SET THEORY IN HIGH SCHOOL:

AN ERROR IN TIMING?

Brian Patrick Kirlin

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ABSTRACT

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This thesis deals with the effect of set theory on the solution of simple equations. The research concerns itself with the real situation within the classroom. Tests were designed and administered to high school students from grades seven through ten. The results were analyzed and they seem to indicate that set theory should not be introduced in high school.

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My sincerest appreciation.

Brian Kirlin

MATHEMATICAL SYMBOLS

$\{\ \}$	set
$\{\ \mid\ \}$	set-builder notation
$\in$	is an element of; belongs to
$\subset$	is a proper subset of
$\cup$	union
$\cap$	intersection
$\emptyset$	empty set

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HOW SET THEORY BECAME PART OF THE CURRICULUM

Instruction in school mathematics from 1900 to the present time may be divided into two distinct periods. From 1900 to the mid 1950's, we have the era of the traditional mathematics. The period from then until now can be described as the era of the new math or modern mathematics.

Starting in 1890, various committees and commissions were prominent in making recommendations for changes in the mathematics curriculum and in methods of teaching mathematics. By the end of the century the curriculum was set. It was set with a view of meeting some reasonably uniform standards. During this first period the aim of the elementary school mathematics was to teach children the skills that would enable them to solve problems of everyday life. Arithmetic was thought of as a tool subject. Speed and accuracy in computation were the chief goals for arithmetic. Psychological experimentation resulting in the advent of intelligence tests at the beginning of the century further affected schools and mathematics teaching. Standard achievement tests were administered in order to determine how many correct answers a child could obtain in a given time.

Norms were set to classify the children according to this ability. Children who did not do well on these tests were given more drill exercises in order to bring them up to the "norm".

The most important committee report of the first half of the twentieth century was the 1923 Report[1] published by the National Committee on Mathematical Requirements. It established three aims for the teaching of mathematics:

1. the practical aims for cultivation of an understanding of the fundamental concepts and processes of mathematics;
2. the disciplinary aims for the development of the power to think logically; and
3. the cultural aims for the acquisition of an appreciation of mathematics.

The curriculum remained basically the same until the mid 1950's when a growing dissatisfaction began to emerge. This dissatisfaction was brought on by the fact that students in general were not doing well in mathematics. Defects in the curriculum were blamed. There were several.[2]

[1] DAVIS, D.R. The Teaching of Mathematics, p.2

[2] KLINE, Morris. Why Johnny Can't Add, p.17

First, there was the problem of deciding to whom we would gear our teaching; do we teach to the college-bound student or to the student who needs a practical knowledge of arithmetic.

Second, there was the problem of the disjointed presentation of mathematics; even in arithmetic, for example, some students could not see the connection between decimals and fractions.

Third, there was the problem of dating; many topics were outmoded. Topics which had been significant for generations no longer carried the same significance.

Fourth, there was a problem of motivation and appeal.

Fifth, there was the problem of reliance on memorization.

Clearly, reform was necessary. And with the launching in 1957 of Sputnik, even the politicans were convinced. As it turned out, the reform offered as much a new approach to the traditional curriculum as it did new contents. It is interesting to note that the impetus to change the mathematics curriculum came from above, from the university mathematics teachers. They had complained that high school students were not adequately prepared to learn university mathematics. The basis of their complaint was that the high school mathematics teachers were unaware of the tremendous changes in mathematics that had taken place in the last fifty years, They continued to teach the same mathematics that they had been taught. On the university level, there was a tremendous development in mathematics. On the high school level, the mathematics program remained static. The University of Illinois became the pioneer in the attempt to revise the high school curriculum. Vaughn, Beberman and Henderson were responsible for the early UICSM (University of Illinois Committee on School Mathematics) program which began in 1952, five years before Sputnik. Other projects soon followed the lead of the University of Illinois. There was the Commission on Mathematics set up by the College Entrance Examination Board, the Secondary School Curriculum Committee founded by the National Council of Teachers of Mathematics, and the Secondary School Mathematics Curriculum Improvement Study organized by Professor Howard Fehr of Columbia University.

Since the reform offered as much a new approach to the traditional curriculum as it did new contents, it was felt the contents should contribute to the goals of elementary and secondary school education; and the approach to the material should make the content inviting and aid comprehension as far as possible.[3] As history will corroborate, curriculum reform received priority.

SMSG (School Mathematics Study Group) became the largest project to revise the high school curriculum. A team of writers prepared books for the ninth, tenth and eleventh years of the high school curriculum. It soon became evident that, just as previously the high school graduates were not ready for university mathematics, the ninth grade students were unable to cope with the new mathematics. It became necessary to prepare the students at an earlier age for the high school mathematics. The result was a new junior high school mathematics program. But the seventh grade students were not prepared for the new junior mathematics program. SMSG then assembled a team of writers to prepare a program for the fourth, fifth and sixth grades. This was soon followed by a series of texts for kindergarten,

3 KLINE, p. 27

first, second and third grades. The final outcome was a mathematics program for kindergarten through twelfth grade.

The end result of this reform was a new mathematics program that discarded several topics and added new ones. Among the new ones, we find set theory. To quote Rene Thom: "Set theory is the essential litany intoned by those who advocate the so-called modern mathematics".[4]

[4] THOM, Rene. "Modern Mathematics: An Educational and Philosophic Error?" American Scientist, p. 698

WHAT IS SET THEORY?

Set theory is associated closely with the name of George Cantor. He was not the first mathematician to consider such ideas but it was he who developed them far more extensively than anyone else. He produced a subject with a character all of its own. It was chiefly under the stimulus of his work that the fundamental role for set theory was propounded in research mathematics and ultimately in mathematical education.

However, Cantor's naive set theory developed logical difficulties and paradoxes appeared. Because of this, an axiomatic approach to set theory was adopted. This was more in line with ideas developed several years before Cantor by George Boole. His view was that mathematics and logic should be associated. Boole established a new algebra known as Boolean Algebra or the Algebra of Sets. Notations have changed somewhat since Boole's day, but the fundamental principles are those that were laid down by Boole more than a century ago.

Boolean set theory describes the following ideas: set, subset, empty set, universal set, operations on sets including the logical product (intersection) and the logical sum (union). Boole's work was continued by several people including DeMorgan who discovered the law of duality - for every proposition involving logical addition and multiplication, there is a corresponding proposition in which the words addition and multiplication are interchanged. In particular, we have DeMorgan's formula for complementation which will be described later. The elementary and high school curricula use Boolean set theory and DeMorgan's laws.

The basic notion of set theory is the concept of set. This basic concept is, in turn, a product of historical evolution. Originally, the theory of sets made use of an intuitive concept of set, characteristic of the so-called "naive" set theory. At that time, the word "set" had the same imprecisely defined meaning as in everyday language. Such, in particular, was the concept of set held by Cantor.

"A set is announced at the beginning as 'any collection M of definitely well-distinguished objects m of our thought or our intuition...into a whole' "[5]

[5] GRATTAN-GUINNESS. "Not From Nowehere, History Behind Mathematical Education," int. J. Math. Educ. Sci. Technol, p 426

Such a view became untenable, as in certain cases the intuitive concept proved to be unreliable. To overcome the vagueness of intuition associated with the concept of set in certain more complicated cases, school set theory was presented in two ways: axiomatically and with Venn diagrams, two extremes to say the least.[6] Under the simplest axiomatization, the primitive notions of set theory are "set" and the relation "to be an element of".

Freudenthal has examined several textbooks written by people who claim little children can learn set theory. He feels that most of these texts probably could not be understood by university mathematicians. However, some texts are better than others. One text entitled "Algebraic and Circular Functions" in current use (at the grade 11 level) in the Protestant School Board of Greater Montreal is a typical one; it describes sets in this way: "The term set is regarded as one of the basic undefined ideas of mathematics. We find it useful to establish an intuitive idea of what we mean by a set by using what we might call a "pseudo-definition".

> "A set is a collection of objects from a specified universe.
>
> A set is well-defined if each member of the universe is either in the set or not in the set.

[6] FREUDENTHAL, HANS. __Mathematics as an Educational Task__, p 334

The complement of a set A is that set of elements in $\mathcal{U}$ which are not members of A.

On these sets we define the following operations:

Union: The union of sets A and B, symbolized $A \cup B$, is the set of all elements that are members of either A or B.

Intersection: The intersection of sets A and B, symbolized by $A \cap B$, is the set of all elements that are members of A and B.

Complementation: i. The complement of a union is the intersection of the complements.. Symbolically: $(A \cup B)' = A' \cap B'$.

ii. The complement of an intersection is the union of the complements. Symbolically: $(A \cap B)' = A' \cup B'$. "

(The previous heading is a statement of DeMorgan's laws.) The text then gives the following as the basic axioms of an algebra of sets. (note: for the pseudo-definition of set, these would be theorems.)

1. a. $A \cap B = B \cap A$ b. $A \cup B = B \cup A$
2. a. $(A \cap B) \cap C = A \cap (B \cap C)$ b. $(A \cup B) \cup C = A \cup (B \cup C)$
3. a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. a. $A \cap \emptyset = \emptyset$ b. $A \cup \emptyset = A$
5. a. $A \cap U = A$ b. $A \cup U = U$
6. a. $A \cap A' = \emptyset$ b. $A \cup A' = U$
7. a. $A \cap A = A$ b. $A \cup A = A$
8. a. $A \cap (A \cup B) = A$ b. $A \cup (A \cap B) = A$
9. a. $(A \cap B)' = A' \cup B'$ b. $(A \cup B)' = A' \cap B'$
10. a. $U' = \emptyset$ b. $\emptyset' = U$
11. $(A')' = A$
12. $A \subseteq B$ if and only if $A \cap B = A$ and
$A \subseteq B$ if and only if $A \cup B = B$.

A Venn diagram is the picture of the interior of a simple closed plane curve. Today's use of these diagrams differs from their original design. Certain theoretical difficulties exist such as how one could represent one set being an element of another or the properties of a set being finite or infinite.[7] Therefore, many textbooks contain a wording such as the following: "Venn diagrams do not prove the truth of a relationship between sets; they only illustrate its plausibility".[8]

[7] FREUDENTHAL, p 342

[8] PATERSON, CLARK, et al. Algebraic and Circular Functions, p 6

WHY TEACH SET THEORY?

The importance of the theory of sets in all present-day mathematics is due to the fact that, as experience has shown, practically every mathematical concept can be defined conveniently in terms of set-theoretical concepts; and every mathematical sentence can conveniently be made into a set-theoretical sentence.

Robert E. K. Rourke presented a paper in 1957 entitled "Some Implications of 20th Century Mathematics for High Schools", at the annual general meeting of the National Council of Teachers of Mathematics. According to Rourke, to create a new high school curriculum, one does not have to discard all the traditional topics but rather enliven them with a new point of view. He presented what he believed to be five sound reasons for using certain ideas of modern mathematics in building a high school curriculum.

"1. TO CLARIFY. We have a lot of fuzz in our texts and in our teaching. At many points, modern thinking can help us to sharpen the ideas and bring them into better focus.

2. TO SIMPLIFY. We need to delete many

useless and vague terms. Modern notions can help us trim off a lot of the fat.

3. TO UNIFY. We have long talked about the need for integration and integrating concepts. Where can we find a better integrator than the concept of a set? Sets give a new unity to the study of equations, inequalities, relations, functions, sample spaces and so forth.

4. TO BROADEN OLD IDEAS. We can gain and give new insights into many topics, such as, for example, graphs and loci, thus developing new interest and enthusiasm.

5. TO INTRODUCE IMPORTANT NEW IDEAS. A great portion of the modern vocabulary and many modern concepts are quite within the range of abilities of high school students. Often these notions are easier than traditional material. Here is an

> opportunity to help our able students prepare for college more effectively."[9]

As for the role sets should assume in high school algebra, Rourke feels that there are certain definite advantages in the use of sets in high school algebra.

First, it enables us to approach equalities and inequalities together. The neglect of inequalities has been unfortunate in our teaching; some of our best students are handicapped in the calculus because their introduction to inequalities has been too little, too late. (One should note that Rourke was talking about college-bound students.) Moreover, in some important modern applications of mathematics (linear programming, for example) inequalities are more important than equalities.

Second, by identifying the points corresponding to the solution set of a sentence as the graph of the sentence, we broaden

9 ROURKE, R.E.K. "Some Implications of 20th Century Mathematics For High Schools," The Mathematics Teacher, p 74-75.

and clarify the concept of the graph.
Third, before becoming immersed in the manipulative aspects of solving equations and inequalities, we gain insight into the meaning of the process.
Fourth, we gain some experience in the language of sets.

An article by R. L. Wilder goes even further. The real power of the theory of sets, and the circumstances that gave it ultimately its key position in modern mathematics, may be found in the extension of Cantor's researches into the real number continuum. During the mathematics revolution, set theory was applied to the teaching of fundamental operations. It was chiefly to elucidate the nature of number that the teacher used the set concept in modern systems of teaching arithmetic. For, by consideration of sets and operations with them (union, intersection, etc.) one could arrive at a much better intuitive understanding of the nature of a number and operations with them.[10]

10 WILDER, R.L. "Development of Modern Mathematics," Historical Topics for the Mathematics Classroom, p 466.

WHAT IS ACTUALLY TAUGHT?

What parts of set theory actually appear on the curriculum?

The following document has been produced by the Department of Education's Curriculum Service of the government of Quebec for use by teachers in the elementary school. In essence, it is an updating of the 1972 document entitled ELEMENTARY MATHEMATICS - LEVELS K-SIX GRADE GUIDELINES.

LEVEL K

SETS	Use terms: set, subset member, empty set. Sort and classify objects. Compare number of objects. Match numeral to number of a set, 0 to 10.
OPERATIONS and PROPERTIES	Separate sets to 10 Join sets to 10.

LEVEL 1

SETS	Use terms: set, subset, member, empty set.

Demonstrate ability to sort, match, classify and count.

LEVEL 2

SETS

Use terms: set, subset, member, empty set.

LEVEL 3

SETS

Use terms: set, subset, member, empty set.

Demonstrate the written use of { }

LEVEL 4

SETS

Use terms: set, subset, member, empty set.

Demonstrate the written use of { }

LEVEL 5

SETS

Use terms: set, subset, member, empty set.

Demonstrate solution sets for equalities and inequalities. Demonstrate the written use of { }

LEVEL 6

SETS	Use terms: set, subset, member, empty set, union, intersection. Demonstrate solution sets for equalities and inequalities. Demonstrate the written use of { }, $\cap$, $\cup$

According to a document entitled "MATHEMATICS IN SECONDARY I-II-III-IV REGULAR AND ENRICHED STREAMS" published in June 1972 by the Department of Education's Curriculum Service of the government of Quebec, the following parts of set theory must be taught in Grades 7 through 10.

SETS

Notion of set; membership	- examples of the use of the term "Set" in mathematics - $\in$, $\notin$

Description of a set	- description by listing; for example: $\{a,b,c,d\}$; $\{1,2,3,\ldots 9,10\}$; $\{1,2,3,\ldots\}$, $\{1,2,11,12,21,22,\ldots\}$ - cardinality of a finite set - description by set-builder notation, for example, $E=\{x \mid \underbrace{x+3=10}_{\text{characteristic (or defining) property}},\ \underbrace{x \in N}_{\text{domain of variable}}\}$
Intuitive notion of finite set and infinite set; empty set	- examples of finite sets: empty set, denoted by $\emptyset$ or $\{\ \}$, and other finite sets: for example, $\{x \mid x < 1{,}000{,}000,\ x \in N\}$ - examples of infinite sets: N, the set of odd numbers, the multiples of 5, $\{x \mid 1 < x < 2,\ x \in \mathbb{R}\}$, etc.

Universe, subset and complement; equality of sets		- complement of a set in a given universe.
		- Remark: There are several types of notations, but A' will be the one used in exams for complement of A.
		- $\emptyset'$ and U' (where U is the universe).
		- $(A')' = A$, whatever A is, in a given universe.
		- subset of a set; definition, notation $\subseteq$
		- find all the subsets of a given set (up to 4 elements)
		- equality of sets: $A=B$ iff $A \subseteq B$ and $B \subseteq A$.
	ENR	- proper subset: definition, notation $\subset$
	ENR	- number of subsets of a finite set.
	ENR	- applications to simple problems of listing outcomes and counting events.

Venn Diagrams		- representation of 1, 2 and 3 sets in Venn diagrams. - simple problems involving these diagrams.
Operations and properties		- definition of union and intersection (with the set-builder notation) - applications and exercises - Venn diagrams and their use to illustrate certain properties of the operations of intersection and union (commutative, associative, distributive) - roles of $\emptyset$ and U with respect to intersection and union in the universe U
	ENR	- DeMorgan's laws: $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.

A separate document entitled "MATHEMATICS IN SECONDARY V (REGULAR STREAM)" published in March 1972, again by the Education Curriculum Service of the government of Quebec sets out the following program for set theory in grade 11.

SETS

SETS	Remark: you might begin by discussing sets, Venn diagrams, and operations on sets. This would be followed by a discussion on statements which are related to set operations.
Concept of set and of membership in a set	"Set" is an undefined term. Correct usage of this term should be developed: - intuitively, by means of examples - more formally, by a discussion of "well-defined" sets: for a given object x, exactly one of the following statements is true: $x \in A$, $x \notin A$.

Universe	Pupils should be familiar with the concept of a universe (or domain) for a variable. Insofar as possible, the universe should be indicated whenever a variable is used.
Description of a set	Tabulation (listing) is used for finite sets and some infinite sets: e.g. $N = \{0,1,2,\ldots\}$. Set-builder notation (standard description) should be used as frequently as possible.
Number of elements in a set	Intuitive approach to the concepts of finite set and infinite set. The empty set is, by definition, a subset of every set. Pupils should be familiar

with the notation N(A) to denote the cardinality of a finite set A.

Operations on sets.	Intersection, union, and complement. A distinction should be made between the operations and the result of the operation.
Properties of set operations.	The cases of commutativity, associativity, and distributivity should be verified using Venn diagrams. Similarly, properties of taking the complement ($(A')'$, $(A \cap B)'$. $(A \cup B)'$) should be verified on Venn diagrams. The roles of $\emptyset$ and U with respect to union, intersection, and taking the complement should be stressed. Pupils should have experience

	with problems dealing with the cardinality of finite sets. These might include finding the numerical relationship between $n(A \cup B \cup C)$ and the cardinals of $A, B, C, A \cap B$, $A \cap C$, $B \cap C$, $A \cap B \cap C$.
Relations between sets.	Proper ($\subset$) and improper ($\subseteq$) subset. Equality of sets Equivalent sets: two sets are equivalent iff there exists a one-to-one correspondence between their elements. In the case of finite sets, this means the two sets have the same number of elements.

To what extent is this curriculum followed? Extensive consultation with teachers and mathematics supervisors has convinced me that within the Protestant School Board of Greater Montreal, it is followed closely.

CRITICISM

During the past twenty years there has existed a sharp division of opinion regarding the new mathematics curricula. Attitudes have ranged from strong commitment to sweeping condemnation. We have heard from several people who heartily endorse the addition of set theory to the elementary and secondary school curriculum. But just as many feel that it was a mistake. They have directed much criticism upon the new math and, in particular, upon set theory.

One of the most vocal critics of the new math from its very inception has been Morris Kline. At meetings of the National Council of Teachers of Mathematics, he attracted large audiences of frustrated teachers who were encouraged by the criticisms of someone with Kline's national reputation.

According to Kline, although the word"set" is useful, learning the associated concepts (union, intersection, subsets, etc.) is a sheer waste of time. Set theory plays a role in very advanced and sophisticated theories of mathematics, but in elementary mathematics it plays none. In fact, it is almost certain that set theory was introduced to give the new math the air of being sophisticated and advanced rather than because it is helpful. It happens to be one of the few topics of advanced mathematics that can be presented without requiring prohibitive background. Beyond using it artifically to define concepts, Kline feels that no significant use is made of the subject and that, in fact, only the vocabulary survives.

One of the defects of the traditional curriculum, according to the modern mathematics leaders, is its imprecise language. The looseness and ambiguities are supposedly so numerous and so deplorable that students are seriously handicapped. The new curriculum claims to eradicate these defects by introducing precise language.

To illustrate the inaccuracy of the traditional language, the modernists give the following illustration. "Peter has four balloons and Joe has five balloons. How many balloons do both have?" Almost everyone would understand the language to mean "what is the sum of the number of balloons Peter has and the number Joe has?" and would answer nine. Not so, say the modernists. Both boys do not have any balloons, and they mean, of course, that they have no balloons in common.[11]

11 KLINE, p 74

It is claimed that precision of language is insured by using the language of sets. By using the concept of set, one can rephrase and presumably make precise many mathematical statements.

Kline feels that to secure precision, the modernists have replaced many definitions in the traditional texts with their own versions. They carefully define every concept that is used, thereby creating an immense amount of terminology. Much of the terminology is abstract and much is totally unnecessary. As an example, we place labels on things that need not be labelled, for example, binary operation. Or we replace older terminology with newer terminology to no particular advantage, for example, $x + 2 = 0$ had been called an equation; now it is an open sentence, open because the value of x is not specified.

Further light is shed on the problems of terminology by René Thom in an article entitled "Modern Mathematics: An Educational or Philosophic Error?" Some people affirm that the use of set theory permits an entire renovation of mathematics teaching and that, because of this, the average student will be able to achieve mastery of the curriculum. Thom believes that this is pure illusion. As long as it is a matter of handling the obvious facts of naive set theory, anyone can get by. But this is neither mathematics nor even logic. In the early years of secondary school, students should learn the use of the symbols $\in, \cap, \cup, \subset$; later, they should be introduced to the quantifiers, and that should be the end of it.[12]

Many critics have charged that the new mathematics texts are nothing more than dictionaries. The excessive terminology has been criticized by R.P. Feynman, professor of physics at the California Institute of Technology. He served as a member of the California State Curriculum Commission when it examined texts in use in California schools. In an article "New Textbooks for the New Mathematics", he attacked the excessive use of terminology. He also criticized the precision sought by using set language. He mimics the precision by pointing out "a zookkeeper, instructing his assistant to take the sick lizards out of the cage, could say, 'take that set of animals which is the intersection of the set of lizards with the set of sick animals out of the cage.' This language is correct,precise, set theoretical language, but it says no more than

[12] THOM, p 699

'take the sick lizards out of the cage.'..."
Further, in his article he states that "...the real problem in speech is not precise language. The problem is clear language".

Freudenthal feels that mathematics has been struck by another evil: the mathematics the pupil has to learn after set theory is determined, both as regards contents and didactics, by set theory rather than the other way around. This is especially awkward if set theory, as often happens, is taught in a way that afterwards makes any further mathematics teaching impossible. If only finite sets are considered, if sets are incessantly illustrated by (often wrong) Venn diagrams, if the teacher behaves as though letters do not mean anything but themselves or mean spots in a Venn diagram, then the way to understanding variables is blocked. This prevents pupils from learning that letters are used to indicate variables, and that, in particular, a letter in algebra can mean an integer, a rational number, or a real number. The student must learn that mathematics is not a game with letters.

Freudenthal believes that explanations of set theoretic concepts move between two extremes [13]

[13] FREUDENTHAL, p 334

which are approximately as follows: One is a formal definition of what a set is - the formal sham definition. Concepts such as "set", "element of..." mark a threshold you cannot transgress with fresh questions - it is the same with the concept of natural number though some people mistakenly believe that they can define it by pure set theory. This does not mean that set theory should be started by axioms, although there are school texts for 11-12 year-olds that introduce sets axiomatically. This is a pseudo-approach, not unlike the traditional approach to Euclidean geometry, with some trivialities worded axiomatically, while decisive suppositions are not even mentioned. Set theory, like any other field, cannot be conceived axiomatically unless you already know it, which in the case of set theory, includes knowing its traps and ambushes. Freudenthal believes that this is quite impossible at school. Having examined the axioms of the usual axiom systems of set theory, he could not discover one axiom for which he could explain what purpose it serves to someone with as little mathematics as a pupil of even the higher grades at high school.

If it is true that there is no way, explicitly or axiomatically, to tell what a set is, how can you start set theory? Freudenthal believes the answer is obvious - the same way numbers and geometrical shapes are dealt with. Just as little as you need to explain to people what a number is, as little as you need a diagram to explain what a point is, so is there just as little a need to define what a set is. One learns through doing and observing Presuming explicit definitions is a symptom of old fashioned methodology.

The other extreme, the use of Venn diagrams, is just as objectionable. Almost all textbooks are prone to make the readers believe that it is a set if a closed curve is drawn around a number of letters, figures or meaningless symbols.

On Venn diagrams René Thom was clear: "The fox knows that is the hens are in the hen-house and the hen-house is in the yard; then the hens are in the yard; he does not bother with set theory."

Freudenthal believes that sets are introduced more often because it is fashionable or because everybody does so, rather than after a conscious deliberation. He believes that sets should not be introduced until one has thought about the specific purposes sets should serve, that is, how they should be related to lived-through reality and to the mathematical content.[14]

[14] FREUDENTHAL, p 354

MY OPINION: FOR OR AGAINST

The opinions reported so far by no means form a comprehensive survey of the history of the debate on the role of set theory in school mathematics. There were many opinions favoring both sides of the argument; I have quoted only a few. Moreover, all of these have been "answered". However, I do not intend to pursue this matter further because I believe that after 20 years the theoretical debate on this issue has lost much of its former importance. Nevertheless, I will venture to express my own opinion based on my teaching experience.

Personally, I oppose the teaching of set theory to elementary and high school students. The proponents of set theory claimed that it unified, clarified and simplified mathematics in elementary and secondary education. I believe that school set theory has not done so, thus nullifying its "raison d'être".

Over the last several years I have become conscious of growing dissatisfaction on my part over the use of sets. I spend much time and effort teaching sets and set terminology and still end up with students making many errors in their use. Rather than succumb to a feeling of frustration I decided, after much thought, to research the effect of sets on a topic familiar to the students, "to see" if indeed, sets unified, clarified and simplified this topic. In particular, I decided to study the effect of set theory on the solution to simple equations. For example: most junior high school students, given the following algebraic problem: $3 + x = 5$, solve for x, would have little difficulty in telling you that $x = 2$. But, expressed in the following context: $\{x \mid 3 + x = 5, x \in \mathbb{R}\}$, how many would be able to give you the answer? Has the terminology used (and it is precise, set theoretical language) done anything to simplify the problem, or clarify it for that matter? Or has it changed a simple problem into a difficult one?

It is my opinion that the difference in terminology will affect the outcome. I have come to believe that the student, when presented with the latter form of the equation, either cannot write down his answer correctly or simply does not follow the instructions.

One must surely agree that the method of asking a question will have a direct hearing on the response. I will present two examples to explain what I mean. First, if a student had no knowledge of Roman numerals, one would not ask him to add three plus five by writing VIII. The odds of this student answering the question correctly are significantly reduced. The second is given in a work by R. E. Mayer entitled "Thinking and Problem Solving". Some research indicates that apparently even very minor differences in representation, can influence how a problem is assimilated. Mayer and Burke in 1967 investigated different ways of representing story problems and found that minor changes in wording had important effects. They presented the following problem.

THE HORSE PROBLEM

REPRESENTATION 1

A man bought a horse for $60 and sold it for $70. Then he bought it back again for $80 and sold it for $90. How much money did he make in the horse business?

a. lost $10
b. broke even
c. made $10
d. made $20
e. made $30

REPRESENTATION 2

A man bought a white horse for $60 and sold it for $70. Then he bought a black horse for $80 and sold it for $90. How much money did he make in the horse business?

a. lost $10
b. broke even
c. made $10
d. made $20
e. made $30

In this experiment, subjects performed quite poorly when given the first representation, getting the correct answer less than 40% of the time. When the second representation was tested, the solution rate was 100%. This example suggests that seemingly minor changes influence how a subject represents a problem and thus affect its solution. I believe that this is the case with set theory in elementary and high school.

To re-iterate, whatever the reformists had in mind, my concern is with the real situation, with what actually is happening (not what the teachers may have done to either create or compound the problem, not how textbooks may have influenced the problem -- these questions are important and are definitely worthy of investigation but not here). I am interested in whether or not pupils are able to use sets in solving equations when they are requested to do so - a reasonable request when one considers that students have been using the notation since grade five and my first tests were administered in grade seven.

METHOD

OBJECT

The object of this research is to test the effect of set theory on the solution of simple equations. The exact effect being tested was the student's ability to vary the form of his answer on the basis of given instructions; not his ability to solve the equations. This formulation of the question was suggested by the distinction between content and form. This distinction leads to the conclusion that sets enter into the solution of equations in two ways: firstly, in the ability to comprehend and use the set form in the communication of problems and of the obtained answers; secondly, in modifying the content of the solution (omitting roots) in accordance with the number system over which an equation has to be solved.[15] This line of reasoning leads to an investigation of the role of sets in the solution of equations by separating errors in content from errors of form.

DESIGN

It was decided that equations of a relatively simple nature would be used as the basis for the

15 BYERS, V. and HERSCOVICS, N. "Understanding School Mathematics", Mathematics Teaching #81., p 24-27

research. This topic was familiar to the students being tested and it is a topic which is studied in varying degrees of difficulty every year from grade seven on. To avoid incorrect content in as far as was possible, equations were selected which could be solved easily by the group being tested.

It was felt that a group experiment would be adequate. It was decided that an examination-type test would be administered. Answers were to be written down by the students and collected at the end of the test. Students were given as much time as they required.

Since form was the ultimate consideration, content has been given as minor a role as possible. Hence, the simple equations, most of which could be solved by inspection. Because of this, it was not expected that it would be possible to draw conclusions regarding the effect of sets on content. It would be interesting to study the effect of set terminology on learning to solve equations as the content is made more difficult. But this question is beyond the scope of this research.

Only one kind of question was to be tested but it was presented in three different forms. The first form described an equation without the use of sets. As an example we have: find the value of x if $x + 3 = 5$. The second form presented the equation verbally using the vocabulary of sets. As an example we have: find the solution set of $x + 3 = 5$, if the replacement set is the set of natural numbers. The third form used set-builder notation. As an example witness the following: $\{x \mid x + 3 = 5, x \in N\}$. The second and third forms required the solution using the proper set notation. The pupils who were tested had already encountered the various forms of these questions; all the terminology, both symbolic and verbal had been explained and was used regularly throughout the year. It was decided that the tests would be scored in terms of the answers that were actually given by those tested.

SUBJECTS

The subjects for this experiment were secondary school students from grades seven through ten. The grade seven students were from Mountrose School and the others were students of Rosemount High School. Both schools are under the jurisdiction of the

Protestant School Board of Greater Montreal.

It was decided to first test these ideas in a pilot experiment.

PILOT TEST

SUBJECTS AND CONTENT

There were thirty-two subjects, all classified as regular (average) grade 9 pupils. Their instruction in mathematics for the currect academic year consisted of three fifty-minute periods every two days.

In this experiment there were six questions. These could be divided into two groups of three. Within each group, each question was presented using a different form. The first form described an equation in set-builder notation. The second form described an equation using the vocabulary of sets, and finally, the third form described an equation without any use of sets. Please recall that the first and second forms have been a part of the curriculum since grade seven and therefore, a certain familiarity was expected.

Consideration was given to the order of presentation. Questions were not grouped according to the three forms just discussed. The first question did not use sets. It was so located because it was felt that (i) it was not too difficult and could probably be solved by every one, and (ii) it would not frighten the student and would encourage him to continue. No thought was given to the placement of questions two through six. The actual test follows:

PILOT TEST

PLEASE READ EACH QUESTION CAREFULLY AND ANSWER EXACTLY WHAT IS ASKED. SHOW WHATEVER WORK YOU THINK IS NECESSARY.

1. FIND x IF $4X + 3 = 7$ AND x IS A NATURAL NUMBER.

2. FIND THE SOLUTION SET OF $5x + 3 = 8$, IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.

3. FIND $\{x \mid 3x + 7 = 13, x \in N\}$.

4. FIND THE SOLUTION SET OF $4x - 3 = 2x + 5$, IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.

5. FIND $\{X \mid 6x + 4 = 3x - 11, x \in N\}$.

6. FIND x IF $9x - 3 = 6x + 9$ AND x IS A NATURAL NUMBER.

The correct answers to all questions follow. Alternative answers are presented in parenthesis.

1. $x = 1$ (1)
2. $\{x \mid x = 1, \ x \in N\}$ $(\{1\}$ or $x \in \{1\})$
3. $\{x \mid x = 2, \ x \in N\}$ $(\{2\}$ or $x \in \{2\})$
4. $\{x \mid x = 4, \ x \in N\}$ $(\{4\}$ or $x \in \{4\})$
5. $\{ \ \}$ $(\emptyset)$
6. $x = 4$ (4)

<u>RESULTS</u>

It was observed that there were few wrong roots but that a student tended to answer all questions in the same way regardless of the required form.

Inspection of the answers that were actually given indicated that they were of four types.

First, either the answer was left out completely or it was incorrect.

Second, using sets, x = {number}

Third, also using sets, x ∈ {number} or simply {number}

Fourth, x = number.

Please note, where sets were required only the third form is correct. In order to examine the answers more carefully, table 1 was constructed from the answers that actually appeared on the tests. Under each heading appears the number of times the answer was given.

Table 1 can be found following p 44.

DISCUSSION

One must surely have noticed that question 5 has been omitted from the table above. This was done deliberately. When the pilot was designed, it was strongly felt that the students' ability or inability to handle the empty set would be of enormous value to this study. It was the only question requiring a negative response. Of the thirty-two students tested, 8 gave a wrong root. Of the remaining group, 19 students left their answer as $x = -5$, 4 wrote $x = \{-5\}$ and 1 wrote $\{-5\}$. As one can see, not one student recognized that the answer was not a natural number, as required by the condition set down in the problem. It was clear that further study must be undertaken on this particular aspect of sets. This question will be discussed again shortly.

We shall now study the answers to the other five questions. If we examine the table of answers given and compare the number of omitted and wrong

Table 1

PILOT TEST: 32 Subjects

Question number	Type of response							
	Omitted, wrong		$x=\{number\}$		$x \in \{number\}$, or $\{number\}$		x=number	
	no.	%	no.	%	no.	%	no.	%
1	1	3	1	3	1	3	29*	91
2	3	9	9	28	7*	22	13	41
3	1	3	6	19	2*	6	23	72
4	4	12½	8	25	4*	12½	16	50
6	7	22	2	6	1	3	22*	69

*Correct answer

roots to all others, we see that a total of 16 out of a possible 160 answers (32 x 5) were omitted or wrong. This represents only a 10% error. This statistic certainly points out that the students were indeed capable of solving the equations, no matter what form was used to present them. This was expected.

But it is those answers which were <u>not</u> considered incorrect in content which are of interest here. For it is <u>not</u> content which was being tested, but form.

For questions 2, 3 and 4 which deal with sets in their various forms, only 8 answers were omitted or wrong of a possible 96, but there were 23 answers of the form $x = \{number\}$; this form is technically incorrect. This means that almost one out of four answers use an incorrect form of set notation. Further examination of the answers given revealed that 52 answers used no set notation whatsoever, for whatever the reason (either the student could not use sets, forgot how to use them, misread the instructions, etc.). This figure represents 54%. Thus almost four out of every five answers given, where sets were required, were wrong! (This excludes those which content-wise were incorrect). If one considers only the correct answers to questions 2, 3 and 4, one finds that the correct answer is given

13 times out of 96 or approximately 13 1/2% of the time.

In order to examine the results more thoroughly, a frequency curve was designed¹. The abscissa represents the type of answer given, while the ordinate represents the number of times that particular answer occurred. On the right of the graph, the frequency of response is given as a percentage. The plotted points were joined by straight lines to make the pattern more visible.

¹This graph is graph 1 and it can be found following p 46.

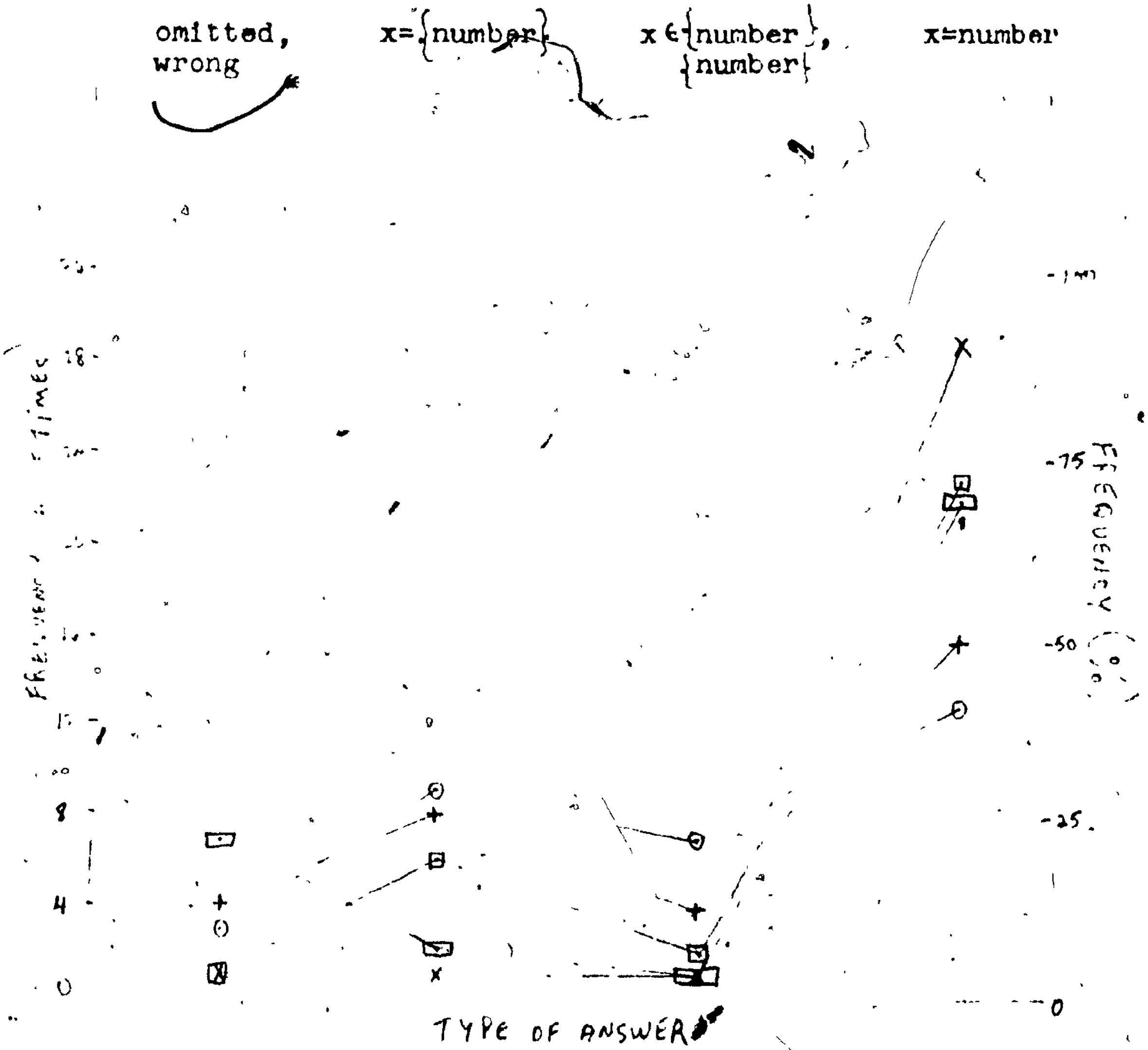
When referring to the graph, one is immediately drawn to the very obvious difference in the percentage of correct solution, between those questions requiring sets and those requiring none. Although this result was expected, such a remarkable distinction was not anticipated. Further, one notices that in those answers requiring sets, sets were seldom used correctly and in most cases, they were not used at all. This seems to indicate that the concept of set has not been mastered

Graph 1

PILOT EXPERIMENT

(Frequency curve)

Grade 9:32 Subjects



Legend:

X 1. Find x if $4x+3=7$ and x is a natural number.

O 2. Find the solution set of $5x+3=8$ if the replacement set is the set of natural numbers.

□ 3. Find $\{x \mid 3x+7=13, x \in N\}$.

+ 4. Find the solution set of $4x-3=2x+5$ if the replacement set is the set of natural numbers.

▭ 6. Find x if $9x-3=6x+9$ and x is a natural number.

at this level even though it was introduced many years previously. Finally, one notices that the questions which were presented in verbal form were answered correctly more often than those which were presented in set-builder notation. This suggests the importance of what is required being explicitly stated (ie. find the solution set).

A further examination of the answers led to the following:

5 students tried to use set notation throughout.

4 students consistently wrote x = {number} (This included one person who had a wrong answer).

14 students did not use sets at all (this included eight who omitted questions or got wrong answers). This represents almost 44% of those tested.

5 students dropped sets after question 2 (including one who wrote x = {number}).

2 students dropped sets after question 3 (including one who used x = {number} once and another who used {x = number} once.

The rest followed no apparent pattern.

CONCLUSIONS

I. The method used in the pilot (including the method of scoring) is adequate to exhibit the effects of set terminology and notation on the solution of simple equations. No statistical analysis seems to be required. * The question of the role of sets in the solution of more difficult equations remains open.

II. The use of equations whose roots could be found easily, effectively separates errors in the form of solution from those of content. An apparent exception is provided by an equation whose solution set is empty.

III. It was therefore decided to use the method of the pilot experiment in the main experiment, subject to the following modifications:

1. As a result of the inconclusive results obtained with the empty set, it was felt that further research was necessary.
2. The "aura effect" must be considered. One must accept the fact that the order of presentation could greatly influence the outcome . To test this possibility, a

* see section 15

subsidiary experiment was designed in which questions from the main experiment were reordered and a different group of subjects tested.

3. In the pilot, since linear equations were being tested, each response had only one root. Because a large percentage of students failed to use set notation, it was felt that perhaps they thought the notation was unnecessary for sets containing one element. Accordingly, a part of the main experiment was devoted to the study of how students would respond to questions requiring two elements in the solution set.

4. Since all students participating in this research to date have been students of average ability, an interesting question arose: how would brighter students handle equations presented in set terminology? A part of the main experiment was devoted to an examination of this question.

5. Finally, would there be any marked improvement in a group, if, after testing, this group was re-taught and tested again? An opportunity presented itself to follow up on this question.

MAIN EXPERIMENT

METHOD

All parts of the main experiment used the method of the pilot experiment. To repeat, the experiment is a group experiment using examination-type tests. Students were given as much time as they required. Answers were written down by the students and were collected at the end of the test for examination and classification according to form. The test was scored in terms of these answers.

The main experiment has been divided into five parts with the following objectives:

Part 1 to exhibit the effects of set terminology and notation on the solution of simple, linear equations.

Part 2 to study the effect of varying the order of presentation of the equations.

Part 3 to determine the effect of a solution set containing a doubleton as opposed to a singleton.

Part 4 to study the effect of ability grouping.

Part 5 to determine the effect of reteaching a group.

Each part of the experiment used the same method and the same type of question. Questions were designed to reflect three forms: 1. the first form described an equation in set-builder notation; 2. the second form described an equation using the vocabulary of sets; and 3. the third form described an equation without any use of sets.

The subjects changed from one part of the experiment to the next; the content of the questions changed when it became necessary to proceed from linear to quadratic equations.

CONTENT AND SUBJECTS

Part 1 In keeping with the main objective of this research which was that of testing form, all the equations presented to this group were linear of the general form! $ax+b=c$; their solutions could all be found by inspection. For reasons explained earlier, one question was presented in which the answer did not satisfy the condition set down in the problem. It was hoped that the student would recognize that it was the empty set which was required.

Three groups of students took part in this test. In one group there were forty-one grade seven students; the other two groups were made up of twenty-nine grade eight students and twenty-five grade nine students. All the students were classified as regular (average). All three groups were receiving mathematics instruction for one fifty-minute period every day.

Part 2 The questions used in this part of the experiment were the same as those used in the previous part. Only the order of presentation was changed. The subjects for this test were twenty-three regular grade nine pupils.

Part 3 Since the object of this experiment was to determine the effect of a solution set containing a doubleton, equations were required that would provide two answers. Thus, quadratics over I (ie. no roots had to be rejected) were selected as the basis for this test. Appropriate subjects were now needed. Grade ten students were selected because at this level quadratics are studied in great detail. There were thirty-nine regular students. They were taught mathematics for fifty minutes once a day throughout the year.

Part 4 The questions for this part were similar in design to those in part 3. Quadratic equations were presented which were easily factorable over I but this time it was required that they be solved over N. Two groups of students were tested. The first was a group of twenty regular grade ten students. The second was a group of twenty-four grade ten accelerated students. We pause to explain the difference between regular and accelerated: anyone who passes the previous level is placed into a 'regular' mathematics class. This would cover the majority of students and most would be of average ability. Those who pass with an average of 80% or better and have their teacher's recommendation are placed into an 'accelerated' math class. This class would cater to the 'cream of the math students' at a particular level.

Part 5 An opportunity presented itself to re-teach sets to the group of accelerated students.

Their teacher re-explained sets and set-builder notation and he explained why several of their answers were incorrect on the test. Approximately one month later, the group was re-tested using the same questions as on the original test. Unfortunately, the same opportunity did not materialize for the regular group because of the demands of the curriculum.

MAIN EXPERIMENT: Part 1 (form)

TEST

There were five questions on this test designed to reflect the three forms described earlier. The test follows.

TEST

PLEASE READ EACH QUESTION CAREFULLY AND ANSWER EXACTLY WHAT IS ASKED. SHOW WHATEVER WORK YOU THINK IS NECESSARY.

1. FIND x IF $3x + 8 = 11$.
2. FIND x IF $2x + 7 = 9$, AND x IS A NATURAL NUMBER.
3. FIND x IF $2x + 3 = 1$, AND x IS A NATURAL NUMBER.
4. FIND THE SOLUTION SET OF $4x + 3 = 7$ IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.
5. FIND $\{x \mid 5x + 3 = 8, x \in N\}$.

As a result of earlier discussion concerning the empty set, question 3 has been included to enable further study of this concept. It was not presented in set notation. All that was required here was for a student to recognize that the answer did not fit the condition of the problem. Several forms of the answer would be acceptable.

The correct answers to all questions follow. Alternative answers are presented in parenthesis.

1. $x = 1$ (1)
2. $x = 1$ (1)
3. There is no answer to this question ($\emptyset$, $\{\ \}$).
4. $\{x \mid x = 1,\ x \in N\}$ ($\{1\}$ or $x \in \{1\}$)
5. $\{x \mid x = 1,\ x \in N\}$ ($\{1\}$ or $x \in \{1\}$)

RESULTS

It was observed that the answers could again be classified into four distinct categories.

First, the question was not answered or it was answered incorrectly (ie. the answer given was not a root)

Second, using sets, $x = \{\text{number}\}$.

Third, also using sets, $x \in \{\text{number}\}$ or $\{\text{number}\}$

Fourth, $x =$ number

Where sets were required, only the third form is correct. The results for question 3 will be reported separately later.

Three tables were constructed from the data, one table for each grade level. These tables indicate the number of times each type of answer was given. The only difference between the following tables and the one in the pilot experiment is that this time the columns entitled ' $x \in \{\text{number}\}$ ' and ' $x = \{\text{number}\}$ ' have been interchanged. Tables 2, 3 and 4 for grades 7, 8 and 9 respectively, follow p 58.

DISCUSSION

As in the pilot experiment, there were very few wrong or omitted answers. In fact, there were fewer. Of a possible 380 answers (25 x 4 grade 9's, 29 x 4 grades 8's, 41 x 4 grade 7's), there were only 31 which fell into this category. This represents only 8%. On the pilot this figure was 10%. Furthermore, there were five students in grade 7 who answered every question incorrectly (as opposed to those who made an occasional error). It was obvious that they could not solve elementary linear equations. But it is not content which is being tested, is it? If the answers of these five students were excluded from the sample, the percentage of wrong and omitted roots would have dropped to 3%. It was decided, however, not to adopt this procedure.

Table 2

MAIN EXPERIMENT(part 1)

Grade 7:41 Subjects

Question number	Type of Response							
	Omitted, wrong		$x \in$ {number}, or {number}		x= {number}		x=number	
	no.	%	no.	%	no.	%	no.	%
1	6	15	0	0	0	0	35*	85
2	6	15	0	0	1	2	34*	83
4	7	17	0*	0	21	51	13	32
5	8	19½	0*	0	25	61	8	19½

* Correct answer

Table 3

MAIN EXPERIMENT(part 1)

Grade 8:29 Subjects

Question number	Type of Response							
	Omitted, wrong		x ∈{number}, or {number}		x={number}		x=number	
	no.	%	no.	%	no.	%	no.	%
1	0	0	1	3	2	7	26*	90
2	1	3	1	3	2	7	25*	86
4	1	3	5*	17	14	48	9	31
5	1	3	5*	17	12	41	11	38

*Correct answer.

Table 4

MAIN EXPERIMENT(part 1)

Grade 9:25 Subjects

Question number	Type of Response							
	Omitted, wrong		x ∈ {number}, or {number}		x = {number}		x=number	
	no.	%	no.	%	no.	%	no.	%
1	0	0	0	0	0	0	25*	100
2	1	4	0	0	2	8	22*	88
4	0	0	4*	16	10	40	11	44
5	0	0	4*	16	8	32	13	52

*Correct answer

At this point responses to questions 4 and 5 which required set notation, will be examined more closely. There are 190 possible answers for these two questions. It was found that there were 90 answers which fell into the category of incorrect form. This represents 47% of the possible answers. There were also 65 answers for which set notation was required but not used. This means 34% have no set notation. If one looks at these figures another way, one can see that 4 answers out of every 5 do not use sets where they are required or use them incorrectly. One must admit that this is astonishingly high, considering that these students have been using the set symbolism since grade 3. They are now in high school and they still appear to be having immense difficulty.

To see if there is any pattern developing from grade 7 through 9, table 5 looks at each grade separately. The table considers questions 4 and 5 together. The answers have been characterized as right or wrong, while the "wrong answer" column has been subdivided into three parts: (a) incorrect form, (b) no set notation where it was required and (c) other errors (ie. incorrect content, omission). Table 5 follows this page.

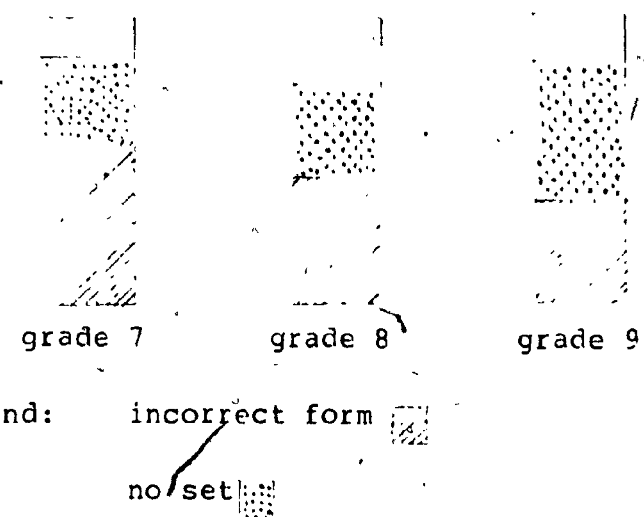
Table 5

Questions 4 and 5

Grade	Right		Wrong					
			Form		No Set		Other	
	no.	%	no.	%	no.	%	no.	%
7	0	0	46	56	21	26	15	18
8	10	18	26	45	20	34	2	3
9	8	16	18	36	24	48	0	0

The wrong answers demonstrate quite clearly a pattern which is illustrated by the following bar graph.

GRAPH 2



It would appear from these results that as the students pass from grade 7 to grade 9, the incorrect form of set notation appears less often, but at the same time, the set notation is dropped more and more.

To get an even closer look at what is happening, questions 4 and 5 were examined separately for each grade level. These are tables 6 and 7 and they can be found following p 60.

It will be observed that the pattern of response for both questions is identical. Furthermore, there appears to be very little difference between grade 8

Table 6

Question 4

Grade	Right		Wrong					
			Form		No Set		Other	
	no.	%	no.	%	no.	%	no.	%
7	0	0	21	51	13	32	7	17
8	5	17	14	48	9	31	1	3
9	4	16	10	40	11	44	0	0
Average		9.5		47.4		34.7		8.4

Note: In the 95 cases studied, there were no answers with wrong content and correct form.

Table 7

Question 5

Grade	Right		Wrong					
			Form		No Set		Other	
	no.	%	no.	%	no.	%	no.	%
7	0	0	25	61	8	19½	8	19½
8	5	17	12	41	11	38	1	3
9	4	16	8	32	13	52	0	0
Average		9.5		47.4		33.7		9.4

Note: In the 95 cases studied, there were no answers with wrong content and correct form.

and grade 9 overall. Both questions follow a similar pattern for incorrect form - a decrease in incorrect usage as a student graduates to the next higher grade. But one should still notice that even in grade 9, roughly one-third of the answers are still given with incorrect notation.

If we assume, for the sake of argument, that the students have been taught the incorrect form (which may definitely have been the case in grade 7), then concentrating our attention on the other wrong answers, we see that there are still one-third of the answers having no set notation when there should be.

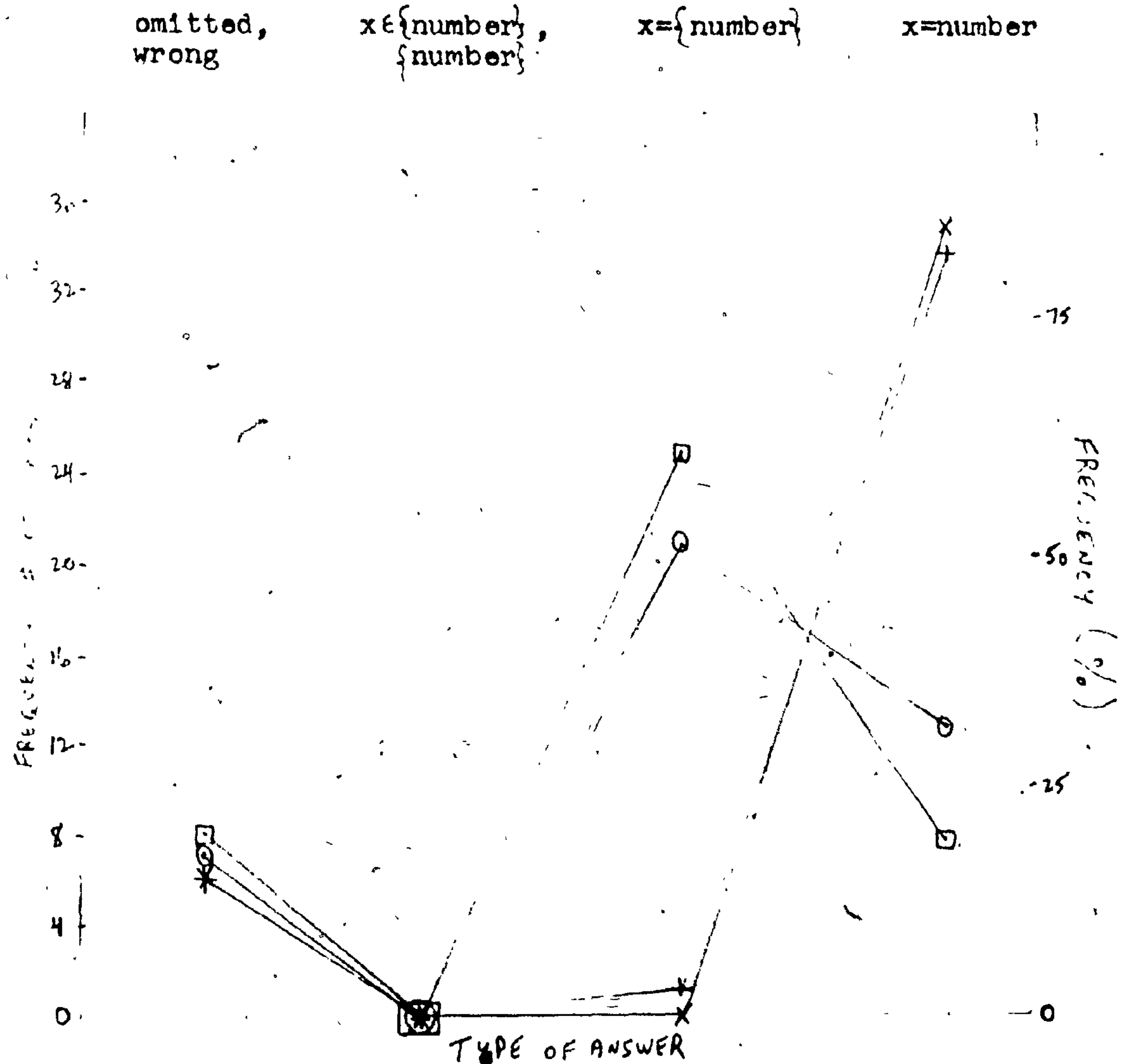
As in the pilot experiment, responses were exhibited on a graph. Separate graphs were prepared for grades 7, 8 and 9, plotting frequency of response versus the type of response. For these graphs, the columns entitled '$x \in \{number\}$' and '$x = \{number\}$' were interchanged. The graphs are numbered 3, 4 and 5 for grades 7, 8 and 9 respectively and they can be found following p 61.

The curves for questions 1 and 2 are similar in appearance for all three grades. The incidence of correct response is very high for both questions.

Graph 3

(Frequency curve)

Grade 7:41 Subjects



Legend:

X 1. Find x if 3x+8=11.

\+ 2. Find x if 2x+7=9 and x is a natural number.

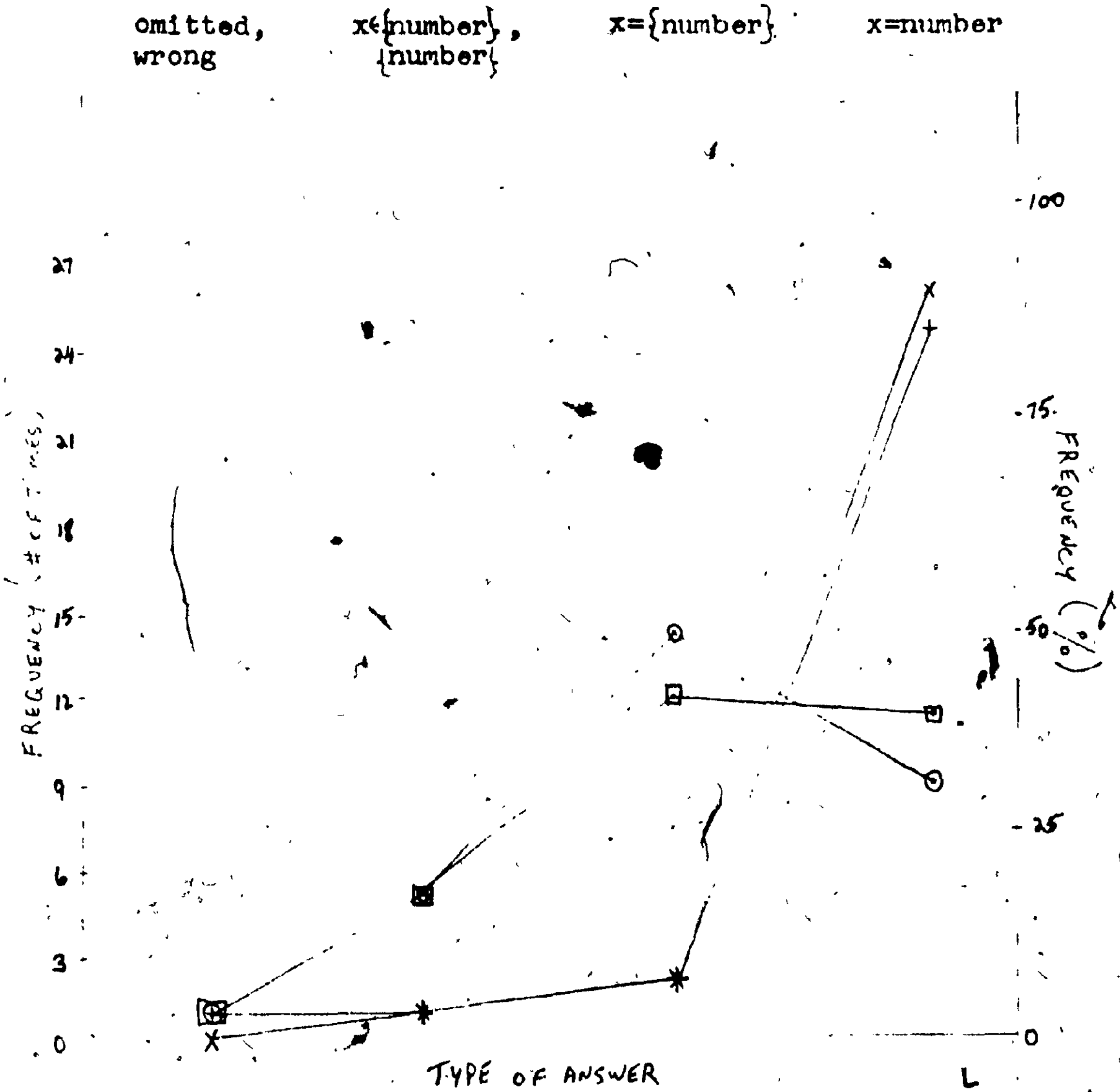
O 4. Find the solution set of 4x+3=7 if the replacement set is the set of natural numbers.

□ 5. Find $\{x \mid 5x+3=8, x \in N\}$.

Graph 4

(Frequency curve)

Grade 8:29 Subjects



Legend:

X 1. Find x if 3x+8=11.

+ 2. Find x if 2x+7=9 and x is a natural number.

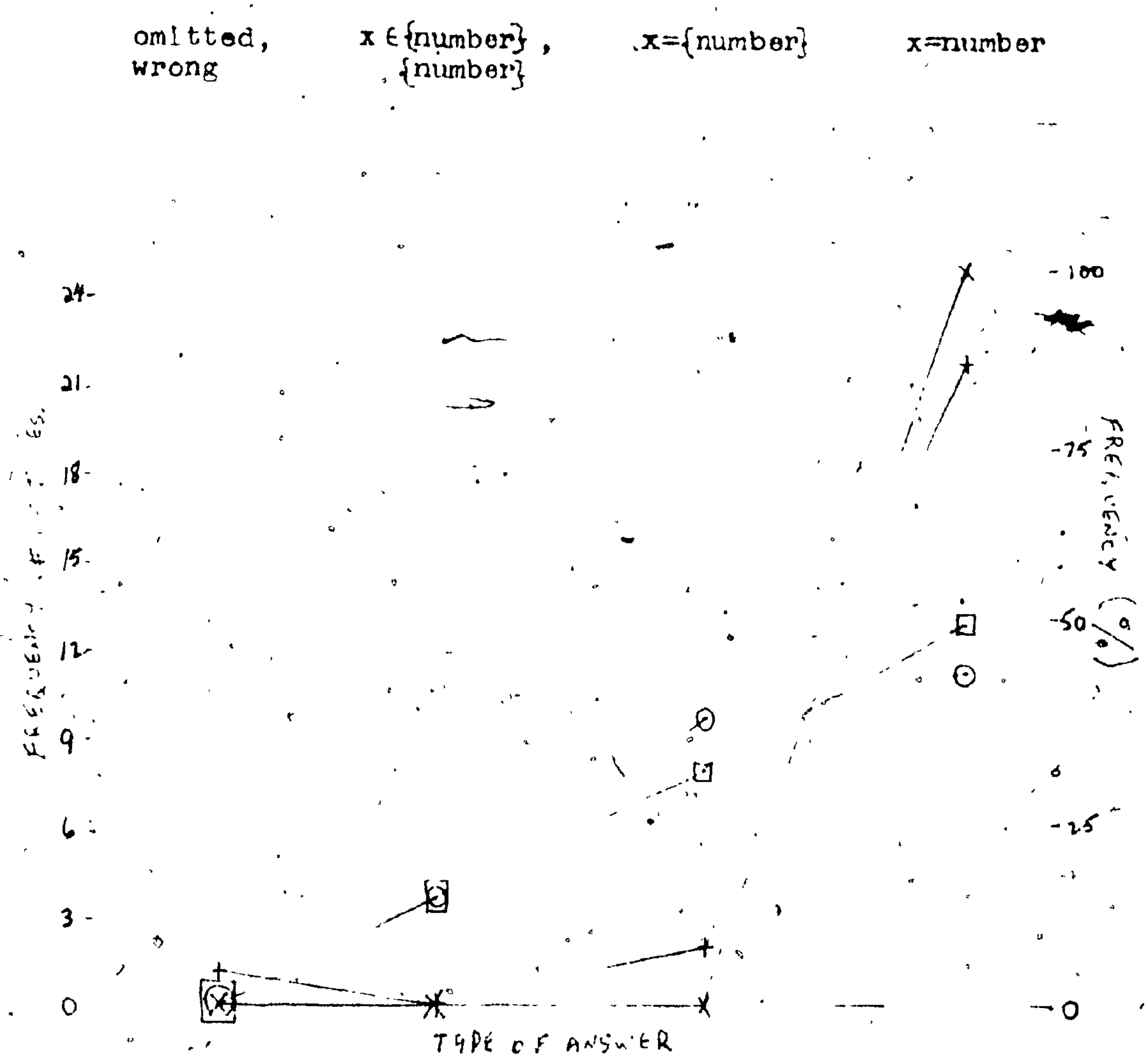
○ 4. Find the solution set of 4x+3=7 if the replacement set is the set of natural numbers.

□ 5. Find $\{x \mid 5x+3=8, x \in N\}$.

Graph 5

(Frequency curve)

Grade 9:25 Subjects



Legend:

X 1. Find x if $3x+8=11$.

+ 2. Find x if $2x+7=9$ and x is a natural number.

O 4. Find the solution set of $4x+3=7$ if the replacement set is the set of natural numbers.

□ 5. Find $\{x \mid 5x+3=8, x \in N\}$.

Although question 1 has been answered correctly slightly more often than question 2, there were only 5 answers preventing these curves from being identical.

The graphs for questions 4 and 5 are similar to each other but very different from those for 1 and 2. The percentage of correct response is also much less for these questions and in this regard, the graphs for grades 7, 8 and 9 differ from each other; for grades 8 and 9, correct solutions are given approximately 16% of the time while for grade 7, the correct solution is <u>never</u> given.

Now, let us turn to question 3 which involved the empty set concept. The question, if you recall, was stated as follows: Find x if 2x + 3 = 1, and x is a natural number. There was no set notation in the question. It was hoped that the students would recognize that -1 did not satisfy the condition set down in the problem. The students' reactions were noted. Their answers were classified as follows: a right answer constituted recognizing that -1 was incorrect as well as that no other number would do. The form of the right answer was then checked. Two forms were used in general (i) the empty set notation and (ii) a worded response to the effect 'this answer won't work'. If the empty set was used, it was expected it would be used correctly. If it was not, it was noted as an incorrect form. A wrong answer was one which was not right and these were divided into(i) an omitted answer, (ii) one which had an incorrect value (ie. content) and (iii) leaving the answer as $x= -1$ or $x= \{-1\}$. Table 8 lists the number of times each answer was given.

TABLE 8

Question 3: 95 students

	OMISSION : 15		CORRECT FORM: 11
WRONG: 71	CONTENT : 31	RIGHT: 24	
	$x= -1$ / $x= \{-1\}$: 25		INCORRECT FORM: 13

Only 24 students recognized that $x = -1$ was NOT the answer. But only 11 of them were able to write this correctly. Several of those who attempted to use the empty set notation (which was not required) used it incorrectly (ie. $\{\emptyset\}$). There were several interesting answers given. One student wrote that the equation "won't possibly work". Others wrote "$x = 0$". It is felt that they knew $x = -1$ was wrong, and for them, x = 0 meant that there was no answer. But this is purely speculation.

What all this means is that only one student in four recognized that the root was inadmissible and only one student in nine could write this statement correctly. In other words, solving an equation over a specified number system introduces errors of content as well as those of form.

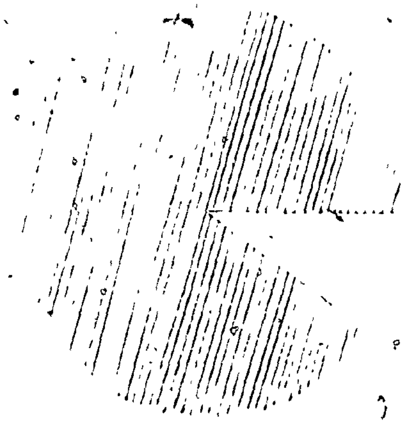
CONCLUSIONS

1. There is very little difference between questions 4 and 5; in other words, the students seem to experience as much difficulty with the vocabulary of sets as they do with set-builder notation.
2. The students omit the set notation almost as often as they use it incorrectly.
3. Over 90% of the answers are not correct if that answer must be given using set notation. This can be seen pictorially in graph (following p 64) where circles are drawn for each question and the area representing the correct solution is darkened.
4. Since the results indicate that the empty set

Graph 6

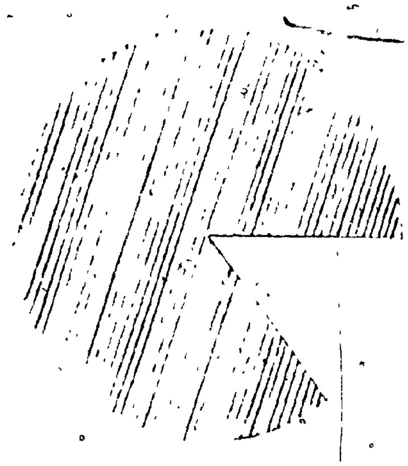
MAIN EXPERIMENT(part 1)

1. Find x if $3x + 8 = 11$.



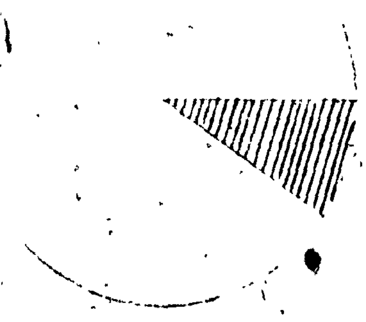
90.5%

2. Find x if $2x + 7 = 9$ and x is a natural number.



85.3%

4. Find the solution set of $4x + 3 = 7$ if the replacement set is the set of natural numbers.



9.5%

5. Find $\{x \mid 5x + 3 = 8, x \in N\}$.

9.5%

Legend: The lined region represents the percentage of correct solutions. (combined results of grades 7, 8, 9) concept deals more with a question of content than it does with form, it is felt that this type of problem is outside the scope of this research. It is left to a future project.

MAIN EXPERIMENT: PART 2
(re-ordering)

TEST

The questions given to the pupils of grade 7, 8 and 9 in part I have been re-ordered. The new order is presented below.

TEST

PLEASE READ EACH QUESTION CAREFULLY AND ANSWER EXACTLY WHAT IS ASKED. SHOW WHATEVER WORK YOU THINK IS NECESSARY.

(4) 1. FIND $\{x \mid 5x + 3 = 8, x \in N\}$

(5) 2. FIND THE SOLUTION SET OF $4x + 3 = 7$, IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.

(1) 3. FIND x IF $3x + 8 = 11$.

(2) 4. FIND x IF $2x + 7 = 9$, AND x IS A NATURAL NUMBER.

The numbers in parenthesis on the left are the original question numbers as they appeared in part 1. The question on the empty set was not included since it would not effect the re-ordering in any event.

RESULTS

The correct answers to these questions were the same as before. And the responses given by the students fell into the same four categories as before. Table 9 gives the number of times each answer was given. It can be found following p 67.

DISCUSSION

The results of this test were compared with the results obtained for the grade 9 students tested in part 1 (see table 4). It will be observed that:

1. Originally, there was 1 content error committed on the four questions: this time there was 1 error.
2. Originally, there were 28 attempts to use sets out of 100 for 28%; this time there were 33 attempts out of 92 for 36%.
3. On those questions requiring sets, originally,

Table 9

(re-ordering)

MAIN EXPERIMENT (part 2)

Grade 9:23 Subjects

Question number	Type of Response							
	Omitted, wrong		$x \in \{number\}$, or $\{number\}$		$x = \{number\}$		x=number	
	no.	%	no.	%	no.	%	no.	%
1	1	4	6*	26	3	13	13	57
2	0	0	9*	39	9	39	5	22
3	0	0	0	0	0	0	23*	100
4	0	0	3	13	3	13	17*	74

*Correct answer

there were 26 answers out of 50 or 52% which used sets correctly or incorrectly; this time there were 27 answers out of 46 or 59%.

CONCLUSION

The results seem to imply that by re-ordering the questions and placing those questions dealing with sets first, this has produced an overall increase of 8% in the use of sets. But how significant is this increase? We are drawn to the fact that on those questions requiring sets both figures, (originally and after re-ordering) fall within the 50% to 60% range. Thus while re-ordering would change the actual figures of the original experiment, these refinements would hardly change the conclusions. The "aura effect" is not large enough to account for students' reluctance to use sets in the solution of equations. In fact, this experiment has served to confirm the results obtained in part 1 in that (within reasonable limits) these results have been replicated under different conditions. While the new group appears to be more successful than the previous group (there could be several reasons for this, for example, ability), there are still many more correct answers for those questions which do not require sets, than for those questions which do.

Therefore, it is felt that the ordering of the questions did not prejudice the design of the main experiment.

MAIN EXPERIMENT: PART 3 (quadratics)

TEST

There were six questions on this test; three were trinomials and three were differences of squares. All six were easily factorable in the integer number system. As before, questions were presented in three different forms.

TEST

PLEASE READ EACH QUESTION CAREFULLY AND ANSWER EXACTLY WHAT IS ASKED. SHOW WHATEVER WORK YOU THINK IS NECESSARY.

1. FIND x IF $x^2 - 6x + 8 = 0$
2. FIND THE SOLUTION SET OF $x^2 - 7x + 12 = 0$, IF THE REPLACEMENT SET IS THE SET OF INTEGERS.
3. FIND $\{x \mid x^2 - 5x + 6 = 0, \quad x \in I\}$
4. FIND $\{x \mid x^2 - 16 = 0, \quad x \in I\}$
5. FIND x IF $x^2 - 36 = 0$
6. FIND THE SOLUTION SET OF $x^2 - 25 = 0$, IF THE REPLACEMENT SET IS THE SET OF INTEGERS.

The correct answers to these questions follow. Alternative answers are given in parenthesis.

1. x = 2, 4 (2,4)
2. $\{x \mid x = 3 \text{ or } 4,\ x \in I\}$ ($x \in \{3, 4\}$ or $\{3, 4\}$)
3. $\{x \mid x = 2 \text{ or } 3,\ x \in I\}$ ($x \in \{2, 3\}$ or $\{2, 3\}$)
4. $\{x \mid x = 4 \text{ or } -4,\ x \in I\}$ ($x \in \{4, -4\}$ or $\{4, -4\}$)
5. $x = \pm 6$ (± 6)
6. $\{x \mid x = 5 \text{ or } -5,\ x \in I\}$ ($x \in \{5, -5\}$ or $\{5, -5\}$)

In this experiment it was required that the answers to questions 2, 3, 4 and 6 would be given using the proper set notation.

RESULTS

It was observed that the answers that were actually given could be classified according to the following forms:

First, the answer was incorrect; these were subdivided into those where both roots were wrong and those where only one root was wrong.

Second, the answer was written as follows:
$x = \{\text{number, number}\}$

Third, $x \in \{\text{number, number}\}$ or $\{\text{number, number}\}$

Fourth, the answer was written as x = number, number.

No answers were written in set-builder notation. Where sets were required only the third form is correct. The number of times each type of answer was given was recorded in TABLE 10 which follows p72 .

DISCUSSION

The incorrect solutions were also subdivided into those that used sets and those that did not use sets. As will be seen, the percentage of answers in this category was considerably higher than in the case of linear equations. However, the split between errors in content and those in form was as striking as it was in the previous case.

In the previous parts of this experiment, answers which were wrong in content were checked for correct form. There were none. This time there were some cases in which "wrong answers" were written in correct form. A careful examination of the 59 wrong answers to questions 2, 3, 4 and 6 where sets were required, indicated that 44 of these (out of a possible 4 x 39 = 156 answers) did not use any set notation. Of the 15 answers that were reported in set notation, only 8 were written correctly. It would seem that there was no point in separating wrong answers into those which used correct set notation

Table 10

(quadratics)

MAIN EXPERIMENT(part 3)

Grade 10:39 Subjects

Question number	Type of Response: Omitted/wrong						$x \in \{-,-\}$, or $\{-,-\}$		$x=\{-,-\}$		$x=-,-$	
	both	one set	no set		total no.	total %	no.	%	no.	%	no.	%
1	8	6	1	13	14	36	2	5	1	3	22*	56
2	10	5	6	9	15	38	3*	8	9	23	12	31
3	12	4	3	13	16	41	2*	5	4	10	17	44
4	6	8	2	12	14	36	1*	3	7	18	17	44
5	5	8	0	13	13	33	0	0	5	13	21*	54
6	5	9	4	10	14	36	6*	15	8	21	11	28

*Correct answer

and those which did not. For the total picture is altered only slightly by this distinction.

In order to examine questions 2, 3, 4 and 6 more closely, TABLE 10 has been re-arranged as follows:

(i) there are two headings only, right answer and wrong answer.

(ii) the "wrong answer" column has been subdivided into form, no set and other. This reorganization is found in TABLE 11 following p 73.

If we compare the percentage of correct response for this test on questions involving sets (7.7%) with that of the previous tests (main experiment part 1 - 9.4% and pilot experiment 13.5%) one notices only a slight difference. Moreover, whatever difference exists, it is in the wrong direction. This would seem to indicate that it is not the number of elements in the solution set which creates the problem, but rather the set form itself. If the 8 answers mentioned at the beginning of the experiment (those which were correct in form but wrong in content) are included, the 7.7% figures jumps to 12.8%. This is still close to the results of the previous experiment.

Table 11

Grade 10:39 Subjects

Question number	Right		Wrong					
			Form		No Set		Other	
	no.	%	no.	%	no.	%	no.	%
2	3	8	9	23	21	54	6	16
6	6	15	8	20	21	54	4	10
3	2	5	4	10	30	77	3	8
4	1	3	7	18	29	74	2	5
Average		7.7		17.9		64.7		9.7

From TABLE 11 it was observed that 28 answers out of 156 or 17.9% used sets incorrectly. This is entirely in keeping with the observations made earlier: that as the student graduates to the next higher level the tendency is for the incorrect form to decrease. On the other hand, the trend of dropping the notation continues as a student passes from grade 7 through to grade 10.

In order to visualize the pattern of response, graph 7 was constructed similar to those constructed earlier. The frequency of response was plotted against the type of response. There were four types of response, outlined above. The graph can be found following p 74.

Each response had two roots. In order to reduce the number of curves appearing on the same graph, the questions were paired and averaged as follows: questions 1 and 5 were graphed together - these had no set notation; questions 2 and 6 were paired - these used the set vocabulary; and questions 3 and 4 were graphed as one - these used set-builder notation.

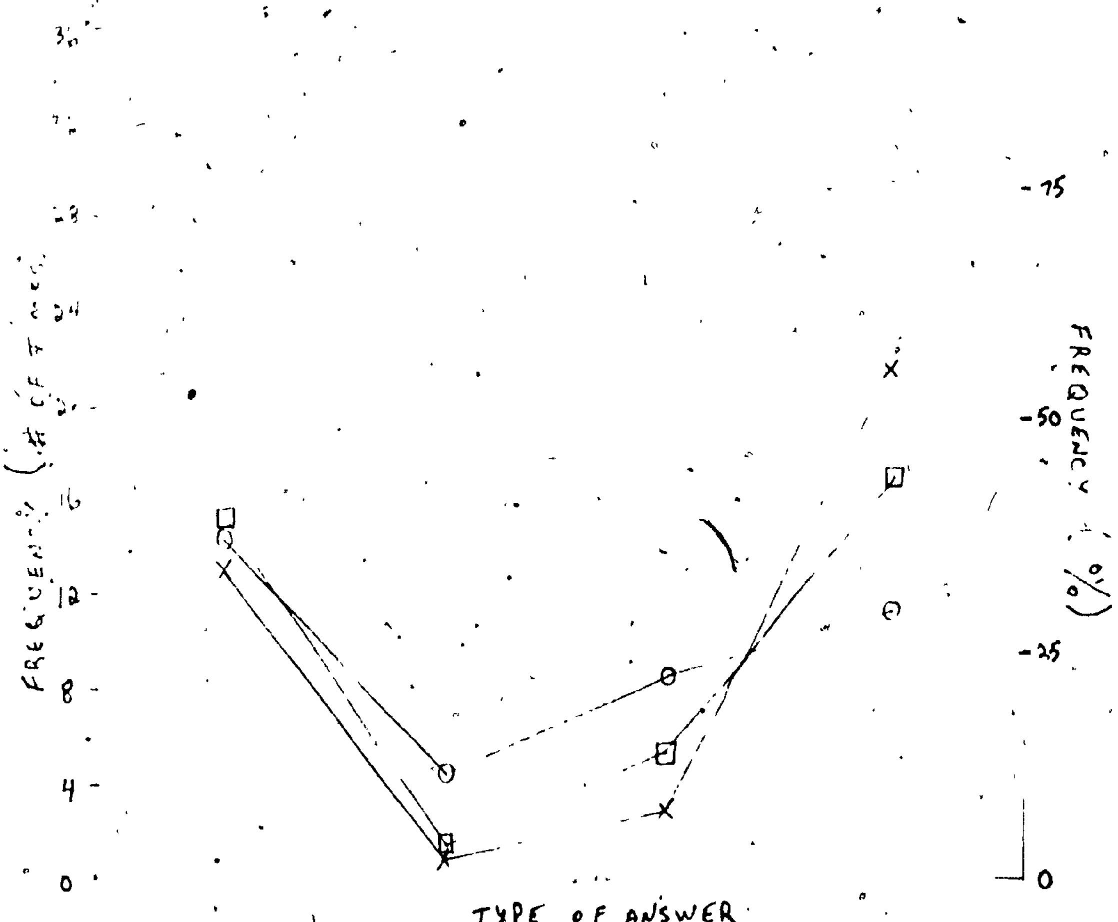
Upon examination of this graph, one observes that

Graph 7

Frequency curve

Grade 10:39 Subjects

omitted, wrong	$\{-,-\}$, $x\in\{-,-\}$	$x=\{-,-\}$	$x=-,-$



Legend:

X 1. Find x if $x^2-6x+8=0$.

X 5. Find x if $x^2-36=0$

O 2. Find the solution set of $x^2-7x+12=0$ if the replacement set is the set of integers.

O 6. Find the solution set of $x^2-25=0$ if the replacement set is the set of integers.

□ 3. Find $\{x|x^2-5x+6=0, x\in I\}$.

□ 4. Find $\{x|x^2-16=0, x\in I\}$.

if form is taken into consideration, those questions involving set-builder notation were not answered as well as those questions which asked for solution sets; and neither type was answered even half as well as those questions which were presented without the use of sets.

CONCLUSIONS

This experiment reinforces previous conclusions: questions presented using sets, whether it was set notation or set vocabulary, were not answered nearly as well as those which were presented without. And it does not matter whether the solution set is a singleton or a doubleton. There are still over 90% of the answers that are wrong if the answer requires the use of set terminology.

MAIN EXPERIMENT: PART 4 (ability grouping)

TEST

PLEASE READ EACH QUESTION CAREFULLY AND ANSWER EXACTLY WHAT IS ASKED. SHOW WHATEVER WORK YOU THINK IS NECESSARY.

1. FIND x IF $x^2 + x - 12 = 0$ AND x IS A NATURAL NUMBER.
2. FIND THE SOLUTION SET OF $x^2 - 5x + 6 = 0$ IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.
3. FIND $\{x \mid x^2 - 7x + 12 = 0, \quad x \in N\}$
4. FIND x IF $x^2 - 3x + 2 = 0$ AND x IS A NATURAL NUMBER.
5. FIND $\{x \mid x^2 + x - 6 = 0, \quad x \in N\}$
6. FIND THE SOLUTION SET OF $x^2 + x - 2 = 0$ IF THE REPLACEMENT SET IS THE SET OF NATURAL NUMBERS.

The following represent the correct answers. Alternative answers appear in parenthesis.

1. $x = 3$ (3)
2. $\{x \mid x = 2, 3, \; x \in N\}$ ($\{2, 3\}$ or $x \in \{2, 3\}$)
3. $\{x \mid x = 3, 4, \; x \in N\}$ ($\{3, 4\}$ or $x \in \{3, 4\}$)
4. x 1 or 2 (1 or 2)
5. $\{x \mid x = 2, \quad x \in N\}$ ($\{2\}$ or $x \in \{2\}$)
6. $\{x \mid x = 1, \quad x \in N\}$ ($\{1\}$ or $x \in \{1\}$)

It was required that the answers to questions 2, 3, 5 and 6 would be written using the proper form of set notation.

RESULTS

The answers appearing on the test were studied and classified according to the following types:

First, the answer was incorrect; these were divided into those where both roots were wrong or only one root was wrong.

Second, using sets, $x = \{ \quad \}$

Third, $\{x \mid x = \quad , \ x \in N\}$ or $\{ \quad \}$.

Fourth, x = number

Where sets were required, only the third form is correct. Tables 12 and 13 record the number of times each answer was given. These can be found following p 77.

DISCUSSION

The discussion will be divided into two parts; a. questions 2, 3 and 4 with two roots and b. questions 1, 5 and 6 with one root in N.

a. The reader will notice that in this experiment, the form $x \in \{$number, number$\}$ has been dropped. Not one

Table 12

(ability grouping)

MAIN EXPERIMENT(part 4)

Grade 10 regular:20 Subjects

Question number	Type of Response										
	Omitted/ wrong						{x|x=-,-}, or {-,-}		x={-,-}		x=-,-
	both	one set	no set		total						
					no.	%	no.	%	no.	%	no.
1	0	13	0	13	13	65	0	0	0	0	7*
2	5	4	3	6	9	45	2*	10	1	5	8
3	2	1	0	3	3	15	1*	5	1	5	15
4	2	3	1	4	5	25	0	0	0	0	15*
5	0	13	1	12	13	65	0*	0	0	0	7
6	0	14	3	11	14	70	1*	5	1	5	4

Question number	x=-,- %
1	35
2	40
3	75
4	75
5	35
6	20

*Correct answer

Table 13

MAIN EXPERIMENT(part 4) (ability grouping)

Grade 10 accelerated:24 Subjects

Question number	Type of Response: Omitted/wrong						$\{x \mid x = -,-\}$ or $\{-,-\}$		$x=\{-,-\}$		$x=-,-$	
	both	one set	no set	total								
					no.	%	no.	%	no.	%	no.	%
1	0	18	0	18	18	75	0	0	0	0	6*	25
2	2	0	2	0	2	8	5*	21	14	58	3	13
3	0	0	0	0	0	0	4*	17	7	29	13	54
4	0	0	0	0	0	0	1	4	1	4	22*	92
5	0	18	7	11	18	75	3*	13	0	0	3	13
6	1	15	14	2	16	67	3*	13	4	17	1	4

*Correct answer

student employed this form when responding to the questions involving sets. One would expect most answers to be given using set-builder notation. But this has not been the case. Among the group of regular students there was not one instance where set-builder notation was used and among the accelerated group there were only three cases.

To compare the two groups more easily, table 14 rearranges the data for questions 2, 3 and 4 provided in tables 12 and 13. This table can be found following p 78.

A comparison of the two groups on question 4, the question without any set notation, immediately points out their difference in ability. The regular class was successful 75% of the time while the accelerated class was successful 92% of the time. However, one notices that even this group of brighter students mishandled the questions involving sets. On question 3 with set-builder notation, only 4 out of 24 answers use the notation correctly. In other words, only approximately 17% of the answers are correct.

On question 2 using set vocabulary, 5 out 24 answers were correct. If we include two wrong answers (content-wise) which used correct notation, approximately 29% of the answers are correct. Compared to the

Table 14

Comparison: regular versus accelerated

Question number		Type of Response							
		Omitted, wrong		$\{x \mid x=-,-\}$, or $\{-,-\}$		$x=\{-,-\}$		$x=-,-$	
		no.	%	no.	%	no.	%	no.	%
R	2	9	45	2*	10	1	5	8	40
A	2	2	8	5*	21	14	58	3	13
R	3	3	15	1*	5	1	5	15	75
A	3	0	0	4*	17	7	29	13	54
R	4	5	25	0	0	0	0	15*	75
A	4	0	0	1	4	1	4	22*	92

*Correct answer

Legend: R-regular grade 10: 20 Subjects

A-accelerated grade 10: 24 Subjects

2. Find the solution set of $x^2 -5x+6=0$ if the replacement set is the set of natural numbers.

3. Find $\{x \mid x^2 -7x+12=0, x \in N\}$.

4. Find x if $x^2 -3x+2=0$ and x is a natural number.

regular group where 3 out of 20 answers were correct on question 2 (this included 1 wrong answer that used correct notation) for 15%, the correct output is doubled but there is still less than 1 in 3 correct answers from the brighter group on questions where sets are concerned.

The data collected in this experiment confirms previous data. In other words, questions in set-builder notation are not answered as well as those using set vocabulary which in turn are not answered as well as those without sets.

b. For questions 1, 5 and 6 the scoring was as follows: one root was extraneous and had to be discarded. In tables 12 and 13 under the column entitled 'omitted and wrong', entries were made as follows - if the equation was solved incorrectly both roots were wrong and an entry was made under "both". This occurred in one case only. If the extraneous root was not rejected, one root was wrong and an entry was made under "one". All other entries were made as before.

It was observed that the correct form for these three questions appeared less often than for the corresponding questions of the first group. There was also a significant increase in the number of wrong answers apparently brought on by the student's inability to reject extraneous roots. This, however, leads to a discussion of content. Further research is necessary on this and other related questions. For example, what happens when a student is presented with a problem where both roots are inadmissible? In any case, as was stated earlier, this question is beyond the scope of the present investigation. One final observation before concluding this discussion. As was done in earlier experiments, an analysis of the wrong answers was made for the brighter group on questions 2 and 3. Table 15 reports those results. It can be found following p 81.

Although there is a strong tendency in this group to use set notation, almost half of the answers are technically incorrect. And one-third of them are presented without any set notation.

CONCLUSION

Based on the findings of this experiment, less than one in three answers given by students classified as above average are correct when set notation is required. The rest either use the notation incorrectly or not at all. One can thus well understand the difficulty encountered by average students when confronted with set notation.

Table 15

Grade 10 accelerated: 24 Subjects

Question number	Right		Wrong: Form		Wrong: no set		Wrong: Other	
	no.	%	no.	%	no.	%	no.	%
2	5	21	14	58	3	13	2	8
3	4	17	7	29	13	54	0	0
Average		19		44		33		4

MAIN EXPERIMENT: PART 5 (re-teaching)

TEST

The same test was given to this group as was originally used. The test can be found in part 4. No changes were made.

RESULTS

The answers that were given were tabulated and they appear in table 16 following p 82.

DISCUSSION

To insure that the re-teaching had a maximum effect, the entire exercise was geared (contrary to good pedagogy) to a rewrite of the same test at a later time. The re-teaching consisted of explaining the errors on the test that the group had just written with further examples inserted to emphasize the concepts covered. The time spent was the equivalent of one math period (45-50 minutes).

The following observations are pertinent.

1. Since it is felt that questions 1, 5 and 6 deal with content, they will not be discussed here

Table 16

(re-teaching)

MAIN EXPERIMENT(part 5)

Grade 10 accelerated: 24 Subjects

Question number	Omitted/wrong						Type of Response					
							{x | x=-,-}, or {-,-}		x={-,-}		x=-,-	
	both	one set	no set		total							
					no.	%	no.	%	no.	%	no.	%
1	0	2	1	1	2	8	2	8	1	4	19*	79
2	1	0	1	0	1	4	15*	62	6	25	2	8
3	0	0	0	0	0	0	13*	54	4	17	7	29
4	0	0	0	0	0	0	3	12	2	8	19*	79
5	0	5	4	1	5	21	13*	54	3	12	3	12
6	0	3	3	0	3	12	15*	62	4	17	2	8

*Correct answer

other than to point out that there was an overall improvement in performance on these three questions after re-teaching.

2. On question 4 where no sets were required, only 79% gave the correct solution as opposed to 92% earlier. But, 5 answers attempted to use sets where only 2 attempted to use sets earlier. No answer was wrong because of content.

3. On questions 2 and 3, there were significantly more correct answers using sets than previously.

4. As before, questions on set-builder notation were answered correctly less often than the questions on set vocabulary.

5. The form of set-builder notation was used 25 times (including two cases where it was not required) after reteaching as opposed to the original test where it was used 3 times.

6. On questions 2 and 3, the incorrect form appears 10 times out of 48 answers for 21%; and 9 answers out of 48 or 19% do not use any set notation although it is required.

7. There is a significant increase in correct usage of sets from the original experiment, but there are still less than two out of three answers that are correct.

CONCLUSION

With the benefit of more teaching, to a group of bright students, the group most likely to improve, a certain measure of success can be claimed. After all, there were twice as many correct answers after additional teaching. But there were still some 40% of the answers that were incorrect. How much more teaching would be necessary for further improvement? Please keep in mind that these students are carrying averages of 80% or better in mathematics. How well would the average student do upon re-teaching?

SUMMARY AND CONCLUSIONS

Recall the reasons advanced by Rourke for teaching set theory to high school students: 1. to clarify; 2. to simplify; 3. to unify; 4. to broaden old ideas; and 5. to introduce important new ideas. So convinced that these reasons were sound, curriculum commissions and committees were established and the high school curriculum was renovated to include set theory. Not content to alter only the high school curriculum, these modernists proceeded to modify the elementary school curriculum as well. Set theory took the schools by storm. Set, subset, member and empty set became part of the mathematics vocabulary starting in kindergarten. By grade six, symbols such as $\{\ \}$, $\cap$ and $\cup$ were prominent in the mathematics course. By grade eight, most equations and inequalities were presented in set-builder notation. Set theory had literally permeated the mathematics curricula at every level from kindergarten through grade eleven.

Exactly what are the effects of set theory on the learning of algebra in elementary and high school, particularly high school? Does set theory really fulfill its purpose as claimed by the modernists? Does it go beyond or does it fall short? Does it have an adverse effect on the learning of a particular concept? Or does it have any effect at all? These questions are valid and there are several more which have not even been mentioned. But the scope of this research does not allow me to answer them. What this research attempts to determine is whether set theory does satisfy, in some measure, its reasons for being taught in school. With this in mind, it was decided to test the effect of sets in one area; that area was the equation.

Groups of students were presented with equations in different forms. Some equations were presented using sets, either in set-builder notation or with the vocabulary of sets. Other equations were presented without the use of sets. It was the form of the answer which was of importance. Therefore, in examining the answers that were given, answers where content was incorrect were noted, but priority in analysis was given to the way the content was presented. To avoid incorrect content in as far as it was possible, equations were selected which could be solved successfully by the group being tested. The form of the answer was then noted.

The reliability of the method used was established by replication, obviating the need for formal statistical analysis. Eight groups of students for a total of 233 subjects were tested with slightly differing equations which satisfied the criterion that their roots could be easily obtained. The groups attended two schools, had different teachers and varied both in grade level and ability. The results in all cases were clear-cut and entirely consistent. In fact, the simplicity of the experimental design puts a replication of the experiment within the reach of any classroom teacher who should desire to do so.

On the pilot experiment, it was immediately apparent that there was much confusion created in the minds of the students when they were presented with set notation. Students were able to 'understand the problem' (ie. they knew that they had to solve an equation); but they tended to disregard instructions and report their answers in the same manner regardless of what was required. Over 40% of those tested did not use sets to describe their answers. Several of the others followed various patterns which ranged from dropping the notation early during the test to consistently using an incorrect form of the notation.

An examination of the answers given for the questions where set notation was required, indicated that the students definitely have not learned the proper use of set notation. There were more answers using sets incorrectly than there were using the notation correctly.

On the main experiment with students of grades 7, 8 and 9, 90% of the answers that required set notation were incorrect. It is clear that the students experienced enormous difficulty reporting their answers.

On the test given to the grade 10 students, the above results were confirmed and a very interesting trend became evident. As the student passes from one level to the next (ie. as the student goes from grade 7 to grade 10), there is a tendency for the incorrect form to be given less and less often. But, at the same time, more and more answers are given without any set notation at all. On passage from grade 9 to grade 10 there is a marked drop in the use of incorrect forms of set notation. But increased mastery of notation does not result in it being used more frequently; on the contrary, the incidence of its use decreases. It is as if students at that level are able to learn the set form but do not see much point in using it. Even the group of brighter students could do no better than one correct answer in three when sets were required.

Since the students could solve the equations without using set notation, it must follow that sets and set terminology do not clarify the mathematics involved. For, if the notation did simplify the problem, it would follow that the students would use it more often, NOT less often. The set notation has simply added a new dimension to the problem, a dimension which has not been mastered. To put it another way, the concept of the solution set does not seem to be grasped even by above average students at the grade 10 level. Thus the introduction of set notation has merely added another abstract concept (that of a set) to an abstract problem (the equation). It is easy to see in retrospect that such an addition could not simplify the problem faced by the student.

The present investigation throws little light on the unifying aspect of set theory. It is therefore very difficult to dispute this reason for introducing sets. According to Fehr: "No longer do we think of mathematics as a collection of disjoint branches - arithmetic, algebra, geometry, analysis - having no inner relations to each other. We think of it as a set of structures, all intimately related, and all applicable to many diverse situations. In this context, sets, mappings, relations and functions are fundamental to the study of all mathematics - they are the unifying elements. Built on these elements, and basic to all secondary school mathematics, are the algebraic structures - the group, the ring or integral domain and the field".[16]

But one must consider the entire programe. Perhaps sets do unify the entire mathematics curriculum. This is not disputed. What is disputed however, is the need to begin this unification so early in the training of our students.

The same can be said for broadening old ideas and introducing important new ones; these appear to be very valid reasons for introducing set theory - at some time. However, it has been demonstrated that its introduction is far too early for these young minds to grasp and the topic would be better presented at a later time. In order for set theory to satisfy any of the above reasons (to clarify, to simplify, etc.) it must first be learned! Evidently this

[16] See FEHR, HOWARD, R. in Modern Mathematics - Volume 1, p. vi

requires a major effort, for I believe I have demonstrated that students are unable or unwilling to use the language of sets in the solution of equations. After ten years (eleven if we include kindergarten) of set vocabulary and four years of set notation, regular students give less than one correct formal answer in ten on equations presented with set terminology. Brighter students do slightly better.

Premature exposure, while not always detrimental, is not always advantageous. We should teach the language of sets when the student is mature enough to handle it. In particular, he should be able to use it for content as well as for formal purposes, ie. he should be able to appreciate the effect on a solution set of the number system over which the equation is to be solved. To grasp the purpose and generality of sets requires precisely those qualities of maturity and self-analysis in the learning process that children usually lack.

When is the student mature enough to handle this topic? I think that set theory, with all its notation and vocabulary, should be introduced no earlier than grade ten and preferably in grade eleven. And it should not be taught as a separate package and then forgotten. Continued use is a necessary condition throughout the balance of high school and into higher mathematics. In my opinion part of the reason students have difficulty with sets may be attributed to this closed package presentation. Add to this the fact that the presentation is basically superficial at the early stages and one can well understand why the results are less than satisfactory.

Sets must be taught in such a way as to make them useful. How this can best be done, I am not prepared to say.

I believe I have succeeded in demonstrating that the introduction of set theory to elementary and high school students has not satisfied its 'raison d'être'. After several years of practice, the results have been the same year after year: a poor command of the notation and the vocabulary when applied to the solution of equations; and whatever time has been taken to introduce and teach set terminology has ultimately delayed the introduction of whatever has been replaced. The presentation of sets at these levels is not worthwhile because of the resulting confusion. We should simply abandon the teaching of sets below the grade 10 level.

One must agree with Morris Kline that set theory is the foundation of a sophisticated and rigorous approach to mathematics; but, if introduced too early, it is a hollow formalism that encompasses ideas which are far more easily understood intuitively.[17]

17 KLINE, p 114

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