left(\mathrm{Cosh}\,\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\mathrm{Sinh}\,\alpha\frac{t}{2}\right)$$

$$g_3 = \frac{e^{3\lambda t}}{2i\alpha}(\alpha^2 - \lambda^2)\mathrm{Sinh}\,\alpha\frac{t}{2}\left(\mathrm{Cosh}\,\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\,\mathrm{Sinh}\,\alpha\frac{t}{2}\right)$$

$$g_4 = \frac{e^{3\lambda t}}{4\alpha^2}(\lambda^2 - \alpha^2)^2\mathrm{Sinh}^2\alpha\frac{t}{2}$$

and

$J(\varrho) = P_1 \varrho P_1 + P_2 \varrho P_2$, where P_1 and P_2 are two projection operators on $\mathcal{H}$ such that

(i) $P_1 + P_2 = I$

(ii) $J(P_1 + P_2) = J(P_1) + J(P_2)$.

P_1 and P_2 are given as following:

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

thus,

$$J(\varrho) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

therefore, J_t is as:

$$J_t = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$$

where,

$$J_1 = \left(\cosh^2\alpha\frac{t}{2} - \frac{\lambda^2}{\alpha^2}\sinh^2\alpha\frac{t}{2}\right)^2$$

$$J_2 = \frac{(\alpha^2-\lambda^2)}{2i\alpha}\sinh\alpha\frac{t}{2}\left(\cosh\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2}\right)\left(\cosh\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2}\right)^2$$

$$J_3 = \frac{(\alpha^2-\lambda^2)}{2i\alpha}\sinh\alpha\frac{t}{2})\cosh\alpha\frac{t}{2} + \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2})\left(\cosh\alpha\frac{t}{2} - \frac{\lambda}{\alpha}\sinh\alpha\frac{t}{2}\right)^2$$

$$J_4 = \frac{-(\alpha^2-\lambda^2)^2}{4\alpha^2}\cdot\sinh^2\alpha\frac{t}{2}\left(\cosh^2\alpha\frac{t}{2} - \frac{\lambda^2}{\alpha^2}\sinh^2\alpha\frac{t}{2}\right)$$

Changing t by t_1 in the above matrix, we can get J_{t_1}.

$$J_{t_1} = \begin{pmatrix} J_5 & J_6 \\ J_7 & J_8 \end{pmatrix}$$

$$J_5 = \left(\cosh^2\alpha\, t_1/2 - \frac{\lambda^2}{\alpha^2}\sinh^2\alpha\, t_1/2\right)^2$$

$$J_6 = \frac{(\alpha^2-\lambda^2)}{2i\alpha}\sinh\alpha\, t_1/2\left(\cosh\alpha\, t_1/2 - -\sinh\alpha\, t_1/2\right)\left(\cosh\alpha\, t_1/2 + \frac{\lambda}{\alpha}\sinh\alpha\, t_1/2\right)^2$$

$$J_7 = \frac{\alpha^2-\lambda^2}{2i\alpha}\sinh\alpha\, t_1/2\left(\cosh\alpha\, t_1/2 + \frac{\lambda}{\alpha}\sinh\alpha\, t_1/2\right)\left(\cosh\alpha\, t_1/2 - \frac{\lambda}{\alpha}\sinh\alpha\, t_1/2\right)^2$$

$$J_8 = \frac{-(\alpha^2-\lambda^2)^2}{4\alpha^2}\sinh^2\alpha\, t_1/2\left(\cosh^2\alpha\, t_1/2 - \frac{\lambda^2}{\alpha^2}\sinh^2\alpha\, t_1/2\right)$$

Secondly, to complete $\sum_t^{II}$, we have to find $S_t\, J_{t_1}$ for the integral part of (5.4).
For the convenience, we set:

$$\cosh\alpha t/2 = A, \qquad \cosh\alpha\, t_1/2 = A_1$$

$\sinh \alpha t/2 = B, \qquad \sinh \alpha t_{1/2} = B_1$

So,

$$S_t J_{t_1} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

$$C_1 = e^{-3\lambda t}\Big[(A-\tfrac{\lambda}{\alpha}B)^2(A_1-\tfrac{\lambda^2}{\alpha^2}B_1^2)^2 - \frac{(\alpha^2-\lambda^2)^2}{4\alpha^2}B(A-\tfrac{\lambda}{\alpha}B)B_1(A_1+\tfrac{\lambda}{\alpha}B_1)(A_1-\tfrac{\lambda}{\alpha}B_1)^2\Big]$$

$$C_2 = \frac{e^{-3\lambda t}}{2i\alpha}(\alpha^2-\lambda^2)\Big[(A-\tfrac{\lambda}{\alpha}B)^2B_1(A_1-\tfrac{\lambda}{\alpha}B_1)(A_1+\tfrac{\lambda}{\alpha}B_1)^2 - \frac{(\alpha^2-\lambda^2)^2}{4\alpha^2}$$

$$B(A-\tfrac{\lambda}{\alpha}B)B_1^2(A_1^2-\tfrac{\lambda^2}{\alpha^2}B_1^2)\Big]$$

$$C_3 = \frac{e^{-3\lambda t}}{2i\alpha}(\lambda^2-\alpha^2)\Big[B(A-\tfrac{\lambda}{\alpha}B)(A_1^2-\tfrac{\lambda^2}{\alpha^2}B_1^2)^2 - \frac{(\lambda^2-\alpha^2)^2}{4\alpha^2}$$

$$B^2B_1(A_1+\tfrac{\lambda}{\alpha}B_1)(A_1-\tfrac{\lambda}{\alpha}B_1)^2\Big]$$

$$C_4 = \frac{-e^{-3\lambda t}}{4\alpha^2}(\lambda^2-\alpha^2)^2\Big[B(A-\tfrac{\lambda}{\alpha}B)B_1(A_1-\tfrac{\lambda}{\alpha}B_1)(A_1+\tfrac{\lambda}{\alpha}B_1)^2 + \frac{(\alpha^2-\lambda^2)^2}{4\alpha^2}B^2$$

$$B_1^2(A_1^2-\tfrac{\lambda^2}{\alpha^2}B_1^2)\Big]$$

therefore,

$$\sum\nolimits_t^{II}(\rho) = \lambda\int_0^t dt_1\, S_t\, J_{t_1} + \ldots$$