# Regions of Stability
# in
# Mathematical Programming Models

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# ABSTRACT

## Regions of Stability in Mathematical Programming Models

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This thesis is a study on stability of convex mathematical programming models. The models are expressed in terms of decision variables $x$ and parameters $\theta$. Roughly speaking, the regions of stability are chunks of the parameter space, where the feasible-set point-to-set mapping is continuous as a function of $\theta$. Moreover, in these regions, the set of optimal solutions and the optimal value function are continuous.

The original contribution of the thesis is discovery of several new regions of stability for convex mathematical programming models. We also show how various results from perturbed convex programming and input optimization can be extended over the new regions. These extensions include reformulations of certain optimality conditions and new conditions for validity of a marginal value formula. We also study the continuity of the Lagrange multipliers.

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# CHAPTER 1

# INTRODUCTION

This thesis is concerned with the stability of convex mathematical programming models in input optimization. The mathematical programming models we will consider are of the form

$$(P,\theta) \qquad \begin{aligned} &\min_{(x)} f^0(x,\theta) \\ &\text{s.t.} \\ &f^i(x,\theta) \le 0, \quad i \in \varphi \triangleq \{1,\ldots,m\} \\ &\theta \in I \end{aligned}$$

where all functions $f^i : R^n \times R^p \to R$ are continuous and $f^i(\cdot,\theta) : R^n \to R$ are convex, $i \in \{0\} \cup \varphi$. The set $I \subset R^p$ is assumed convex . Such a model is termed convex. Note that for a fixed parameter $\theta \in I$, $(P,\theta)$ is a "usual" program.

$$(P) \qquad \begin{aligned} &min f^0(z) \\ &\text{s.t.} \\ &f^i(z) \le 0, \quad i \in \varphi \end{aligned}$$

The models are considered as input-ouput system. The "input" is the parameter vector $\theta$ and "output" is the feasible set $F(\theta) = \{x \in R^n : f^i(x,\theta) \le 0, i \in \varphi\}$, the set of optimal solutions $\tilde{F}(\theta) = \{\tilde{x}(\theta)\}$, and the optimal value $\tilde{f}(\theta) = f^0(\tilde{x}(\theta),\theta)$. Primarily, we study situations when the output $\{F(\theta), \tilde{F}(\theta), \tilde{f}(\theta)\}$ is a "continuous" function of the input. (Formal definitions are given in the text below)

Regions of stability are chunks of the parameter space where the above output triple is continuous. The optimal value function $\tilde{f}(\theta)$ is optimized over such regions, starting from some $\theta^0 \in I$, until an "optimal input" $\theta^*$ and an "optimal realization" $(P, \theta^*)$ of the mathematical model $(P, \theta)$ is achieved. (See [48, 50].) Note that (unlike in usual mathematical programming) the optimal input $\theta^*$ depends on the initial input $\theta^0$, from where, the minimization of $\tilde{f}(\theta)$ started. This is the subject of the so-called " input optimization".

Regions of stability are also of interest outside input optimization, e.g., in linear programming, in data envelopment analysis, and in the study of random decision systems with complete connections.

The thesis is organized as follows. We begin with Chapter 2, where the relevant background material on point-to-set mappings is recollected. In Chapter 3, we identify three new regions of stability and also show that two familiar sets are regions of stability under weaker hypotheses than previously known. Also, we give a necessary condition for stability for an arbitrary region of stability under somewhat weaker assumptions than those given in the literature. Then we turn our attention to optimality conditions in Chapter 4. First we show that the optimality condition for an optimal input from [44, 46] extends to two of the new regions. An interesting observation is that an input constraint qualification is not required for optimality conditions stated over a subset of a new region of stability denoted by $H(\theta^*)$. This subset includes some well-known regions from the literature such as $W(\theta^*)$, and $V_1(\theta^*)$ [36, 48, 50]. In Chapter 5, continuity of the restricted Lagrangian multiplier functions is established on any region of stability under an additional assumption. Here we also show that the previously known necessary conditions for differentiable functions extend to the region of stability $H(\theta^*)$. In Chapter 6, we show that the known optimality conditions for bi-convex models ([43, 45, 48]) and the marginal value formula ([43, 45, 48]) can be extended to the new regions of stability. In

Chapter 7, we give a schematic comparison of all presently used regions of stability in the literature and compare them with the ones introduced in this thesis. It turns out that our new regions of stability are among the largest ones presently available. At this point we also discuss some applications of regions of stability and input optimization, especially in data envelopment analysis ([8]). Finally, we pose several open questions. It should be pointed out that by fixing the parameter $\theta$, our results recover many recent and classical results in convex programming, e. g. , from [1, 5, 12, 33].

Some of the results of this thesis are to appear in the article: " New regions of stability in input optimization " that has been accepted for publication in Aplikace Matematiky [19].

## Chapter 2

## Mathematical Background

### 2.1 Point-To-Set Mappings

A point-to-set mapping $\Gamma : Z \to X$ between two topological vector spaces is a mapping which associates a subset of X with each point of Z. The properties of point-to-set mappings have been studied from a number of different viewpoints (See, e.g., [7, 18, 40, 41].)

The study of such mappings has been motivated by numerous applications in different fields. Zangwill [40, 41] seems to have been the first to fully exploit the idea of point-to-set mappings in the field of mathematical programming.

The purpose of this section is to recollect properties of point-to-set mappings relevant to the study of mathematical programming problems.

**2.1 Definition.** *Consider the point-to-set mapping $\Gamma : Z \to X$, between two vector spaces $Z$ and $X$. Let $S$ be a subset of $Z$ and $\theta^* \in S$. Then $\Gamma$ is lower semicontinuous (l.s.c.) at $\theta^*$ relative to $S$, if for every open set $A \subset X$ such that $A \cap \Gamma(\theta^*) \neq \emptyset$, there exists a neighborhood $N(\theta^*)$ such that $A \cap \Gamma(\theta) \neq \emptyset$ for every $\theta \in N(\theta^*) \cap S$.*

**2.2 Definition.** *Consider the point-to-set mapping $\Gamma : Z \to X$, let $S$ be a subset of $Z$ and $\theta^* \in S$. Then $\Gamma$ is upper semicontinuous (u.s.c.) at $\theta^*$ relative to $S$, if for every open set $A \subset X$ such that $\Gamma(\theta^*) \subset A$, there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that $\Gamma(\theta) \subset A$, for every $\theta \in N(\theta^*)$.*

**2.3 Definition.** *The point-to-set mapping $\Gamma : Z \to X$ is continuous at $\theta^*$ relative to $S$, if $\Gamma$ is both lower and upper semicontinuous at $\theta^*$.*

The above definitions are introduced by Berge [7]. Similar definitions were given by Hogan [18] as follows.

**2.4 Definition.** *The point-to-set mapping $\Gamma : Z \to X$ is open at $\theta^*$ relative to S, if given any sequence $\theta^k \in S, \theta^k \to \theta^*$ and $x^* \in \Gamma(\theta^*)$, there exists an integer m and* a sequence $\{x^k\} \subset X$ such that $x^k \in \Gamma(\theta^k)$ for each $k \geq m$, and $x^k \to x^*$.

**2.5 Definition.** *The point-to-set mapping $\Gamma : Z \to X$ is closed at $\theta^*$ relative to S, if given any sequence $\theta^k \in S, \theta^k \to \theta^*$ and $x^k \in \Gamma(\theta^k)$ such that $x^k \to x^*$, it follows that $x^* \in \Gamma(\theta^*)$.*

**2.6 Definition.** *The point-to-set mapping $\Gamma : Z \to X$ is continuous at $\theta^*$ relative to S, if it is both open and closed at $\theta^*$.*

The terms "lower" and "upper semicontinuous" have meanings similar to "open" and "closed" point-to-set mappings, respectively. It was shown by Hogan [18] that some of these definitions are in fact equivalent, under fairly weak assumptions.

**2.7 Theorem.** *The point-to-set mapping $\Gamma : Z \to X$ is lower semicontinuous at $\theta^* \in S$ iff $\Gamma$ is open at $\theta^*$.*

**2.8 Definition.** *The point-to-set mapping $\Gamma : Z \to X$ is uniformly compact at $\theta^*$ relative to S, if there is a neighborhood $N(\theta^*)$ of $\theta^*$ such that the closure of the set*

$$\bigcup_{\theta \in N(\theta^*)} \Gamma(\theta)$$

*is compact.*

**2.9 Theorem.** *Suppose that a point-to-set mapping $\Gamma : Z \to X$ is uniformly compact at $\theta^*$, Then both $\Gamma(\theta^*)$ is compact and $\Gamma$ is upper semicontinuous at $\theta^*$ iff $\Gamma$ is closed at $\theta^*$.*

**Further, if $\Gamma$ is lower semicontinuous at each point of Z, then $\Gamma$ is lower semicontinuous in Z. If $\Gamma$ is upper semicontinuous at each point of Z with $\Gamma(\theta)$ being compact, then we say that $\Gamma$ is upper semicontinuous in Z. $\Gamma$ is continuous in Z, if it is both lower and upper semicontinuous in Z.**

## 2.2 Input Optimization

Input optimization deals with mathematical programming models of the form

$$\min_{(x)} f^0(x,\theta)$$

$$(P,\theta) \qquad \text{s.t.}$$

$$f^i(x,\theta) \leq 0, \quad i \in \varphi = \{1,\ldots,m\}$$

$$\theta \in I$$

Here the function $f^i : R^n \times R^p \to R$ are assumed to be continuous and $f^i(\cdot,\theta) : R^n \to R$ are assumed to be convex for every $\theta \in R^p, i \in \{0\} \cup \varphi$; $I \subset R^p$ is a convex set. Note that, for a fixed vector $\theta$, the model $(P,\theta)$ is a usual convex program.

With each $\theta \in I$ we associate the triple (output)

$$F(\theta) = \{x \in R^n : f^i(x,\theta) \leq 0, i \in \varphi\}$$

the feasible set,

$$\tilde{F}(\theta) = \{\tilde{x}(\theta)\}$$

the set of all optimal solutions $\tilde{x}(\theta)$, and

$$\tilde{f}(\theta) = f^0(\tilde{x}(\theta),\theta)$$

the optimal value.

We will study perturbations of the output $\{F(\theta), \tilde{F}(\theta), \tilde{f}(\theta)\}$ in a neighborhood $N(\theta^*)$ of an arbitrary but fixed $\theta^* \in I$. In many models describing real-life situations (typically in multi-objective models, but also in linear programming), "continuity" of the output is not guaranteed for arbitrary perturbations in a neighborhood of $\theta^*$. (See., e.g., [5, 47].)

However, the continuity is preserved on "regions of stability".

The regions of stability can be expressed in terms of constructive objects of convex analysis such as ( defined for each $\theta \in I$)

$$\varphi^{=}(\theta) = \{i \in \varphi : x \in F(\theta) \Longrightarrow f^{i}(x,\theta) = 0\}$$

called the minimal index set of active constraints (See [5, 44, 48]) and the corresponding set in $R^n$:

$$F^{=}(\theta) = \{x \in R^n : f^{i}(x,\theta) = 0, i \in \varphi^{=}(\theta)\}$$

These objects have been studied and one can calculate them, at least for analytic convex function (See [5]). We recall that the classical Lagrangian, associated with $(P,\theta)$, is

$$L(x,u;\theta) = f^{0}(x,\theta) + \sum_{i \in \varphi} u_i f^{i}(x,\theta).$$

We will also use the "restricted" Lagrangians

$$L^{<}(x,u;\theta) = f^{0}(x,\theta) + \sum_{i \in \varphi^{<}(\theta)} u_i f^{i}(x,\theta)$$

and

$$L_{*}^{<}(x,u;\theta) = f^{0}(x,\theta) + \sum_{i \in \varphi^{<}(\theta^{*})} u_i f^{i}(x,\theta)$$

for some $\theta^{*} \in R^p$. (See, e.g., [44, 53, 55].) Here $\varphi^{<}(\theta) \triangleq \varphi \setminus \varphi^{=}(\theta)$. Denote $q(\theta) = card\ \varphi^{<}(\theta)$ and let $R^q_+$ denote the nonnegative orthant of $R^q$. Restricted Lagrangians are used in characterizations of optimality and in stability. In particular, we recall the following result ( [36, 44]) "Consider the convex model $(P,\theta)$ at some arbitrary $\theta$. Then $x^{*}(\theta) \in F^{=}(\theta)$ is an optimal solution if, and only if there exists a nonnegative vector function $u^{*} = u^{*}(\theta) \in R_+^{q(\theta)}$ such that

$$L^{<}(x^{*}(\theta),u;\theta) \leq L^{<}(x^{*}(\theta),u^{*};\theta) \leq L^{<}(x,u^{*};\theta)$$

for every $u \in R_+^{q(\theta)}$ and $x \in F^{=}(\theta)$."

Throughout this thesis we will always assume that the set of optimal solutions is non-empty and bounded. The objective functions with this property deserve a special name :

**2.10 Definition.** *[48] An objective $f^0(x,\theta)$, in the convex model $(P,\theta)$, is said to be realistic at $\theta^* \in I$ if $\vec{F}(\theta^*) \neq \emptyset$ and bounded.*

Let us recall the classical condition of convex programming known as " Slater's condition". It is said that Slater's condition holds for the contraints of the model $(P,\theta)$, if there exists a point $\hat{x} \in R^n$ such that $f^i(\hat{x},\theta) < 0$, for every $i \in \varphi$. Under this condition $\varphi^=(\theta)$ is an empty set, $F^=(\theta) = R^n$ and the restricted Lagrangians become the usual classical Lagrangians.

# CHAPTER 3

## Regions of Stability

### 3.1 Basic Results on Stability

Stability in mathematical programming has been considered by many authors from different points of view (See, e.g., [3, 4, 5, 6, 11, 13, 14, 22, 31, 32, 57].) A constructive approach to stability of convex models has been recently developed in a series of papers by Zlobec and his colleagues and students (e.g. [43 44, 45, 46, 47, 48, 50, 52, 54, 56].) A basic notion in this approach is a " region of stability". In order to introduce and study this notion, let us first recollect some basic facts.

**3.1 Theorem.** *[5,6] Consider the convex model $(P, \theta)$ at $\theta = \theta^* \in I$. Then the following statements are equivalent.*

*(i) The point-to-set mapping $F : \theta \to F(\theta)$ is continuous at $\theta^*$ (in the sense of Definition 2.3)*

*(ii) For every realistic objective function $f^0$ there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that both*

$$\tilde{F}(\theta) \neq \emptyset \text{ for every } \theta \in N(\theta^*), \text{ and}$$

$$\theta \in N(\theta^*) \text{ and } \theta \to \theta^* \implies \tilde{F}(\theta) \text{ is bounded and all its limit points are in } \tilde{F}(\theta^*).$$

*(iii) For every realistic objective function $f^0$ there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that both*

$$\tilde{F}(\theta) \neq \emptyset \text{ for every } \theta \in N(\theta^*), \text{ and}$$

$$\theta \in N(\theta^*) \text{ and } \theta \to \theta^* \text{ imply that } \tilde{f}(\theta) \to \tilde{f}(\theta^*).$$

**The above theorem provides three options for the definition of stability. For our definition we shall use the first one.**

**3.2 Definition.** *[43, 54, 56] Consider a convex model $(P,\theta)$ at $\theta = \theta^* \in I$ with a realistic objective function $f^0$. We say that the model is stable in a region $S \subset I \subset R^p$ at $\theta^* \in S$ if the point-to-set mapping $F : \theta \to F(\theta)$ is continuous at $\theta^*$ over S.*

*In particular, we say that the model is stable at $\theta^*$, if one can specify $S = N(\theta^*)$.*

In order to establish continuity of $F : \theta \to F(\theta)$, it is enough to find condition for lower semicontinuity of F. This is a consequence of the following lemma (See, e.g., [2, 7, 18, 20, 21].)

**3.3 Lemma.** *Consider the convex model $(P, \theta)$, where all functions $f^i : R^n \times R^p \to R, i \in \varphi$ are continuous. Then the mapping $\Gamma : \theta \to F(\theta)$ is closed in I.*

In view of Theorem 3.1 and Lemma 3.3, in order to construct regions of stability, it is sufficient to find conditions that guarantee lower semicontinuity of the point-to-set mapping F at $\theta^*$ over S.

The following theorem gives us three " classical" regions of stability. (For all three regions the mapping $F : \theta \to F(\theta)$ is lower semicontinuous. )

**3.4 Theorem.** *[43, 53, 56] Consider the convex model $(P, \theta)$ at some $\theta^*$. Then the following sets are regions of stability at $\theta^*$ for every realistic objective function.*

$$M(\theta^*) = \{\theta : F(\theta^*) \subset F(\theta)\}$$

$$V(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta), \text{ and } f^i(x,\theta) \leq 0, \forall x \in F(\theta^*), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)\}$$

$$W(\theta) = \{\theta : F^=(\theta^*) \subset F^=(\theta) \text{ and } \varphi^=(\theta^*) = \varphi^=(\theta)\}.$$

To simplify the notation we denote by

$$R_1(\theta^*) = \{\theta : \varphi^=(\theta^*) = \varphi^=(\theta)\}$$

$$R_2(\theta^*) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F^=(\theta), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)\}$$

$$R_3(\theta^*) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F^=(\theta^*), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)\}$$

$$R_4(\theta^*) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F(\theta^*), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)\}.$$

If $F^{=} : \theta \to F^{=}(\theta)$ is lower semicontinuous at $\theta^*$, then $R_1(\theta^*)$ and $R_2(\theta^*)$ are regions of stability at $\theta^*$. (See [37, 48].)

It was shown in [34] that there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that

$$F^{=}(\theta) = F_{\star}^{=}(\theta)$$

for every $\theta \in N(\theta^*) \cap R_2(\theta^*)$. Here $F_{\star}^{=}(\theta) = \{x : f^i(x,\theta) \leq 0, i \in \varphi^{=}(\theta^*)\}$. As a matter of fact

$$R_2(\theta^*) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F^{=}(\theta), i \in \varphi^{=}(\theta^*)\} = \{\theta : F^{=}(\theta) = F_{\star}^{=}(\theta)\}$$

Let us note that $M(\theta^*)$, $V(\theta^*)$ and $R_2(\theta^*)$ are generally incomparable, while $W(\theta^*) \subset V(\theta^*)$ and $R_1(\theta^*) \subset R_2(\theta^*)$. ( Also see Section 7.1.) However, since it is usually easier to calculate $W(\theta^*)$ than $V(\theta^*)$and $R_1(\theta^*)$ than $R_2(\theta^*)$, we will occasionally state our results also for $W(\theta^*)$ or $R_1(\theta^*)$ or the following subsets of the region of stability $V(\theta^*)$.

$$V_1(\theta^*) = \{\theta : F^{=}(\theta^*) \subset F^{=}(\theta)\} \cap R_3(\theta^*)$$

$$V_2(\theta^*) = \{\theta : F^{=}(\theta^*) = F^{=}(\theta)\} \cap R_4(\theta^*)$$

$$V_3(\theta^*) = \{\theta : F^{=}(\theta^*) = F^{=}(\theta)\} \cap R_3(\theta^*)$$

(See, e.g., [48, 50].)

## 3.2 New Regions of Stability

Take a $\theta^\star \in I$ and denote

$$Z(\theta^\star) = \{\theta : F(\theta^\star) \subset F^=(\theta)\} \cap R_4(\theta^\star)$$
$$H(\theta^\star) = \{\theta : f^i(x,\theta) \lesssim 0, \forall x \in F(\theta^\star), i \in \varphi^=(\theta^\star)\}$$

Note that the latter can also be written as

$$H(\theta^\star) = \{\theta : F(\theta^\star) \subset F_\star^=(\theta)\}$$

where $F_\star^=(\theta)$ is defined in Section 3.1.We claim that these two sets are regions of stability. Since $Z(\theta^\star)$ is a subset of $H(\theta^\star)$, it is enough to show that $H(\theta^\star)$ is a region of stability.

**3.5 Theorem.** *Consider the convex model* $(P,\theta)$ *at some* $\theta = \theta^\star$, *where* $F(\theta^\star) \neq \emptyset$. *Then* $H(\theta^\star)$ *is a region of stability at* $\theta^\star$ *for every realistic objective function.*

*Proof* It is enough to show that the point-to-set mapping $F : \theta \to F(\theta)$ is lower semicontinuous at $\theta^\star$, relative to the set $H(\theta^\star)$. If this were not true, then there would exist an open set $A \subset R^n$ such that

$$A \cap F(\theta^\star) \neq \emptyset$$

but

$$A \cap F(\theta^k) = \emptyset \tag{3.1}$$

for a sequence $\theta^k \in H(\theta^\star), \theta^k \to \theta^\star$. Now, choose an arbitrary

$$\hat{x} \in \textit{relint}\ \{A \cap F(\theta^\star)\}.$$

Clearly

$$f^i(\hat{x},\theta^\star) < 0, \qquad i \in \varphi^<(\theta^\star)$$

and hence

$$f^i(\hat{x}, \theta^k) < 0, \qquad i \in \varphi^<(\theta^*) \tag{3..2}$$

for all sufficiently large $k's$. In particular, since $\theta^k \in H(\theta^*)$,

$$f^i(\hat{x}, \theta^k) \leq 0, \qquad i \in \varphi^=(\theta^*)$$

This with (3.2) implies that

$$\hat{x} \in F(\theta^k)$$

which contradicts (3.1). ∎

Since

$$F(\theta^*) \subset F^=(\theta^*) \subset F^=(\theta)$$

and

$$f^i(x, \theta) \leq 0, \quad \forall x \in F(\theta^*), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)$$

for every $\theta \in V(\theta^*)$ and

$$F(\theta^*) \subset F(\theta) \subset F^=(\theta)$$

for $\theta \in M(\theta^*)$, we note that $H(\theta^*)$ is a bigger region of stability than $M(\theta^*)$ and $V(\theta^*)$, i.e.

$$\{M(\theta^*) \cup V(\theta^*)\} \subset H(\theta^*).$$

Also, it is easy to show that

$$\{M(\theta^*) \cup V(\theta^*)\} \subset Z(\theta^*).$$

The construction of $V(\theta^*)$ suggests that $R_4(\theta^*)$ may be a region of stability , under the assumption on lower semicontinuity of the mapping $F^=$. Unfortunately, as the following example shows, this is not enough.

**3.6 Example.** *(Communicated by Semple)*

Consider the constriants

$$f^1 = |x_1 - \theta| \le 0$$

$$f^2 = x_1 - \theta x_2 \le 0$$

$$f^3 = x_2 - 2 \le 0$$

$$f^4 = -x_2 \le 0$$

around $\theta^* = 0$. Suppose that $\theta$ is taken from $\theta \in I = [0, 1]$.

Here

$$F(\theta) = \{\begin{pmatrix} \theta \\ x_2 \end{pmatrix} : 1 \le x_2 \le 2\}$$

for $\theta > 0$, while

$$F(\theta^*) = \{\begin{pmatrix} 0 \\ x_2 \end{pmatrix} : 0 \le x_2 \le 2\} .$$

So the mapping F is not lower semicontinuous at $\theta^*$. But

$$F^=(\theta) = \{\begin{pmatrix} \theta \\ x_2 \end{pmatrix} : x_2 \in R\}$$

for every $\theta \ge 0$, clearly the mapping $F^=$ is lower semicontinuous, and

$$R_4(\theta^*) = [0, 1]. \quad \blacksquare$$

However, the conjecture is valid with the additional assumption that the feasible set $F(\theta^*)$ has interior points. The latter is typically satisfied in the so-called " lexicographic optimization ", In this kind of optimization Slater's condition cannot be satisfied. (See [5, 47, 48].)

**3.7 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta = \theta^*$. If $F(\theta^*)$ has non-empty interior and if the point-to-set mapping $F^=$ is lower semicontinuous at $\theta^*$, then $R_4(\theta^*)$ is a region of stability at $\theta^*$ for every realistic objective function.*

*Proof* The result is proven by contradiction. Suppose that the mapping F is not lower semicontinuous at $\theta^*$. Then there exists an open set $\mathcal{A}$, such that

$$\mathcal{A} \cap F(\theta^*) \neq \emptyset$$

but

$$\mathcal{A} \cap F(\theta^k) = \emptyset \tag{3.3}$$

for a sequence $\theta^k \in R_4(\theta^*), \theta^k \to \theta^*$. Now choose an arbitrary

$$\hat{x} \in \ int\ \{\mathcal{A} \cap F(\theta^*)\}.$$

Clearly

$$f^i(\hat{x}, \theta^*) < 0, \quad i \in \wp^<(\theta^*)$$

and further, by continuity

$$f^i(\hat{x}, \theta^k) < 0, \quad i \in \wp^<(\theta^*) \tag{3.4}$$

for all sufficiently large k's. Since $\hat{x} \in F(\theta^*)$ and $\theta^k \in R_4(\theta^*)$, it follows that

$$f^i(\hat{x}, \theta^k) \leq 0, \qquad i \in \wp^=(\theta^*) \setminus \wp^=(\theta^k)$$

which, together with (3.4) gives

$$f^i(\hat{x}, \theta^k) \leq 0, \quad i \in \wp^<(\theta^k) \tag{3.5}$$

This means that $\hat{x} \notin F^=(\theta^k)$. ( Otherwise, $\hat{x} \in F^=(\theta^k)$ and (3.5) would imply $\hat{x} \in F(\theta^k)$, contradicting (3.3). ) Since $F(\theta^*)$ has interior, we can place a small open ball $B(\hat{x})$, centered at $\hat{x}$, inside $\mathcal{A} \cap F(\theta^*)$, such that

$$B(\hat{x}) \cap F^=(\theta^k) = \emptyset \tag{3.6}$$

But

$$\hat{x} \in F(\theta^*) \subset F^=(\theta^*)$$

ans hence

$$B(\hat{x}) \cap F^{=}(\theta^*) \neq \emptyset \tag{3.7}$$

Therefore, now we have an open set $B(\hat{x})$ such that (3.7) holds but, for a sequence $\theta^k \to \theta^*$ also, (3.6) holds. This contradicts the assumption on lower semicontinuity of the mapping $F^{=}$. ∎

It was shown in the literature that $R_1(\theta^*)$ and $R_2(\theta^*)$ are regions of stability if the mapping $F^{=}$ is lower semicontinuous. ( [37, 48]:) The construction of $Z(\theta^*)$ suggests that also $R_2(\theta^*)$ may be a region of stability under somewhat weaker hypothesis. Unfortunately the weaker hypothesis is not easy to verify.

**3.8 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta = \theta^*$. If for every open set $\mathcal{A}$ and $\mathcal{A} \cap F(\theta^*) \neq \emptyset$, there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that $\mathcal{A} \cap F^{=}(\theta) \neq \emptyset$ for every $\theta \in N(\theta^*)$, then $R_2(\theta^*)$ is a region of stability at $\theta^*$ for every realistic objective function.*

<u>*Proof*</u> We will show that F is lower semicontinuous at $\theta^*$ over $R_2(\theta^*)$.

First we choose an arbitrary

$$\hat{x} \in \textit{relint}\ \{F(\theta^*)\}$$

Clearly

$$f^i(\hat{x}, \theta^*) < 0, \quad i \in \varphi^{<}(\theta^*) \tag{3.8}$$

By the assumption, there exists $x^k = x^k(\theta^k) \in F^{=}(\theta^k)$ such that $x^k \to \hat{x}$. Now by joint continuity (3.8) gives

$$f^i(x^k, \theta^k) < 0, \quad i \in \varphi^{<}(\theta^*) \tag{3.9}$$

But $\theta^k \in R_2(\theta^*)$, it follows that

$$f^i(x^k, \theta^k) \leq 0 \quad i \in \varphi^{<}(\theta^k)$$

also, $x^k \in F^=(\theta^k)$. Hence

$$x^k \in F(\theta^k)$$

which is what we want to show. ∎

As we noted earlier, the new condition is somewhat restrictive. However, it is satisfied if $F^=$ is lower semicontinuous or $F(\theta^*) \subset F^=(\theta)$. It can be shown that if $F(\theta^*)$ has non-empty interior and the new condition is satisfied, then $R_4(\theta^*)$ is a region of stability.

Now we show by examples that the new regions of stability given in this section are indeed larger than the ones known in the literature.

**3.9 Example.** *The following example shows that there exists situations where* $\{M(\theta^*) \cup V(\theta^*)\} \subset Z(\theta^*)$ *with a strict inclusion.*

Consider a convex model with the two constraints

$$f^1 = -x + \theta + 1 \leq 0$$

$$f^2 = \max\{0, x - \theta\} - x + \theta \leq 0$$

around $\theta^* = 0$

Since $\widehat{F}(\theta) = [\theta + 1, \infty)$, we have

$$M(\theta^*) = \{\theta : [1, \infty) \subset [\theta + 1, \infty)\} = (-\infty, 0].$$

Further

$$\wp^=(\theta) = \{2\}$$

for every $\theta$, while

$$F^=(\theta) = [\theta, \infty)$$

Hence

$$V(\theta^*) = \{\theta : [0, \infty) \subset [\theta, \infty)\} \cap R = (-\infty, 0]$$

But the new region of stability is bigger:

$$Z(\theta^\star) = \{\theta : [1, \infty) \subset [\theta, \infty)\} = (\infty, 1].$$

(Using $Z(\theta^\star)$ we conclude that the model is, in fact, stable at $\theta^\star$ for every realistic objective function.)

**3.10 Example.** *The example below shows that $Z(\theta^\star) \subset H(\theta^\star)$ with a strict inclusion.*

Consider

$$f^1 = -\theta^2 x + |x| - x \leq 0$$

$$f^2 = -(|\theta - 1| - (\theta - 1))(x - \theta) + |x - \theta| - (x - \theta) \leq 0$$

around $\theta^\star = 0$. For every $\theta, F(\theta) = [\theta, \infty) \cap [0, \infty)$, but

$$\varphi^{=}(\theta) = \begin{cases} \{1\}, & \theta = 0 \\ \{2\}, & \theta \geq 1 \\ \emptyset, & \theta < 1 (\theta \neq 0). \end{cases}$$

and

$$F^{=}(\theta) = \begin{cases} [0, \infty), & \theta = 0 \\ [\theta, \infty), & \theta \geq 1 \\ \infty, & \theta < 1 (\theta \neq 0)\ . \end{cases}$$

Therefore $H(\theta^\star) = \{\theta : f^i(x, \theta) \leq 0, \forall x \in F(\theta^\star), i \in \varphi^{=}(\theta^\star)\} = R$, while $Z(\theta^\star) = \{\theta : F(\theta^\star) \subset F^{=}(\theta)\} \cap H(\theta^\star) = (-\infty, 1)$

One might think that in a neighborhood $N(\theta^\star)$ of $\theta^\star, M(\theta^\star)$ is the same as $H(\theta^\star)$. We will show by example that this may not be the case, even if the feasible set $F(\theta^\star)$ is compact.

**3.11 Example.** *The example below shows that $F(\theta^\star)$ is compact, but $M(\theta^\star) \subset H(\theta^\star)$ with a strict inclusion.*

Consider a convex model with the two constraints

$$f^1 = -x + \theta + 1 \leq 0$$

$$f^2 = x \leq 0$$

around $\theta^* = 0$

Since $F(\theta) = [\theta + 1, 100]$, it's clear that $F(\theta^*) = [1, \infty)$ is compact and

$$M(\theta^*) = \{\theta : [1, 100] \subset [\theta + 1, 100]\} = (-\infty, 0]$$

Further

$$\varphi^{=}(\theta) = \emptyset$$

for every $\theta$.

Hence $H(\theta^*) = R$.

### 3.3 A Necessary Condition for Stability

A necessary condition for stability was proved in [31, 50] for path-connected region of stability. Now we show its validity for an arbitrary region of stability.

**3.12 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta = \theta^* \in I$. Let $S$ be an arbitrary region of stability at $\theta^*$ for every realistic objective function. Then there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that*

$$\varphi^{=}(\theta) \subset \varphi^{=}(\theta^*)$$

*for every $\theta \in N(\theta^*) \cap S$.*

*Proof*. If the set S consists of the single point $\theta^*$, or if $\varphi^{=}(\theta^*) = \varphi$, then the proof is trivial, So, we suppose that $\varphi^{<}(\theta^*) \triangleq \varphi \setminus \varphi^{=}(\theta^*) \neq \emptyset$ and that the claim is not true. There is an index $j \notin \varphi^{=}(\theta^*)$ and a sequence $\theta^k \in S, \theta^k \to \theta^*$ such that

$$f^j(x, \theta^k) = 0 \tag{3.10}$$

for all $x \in F(\theta^k)$. It is clear that

$$f^j(\hat{x}, \theta^*) \neq 0$$

for some $\hat{x}$ in $F(\theta^*)$, since $j \notin \wp^=(\theta^*)$. Therefore

$$f^j(\hat{x},\theta^*) < 0 \tag{3.11}$$

since $\hat{x} \in F(\theta^*)$.

By lower semicontinuity of $F : \theta \to F(\theta)$, (3.11) implies

$$f^j(x^k,\theta^k) \leq 0 \tag{3.12}$$

for some $\theta^k \in S, \theta^k \to \theta^*$ and $x^k \in F(\theta^k), x^k \to \hat{x}$. It is clear that (3.12) contradicts (3.10). ∎

By using the above theorem, many results in the literature can be restated on an arbitrary region of stability instead of a path-connected one.

The above theorem is used in proof of optimality condition later. Also, it has a nice economic interpretation [50].

**3.13 Example** *The following example shows that there exists not-path-connected region of stability.*

Consider a convex model with one constraint

$$f^1 = |x| - x + \theta \sin \frac{1}{\theta} \leq 0$$

around $\theta^* = 0$. Here $0 \sin \frac{1}{0}$ is defined to be 0.

We find that

$$F(\theta^*) = [0, \infty)$$

and

$$\wp^=(\theta^*) = \{1\}$$

Thus

$$\begin{aligned} H(\theta^*) &= \{\theta : f^i(x,\theta) \leq 0, \forall \in F(\theta^*), i \in \wp^=(\theta^*)\} \\ &= \{\theta : \theta \sin \frac{1}{\theta} \leq 0\} \\ &= \{\theta : \frac{1}{\theta} \in [(2k+1)\pi, (2k+2)\pi], k \geq 0\} \cup \{\theta : \frac{1}{\theta} \in [2k\pi, (2k+1)\pi], k < 0\} \end{aligned}$$

which is clearly not-path-connected.

# CHAPTER 4

## Optimality Conditions for Convex Models

### 4.1. Characterizing an Optimal Input

Optimality conditions for a convex model are stated in the literature mostly for the regions of stability $M(\theta^\star), V(\theta^\star)$ and $Z(\theta^\star)$, e.g., [44, 46, 48]. We now extend these conditions to the larger region of stability $H(\theta^\star)$. First we recall the notions of an " optimal input " and an " input constraint qualification " (abbreviation : ICQ).

**4.1 Definition.** *[44] Consider the convex model $(P,\theta)$ at some $\theta^\star \in I$ with a realistic objective function. We say that $\theta^\star$ is a locally optimal input for the model, with respect to a region of stability $S(\theta^\star)$, if*

$$\tilde{f}(\theta^\star) \le \tilde{f}(\theta)$$

*for every $\theta \in N(\theta^\star) \cap S(\theta^\star)$, where $N(\theta^\star)$ is some neighborhood of $\theta^\star$. The corresponding program $(P,\theta^\star)$ is a locally optimal realization and $\tilde{f}(\theta^\star)$ is a locally optimal value of the model $(P,\theta)$.*

Recall that the optimal input $\theta^\star$ depends on the choice of the initial input $\theta^0$. In order to formulate a necessary condition for optimally, we generally need an ICQ.

**4.2 Definition.** *[46] An input constraint qualification for the convex model $(P,\theta)$ at $\theta^\star \in I$, with respect to a region of stability $S(\theta^\star)$, is a condition on the constraints of the model with the property that for every $\theta \in N(\theta^\star) \cap S(\theta^\star)$, where $N(\theta^\star)$ is a neighborhood of $\theta^\star$, the system*

$$(C,\theta) \qquad \begin{aligned} & f^0(x,\theta) < \tilde{f}(\theta^\star) \\ & f^i(x,\theta) < 0, \quad i \in \wp^<(\theta^\star) \\ & x \in F^=(\theta^\star) \end{aligned}$$

*is inconsistent.* ∎

For some regions of stability, such as

$$W(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta), \varphi^=(\theta^*) = \varphi^=(\theta)\}$$

and

$$V_1(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta) \text{ and } f^i(x,\theta) \leq 0 \; \forall x \in F^=(\theta^*), i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)\}$$

it is easy to show that the system $(C,\theta)$ is always inconsistent at an optimal input $\theta^*$. (Clearly, the system $(C,\theta^*)$ is also inconsistent for every mathematical program $(P,\theta^*)$ at an optimal solution $x = \tilde{x}(\theta^*)$.) We will now show that the system $(C,\theta)$ is inconsistent also for the set

$$H_1(\theta^*) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F^=(\theta^*), i \in \varphi^=(\theta^*)\}$$

which is a bigger region of stability than $W(\theta^*)$ and $V_1(\theta^*)$.

**4.3 Theorem.** *Consider the convex model $(P,\theta)$ and an optimal input $\theta^* \in I$, with respect to the region of stability $H_1(\theta^*)$. Then there exists a neighborhood $N(\theta^*)$ of $\theta^*$ such that for every $\theta \in N(\theta^*) \cap H_1(\theta^*)$ the system $(C,\theta)$ is inconsistent.*

*Proof* If the result were not true, then there would exist $\theta^k \in H_1(\theta^*)$, $\theta^k \to \theta^*$ and $x^k \in F^=(\theta^*)$, such that $(C,\theta)$ is consistent. Hence

$$f^0(x^k,\theta^k) < \bar{f}(\theta^*) \tag{4.1}$$

$$f^i(x^k,\theta^k) < 0, \quad i \in \varphi^<(\theta^*) \tag{4.2}$$

By the definition of $H_1(\theta^*)$, we have

$$f^i(x^k,\theta^k) \leq 0, \quad i \in \varphi^=(\theta^*) \tag{4.3}$$

The two relations (4.2) and (4.3) imply $x^k \in F(\theta^k)$. We have a contradiction to (4.1), since $\theta^*$ is an optimal input. ∎

Note that Example 3.9 also shows that $V_1(\theta^*) \subset H_1(\theta^*)$ with a strict inclusion, since $H_1(\theta^*) = R$ while $V_1(\theta^*) = (-\infty, 1]$.

An $ICQ$ guarantees the existence of a " saddle point " for the " restricted Lagrangian "

$$L_*^<(x,u;\theta) = f^0(x,\theta) + \sum_{i\in\varphi^<(\theta^*)} u_i f^i(x,\theta).$$

One such $ICQ$, for the region of stability $Z(\theta^*)$, is the following condition:

" For every $\theta \in N(\theta^*) \cap S(\theta^*)$, where $N(\theta^*)$ is a neighborhood of $\theta^*$, and for every $x \in F^=(\theta^*)$ such that

$$f^i(x,\theta) < 0, \quad i \in \varphi^<(\theta^*)$$

it follows that

$$f^i(x,\theta^*) \leq 0, \quad i \in \varphi^<(\theta^*)."$$

This condition is referred to as $ICQ1$ in [46, 48]. $ICQ1$ is $ICQ$ for the region of stability $Z(\theta^*)$ was proven in [19]. We will now show that $ICQ1$ is $ICQ$ also for the region of stability $H(\theta^*)$.

**4.4 Lemma.** *Consider the convex model $(P,\theta)$ at an optimal input $\theta^* \in I$ with respect to the region of stability $H(\theta^*)$. Then $ICQ1$ is $ICQ$.*

*Proof* Suppose that the condition $ICQ1$ holds, but not ICQ. Then there exist a sequence $\theta^k \in H(\theta^*), \theta^k \to \theta^*$ and $x^k = x^k(\theta^k) \in F^=(\theta^*)$ such that

$$f^0(x^k,\theta^k) < \tilde{f}(\theta^*) \tag{4.4}$$

$$f^i(x^k,\theta^k) < 0, \quad i \in \varphi^<(\theta^*) \tag{4.5}$$

Since ICQ1 holds, (4.5) implies

$$f^i(x^k, \theta^*) < 0, \quad i \in \wp^<(\theta^*)$$

and hence $x^k \in F(\theta^*)$. On the other hand, $\theta^k \in H(\theta^*)$ implies

$$f^i(x^k, \theta^k) \lesseqgtr 0, \quad i \in \wp^=(\theta^*)$$

which together with (4.5) gives $x^k \in F(\theta^k)$. Since $\theta^*$ is a locally optimal input, we have a contradiction to (4.4). ∎

An optimality condition over the region of stability $H(\theta^*)$ follows:

**4.5 Theorem.** *Consider the convex model $(P, \theta)$ with a realistic objective function at some $\theta^* \in I$. Suppose that $\theta^*$ is a locally optimal input with respect to the region of stability $H(\theta^*)$ and that the condition ICQ1 is satisfied at $\theta^*$ with respect to $H(\theta^*)$. Let $\tilde{x}(\theta^*)$ be a corresponding optimal solution. Then there exists a neighborhood $N(\theta^*)$ and a non-negative vector function*

$$\Phi : N(\theta^*) \cap H(\theta^*) \to R_+^{q(\theta^*)}$$

*such that , whenever $\theta \in N(\theta^*) \cap H(\theta^*)$,*

$$L_*^<(\tilde{x}(\theta^*), u; \theta^*) \leq L_*^<(\tilde{x}(\theta^*), \Phi(\theta^*); \theta^*) \leq L_*^<(x, \Phi(\theta); \theta) \tag{4.6}$$

*for every $u \in R_+^{q(\theta^*)}$ ( the non-negative orthant in $R^{q(\theta^*)}$, where $q(\theta^*)$ is the cardinality of $\wp^<(\theta^*)$) and every $x \in F^=(\theta^*)$.*

*Proof* Since ICQ1 is indeed ICQ for $H(\theta^*)$, the result is an immediate consequence of, say, [48, Theorem 7.4]. ∎

The importance of Theorem 4.5 is that a necessary condition for an optimal input is now stated over a larger region of stability than $Z(\theta^*)$. Of course, the result also holds under some more restrictive ICQ's such as ICQ2 or Slater's condition. (see [46].)

The above result is also sufficient for optimality if the saddle-point inequality holds for every $x \in F^{=}(\theta)$ (See [44] for details.)

It is easy to show that (4.6) is true for every $u \in R_{+}^{q(\theta^{*})}$ and every $x \in \tilde{F}(\theta^{*})$, if we replace ICQ by the following condition :

" For every $\theta \in N(\theta^{*}) \cap S(\theta^{*})$, where $N(\theta^{*})$ is a neighborhood of $\theta^{*}$., the system

$$f^{0}(x,\theta) < \tilde{f}(\theta^{*})$$

$$f^{i}(x,\theta) < 0, \quad i \in \wp^{<}(\theta^{*})$$

$$x \in F(\theta^{*})$$

is inconsistent."

Let us note that the above system is inconsistent if ICQ is satisfied. So the above result is more general than Theorem 4.5. On the other hand, the set $F^{=}(\theta^{*})$ can be often constructed, so Theorem 4.5 seems more useful. Note that, when Slater's condition holds, then $L^{<}$ becomes the usual Lagrangian, $F^{=}(\theta^{*}) = R^{n}$ and we recover the classical saddle point characterization of optimality in convex programming of the Karush-Kuhn-Tucker type (See [1, 12, 27, 33 ].).

## 4.2. Modified Input Constraint Qualifications

Recently, a new kind of ICQ's referred to as " modified input constraint qualifications " (MICQ), has been introduced in [35].

**4.6 Definition.** *Consider the convex model $(P,\theta)$, with a realistic objective function at some $\theta^* \in I$, and let $S(\theta^*)$ be a region of stability of $\theta^*$. A condition on the constraints of the model $(P,\theta)$, with the property that for every $\theta \in N(\theta^*) \cap S(\theta^*)$, where $N(\theta^*)$ is a neighborhood of $\theta^*$, the system*

$$(MC,\theta) \qquad \begin{aligned} f^0(x,\theta) &< \tilde{f}(\theta^*) \\ f^i(x,\theta) &< 0, \quad i \in \varphi^<(\theta^*) \\ x &\in F^=(\theta) \end{aligned}$$

*is inconsistent, is called a modified input constraint qualification.*

The following three conditions can be verified to be MICQ's : There is a neighborhood $N(\theta^*)$ of $\theta^*$ such that for every $\theta \in N(\theta^*) \cap S(\theta^*)$ :

$MICQ1$ : "$f^i(x,\theta) \leq 0, i \in \varphi^=(\theta^*) \setminus \varphi^=(\theta)$ for every $x \in F^=(\theta)$" ;

$MICQ2$ : "$\varphi^=(\theta) = \varphi^=(\theta^*)$";

$MICQ3$ : " The constraints $f^i, i \in \varphi^=(\theta^*)$ do not depend on $\theta$".

By using an MICQ, a new necessary condition for an optimal input was stated in [35] for an arbitrary region of stability as follows :

**4.7 Theorem.** *Consider the convex model $(P,\theta)$ with a realistic objective function at some $\theta^* \in I$. Let $\tilde{x}(\theta^*)$ be a corresponding optimal solution and let $S(\theta^*)$ be an arbitrary region of stability at $\theta^*$. If $\theta^*$ is a locally optimal input relative to $S(\theta^*)$, and if a modified input constraint qualification holds at $\theta^*$ relative to $S(\theta^*)$, then there exists a neighborhood $N(\theta^*)$ of $\theta^*$ and a non-negative vector function*

$$U : N(\theta^*) \cap S(\theta^*) \to R_+^{q(\theta^*)}$$

*such that , whenever $\theta \in N(\theta^*) \cap S(\theta^*)$,*

$$L_*^{<}(\tilde{x}(\theta^*), u; \theta^*) \le L_*^{<}(\tilde{x}(\theta^*), U(\theta^*); \theta^*) \le L_*^{<}(x, U(\theta); \theta) \tag{4.7}$$

*for every $u \in R_+^{q(\theta^*)}$ and every $x \in F^=(\theta)$* ∎

It's easy to show that (4.7) is true for every $u \in R_+^{q(\theta^*)}$ and every $x \in F(\theta)$, without assuming that MICQ is satisfied. Therefore, the above result is more general than Theorem 4.7, but the information on $F^=(\theta)$ is more readily available than for $F(\theta)$.

Also note that Theorem 4.7 doesn't require the region of stability S to be path-connected.

### 4.3 A Complete Charaterization of Optimality

Recently, a complete characterization of an optimal input without any ICQ or MICQ whatever was given in [34] for convex models. This characterization also uses the restricted Lagrangian $L_*^{<}$, the same saddle-point inequality as in Theorem 4.5, but it is restricted to $x's$ belonging to the sets

$$F_*^=(\theta) = \{x : f^i(x,\theta) \le 0, i \in \wp^=(\theta^*)\}$$

as $\theta \in N(\theta^*) \cap S(\theta^*)$. Since $\wp^=(\theta) \subset \wp^=(\theta^*)$ for every $\theta \in N(\theta^*) \cap S(\theta^*)$, where $N(\theta^*)$ is a neighborhood of $\theta^*$ and $S(\theta^*)$ is an arbitrary region of stability (see Section 3.3 for details.), we note that , for such $\theta's$

$$F(\theta) \subset F_*^=(\theta) \subset F^=(\theta) \tag{4.8}$$

For the proof, see Proposition 2.2 of [34].

However, $F^=(\theta^*)$ and $F^=(\theta)$ are generally incomparable. The complete characterization follows. The novelty here is that the set S is not assumed to be path-connected, as assumed in [34].

**4.8 Theorem.** *Consider the convex model $(P, \theta)$ with a realistic objective function at some $\theta^* \in I$. Denote by $\tilde{x}(\theta^*)$ a corresponding optimal solution. Then $\theta^*$ is a locally optimal input with respect to a region of stability $S(\theta^*)$ if, and only if, there exists a neighborhood $N(\theta^*)$ of $\theta^*$ and a non-negative vector function*

$$\Lambda : N(\theta^*) \cap S(\theta^*) \to R_+^{q(\theta^*)}$$

*such that, whenever $\theta \in N(\theta^*) \cap S(\theta^*)$,*

$$L_*^<(\tilde{x}(\theta^*), u; \theta^*) \leq L_*^<(\tilde{x}(\theta^*), \Lambda(\theta^*); \theta^*) \leq L_*^<(x, \Lambda(\theta); \theta)$$

*for every $u \in R_+^{q(\theta^*)}$ and every $x \in F_*^=(\theta)$.* ∎

Note that $\Lambda, \Phi$ and $U$ are three essentially different mappings.

Remark: The optimal input $\theta^*$, and the corresponding optimal solutions $\tilde{x}(\theta^*)$,do not necessarily correspond to the optimal solution of the corresponding "usual" program (P).(using substitution $z = (x, \theta)$). Every optimal solution $z^* = (x^*, \theta^*)$ of $(P)$ contains an optimal input $\theta^*$, but there is optimal input $\theta^*$ such that $z^* = (\tilde{x}(\theta^*), \theta^*)$ is not an optimal for $(P)$.

# CHAPTER 5

## Continuity of the Lagrange Multipliers
## and
## Necessary Conditions for Differentiable Functions

### 5.1 Continuity of the three Lagrange Multiplier Functions

For a fixed $\theta$, consider the restricted Lagrangians

$$L^{<}(x,u;\theta) = f^0(x,\theta) + \sum_{i\in\varphi^{<}(\theta)} u_i f^i(x,\theta) \tag{5.1}$$

and

$$L_*^{<}(x,u;\theta) = f^0(x,\theta) + \sum_{i\in\varphi^{<}(\theta^*)} u_i f^i(x,\theta) \tag{5.2}$$

The corresponding Lagrange multipliers to (5.1) is

$$U^{<}(\theta) = \{u_i^*(\theta) : i \in \varphi^{<}(\theta)\}.$$

But when (5.2) is used in the three necessary conditions for an optimal input, we get three Lagrange multipliers

$$\Phi(\theta) = \{\phi_i(\theta) : i \in \varphi^{<}(\theta^*)\}$$

$$\Lambda(\theta) = \{\lambda_i(\theta) : i \in \varphi^{<}(\theta^*)\}$$

and

$$U(\theta) = \{u_i(\theta) : i \in \varphi^{<}(\theta^*)\}.$$

Now we discuss the continuity of $\Phi(\theta)$ first.

On the regions of stability, we have $\varphi^{<}(\theta^*) \subset \varphi^{<}(\theta)$ for every $\theta$ in a neighborhood $N(\theta^*)$ of $\theta^*$; (See [13] and Theorem 3.12). Therefore it follows that $\Phi(\theta)$ is a subset of $U^{<}(\theta)$. It is known that $U^{<}(\theta)$ is generally discontinuous on regions of stability, and so is $\Phi(\theta)$ (See [36, 38, 45]).

However, if $F^=$ is lower semicontinuous at $\theta^*$, then $\Phi(\theta)$ is continuous over the region of stability $R_2(\theta^*)$. (See [38]) Continuity is also proven on

$$V_1(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta)\} \cap R_3(\theta^*)$$

(See [36])

In this section we will establish the continuity of $\Phi(\theta)$ on every subset of an arbitrary region of stability, provided that

$$F(\theta^*) = F^=(\theta^*) \tag{5.3}$$

The condition (5.3) looks somewhat restrictive. In particular, if Slater's condition holds at $\theta^*$, then $F^=(\theta^*) = R^n$ and the condition holds only for unconstrained models. However, another extreme case is when the feasible set is determined only by linear equations, in which case the condition is trivially satisfied. A more general situation, where (5.3) holds, is described by the example below.

**5.1 Example.**

Consider a convex model with only one constraint

$$f^1 = |x| - (1 + \theta^2)x \le 0$$

around $\theta = \theta^* = 0$.

Here $F(\theta) = [0, \infty)$ for every $\theta$, but

$$\varphi^=(\theta) = \begin{cases} \{1\} & \text{if } \theta = 0 \\ \emptyset & \text{if } \theta \neq 0 \end{cases}$$

and hence

$$F^=(\theta) = \begin{cases} [0, \infty) & \text{if } \theta = 0 \\ R & \text{if } \theta \neq 0 \end{cases}$$

The point-to set mapping $F^=$ is lower semicontinuous at $\theta^*$ and

$$R_2(\theta^*) = \{\theta^*\}$$

Hence the result from the literature (such as Theorem 6.1 [48]) on continuity of the Lagrange multiplier function $\Phi(\theta)$ is not useful here. However, since the requirement (5.3) is satisfied, and *int* $F(\theta^\star) \neq \emptyset$, we are able to establish continuity on every region of stability, say, $R_4(\theta^\star) = R$.

The result on continuity of the Lagrange multiplier function follows. It is a new result (forthcoming in [19].).

**5.2 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta^\star \in I$ with a realistic objective function and let $F(\theta^\star) = F^{=}(\theta^\star)$. If $S(\theta^\star)$ is an arbitrary region of stability at $\theta^\star$ then, for an arbitrary sequence $\theta^k \in S(\theta^\star), \theta^k \to \theta^\star$:*

*(i) The sequence $\Phi(\theta^k)$ is bounded for all sufficiently large k's and*

*(ii) the set of all limit points of $\Phi(\theta^k)$, as $\theta^k \to \theta^\star$, is nonempty and it is contained in $\Phi(\theta^\star)$.*

<u>*Proof*</u> (i) This statement holds for any region of stability, as one can see from the proofs of Theorem 3.1 [35] or Theorem 2.1 [37].

This statement also holds for the other two Lagrange multiplier function $\Lambda$ and $U$. The proof is essentially the same.

Since both sequence $\{\phi_i(\theta^k) : i \in \varphi^{<}(\theta^\star)\}$ and $\{\tilde{x}(\theta^k)\}$ are bounded (the first by (i), the second because $S(\theta^\star)$ is a region of stability) as $\theta^k \to \theta^\star$, there exists a subsequence $\{\theta^{k,s}\}$ of $\{\theta^k\}$ such that $\{\phi_i(\theta^{k,s}) : i \in \varphi^{<}(\theta^\star)\}$ and $\{\tilde{x}(\theta^{k,s})\}$ are convergent. We claim that $\phi_i(\theta^{k,s}) \to \phi_i(\theta^\star), i \in \varphi^{<}(\theta^\star)$. Without loss of generality, we start the proof with $\theta^{k,s} = \theta^k$. From the familiar saddle point characterization of optimality we have

$$f^0(\tilde{x}(\theta^k), \theta^k) + \sum_{i \in \varphi^{<}(\theta^k)} v_i f^i(\tilde{x}(\theta^k), \theta^k) \leq \tilde{f}(\theta^k) \tag{5.4}$$

for every $v_i \geq 0, i \in \varphi^{<}(\theta^k)$ and

$$\tilde{f}(\theta^k) \leq f^0(x, \theta^k) + \sum_{i \in \varphi^{<}(\theta^k)} \phi_i(\theta^k) f^i(x, \theta^k) \tag{5.5}$$

for every $x \in F^{=}(\theta^k)$. First , by specifying

$$v_i = 0, \quad i \in \varphi^{<}(\theta^k) \setminus \varphi^{<}(\theta^*)$$

in (5.4), we can replace the summation by $i \in \varphi^{<}(\theta^*)$. In the limit $\theta^k \to \theta^*$, (5.4) gives

$$f^0(\tilde{x}(\theta^*), \theta^*) + \sum_{i \in \varphi^{<}(\theta^*)} v_i f^i(\tilde{x}(\theta^*); \theta^*) \leq \tilde{f}(\theta^*) \tag{5.6}$$

for every $v_i \geq 0, i \in \varphi^{<}(\theta)$. This is the left-hand side of the saddle-point inequality at $\theta^*$. On the other hand, since for every $x \in F(\theta^k)$ we have

$$f^i(x, \theta^k) \leq 0, \quad i \in \varphi^{<}(\theta^k) \setminus \varphi^{<}(\theta^*)$$

it follows that the right hand side of (5.5) can be modified as follows

$$\tilde{f}(\theta^k) \leq L_*^{<}(x, \phi(\theta^k); \theta^k) \tag{5.7}$$

for every $x \in F(\theta^k)$. Now we choose an arbitrary $x \in F(\theta^*)$. Since $\theta^k$ is chosen from a region of stability $S(\theta^*)$, the mapping F is lower semicontinuous at $\theta^*$ with respect to $S(\theta^*)$. Therefore there exists a sequence $x^k = x^k(\theta^k) \in F(\theta^k)$ such that $x^k \to x$ as $\theta^k \to \theta^*$. The inequality (5.7) gives in the limit

$$\tilde{f}(\theta^*) \leq L_*^{<}(x, \phi(\theta^*); \theta^*) \tag{5.8}$$

for every $x \in F(\theta^*)$. (Here $\phi(\theta^*)$ is an arbitrary limit point of the sequence $\Phi(\theta^k)$ that generates the above $\theta^k \in S(\theta^*), \theta^k \to \theta^*$. ) Since $F(\theta^*) = F^{=}(\theta^*)$, the inequality (5.8) also holds for every $x \in F^{=}(\theta^*)$. This is the right-hand side of the saddle-point inequality at $\theta^*$. So we can conclude that the limit point $\phi(\theta^*)$ of $\phi(\theta^k)$, as $\theta^k \in S(\theta^*), \theta^k \to \theta^*$, is indeed in $\Phi(\theta^*)$. ∎

Following definition 2.2 (see also Berge [7]), we can reformulate Theorem 5.2 as follows

**5.3 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta^* \in I$ with $\tilde{F}(\theta^*) \neq \emptyset$ and bounded and let $F(\theta^*) = F^=(\theta^*)$. If $S(\theta^*)$ is an arbitrary region of stability at a locally optimal input $\theta^*$, then the point-to-set map $\Phi : \theta \to \Phi(\theta)$ is upper semicontinuous at $\theta^*$ over $S(\theta^*)$.* ∎

Recently, a different condition for the continuity of $\Phi(\theta)$ was given in [35]. The condition $F(\theta^*) = F^=(\theta^*)$ can be omitted, while the requirement is replaced by an input constraint qualification. Also, continuity of the Lagrange multipliers U and $\Lambda$, appearing in Theorem 4.7 and 4.8 respectively, was considered in [35]. The conditions for continuity are different from those for $\Phi$, but the proof is similar. Let us recall the results for the sake of comparision.

**5.4 Theorem.** *[35] Consider the convex model $(P, \theta)$ with a realistic objective function at a locally optimal input $\theta^* \in I$ with respect to a region of stability $S(\theta^*)$. If the point-to-set mapping $\gamma_* : \theta, \to F_*^=(\theta)$ is lower semicontinuous at $\theta^*$, relative to $S(\theta^*)$, then the point-to-set mapping $\Lambda : \theta \to \Lambda(\theta)$ is upper semicontinuous.*

**5.5 Theorem.** *[35] Consider the convex model $(P, \theta)$ with a realistic objective function at a locally optimal input $\theta^* \in I$ with respect to a region of stability $S(\theta^*)$. If the point-to-set mapping $\gamma : \theta \to F^=(\theta)$ is lower semicontinuous at $\theta^*$, relative to $S(\theta^*)$, then the point-to-set mapping $U : \theta \to U(\theta)$ is upper semicontinuous.*

Note that the set of Lagrange multiplier $\Lambda(\theta)$ and $U(\theta)$ are comparable for $\theta \in N(\theta^*) \cap S(\theta^*)$, where $N(\theta^*)$ is a neighborhood of $\theta^*$. Since on the region of stability $S(\theta^*), \wp^=(\theta) \subset \wp^=(\theta^*)$, it follows that $F_*^=(\theta) \subset F^=(\theta)$ and hence $U(\theta) \subset \Lambda(\theta)$.

## 5.2. Necessary Conditions for Models with Differentiable Functions

By using the upper semicontinuity of the three kinds of Lagrange multipliers, which were discussed in Section 5.1, necessary conditions for differentiable functions were proven on particular subsets of an arbitrary region of stability $S(\theta^*)$ (See Section 5 of [35] for details ), such as

$$S_1(\theta^*) = \{\theta : F(\theta^*) \subset F_*^=(\theta)\} \cap S(\theta^*)$$

and

$$S_2(\theta^*) = \{\theta : F(\theta^*) \subset F^=(\theta)\} \cap S(\theta^*)$$

It is clear that $S_1(\theta^*) \subset H(\theta^*)$ and it is also easy to show that $S_2(\theta^*) \subset H(\theta^*)$, since $R_2(\theta^*), R_4(\theta^*)$ and $H(\theta^*)$ are among the " largest " regions of stability and

$$\{\theta : F(\theta^*) \subset F^=(\theta)\} \cap \{R_2(\theta^*) \cup R_4(\theta^*)\} \subset H(\theta^*)$$

We will now show that two results from [35] remain valid over the region of stability $H(\theta^*)$.

We need more notation. We use a part of the unit balls in $R^p$ defined by

$$B(\theta^*) = \{\frac{\theta - \theta^*}{||\theta - \theta^*||} : \theta \in S(\theta^*), \theta \neq \theta^*\}$$

and

$$B_1(\theta^*) = \{\frac{\theta - \theta^*}{||\theta - \theta^*||} : \theta \in H(\theta^*), \theta \neq \theta^*\}$$

where $S(\theta^*)$ is an arbitrary region of stability. We denote by $B^0$ the derived set of B, i.e., the set of all limit points as $\theta \in H(\theta^*), \theta \neq \theta^*, \theta \to \theta^*$. Also

$$(B^0)^+ = \{u : u^T b \geq 0, \quad \forall\, b \in B^0\}$$

is called the polar of $B^0$.

**5.6 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta^* \in I$ with a realistic objective function. Let $\{\tilde{x}(\theta^*), \Lambda(\theta^*)\}$ be a unique corresponding saddle point and let the functions $f^i(\tilde{x}(\theta^*), \cdot), j \in \{0\} \cup \wp^<(\theta^*)$ be differentiable . Let the point-to-set mapping $\gamma_* : \theta \to F_*^=(\theta)$ be lower semicontinuous at $\theta^*$ relative to $H(\theta^*)$. If $\theta^*$ is a locally opimal input with respect to $H(\theta^*)$, then*

$$\nabla_\theta L_*^<(\tilde{x}(\theta^*), \Lambda(\theta^*); \theta)|_{\theta=\theta^*} \in \{B_1^0(\theta^*)\}^+.$$

*Proof* Since $H(\theta^*)$ has the property of $S_1(\theta^*)$, the proof is exactly the same as that given in [34].

**5.7 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta^* \in I$ with a realistic objective function. Suppose that $\theta^*$ is a locally optimal input with respect to $H(\theta^*)$ and suppose that a modified input constraint qualification holds at $\theta^*$ relative to $H(\theta^*)$ . Let $\{\tilde{x}(\theta^*), U(\theta^*)\}$ be a unique corresponding saddle point and let the functions $f^j(\tilde{x}(\theta^*), \cdot), j \in \{0\} \cap \wp^<(\theta^*)$ be differentiable. Also assume that the point-to-set mapping $F^= : \theta \to F^=(\theta)$ is lower semicontinuous at $\theta^*$ relative to $H(\theta^*)$. Then*

$$\nabla_\theta L_*^<(\tilde{x}(\theta^*), U(\theta^*); \theta)|_{\theta=\theta^*} \in \{B_1^0(\theta^*)\}^+.$$

*Proof* Since $H(\theta^*)$ has the property of $S_2(\theta^*)$, the proof is the same as that given in [35].

For the sake of completeness, Let us recall another necessary condition which was obtained in the presence of an ICQ and no assumptions on $F^=$ or $\gamma_*$ are needed. ( See [35] for the proof ).

**5.8 Theorem.** *Consider the convex model $(P, \theta)$ at some $\theta^* \in I$ with a realistic objective function. Suppose that $\theta^* \in I$ is a locally optimal input with respect to a region of stability $S(\theta^*)$ and suppose that an input constraint qualification holds at $\theta^*$ relative to $S(\theta^*)$ . Let $\{\tilde{x}(\theta^*), \Phi(\theta^*)\}$ be a unique corresponding saddle point* *and let the functions $f^j(\tilde{x}(\theta^*), \cdot), j \in \{0\} \cap \varphi^<(\theta^*)$ be differentiable. Then*

$$\nabla_\theta L_*^<(\tilde{x}(\theta^*), \Phi(\theta^*); \theta)|_{\theta=\theta^*} \in \{B^0(\theta^*)\}^+.$$

# CHAPTER 6

## Optimality Conditions for Bi-Convex Models and a Marginal Value Formula

### 6.1 A Necessary Condition for Bi-Convex Models

The necessary condition for an optimal input for general convex models can be strengthened for bi-convex models , i.e., for the models $(P, \theta)$ where both $f^i(\cdot, \theta) : R^n \to R$ and $f^i(x, \cdot) : R^p \to R, i \in \{0\} \cap \varphi$ are convex functions. The result, given below, was proven in [19] for a subset of the region of stability $Z(\theta^\star)$, namely for

$$Z_1(\theta^\star) = \{\theta : F(\theta^\star) \subset F^=(\theta)\} \cap R_2(\theta^\star).$$

Following the ideas from [50], we use a part of the unit ball $B = B(\theta^\star)$ in $R^p$ defined by

$$B = \{\frac{\theta - \theta^\star}{||\theta - \theta^\star||} : \theta \in N(\theta^\star) \cap Z_1(\theta^\star), \theta \neq \theta^\star\}$$

for some fixed neighborhood $N(\theta^\star)$ of $\theta^\star$. The polar of $B^0$ is defined in Section 5.2.

We also need a condition on indices of the constraints, known as the " index condition " (See [48, 49] ). First we denote by

$$\varphi(\tilde{x}(\theta^\star), \theta^\star) = \{i \in \varphi : f^i(\tilde{x}(\theta^\star), \theta^\star) = 0\}$$

the set of active constraints for $\theta^\star$ at the optimal solution $\tilde{x}(\theta^\star)$. We recall that, for differentiable functions, the index condition is said to hold at $\theta^\star$, with respect to a region of stability $S(\theta^\star)$, if

$$\text{(IND)} \qquad \{\varphi^<(\theta^k) \cap \varphi(\tilde{x}(\theta^\star), \theta^\star)\} \subset \varphi^<(\theta^\star)$$

for all but possible finitely many k's, for every sequence $\theta^k \in S(\theta^\star), \theta^k \to \theta^\star$.

To simplify the notation, we introduce the abbreviation

$$g(\theta) = L_\star^<(\tilde{x}(\theta^\star), \tilde{u}(\theta^\star); \theta)$$

where $(\tilde{x}(\theta^\star), \tilde{u}(\theta^\star))$ is a restricted saddle point. Note that $g(\theta)$ is a convex function.

**6.1 Theorem.** *Consider the bi-convex model $(P,\theta)$ at $\theta = \theta^* \in I$ with a realistic objective function. Suppose that the corresponding saddle point $\{\tilde{x}(\theta^*), \tilde{u}_i(\theta^*) : i \in \varphi^<(\theta^*)\}$ of the above restricted Lagrangian is unique and that the index condition (IND) holds relative to $Z_1(\theta^*)$. We also assume that the point-to-set mapping $F^=$ is lower semicontinuous at $\theta^*$ and that all functions $f^i(\tilde{x}(\theta^*),\theta), i \in \{0\} \cup \varphi^<(\theta^*)$ are differentiable at $\theta^*$. If $\theta^*$ is a locally optimal input with respect to $Z_1(\theta^*)$, then*

$$\nabla_\theta g(\theta^*) \in (B^0)^+.$$

<u>*Proof*</u> In the proof we use the fact that $Z_1(\theta^*) \subset R_2(\theta^*)$ and the result on continuity of the restricted Lagrange multipliers with respect to the set $R_2(\theta^*)$. (See [50]. ) So, take an arbitrary $l \in B^0$. This point is generated by some $\theta^k \in Z_1(\theta^*), \theta^k \to \theta^*$. By the continuity of Lagrange multipliers, there exist $u_i(\theta^k) \to \tilde{u}_i(\theta^*), i \in \varphi^<(\theta^*)$. For this sequence $\theta^k$ define

$$E(\theta^k) = L^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^k) - L^<(\tilde{x}(\theta^*), \tilde{u}(\theta^k); \theta^k) \tag{6.1}$$

and, using the gradient inequality for $g(\theta)$, we find that

$$(\nabla g(\theta^k), \theta^k - \theta^*) \geq L^<(\tilde{x}(\theta^*), \tilde{u}(\theta^k); \theta^k) - g(\theta^*) + E(\theta^k) \tag{6.2}$$

now we invoke the estimate

$$\tilde{f}(\theta) - \tilde{f}(\theta^*) \leq L^<(z, \tilde{u}(\theta); \theta) - L^<_*(\tilde{x}(\theta^*), v; \theta^*) \tag{6.3}$$

holding for every $z \in F^=(\theta)$ and $v \in R_+^{q(\theta^*)}$, as it is known from [43]. Using the fact that $\theta^k \in Z_1(\theta^*)$, we note that

$$\tilde{x}(\theta^*) \in F(\theta^*) \subset F^=(\theta^k).$$

So we can specify $z = \tilde{x}(\theta^*)$ and $v = \tilde{u}(\theta^*)$ in (6.3) and (6.2) now gives

$$(\nabla g(\theta^k), \theta^k - \theta^*) \geq \tilde{f}(\theta^k) - \tilde{f}(\theta^*) + E(\theta^k) \geq E(\theta^k) \tag{6.4}$$

for all sufficiently large k's, because $\theta^*$ is a locally optimal input. Hence

$$[\nabla g(\theta^k), \frac{\theta^k - \theta^*}{||\theta^k - \theta^*||}] \geq \frac{E(\theta^k)}{||\theta^k - \theta^*||} \tag{6.5}$$

Let us note that

$$\begin{aligned} E(\theta^k) &= \sum_{i \in \varphi^<(\theta^k)} [\tilde{u}_i(\theta^*) - \tilde{u}_i(\theta^k)] f^i(\tilde{x}(\theta^*), \theta^k) \\ &\geq \sum_{i \in \varphi^<(\theta^k) \cap \varphi(\tilde{x}(\theta^*), \theta^*)} [\tilde{u}(\theta^*) - \tilde{u}_i(\theta^k)] f^i(\tilde{x}(\theta^*), \theta^k) \end{aligned} \tag{6.6}$$

since the terms with nonactive indices are nonnegative. Now the index condition guarantees non-negativity of the limit when $k \to \infty$ of the right-hand side term in (6.5). This, together with continuous differentiability property of the differentiable convex function $g(\theta)$, gives the desired result. ∎

A result of the above kind was proven in the literature (See [49,50]. ) but only for the region of stability $V_1(\theta^*)$ which is a subset of $Z_1(\theta^*)$.

**6.2 Example.** *The following example, which is from Section 3.2 , shows that one may have $V_1(\theta^*) \subset Z_1(\theta^*)$ with a strict inclusion.*

Consider a bi-convex model with the two constraints

$$f^1 = \theta + 1 - x \leq 0$$

$$f^2 = max(0, x - \theta) + \theta - x \leq 0$$

for every $\theta, F(\theta) = [\theta + 1, \infty), \varphi^=(\theta) = \{2\}$ and $F^=(\theta) = [\theta, \infty)$

Therefore

$$V_1(\theta^*) = (-\infty, 0]$$

while

$$Z_1(\theta^*) = (-\infty, 1].$$

As we see that the above necessary condition for optimality requires the " index condition ". It's easy to find examples which show that the index condition is not satisfied, though it is satisfied when " Slater's condition " holds. Now we are going to state a necessary condition without the index condition. The condition

"All functions $f^i(\tilde{x}(\theta^*),\theta)$, $i \in \{0\} \cup \varphi^<(\theta^*)$ are differentiable at $\theta^*$." (6.7)

can also be omitted , but these two requirements are replaced by the following two conditions:

"All functions $f^i(\tilde{x}(\theta^*),\theta)$, $i \in \{0\} \cup \varphi(\tilde{x}(\theta^*),\theta^*)$ are differentiable at $\theta^*$." (6.8)

and

"For every fixed path $\theta^k \in S(\theta^*), \theta^k \to \theta^*$ the limit

$$\lim_{\substack{\theta^k \in S(\theta^*) \\ \theta^k \to \theta^*}} \frac{\theta^k - \theta^*}{||\theta^k - \theta^*||} = l \tag{6.9}$$

exists."

The necessary condition for optimality follows

**6.3 Theorem.** *Consider the bi-convex model $(P,\theta)$ at $\theta = \theta^* \in I$ with a realistic objective function. Suppose that the corresponding saddle point $\{\tilde{x}(\theta^*), \tilde{u}_i(\theta^*) : i \in \varphi^<(\theta^*)\}$ of the restricted Lagrangian is unique. We also assume that the point-to-set mapping $F^=$ is lower semicontinuous at $\theta^*$ and that conditions (6.8) and (6.9) are satisfied. If $\theta^*$ is a locally optimal input with respect to $Z_1(\theta^*)$, then*

$$\nabla_\theta g(\theta^*) \in (B^0)^+.$$

Proof Follow the same proof as given in Theorem 6.1 and arrive at the inequality

$$E(\theta^k) \geq \sum_{i \in \varphi^<(\theta^k) \cap \varphi(\tilde{x}(\theta^*),\theta^*)} [\tilde{u}_i(\theta^*) - \tilde{u}_i(\theta^k)] f^i(\tilde{x}(\theta^*),\theta^k)$$

Dividing both sides by $||\theta - \theta^*||$ , now condition (6.8) and (6.9) guarantee non-negativity of the limit when $k \to \infty$ of the right-hand side term. This, together with continuous differentiability property (of the differentiable convex function) $g(\theta)$, gives the desired result.

## 6.2 A Sufficient Condition for Bi-Convex Models

A sufficient condition for an optimal input was proven in the literature for the region of stability $M_1(\theta^\star) \cup V_2(\theta^\star)$ (See [48, 50].) We will now show that the result remains valid for the region of stability

$$S_3(\theta^\star) = \{\theta : F(\theta) \subset F^=(\theta^\star)\} \cap S(\theta^\star)$$

where $S(\theta^\star)$ is an arbitrary region of stability.

Denote by $B = B(\theta^\star)$ the set

$$B = \{\frac{\theta - \theta^\star}{\|\theta - \theta^\star\|} : \theta \in \{M_1(\theta^\star) \cup V_2(\theta^\star)\} \cap I, \theta \neq \theta^\star\}.$$

A sufficient condition for optimality follows

**6.4 Theorem.** *Consider the differentiable bi-convex model $(P, \theta)$ with a realistic objective function at $\theta = \theta^\star \in I$. Suppose that the corresponding saddle point $\{\tilde{x}(\theta^\star), \tilde{u}_i(\theta^\star) : i \in \wp^<(\theta^\star)\}$ is unique. If*

$$\nabla_\theta L_\star^<(\tilde{x}(\theta^\star), \tilde{u}(\theta^\star); \theta)|_{\theta=\theta^\star} \in \ int\ \{B^0(\theta^\star)\}^+ \tag{6.10}$$

*then $\theta^\star$ is an isolated locally optimal input over the region of stability $S_3(\theta^\star)$.*

*Proof* First we invoke the estimate

$$\tilde{f}(\theta) - \tilde{f}(\theta^\star) \geq L^<(\tilde{x}(\theta), u; \theta) - L_\star^<(x, \tilde{u}(\theta^\star); \theta^\star)$$

holding for every $x \in F^=(\theta^\star)$ and $u \in R_+^{q(\theta^\star)}$, as it is given in [Lemma 4.3 of [50]). Specifying $u = (u_i) \in R_+^{q(\theta)}$ as follows

$$u_i = \begin{cases} \tilde{u}_i(\theta^\star) & i \in \wp^<(\theta^\star) \\ 0 & i \in \wp^<(\theta) \setminus \wp^<(\theta^\star). \end{cases}$$

we find that, for every $\theta \in S_3(\theta^\star)$ and every $x \in F^=(\theta^\star)$,

$$\tilde{f}(\theta) - \tilde{f}(\theta^\star) \geq L_\star^<(\tilde{x}(\theta), \tilde{u}(\theta^\star); \theta) - L_\star^<(x, \tilde{u}(\theta^\star); \theta^\star) \ . \tag{6.11}$$

If $\theta \in S_3(\theta^*)$, then $\tilde{x}(\theta) \in F(\theta) \subset F^=(\theta^*)$. After specifying $x = \tilde{x}(\theta)$ in (6.11) we obtain

$$\tilde{f}(\theta) - \tilde{f}(\theta^*) \geq L_*^<(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta) - L_*^<(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta^*)$$

and further, by the gradient inequality,

$$\tilde{f}(\theta) - \tilde{f}(\theta^*) \geq ([\nabla_\theta L_*^<(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta)]|_{\theta=\theta^*}, \theta - \theta^*)$$

Finally, let $\theta^i \in S_3(\theta^*)$ and $\theta^i \to \theta^*$, then $\tilde{x}(\theta^i) \to \tilde{x}(\theta^*)$, as $i \to \infty$, by uniqueness of $\tilde{x}(\theta^*)$ and since $S_3(\theta^*)$ is a region of stability. All functions $f^i(\cdot, \theta), i \in \{0\} \cup \varphi$ are convex and differentiable, so the gradient is a continuous function in $x$. Let $l$ be in the set $B^0$. Then, for a subsequence $\{\theta^{i,j}\}$, we find that

$$\lim_{j\to\infty} \frac{\tilde{f}(\theta^{i,j}) - \tilde{f}(\theta^*)}{||\theta^{i,j} - \theta^*||} > 0$$

by (6.10). Hence $\tilde{f}(\theta) > \tilde{f}(\theta^*)$ for $\theta \in S_3(\theta^*) \cap N(\theta^*), \theta \neq \theta^*$, where $N(\theta^*)$ is some neighborhood of $\theta^*$, by continuity of $\tilde{f}(\theta)$ in a region of stability.

**6.5 Example.** *Example 3.11 also shows that one may have $\{M_1(\theta^*) \cup V_2(\theta^*)\} \subset S_3(\theta^*)$ with a strict inclusion.*

We find

$$M_1(\theta^*) = \{\theta : F(\theta) \subset F^=(\theta^*)\} \cap M(\theta^*) = [-1, 0]$$

and

$$V_2(\theta^*) = \{\theta : F^=(\theta^*) = F^=(\theta)\} \cap R_4(\theta^*) = \{\theta^*\}$$

while

$$S_3(\theta^*) = \{\theta : F(\theta) \subset F^=(\theta^*)\} \cap S(\theta^*) = [-1, 1]$$

for $S(\theta^*) = H(\theta^*)$.

## 6.3 A Marginal Value Formula

The marginal value formula ( i.e., the path derivative of the optimal value function) was first proven in the literature in the absence of Slater's condition on the region of stability $V_3(\theta^\star)$ (See [43,45,49].). Here we extend its validity to the five different regions:

$$Z_2(\theta^\star) = \{\theta : F(\theta) \subset F^=(\theta^\star)\} \cap Z_1(\theta^\star)$$

$$Z_3(\theta^\star) = \{\theta : F(\theta) \subset F^=(\theta^\star) \subset F^=(\theta)\} \cap R_3(\theta^\star)$$

$$R_5(\theta^\star) = \{\theta : F(\theta) \subset F(\theta^\star)\} \cap R_3(\theta^\star)$$

$$R_6(\theta^\star) = \{\theta : F(\theta) \subset F^=(\theta^\star)\} \cap R_1(\theta^\star)$$

and

$$H_2(\theta^\star) = \{\theta : F(\theta) \subset F(\theta^\star)\} \cap H(\theta^\star)$$

Note that $V_3(\theta^\star) \subset \{Z_2(\theta^\star) \cap Z_3(\theta^\star)\}$ but, generally, $V_3(\theta^\star)$ is different from $R_5(\theta^\star), R_6(\theta^\star)$ and $H_2(\theta^\star)$. However, some extra assumptions are needed for the extension : The point-to-set mapping $F^=$ willl always be assumed lower semicontinuous at $\theta^\star$; also, for $R_5(\theta^\star)$, we have to assume that int $F(\theta^\star) \neq \emptyset$ (in which case $R_3(\theta^\star) \subset R_4(\theta^\star)$ is indeed a region of stability ) also, for $R_5(\theta^\star)$ and $H_2(\theta^\star)$, we have to assume that $F(\theta^\star) = F^=(\theta^\star)$ ( in order to apply Theorem 4.3 ).

Two crucial arguments used, in deriving the marginal value formula, are :

(i) Continuity of the restricted Lagrange multiplier function $\Phi(\theta)$ and

(ii) Showing that $x = \tilde{x}(\theta^k) \in F(\theta^k)$ implies $x \in F^=(\theta^\star)$. The latter argument is obviously valid for the above five regions , so we verify validity only of the continuity argument : The argument is valid for $Z_2(\theta^\star)$ because

$$Z_2(\theta^\star) \subset Z_1(\theta^\star) \subset R_2(\theta^\star)$$

and $\Phi(\theta)$ is continuous on $R_2(\theta^\star)$, under the lower semicontinuity assumption on $F^=$. (See [37, Theorem 2.1].) Since $R_5(\theta^\star) \subset R_3(\theta^\star)$, and the latter is a region of stability, if $F^=$ is lower semicontinuous and if $F(\theta^*)$ has interior points, it follows that $R_5(\theta^*)$ itself is a region of stability. The additional assumption $F(\theta^*) = F^=(\theta^*)$ on $R_5(\theta^*)$ and $H_2(\theta^*)$ guarantees continuity of $\Phi(\theta)$, by our Theorem 5.3. The region $Z_3(\theta^*)$ enjoys continuity because $Z_3(\theta^*) \subset V_1(\theta^*)$ and continuity is established on $V_1(\theta^*)$ in [36, Theorem 3.1]. Note that $F^=$ is lower semicontinuous on $Z_3(\theta^*)$, because of the requirement $F^=(\theta^*) \subset F^=(\theta)$. Finally, $R_6(\theta^*) \subset R_1(\theta^*) \subset R_2(\theta^*)$ and continuity, being established on $R_2(\theta^*)$, guarantees continuity of $R_6(\theta^*)$, provided, of course, that $F^=$ is lower semicontinuous.

To simplify notation, we use again the abbreviation $g(\theta)$ for the Langrangian $L_*^<$ at $x = \tilde{x}(\theta^*)$ and $u = \tilde{u}(\theta^*)$.

**6.6 Theorem.** *Consider the bi-convex model $(P, \theta)$ at $\theta = \theta^* \in I$ with a realistic objective function. Suppose that the corresponding saddle point $\{\tilde{x}(\theta^*), \tilde{u}_i(\theta^*) : i \in \varphi^<(\theta^*)\}$ is unique and that the index condition (IND) holds at $\theta^*$ with respect to a region of stability $S(\theta^*) = Z_i(\theta^*), i = 2,3$ or $S(\theta^*) = R_i(\theta^*), i = 5,6$ or $S(\theta^*) = H_2(\theta^*)$ and it is assumed that the mapping $F^=$ is lower semicontinuous at $\theta^*$, and in the case of $R_5(\theta^*)$, that $F(\theta^*)$ has an interior point, and that $F(\theta^*) = F^=(\theta^*)$ on $R_5(\theta^*)$ and $H_2(\theta^*)$. Also, suppose that $f^i(x,\cdot), i \in \{0\} \cup \varphi^<(\theta^*)$ are differentiable functions in $S(\theta^*) \cap N(\theta^*)$, where $N(\theta^*)$ is some neighborhood of $\theta^*$, and that the derivatives $\nabla_\theta f^i(x,\theta)|_{\theta=\theta^*}, i \in \{0\} \cup \varphi^<(\theta^*)$ are continuous functions in x at $\tilde{x}(\theta^*)$. Then for every fixed path $\theta^k \in S(\theta^*), \theta^k \to \theta^*$ such that the limit*

$$\lim_{\substack{\theta^k \in S(\theta^*) \\ \theta^k \to \theta^*}} \frac{\theta^k - \theta^*}{||\theta^k - \theta^*||} = l \tag{6.12}$$

*exists, we have*

$$\lim_{\substack{\theta^k \in S(\theta^*) \\ \theta^k \to \theta^*}} \frac{\tilde{f}(\theta^k) - \tilde{f}(\theta^*)}{||\theta^k - \theta^*||} = (\nabla g(\theta^*), l) \tag{6.13}$$

*Proof* Take a sequence $\theta^k \in S(\theta^*), \theta^k \to \theta^*$ such that the limit $l$ in (6.12) exists. Without loss of generality, we can assume that for this sequence

$$u_i(\theta^k) \to \tilde{u}_i(\theta^*), \quad i \in \varphi^<(\theta^*)$$

by the continuity argument given prior to the theorem. We now follow the proof of Theorem 6.1 and arrive at the inequality

$$(\nabla g(\theta^k), \theta^k - \theta^\star) \geq \tilde{f}(\theta^k) - \tilde{f}(\theta^\star) + E(\theta^k) \tag{6.4}$$

for all sufficiently large $k's$ . This gives an upper bound for $\tilde{f}(\theta^k) - \tilde{f}(\theta^\star)$. A lower bound is obtained from the estimate

$$\tilde{f}(\theta^k) - \tilde{f}(\theta^\star) \geq L^{<}(\tilde{x}(\theta^k), u; \theta^k) - L^{<}_{\star}(x, \tilde{u}(\theta^\star); \theta^\star) \tag{6.14}$$

valid for every $x \in F^{=}(\theta^\star)$ and every $u \in R_+^{q(\theta)}$, where $q(\theta) = card\ \varphi^{<}(\theta)$. (See [43].) Here we can specify $x = \tilde{x}(\theta^k) \in F(\theta^k) \subset F^{=}(\theta^\star)$, by the second argument given prior to the theorem, and

$$u_i = \begin{cases} \tilde{u}_i(\theta^\star), & \text{if } i \in \varphi^{<}(\theta^\star) \\ 0, & \text{if } i \in \varphi^{<}(\theta^k) \setminus \varphi^{<}(\theta^\star). \end{cases}$$

Now the lower bound

$$\tilde{f}(\theta^k) - \tilde{f}(\theta^\star) \geq (\nabla_\theta L^{<}_{\star}(\tilde{x}(\theta^k), \tilde{u}(\theta^\star); \theta)|_{\theta=\theta^\star}, \theta^k - \theta^\star)$$

follows from (6.15), convexity of $g(\theta)$, and the gradient inequality applied to $g(\theta)$ at $\theta^\star$. Therefore, after division by $||\theta - \theta^\star||$,

$$\begin{aligned}(\nabla g(\theta^k), \frac{\theta^k - \theta^\star}{||\theta^k - \theta^\star||}) &\geq \frac{\tilde{f}(\theta^k) - \tilde{f}(\theta^\star)}{||\theta^k - \theta^\star||} \\ &\geq (\nabla_\theta L^{<}_{\star}(\tilde{x}(\theta^k), \tilde{u}(\theta^\star); \theta)|_{\theta=\theta^\star}, \frac{\theta^k - \theta^\star}{||\theta^k - \theta^\star||})\end{aligned}$$

Hence, by continuity properties of both the Lagrangian multiplier functions and contraints, the marginal value formula (6.13) is obtained in the limit. ∎

Similarly to the necessary condition for optimality given in Section 6.1, we will derive a marginal value formula without the index condition.

**6.7 Theorem.** *Consider the bi-convex model $(P, \theta)$ at $\theta = \theta^* \in I$ with a realistic objective function. Suppose that the corresponding saddle point $\{\tilde{x}(\theta^*), \tilde{u}_i(\theta^*) : i \in \varphi^<(\theta^*)\}$ is unique and that condition (6.8) holds, where $S(\theta^*) = Z_i(\theta^*), i = 2, 3$ or $S(\theta^*) = R_i(\theta^*), i = 5, 6$ or $S(\theta^*) = H_2(\theta^*)$ and it is assumed that the mapping $F^=$ is lower semicontinuous at $\theta^*$, and in the case of $R_5(\theta^*)$, that $F(\theta^*)$ has interior points, and that $F(\theta^*) = F^=(\theta^*)$ on $R_5(\theta^*)$ and $H_2(\theta^*)$. Also, suppose that the derivatives $\nabla_\theta f^i(x, \theta)|_{\theta=\theta^*}, i \in \{0\} \cup \varphi^<(\theta^*)$ are continuous functions in $x$ at $\tilde{x}(\theta^*)$. Then for every fixed path $\theta^k \in S(\theta^*), \theta^k \to \theta^*$ such that the limit*

$$\lim_{\substack{\theta^k \in S(\theta^*) \\ \theta^k \to \theta^*}} \frac{\theta^k - \theta^*}{||\theta^k - \theta^*||} = l$$

*exists , we have*

$$\lim_{\substack{\theta^k \in S(\theta^*) \\ \theta^k \to \theta^*}} \frac{\tilde{f}(\theta^k) - \tilde{f}(\theta^*)}{||\theta^k - \theta^*||} = (\nabla g(\theta^*), l)$$

Proof Follow the same proof as given in Theorem 6.6 and arrive at the inequality

$$(\nabla g(\theta^k), \theta^k - \theta^*) \geq \tilde{f}(\theta^k) - \tilde{f}(\theta^*) + E(\theta^k)$$

for all sufficiently large k's. The point now is to show the non-negativity of the limit when $k \to \infty$ of $E(\theta^k)$. This was shown in Theorem 6.1 by using the index condition. It was also shown in Theorem 6.3 by using condition (6.8). The rest of the proof is exactly the same as given in Theorem 6.6. ∎

Note that the index condition is different from condition (6.8) ; therefore Theorem 6.7 is different from Theorem 6.6.

# CHAPTER 7

## Comparison of Regions of Stability
## and
## Selected Applications

### 7.1 Comparison of Regions of Stability

In order to compare all presently used regions of stability, we will first recollect them all here.

If the point-to-set mapping $F^{=} : \theta \to F^{=}(\theta)$ is lower semicontinuous, then we know that the following sets are regions of stability

$$R_1(\theta^\star) = \{\theta : \varphi^{=}(\theta^\star) = \varphi^{=}(\theta)\}$$
$$R_2(\theta^\star) = \{\theta : F_\star^{=}(\theta) = F^{=}(\theta)\}$$
$$R_6(\theta^\star) = \{\theta : F(\theta) \subset F^{=}(\theta^\star)\} \cap R_1(\theta^\star).$$

If $F^{=}$ is lower semicontinuous and int $F(\theta^\star) \neq \emptyset$ , then also

$$R_3(\theta^\star) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F^{=}(\theta^\star), i \in \varphi^{=}(\theta^\star) \setminus \varphi^{=}(\theta)\}$$
$$R_4(\theta^\star) = \{\theta : f^i(x,\theta) \leq 0, \forall x \in F(\theta^\star), i \in \varphi^{=}(\theta^\star) \setminus \varphi^{=}(\theta)\}$$
$$R_5(\theta^\star) = \{\theta : F(\theta) \subset F(\theta^\star)\} \cap R_3(\theta^\star).$$

are regions of stability.

The following sets are regions of stability without any assumptions

$$M(\theta^*) = \{\theta : F(\theta^*) \subset F(\theta)\}$$

$$M_1(\theta^*) = \{\theta : F(\theta) \subset F^=(\theta^*)\} \cap M(\theta^*)$$

$$W(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta)\} \cap R_1(\theta^*)$$

$$V(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta)\} \cap R_4(\theta^*)$$

$$V_1(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta)\} \cap R_3(\theta^*)$$

$$V_2(\theta^*) = \{\theta : F^=(\theta^*) = F^=(\theta)\} \cap R_4(\theta^*)$$

$$V_3(\theta^*) = \{\theta : F^=(\theta^*) = F^=(\theta)\} \cap R_3(\theta^*)$$

$$V_4(\theta^*) = \{\theta : F^=(\theta^*) \subset F^=(\theta)\} \cap R_2(\theta^*).$$

$$Z(\theta^*) = \{\theta : F(\theta^*) \subset F^=(\theta)\} \cap R_4(\theta^*)$$

$$Z_1(\theta^*) = \{\theta : F(\theta^*) \subset F^=(\theta)\} \cap R_2(\theta^*)$$

$$Z_2(\theta^*) = \{\theta : F(\theta) \subset F^=(\theta^*)\} \cap Z_1(\theta^*)$$

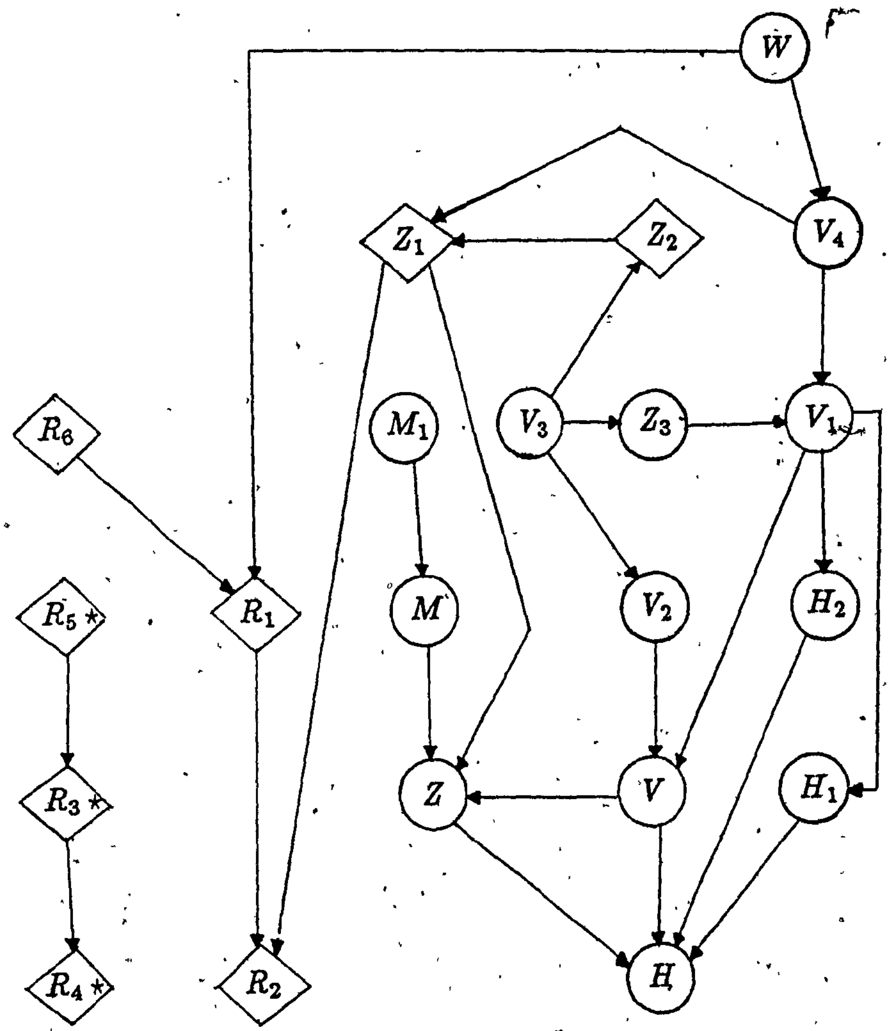
$$Z_3(\theta^*) = \{\theta : F(\theta) \subset F^=(\theta^*)\} \cap R_3(\theta^*)$$

$$H(\theta^*) = \{\theta : F(\theta^*) \subset F_*^=(\theta)\}$$

$$H_1(\theta^*) = \{\theta : F^=(\theta^*) \subset F_*^=(\theta)\}$$

$$H_2(\theta^*) = \{\theta : F(\theta) \subset F(\theta^*)\} \cap H(\theta^*).$$

All these presently used regions of stability are compared by inclusion below. The arrows mean inclusion. Thus $M_1 \to M$ means $M_1 \subset M$, etc. The sets in the " diamond shapes" are regions of stability under the additional assumption that the point-to-set mapping $F^= : \theta \to F^=(\theta)$ be lower semicontinuous at $\theta^*$. The sets with an asterisk are regions of stability if, in addition to lower semicontinuity of $F^=$, the feasible set $F(\theta^*)$ has nonempty interior. Some regions are generally incomparable, such as M and V. Among " smallest regions of stability are $R_0$ and $W$, while some of the " largest " ones are $R_2, R_4$ and $H$.



W
$Z_1$
$Z_2$
$V_4$
$R_6$
$M_1$
$V_3$
$Z_3$
$V_1$
$R_5$ *
$R_1$
$M$
$V_2$
$H_2$
$R_3$ *
$Z$
$V$
$H_1$
$R_4$ *
$R_2$
$H$

## 7.2 Applications of Region of Stability

Regions of stability often have simple economic interpretations.. For example consider the linear program

$$\max \ (c, x)$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

Assume that all the elements of A and b are positive. The region of stability $M(\theta^*)$ consists of all perturbation that increase the feasible set, i. e., local increases of $b_i$ and decreases of $a_{ij} > 0$. Economically, this may mean that purchasing extra units of energy (expansion) and improvement of the efficiency of the machines are " stable " processes in an economic development. More about this and related economic principle, the reader can find in [48].

Recently, stability of the efficiency tests in data envelopment analysis (DEA) was established in [8]. In this section, we will translate some of the results on stability region into the framework of DEA.

Data envelopment analysis was initiated by Charnes ,Copper, and Rhodes [9] and it has been successfully used in many situations with not-for-profit entities. (See ,e.g., [9, 25].). Recently, it was shown in [8] that the efficiency tests in DEA are indeed stable for every perturbation in the convex hulls of the input and output data. This suggests that the cause of possible discrepancies, between the estimated inefficiencies and the real-life situations, should be sought in deficiencies of the (analytically underscribable ) data generators and not necessarily in DEA.

The primal form of the Charnes-Cooper efficiency test can be formulated as follows

$$\max_{(x,y)} \ (x,Y)$$

s.t.

$$(y,X) = 1$$

$$(x,Y^i) \le (y,X^i), \quad i = 1,...,N$$

$$(x,Y) \le (y,X)$$

$$x \ge \epsilon e,$$

$$y \ge \epsilon e$$

for some unspecified small $\epsilon > 0$. Here e is the vector of ones. Vector $X^i \in R^m$ is considered as " input " ($X^i \ge 0$, but $X^i \ne 0$) and $Y^i \in R^s$ as " output " $(Y^i \ge 0), i = 1,..,N$. $(X^i, Y^i)$ are the empirical points. After substitution

$$X = \sum_{i=1}^{N} \theta_i X^i, \quad \sum_{i=1}^{N} \theta_i = 1, \quad \theta_i \ge 0, i = 1,...,N$$

and

$$Y = \sum_{i=1}^{N} \theta_i' Y^i, \quad \sum_{i=1}^{N} \theta_i' = 1, \quad \theta_i' \ge 0, i = 1,...,N.$$

The above program becomes the " bi-linear model "

$$\min \ f^o = -\sum_{i=1}^{N} \theta_i'(Y^i, x)$$

$$f^1 = \sum_{i=1}^{N} \theta_i(X^i, y) - 1 \le 0$$

$$f^2 = -\sum_{i=1}^{N} \theta_i(X^i, y) + 1 \le 0$$

$$(BL, \theta, \theta') \qquad f^{2+i} = (Y^i, x) - (X^i, y) \le 0, i = 1,...,N$$

$$f^{N+3} = \sum_{i=1}^{N} \theta_i'(Y^i, x) - \sum_{I=1}^{N} \theta_i(X^i, y) \le 0$$

$$f^{N+3+i} = -x_i + \epsilon \le 0, \quad i = 1,...,s$$

$$f^{N+3+S+i} = -y_i + \epsilon, \quad i = 1,...,m.$$

Hence the entire $(x,y) \in R^{m+s}$ is identified as an x in $(BL,\theta,\theta')$, $\theta$ and $\theta'$ are fixed and each restricted to the region

$$I = \{\theta \in R^N ; \sum_{i=1}^{N} \theta_i = 1, \quad \theta_i \geq 0, i = 1, ..., N\}.$$

It was shown in [8] that

$$\varphi^=(\theta,\theta') = \{1,2\}$$

for every $(\theta,\theta') \in I \times I$, the symbol " $\times$ " denotes the Cartesian product. and that the point-to-set mapping

$$F^= : (\theta,\theta') \rightarrow F^= (\theta,\theta')$$

is lower semicontinuous. Hence it follows that

$$R_1(\bar{\theta},\bar{\theta}') = I \times I,$$

is a region of stability at every $(\bar{\theta},\bar{\theta}') \in I \times I$ .

Corresponding to Theorem 6.1 of [48], also convergence of the restricted Lagrange multiplier functions $\Phi$ over the set $I \times I$ was established by [8]. We will now show that some of the important results in input optimization can be translated into the framework of DEA.

Since $\varphi^=(\theta,\theta') = \{1,2\}$, for every $(\theta,\theta') \in I \times I$, it follows that $F_*^=(\theta,\theta')$ $F^=(\theta,\theta')$. Also, a modified input constraint qualification holds at $\theta^*$ relative to $R_2(\theta^*)$, it follows that Theorem 4.7 is a special case of Theorem 4.8, when we state the corresponding results over the set $I \times I$. Moreover, the set of Lagrange multipliers $\Lambda(\theta)$ and $U(\theta)$ given in Section 5.1 coincides and the point-to-set mapping $\gamma_* : (\theta,\theta') \rightarrow F_*^=(\theta,\theta')$ is lower semicontinuous. Therefore the convergence of the other two Lagrange multiplier functions over the set $I \times I$ follows

**7.1 Theorem.** *Consider the model $(BL,\theta,\theta')$ at some $(\hat{\theta},\hat{\theta}') \in I \times I$ with a realistic objective function and an arbitrary sequence $(\theta,\theta') \in I \times I$, $(\theta,\theta') \rightarrow (\hat{\theta},\hat{\theta}')$. Then* *(i) The sequence of the corresponding Lagrange multipliers*

$$\Lambda(\theta,\theta') = U(\theta,\theta') = \{u_i(\theta,\theta') : i = 3, \ldots, N+3+s+m\}$$

*is bounded and*

*(ii) the set of all limit points of* $U(\theta,\theta')$, *as* $(\theta,\theta') \to (\bar{\theta},\bar{\theta}')$, *is non-empty and is contained in* $U(\bar{\theta},\bar{\theta}')$. ∎

It is also easy to show that the neccssary conditions for both convex and bi-convex models can be stated over the region of stability

$$S(\bar{\theta},\bar{\theta}') = \{(\theta,\theta') : F(\bar{\theta},\bar{\theta}') \subset F^{=}(\theta,\theta')\} \cap I \times I,$$

while a sufficient condition for bi-convex model and the marginal value formula can be stated over the set

$$S(\bar{\theta},\bar{\theta}') = \{(\theta,\theta') : F(\theta,\theta') \subset F^{=}(\bar{\theta},\bar{\theta}')\} \cap I \times I$$

Since our regions of stability are independent of the objective function, the results obtained for single-objective input optimization can be extended to the multi-objective situation in a rather straightforward manner. (For details see [47].). Recently regions of stability were also used for random decision systems with complete connections. (See [42]). Other applications include linear modelling and goal programming. (See [10, 52].)

## Open Questions

In this thesis, we have studied the stability in finite dimensional space. Different stability questions in infinite dimensional spaces have been considered by many authors. (See, e.g. ; [15, 23, 24, 26, 39, 58].) Some regions of stability have been indentified in infinite dimensional spaces, under some hypotheses (See [39, 42].). Our region of stability $H(\theta^*)$ is one of the largest regions of stability presently known, but it is easy to construct examples in one dimension which show that the point-to-set mapping $F : \theta \to F(\theta)$ is lower semicontinuous for every $\theta$ while $H(\theta^*) \neq R$. Therefore, in some situations one may possibly construct a larger region of stability than $H(\theta^*)$ or the " largest " region of stability. Development of numerical methods for calculating regions of stability and in input optimization is a topic suggested for future research.

All our results are stated for finite-dimensional spaces and for convex models, the extension to abstract optimization (e.g., semi-infinite and infinite programming) or to generalized ~~convex~~ and nonconvex models are directions for future research. Also, possible connections between input optimization and other optimization areas (e.g., integer programming models, optimal control theory, nonsmooth analysis ) remain open for research.

# APPENDIX

## Survey of New Results

Here all new results are listed

1. Theorem 3.5 : $H(\theta^*)$ is a region of stability.

2. Theorem 3.7: $R_4(\theta^*)$ is a region of stability, provided that $F^=$ is lower semicontinuous and $F(\theta^*)$ has non-empty interior.

3. Theorem 3.8: $R_2(\theta^*)$ is a region of stability under a weaker hypothesis.

4. THeorem 3.12: A necessary condition for stability is true for an arbitrary region of stability.

5. Theorem 4.3: ICQ is satisfied over $H_1(\theta^*)$.

6. Theorem 4.5: An optimality condition is true over $H(\theta^*)$.

7. Theorem 5.2: The Lagrange multiplier function $\Phi(\theta)$ is continuous over an arbitrary region of stability, provided that $F(\theta^*) = F^=(\theta^*)$.

8. Theorem 5.6 : A neccary condition for differentiable function is true over $H(\theta^*)$.

9. Theorem 5.7: Another necessary condition for differentiable function is true over $H(\theta^*)$.

10. Theorem 6.3 : A necessary condition for bi-convex model is true without the index condition.

11. Theorem 6.4: A sufficient condition for bi-convex model is true over $S_3(\theta^*)$.

12. Theorem 6.6: The marginal value formula is valid over five different regions of stability.

13. Theorem 6.7: The marginal value formula is valid without the index condition.

14. Theorem 7.1: Convergence of two Lagrange multiplier functions over the set $I \times I$.

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