

STATIC AND DYNAMIC BUCKLING ANALYSIS OF
GUYED STACKS AND MASTS WITH VARIABLE INERTIA

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ABSTRACT.

STATIC AND DYNAMIC BUCKLING ANALYSIS OF GUYED
STACKS AND MASTS WITH VARIABLE INERTIA

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A method of analysis of the static and dynamic critical load of the stack with variable inertia is proposed in this thesis. Along the total height of the chimney, the inertia $I(x)$ is assumed to be a uniformly continuous function of x . But it can also be a piecewise continuous function.

Because the guys serve as supports, the stack is assumed to be a bar on elastic supports. So, formulae leading to the static bending are derived. The influence of the own weight distributed along the vertical axis is taken into account in the search for the critical load. There results a differential equation of the fifth order with variable coefficients. This differential equation is transformed into a perturbation-type equation with formulae of limitation of the diameter and thickness at the top with respect to the diameter and thickness at the bottom.

The critical load of the bar with constant inertia is also analyzed. By the concept of fictitious length, the differential equation of the third order, derived by Euler in the case of the cantilever column, is transformed into a non-homogeneous differential equation. A new operator attached to Bessel functions and considered as a ring homomorphism is introduced. If one uses this operator, Bessel functions can be handled with elegance and simplicity as in ordinary operations on polynomials. Completely new formulae are proposed for the critical load of the bar,

when one considers the influence of the own weight.

The dynamic analysis of the stack with variable inertia lead to partial differential equations of the fifth and sixth orders. As a result, algebraic polynomials of the fifth degree are derived and analyzed. The undamped motion, in the case of the bar with constant inertia, leads to partial differential equations of the fourth order. An expansion in power series is proposed.

In the damped motion, the proposed method of solution leads also to the equation of the plate on elastic foundations in the old Winkler hypothesis.

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	1
ACKNOWLEDGEMENTS	iii
TABLE OF CONTENTS	v
LIST OF FIGURES	ix
LIST OF TABLES	x
LIST OF APPENDICES	xi
NOMENCLATURE	xii

CHAPTER I

INTRODUCTION

1.1 The Chimney	1
1.2 Statement of the Problem	2
1.3 Hypotheses	2
1.4 Origin	3
1.5 Review of the Literature	3
1.6 Proposed Solution	7

CHAPTER II

STATIC BEHAVIOUR OF CABLES

2.1 Elastic Support Displacement	9
2.2 Support Flexibility	10

CHAPTER III

BENDING OF THE STACK UNDER STATIC LOAD

3.1 Equilibrium Equations	14
3.2 Hypotheses and Solution	15
3.3 The Law of Inertia	16
3.4 Element Subjected to a Lateral Displacement and to a Flexural Bending at its Ends	17

	<u>Page</u>
3.5 Elements Subjected to a Flexural Bending at End 1	19
3.6 Remark	19
3.6.1 Remark	19
3.6.2 Remark	20
3.7 Effect of Transversal Loads	20
3.8 Remark	22

CHAPTER IV

STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIA

4.1 Stability Under Axial Load	24
4.2 Differential Equation of the Neutral Axis	25
4.3 Proposed Method of Solution	28
4.4 Remarks	31
4.5 Element With a Constant Inside Diameter	31
4.6 Bar of Constant Thickness	32
4.7 Solution to Differential Equation (4.24)	33

CHAPTER V

BAR WITH CONSTANT INERTIA - STATIC BUCKLING

5.1 Introduction	36
5.2 Formulation of the Problem	37
5.3 General Solution to Equation (5.6)	39
5.3.1 On the Existence of a Linear Operator T_n Attached to Bessel Functions	39
5.3.2 Solution to Equation (5.6)	40
5.3.3 Remark	42
5.4 Critical Load	43
5.4.1 Bar Hinged at Both Ends	43
5.4.2 Bar as Cantilever	43
5.4.3 Bar Fixed at Both Ends	44
5.4.4 Bar Fixed at One End and Hinged at the Other	44

	<u>Page</u>
5.5 Continuous Beam on Fixed or Elastic Supports	45
5.5.1 Uniform Distributed Load Acting Alone	46
5.5.2 Column as a Cantilever	46
5.5.3 Bar Hinged at Both Ends	46
5.5.4 Bar fixed at Both Ends	47
5.5.5 Bar fixed at One End and Hinged at the Other	47

CHAPTER VI

DYNAMIC ANALYSIS. BAR WITH VARIABLE INERTIA

6.1 Equation of Motion	50
6.2 Natural Frequency of the System	53
6.3 Flexibility of the Supports	55
6.4 Influence of the Motion of the Guys	55
6.5 Mode of Vibration	57
6.6 Dynamic Buckling	57
6.7 Forced Vibration and Damping	58
6.8 Solution to Equation of Motion	61
6.9 Remark	62
6.9.1 Remark	62
6.9.2 Remark	62
6.10 Bar With Constant Inside Diameter	62
6.10.1 Undamped Motion	62
6.10.2 Natural Frequency of the System	63
6.10.3 Forced Vibration and Damping	64
6.11 Constant Wall Thickness and Variable Diameter	65
6.11.1 Undamped Motion	65
6.11.2 Damped Vibration	66

CHAPTER VII

CONSTANT INERTIA DYNAMIC ANALYSIS

7.1 Undamped Vibration	70
7.2 Solution to the Equation of Motion	70

	<u>Page</u>
7.3 Remark	74
7.4 Remark	74
7.5 Damped Vibration	75
CHAPTER VIII	
8.1 Horizontal Loads	78
8.2 Properties of the Stack	78
8.3 Remark on Cables	78
CHAPTER IX	
CONCLUSIONS AND RECOMMENDATIONS	98
REFERENCES	103

LIST OF FIGURES

<u>Figure</u>	<u>Page</u>
2.1 Wind Direction.	12
2.2 Loads on Guys.	12
2.3 Tensile Forces on Guys.	13
2.4 Variation of Horizontal Reaction.	13
3.1 Path of the Arc Length.	23
3.2 Forces Acting at a Point.	23
3.3 Forces and Displacements at the Ends of an Element of Stack.	23
4.1 Set of Acting Forces.	35
6.1 Displacement of Guy During Vibration.	69
6.2 Displacement Diagram for Guy Anchor When the Guy and Vibration of the Stack are not Coplanar.	69
8.1 Tensile Forces in Guys.	94
8.2 Variation of the Horizontal Load.	94
8.3 First Critical Load. Stack of Constant Inertia.	95
8.4 First Critical Load. Stack of Variable Inertia.	95
8.5 Stack of Constant Inertia. First Mode Shape.	96
8.6 Stack of Variable Inertia. First Mode Shape.	96
A.1 Variation of Sections.	A.3

LIST OF TABLES

<u>Table</u>		<u>Page</u>
1	Cable Properties.	80
2	Cable Properties (use of the multiplier).	81
3	Forces and Displacements of Stack (with axial load).	82
4	Forces and Displacements of Stack (without axial load).	83
5	Forces and Displacements of Stack (axial load and use of the multiplier).	84
6	Forces and Displacements of Stack (without axial load, use of the multiplier).	85
7	Forces and Displacements of Stack (with axial load).	86
8	Forces and Displacements of Stack (without axial load).	87
9	Forces and Displacements of Stack (axial load and use of the multiplier).	88
10	Forces and Displacements of Stack (without axial load, use of the multiplier).	89
11	Stack With Variable Inertia. Undamped Motion.	90
12	Stack With Variable Inertia. Damped Motion.	91
13	Stack With Constant Inertia. Undamped Motion.	92
14	Stack With Constant Inertia. Damped Motion.	93

LIST OF APPENDICES

	<u>Page</u>
APPENDIX A	
FUNDAMENTAL PARAMETERS OF THE CHIMNEY WITH VARIABLE INERTIA	A. 1
A.1 Law of Inertia	A. 1
APPENDIX B	
STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIA	B. 1
FUNDAMENTAL MATRIX	
B.1 Differential Equation of the Elastic Line	B. 1
B.2 Fundamental Matrix of (B.3)	B. 3
B.2.1 Case of Four Real Roots	B. 4
B.2.2 Case of Four Complex Roots	B. 5
B.2.3 Two Roots are Real and the Two Others are Complex	B. 7
APPENDIX C	
CHIMNEY OF CONSTANT INSIDE DIAMETER AND VARIABLE THICKNESS	C. 1
C.1 Differential Equation of the Elastic Line	C. 1
APPENDIX D	
CHIMNEY OF CONSTANT THICKNESS	D. 1
D.1 Differential Equation of the Elastic Line	D. 1
APPENDIX E	
DYNAMIC ANALYSIS OF THE CHIMNEY WITH VARIABLE INERTIA	E. 1
E.1 Undamped Motion	E. 1
E.2 General Solutions to Equation (6.5)	E. 2
E.2.1 Case of One Real Root	E. 3
E.2.2 Case of Three Real Roots	E. 7
E.2.3 Case of Five Real Roots	E. 10
E.3 General Solution to Equation (6.5)	E. 13

	<u>APPENDIX F</u>	<u>Page</u>
FORCED VIBRATION AND DAMPED MOTION OF THE STACK WITH VARIABLE INERTIA		F.1
F.1 Damped Motion		F.1
APPENDIX G		
CHIMNEY WITH CONSTANT DIAMETER		G.1
6.1 Undamped Motion		G.1
APPENDIX H		
CHIMNEY OF CONSTANT INSIDE DIAMETER FORCED VIBRATION AND DAMPED MOTION		H.1
H.1 Equation of Motion		H.1
APPENDIX I		
CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETER		I.1
I.1 Undamped Motion		I.1
I.1.1 Equation of Motion		I.1
APPENDIX J		
CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETER. DAMPED MOTION		J.1
J.1 Equation of Motion		J.1
APPENDIX K		
INFLUENCE OF THE AXIAL FORCE ON THE STATIC FLEXURAL BENDING MOMENT		K.1
K.1 Differential Equation of the Elastic Line		K.1

NOMENCLATURE

Unless specified and used temporarily otherwise, the notations in the thesis have the following meaning:

a,b,c,d	parameters related to the geometry of the stack
a_1, a_2, a_3, a_4, a_5	parameters used as coefficients in the differential equation of the elastic line
f	load vector
f_d	damping force
g	acceleration due to gravity
l	length of an element of stack
m(x)	unit weight of an element of stack
p(x)	intensity of the distributed horizontal forces
q, q_0	unit own weight of stack along the vertical axis
s	length of the chord of cable
t	time
t', t'_1, t'_2	thickness of an element of stack
$u'(x)$	component of the vector displacement
U_{ij}	elements of the fundamental matrix
w	unit weight of cable
x,y,z	coordinates
A	linear operator
B	perturbation matrix
A_c	area of a section of cable
A', B', C	parameters in the cable equation
D, D_1, D_2	diameters of an element of stack
E, E', E'', E_c	Young moduli
A_{ijk}	parameters in the search for the fundamental matrix

C	resistance to transverse velocity
C_s	resistance to strain velocity
F	parameter in the evaluation of the dynamic elasticity of cable
G	factor of safety
$I(x), I_0$	inertial of an element of stack
K	flexural stiffness
M_1, M_2	bending moments at ends of an element of stack
P	axial load
V_1, V_2	shear forces at ends of an element of stack
Q, Q_0	vertical components of tensile force of guy
S, S_0	tensile forces of guy
$T(t)$	time function
U	vector displacement
V	horizontal shear force
U	fundamental matrix
γ	matrix
$[f]$	wrench of acceleration
$E_{\lambda i}$	projector of spectral decomposition
$B(p,q)$	Eulerian Beta function
C	set of complex numbers
$[F]$	wrench at ends of an element of stack
γ	parameter in the evaluation of dynamic elasticity of guy
J_v	Bessel function of order v
K	coefficient of flexibility
L	mapping
M	bending moment

$M_{\lambda_i}^v$	subspace relatively to the eigenvalue λ_i
D	domain,
(D)	interior domain to D
$M(D)$	mass of D
U_1, U_2	parameters in the equation of the elastic line of the stack with constant inertia
\mathbb{R}	set of real numbers
T_n	operator attached to Bessel functions
Z	set of integers
$\alpha, \beta, \theta, \delta, \eta$	parameters in the search for the fundamental matrix
δ_a	damping coefficient
σ	angular displacement of guys
θ'	angular displacement of an element of stack
γ	density of acceleration
ρ	specific weight
λ_i	eigenvalue
λ'	constant of integration
v_i	index of the complex number λ_i
$n'(t)$	guy displacement with respect to the chord
$v'(t)$	guy displacement at the top
ϵ	strain
Σ	set of eigenvalues
ϕ	function of perturbation
ϕ_i, θ_i	parameter in the elements of the perturbation matrix
w	variable
x	coefficient of flexibility
$\Omega, \phi(\Omega)$	parameters in the evaluation of the dynamic elasticity of guy

w	frequency
ω	rate of rotation
t	unit vector
C	external forces
$\Gamma(x)$	Gamma Function
y	forces matrix at ends 1 and 2 of an element of stack
u	vector displacements at ends 1 and 2 of an element of stack
	Whatever

CHAPTER I

INTRODUCTION

CHAPTER I

INTRODUCTION

1.1 The Chimney

As its name indicates, the chimney is a pipe designed to expel into the atmosphere the burning gas of plants. Since the dimensions of plants keep getting larger and larger, there results a constant increase in the size of the chimneys.

For both the manufacturer and the engineer, this gives rise to major conceptual problems of increasing difficulty. So, the height of the chimney is considered to be one of the most important parameters in its design.

In fact, since man is systematically modifying the environment, pollution now poses a permanent threat to health, so it is imperative to reduce the volume and varieties of gas and smoke that threaten the immediate environment. Thus, the ever increasing regulations imposed by the Environmental Protection Agency have led, in the last two decades, to the designing of chimneys of more than two hundred to three hundred meters tall.

Unfortunately, nature is not so easily dominated; stability and economics become concomitant elements. As a result, reinforced concrete was the solution to the problems encountered in the construction of tall chimneys. However, steel is used with increasing frequency. This thesis is intended to be a contribution to the static and dynamic buckling analysis of guyed stacks with variable inertia.

1.2 Statement of the Problem

The formulation of the problem is as follows: consider a simply connected system in an Euclidian Space, schematized by a revolution surface with a vertical axis, whether or not stayed by elastic supports constituted by a set of guys. It is required to analyze the stability of this system under the set of the most unfavourable vertical and horizontal forces to which it can be submitted during its life-time.

1.3 Hypotheses

The guyed stack with variable inertia is one of the most difficult problems of the science of construction.

In fact, the system at rest is set in motion by variable horizontal forces according to a certain direction, such as wind forces. However, the stream of airflow in its motion generates secondary effects, such as Von Karman vortices transmitting to the body oscillation motions in a plane perpendicular to the initial direction.

In its displacement, the surface compels the guys to move in planes which are not necessarily the same as their vertical planes. The guys, whose purpose is to minimize displacements, develop compression forces on the surface. This inevitably generates variable vertical displacements, along with the temperature effects. As a result, engineers are confronted with a problem which, by reason of its tridimensional nature, is a complicated one. It is therefore necessary to put forward some simplifying hypotheses.

- a) Depending upon the case, motion will be one-dimensional.
- b) The effect due to temperature is neglected, but it can always be introduced into the theory which has been developed.

- c) The shortening of the stack, under axial load, will not be taken into account, the assumption being that it does not affect the behaviour of the structure.
- d) Regarding the static and dynamic behaviour of cables, the author has accepted as valid, and therefore uses where necessary, the findings of Kolousek [1] or Davenport [2] in their research on guys. He has preferred to focus his attention on the derivation of motion equations of the stack itself.
- e) The geometry of the system is considered as that of a curvilinear medium, to which can be applied the fundamental equation of Continuum Mechanics, by restricting the design to the linear field and taking into account the fundamental hypotheses of Strength of Materials.
- f) The random nature of the phenomena generating the exciting forces has not been specifically considered. But it can possibly be introduced in the derived solution.
- g) Only the elastic buckling is analyzed, and no mention will be made of the devices used to reduce cylindrical vibration.

Despite these limitations, the development of the equations has led to formulae which can appear very sophisticated.

1.4 Origin

The origin of the problem is as follows: As a doctoral thesis, Professor Troitsky proposed that the author conduct research on the problem of the guyed stack.

1.5 Review of the Literature

The behaviour of cables is one of the most important factors in the design of guyed masts or chimneys. However, their analysis is not

very simple. In the past century, this analysis has been the object of extensive study by many researchers, as outlined below.

Contamin [3] mentions the ministerial circular of the French Government dated August 12th, 1852 concerning the use of cables in suspension bridges, and gives formulae for the utilization of chains and cables.

Pigeaud [4] analyses the role of cables in suspension bridges and formulates a method for finding, by trial and error, the displacement of two cables symmetrically fastened at a point C and subjected to a horizontal force at this point.

Relf and Powell [5] conducted experiments on cables of different sizes inclined at an angle α in the direction of the stream of airflow. They verified that the normal force on the cable is proportional to the square of cosines of the inclined angle α , as it is expressed in Hoerner's [6] book.

But it must be pointed out that the problem of guyed masts has been clearly recognized by Bourseire [7]. He mentions in his article that the literature on this problem was sparse. By using descriptive geometry, he presented an excellent method for the analysis of cables, their initial tension and also for the design of pylons. The problem was treated as a beam on elastic supports. He then derived the famous formula relating horizontal force at the connection of guys to the tension of hard and slack guys.

Sainflou [8] continued Bourseire's [7] research. Using the results published by Pigeaud [4], he formulated the classic third-degree equation relating geometry, displacement and the initial and final tension of

leeward and windward cables.

In the technical notes of the "C.E.C.M." written by Joukoff and Massonet [9], wind action on cables was considered by the use of spherical trigonometry. Using the formulae of Bleich [10] and Agyris [11], they put forward their solution in a graphical form.

Different techniques have been used by researchers to analyse the static and the dynamic behaviour of guys, as quoted in references [12,13, 14,15,16,17,18,19,20,21,22,23].

Some of these techniques are: implicit functions, successive approximations, expansion in power series from the Catenary equation, the technique of linearization and the technique of optimization from the Powell quadratic interpolation theory.

Davenport [2] studied the behaviour of cylindrical columns, whether or not stayed by guys. By using and simplifying Kolouseck's [1] work on cables, he proposed a formula expressing the dynamic elasticity of guys, whether or not they oscillate in their vertical plane.

Avaling himself of Davenport's [2] research, Addie Robert [23] studied the tridimensional behaviour of cables under the effect of horizontal forces.

Concerning the design of the mast itself: Bourseire [7], neglecting the shortening of the mast, expressed the three equilibrium equations at the connection of each set of guys. Thus, forces and bending moments were determined by assuming a linear displacement along the height of the mast.

Sainflou [8] applied Bresse's fundamental equations and arrived at

a system of five unknowns at each node: vertical and horizontal displacements, rotations, tension of hard and slack guys.

Kolousek [1], after neglecting the influence of the vertical displacement, wrote in his analysis of the bending moment the equation of the uneven beam, by taking into account at each joint the influence of the elastic support settlement. He proceeded in the same way when analyzing the static and dynamic buckling load.

Joukoff and Massonet [9] indicated a procedure to be followed in the design of guyed masts, and mentioned that, despite the complexity of the computation, the normal procedure should be to consider the horizontal and vertical forces acting together, as well as the secondary effects due to the elastic deformation of the mast.

Schott and Thurston [24] designed a tubular mast by using the beam-column equation as it is expressed by Timoshenko [25].

Davenport [2] used the generalized coordinates inferred from the classic solution of the differential equation of the beam-column, and presented formulae leading to the determination of forces and bending moments of guyed masts and chimneys.

Meyers [16] used the theorem of implicit functions to study mast deformation, but the critical load has not been determined in the analysis.

Grabowsky [26] used Clapeyron's equation and by taking into account the influence of the axial force concentrated at each joint of the mast, developed two approximate methods from the determinant he introduced into the theory. This determinant was divided into four sub-determinants. The first element of the main diagonal, called the "reduced determinant", gives the critical load.

Mac Cann [22] developed the method proposed by Hetenyi [27]. By using Mac-Laurin's series, he derived a set of formulae relating together displacements, bending moments and shear force, the distributed external force appearing in the form of a multiple integral of the fourth order.

Goldberg and Gaunt [28] studied the stability of the guyed mast. The formulation of their method, a similar one to Meyer's, led them to a set of non-linear equations, numerically solved by the Runge-Kutta method.

Addie Robert [23] analyzed the non-linearity of the static loads. He also considered vibration under dynamic loads from the deterministic and random points of view.

1.6 Proposed Solution

The procedure outlined in this thesis is in line with that of Kolousek's memoir, and the proposed solution consists in the analysis of an element of the system subjected to any set of loadings, without limiting its generality. Therefore, the loadings may be continuous or non-continuous.

After the derivation of the equations of motion, the choice has to be made between a numerical solution, the finite Elements Method and a mathematical solution. The third solution has been selected because it has appeared the simplest, and apparently nothing has yet been published that is relevant to the research which is being conducted.

In Chapter II, the classic equation of cables is restated, as it was expressed by Kolousek [1]. The bending moment of the system under static loads is analyzed in Chapter III, and static buckling in Chapter IV, allowance being made for the influence of the structure's own weight,

including the weight of the lining.

Due to the complexity of the differential equation of the fifth order obtained by derivation under the sign \int , an expansion in power series of analytic functions has been proposed, leading to a perturbation-type equation with characteristic polynomial of the fourth degree. As a result, it seemed more elegant and more practical to introduce into the thesis the fundamental matrix of the auxiliary system associated with the differential equation which has been derived.

In Chapter V, the case of the bar with constant inertia is analyzed with the help of Bessel functions by extending the Greenhill's [29] solution.

Chapter VI is devoted to the dynamic analysis of the beam with a variable section, and Chapter VII to the dynamic analysis of the beam with constant inertia. For the beam of variable inertia, a partial differential equation of the sixth order has been derived, in the case of damped motion. This partial differential equation reduces to the fifth order when undamped motion is considered. The method used in the static part has led to a characteristic polynomial of the fifth degree.

In Chapter VIII, numerical examples are solved by the use of a computer program developed for the purpose. Chapter IX contains a brief summary of the study, together with concluding remarks.

CHAPTER II
STATIC BEHAVIOUR OF CABLES

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STATIC BEHAVIOUR OF CABLES

2.1 Elastic Support Displacement

Each support of the system is protected against excessive displacement by a set of guys. Under action of the uniform load per unit length, the guy assumes the shape of the catenary, which does not differ much from the parabolic shape. However, the parabolic shape is used (in accordance with general practice in design), since the guy is not only loaded by its own weight and wind, but also by concentrated loads such as isolators, tenders, etc. So, the catenary does not correspond to the real shape of the curve assumed by the guy under loads.

Generally, in the case of four guys symmetrically fastened and opposite, two by two at a point, the most unfavourable imposed conditions for which the structure is to be built are those schematized in Fig. 2.1

At the laying of the cables, the guys being adjusted in calm weather, one will call the state of fixing of the cables that at which wind action and the force at the top of the guy are zero. Thus, only permanent loads are acting upon the system.

In this case, the guy is stretched by the fixing axial force S_0 and acted upon by the uniform load q_0 whose resultant is Q_0 along the chord of the guy and, the maximum displacement is f_0 (Fig. 2.2). Under the wind action, tensile force S is developed in the guy, the transversal load component being Q and the displacement f (Fig. 2.2b).

$$\Delta S = -\frac{8}{3} \frac{f^2}{S} + \frac{Ss}{A_c E_c} + \frac{8}{3} \frac{f_0^2}{S} - \frac{S_0 s}{A_c E_c} \quad (2.1)$$

If

$$\frac{2f}{\frac{s}{2}} = \frac{0}{2S}$$

then:

$$\Delta s = -\frac{Q^2 s}{24 S^2} + \frac{S_s}{A_c E_c} + \frac{Q^2 s}{24 S_0^2} - \frac{S_0 s}{A_c E_c} \quad (2.2)$$

The system being symmetrically stayed and loaded according to the symmetrical plane A.A. (see Fig.2.1), one will designate y as the displacement of the support or joint connection of guys. By considering Fig.2.1 and Fig.2.2, relation (2.3) can be easily derived from y and Δs .

$$y = \frac{\Delta s}{\cos\alpha \cos\beta} \quad (2.3)$$

Thus

$$y = \frac{A'}{S^2} + B' S + C \quad (2.4)$$

In (2.4), the different parameters have the following values:

$$A' = -\frac{Q^2 s}{24 \cos\alpha \cos\beta} \quad (2.5)$$

$$B' = \frac{s}{A_c E_c \cos\alpha \cos\beta} \quad (2.6)$$

$$C = \left(\frac{Q^2 s}{24 S_0} - \frac{S_0 s}{A_c E_c} \right) \frac{1}{\cos\alpha \cos\beta} \quad (2.7)$$

2.2 Support Flexibility

By considering formula (2.4), for a given value of y , one can easily determine the tensile cable force. A third degree equation is then obtained; according to the case, the tensile force is increasing or

decreasing (Fig.2.3). Thus, for a given equilibrium state, the horizontal reaction V at the support numbered k of the continuous beam schematized in Fig. 2.1 is equal to the horizontal component of the tensile force of guys at support k . In the case under consideration, if v_k is the horizontal reaction, then:

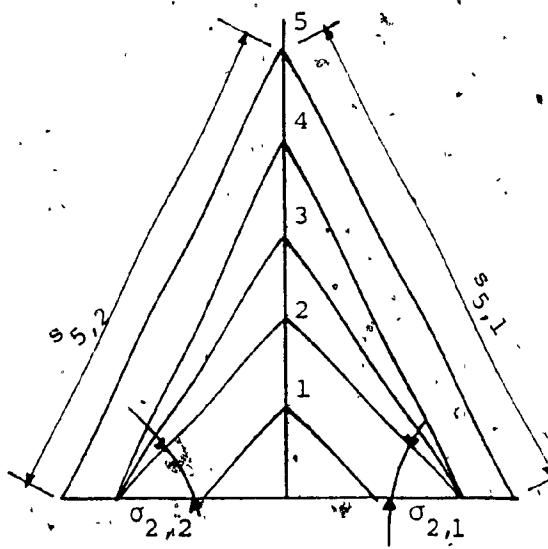
$$v_k = 2(s_{k,I} - s_{k,II}) \cdot \frac{a}{s_{k,I}} \quad (2.8)$$

This reaction acts in the symmetrical plane A-A. Then equations (2.4) and (2.8) establish a non-linear relation between y and V since (2.4) is not a linear one. In order to consider the system as a continuous beam on elastic supports one will assume the linear relation (2.9)

$$y = K + \alpha V \quad (2.9)$$

K and α are constant parameters which can be designated as coefficients of support flexibility.

Then, the representative curve $V(y)$ of equation (2.8) is replaced by a straight line whose equation is formula (2.9). This hypothesis is true if the straight line is tangent to the curve at the point of support (y, V) whose coordinates are exactly the displacement and horizontal reaction at this point of the continuous beam acted upon by the given loading. Since the location of this point is not known a priori, one has to proceed by trial and error.



View

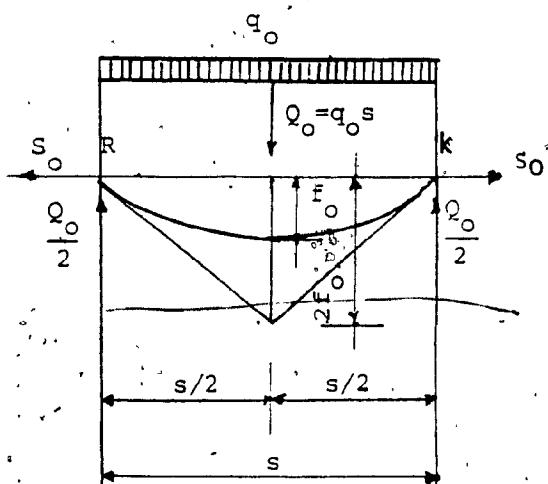


Fig. 2.2: Loads on Guy

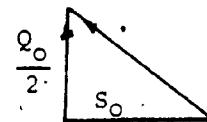


Fig. 2.2a

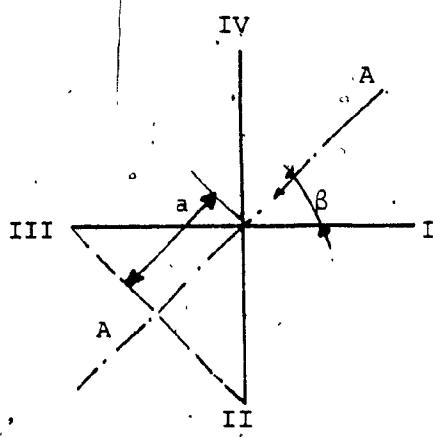


Fig. 2.1: Wind Direction Plan

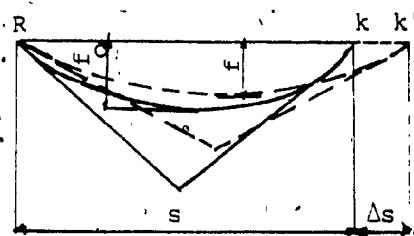


Fig. 2.2b

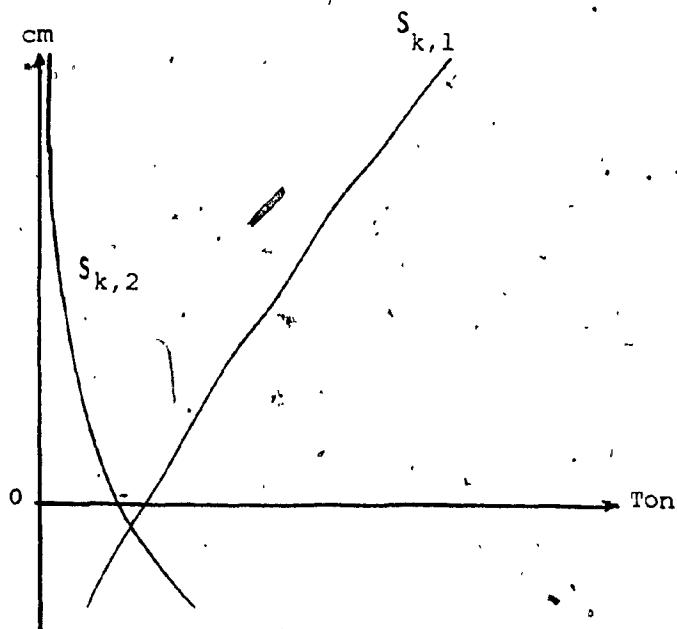


Fig. 2.3: Tensile Forces in Guys

Horizontal scale 1 mm = 500 kg

Vertical scale 1 cm = .5 cm

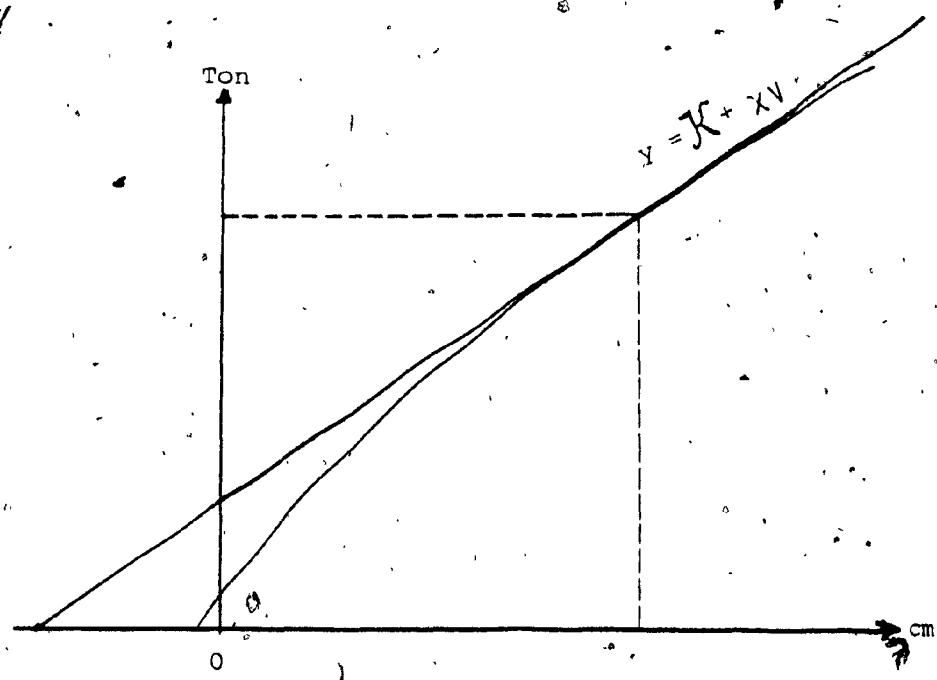


Fig. 2.4: Variations of the Horizontal Reaction V.

Horizontal scale 1 cm = .5 cm

Vertical scale 1 mm = .500 kg

CHAPTER III

BENDING OF THE STACK UNDER STATIC LOAD

CHAPTER III

BENDING OF THE STACK UNDER STATIC LOAD

This chapter is an attempt to analyze the flexural bending of the system subjected to a set of horizontal forces assumed to be static.

3.1 Equilibrium Equations

The system under consideration is schematized by an arc length whose extremities are designated by 1 and 2. The path of this arc length is followed in its motion, and its points M are located by a curvilinear abscissa.

The two vector functions $\zeta(s)$ and $M(s)$, defined along the arc 1-2 (see Fig. 3.1), determine the internal forces in the medium in its motion, and will be considered to be continuously differentiable. They represent the external forces exerted by the part of the body $s \geq s_0$ upon the part of the body $s \leq s_0$, at a point M_0 whose abscissa lies between 1 and 2.

The external forces applied to the system will be defined:

- by distributed forces on 1-2 defined by the linear density $f(s)$ (applied in the domain of definition of the system)
- by forces applied on terminal sections represented by elements of reduction at points 1 and 2 of the wrench that they constitute, namely:

$[F_1]$ with components (V_1, M_1) at end 1

$[F_2]$ with components (V_2, M_2) at end 2

These wrenches correspond to the forces applied at the boundary.

Therefore, one can apply between points 1 and 2, in agreement with Germain [30], the fundamental lemma of Continuum Mechanics, stated below:

Let $\phi(M)$ be a continuous function defined in a domain (\bar{D}) , let (D) be an interior domain to (\bar{D}) ; if

$$\iiint_D \phi(M) dV = 0 \quad (3.1)$$

whatever be the subdomain (D) interior to (\bar{D}) , the function $\phi(M)$ is identically equal to zero in (D)

This leads to the two equilibrium equations:

$$f + \frac{dZ}{ds} = 0$$

$$\frac{d}{ds}(M) + \tau \bar{G} = 0 \quad (3.2)$$

τ : unit vector of the tangent at M to the arc-curve 1-2. Similarly, the wrench of the rate of deformation at each point and at each time will be defined by its elements of reduction at M as $\gamma(s,t)$ and $E(s,t)$. If $U(s,t)$ and $W(s,t)$ designate the speed and the rate of rotation at point M , it follows that formulae (3.3) can be written:

$$\gamma = \frac{\partial W}{\partial s} \quad (3.3)$$

$$E = \frac{\partial U}{\partial s} + \tau \wedge \omega$$

3.2 Hypotheses and Solution

The motion of the elastic body will be located in an Euclidian space. The entire elastic body stayed by guys is considered to be a simply connected system analyzed with all the facilities provided by the

StVenant principle. The behaviour law of elastic media will also be applied to any point of the moving body. The locus of centers of inertia of the right cross-sections is defined by the x-axis, while the directions of y and z axes are those of two principal axes of these sections.

The effects due to shortening or extension of the solid body will be neglected. Temperature effects will not be taken into account.

If each part of the stack lying between two sets of guys is considered to be an element, the system will then be made up of n elements. Therefore, the solution to the flexural bending problem due to static loads is given by the mapping:

$$\mathcal{L} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \quad (3.4)$$

This mapping \mathcal{L} defines the constraints at contiguous or non-contiguous extremities, that is to say the equilibrium equations, which means that the wrenches at extremity i or extremity i+1 form a set equivalent to zero.

3.3 The Law of Inertia

Because the cables serve as supports, the system with variable inertia is analyzed as a bar on elastic supports, having the shape of frustum of a cone. The law of inertia $I(x)$ which is adopted in the present thesis is that proposed by Pschunder [31], so that $I(x)$, as indicated in Appendix A, is a continuous differentiable function. Assuming that the thickness $t'(x)$ verifies at each point of the bar the following inequality (3.5A)

$$\frac{t'(x)}{D(x)} < \frac{1}{20} \quad (3.5A)$$

then

$$I(x) = I_0(1+bx)(1+ax)^3 \quad (3.5)$$

3.4 Element Subjected to a Lateral Displacement and to a Flexural Bending at its Ends

The curve (c) is assumed to be plane and located in the plane xoy. The curvilinear abscissa s is the abscissa x measured along the axis of the element. It is assumed that the bar is subjected to a displacement field at extremities 1 and 2. For the flexural bending analysis, it will be assumed that the external forces are normal to the curve (c) Fig. 3.1. (However, influence of axial load on bending is considered in Appendix K).

By designating $\theta'(x)$ as angular rotation of the section at abscissa x, $u'(x)$ and $y(x)$ as components of the displacement function $U'(x)$ according to ox and oy, the bending moment M being carried by the x-axis, equations (3.5A) and (3.5) allow one to write at any point of the bar formula (3.6)

$$EI(x) \frac{d^2y}{dx^2} = M_2 + V_2x \quad (3.6)$$

If K is the flexural rigidity of the element with constant Inertia I_0 defined by:

$$K = EI_0$$

one will consider the mapping ϕ such that:

$$\phi : x \rightarrow \phi(x) \quad \forall x \in \mathbb{R}^+ \cup \{0\}$$

and defined by:

$$\phi(x) = \frac{M_2}{(1+ax)^3(1+bx)} + \frac{V_2 x}{(1+ax)^3(1+bx)} \quad (3.8)$$

Moreover, one will assign as domain to ϕ the compact set $[0, \ell]$.

Therefore, $\phi(x)$ is uniformly continuous, and the indefinite integral (3.9) is the solution to equation (3.6)

$$Ky(x) = \int \int \phi(x) dx dx' \quad (3.9)$$

If one uses the boundary conditions at extremities 1 and 2, equation (3.9) is expressed in an explicit form by (3.10).

$$\begin{aligned}
 Ky(x) = & V_2 \left[-\frac{b}{(b-a)^3} \left\{ x \log(1+bx) - x + \right. \right. \\
 & \left. \left. \frac{1}{b} \log(1+bx) \right\} + \frac{1}{2a^2(b-a)(1+ax)} - \frac{1}{a(b-a)^2} \log(1+ax) \right. \\
 & \left. + \frac{b}{(b-a)^3} \left\{ x \log(1+ax) - x + \frac{1}{a} \log(1+ax) \right\} + \right. \\
 & \left. \left. \frac{(b+a)x}{2a(b-a)^2} - \frac{1}{2a^2(b-a)} \right] + \right. \\
 & M \left[\frac{b^2}{(b-a)^3} \left\{ x \log(1+bx) - x + \frac{1}{b} \log(1+bx) \right\} \right. \\
 & \left. - \frac{1}{2a(b-a)(1+ax)} + \frac{b}{a(b-a)^2} \log(1+ax) \right. \\
 & \left. - \frac{b^2}{(b-a)^3} \left\{ x \log(1+ax) - x + \frac{1}{a} \log(1+ax) \right\} - \right]
 \end{aligned}$$

$$\frac{3b-a}{2(b-a)^2} + \frac{1}{2a(b-a)} + K\theta'_{2x} + Ky_2 \quad (3.10)$$

Therefore, the hypotheses assumed in paragraph (3.2) are expressed by

(3.11)

$$F = \mathcal{Y} \mathcal{U} \quad (3.11)$$

In this formula \mathcal{Y} is a 4×4 matrix, and \mathcal{U} is the displacement vector at the ends, including effects due to the elastic supports.

3.5 Elements Subjected to a Flexural Bending at End 1

In this case $M_2 = 0$. Then:

$$\phi(x) = \frac{V_2 x}{(1+ax)^3(1+bx)} \quad (3.12)$$

The solution is given by (3.10) in which $M_2 = 0$. Also (3.11) becomes a 3×3 matrix. Note that all conditions of support at extremities 1 and 2 can be considered in (3.10) or (3.11).

3.6 Remark

Note that the preceding formulae are also valid in the case of a hollow section of cylindrical shape (see paragraph 4.5) whose internal diameter is fixed, while the external diameter varies linearly with respect to its thickness.

3.6.1 Remark

If one assumes the cone to be of constant thickness, the preceding equations are applicable in all their generalities. With the definition of the parameters listed in Appendix A, it will suffice

to write: $b = 0$ and $c = 1$.

Therefore, the equation given $y(x)$ can be written as follows:

$$Ky(x) = V_2 \left[-\frac{1}{2a(1+ax)} + \frac{1}{a^3} \log(1+ax) \right]$$

$$- \frac{x}{2a^2} + \frac{1}{2a^3} \right] + M_2 \left[\frac{1}{2a^2(1+ax)} + \right.$$

$$\left. \frac{x}{2a} - \frac{1}{2a^2} \right] + K\theta' V_2 x + Ky_2$$

3.6.2 Remark

If the cross-section is uniform over the entire length of the chimney, one deals with the case of the cylindrical bar. Moreover, in the preceding paragraphs, the lateral displacements at extremities 1 and 2 were considered. One can also take into account the influence of the axial load, if it exists, applied at the ends of the bar (Appendix K). In the case of a tensile force, the field of displacement decreases, and it increases in the case of a compressive force.

3.7 Effect of Transversal Loads

The principle of superposition being applied, the bar is considered to be under fixed supports, preventing any rotation or translation at the extremities, depending on the case under consideration. Given the transverse load function $p(x)$, and neglecting the influence of the axial force, $\phi(x)$ is expressed by (3.14) in the case of a fixed-ends element.

$$\phi(x) = \frac{M_2}{(1+ax)^3(1+bx)} + \frac{V_2 x}{(1+ax)^3(1+bx)} - \frac{p(x) x^2}{2(1+ax)^3(1+bx)}$$

If $p(x)$ is the constant function for a given element, then equation (3.14) is expressed in explicit form by (3.15)

$$\begin{aligned}
 K_y(x) &= V_2 \left[-\frac{b}{(b-a)^3} \left\{ x \log(1+bx) - x + \frac{1}{b} \log(1+bx) \right\} \right. \\
 &\quad + \frac{1}{2a^2(b-a)(1+ax)} - \frac{\log(1+ax)}{a(b-a)^2} + \frac{b^2}{(b-a)^3} \left\{ x \log(1+ax) \right. \\
 &\quad \left. - x + \frac{1}{a} \log(1+ax) + \frac{(b+a)x}{2a(b-a)^2} - \frac{1}{2a^2(b-a)} \right] \\
 &\quad + M_2 \frac{b^2}{(b-a)^3} \left\{ x \log(1+bx) - x + \frac{1}{b} \log(1+bx) \right\} \\
 &\quad - \frac{1}{2a(b-a)(1+ax)} + \frac{b}{a(b-a)^2} \log(1+ax) \\
 &\quad - \frac{b^2}{(b-a)^3} \left\{ x \log(1+ax) - x + \frac{1}{a} \log(1+ax) \right. \\
 &\quad \left. - \frac{(3b-a)x}{2(b-a)^2} + \frac{1}{2a(b-a)} \right] \\
 &\quad + \frac{p}{2} \left[-\frac{1}{(b-a)^3} \left\{ x \log(1+bx) - x + \frac{1}{b} \log(1+bx) \right\} \right. \\
 &\quad \left. + \frac{1}{2a^3(b-a)(1+ax)} - \frac{(-2a^2 + 3ab - b^2)}{a^3(b-a)^3} \log(1+ax) \right]
 \end{aligned}$$

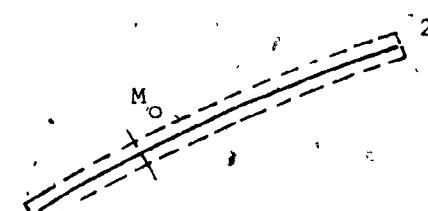
$$\frac{1}{(b-a)^3} \left\{ x \log(1+ax) - x + \frac{1}{a} \log(1+ax) \right\} + \frac{x}{a^2} \left\{ \frac{1}{2(b-a)} + \frac{(-2a^2 + 3ab - b^2)}{(b-a)^3} \right\} \frac{1}{2a^3(b-a)} \quad (3.15)$$

It is sufficient to proceed as in paragraph (3.4) in order to determine the reaction forces, and \mathcal{U} is a function of the lateral load.

In the case where end 2 is hinged and 1 fixed, it is sufficient to write in (3.15): $M_2 = 0$.

3.8. Remark

The remarks stated in paragraph (3.5) are also applicable in the case where the effects of the transverse loadings are considered. A uniform load has been analyzed, but any other law of variation of $p(x)$ can be considered by the previous method, since one is confronted with a double integral and boundaries problem.



Arc Length (c)

Fig. 3.1

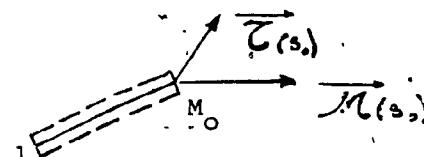


Fig. 3.2

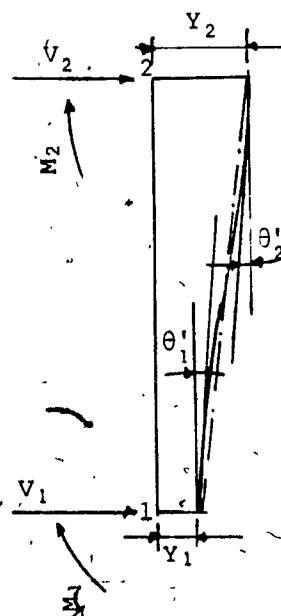


Fig. 3.3

CHAPTER IV

STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIA

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STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIA

4.1 Stability Under Axial Load

The stability of the bar with variable inertia is a complex problem, since the own weight of the structure increases with height and constitutes a non-uniform loading.

In addition, lateral displacement of the body modifies the tension of the guys, which introduce, as reaction, compressive force on the structure, and as was seen above, the support flexibility is a non-linear function. Therefore, it becomes important to take into account the influence of moments introduced by the vertical variable forces according to the vertical axis.

As a result, the analysis of the stack's stability under the set of applied loads becomes more difficult.

In spite of all, stability is an eigenvalue problem. A safety factor 'G' against buckling will be introduced in the search for the eigenvalues, equilibrium conditions at the ends of the bar being expressed by the mapping \mathcal{L} such that:

$$\mathcal{L} : \mathbb{R}^{2 \times n} \rightarrow \mathbb{R}^n$$

The problem to be solved is the following: << for what value of 'G' is the operator \mathcal{L} a singular one? >>

The equation obtained being transcendental, one will assign different values to 'G'. The lowest value of 'G' for which \mathcal{L} is a singular operator gives the critical load.

4.2 Differential Equation of the Neutral Axis

The reference under consideration is the reference oxyz. In the state of deformation of the bar, a bending moment carried by \vec{k} appears in each section. The stability analysis will be performed by neglecting infinitesimal values of order at least equal to two with respect to $y(x)$ and $y'(x)$.

Then, the components of τ , n and the radius of curvature (Fig. 4.1) are written:

$$\tau : (1, y') , \quad n : (-y', 1)$$

$$\frac{1}{R} = y'' \quad \frac{ds}{dx} = 1 \quad (4.1)$$

It can be noted that the own weight, per unit length of the element, is a continuous function of x given by equation (4.2)

$$q(x) = q_0(4 + (a + b)x + 2(1 + ax)(1 + bx)) \quad (4.2)$$

In this formula, q_0 is written

$$q_0 = \frac{\pi}{6} \rho D_2 t_2$$

This load $q(x)$ is distributed along the vertical axis ox. By considering the set of vertical and horizontal loads, the forces acting at ends 1 and 2 of the element, the previous equations, (3.2) and (3.3), are expressed by the integro-differential equation (4.3) which translates the differential equation of the elastic line of the element, in a state of deformation.

$$EI(x) \frac{d^2y}{dx^2} = q_0 \int_x^l (4 + (a+b)x + 2(1+ax)(1+bx))(y(\eta) - y(x))d\eta$$

$$- Py + M_2 + V_2x - p \frac{x^2}{2} \quad (4.3)$$

The inertia function $I(x)$ verifies the restricted inequality $I(x) > 0$. Moreover, on the compact set $[x, l]$, $q(x)$ is uniformly continuous. Therefore, one can apply the following theorem: << If $f(x, n)$ is a function, depending on the parameter n , and if this function is continuous with respect to both of the variables x and n over a certain compact set D , then it is a continuous function of x , and uniformly with respect to the parameter n . >>

Therefore, equation (4.3) is continuously differentiable. Then, by the definition of the bending moments, the shear force, the pressure at any point of the bar, the differential equation of the fifth order (4.4) is obtained.

It indicates the gradient of pressure at point x , and is also the differential equation satisfied by the elastic lines in its deformation. The integral equation has thus been transformed into a differential equation with variable coefficients:

$$\begin{aligned} \frac{d^5y}{dx^5} + \left(\frac{9a}{1+ax} + \frac{3b}{1+bx} \right) \frac{d^4y}{dx^4} + \left(\frac{18a^2}{(1+ax)^2} + \right. \\ \left. + \frac{18ab}{(1+ax)(1+bx)} + \frac{G(P + 6l q_0)}{K(1+ax)^3(1+bx)} - \frac{q_0}{K(1+ax)^3(1+bx)} \right] \cdot \\ \left. \left(6 - 3l(a+b)x + (-12abl + 3a + 3b)x^2 + 2abx^3 \right) \right) \frac{d^3y}{dx^3} + \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{6a^3}{(1+ax)^3} + \frac{18a^2b}{(1+ax)^2(1+bx)} - \frac{q_0}{K(1+ax)^3(1+bx)} \right) [12 - 9\ell(a+b) \\
 & + 15(a+b)x - 12ab\ell x + 16abx^2] \right) \frac{d^2y}{dx^2} \\
 & - \frac{q_0}{K(1+ax)^3(1+bx)} (9(a+b) - 12ab\ell + 24abx) \frac{dy}{dx} = 0 \quad (4.4)
 \end{aligned}$$

4.3 Proposed Method of Solution

The proposed method is related to the one evolved by Fuchs in the study of differential equations of the second order in the neighbourhood of a "singular point". The problem is thus transformed into a perturbation problem by an expansion in series of analytic functions.

In fact, the following hypothesis can be assumed on the value of K :

$$\frac{\ell^2}{K} \ll 1$$

ℓ : length of the element.

Moreover, the two functions $\frac{1}{1+ax}$ and $\frac{1}{1+bx}$ can be expanded into a series of analytic functions inside the disk of radius of convergence:

$$R = \inf\left(\frac{1}{a}, \frac{1}{b}\right) \quad (4.5)$$

this necessarily implies:

$$|ax| < 1 \text{ and } |bx| < 1$$

From the definition of a and b , it follows that:

$$\left| \frac{1-\Delta}{\Delta} \frac{x}{l} \right| < 1 \quad (4.6)$$

$$\left| \frac{1-c}{c} \frac{x}{l} \right| < 1$$

The set of relations (4.5) and (4.6) lead to the two inequalities:

$$\frac{1}{2} < \frac{D_2}{D_1} < 1 \quad (4.7)$$

$$1/2 < t_2/t_1 < 1$$

The inequalities (4.7) are called: << mathematical conditions which limit the diameter and the wall thickness at the top with respect to the diameter and thickness at the bottom of the element of length l >>.

If, for example, the following series is considered:

$$\frac{2 q_0 ab}{K} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+n} (n+1)(n+2) a^n b^m x^{(m+n+3)}$$

With the hypothesis made on $\frac{l}{K}$, it verifies the inequality:

$$\left| \frac{2 q_0 ab}{K} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+n} (n+1)(n+2) a^n b^m x^{(m+n+3)} \right| \leq$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n+1)(n+2) a^{(n+1)} b^{(m+1)} l^{(m+n+2)}$$

equivalent to the estimate of the term of upper degree in the coefficients of $\frac{d^3y}{dx^3}$. This series is absolutely convergent inside the disk of radius of convergence defined by inequality (4.5).

So, if we expand in entire functions the coefficients of equation (4.4), the series is convergent. Therefore, the state of buckling of

the element is given by equation (4.8) below, the parameters a_i , θ_i , $i=1,\dots,4$, being defined in Appendix B.

$$\begin{aligned} \frac{d^5y}{dx^5} + a_1 \frac{d^4y}{dx^4} + a_2 \frac{d^3y}{dx^3} + a_3 \frac{d^2y}{dx^2} + a_4 \frac{dy}{dx} = & \theta_1(n,m) \frac{d^4y}{dx^4} + \\ & \theta_2(m,n,x) \frac{d^3y}{dx^3} + \theta_3(m,n,x) \frac{d^2y}{dx^2} + \theta_4(m,n,x) \frac{dy}{dx} \end{aligned} \quad (4.8)$$

As indicated in Appendix B, this is a perturbation equation. So, the differential equation of the fifth order can be transformed into the differential system (4.9).

$$\frac{du}{dx} = [A + B(x)]u \quad (4.9)$$

In this formula, A is a 4×4 constant matrix, while $B(x)$ is a 4×4 perturbation matrix whose elements $\rightarrow 0$ when both n and $m \rightarrow \infty$, $x \in [0, l]$.

Under the form (4.9), the search for the solution to the problem under consideration is referred to a problem of spectral theory.

In fact, given the set of complex numbers \mathbb{C} and \mathbb{Z} , the set of integers, one defines by \mathcal{M}_λ^v the subspace associated with λ (an eigenvalue) such that:

$$\begin{aligned} \mathcal{M}_\lambda^v = \{x: (\lambda I - A)^v x = 0\} \\ \lambda \in \mathbb{C} \text{ and } v \in \mathbb{Z}^+ \cup \{0\} \end{aligned} \quad (4.10)$$

v : index of the complex number λ relative to the linear operator A .

$E_{\lambda i}$ being projectors of the spectral decomposition associated with A , the solution to the homogeneous equation can be written:

$$u(x) = \sum_{i=1}^v \sum_{m=0}^{v_i-1} (A - \lambda_i I)^m x^m e^{(\lambda_i x)} E_{\lambda_i} u(0) \quad (4.11)$$

I : identity matrix

λ_i : solution to the following equation:

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -a_4 & -a_3 & -a_2 & -(a_1 + \lambda) \end{vmatrix} = 0 \quad (4.12)$$

a_1, a_2, a_3, a_4 : parameters defined in Appendix B. So, in agreement with Roseau [32], any solution to the differential equation (4.9) conforms to a Volterra's [33] integral equation of the second kind

$$u(x) = U(x)u(0) + \int_0^x U(x-\tau)B(\tau)u(\tau) d\tau \quad (4.13)$$

In this formula, $U(x)$ is the fundamental matrix of (4.9), such that:

$$U(0) = I$$

Therefore, the equation satisfied by the deflected shape will be written under the form given by (4.14)

$$y(x) = Y(x)y(0) + \iint_0^x U(x'-\tau)B(\tau)u(\tau)d\tau dx' \quad (4.14)$$

In this formula, $Y(x)$ is a 5×5 matrix defined in Appendix B, and the double integral is reduced to a simple integral for the lines different from the first line.

4.4 Remarks

The knowledge of $y(x)$ implies the complete solution to the problem under consideration. As one is interested only in the critical load, the integral part of equation (4.14) will be ignored. So, (3.4) becomes a system of homogeneous equations with respect to the parameters of deformation.

4.5 Element With a Constant Inside Diameter

The same method is again applied. Since the parameters defined in Appendix A remain the same, a slight modification is introduced by putting:

$$d = \frac{t_2}{D_2} \quad (4.15)$$

The own weight distributed along the vertical axis is given by:

$$q(x) = q_0(6 + 3(1+d)bx + 2db^2x^2) \quad (4.16)$$

In this formula q_0 is defined by:

$$q_0 = \frac{\pi \rho D_2 t_2}{6} \quad (4.17)$$

The differential equation expressing the stability of the element is written:

$$\begin{aligned} & \frac{d^5y}{dx^5} + \left(\frac{9a}{1+ax} + \frac{3b}{1+bx} \right) \frac{d^4y}{dx^4} + \\ & \left\{ \frac{18a^2}{(1+ax)^2} + \frac{18ab}{(1+ax)(1+bx)} + \frac{G(P + 6\ell q_0)}{K(1+ax)^3(1+bx)} - \frac{q_0}{K(1+ax)^3(1+bx)} \right. \\ & \left. (6 - 6b\ell(1+d))x + (3b(1+d) - 2\ell db^2)x^2 + 2dbx^3 \right\} \frac{d^3y}{dx^3} \end{aligned}$$

$$+\left\{\frac{6a^3}{(1+ax)^3} + \frac{18a^2b}{(1+ax)^2(1+bx)} - \frac{q_0}{K(1+ax)^3(1+bx)}\right\} [12 - \\ 9b\ell(1+d) + (15b(1+d) - 12d\ell b^2)x + 16db^2x^2] \left\{\frac{d^2y}{dx^2}\right\} \\ - \frac{q_0}{K(1+ax)^3(1+bx)} \left\{9b(1+d) - 12d\ell b^2 + 24db^2x\right\} \frac{dy}{dx} = 0 \quad (4.18)$$

The new values of parameters a_i and θ_i , $i=1,\dots,4$, being defined in Appendix C, the solution to equation (4.18) is of the same form as that of relation (4.14).

4.6 Bar of Constant Thickness

Consider now the case of the conic bar of constant thickness.

This case can be solved if one uses Lagrange's solution. But, the method described above will be applied. The thickness being constant, $b=0$, $c=1$; the distributed load $q(x)$ is written:

$$q(x) = 3q_0(2+ax) \quad (4.19)$$

In (4.19) q_0 is defined by:

$$q_0 = \frac{\pi \rho D_2 t}{6}$$

The bending moment at a point of ordinate x is given by (4.20)

$$K(1+ax)^3 \frac{d^2y}{dx^2} = 3q_0 \int_x^l (2+ax)[y(n) - y(x)] dn \\ - Py + M_2 + V_2x - px^2/2 \quad (4.20)$$

Whence the differential equation (4.21) verified by $y(x)$:

$$\frac{d^4y}{dx^4} + \frac{6a}{1+ax} \frac{d^3y}{dx^3} + \left[\frac{6a^2}{(1+ax)^2} + \frac{P + 3q_0 \{2\ell - (a\ell - 2)x - ax^2\}}{K(1+ax)^3} \right] \frac{d^2y}{dx^2} + \frac{q_0}{K(1+ax)^3} [6a\ell - 2 - 7ax] \frac{dy}{dx} = p - \frac{1}{K(1+ax)^3} \quad (4.21)$$

If, as was the case earlier, there is to be an expansion in series of analytic functions, the first condition (4.7) is written:

$$1/2 < \frac{D_2}{D_1} \leq 1 \quad (4.22)$$

Thus, the following equation is obtained, when the safety factor 'G' is introduced:

$$\frac{d^4y}{dx^4} + 6a \frac{d^3y}{dx^3} + \left(6a^2 + \frac{G(P + 6\ell q_0)}{K} \right) \frac{d^2y}{dx^2} + \frac{q_0}{K} (6a - 2) \frac{dy}{dx} = \theta_1(x, n) \frac{d^3y}{dx^3} + \theta_2(x, n) \frac{d^2y}{dx^2} + \theta_3(x, n) \frac{dy}{dx} + \frac{P}{K(1+ax)^3} \quad (4.23)$$

$\theta_1, \theta_2, \theta_3$ being defined in Appendix D, the linear system associated with the differential equation (4.23) is written:

$$\frac{du}{dx} = [A = B(\lambda)] u + f(x) \quad (4.24)$$

4.7 Solution to Differential Equation (4.24)

If one considers the homogeneous equation associated with equation (4.24), the characteristic equation is written:

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ \frac{-q_0(6a\ell - 2)}{K} & \frac{-(a + G(\mu + 6\ell q_0))}{K} & -(6a + \lambda) \end{vmatrix} \quad (4.25)$$

So the solution is written:

$$u(x) = U(x)u(0) + \int_0^x U(x-\tau)B(\tau)u(\tau)d\tau + \int_0^x U(x-\tau)f(\tau)d\tau \quad (4.26)$$

In (4.26) $U(x)$ is the fundamental matrix defined in Appendix D.

Therefore, $y(x)$ is written:

$$y(x) = Y(x)y(0) + \int_0^x \int_0^{x'} U(x' - \tau)B(\tau)u(\tau)d\tau dx' + \int_0^x \int_0^{x'} U(x' - \tau)f(\tau)d\tau dx' \quad (4.27)$$

In (4.27), $Y(x)$ is a 4×4 matrix defined in Appendix Q; the double integral is reduced to a simple integral for the lines different from the first line.

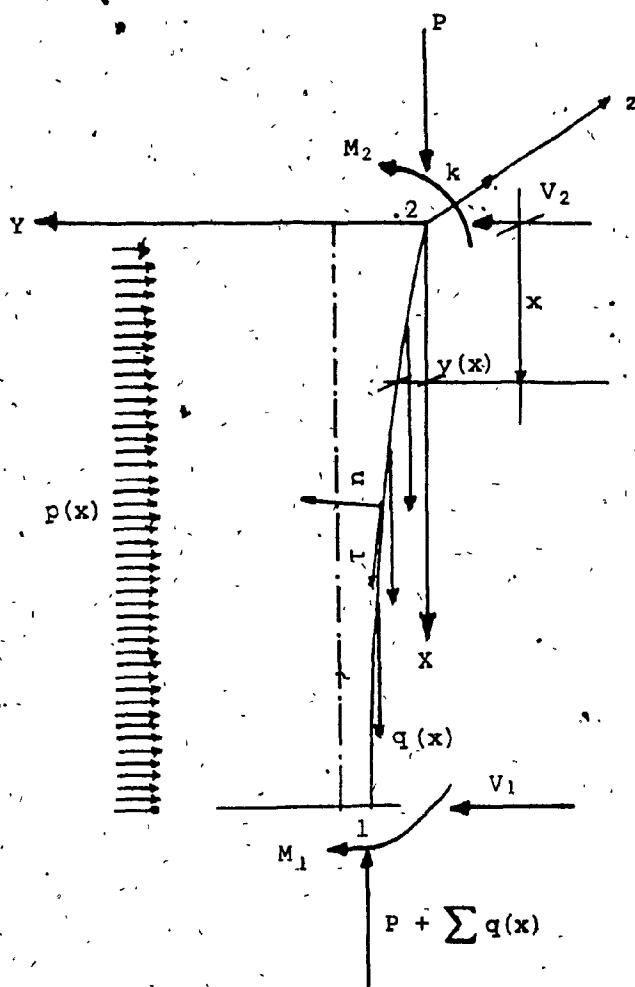


Fig. 4.1: Set of Acting Loads

CHAPTER V

BAR WITH CONSTANT INERTIA - STATIC BUCKLING

CHAPTER V

BAR WITH CONSTANT INERTIA - STATIC BUCKLING

5.1 Introduction

The case of the chimney with constant inertia does not differ much from the well-known case of the guyed tubular mast. However, the formulation of the problem is quite another matter if, in addition to the axial load, one takes into account stresses introduced in the member by the bending moment due to the own weight, uniformly distributed along the vertical axis.

Euler, was the first to state the stability problem of the bar with constant inertia fixed at one end and free at the other, acted upon by its own weight uniformly distributed along the vertical axis. According to Todhunter [34], Euler did not seem to be satisfied with his own solutions.

Greenhill [29], using Bessel functions, carried out the final solution of this problem.

According to Timoshenko [25], Grischoff [35], by using two parameters, tackled the problem of the cantilever acted upon by the axial load and the uniform load. One parameter was related to the axial load, and the other to the uniform load. He set up a table of critical loads of the column. The same method has been extended to the bar hinged at both ends.

Based on Greenhill's [29] solution, and introducing a new operator attached to the Bessel functions and which can be, certainly, extended to the other mathematical functions, a complete solution is proposed, in

the following paragraphs, to the problem stated by Euler.

5.2 Formulation of the Problem

The bar with a vertical axis, acted upon by a set of vertical and horizontal loads, is schematized in Fig. 4.1. The differential equation of the elastic line is expressed by equation (5.1)

$$EI \frac{d^2y}{dx^2} = \int_x^l q_0(y(\eta) - y(x)) d\eta - Py + V_2 x + M_2 - \frac{Px^2}{2} \quad (5.1)$$

If one defines as fictitious length ℓ' the quantity below:

$$\ell' = \frac{P}{q_0} + l \quad (5.2)$$

then, by derivation under the sign \int , the integral equation is transformed into the non-homogeneous differential equation (5.3), of the third order whose left-hand side is similar to the equation derived by Euler:

$$EI \frac{d^3y}{dx^3} + q_0(\ell' - x) \frac{dy}{dx} = V_2 - px \quad (5.3)$$

The solution to the homogeneous equation is of the same form as that of Greenhill [29], as expressed by Timoshenko [25] or Courbon [36].

Then, it is sufficient to solve the equation with the right-hand side, the quantity $q_0(\ell' - x)$ being replaced by $G q_0(\ell' - x)$, where 'G' is the safety factor against buckling.

If one considers the change of variable,

$$z = \frac{2}{3} \sqrt{\frac{G q_0 (\ell^3 - x^3)}{EI}} \quad (5.4)$$

the boundary conditions are written:

$$x = 0 \quad z(0) = \frac{2}{3} \sqrt{\frac{G q_0 \ell^3}{EI}} \quad (5.5)$$

$$x = \ell \quad z(\ell) = \frac{2}{3q_0} \sqrt{\frac{GP^3}{EI}}$$

By the transformation

$$u = \frac{dy}{dz}$$

equation (5.3) is further expressed by equation (5.6) below:

$$\frac{d^2u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \left(1 - \frac{1}{9z^2}\right) u = \frac{2}{3} \left[v_2 + p \left\{ \frac{P}{q_0} + \ell - \left(\frac{3}{2}\right)^{2/3} \left(\frac{EI}{Gq_0}\right)^{1/3} \right\} \right] \frac{1}{Gq_0 z} \quad (5.6)$$

The solution to the homogeneous equation is the Bessel function of the first kind of order $v = 1/3$.

This solution is written:

$$U = C_1 J_{-1/3}(z) + C_2 J_{1/3}(z) \quad (5.7)$$

In equation (5.7), C_1 and C_2 are constants of integration.

The functions J_v and J_{-v} being solutions of the homogeneous

differential equation, the Wronskian is written:

$$\left\{ J_v(z), J_{v'}(z) \right\} = -\frac{2 \sin v\pi}{\pi z} \quad (5.8)$$

5.3 General Solution to Equation (5.6)

5.3.1 On the Existence of a Linear Operator T_n Attached to Bessel Functions

Considering Bessel function J_v , one defines T_n as an operator attached to J_v . This operator designates the only operation of multiplication. Subscript 'n' varying from 0 to ∞ is the integer appearing in the expansion series of $J_v(z)$ such that

$$T_n = 1 \quad \forall n \quad (5.9)$$

By the definition of this symbol, Bessel functions can be handled with more elegance and facility, mainly operations of integration and differentiation since, keeping as denominator the initial Gamma function corresponding to $J_v(z)$, one proceeds as in the elementary operations on polynomials.

Moreover, T_n is a linear operator. It therefore has the following properties:

$$1) T_n(\lambda J_v) = \lambda T_n(J_v)$$

$$2) T_n(J_v + J_\mu) = T_n(J_v) + T_n(J_\mu)$$

3) T_n verifies also (recall that J_v is a special function):

$$T_n(J_v(z)J_\mu(z)) = T_n(J_v(z)) T_n(J_\mu(z))$$

That is, T_n is a ring homomorphism, and:

$$T_n(J_v(z)J_\mu(z)) = \sum_0^\infty \frac{(-1)^m T_n \Gamma(v + \mu + 1 + n + 2m) J_{v+\mu+2m}(z)}{m! \Gamma(\mu + m + 1) \Gamma(v + n + 1)}$$

Then, the following identities hold:

$$T_n(J_v(z)J_\mu(z)) = \sum_{n=0}^\infty \frac{(-1)^n (\mu + v + 2n)_n (1/2z)^{\mu + v + 2n}}{n! \Gamma(\mu + n + 1) \Gamma(v + n + 1)}$$

In this formula, $(\mu + v + 2n)_n$ is obtained when Vandermonde's theorem is used, as stated by Watson [37].

$$T_n(J_v(z)J_\mu(z)) = \frac{(1/2z)^{\mu+v}}{\Gamma(v+1)} \sum_{n=0}^\infty (-1)^n \frac{F(-n; -\mu - n; v + 1, 1)}{n! \Gamma(\mu + n + 1)} (1/2z)^{2n}$$

In this formula, F is the hypergeometric function of Gauss.

In the ordinary conditions of integration and differentiation:

$$4) \int z^t T_n J_v(z) dz = \frac{2^{t+1} T_n \Gamma(t + v + n + 2) J_{v+t+1}(z)}{(t + v + 2n + 1) \Gamma(n + v + 1)}$$

$$5) \frac{d^k}{dz^k} (z^t T_n J_v(z)) = \frac{2^{t-k} T_n \frac{k}{1} (\pi(t + v + 2n + 1 - k)) \Gamma(t + v + n - k) J_{t+v-k}(z)}{\Gamma(1 + v + n)}$$

5.3.2 Solution to Equation (5.6)

Using the ordinary relations between Bessel functions and the properties of the operator T_n , the general solution to equation (5.6) is written:

$$U(z) = C_1 J_{1/3}(z) + C_2 J_{1/3}(z) + \frac{\pi}{\sqrt{3}} V_1 \sum_{k=0}^{\infty} \frac{(-1)^k T_n \Gamma(2+n+2k) J_{1+2k}(z)}{k! \Gamma(2+n+k)}$$

$$\left[\frac{1}{(4/3+2n) B(2/3+k, 4/3+n)} - \frac{1}{(2/3+2n) B(2/3+k, 2/3+n)} \right] +$$

$$2^{5/3} \frac{\pi}{\sqrt{3}} V_2 \sum_{k=0}^{\infty} \frac{(-1)^k T_n \Gamma(8/3+2k+n) J_{5/3+2k}(z)}{k! (1+k+n)!}$$

$$\left[\frac{1}{(2+2n) B(2/3+k, 4/3+n)} - \frac{1}{(4/3+2n) B(4/3+k, 2/3+n)} \right] \quad (5.10)$$

In formula (5.10):

$$V_1 = \frac{2}{3} \left[V_1 + p \left(\frac{P}{q_0} + \lambda \right) \frac{1}{Gq_0} \right]$$

$$V_2 = -\frac{2}{3Gq_0} \left(P \left(\frac{3}{2} \right)^{2/3} \left(\frac{EI}{Gq_0} \right)^{1/3} \right)$$

and $B(p;q)$ is the Eulerian Beta function.

So, it is easy to see that the displacement function $y(x)$ is written:

$$y(x) = C_3 + \frac{3}{2} C_1 \frac{2^{-1/3+1}}{2^{-1/3}} \frac{T_n \Gamma(5/3+n) J_{2/3}(z)}{(3n+1) \Gamma(2/3+n)} +$$

$$\frac{3}{2} C_2 \frac{2^{1/3+1}}{2^{1/3}} \frac{T_n \Gamma(7/3+n) J_{4/3}(z)}{(3n-1) \Gamma(4/3+n)} +$$

$$\frac{3}{2} \sqrt{\frac{\pi}{3}} V_1 \sum_{k=0}^{\infty} \frac{(-1)^k z^2 (2+2k+n)! T_n J_{2+2k}(z)}{k! (1+k+n)! (3+3k+3n)}$$

$$\left[\frac{1}{(4/3+2n) B (2/3+k, 4/3+n)} - \frac{1}{(2/3+2n) B (4/3+k, 2/3+n)} \right]$$

$$+ \frac{3}{2} V_2 \sqrt{\frac{\pi}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{7/3} \Gamma(11/3+2k+n) T_n J_{8/3+2k}(z)}{k! (1+k+n)! (4+3k+n)} \left[$$

$$\frac{1}{(2+2n) B (2/3+k, 4/3+n)} - \frac{1}{(4/3+n) B (4/3+k, 2/3+n)} \right] \quad (5.11)$$

In (5.11), C_3 is a third constant of integration.

5.3.3 Remark

In the preceding paragraphs, one was interested in a factor of safety 'G' such that:

$$1 < G < \infty$$

Then, the series defined by (5.11) is absolutely convergent, since Bessel function J_V is also convergent. This equation is the general solution when four unknowns of that equation are determined.

So, this solution can be used to analyze the bar with constant inertia under all the different support conditions that one might encounter in practice.

5.4 Critical Load

Equation (5.11) leads to the critical load of the bar, when one considers the combined action of the horizontal load, the uniform vertical load and an axial load. The ordinary boundary conditions are as follows:

5.4.1 Bar Hinged at Both Ends

$$\begin{aligned} (y(x))_{z=z(0)} &= 0 \\ x = 0 & \\ \left(\frac{dy(x)}{dx}\right)_{z=z(0)} &= 0 \end{aligned} \tag{5.12}$$

$$\begin{aligned} (y(x))_{z=z(\ell)} &= 0 \\ x = \ell & \\ \left(\frac{dy(x)}{dx}\right)_{z=z(\ell)} &= 0 \end{aligned}$$

5.4.2 Bar as Cantilever

$$\begin{aligned} \left(\frac{d^3y(x)}{dx^3}\right)_{z=z(\ell)} &= 0 \\ x = 0 & \\ \left(\frac{d^2y(x)}{dx^2}\right)_{z=z(\ell)} &= 0 \\ (y(x))_{z=z(\ell)} &= 0 \\ x = \ell & \\ \left(\frac{dy(x)}{dx}\right)_{z=z(\ell)} &= 0 \end{aligned} \tag{5.13}$$

5.4.3 Bar Fixed at Both Ends

$$(y(x))_{z=z(0)} = 0$$

$$x = 0 \quad \left(\frac{dy(x)}{dx} \right)_{z=z(0)} = 0$$

$$(y(x))_{z=z(l)} = 0$$

$$x = l \quad \left(\frac{dy(x)}{dx} \right)_{z=z(l)} = 0$$

(5.14)

5.4.4 Bar fixed at One End and Hinged at the Other

For example:

$$(y(z))_{z=z(0)} = 0$$

$$x = 0 \quad \left(\frac{dy(x)}{dx} \right)_{z=z(0)} = 0$$

$$(y(x))_{z=z(l)} = 0$$

$$x = l \quad \left(\frac{d^2y(x)}{dx^2} \right)_{z=z(l)} = 0$$

(5.15)

The set of conditions (5.12), (5.13), (5.14), (5.15) is expressed by the homogeneous system of the linear equation:

$$AC = 0 \quad (5.16)$$

A and C represent a 4×4 matrix and array column

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ V_1 \end{bmatrix}$$

respectively.

5.5 Continuous Beam on Fixed or Elastic Supports

Considering equation (5.11), the slope deflection method can be used to analyze the bar under multiple fixed or uneven supports. If the number of spans is 'n', the equilibrium conditions at each support are expressed by the mapping \mathcal{L} such that:

$$\mathcal{L} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \quad (5.17)$$

Then, the operator A expressing (5.17) is a $2n \times 2n$ matrix. In (5.16), as in (5.17), the homogeneous system of linear equations obtained has a solution if the determinant associated with A is equal to zero.

The equation $f(z)$ obtained being transcendental, different values will be assigned to ' G '.

If:

$$\det A = 0 \quad (5.18)$$

then the least value of ' G ' which satisfies (5.18) leads to the first value of the critical load. To this value of ' G ' there corresponds a specific value of $Z(0)$ and $Z(l)$.

Considering the set of relations (5.5), a simple algebraic manipulation leads to formula (5.19) which expresses the critical load:

$$(P + q_0 l/3)_{cr} = \frac{3}{4} \frac{EI}{l^2} (z^2(0) - z(l)^2) - \frac{q_0^{1/3}}{l} \left(\left(\frac{3}{2} z(l) \right)^{2/3} (EI) \right)^{2/3} \quad (5.19)$$

5.5.1 Uniform Distributed Load Acting Alone

It is interesting to ask at what height the failure of the column will occur under the uniform load acting alone and according to various conditions of support. In this case

$$P = 0, \quad l = l'$$

5.5.2 Column as a Cantilever

This case is the Greenhill [35] solution. One knows that:

$$z = 1.868$$

So, the following identities hold:

$$(q_0 l)_{cr} = \begin{bmatrix} 7.83 \frac{EI}{l^2} \\ 3.1734 \frac{(\pi^2 EI)}{4l^2} \\ \frac{\pi^2 EI}{(1.222 l)^2} \end{bmatrix}$$

5.5.3 Bar Hinged at Both Ends

As a supplementary condition to those expressed by (5.13), one imposes

$$C_1 = 0$$

Then, conditions of support imply:

$$C_3 = 0$$

In this case A is a 2×2 matrix. The least value of z satisfying (5.12) is given by:

$$z = 3.56665$$

Then, the following identities hold:

$$(q_0 \ell)_{cr} = \begin{bmatrix} 28.662 \frac{EI}{\ell^2} \\ * \\ 2.9002 \frac{(\pi^2 EI)}{\ell^2} \\ * \\ \frac{\pi^2 EI}{(0.5872 \ell)^2} \end{bmatrix}$$

5.5.4 Bar fixed at Both Ends

A is again a 2×2 matrix; the least value of z satisfying conditions (5.14) is given by

$$z = 4.81165$$

Then the following identities hold.

$$(q_0 \ell)_{cr} = \begin{bmatrix} 52.09194 \frac{EI}{\ell^2} \\ * \\ 1.3195 \frac{(4\pi^2 EI)}{\ell^2} \\ * \\ \frac{\pi^2 EI}{(0.43428\ell)^2} \end{bmatrix}$$

5.5.5 Bar fixed at One End and Hinged at the Other

For example, the origin is assumed to be fixed. It seems that this problem can be solved only by using an arbitrary function. So one imposes:

$$C_2 = 0$$

and defines the following arbitrary function:

$$\mu(z) = z^{1/3} C_1 \left[J_{-1/3}(z) + \frac{1}{c} \right] + \frac{\pi}{\sqrt{3}} V_1 \sum_{k=0}^{\infty} \frac{T_n J_{2k+1}(z)}{(2n+2k+1)}$$

$$\left[\frac{1}{B(2/3+n, 7/3+n+2k)} - \frac{1}{B(4/3+n, 5/3+n+2k)} \right] +$$

$$V_1 \frac{\pi}{\sqrt{3}} T_n \Gamma \left(\frac{8}{3} + 2n \right) J_{5/3}(z) \left[\frac{2^2}{2^{1/3} (2+2n) \Gamma (7/3+2n) B(2/3+n, 5/3+n)} \right]$$

$$- \frac{2^{4/3}}{2^{-1/3} (4/3+2n) \Gamma (6/3+2n) B(2/3+n, 4/3+n)} \quad (5.21)$$

c is an arbitrary constant such that:

$$5 < c < 6$$

$$28.622 \frac{EI}{l^2} < (q_0 l)_{cr} < 52.09194 \frac{EI}{l^2} \quad (5.22)$$

In the second inequality (5.22), the lower limit and the upper limit correspond respectively to the case of the bar hinged or built at both ends.

Then:

$$y(x) = \frac{3C_1}{2} \left[\frac{2^{4/3} T_n \Gamma (2+n) J_1(z)}{(3/2+3n) \Gamma (2/3+n)} + \frac{z^{4/3}}{2c} \right] +$$

$$\frac{3}{2} V_1 \frac{\pi}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^2 (2+2k+n)! T_n J_{2k+2}(z)}{k! (1+k+n)! (3+3k+3n)}$$

$$\left[\frac{1}{(4/3+2n) B(2/3+k, 4/3+n)} - \frac{1}{(2/3+2n) B(4/3+k, 2/3+n)} \right]$$

$$+ \frac{3}{2} \sqrt{\frac{\pi}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{7/3} \Gamma(11/3 + 2k + n) T_n J_{8/3 + 2k}(z)}{k! (1+k+n)! (4+3k+n)}$$
$$\left\{ \left[\frac{1}{(2+2n) B(2/3+k, 4/3+n)} - \frac{1}{(4/3+2n) B(4/3+k, 2/3+n)} \right] \right\} \quad (5.25)$$

If $c = 5$ the set of relations (5.15)

$$z = 4.05026$$

Then:

$$(q_0 \ell)_{cr} = \begin{bmatrix} 36.91037 \frac{EI}{\ell^2} \\ 1.8693 \frac{(\pi^2 EI)}{(0.707 \ell)^2} \\ \frac{\pi^2 EI}{(0.5171 \ell)^2} \end{bmatrix}$$

If $c = 6$ then:

$$z = 4.35618$$

And:

$$(q_0 \ell)_{cr} = \begin{bmatrix} 42.69668 \frac{EI}{\ell^2} \\ 2.16277 \frac{(\pi^2 EI)}{(0.707 \ell)^2} \\ \frac{\pi^2 EI}{(0.480788 \ell)^2} \end{bmatrix}$$

CHAPTER VI

DYNAMIC ANALYSIS. BAR WITH VARIABLE INERTIA

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DYNAMIC ANALYSIS. BAR WITH VARIABLE INERTIA

The above static analysis emphasizes the difficulty of analysis of the stack with variable inertia or of any similar structure (according to the assumed law of inertia). In order to analyze the dynamic behaviour of the system, the same hypotheses assumed in paragraphs (3.1) and (3.2) are considered. Moreover, it is commonly accepted in practice that oscillation under Karman vortices is the most dangerous for stacks.

The design one is working on will therefore be valid for both the plane of action of horizontal forces and the plane perpendicular to it, if one ignores the bidimensional or tridimensional nature of the problem. The only thing subject to change is the intensity of the exciting forces, according to the case under consideration.

First of all, the motion of an element of the system is analyzed and, secondly, the influence of the motion of the guys is determined by using the findings of Kolousek [1] or Davenport [2] on guys. The method described in Chapter IV will be applied.

6.1 Equation of Motion

The dynamic behaviour of the guyed chimney is a nonlinear problem. That can be seen if one takes into account the shortening or elongation of the element under the axial load. However, these effects and those due to temperature will not be considered. Then, the chimney and the set of guys constitute a system (S) in motion in the reference previously defined (in paragraph 4.2). At time t , the mass M of the

part (Ω) of (S) whose density is $\rho(M, t)$, M being a point of the system. This mass $M(\Omega)$ is given by formula (6.1) below:

$$M(\Omega) = \iiint_{\Omega} \rho(M, t) dv \quad (6.1)$$

The field of velocity vectors of the point M of (Ω) being designated by $V(M)$, (Ω) and (A) being respectively the wrenches of momentum and acceleration, in the neighbourhood of M , the density of acceleration is defined by $\gamma(M)$:

Then, the fundamental law of dynamics is written:

$$[A] = \frac{d}{dt} [\Omega] = [F] \quad (6.2)$$

The equilibrium equations of the system in its displacement around the vertical axis are given by the integro-differential equation (6.3), $m(x)$ being the system's unit mass.

$$\begin{aligned} \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 y}{\partial x^2} + m(x) \frac{\partial^2 y}{\partial t^2} &= \int_x^l 4ab[y(n, t) - y(x, t)] \left(\frac{q_0}{g} \right) dn + \\ \frac{q_0}{g} [6 - 6\ell(a+b) + \{9(a+b) - 8ab\ell\}x + 10abx^2] \frac{\partial y(x, t)}{\partial x} + \\ \frac{q_0}{g} [-6\ell + \{6 - 3\ell(a+b)\}x + \{3(a+b) - 2ab\ell\}x^2 + \\ 2abx^3] \frac{\partial^2 y(x, t)}{\partial x^2} + p(x, t) - \frac{P}{g} \frac{\partial^2 y(x, t)}{\partial x^2} \end{aligned} \quad (6.3)$$

One can transform (6.3) into (6.4):

$$\frac{\partial^5 y(x,t)}{\partial x^5} + \left[\frac{9a}{1+ax} + \frac{3b}{1+bx} \right] \frac{\partial^4 y(x,t)}{\partial x^4} +$$

$$\left[\frac{P}{Kg(1+ax)^3(1+bx)} + \frac{18a^2}{(1+ax)^2} + \frac{18ab}{(1+ax)(1+bx)} - \right.$$

$$\left. \frac{q_0}{Kg(1+ax)^3(1+bx)} \right\} -6\ell + [6 - 3\ell(a+b)]x + [3(a+b) - 2ab\ell]x^2 +$$

$$2abx^3] \frac{\partial^3 y(x,t)}{\partial x^3} + \left[\frac{6a^3}{(1+ax)^3} + \frac{18a^2b}{(1+ax)^2(1+bx)} - \right.$$

$$\left. \frac{q_0}{Kg(1+ax)^3(1+bx)} \right\} 12 - 9\ell(a+b) + [15(a+b) - 12ab\ell]x +$$

$$16abx^2] \frac{\partial^2 y(x,t)}{\partial x^2} - \frac{q_0}{Kg(1+ax)^3(1+bx)} \left[$$

$$9(a+b) - 12ab\ell + 24abx] \frac{\partial y(x,t)}{\partial x} +$$

$$\frac{q_0}{Kg(1+ax)^3(1+bx)} \left[6 + 3(a+b)x + 2abx^2 \right] \frac{\partial^3 y(x,t)}{\partial x \partial t^2} -$$

$$\frac{q_0}{Kg(1+ax)^3(1+bx)} \left[3(a+b) + 4abx \right] \frac{\partial^2 y(x,t)}{\partial t^2} =$$

$$\left(\frac{\partial}{\partial x} p(x,t) \right) \frac{1}{K(1+ax)^3(1+bx)} \quad (6.4)$$

Inside the disk of radius $\inf\left(\frac{1}{a}, \frac{1}{b}\right)$, let us perform an expansion in series of analytic functions. Then, the partial differential equation (6.5) of the fifth order, with two variables, is obtained (see Appendix E):

$$\begin{aligned} & \frac{\partial^5 y}{\partial x^5} + a_1 \frac{\partial^4 y}{\partial x^4} + a_2 \frac{\partial^3 y}{\partial x^3} + a_3 \frac{\partial^2 y}{\partial x^2} - \\ & \frac{q_0}{Kg} [9(a+b) - 12ab\ell] \frac{\partial y}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^3 y}{\partial x \partial t^2} + \frac{3q_0(a+b)}{Kg} \frac{\partial^2 y}{\partial t^2} = \\ & \frac{\partial}{\partial x} (p(x,t)) \frac{1}{Kg(1+ax)^3(1+bx)} + \theta_1 \frac{\partial^4 y}{\partial x^4} + \theta_2 \frac{\partial^3 y}{\partial x^3} \\ & \theta_3 \frac{\partial^2 y}{\partial x^2} + \theta_4 \frac{\partial y}{\partial x} + \theta_5 \frac{\partial^3 y}{\partial x \partial t^2} + \theta_6 \frac{\partial^2 y}{\partial t^2} \end{aligned} \quad (6.5)$$

In the set of indefinitely differentiable functions, let us choose $y(x,t)$ as being the product of two independent functions; this product means that the shape of the curve taken by the system does not vary with time, and that amplitude is the only thing that varies harmonically under conditions of free vibration.

Therefore:

$$y(x,t) = Y(x) \cdot T(t) \quad (6.6)$$

6.2 Natural Frequency of the System

The amplitude is assumed to be small in the undamped motion. So, what is the natural frequency of the structure under the action of Karman vortices or the variable thrust due to any set of horizontal forces, such that it is possible to avoid resonance, when the axial load is identical to zero?

First of all, the influence of the mass of guys is neglected. Therefore, a beam-element and equation (6.5) are considered. One applies (6.6) to the homogeneous part of (6.5). The two following differential equations are obtained:

$$y^{(5)}(x) + a_1 y^{(4)}(x) + a_2 y^{(3)}(x) + a_3 y^{(2)}(x) + a_4 y^{(1)}(x) + a_5 y(x) = 0 \quad (6.7)$$

$$\ddot{T} + \omega^2 T = 0 \quad (6.8)$$

Parameters $a_1, a_2, a_3, a_4, a_5, \omega$ are defined in Appendix E.

Note that the characteristic polynomial leading to the eigenvalues is of the fifth degree. Any numerical method or the new one proposed by the author* for solution to algebraic equations of order ≥ 5 , can be used in order to find the eigenvalues.

Then, using Fubini's theorem, the general solution to (6.5) is written (see Appendix E):

$$y(x,t) = Y(x,t) y(0) + \int_0^x \int_0^t Y(x-\mu, t-\tau) B(\mu, \tau) y(\mu, \tau) d\mu d\tau \quad (6.9)$$

So, with the aid of Gronwall's lemma, and since one is interested only in the homogeneous system of equations, the first term in the right-hand side of (6.9) will be used.

So,

$$y(x,t) = Y(x,t) y(0) \quad (6.10)$$

Then index v_i of the complex number λ_i has been taken to be equal to

* To appear: <<Contribution à la solution par radicaux des polynomes algébriques de degré ≥ 5 .>> Il a été découvert que les équations dérivées forment un groupe à 4 éléments de même structure que le groupe de Klein.

1; the case where $v_i > 1$ has been excluded for the problem under consideration, since the partial derivatives of the second, third and fourth orders are different from zero.

6.3 Flexibility of the Supports

Large deformations not being considered, one agrees with Kolousek when he says that, for minor displacements, the support flexibility of the symmetrically guyed pylons (the pylon not being loaded) is the same in any direction.

Therefore, oscillations are the same whatever be the plane of vibration under consideration. In this case, the flexibility of the support and the horizontal force at the level k of the set of guys are given by Kolousek's [1] formulae (6.11) and (6.12).

$$\frac{1}{X} = \frac{2Eu}{s} \cos^2\sigma + 2Eu \cos\sigma \left(\frac{s^2}{8f} - \frac{2fs}{3} \frac{Eu}{S_0} \cos\sigma \right) \quad (6.11)$$

$$V = \frac{1}{X} y_k$$

6.4 Influence of the Motion of the Guys

The results quoted in this paragraph are excerpted from Davenport's [2] publication.

Considering Fig. 6.1, one formulates, with Kolousek [1] and Davenport [2], equation (6.13) in order to find the modes of vibration of guys.

$$\frac{w}{g} \left(\frac{x}{s} \sin\sigma \frac{\partial^2 v(t)}{\partial t^2} + \frac{\partial^2 n(t)}{\partial t^2} \right) = \Delta T(t)y'' + Tn''(t) \quad (6.13)$$

The top displacement of the guy being given by:

$$v(t) \cos\sigma = \frac{\Delta T s}{A_c E_c} + \int_0^s y'' n(t) dx \quad (6.14)$$

when guys and column are vibrating in the same plane, the introduction of (6.14) in (6.13), and the expansion in series of Fourier lead to formula (6.15), which expresses the dynamic modulus of deflection of the elastic support.

$$\frac{K}{x} = \frac{E_c A_c \cos^2\sigma}{s} \left[1 - \frac{F \Omega^2 - 1}{g \cdot \phi(\Omega) - 1} \right] \quad (6.15)$$

In (6.18), parameters have the following values:

$$F = \frac{\pi^2 T^2 \sin\sigma}{2ws^2 E_c A_c \cos^2\sigma}$$

$$g = \frac{\pi^2 T^3}{s^3 w^2 E_c A_c \cos^2\sigma}$$

$$\Omega^2 = \frac{s^2 w \omega^2}{\pi^2 g T}$$

$$\phi(\Omega) = \frac{\Omega^2}{1 - \tan(\frac{\pi\Omega}{2})} \quad (6.16)$$

In the case where cables and mast are not vibrating in the same plane, it is assumed that the vertical plane of the guys forms the angle θ' with the plane of vibration of the column (see Fig. 6.2). Like

Davenport [2], one defines in this case:

$$\cos\phi = \frac{\cos\theta' \cdot \cos\sigma}{1 - \cos^2\theta' \cos^2\sigma}$$

(6.17)

Then, the dynamic modulus of deflection of the elastic support is expressed by (6.18)

$$\frac{1}{X} = \frac{E_c A_c \cos^2\sigma \cos^2\theta'}{s} \left[1 + \frac{\cos\theta'}{\cos\phi} \frac{F \cdot \Omega^2 - \cos^2\phi}{G_\phi(\Omega) - \cos^2\phi} \right] \quad (6.18)$$

The parameters have the same significance as in (6.15). Note also that the dynamic modulus of the set of guys is obtained by algebraic summation.

6.5 Mode of Vibration

The system being continuous, there exists an infinite number of modes of vibration obtained from equilibrium equations. The frequency equation ($f(\omega)$) being transcendental, it will suffice to plot the curve by assigning values to ω .

If one defines as Σ the set $\{\omega_i\}$ such that:

$$f(\omega_i) = 0$$

It is easy to find the subspace $M_{\omega_i}^{v_i}$, that is to say the fundamental modes of vibration. Note that, from the definition of Σ , $\{\omega_i\}$ is an infinite set.

6.6 Dynamic Buckling

The mass is, in vibration problems, a factor of paramount importance, since it tends to decrease the amplitude of oscillation of the structure.

So, the set of gravity loads constituted by elements of the system

and the vertical force transmitted by guys under horizontal thrusts, slows down vibration. There follows a decrease in the critical buckling load. Only the first critical force is of interest, since, beyond it, the system is unstable. This dynamic critical load is obtained when inertia forces and the set of vertical forces form a system equivalent to zero. Then, P is different from zero in equation (6.4).

6.7 Forced Vibration and Damping

One will now consider vibration with the presence of viscous forces. Two types of viscous damping can be incorporated into the formulation: resistance to transverse displacement of the element and a viscous resistance to straining of the beam material. They limit the amplitude of vibration of the structure. The linear behaviour of the material is applied below. Moreover, it is known that the complex Young modulus of a material is given by:

$$E^* = E' + i E''$$

where E' is the elastic or storage moduli, and E'' , the loss moduli.

For a material such as steel, it is accepted that

$$E \approx |E^*| \approx E'$$

So,

$$C_s = (1 + \delta_a^2) I(x) E \quad (6.19)$$

where C_s is the resistance to strain velocity, and δ_a : the damping.

The resistance to transverse velocity is commonly represented by $C'(x)$.

Then, the damping force corresponding to $C'(x)$ and the damping stress corresponding to C_s are given by formulae (6.20).

$$f_d = C_1' y \quad (6.20)$$

$$\sigma = C_s \dot{\varepsilon}$$

If $p(x,t)$ is the density of the applied forces, then one obtains below the partial differential equation (6.21) of the sixth order with variable coefficients

$$C_s \frac{\partial^6 y(x,t)}{\partial x^5 \partial t} + \frac{\partial^5 y(x,t)}{\partial x^5} + \left(\frac{9a}{1+ax} + \frac{3b}{1+bx} \right) \cdot$$

$$\left[C_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right] + \left[\frac{18a^2}{(1+ax)^2} + \frac{18ab}{(1+ax)(1+bx)} \right] \cdot$$

$$\left[C_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right] + \frac{1}{Kg(1+ax)^3(1+bx)} \left[p \right]$$

$$-q_0 \{-6\ell + [6 - 3\ell(a+b)]x + [3(a+b) - 2ab\ell]x^2 +$$

$$2abx^3\} \frac{\partial^3 y(x,t)}{\partial x^3} - \frac{q_0}{Kg(1+ax)^3(1+bx)} \left[12 - 9\ell(a+b) \right]$$

$$+ \{15(a+b) - 12ab\ell\}x + 16abx^2 \left[\frac{\partial^2 y(x,t)}{\partial x^2} \right]$$

$$\frac{q_0}{Kg(1+ax)^3(1+bx)} \left[9(a+b) - 12ab\ell + 24abx \right] \frac{\partial y(x,t)}{\partial x} +$$

$$\frac{q_0}{Kg(1+ax)^3(1+bx)} \left[6 + 3(a+b)x + 2abx^2 \right] \frac{\partial^3 y(x,t)}{\partial x \partial t^2} +$$

$$\frac{q_0}{Kg(1+ax)^3(1+bx)} [3(a+b) + 4abx] \frac{\partial^2 y(x,t)}{\partial t^2} +$$

$$\frac{C'}{K(1+ax)^3(1+bx)} \frac{\partial^2 y(x,t)}{\partial x \partial t} = \left(\frac{\partial}{\partial x} p(x,t) \right) \frac{1}{K(1+ax)^3(1+bx)} \quad (6.21)$$

In (6.21):

$$C'_s = 1 + \frac{\delta^2}{a}$$

An expansion in series of analytic functions leads to equation (6.22), as indicated in Appendix F.

$$\frac{C'_s \partial^6 y(x,t)}{\partial x^5 \partial t} + (9a + 3b) \left[\frac{C'_s \partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right] +$$

$$\frac{\partial^5 y(x,t)}{\partial x^5} + (18a^2 + 18ab) \left[\frac{C'_s \partial^4 y(x,t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x,t)}{\partial x^3} \right] +$$

$$(6a^3 + 12a^2b) \left[\frac{C'_s \partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right] -$$

$$\frac{q_0}{Kg} (12 - 9\ell(a+b)) \frac{\partial^2 y}{\partial x^2} + \frac{(6q_0 \ell + b)}{Kg} \frac{\partial^3 y(x,t)}{\partial x^3} +$$

$$\frac{q_0}{Kg} [9(a+b) - 12ab\ell] \frac{\partial y}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^3 y(x,t)}{\partial x \partial t^2} +$$

$$\frac{3(a+b)q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{C'}{K} \frac{\partial^2 y(x,t)}{\partial x \partial t} =$$

$$\frac{1}{K(1+ax)^3(1+bx)} \cdot \frac{\partial}{\partial x} p(x,t) + \phi\left(x, \frac{\partial^k y(x,t)}{\partial x^s \partial t^u}, t\right) \quad (6.22)$$

$$k = 1,5 \quad s = 0,5 \quad u = 0,2 \quad s+u = k$$

6.8 Solution to Equation of Motion

As formulated by equation (6.22), the problem becomes easy to solve. So, it suffices to use one of the common methods of mathematical physics, with certain modifications. The solution proposed here involves the utilization of the classical method of d'Alembert or the method of separation of variables, and the derivation with respect to the time variable.

The two operations lead to the fundamental differential equations (6.23) and (6.24)

$$y^{(5)}(x) + a_1 y^{(4)}(x) + a_2 y^{(3)}(x) + a_3 y^{(2)}(x) + a_5 y(x) + a_6 = 0 \quad (6.23)$$

$$T + \omega^2 T - \lambda' T = 0 \quad (6.24)$$

λ' being a constant of integration and ω the frequency.

Note that the axial force does not appear in equation (6.23), whether or not the influence of this force is taken into account. Eight constants of integration appear in the solution. However, they are not independent, since one is dealing with a partial differential equation of the sixth order.

As indicated in Appendix E, the general solution to (6.22) is expressed by (6.25), using Fubini's theorem.

$$y(x,t) = Y(x,t)y(0) + \iint_0^x \int_0^t Y(x-\mu, t-\tau)B(\mu, \tau)y(\mu, \tau)d\mu d\tau \\ + \iint_0^x \int_0^t Y(x-\mu, t-\tau)f(\tau, \mu)d\mu d\tau \quad (6.25)$$

6.9 Remark

When the axial force is equal to or different from zero, one obtains either free vibration in the presence of viscous forces or the dynamic buckling load, by applying the equilibrium equation.

6.9.1 Remark

The boundary conditions applied to $y(x,t)$ and to its derivatives with respect to the two variables x and t , provide the complete solution to the problem stated above.

6.9.2 Remark

The method used in paragraphs 6.1 and 6.8 leads to the terms

$\frac{\partial}{\partial x} p(x,t)$ instead of $p(x,t)$. According to the expression of $p(x,t)$, the third term in the right-hand side of (6.25) can, therefore, be equal to or different from zero. However, with the help of the initial conditions, axial forces (according to the case under consideration) and unit load density will be incorporated in the general solution.

6.10 Bar With Constant Inside Diameter

6.10.1 Undamped Motion

The unit mass being obtained from relation (4.16), the equation of motion is expressed by relation (6.26)

$$\frac{\partial^5 y(x,t)}{\partial x^5} + \left[\frac{9a}{1+ax} + \frac{3b}{1+bx} \right] \frac{\partial^4 y(x,t)}{\partial x^4}$$

$$\left[\frac{p}{Kg(1+ax)^3(1+bx)} + \frac{18a^2}{(1+ax)^2} + \frac{18ab}{(1+ax)(1+bx)} - \frac{q_0}{Kg(1+ax)^3(1+bx)} \right]$$

$$\left\{ -6\ell + [6 - 6b(1+d)\ell]x + [3b(1+d) - 12d\ell b^2]x^2 + \right.$$

$$2db^3x^3 \left] \frac{\partial^3 y}{\partial x^3}(x,t) + \left[\frac{6a^3}{(1+ax)^3} + \frac{18a^2b}{(1+ax)^2(1+bx)} \right] \right.$$

$$\left. \frac{q_0}{Kg(1+ax)^3(1+bx)} \right\} 12 - 9b\ell(1+d) + [15b(1+d) - 12d\ell b^2]x +$$

$$16db^2x^2 \left] \frac{\partial^2 y}{\partial x^2}(x,t) - \frac{q_0}{Kg(1+ax)^3(1+bx)} \right[9b(1+d)$$

$$12d\ell b^2 + 24db^2x \left] \frac{\partial y}{\partial x}(x,t) + \frac{q_0}{Kg(1+ax)^3(1+bx)} \right[$$

$$6 + 3b(1+d)x + 2db^2x^2 \left] \frac{\partial^3 y}{\partial x \partial t^2}(x,t) + \frac{q_0}{Kg(1+ax)^3(1+bx)} \right[$$

$$3b(1+d) + 4db^2x \left] \frac{\partial^2 y}{\partial t^2}(x,t) = \frac{1}{K(1+ax)^3(1+bx)} \frac{\partial}{\partial x} p(x,t) \right. \quad (6.26)$$

6.10.2 Natural Frequency of the System

As in paragrph (6.2), one will set $P=0$ and will search for the solution as product of two functions defined by (6.6). Then, one obtains the following equation:

$$y^{(5)} + a_1 y^{(4)}(x) + a_2 y^{(3)}(x) + a_3 y^{(2)}(x) + a_4 y^{(1)}(x) + a_5 y(x) = 0 \quad (6.27)$$

$$\ddot{T} + \omega^2 T = 0 \quad (6.8)$$

The coefficients are defined in Appendix C.

So, the solution is expressed by (6.25).

6.10.3 Forced Vibration and Damping

In this case, one obtains the partial differential equation of the sixth order similar to (6.21). The expansion in series of analytic functions leads to (6.28), the function $\phi(x, \frac{\partial^k y}{\partial x^s \partial t^u}, t)$ being defined in Appendix G.

$$\begin{aligned}
 & C'_s \frac{\partial^6 y(x,t)}{\partial x^5 \partial t} + (9a + 3b) \left[C'_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right] + \\
 & \frac{\partial^5 y(x,t)}{\partial x^5} + [18a^2 + 18a^2 b] \left[C'_s \frac{\partial^4 y(x,t)}{\partial x^4} + \frac{\partial^3 y(x,t)}{\partial x^3} \right] + \\
 & (6a^3 + 12a^2 b) \left[C'_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right] + \\
 & \frac{1}{Kg} (P + 6q_0 \ell) \frac{\partial^3 y(x,t)}{\partial x^3} - \frac{q_0}{Kg} [12 - 9\ell b(1+d)] \frac{\partial^2 y(x,t)}{\partial x^2} - \\
 & \frac{q_0}{Kg} [9b(1+d) - 12d\ell b^2] \frac{\partial y(x,t)}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^3 y(x,t)}{\partial x \partial t^2} + \\
 & \frac{3b(1+d) q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{C'}{K} \frac{\partial^2 y(x,t)}{\partial x \partial t} = \\
 & \left(\frac{\partial}{\partial x} p(x,t) \right) \frac{1}{K(1+dx)^3(1+bx)} + \phi \left(x, \frac{\partial^k y(x,t)}{\partial x^s \partial t^u}, t \right) \quad (6.28)
 \end{aligned}$$

$$k = 1,5 \quad s = 0,5 \quad u = 0,2 \quad s + u = k$$

Then, the solution is expressed by equation (6.29)

$$y(x,t) = Y(x,t)y(0) + \iint_{0 0}^{x t} (x - \mu, t - \tau)B(\mu, \tau)y(\mu, \tau)d\mu d\tau \\ + \iint_{0 0}^{x t} Y(x - \mu, t - \tau)f(\tau, \mu)d\mu d\tau \quad (6.29)$$

6.11 Constant Wall Thickness and Variable Diameter

6.11.1 Undamped Motion

The step followed in paragraphs (6.1) and (6.2) is again the same, but the order of the partial differential equation to be solved decreases by 1.

As in paragraph (6.2), neglecting the influence of motion due to the guys in the analysis of free vibration, the external load will be taken as equivalent to zero for the element under consideration. By using equation (4.23), which expresses the static buckling load, and by using the method indicated in paragraph (6.1), one obtains the following partial differential equation of the fourth order (see Appendix I):

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{6a\partial^3 y(x,t)}{\partial x^3} + [6a^2 + \frac{1}{gK}(P + 6\ell q_0)] \frac{\partial^2 y(x,t)}{\partial x^2} \\ \frac{3q_0}{Kg} (-2 + 2a\ell) \frac{\partial y(x,t)}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} = \\ \frac{p(x,t)}{K(1+ax)^3} + \phi(x, \frac{\partial^k y(x,t)}{\partial x^s \partial t^u}, t) \quad (6.30)$$

$$s = 1, 3 \quad u = 0, 2 \quad s + u = k, \quad k = 1, 3$$

By considering the homogeneous equation and the solution as given by (6.6), one arrives at the two differential equations (6.31) and (6.32).

$$y^{(4)}(x) + 6ay^{(3)}(x) + [6a^2 + \frac{1}{gK} (P + 6q_0\ell)]y^{(2)}(x) - \\ \ddot{T} + \omega^2 T = 0 \quad (6.32)$$

Then the general solution to equation (6.30) is given by (6.33), as in Appendix H.

$$y(x,t) = Y(x,t)y(0) + \int_0^x \int_0^t Y(x-\mu, t-\tau)B(\mu, \tau)y(\mu, \tau)d\mu d\tau \\ + \int_0^x \int_0^t Y(x-\mu, t-\tau)f(\mu, \tau)d\mu d\tau \quad (6.33)$$

In equation (6.33) Y and B are two 4×4 matrices. This equation can be used to analyze either free or forced vibration.

6.11.2 Damped Vibration

As in paragraph (6.7), the two types of viscous forces, given by (6.20), are considered in the course of the analysis.

In this case, the equation of motion is given by the partial differential equation (6.34) of the fifth order.

$$C_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} + \frac{6a}{1+ax} C_s \left(\frac{\partial^4 y(x,t)}{\partial x^3 \partial t} \right) + \\ \left(\frac{\partial^3 y(x,t)}{\partial x^3} \right) + \frac{6a^2}{(1+ax)^2} \left(C_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right) + \\ \frac{1}{Kg(1+ax)^3} [P + 3q_0(\ell-x)(2+ax)] \frac{\partial^2 y(x,t)}{\partial x^2} +$$

$$\begin{aligned}
 & \frac{3q_0}{Kg(1+ax)^3} [-2 + 2al - 3ax] \frac{\partial y(x,t)}{\partial x} + \\
 & \frac{3q_0(2+ax)}{Kg(1+ax)^3} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{C'}{Kg(1+ax)^3} \frac{\partial y(x,t)}{\partial t} = \\
 & \frac{p(x,t)}{K(1+ax)^3} \tag{6.34}
 \end{aligned}$$

As was the case above, the expansion in series of analytic functions which has been proposed, leads to (6.35)

$$\begin{aligned}
 & C'_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} + 6a \left(C'_s \frac{\partial^4 y(x,t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x,t)}{\partial x^3} \right) + \\
 & 6a^2 \left(C'_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right) + \frac{1}{Kg} [p + 6q_0 l] \frac{\partial^2 y(x,t)}{\partial x^2} + \\
 & \frac{3q_0}{Kg} (-2 + 2al) \frac{\partial y(x,t)}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{C'}{Kg} \frac{\partial y(x,t)}{\partial t} = \\
 & \frac{1}{K(1+ax)^3} p(x,t) + \phi \left(x, \frac{\partial^k y(x,t)}{\partial x^s \partial t^u}, t \right) \tag{6.35}
 \end{aligned}$$

$$k = 1, 4 \quad s = 0, 4 \quad u = 0, 2 \quad s+u=k$$

When the solution to the homogeneous equation is considered, one obtains equations (6.36) and (6.37):

$$y^{(4)}(x) + 6ay^{(3)}(x) + 6a^2y^{(2)}(x) + \left(\frac{C' - 6q_0 \omega^2}{C'_s Kg} \right) y(x) = 0 \tag{6.36}$$

$$\ddot{T} + \omega^2 T - \lambda' T = 0 \tag{6.37}$$

The general solution to (.635) is of the same form as the solution given by (6.33), but seven constants of integration appear in it. So, one can obtain either the free vibration or the dynamic buckling load.

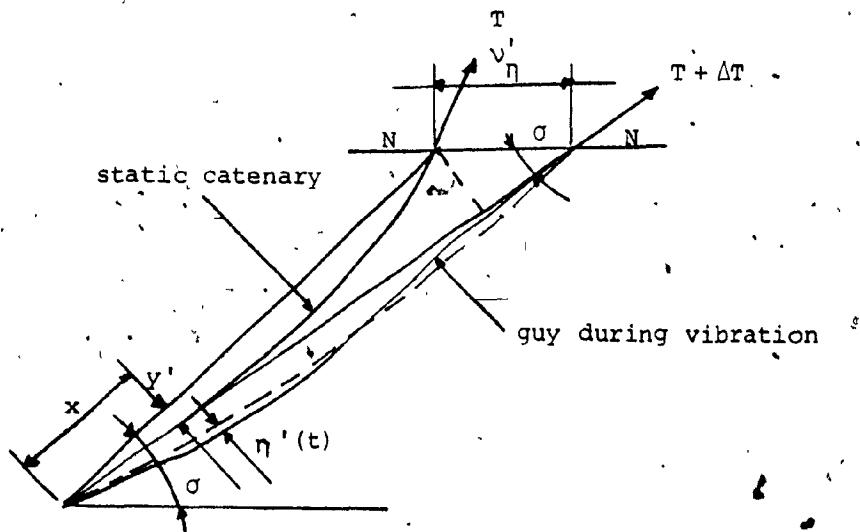


Fig. 6.1: Displacement of Guy During Vibration

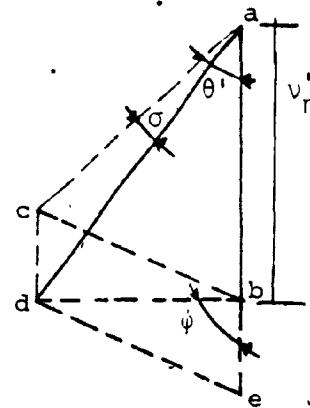


Fig. 6.2: Displacement Diagram for Guy Anchor
When the Guy and Vibration of the Stack
are not Coplanar

CHAPTER VII
CONSTANT INERTIA DYNAMIC ANALYSIS

CHAPTER VII

CONSTANT INERTIA DYNAMIC ANALYSIS

Apparently, the case of the beam with constant inertia is the simplest one. However, the results obtained in the analysis of the static buckling load show that the problem becomes complicated when the influence of the own weight is considered.

7.1 Undamped Vibration

Let us apply the dynamic equation by using the integro-differential equation as formulated in the static part. In this way, one obtains the partial differential equation of the fourth order:

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \frac{1}{gK} [P + q_0(\ell - x)] \frac{\partial^2 y(x,t)}{\partial x^2} - \frac{q_0}{Kg} \frac{\partial y(x,t)}{\partial x} + \frac{q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{1}{K} p(x,t) \quad (7.1)$$

Again, (7.1) is an equation with variable coefficients. The search for the solution in the form given by (6.6) leads to (7.2) and (7.3)

$$y^{(4)}(x) + \frac{1}{gK} [P + q_0(\ell - x)] y^{(2)}(x) - \frac{q_0}{Kg} y^{(1)}(x) - \frac{q_0}{Kg} \omega^2 y = 0$$
$$\ddot{T} + \omega^2 T = 0 \quad (7.3)$$

7.2 Solution to the Equation of Motion

Let us seek a solution as an expansion in series of functions convergent at $x=0$. First of all, by the change of variable

$$x \rightarrow x/\ell$$

one defines:

$$\alpha_1 = \frac{P + q_0 \ell}{\ell^2 g K}$$

$$\alpha_2 = -\frac{q_0}{\ell^2 g K}$$

$$\alpha_3 = -\frac{q_0}{\ell^3 g K}$$

$$\alpha_4 = -\frac{q_0 w^2}{g K \ell^4}$$

Then, (7.2) is written:

$$y^{(4)}(x) + (\alpha_1 + \alpha_2 x) y^{(2)}(x) + \alpha_3 y^{(1)}(x) + \alpha_4 y(x) = 0 \quad (7.4)$$

Let us consider equation (7.4); the method used in the search for a solution to Hermite's differential equation of order 2 is considered, as formulated, for example, in Parodi's [38] work. The solution sought will be of the form:

$$y(x) = x^r + \sum_{n=1}^{\infty} c_n x^{r+n} \quad (7.5)$$

In the case under consideration, the indicial equation $F(r)$ is written:

$$r(r-1)(r-2)(r-3) = 0. \quad (7.6)$$

To the root $r=3$, one obtains the corresponding solution:

$$y_1(x) = x^3 - \frac{\alpha_1}{20} x^5 - \frac{4\alpha_2 + \alpha_3}{120} x^6$$

$$\sum_{p=2}^{\infty} \frac{[\alpha_1(r+p)(r+p-1)c_p + \{\alpha_2(r+p-1) + \alpha_3\}(r+p)c_{p-1} + \alpha_2c_{p-2}]}{(r+p+2)(r+p+1)(r+p-1)} x^{r+p} \quad (7.7)$$

With:

$$c_{r+p} = \frac{[\alpha_1(r+p)(r+p-1)c_{p+1} + \alpha_2(r+p-1)(r+p)c_p + \alpha_4c_{p-1}]}{(r+p)(r+p+1)(r+p+2)(r+p+3)}$$

$$c_0 = 1$$

$$c_1 = 0$$

$$c_2 = -\frac{\alpha_1}{20}$$

$$c_3 = -\frac{4\alpha_2 + \alpha_3}{120}, \quad c_4 = \frac{\alpha_1^2 - \alpha_4}{840}$$

To the root $r = 2$, one obtains the corresponding solution:

$$y_2(x) = x^2 + c_1x^3 - \frac{\alpha_1}{12} x^4 - \frac{1}{60} (3\alpha_1c_1 + \alpha_2 + \alpha_3) x^5$$

$$\sum_{p=2}^{\infty} \frac{[\alpha_1(r+p)(r+p-1)c_p + \{\alpha_2(r+p-2) + \alpha_3\}(r+p-1)c_{p-1} + \alpha_4c_{p-2}]}{(r+p+2)(r+p+1)(r+p)(r+p-1)} x^{4+p} \quad (7.8)$$

With:

$$c_0 = 1$$

c_1 : indeterminate

$$c_2 = -\frac{\alpha_1}{12}$$

$$c_3 = -\frac{1}{60} [3\alpha_1c_1 + \alpha_2 + \alpha_3]$$

$$c_{r+p} = -\frac{[\alpha_1(r+p)(r+p-1)c_p + \{\alpha_2(r+p-2) + \alpha_3\}(r+p-1)c_{p-1} + \alpha_4c_{p-2}]}{(r+p+2)(r+p+1)(r+p)(r+p-1)}$$

To the root $r = 1$, one obtains the corresponding solution:

$$y_3(x) = x + C_1 x^2 + C_2 x^3 - \frac{2\alpha_1 + \alpha_3}{24} x^4 - \frac{6\alpha_1 C_2 + 2\alpha_2 C_1 + \alpha_4}{120} x^5$$

$$\sum_{p=3}^{\infty} \frac{[\alpha_1(r+p)(r+p-1)C_p + \{\alpha_2(r+p-2) + \alpha_3\}(r+p-1)C_{p-1} + \alpha_4 C_{p-2}]}{(r+p+2)(r+p+1)(r+p)(r+p-1)} x^{(3+p)} \quad (7.9)$$

with:

C_1 and C_2 indeterminate.

$$C_0 = 1$$

$$C_{r+p} = - \frac{\alpha_1(r+p)(r+p-1)C_p + \{\alpha_2(r+p-2) + \alpha_3\}(r+p-1)C_{p-1} + \alpha_4 C_{p-2}}{(r+p+2)(r+p+1)(r+p)(r+p-1)}$$

To the root $r = 0$, corresponds the solution:

$$y_4(x) = 1 + C_1 x + C_2 x^2 + C_3 x^3 - \frac{(2\alpha_1 C_2 + \alpha_3 C_1 + \alpha_4)}{24} x^4$$

$$\sum_{p=3}^{\infty} \frac{[\alpha_1 p(p-1)C_p + \{\alpha_2(p-2) + \alpha_3\}(p-1)C_{p-1} + \alpha_4 C_{p-2}]}{(p+2)(p+1)p(p-1)} x^{2+p} \quad (7.10)$$

with:

C_1, C_2, C_3 indeterminate

$$C_4 = - \frac{[2\alpha_1 C_2 + \alpha_3 C_1 + \alpha_4]}{24}$$

$$C_{p+2} = - \frac{[\alpha_1 p(p-1)C_p + \{\alpha_2(p-2) + \alpha_3\}(p-1)C_{p-1} + \alpha_4 C_{p-2}]}{(p+2)(p+1)p(p-1)}$$

Then, the solution to the inhomogeneous equation is written:

$$y(x) = (A_1 y_1(x) + A_2 y_2(x) + A_3 y_3(x) + A_4 y_4(x))(E_1 \cos \omega t + E_2 \sin \omega t) \quad (7.11)$$

That is to say: $y(x,t)$ is expressed as a linear combination of the preceding solutions, and the general solution to (7.1) can always be written:

$$y(x,t) = Y(x,t)y(0) + \int_0^x \int_0^t Y(x-\mu, t-\tau)f(\mu, \tau)d\mu d\tau \quad (7.12)$$

7.3 Remark

Note that, as in the search for solutions to differential equations of the Hermite type, one obtains for each solution that corresponds to the roots of the indicial equation $F(r)$ a number of arbitrary parameters identical to the integer which is the difference between the greatest root and the root under consideration.

In addition to those parameters, there exist six other constants of integration. The number of relations that one can write corresponds to the number of parameters. However, the problem becomes more and more complex, and it seems that we are on our way to obtaining non-linear relations among those constants.

So, for the problem under consideration, one is restricted to the case where the six constants obtained from (7.6) are identical.

They will be taken as 1.

7.4 Remark

Since the objective is the determination of the critical dynamic load, the integral part of equation (7.12) will not be considered. So,

the linear system associated with the equilibrium equations leads to the natural frequency or to the dynamic buckling load.

7.5 Damped Vibration

By using once again relations (6.20), and by applying the equilibrium equation, one gets the partial differential equation (7.13) of the fifth order with variable coefficients.

$$C_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} + \frac{1}{Kg} [P + q_0(\ell - x)] \frac{\partial^2 y}{\partial x^2}(x,t) - \\ \frac{q_0}{Kg} \frac{\partial y(x,t)}{\partial x} + \frac{q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{C'}{Kg} \frac{\partial y(x,t)}{\partial t} = \frac{1}{K} p(x,t) \quad (7.13)$$

The search for the solution in the form (6.6), and the method indicated in paragraph (6.8), lead to (7.14) and (7.15).

$$y^{(4)}(x) + \frac{q_0}{C'Kg} (1 - \omega^2 C') y(x) = 0 \quad (7.14)$$

$$\ddot{T} + \omega^2 \dot{T} - \lambda' T = 0 \quad (7.15)$$

It is interesting to find that, if $\omega^2 C' < 1$, (7.14) is of the same form as the differential equation of the beam on elastic support formulated in the old Winkler hypothesis.

So, according to the value of ω , the quantity $(1 - \omega^2 C')$ can be either positive, negative or zero. The method has therefore led to the formulation of differential equations with constant coefficients. From the boundary values, one obtains the influence of the axial load, whether or not it is considered. The elements of the fundamental matrices are

given below:

a) $1 - \omega^2 C' > 0$

$$i = 1 \quad \alpha = \sqrt{\frac{q_0}{C' \text{Kg}} (1 - \omega^2 C')}$$

$$y_{11}(x) = \frac{1}{(2\alpha)^{j-1}} \text{ch}ax \cdot \cos ax$$

$$y_{12}(x) = \frac{1}{(2\alpha)^{j-1}} \text{sh}ax \cdot \cos ax + \text{ch}ax \cdot \sin ax$$

$$y_{13}(x) = \frac{1}{(2\alpha)^{j-1}} \sin ax \cdot \text{sh}ax$$

$$y_{14}(x) = \frac{1}{(2\alpha)^{j-1}} \text{ch}ax \cdot \sin ax - \text{sh}ax \cdot \cos ax$$

$$i > 1 \quad y_{ij} = \frac{d}{dx} y_{(i-1)j}(x)$$

b) $1 - \omega^2 C < 0$

$$y_{11}(x) = \frac{1}{2} \frac{1}{\alpha^{j-1}} (\text{ch}ax + \cos ax)$$

$$y_{12}(x) = \frac{1}{2\alpha^{j-1}} (\text{sh}ax + \sin ax)$$

$$y_{13}(x) = \frac{1}{2\alpha^{j-1}} (\text{ch}ax - \cos ax)$$

$$y_{14}(x) = \frac{1}{2\alpha^{j-1}} (\sin \alpha x - \sin \alpha x)$$

$$i > 1 \quad y_{ij}(x) = \frac{1}{2\alpha^{j-1}} \frac{d}{dx} (y_{(i-1)j}(x))$$

$$c) \quad i - \omega^2 C = 0$$

$$y_{1j}(x) = \frac{x^{(j-1)}}{(j-1)!}$$

$$i > 1 \quad y_{ij}(x) = \frac{d}{dx} y_{(i-1)j}(x)$$

Then the general solution to (7.13) is written:

$$y(x,t) = Y(x,t)y(0) + \int_0^x \int_0^t Y(x-\mu, t-\tau)f(t, \tau)d\mu d\tau \quad (7.16)$$

CHAPTER VIII

NUMERICAL EXAMPLES

CHAPTER VIII

NUMERICAL EXAMPLES

This chapter sets out to illustrate the theory which has been developed. Some numerical examples related only to the bar with variable inertia and to the bar with constant inertia, are proposed.

8.1 Horizontal Loads

As examples of horizontal loads, the wind effects are analyzed. It is clear that one will proceed in the same way to analyze any other type of horizontal forces such as blasts or earthquakes.

The speed of the wind reference is assumed to be of 100 mph or 160 Km-h at 10 m above ground level. On each element of the chimney, lying between two sets of guys, the wind force is assumed to be uniform and constant.

8.2 Properties of the Stack

As an example, let us analyze a stack 200 m tall stayed by five sets of guys. It is assumed to be hinged at the bottom. The vertical distance between two sets of guys is constant and equal to 38 m. An overhang of 10 m terminates the stack. The lining is assumed to be of light concrete whose specific weight is taken as 1500 Kg/m³.

The different operations and parameters are summarized in the following tables.

8.3 Remark on Cables

In an unpublished memoir, the author [39] has shown the existence of a parameter called multiplier, analogous to Lagrange's Multipliers in Analytical Mechanics and which has been used to analyze the vibration of

continuous beams.

When formulae (2.3) are used to compute the tensile forces in upward and leeward cables for a given value of the displacement, the tensile forces are not the same for the zero displacement and for a certain level of guys. It seems that it is possible to introduce the multiplier in equations (2.3) in order to set tensile force in the cables back to the initial tension, for this zero displacement. The result is shown in Tables 2 to 10 below.

The static critical load obtained is shown in tables 3 to 10, while the different equations of frequency lead to the following values:

ω_1 : Fundamental Frequency (radians per second) P : Axial load at level 38m						
stack with variable inertia $\omega_1 =$	Undamped Motion		Damped Motion			
	$P = 0$	$P = 486$ Tons	$P = 0$	$P = 486$ Tons	$P = 0$	$P = 486$ Tons
	8.7963	8.6091	8.7468	8.5476		
stack with con- stant inertia $\omega_1 =$	66.56	66.22	28.4108	28.0686	66.30	65.36

Note that the magnification factor can be easily obtained from the integral part of the solution to the dynamic motion of the stack.

TABLE NO. 1

Level (m)	Diam. cm	A_c cm^2	E_c t.m.^2	Weight (kg/m)	Windward Cable		Leeward Cable		Displac. (evaluated) (m)
					S_0 (Tons)	S (Tons)	S_0 (Tons)	S (Tons)	
38.	3.016	5.19	16.10^6	4.409	6.00	28.94	6.00	0.75	.35
76.	3.18	5.82	16.10^6	4.85	6.50	27.85	6.50	1.71	.50
114.	3.33	6.50	16.10^6	5.40	7.50	29.75	7.50	3.94	.75
152.	3.49	7.00	16.10^6	5.87	8.50	29.34	8.50	7.09	.95
190.	3.65	7.75	16.10^6	6.47	9.50	32.64	9.50	8.96	1.15

Cable Properties

TABLE NO. 2

Level (m)	Diam. cm	A_c cm^2	E_c T.m.^2	Weight (kg/m)	Windward Cable		Leeward Cable		Displac. (evaluated) (m)
					S_o (Tons)	S (Tons)	S_o (Tons)	S (Tons)	
38.	3.016	5.19	$16 \cdot 10^6$	4.409	6.00	14.40	6.00	2.04	.13
76.	3.18	5.82	$16 \cdot 10^6$	4.85	6.50	15.34	6.50	2.64	.26
114.	3.33	6.50	$16 \cdot 10^6$	5.40	7.50	15.70	7.50	3.34	.39
152.	3.49	7.00	$16 \cdot 10^6$	5.87	8.50	15.27	8.50	6.87	.52
190.	3.65	7.75	$16 \cdot 10^6$	6.47	9.50	16.56	9.50	7.53	.65

Cable Properties (use of the multiplier)

TABLE NO. 3

Level (m)	D (m)	T (m)	EI (T.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (T)	V ₂ (T)	F _{cr} (T)	Displac. (evaluated) (m)
0	3.182	0.02	4.329.886	-0	-0				
38.	3.03076	0.01886	3.489.026	-291.8	-286.4	-18.0	-17.9	11360.	0.273
76.	2.87952	0.01772	2.776.176	-386.5	-386.5	-17.6	14.9	9008.	0.608
114.	2.72828	0.01658	2.179.169	-274.8	-274.8	-14.5	15.5	6815.	1.058
152.	2.57704	0.01544	1.683.376	-96.2	-96.2	-15.1	20.4	4418.	1.607
190.	2.4258	0.0143	1.568.265	-35.35	-35.35	-19.9	7.07	1950.	2.189

Forces and Displacements of Stack (with axial load)

TABLE NO. 4

Level (m)	Diam. (m)	Thick (m)	EI (T.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (Ton)	V ₂ (Ton)	F _{cr} (evaluated) (m)	Displac. (evaluated) (m)
0	3.182	0.02	4.329.886	0	0	-	-	0	0
38	3.03076	0.01986	3.489.026	-259.52	-255.89	-16.62	14.476	11360	300
76	2.87952	0.01772	2.776.716	-286.8	-286.8	-14.09	9.68	9008	.647
114.	2.72828	0.01658	2.179.716	-109.87	-109.87	-9.25	10.12	6815	1.062
152.	2.57704	0.01544	1.683.376	54.05	54.05	-9.67	16.62	4418	1.486
190.	2.4258	0.0143	1.568.265	-35.35	-35.35	-16.15	7.07	1950	1.842

Forces and Displacements of Stack (without axial load)

TABLE NO. 5

Level (m)	Diam. (m)	Thick. (m)	EI (T.m ²)	M ₁ (T.M)	M ₂ (T.m)	V ₁ (Tons)	V ₂ (Tons)	F _{cr} (Tons)	Displac. (evaluated) (m)
0.	3.182	0.02	14.329.886	0.	0.				0.
38.	3.03076	0.01886	3.489.026	268.2	-263.2	-17.2	17.9	11360	0.324
76.0	2.87952	0.01772	2.776.716	-375.3	-375.3	-17.5	12.6	9008	0.703
114.0	2.72828	0.01658	2.179.169	-207.9	-207.9	-12.2	14.5	6815	1.187
152.0	2.57704	0.01544	1.683.376	-48.0	-48.0	-14.03	19.45	4418	1.737
190.	2.4258	0.0143	1.568.265	-35.35	-35.35	-18.9	7.07	1950	2.288

Forces and Displacements of Stack (axial load and use) of the multiplier)

TABLE NO. 6

Level (m)	Diam. (m)	Thick. (m)	EI (Ton.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (Tons)	V ₂ (Tons)	F _{cr} (evaluated) (m)
0.	3.182	0.02	4.329.886	0.	0.			0.
38.	3.03076	0.01886	3.489.06	-245.1	-245.1	-16.24	15.29	11360 .343
76.	2.87952	0.01772	2.776.716	-303.4	-303.4	-14.91	8.74	9008 .731
114.	2.72828	0.01658	2.179.169	-91.06	-91.06	-8.31	10.56	6815 1.188
152.	2.57704	0.01544	1.683.376	56.13	56.13	-10.11	16.76	4418 1.649
190.	2.4258	0.0143	1.568.265	-35.35	-35.35	-16.20	7.07	1950 2.0399

Forces and Displacements of Stack (without axial, use of the multiplier)

TABLE NO. 7

Level (m)	Diam. (m)	Thick (m)	E_1 (T.m ²)	M_1 (T.m)	M_2 (T.m)	V_1 (Ton)	V_2 (Ton)	F_{cr} (Ton)	Displac (evaluated) (m)
0	2.70	0.015	2.434.797	0	0				0
38.	2.70	0.015	2.434.797	-291.8	-286.4	-18.0	18.0	5143	0.273
76.	2.70	0.015	2.434.797	-386.5	-386.5	-17.6	14.9	3202	0.608
114.	2.70	0.015	2.434.797	-274.8	-274.8	-14.5	15.5	1820	1.058
152.	2.70	0.015	2.434.797	-96.21	-96.2	-15.1	20.4	619	1.607
190.	2.70	0.015	2.434.797	-35.35	-35.35	-19.9	7.07	47	2.189

Forces and Displacements of Stack (with axial load)

TABLE NO. 8

Level (m)	Diam. (m)	Thick (m)	EI (T.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (Tons)	V ₂ (Tons)	F _{cr} (Tons)	Displac. (evaluated) (m)
0	2.70	0.015	2.434.797	0	0				0
38.	2.70	0.015	2.434.797	-259.6	-255.9	-16.6	14.5	5142	0.300
76.	2.70	0.015	2.434.797	-286.9	-286.9	-14.1	9.7	3282	0.647
114.	2.70	0.015	2.434.797	-109.9	-109.9	-9.3	10.1	1820	1.062
152.	2.70	0.015	2.434.797	54.9	53.9	-9.7	16.6	619	1.486
190.	2.70	0.015	2.434.797	-35.35	-35.35	-16.1	7.07	-47	1.842

Forces and Displacements of Stack (without axial load)

TABLE NO. 9

Level (m)	Diam. (m)	Thick (m)	EI (T.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (Tons)	V ₂ (Tons)	F _{cr} (Tons)	Displac. (evaluated) (m)
0	2.70	0.015	2.434.797	0	0				0
38.	2.70	0.015	2.434.797	-268.2	-263.2	-17.2	17.9	5143	0.324
76.	2.70	0.015	2.434.797	-375.3	-375.3	-17.5	12.6	3282	0.703
114.	2.70	0.015	2.434.797	-207.9	207.9	-12.2	14.5	1820	1.186
152.	2.70	0.015	2.434.797	-48.0	-48.0	-14.0	19.5	6.9	1.737
190.	2.70	0.015	2.434.797	-35.3	-35.3	-18.9	7.07	-87	2.287

Forces and Displacements of Stack (axial load and use of the multiplier)

TABLE NO. 10

Level (m)	Diam. (m)	Thick. (m)	EI (T.m ²)	M ₁ (T.m)	M ₂ (T.m)	V ₁ (Tons)	V ₂ (Tons)	F _{cr} (Tons)	Displac. (evaluated) (m)
0	2.70	0.015	2.434.797	0	0				0
38	2.70	0.015	2.434.797	-245.2	-241.6	-16.2	15.3	5143	0.343
76	2.70	0.015	2.434.797	-303.5	-303.5	-14.9	8.7	3282	0.730
114	2.70	0.015	2.434.797	-91.1	-91.1	-8.3	10.6	1820	1.188
152	2.70	0.015	2.434.797	56.1	56.1	-10.1	16.6	619	1.645
190	2.70	0.015	2.434.797	-35.35	-35.35	-16.2	7.07	-47	2.040

Forces and Displacements of Stack (without axial load, use of the multiplier)

TABLE NO. 11: Ratios of Deformations ($\omega_1 = 8.7963$)

1	.0048	.0049	.0048	.0048	.0048	.0048	.0049	.0048	.0048
2	.0076	.0076	.0076	.0076	.0076	.0076	.0076	.0076	.0076
3	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100
4	-.0014	-.0014	-.0014	-.0014	-.0014	-.0014	-.0014	-.0014	-.0014
5	.0110	.0110	.0110	.0110	.0110	.0110	.0110	.0110	.0110
6	.2211	.2210	.2211	.2211	.2211	.2211	.2211	.2211	.2211
7	.4035	.4036	.4035	.4035	.4035	.4035	.4036	.4035	.4035
8	.7073	.7073	.7073	.7074	.7073	.7073	.7073	.7073	.7073
9	1.0649	1.0649	1.0651	1.0649	1.0649	1.0649	1.0649	1.0649	1.0650
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Stack With Variable Inertia. Undamped Motion.

TABLE NO. 12: Ratios of Deformations ($\omega_1 = 8.7468$)

1	.0048	.0048	.0048	.0048	.0048	.0048	.0048	.0048
2	.0075	.0075	.0075	.0075	.0075	.0075	.0075	.0075
3	.0097	.0097	.0097	.0097	.0097	.0097	.0097	.0097
4	-.0009	-.0009	-.0009	-.0009	-.0009	-.0009	-.0009	-.0009
5	.0109	.0109	.0109	.0109	.0109	.0109	.0109	.0109
6	.2160	.2160	.2160	.2160	.2160	.2160	.2160	.2160
7	.3952	.3953	.3952	.3952	.3952	.3952	.3952	.3952
8	.6958	.6958	.6938	.6958	.6958	.6958	.6958	.6958
9	1.0432	1.0432	1.0433	1.0432	1.0432	1.0432	1.0432	1.0433
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Stack With Variable Inertia. Damped Motion.

TABLE NO. 13 : Ratios of Deformations ($\omega_1 = 66.6626$)

1	-.1999	-.1999	-.1999	-.1999	-.1999	-.1999	-.1999	-.1999	-.1999
2	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000
3	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000	-.2000
4	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001
5	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001	-.2001
6	.1999	.1999	.1999	.1999	.1999	.1999	.1999	.1999	.1999
7	.3999	.3999	.3999	.3999	.3999	.3999	.3999	.3999	.3999
8	.5999	.5999	.5999	.5999	.5999	.5999	.5999	.5999	.5999
9	.7999	.7999	.7999	.7999	.7999	.7999	.7999	.7999	.7999
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Stack With Constant Inertia. Undamped Motion.

TABLE NO. 14 : Ratios of Deformation ($\omega_1 = 66.30$)

1	.0023	.0023	.0023	.0023	.0023	.0023	.0023
2	.0041	.0041	.0041	.0041	.0041	.0041	.0041
3	.0068	.0068	.0068	.0068	.0068	.0068	.0068
4	.0081	.0081	.0081	.0081	.0081	.0081	.0081
5	.0078	.0078	.0078	.0078	.0078	.0078	.0078
6	.0045	.0045	.0045	.0045	.0046	.0046	.0045
7	.1662	.1661	.1661	.1661	.1662	.1662	.1661
8	.3731	.3731	.3730	.3730	.3732	.3732	.3731
9	.6761	.6761	.6760	.6759	.6761	.6761	.6761
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Stack With Constant Inertia. Damped Motion.

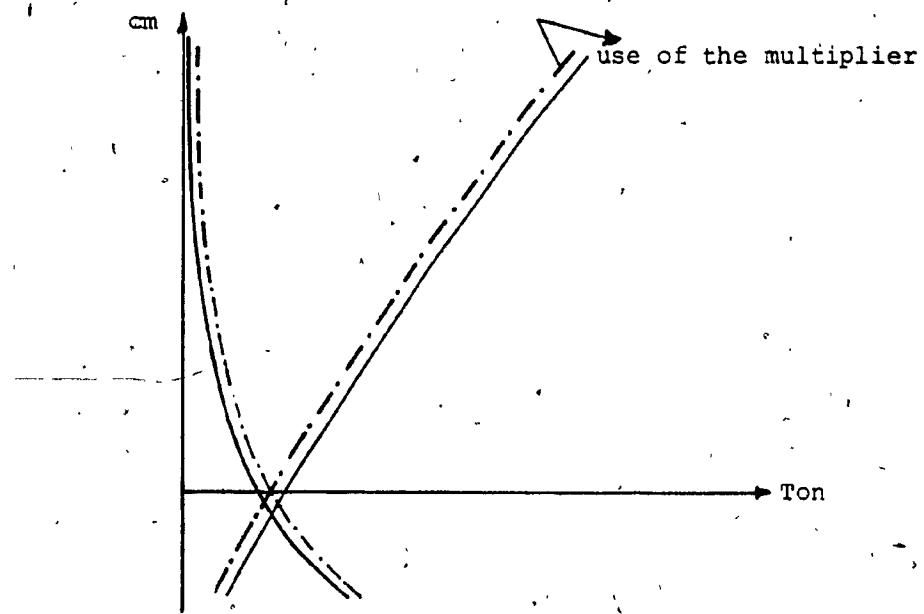


Fig. 8.1: Tensile Forces (Cables No. 1)
Horizontal scale 1 mm = 500 kg
Vertical scale 1 cm = .5 cm

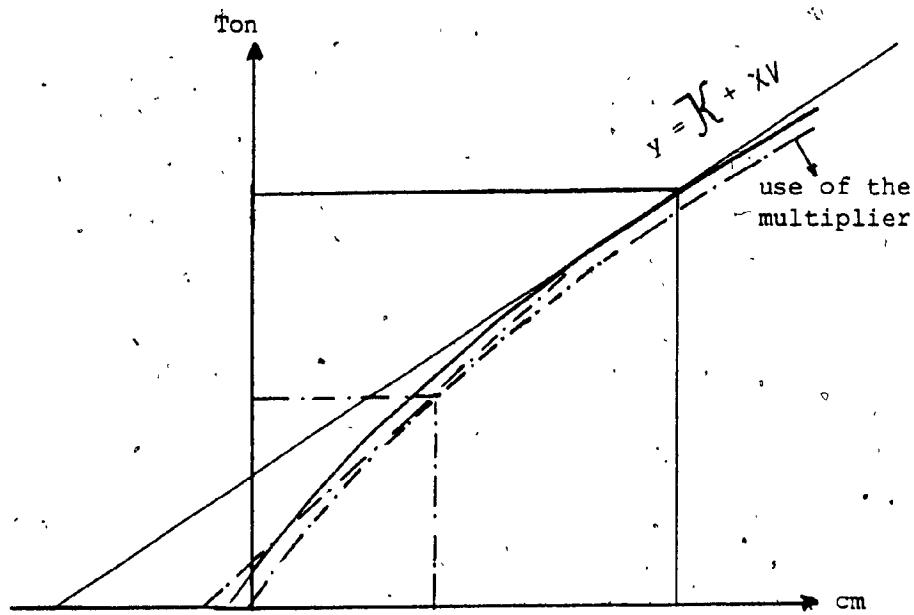


Fig. 8.2: Variation of the Horizontal Reaction V
Horizontal scale 1 cm = .5 cm
Vertical scale 1 mm = 500 kg

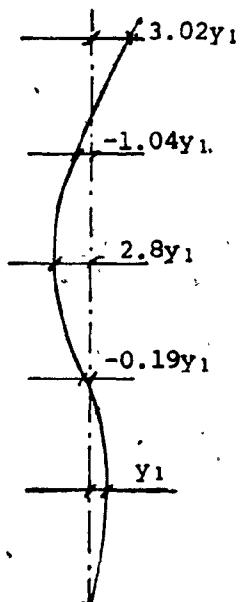


Fig. 8.3: First Critical Load, Stack of Constant Inertia.

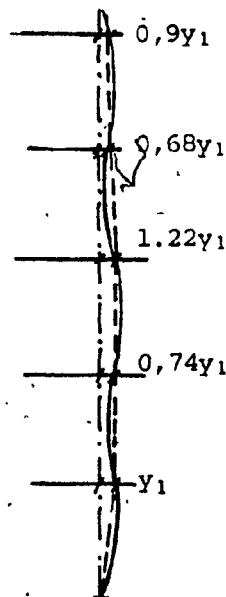
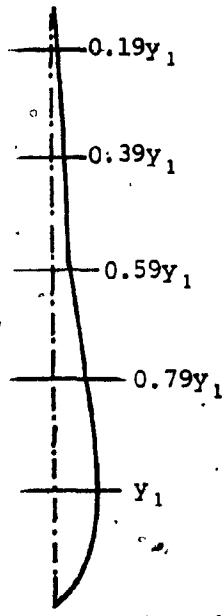
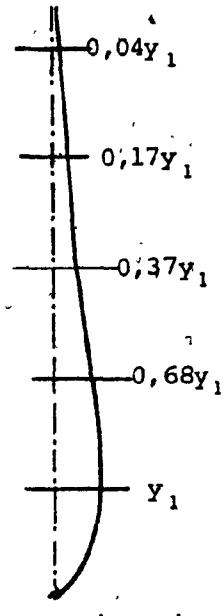


Fig. 8.4: First Critical Load. Stack of Variable Inertia.

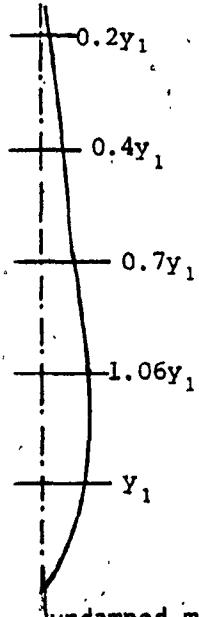


undamped motion

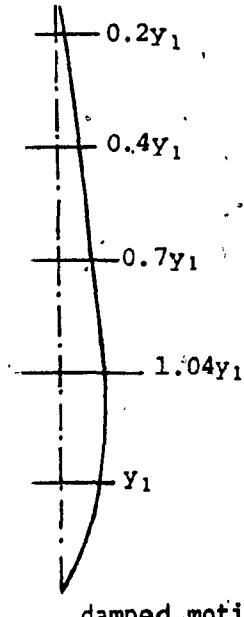


damped motion

Fig. 8.5: Stack of Constant Inertia.
First Mode Shape.



undamped motion



damped motion

Fig. 8.6: Stack of Variable Inertia.
First Mode Shape.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

The theory developed in the outline of this thesis was devoted to the static and dynamic buckling analysis of guyed stacks with constant and variable inertia. It consisted mainly in the linearization of differential and partial differential equations with variable coefficients in the neighbourhood of a singular point in the real plane.

With reference to the present analysis, the following conclusions can be made:

- 1) The equation of cables have been stated in the static part. For the zero displacement, the use of the multiplier allowed the tensile force in the guys to be set back to the initial tension.
- 2) Consider the case of two guys symmetrically fixed at a point located at a certain elevation; consider also the formulae giving the displacements of the upward and leeward guys when they are in the neighbourhood of the rest position. Then the tension of the upward cable is always greater than the initial tension. There exists a limit value of the initial tension for which the tension in the leeward cable is always greater than the initial tension, and, for a given height, this tension is always greater than the initial tension.
- 3) With respect to the bending moment due to the static loads, it can be observed that the bending moment and shear forces of the system are widely modified by the introduction of the axial load; while the increase in the displacement of the lower supports is small, it is important for the upper supports.

4) In the case of the system under the set of applied forces, if an equilibrium position of cables is determined, the search for the diameter of the chimney such that the computed displacements coincide with the initially determined displacements of the cables is a long process, which could lead to an overdimension of the diameter.

5) In the formulation of the problem of the bar with variable inertia, it can be observed that the axial load for an element of length l is equivalent to the concentrated load plus six times the total load of the bar of length l . This total load is that of the bar of uniform thickness, which is equal to the thickness at the top of the element.

In the case of the bar with constant thickness, it has been shown that $1/3$ of the total uniform load is considered to be applied to the top of the bar. By using Bessel functions, an entirely new set of formulae have been stated. These formulae, in the case of combined axial load and uniformly distributed load, are clearly distinct from those of Euler.

For the first time, in the case of the bar hinged or fixed at both ends, the researcher has stated the theoretical value of the critical load when the bar is acted upon by a uniform distributed load along the vertical axis. Through the use of an arbitrary function, a formula is also proposed for the bar considered hinged at one end and fixed at the other.

One should also note that the linear operator leading to the critical load is a non-symmetric matrix.

6) The linearization of the partial differential equation of the sixth order, which expresses the dynamic equilibrium conditions of the

bar, led to the natural frequency and consequently to the forcing frequency. A comparison of the two frequencies shows that the axial load due to tension of cables does not substantially modify the natural frequency of the stack. This is in agreement with Novack [40].

It should also be pointed out that the first frequency obtained in the case of the axial load, in the motion of the stack (with inertia, constant or not) seems to be in accord with the formula of Kolousek [41] which expresses the frequency as a function of the ratio of the dynamic axial load and the static critical load.

7) In the case of the bar with constant inertia, the proposed method of solution to the partial differential equation translating the damped motion led, when the influence of the own weight was considered, to the ordinary differential equation of the fourth order for the computation of the frequency, and to the differential equation of the beam on elastic foundation in the old Winkler hypothesis.

In the undamped motion, a new solution was proposed in order to find the fundamental frequency; this solution seems to be more complete than that proposed by Gluck [42].

8) In the case of the static buckling load and the dynamic load, the first mode shape was plotted by the consideration of the cofactors associated with the determinant of the linear operator expressing the equilibrium conditions. This method could lead to an overdimension of the section. One should also note that the influence of the uniform load led to the formulation of equations distinct from the ordinary static and dynamic buckling equations.

equations $F(r)$. One could undertake a more elaborate investigation, by considering the ordinary boundary conditions which seem to be expressed by non-linear relations between these constants.

6) The theory was elaborated by neglecting the influence of the temperature and shortening of the stack; a more elaborate investigation could be undertaken with a view to including those parameters in the development.

7) The preceding research work was based on the elastic domain. It could be interesting to extend it to the inelastic domain.

8) The deterministic behaviour of the guyed chimney under the exciting forces has been considered in the preceding work. It might be interesting to extend this work to the random aspect of the phenomenon.

9) In the outline of this thesis, the oscillation of the stack according to one plane of vibration was only considered; in fact, under Von Karman vortices, oscillation occurs in two perpendicular planes. One could investigate the problem more thoroughly by considering its bidimensional or tridimensional aspects.

10) By taking into account the influence of damping in the dynamical behaviour of the stack with variable inertia; it is seen that the axial force and the damping do not modify the fundamental frequency, as opposed to the case of the stack with constant inertia. It could be interesting to investigate the reason for this differential behaviour.

11) Given the system stayed by the set of guys, does there exist a limit value of the diameter or of the viscous damping coefficient for which the ratios of cofactors related to the same row of the equilibrium

RECOMMENDATIONS

- 1) With regard to the theory which has been developed, it can be extended to any law of inertia for which the integer $n \geq 2$, in the Pschunder [31] formula; the order of the characteristic polynomial for the search for the critical load or the fundamental frequency becomes $n+4$.
- 2) This thesis offers a theoretical as well as a numerical solution to the static and dynamic buckling analysis of guyed stacks. Further, experimental observation could be done in order to present a comparative approach.
- 3) The slope deflection method, which was used in the present research, led to an homogeneous set of linear equations, in the absence of terms not including the deformation parameters. A more elaborate mathematical investigation could be embarked upon in order to determine the influence of the perturbator terms on the results obtained.
- 4) At first, the energy method was considered as the way to a solution, and Lagrange's equations were written by using the static curves of Chapter V. Due to complications in the computation of the results, the linearization method was used in order to solve the vibration problem of the stack. It might be possible to solve the problem by the energy method, in which case the solution would be merely of academic interest.
- 5) In the search for the frequency of the stack with constant inertial in its undamped motion, the value 1 was arbitrarily attributed to the six constants obtained from the four solutions to the indicial

equations $F(r)$. One could undertake a more elaborate investigation, by considering the ordinary boundary conditions which seem to be expressed by non-linear relations between these constants.

6) The theory was elaborated by neglecting the influence of the temperature and shortening of the stack; a more elaborate investigation could be undertaken with a view to including those parameters in the development.

7) The preceding research work was based on the elastic domain. It could be interesting to extend it to the inelastic domain.

8) The deterministic behaviour of the guyed chimney under the exciting forces has been considered in the preceding work. It might be interesting to extend this work to the random aspect of the phenomenon.

9) In the outline of this thesis, the oscillation of the stack according to one plane of vibration was only considered; in fact, under Von Karman vortices, oscillation occurs in two perpendicular planes. One could investigate the problem more thoroughly by considering its bidimensional or tridimensional aspects.

10) By taking into account the influence of damping in the dynamical behaviour of the stack with variable inertia; it is seen that the axial force and the damping do not modify the fundamental frequency, as opposed to the case of the stack with constant inertia. It could be interesting to investigate the reason for this differential behaviour.

11) Given the system stayed by the set of guys, does there exist a limit value of the diameter of the viscous damping coefficient for which the ratios of cofactors related to the same row of the equilibrium

matrix diverge, leading consequently to an overdimension of this diameter?

Further investigation could be done, based on this criteria.

- 12) In the search for the buckling load or the natural frequency of the stack with variable inertia, it was observed that the operator expressing the equilibrium conditions is a non-symmetric matrix, when one consideres the influence of the own weight. Further analytical investigation could be pursued in order to analyze the exact structure of this matrix.

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REFERENCES

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APPENDIX A

FUNDAMENTAL PARAMETERS OF THE CHIMNEY WITH VARIABLE INERTIA

APPENDIX A

FUNDAMENTAL PARAMETERS OF THE CHIMNEY WITH VARIABLE INERTIAA.1 Law of Inertia

Let us consider a section of chimney of length l lying between two sets of guys. This element constitutes a frustum cone of variable thickness and diameter assimilated to the bar with variable inertia. Let us take as parameters those defined by Pschunder [3].

By considering Fig. A.1, one defines:

$$c = \frac{t'_2}{t'_1} \quad (A.1)$$

At a point of ordinate x , the thickness is written:

$$t'(x) = t'_1 c \left[1 + \left(\frac{1-c}{c} \right) \left(\frac{x}{l} \right)^n \right]$$

If the linear variation of thickness is only considered, one sets up:

$$n = 1 \quad (A.2)$$

Then, if

$$b = \frac{1-c}{cl} \quad (A.3)$$

the thickness t'_x is written:

$$t'(x) = t'_1 c \left(1 + \frac{bx}{l} \right) \quad (A.4)$$

Similarly, if:

$$\Delta = \frac{D_2}{D_1} \quad (A.5)$$

one defines:

$$a = \frac{1-\Delta}{\Delta} \cdot \frac{P}{l} \quad (A.6)$$

Then the diameter at ordinate x is written:

$$D(x) = D_1 \Delta(1+ax) \quad (A.7)$$

If inequality (3.5) is verified at any point of the element, the law of inertia can be written;

$$I(x) = \pi R^3(x)t'(x) \quad (A.8)$$

By setting up:

$$I_0 = \frac{\pi}{8} t'^1 c (\Delta D_1)^3 \quad (A.9)$$

then:

$$I(x) = I_0 (1+bx) (1+ax)^3 \quad (A.10)$$

A.3

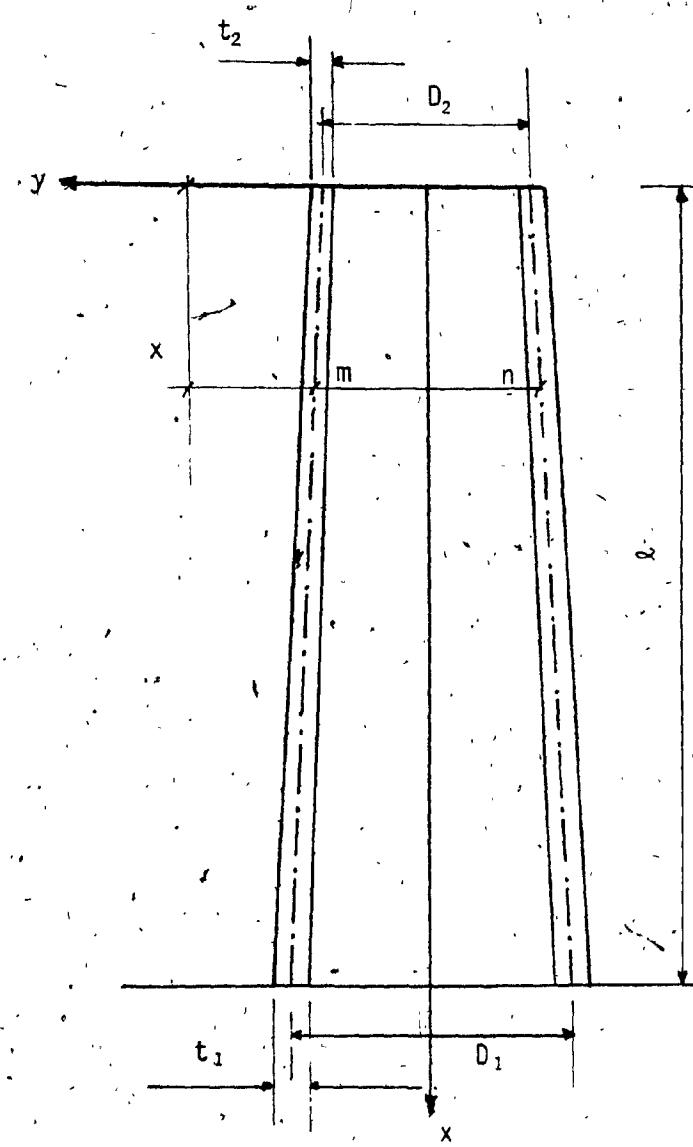


Fig. A.1: Variation of Sections

APPENDIX B

STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIA

FUNDAMENTAL MATRIX

B.1

APPENDIX B

STATIC BUCKLING OF THE STACK WITH VARIABLE INERTIAFUNDAMENTAL MATRIXB.1 Differential Equation of the Elastic Line

Let us consider the differential equation (4.4). Inside the disk of radius of convergence $\inf \left(\frac{1}{a}, \frac{1}{b} \right)$, let us perform an expansion in series of analytic functions of $\frac{1}{1+ax}$ and $\frac{1}{1+bx}$

So, one defines the following parameters:

$$\phi_1(x, m, n) = \sum_{n=1}^{\infty} (-1)^n (ax)^n$$

$$\phi_2(x, m, n) = \sum_{n=1}^{\infty} (-1)^n (bx)^n$$

$$\phi_3(x, m, n) = \sum_{n=1}^{\infty} (-1)^n (n+1)(ax)^n$$

$$\phi_4(x, m, n) = \sum_{n=1}^{\infty} (-1)^n (a^n + b^n) x^n + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+n)} a^n b^m x^{(m+n)}$$

$$\phi_5(x, m, n) = \sum_{n=1}^{\infty} (-1)^n [(n+1) a^n + b^n] x^n + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+n)} (n+1) a^n b^m x^{(m+n)}$$

$$\phi_6(x, m, n) = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n$$

$$\phi_7(x, m, n) = \sum_{n=1}^{\infty} (-1)^n \left[\frac{(n+1)(n+2)}{2} a^n + b^n \right] x^n + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(m+n)} \frac{(n+1)(n+2)}{2} a^n b^m x^{(m+n)}$$

These parameters will remain the same for the other appendices.

B.2

If:

$$\theta_1(x, m, n) = -(9a\phi_1(x, m, n) + 3b\phi_2(x, m, n))$$

$$\theta_2(x, m, n) = -(18a^2\phi_3(x, m, n) + 18ab\phi_4(x, m, n) + \frac{1}{K}[G(P + 6\ell q_0) -$$

$$q_0\{6 - 3\ell(a + b)x + (-12ab\ell + 3a + 3b)x^2 + 2abx^3\}]\phi_7(x, m, n))$$

$$\theta_3(x, m, n) = -(6a^3\phi_6(x, m, n) + 18a^2b\phi_5(x, m, n) - \frac{q_0}{K}[12 - 9\ell(a + b) +$$

$$\{15(a + b) - 12ab\ell\}x + 16abx^2]\phi_7(x, m, n))$$

$$\theta_4(x, m, n) = \frac{q_0}{K}[9(a + b) - 12ab\ell + 24abx]\phi_7(x, m, n)$$

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18ab + G \frac{(P + 6\ell q_0)}{K}$$

$$a_3 = 6a^3 + 18a^2b - \frac{q_0}{K}[12 - 9\ell(a + b)]$$

$$a_4 = \frac{q_0}{K}(12ab\ell - 9(a + b))$$

then, the following equation is obtained:

$$\frac{d^5y}{dx^5} + a_1 \frac{d^4y}{dx^4} + a_2 \frac{d^3y}{dx^3} + a_3 \frac{d^2y}{dx^2} + a_4 \frac{dy}{dx} =$$

$$\theta_1(x, m, n) \frac{d^4y}{dx^4} + \theta_2(m, n, x) \frac{d^3y}{dx^3} + \theta_3(m, n, x) \frac{d^2y}{dx^2}$$

$$\theta_4(m, n, x) \frac{dy}{dx} \quad (B.1)$$

Parameters θ_i ($i = 1, 4$) verify:

$$\begin{aligned} |\theta_1(x, m, n)| &\rightarrow 0 \\ (m, n) &\rightarrow \infty \end{aligned} \quad (B.2)$$

So, equation (B.1) is considered to be a perturbation equation. By the

B.3

change of variable:

$$u = \frac{dy}{dx} \quad (B.3)$$

equation (B.1) is transformed into the linear system

$$\frac{du}{dx} = [A + B(x)] u \quad (B.3)$$

In this equation, A and B are 4×4 matrices. In agreement with Roseau [32], over the compact set $[0, \ell]$, B(x) verifies the following Gronwall's lemma:

<< If u and v are continuous functions of t with positive values, if k is a positive constant and if:

$$u(t) \leq k + \int_0^t u(s) v(s) ds$$

then:

$$u(t) \leq k \exp \int_0^t v(s) ds \quad >>$$

The solution to (B.3) will be written:

$$u(x) = U(x) u(0) + \int_0^x U(x - \tau) B(\tau) u(\tau) d\tau \quad (B.4)$$

In formula (B.4), U(x) is the fundamental matrix of (B.3).

B.2 Fundamental Matrix of (B.3)

The following fourth degree equation is the characteristic polynomial associated with (B.3)

$$\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0 \quad (B.5)$$

Any method can be used to solve the quartic equation. Since the quartic

equation is not bi-quadratic, it is impossible for the four eigenvalues to be purely imaginary numbers. Then, the following cases are considered.

B.2.1 Case of Four Real Roots

If $\alpha, \beta, \theta, \eta$ are the roots, the following parameters are defined:

$$\Delta = (\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\theta - \eta)(\eta - \beta)(\eta - \alpha)$$

$$A_{111} = \beta\theta\eta(\theta - \beta)(\eta - \beta)(\theta - \eta)/\Delta$$

$$A_{112} = -\alpha\eta\theta(\theta - \alpha)(\theta - \eta)(\eta - \alpha)/\Delta$$

$$A_{113} = \alpha\beta\theta(\beta - \alpha)(\theta - \beta)(\theta - \alpha)/\Delta$$

$$A_{114} = -\alpha\beta\eta(\beta - \alpha)(\eta - \beta)(\eta - \alpha)/\Delta$$

$$A_{121} = -\frac{1}{\Delta} [(\eta^2 - \beta^2)(\theta^3 - \beta^3) - (\eta^3 - \beta^3)(\theta^2 - \beta^2)]$$

$$A_{122} = \frac{1}{\Delta} [(\eta^2 - \alpha^2)(\theta^3 - \alpha^3) - (\theta^2 - \alpha^2)(\eta^3 - \alpha^3)]$$

$$A_{123} = -\frac{1}{\Delta} [(\beta^2 - \alpha^2)(\theta^3 - \alpha^3) - (\beta^3 - \alpha^3)(\theta^2 - \alpha^2)]$$

$$A_{124} = \frac{1}{\Delta} [(\beta^2 - \alpha^2)(\eta^3 - \alpha^3) - (\beta^3 - \alpha^3)(\eta^2 - \alpha^2)]$$

$$A_{131} = \frac{1}{\Delta} [(\eta - \beta)(\theta^3 - \beta^3) - (\theta - \beta)(\eta^3 - \beta^3)]$$

$$A_{132} = -\frac{1}{\Delta} [(\eta - \alpha)(\theta^3 - \alpha^3) - (\theta - \alpha)(\beta^3 - \alpha^3)]$$

$$A_{133} = \frac{1}{\Delta} [(\beta - \alpha)(\theta^3 - \alpha^3) - (\theta - \alpha)(\beta^3 - \alpha^3)]$$

$$A_{134} = -\frac{1}{\Delta} [(\beta - \alpha)(\eta^3 - \alpha^3) - (\eta - \alpha)(\beta^3 - \alpha^3)]$$

$$A_{141} = -\frac{1}{\Delta} [(\eta - \beta)(\theta - \beta)(\theta - \eta)]$$

$$A_{142} = \frac{1}{\Delta} [(\eta - \alpha)(\theta - \eta)(\theta - \alpha)]$$

$$A_{143} = -\frac{1}{\Delta} [(\beta - \alpha)(\theta - \beta)(\theta - \alpha)]$$

$$A_{144} = \frac{1}{\Delta} [(\beta - \alpha)(\eta - \beta)(\eta - \alpha)] \quad (B.6)$$

If:

$$1 \leq i \leq 4$$

$$1 \leq j \leq 4$$

$$1 \leq k \leq 4$$

then,

$$A_{ijk} = (\lambda_k)^{i-1} A_{1jk} \quad (B.7)$$

where λ_k , $k=1,4$ designate the four roots.

So, the elements of the fundamental matrix are written:

$$u_{ij} = \sum_{k=1}^4 A_{ijk} \exp(\lambda_k x) \quad (B.8)$$

One obtains immediately:

$$\begin{aligned} i = 1 \quad y_i(j+1) &= \sum_{k=1}^4 \lambda_k^{-1} A_{1jk} \exp(\lambda_k x) \\ y_{11}^{(x)} &= 1 \\ i > 1 \quad y_i^{(x)} &= u_{(i-1)j}(x) \\ y_{i1}^{(x)} &= 0 \end{aligned} \quad (B.9)$$

B.2.2 Case of Four Complex Roots

In this case, the eigenvalues are written:

$$\lambda_1 = \alpha + i\theta$$

$$\lambda_3 = \beta + i\eta$$

$$\lambda_2 = \alpha - i\theta$$

$$\lambda_4 = \beta - i\eta$$

One defines:

$$\Delta = -4\theta\eta [\beta - \alpha]^2 + (\eta + \theta)^2 [(\beta - \alpha)^2 + (\eta - \theta)^2]$$

$$A_{111} = + \frac{2\theta\eta}{\Delta} [\beta^2 + \eta^2] [-(\beta^2 + \eta^2) + 4\alpha\beta - 3\alpha^2 + \theta^2]$$

$$A_{112} = \frac{2\eta}{\Delta} [\beta^2 + \eta^2] [2\beta(\alpha^2 - \theta^2) - \alpha(\beta^2 + \eta^2) - \alpha(\alpha^2 - \theta^2) + 2\alpha\theta^2]$$

$$A_{113} = \frac{2\theta}{\Delta} [\alpha^2 + \theta^2] [-\eta(\alpha^2 + \theta^2) - 3\eta\beta^2 + \eta^3 + 4\alpha\beta\eta]$$

$$A_{114} = \frac{2\theta}{\Delta} [\alpha^2 + \theta^2] [-\beta(\alpha^2 + \theta^2) + 3\beta\eta^2 - \beta^3 + 2\alpha(\beta^2 - \eta^2)]$$

$$A_{121} = \frac{4\theta\eta}{\Delta} [\alpha(\eta^2 - 3\beta^2) - \beta(\theta^2 - 3\alpha^2)]$$

$$A_{122} = \frac{2\eta}{\Delta} [\beta^2 + \eta^2]^2 + (\alpha^2 - \theta^2)(\eta^2 - 3\beta^2) + 2\beta\alpha(\alpha^2 - 3\theta^2)$$

$$A_{123} = \frac{4\eta}{\Delta} [\beta(\theta^2 - 3\alpha^2) - \alpha(\eta^2 - 3\beta^2)]$$

$$A_{124} = \frac{2\theta}{\Delta} [(\alpha^2 + \theta^2)^2 + (\beta^2 - \eta^2)(\theta^2 - 3\alpha^2) + 2\beta\alpha(\beta^2 - 3\eta^2)]$$

$$A_{131} = \frac{2\eta\theta}{\Delta} [\theta^2 - 3\alpha^2 - \eta^2 + 3\beta^2]$$

$$A_{132} = - \frac{2\eta}{\Delta} [2\beta(\beta^2 + \eta^2) + \alpha(\eta^2 - 3\beta^2) + \alpha(\alpha^2 - 3\theta^2)]$$

$$A_{133} = \frac{2\eta\theta}{\Delta} [\eta^2 - 3\beta^2 - \theta^2 + 3\alpha^2]$$

$$A_{134} = - \frac{2\theta}{\Delta} [2\alpha(\alpha^2 + \theta^2) + \beta(\theta^2 - 3\alpha^2) + \beta(\beta^2 - 3\eta^2)]$$

$$A_{141} = \frac{4\eta\theta}{\Delta} (\alpha - \beta)$$

$$A_{142} = \frac{2\eta}{\Delta} [(\beta - \alpha)^2 - \theta^2 + \eta^2]$$

$$A_{143} = \frac{4\eta\theta}{\Delta} (\beta - \alpha)$$

$$A_{144} = \frac{2\theta}{\Delta} [(\beta - \alpha)^2 - \eta^2 + \theta^2]$$

(B.10)

B.7

So, the elements of the fundamental matrix are written:

$$\begin{aligned}
 U_{ij}(x) = & (A_{1j_1} + iA_{1j_2})(\lambda_1)^{i-1} \exp(\lambda_1 x) + (A_{1j_1} - iA_{1j_2}) \\
 & (\lambda_2)^{i-1} \exp(\lambda_2 x) + (A_{1j_3} + iA_{1j_4})(\lambda_3)^{i-1} \exp(\lambda_3 x), \\
 & + (A_{1j_3} - iA_{1j_4})(\lambda_4)^{i-1} \exp(\lambda_4 x),
 \end{aligned} \tag{B.11}$$

While, the elements of $Y(x)$ are written:

$$\begin{aligned}
 i = 1, \quad y_{i(j+1)}(x) & \Rightarrow \int U_{ij}(x) dx \\
 y_{11}(x) & = 1 \\
 i > 1, \quad y_{i(j+1)}(x) & = U_{(i-1)j}(x) \\
 y_{i_1}(x) & = 0
 \end{aligned} \tag{B.12}$$

B.2.3 Two Roots are Real and the Two Others are Complex

In this case, the eigenvalues are written

$$\lambda_1 = \alpha + \beta$$

$$\lambda_2 = \alpha - \beta$$

$$\lambda_3 = \theta + in$$

$$\lambda_4 = \theta - in$$

So, the following parameters are defined:

$$\Delta = 4\beta n[(\theta + \beta - \alpha)^2 + n^2][((\theta - \alpha - \beta)^2 + n^2]$$

$$A_{111} = -\frac{1}{\Delta} [2n(\alpha - \beta)(\theta^2 + n^2)\{\theta^2 + n^2 + (\beta - \alpha)^2 - 2\theta(\beta - \alpha)\}],$$

B.8

$$A_{112} = -\frac{1}{\Delta} [2\eta(\alpha + \beta)(\theta^2 + \eta^2)\{-(\theta^2 + \eta^2) - (\alpha + \beta)^2 + 2\theta(\alpha + \beta)\}]$$

$$A_{113} = \frac{1}{\Delta} [2\beta\eta(\beta^2 - \alpha^2)(\eta^2 + \beta^2 - \alpha^2 - 3\theta^2 + 4\alpha\theta)]$$

$$A_{114} = -\frac{1}{\Delta} [2\beta(\beta^2 - \alpha^2)\{\theta^3 - 2\alpha\theta^2 - \theta(3\eta^2 - \alpha^2) + 2\alpha\eta^2 - \theta\beta^2\}]$$

$$A_{121} = -\frac{2\eta}{\Delta} [-(\theta^2 + \eta^2)^2 + (\alpha - \beta)^2\{3\theta^2 - \eta^2 - 2\theta(\alpha - \beta)\}]$$

$$A_{122} = \frac{2\eta}{\Delta} [-(\theta^2 + \eta^2)^2 + (\alpha + \beta)^2\{3\theta^2 - \eta^2 - 2\theta(\alpha + \beta)\}]$$

$$A_{123} = \frac{4\beta\eta}{\Delta} [\alpha(\eta^2 - 3\theta^2) + \theta(3\alpha^2 + \beta^2)]$$

$$A_{124} = -\frac{2\beta}{\Delta} [(\alpha^2 - \beta^2)^2 + 2\alpha\theta(\theta^2 - 3\eta^2) - (3\alpha^2 + \beta^2)(\theta^2 - \eta^2)]$$

$$A_{131} = \frac{2\eta}{\Delta} [(\alpha - \beta)(3\theta^2 - \eta^2) - 2\theta(\theta^2 + \eta^2) - (\alpha - \beta)^3]$$

$$A_{132} = -\frac{2\eta}{\Delta} [(\alpha + \beta)(3\theta^2 - \eta^2) - 2\theta(\theta^2 + \eta^2) - (\alpha + \beta)^3]$$

$$A_{133} = -\frac{2\beta\eta}{\Delta} [\eta^2 - 3\theta^2 + 3\theta^2 + \beta^2]$$

$$A_{134} = \frac{2\beta}{\Delta} [2\alpha(\alpha^2 - \beta^2) - \theta(3\alpha^2 + \beta^2) + \theta(\theta^2 - 3\eta^2)]$$

$$A_{141} = \frac{2\eta}{\Delta} [\eta^2 + (\theta - \alpha + \beta)^2]$$

$$A_{142} = -\frac{2\eta}{\Delta} [\eta^2 + (\theta - \alpha - \beta)^2]$$

$$A_{143} = \frac{4\beta\eta}{\Delta} [\alpha - \theta]$$

$$A_{144} = -\frac{2\beta}{\Delta} [\theta^2 - \eta^2 + \alpha^2 - 2\alpha\theta - \beta^2] \quad (B.13)$$

The elements of the fundamental matrix are written:

$$U_{ij}(x) = \sum_{k=1}^2 A_{ijk} (\lambda_k x)^{i-1} \exp(\lambda_k x) + \\ (A_{ij3} + iA_{ij4})(\lambda_3)^{i-1} \exp(\lambda_3 x) + (A_{ij3} - iA_{ij4})(\lambda_4)^{i-1} \exp(\lambda_4 x) \quad (B.14)$$

B.9

While, the elements of $\Upsilon(x)$ are written:

$$i = 1 \quad y_{i(j+1)}(x) = \int u_{ij}(x) dx$$

$$y_{11}(x) = 1$$

$$i > 1 \quad y_{i(j+1)}(x) = u_{(i-1)j}(x)$$

$$y_{i1}(x) = 0$$

(B.15)

APPENDIX C

CHIMNEY OF CONSTANT INSIDE DIAMETER AND VARIABLE THICKNESS

C.1

APPENDIX C

CHIMNEY OF CONSTANT INSIDE DIAMETER AND VARIABLE THICKNESSC.1 Differential Equation of the Elastic Line

Given the differential equation (4.18), one performs an expansion in series of the coefficients inside the disk of radius of convergence $\inf(1/a, 1/b)$.

So, the following parameters are defined:

$$\theta_1(x, m, n) = - (9a\phi_1(x, m, n) + 3b\phi_1(x, m, n))$$

$$\theta_2(x, m, n) = - (18a^2\phi_3(x, m, n) + 18ab\phi_4(x, m, n) + \frac{1}{K} [G(P + 6\ell q_0) - q_0 \{6 - 6b\ell(1+d)x + (3b + 3bd - 2\ell db^2)x^2 + 2db^2x^3\}] \phi_7(x, m, n))$$

$$\theta_3(x, m, n) = - (6a^3\phi_6(x, m, n) + 18a^2b\phi_5(x, m, n) - \frac{q_0}{K} [12 - 9b\ell(1+d) + \{15b(1+d) - 12d\ell b^2\}x + 16db^2x^2] \phi_7(x, m, n))$$

$$\theta_4(x, m, n) = \frac{q_0}{K} [9b(1+d) - 12d\ell b^2 + 24db^2x] \phi_7(x, m, n)$$

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18ab + \frac{G(P + 6\ell q_0)}{K}$$

$$a_3 = 6a^3 + 18a^2b - \frac{q_0}{K} [12 - 9b\ell(1+d)]$$

$$a_4 = \frac{q_0}{K} [12d\ell b^2 - 9b(1+d)]$$

Then, (4.18) is of the same form as (4.8). So, the solution is given in Appendix B.

~~APPENDIX D~~

CHIMNEY OF CONSTANT THICKNESS

APPENDIX D

CHIMNEY OF CONSTANT THICKNESSD.1 Differential Equation of the Elastic Line

Given the differential equation (4.21):

$$\frac{d^4y}{dx^4} + \frac{6a}{1+ax} \frac{d^3y}{dx^3} + \left[\frac{6a^2}{(1+ax)^2} + \frac{P+3q_0\{2\ell+(a\ell-2)x-ax^2\}}{K(1+ax)^3} \right] \frac{d^2y}{dx^2}$$

$$+ \frac{Gq_0}{K(1+ax)} (6a\ell - 2 - 7ax) \frac{dy}{dx} = \frac{P}{K(1+ax)^3} \quad (4.21)$$

one performs the expansion in series of analytic functions of the coefficients inside the circle of radius convergence $1/a$.

So, the following parameters are defined:

$$\theta_1(x,n) = -6a \sum_{n=1}^{\infty} (-1)^n (ax)^n$$

$$\theta_2(x,n) = - \left[6a^2 \sum_{n=1}^{\infty} (-1)^n (n+1)(ax)^n + \frac{G(P+6\ell q_0)}{K} \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n \right]$$

$$+ \frac{q_0}{K} (a\ell-2)x \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n - \frac{ax^2 q_0}{K} \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n \quad (D.1)$$

$$\theta_3(x,n) = - \frac{q_0}{K} \left[(6a\ell-2) \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n - 7ax \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (ax)^n \right] \quad (D.1)$$

By the transformation:

$$\frac{dy}{dx} = u$$

the linear system of differential equation associated with (4.21) is

D.2

written:

$$\frac{du}{dx} = [A + B(x)] u + f(x) \quad (4.24)$$

In this equation, A and B are 3×3 constant and perturbation matrices respectively, while $f(x)$ is an array column.

- The characteristic polynomial associated with A is of third degree.
- If the three roots α, β, θ are real, one will define:

$$\lambda_1 = \alpha$$

$$\lambda_2 = \beta$$

$$\lambda_3 = \theta$$

$$\Delta = (\beta - \alpha)(\theta - \beta)(\theta - \alpha)$$

$$A_{111} = \frac{1}{\Delta} \beta \theta (\theta - \beta)$$

$$A_{112} = -\frac{1}{\Delta} \alpha \theta (\theta - \alpha)$$

$$A_{113} = \frac{\alpha \beta}{\Delta} (\beta - \alpha)$$

$$A_{121} = -\frac{1}{\Delta} (\theta^2 - \beta^2)$$

$$A_{122} = \frac{1}{\Delta} (\theta^2 - \alpha^2)$$

$$A_{123} = -\frac{1}{\Delta} (\beta^2 - \alpha^2)$$

$$A_{131} = \frac{1}{\Delta} (\theta - \beta)$$

$$A_{132} = \frac{1}{\Delta} (\alpha - \theta)$$

$$A_{133} = \frac{1}{\Delta} (\beta - \alpha)$$

(D.2)

So, the elements of the fundamental matrix are written:

D.3

$$u_{ij}(x) = \sum_{k=1}^3 A_{ijk} (\lambda_k)^{i-1} \exp(\lambda_k x) \quad (D.3)$$

While,

$$i = 1 \quad y_{i(j+1)}(x) = \int u_{ij}(x) dx$$

$$y_{11}(x) = 1$$

$$i > 1 \quad y_{i(j+1)}(x) = u_{(i-1)j}(x)$$

$$y_{i_1}(x) = 0 \quad (D.4)$$

If the roots are complex, they can be expressed as:

$$\lambda_1 = -(2\alpha + \theta)$$

$$\lambda_2 = \alpha - \theta + i\beta$$

$$\lambda_3 = \alpha - \theta - i\beta$$

So, the following parameters are defined:

$$\Delta = 2\beta(9\alpha^2 + \beta^2)$$

$$A_{111} = \frac{2\beta}{\Delta} [(\alpha - \theta)^2 + \beta^2]$$

$$A_{112} = \frac{2\beta}{\Delta} (\theta + 2\alpha)(4\alpha - \theta)$$

$$A_{113} = -\frac{2}{\Delta} (2\alpha + \theta) [-\beta^2 + 3\alpha(\alpha - \theta)]$$

$$A_{121} = -\frac{4\beta}{\Delta} (\alpha - \theta)$$

$$A_{122} = \frac{4\beta}{\Delta} (\alpha - \theta)$$

$$A_{123} = \frac{2}{\Delta} (\beta^2 + 3\alpha^2 + 6\alpha\theta)$$

$$A_{131} = \frac{2\beta}{\Delta}$$

D.4

$$A_{132} = -\frac{2\beta}{\Delta}$$

$$A_{132} = \frac{2\beta}{\Delta}$$

Then:

$$\begin{aligned} u_{ij} &= A_{1j_1}(\lambda_1)^{i-1} \exp(\lambda_1 x) + (A_{1j_2} + iA_{1j_3})(\lambda_2)^{i-1} \exp(\lambda_2 x) \\ &\quad + (A_{1j_2} - iA_{1j_3})(\lambda_3)^{i-1} \exp(\lambda_3 x) \end{aligned} \quad (D.5)$$

$$i = 1 \quad y_{ij+1}(x) = \int u_{ij}(x) dx$$

$$y_{11}(x) = 1$$

$$i > 1 \quad y_{i(j+1)}(x) = u_{(i-1)j}(x)$$

$$y_{i1}(x) = 0 \quad (D.6)$$

APPENDIX E

DYNAMIC ANALYSIS OF THE CHIMNEY WITH VARIABLE INERTIA

E.1

APPENDIX E

DYNAMIC ANALYSIS OF THE CHIMNEY WITH VARIABLE INERTIAE.1 Undamped Motion

Let us consider the partial differential equation (6.4). As it is difficult to solve this equation, the method developed in the static part will be extended to the dynamic part.

So, one will perform and expansion in series of the coefficients inside the disk of radius of convergence inf (1/a, 1/b). The following parameters are defined:

$$\theta_1(x, m, n) = - (9a\phi_1(x, m, n) + 3b\phi_2(x, m, n))$$

$$\theta_2(x, m, n) = - (18a^2\phi_3(x, m, n) + 18ab\phi_4(x, m, n) + \frac{1}{Kg} [P + 6\ell q_0])$$

$$q_0 \{6 - 3\ell(a+b)x + (-12ab\ell + 3a + 3b)x^2 + 2abx^3\} \phi_7(x, m, n)$$

$$\theta_3(x, m, n) = - (6a^3\phi_6(x, m, n) + 18a^2b\phi_5(x, m, n) - \frac{q_0}{Kg} \{12 - 9\ell(a+b)$$

$$+ [15(a+b) - 12ab\ell]x + 16abx^2\} \phi_7(x, m, n))$$

$$\theta_4(x, m, n) = \frac{q_0}{Kg} [9(a+b) - 12ab\ell + 24abx] \phi_7(x, m, n)$$

$$\theta_5(x, m, n) = - \frac{q_0}{Kg} [6 + 3(a+b)x + 2abx^2] \phi_7(x, m, n)$$

$$\theta_6(x, m, n) = \frac{q_0}{Kg} [3(a+b) + 4abx] \phi_7(x, m, n)$$

$$\phi(x, \frac{\partial^k y(x, t)}{\partial x^k \partial t^0}) = \theta_1(x, m, n) \frac{\partial^4 y(x, t)}{\partial x^4} + \theta_2(x, m, n) \frac{\partial^3 y(x, t)}{\partial x^3} +$$

$$\theta_3(x, m, n) \frac{\partial^2 y(x, t)}{\partial x^2} + \theta_4(x, m, n) \frac{\partial y(x, t)}{\partial x} + \theta_5(x, m, n) \frac{\partial^3 y(x, t)}{\partial x \partial t^2}$$

$$+ \theta_6(x, m, n) \frac{\partial^2 y(x, t)}{\partial t^2}$$

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18ab + \frac{P + 6\ell q_0}{Kg}$$

$$a_3 = 6a^3 + 18a^2b - \frac{q_0}{Kg} [12 - 9\ell(a+b)] \quad (E.1)$$

Then, one obtains the partial differential equation (6.5). In this equation, θ_i , ($i = 1, 6$) are the perturbator terms.

E.2 General Solution to Equation (6.5)

Given equation (6.5), one will tackle the problem by one of the methods used in solving the non-homogeneous partial differential equations.

E.2.1 Solution to the Homogeneous Equation

The motion is assumed to be harmonic, as stated in paragraph (6.1). So, the d'Alembert's method or the method of separation of variables is applied to the homogeneous part of (6.5). The two following differential equations are obtained:

$$\frac{d^5y}{dx^5} + a_1 \frac{d^4y}{dx^4} + a_2 \frac{d^3y}{dx^3} + a_3 \frac{d^2y}{dx^2} + a_4 \frac{dy}{dx} + a_5 y = 0 \quad (E.2)$$

$$\ddot{T} + \omega^2 T = 0 \quad (E.3)$$

where ω is the frequency and:

$$a_4 = -\frac{q_0}{Kg} [9(a+b) - 12ab\ell + 6\omega^2]$$

$$a_5 = -\frac{3q_0\omega^2}{Kg} (a+b)$$

By the transformation

$$y_1 = \frac{dy}{dx}$$

E.3

One gets the linear system of differential equation associated with (E.2)

$$\frac{dy}{dx} = Ay \quad (E.4)$$

In this equation, A is 5×5 matrix.

Then, the characteristic polynomial will be of the fifth order:

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \quad (E.5)$$

Any method can be used in order to get the eigenvalues. When they are obtained, it becomes easy to write the fundamental matrix. The roots are assumed to be distinct.

E.2.1.1 Case of One Real Root

In this case, the eigenvalues are of the form:

$$\begin{aligned}\lambda_1 &= \alpha \\ \lambda_2 &= \beta + i\theta \\ \lambda_3 &= \beta - i\theta \\ \lambda_4 &= \delta + in \\ \lambda_5 &= \delta - in\end{aligned} \quad (E.6)$$

So, the following parameters are defined:

$$\Delta = -4\theta n[(\beta - \alpha)^2 + \theta^2][((\delta - \beta)^2 + (n + \theta)^2)[(\delta - \beta)^2 + (n - \theta)^2][(\delta - \alpha)^2 + n^2]$$

$$A_{111} = -\frac{4\theta n}{\Delta} (\beta^2 + \theta^2)(\delta^2 + n^2)[(\delta - \beta)^2 + (n + \theta)^2][(\delta - \beta)^2 + (n - \theta)^2]$$

$$A_{112} = \frac{4\alpha\theta n}{\Delta} (\delta^2 + n^2)[(\delta - \alpha)^2 + n^2][(2\beta - \alpha)[(\delta - \beta)^2 + n^2 - \theta^2] - 2(\delta - \beta)(\beta^2 - \theta^2 - \alpha\beta)]$$

$$A_{113} = -\frac{4\alpha n}{\Delta} (\delta^2 + n^2)[(\delta - \alpha)^2 + n^2][(\beta^2 - \alpha\beta - \theta^2)[(\delta - \beta)^2 + n^2 - \theta^2] + 2\theta^2(\delta - \beta)(2\beta - \alpha)]$$

E.4

$$A_{114} = \frac{4\alpha\theta\eta}{\Delta} (\beta^2 + \theta^2)[(\beta - \alpha)^2 + \theta^2][(2\delta - \alpha)\{(\delta - \beta)^2 + \theta^2 - \eta^2\} + 2(\delta - \beta)(\delta^2 - \eta^2 - \delta\alpha)]$$

$$A_{115} = -\frac{4\alpha\theta}{\Delta} (\beta^2 + \theta^2)[(\beta - \alpha)^2 + \theta^2]\{[\delta^2 - \eta^2 - \delta\alpha]\{(\delta - \beta)^2 + \theta^2 - \eta^2\} - 2\eta^2(\delta - \beta)(2\delta - \alpha)\}$$

$$A_{121} = \frac{8\theta\eta}{\Delta} (\delta^2 + \eta^2)^2[\beta(\delta^2 + \eta^2) + \delta\theta^2 - 3\delta\beta^2 + 2\beta(\beta^2 - \theta^2)] + \frac{8\theta\eta}{\Delta} (\beta^2 + \theta^2)^2[\delta(\beta^2 + \theta^2) + \beta\eta^2 - 3\beta\delta^2 - 3\beta\delta^2 + 2\delta(\delta^2 - \eta^2)]$$

$$A_{122} = -\frac{8\theta\eta}{\Delta} (\delta^2 + \eta^2)^2[-(\beta^2 - \theta^2)(\delta - \beta) + \beta\{(\delta - \beta)^2 + \eta^2 - \theta^2\}] + \frac{4\eta\theta\alpha^2}{\Delta} [\{(\beta - \alpha)(\beta^2 - \theta^2) - 2\beta\theta^2\}\{3\delta^2 - \eta^2 - 2\delta\alpha\} + \{3\beta^2 - \theta^2 - 2\beta\alpha\}\{4\delta\eta^2 + (\alpha + \beta)(3\delta^2 - \eta^2) - 2\delta(2\delta^2 + \alpha\beta)\}]$$

$$A_{123} = -\frac{4\eta}{\Delta} [\delta^2 + \eta^2]^2\{\alpha^2(\delta - \alpha)^2 + \alpha^2\eta^2 - (\beta^2 - \theta^2)[(\delta - \beta)^2 + \eta^2 - \theta^2] - 4\theta^2\beta(\delta - \beta)\} - \frac{4\eta\alpha^2}{\Delta} \{-\theta^2(3\beta^2 - \theta^2 - 2\alpha\beta)(3\delta^2 - \eta^2 - 2\delta\alpha) + \{(\beta - \alpha)(\beta^2 - \theta^2) - 2\beta\theta^2\}[4\delta\eta^2 + (\alpha + \beta)(3\delta^2 - \eta^2) - 2\delta(2\delta^2 + \alpha\beta)\}]$$

$$A_{124} = -\frac{8\theta\eta}{\Delta} (\beta^2 + \theta^2)^2[(\delta - \beta)(\delta^2 + \eta^2) + \delta\{(\delta - \beta)^2 + \theta^2 - \eta^2\}] + \frac{2\eta\theta\alpha^2}{\Delta} [\{(\delta^2 - \eta^2)(\delta - \alpha) - 2\delta\eta^2\}\{\beta^2 - \theta^2 + 2\beta(\beta - \alpha)\} + \{3\delta^2 - \eta^2 - 2\delta\alpha\}\{(\beta^2 - \theta^2)(-2\beta + \alpha + \delta) + 2\beta[(\beta - \alpha)(\delta - \beta) + \theta^2]\}]$$

$$A_{125} = \frac{4\theta}{\Delta} [(\beta^2 + \theta^2)^2]\{[(\delta - \beta)^2 + \theta^2 - \eta^2](\delta^2 - \eta^2) - 4\delta\eta^2(\delta - \beta)\} - \alpha^2[(\beta - \alpha)^2 + \theta^2] + \frac{4\alpha^2\theta}{\Delta} [\eta^2\{3\delta^2 - \eta^2 - 2\delta\alpha\}\{\beta^2 - \theta^2 + 2\beta(\beta - \alpha)\} - \{(\delta^2 - \eta^2)(\delta - \alpha) - 2\delta\eta^2\}\{(\beta^2 - \theta^2)(-2\beta + \alpha + \delta) + 2\beta[(\beta - \alpha)(\delta - \beta) + \theta^2]\}]$$

E.5

$$A_{131} = \frac{4\eta\theta}{\Delta} [-(\delta^2 + \eta^2)^3 - (\beta^2 + \theta^2)^3 + (\delta^2 + \eta^2 + \beta^2 + \theta^2)(3\beta^2 - \theta^2)(3\delta^2 - \eta^2)]$$

$$A_{132} = \frac{4\eta\theta}{\Delta} [(\delta^2 + \eta^2)^3 + 4\delta\alpha^3(\eta^2 - \delta^2) - \alpha^4(\eta^2 - 3\delta^2)]$$

$$+ \frac{4\eta\theta}{\Delta} [\theta^2 - 3\beta^2] [-(\delta^2 + \eta^2)(\eta^2 - 3\delta^2) + 4\alpha\delta(\eta^2 - \delta^2) + \alpha^4]$$

$$- \frac{16\beta\theta\eta}{\Delta} [\theta^2 - \beta^2] [2\delta(\delta^2 + \eta^2) + \alpha(\eta^2 - 3\delta^2) + \alpha^3]$$

$$A_{133} = -\frac{4\alpha\eta}{\Delta} (\delta^2 + \eta^2) [-(\delta^2 + \eta^2)^2 + \alpha^2(3\delta^2 - \eta^2) - 2\delta\alpha^3]$$

$$- \frac{4\beta\eta}{\Delta} (\delta^2 + \eta^2)^3 - \frac{16\beta\delta\eta\alpha^3}{\Delta} (\eta^2 - \delta^2) + \frac{4\beta\eta\alpha^4}{\Delta} (\eta^2 - 3\delta^2) -$$

$$- \frac{4\beta\eta}{\Delta} [\beta^2 - 3\theta^2] [(\delta^2 + \eta^2)(\eta^2 - 3\delta^2) - 4\alpha\delta(\eta^2 - \delta^2) - \alpha^4] -$$

$$\frac{4\eta}{\Delta} [\beta^4 + \theta^4 - 6\beta^2\theta^2] [2\delta(\delta^2 + \eta^2) + \alpha(\eta^2 - 3\delta^2) + \alpha^3]$$

$$A_{134} = \frac{4\eta\theta}{\Delta} [(\beta^2 + \theta^2)^3 - 4\alpha^3\beta(\beta^2 - \theta^2) - \alpha^4(\theta^2 - 3\beta^2)] - \frac{4\eta\theta}{\Delta} (\eta^2 - 3\delta^2)$$

$$[(\beta^2 + \theta^2)(\theta^2 - 3\beta^2) - 4\alpha\beta(\theta^2 - \beta^2) - \alpha^4] - \frac{16\theta\delta\eta}{\Delta} [2\beta(\beta^2 + \theta^2) +$$

$$\alpha(\theta^2 - 3\beta^2) + \alpha^3]$$

$$A_{135} = \frac{4\alpha\theta}{\Delta} (\beta^2 + \theta^2) [(\beta^2 + \theta^2)^2 + \alpha^2(\theta^2 - 3\beta^2) + 2\beta\alpha^3] - \frac{4\delta\theta}{\Delta} [\beta^2 + \theta^2]^3 -$$

$$4\alpha^3\beta(\beta^2 - \theta^2) - \alpha^4(\theta^2 - 3\beta^2) - \frac{4\delta\theta}{\Delta} [(\delta^2 - 3\eta^2)][(\beta^2 + \theta^2)(\theta^2 - 3\beta^2) -$$

$$4\alpha\beta(\theta^2 - \beta^2) - \alpha^4] - \frac{4\theta}{\Delta} [\delta^4 + \eta^4 - 6\eta^2\delta^2] [2\beta(\beta^2 + \theta^2) + \alpha(\theta^2 - 3\beta^2) +$$

$$\alpha^3]$$

$$A_{141} = \frac{8\delta\eta\theta}{\Delta} [(\delta^2 + \theta^2 - \beta^2 - \eta^2)^2 + 4\delta^2\eta^2 + 4\beta^2(\delta^2 - \eta^2 - \beta^2)]$$

$$+ \frac{8\beta\eta\theta}{\Delta} [(\beta^2 + \eta^2 - \delta^2 - \theta^2)^2 + 4\beta^2\theta^2 + 4\delta^2(\beta^2 - \theta^2 - \delta^2)]$$

E.6

$$A_{1+2} = \frac{1}{\Delta} [-16\alpha\beta\theta\eta\delta(\delta^2 + \theta^2 - \eta^2 - \beta^2) - 4\delta\eta\theta((\delta^2 - \eta^2 - \alpha^2)^2 + 4\delta^2\eta^2)$$

$$+ 4\theta\beta\eta[(2\delta^2(\delta^2 + \eta^2 - \alpha^2) + (\delta^2 + \eta^2 + \alpha^2)(\delta^2 + \alpha^2 + 2\theta^2 - 2\beta^2 - \eta^2)]]$$

$$A_{1+3} = \frac{1}{\Delta} [4\alpha\delta\eta \{(\delta^2 + \theta^2 - \beta^2 - \eta^2)^2 + 4(\delta^2\eta^2 - \beta^2\theta^2)\} - 4\delta\eta\beta((\delta^2 - \eta^2 - \alpha^2)^2$$

$$+ 4\delta^2\eta^2) + 2\eta\{2\delta^2(\delta^2 + \eta^2 - \alpha^2)(\beta^2 - \theta^2 - \alpha^2) + (\delta^2 + \eta^2 + \alpha^2)[4\theta^2\beta^2$$

$$+ (\beta^2 - \theta^2 - \alpha^2)(\delta^2 + \theta^2 - \beta^2 - \eta^2)\}]]$$

$$A_{1+4} = \frac{1}{\Delta} [-16\alpha\beta\delta\theta\eta(\beta^2 + \eta^2 - \theta^2 - \delta^2) - 4\beta\eta\theta((\beta^2 - \theta^2 - \alpha^2)^2 + 4\beta^2\theta^2$$

$$+ 4\theta\delta\eta\{2\beta^2(\beta^2 + \theta^2 - \alpha^2) + (\beta^2 + \theta^2 + \alpha^2)(\beta^2 + \alpha^2 - 2\eta^2 - 2\delta^2 - \theta^2)\}]$$

$$A_{1+5} = \frac{1}{\Delta} [4\alpha\beta\theta\{(\beta^2 + \eta^2 - \delta^2 - \theta^2)^2 + 4(\beta^2\theta^2 - \delta^2\eta^2)\} - 4\beta\delta\theta \{$$

$$(\beta^2 - \theta^2 - \alpha^2)^2 + 4\beta^2\theta^2\} + 2\theta\{2\beta^2(\beta^2 + \theta^2 - \alpha^2)(\delta^2 - \eta^2 - \alpha^2)$$

$$+ (\beta^2 + \theta^2 + \alpha^2)[4\delta^2\eta^2 + (\delta^2 - \eta^2 - \alpha^2)(\beta^2 + \eta^2 - \delta^2 - \theta^2)\}]]$$

$$A_{151} = -\frac{4\theta\eta}{\Delta} [(\delta - \beta)^2 + (\eta + \theta)^2][(\delta - \beta)^2 + (\eta - \theta)^2]$$

$$A_{152} = \frac{4\theta\eta}{\Delta} [(\delta - \alpha)^2 + \eta^2][(\delta - \beta)^2 + (\eta - \theta)^2 - 2(\beta - \alpha)(\delta - \beta)]$$

$$A_{153} = -\frac{4\eta}{\Delta} [(\delta - \alpha)^2 + \eta^2][2\theta^2(\delta - \beta) + (\beta - \alpha)\{(\delta - \beta)^2 + \eta^2 - \theta^2\}]$$

$$A_{154} = \frac{4\theta\eta}{\Delta} [(\beta - \alpha)^2 + \theta^2][(\delta - \beta)^2 + \theta^2 - \eta^2 + 2(\delta - \alpha)(\delta - \beta)]$$

$$A_{155} = -\frac{4\theta}{\Delta} [(\beta - \alpha)^2 + \theta^2][-2\eta^2(\delta - \beta) + (\delta - \alpha)\{(\delta - \beta)^2 + \theta^2 - \eta^2\}]$$

So, the elements of the 5×5 fundamental matrix are written:

$$\begin{aligned} y_{1j}(x) &= A_{1j_1}(\lambda_1)^{i-1} \exp(\lambda_1 x) + (A_{1j_2} + iA_{1j_3})(\lambda_2)^{i-1} \exp(\lambda_2 x) + \\ &\quad (A_{1j_2} - iA_{1j_3})(\lambda_3)^{1-i} \exp(\lambda_3 x) + (A_{1j_4} + iA_{1j_5})(\lambda_4)^{i-1} \exp(\lambda_4 x) + \\ &\quad (A_{1j_4} - iA_{1j_5})(\lambda_5)^{i-1} \exp(\lambda_5 x) \end{aligned} \quad (E.7)$$

E.7

E.2.1.2 Case of Three Real Roots

In this case, the eigenvalues are of the form:

$$\begin{aligned}
 \lambda_1 &= \alpha \\
 \lambda_2 &= \beta \\
 \lambda_3 &= \theta \\
 \lambda_4 &= \delta + i\eta \\
 \lambda_5 &= \delta - i\eta
 \end{aligned} \tag{E.8}$$

So, the following parameters are defined:

$$\Delta = 2\eta(\beta - \alpha)(\theta - \beta)(\theta - \alpha)[(\delta - \theta)^2 + \eta^2][(\delta - \beta)^2 + \eta^2][(\delta - \alpha)^2 + \eta^2]$$

$$A_{111} = \frac{2\beta\theta\eta}{\Delta} (\theta - \beta)[\delta^2 + \eta^2][(\delta - \theta)^2 + \eta^2][(\delta - \beta)^2 + \eta^2]$$

$$A_{112} = -\frac{2\eta\theta}{\Delta} (\theta - \alpha)[\delta^2 + \eta^2][(\delta - \alpha)^2 + \eta^2][(\delta - \theta)^2 + \eta^2]$$

$$A_{113} = \frac{2\eta\alpha\beta}{\Delta} (\beta - \alpha)(\delta^2 + \eta^2)[(\delta - \beta)^2 + \eta^2][(\delta - \alpha)^2 + \eta^2]$$

$$\begin{aligned}
 A_{114} = -\frac{2\eta\alpha\beta\theta}{\Delta} (\beta - \alpha)(\theta - \alpha)(\theta - \beta) &\{ (2\delta - \theta)[(\delta - \beta)(\delta - \alpha) - \eta^2] + \\
 &(2\delta - \alpha - \beta)[\delta(\delta - \theta) - \eta^2] \}
 \end{aligned}$$

$$\begin{aligned}
 A_{115} = \frac{2\alpha\beta\theta}{\Delta} (\beta - \alpha)(\theta - \beta)(\theta - \alpha) &\{ [\delta(\delta - \theta) - \eta^2][(\delta - \beta)(\delta - \alpha) - \eta^2] \\
 &- \eta^2(2\delta - \theta)(2\delta - \alpha - \beta) \}
 \end{aligned}$$

$$A_{121} = \frac{1}{\Delta} \{-2\eta(\delta^2 + \eta^2)^2[\theta^2(\delta - \theta)^2 + \theta^2\eta^2 - \beta^2(\delta - \beta)^2 - \beta^2\eta^2] -$$

$$2\eta(\theta - \beta)\theta^2\beta^2[2\delta(\delta - \beta)(\delta - \theta) - 2\delta\eta^2 + (\delta^2 - \eta^2)(2\delta - \beta - \theta)]\}$$

$$A_{122} = \frac{1}{\Delta} \{2\eta\theta^2(\delta^2 + \eta^2)^2[(\delta - \theta)^2 + \eta^2] - 2\eta\alpha^2(\delta^2 + \eta^2)^2[(\delta - \alpha)^2 + \eta^2] +$$

$$2\eta(\theta - \alpha)\theta^2\alpha^2[2\delta(\delta - \alpha)(\delta - \theta) - 2\delta\eta^2 + (\delta^2 - \eta^2)(2\delta - \alpha - \theta)]\}$$

E.8

$$\begin{aligned} A_{123} &= \frac{1}{\Delta} \{-2\eta(\delta^2 + \eta^2)^2 [\beta^2(\delta - \beta)^2 + \beta^2\eta^2 - \alpha^2(\delta - \alpha)^2 - \alpha^2\eta^2] \\ &\quad - 2\eta(\beta - \alpha)\alpha^2\beta^2 [2\delta(\delta - \alpha)(\delta - \beta) - 2\delta\eta^2 + (\delta^2 - \eta^2)(2\delta - \alpha - \beta)]\} \end{aligned}$$

$$\begin{aligned} A_{124} &= \frac{2}{\Delta} \{\eta(\theta - \beta)\theta^2\beta^2 [2\delta(\delta - \theta)(\delta - \beta) - 2\delta\eta^2 + (\delta^2 - \eta^2)(2\delta - \beta - \theta) \\ &\quad - \eta(\theta - \alpha)\alpha^2\theta^2 [2\delta(\delta - \theta)(\delta - \alpha) - 2\delta\eta^2 + (\delta^2 - \eta^2)(2\delta - \alpha - \beta)]\} \end{aligned}$$

$$\begin{aligned} A_{125} &= \frac{2}{\Delta} \{-\theta^2\beta^2(\theta - \beta) [-\alpha^2(\beta - \alpha)(\theta - \alpha) + (\delta^2 - \eta^2)\{(\delta - \theta)(\delta - \beta) - \eta^2\} \\ &\quad - 2\delta\eta^2(2\delta - \beta - \theta) + \alpha^2\theta^2(\theta - \alpha)[(\delta^2 - \eta^2)\{(\delta - \theta)(\delta - \alpha) - \eta^2\} \\ &\quad - 2\delta\eta^2(2\delta - \alpha - \theta)] - (\beta - \alpha)\alpha^2\beta^2[(\delta^2 - \eta^2)\{(\delta - \beta)(\delta - \alpha) - \eta^2\} \\ &\quad - 2\delta\eta^2(2\delta - \alpha - \beta)]\} \end{aligned}$$

$$\begin{aligned} A_{131} &= \frac{1}{\Delta} \{2\theta\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \theta^2(\eta^2 - 3\delta^2) + 2\delta\theta^3] - \\ &\quad 2\beta\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \beta^2(\eta^2 - 3\delta^2) + 2\delta\beta^3] \\ &\quad - 2\eta\beta\theta[-\beta^2\theta^2(\theta - \beta) + (\theta^3 - \beta^3)(3\delta^2 - \eta^2) + 4\delta(\theta^2 - \beta^2)(\delta^2 - \eta^2)]\} \end{aligned}$$

$$\begin{aligned} A_{132} &= \frac{1}{\Delta} \{-2\theta\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \theta^2(\eta^2 - 3\delta^2) + 2\delta\theta^3] \\ &\quad + 2\alpha\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \alpha^2(\eta^2 - 3\delta^2) + 2\delta\alpha^3] \\ &\quad + 2\alpha\eta\theta[-\alpha^2\theta^2(\theta - \alpha) + (\theta^3 - \alpha^3)(3\delta^2 - \eta^2) + 4\delta(\theta^2 - \alpha^2)(\delta^2 - \eta^2)]\} \end{aligned}$$

$$\begin{aligned} A_{133} &= \frac{1}{\Delta} \{2\beta\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \beta^2(\eta^2 - 3\delta^2) + 2\delta\beta^3] \\ &\quad - 2\alpha\eta(\delta^2 + \eta^2)[(\delta^2 + \eta^2)^2 + \alpha^2(\eta^2 - 3\delta^2) + 2\delta\alpha^3] \\ &\quad - 2\eta\alpha\beta[-\alpha^2\beta^2(\beta - \alpha) + (\beta^3 - \alpha^3)(3\delta^2 - \eta^2) + 4\delta(\beta^2 - \alpha^2)(\delta^2 - \eta^2)]\} \end{aligned}$$

$$A_{134} = \frac{2}{\Delta} \{ -\eta [\beta^3 \theta^3 (\theta - \beta) - \alpha^3 (\theta^4 - \beta^4) + \alpha^4 (\theta^3 - \beta^3)] - \\ \eta (\eta^2 - 3\delta^2) [\beta \theta (\theta^3 - \beta^3) - \alpha (\theta^4 - \beta^4) + \alpha^4 (\theta - \beta)] + \\ 4\delta\eta (\eta^2 - \delta^2) [\theta \beta (\theta^2 - \beta^2) - \alpha (\theta^3 - \beta^3) + \alpha^3 (\theta - \beta)] \}$$

$$A_{135} = \frac{2}{\Delta} \{ -\alpha\beta\theta [\beta^2 \theta^2 (\theta - \beta) - \alpha^2 (\theta^3 - \beta^3) + \alpha^3 (\theta^2 - \beta^2)] + \\ \delta [\beta^3 \theta^3 (\theta - \beta) - \alpha^3 (\theta^4 - \beta^4) + \alpha^4 (\theta^3 - \beta^3)] \\ - \delta (\delta^2 - 3\eta^2) [\beta \theta (\theta^3 - \beta^3) - \alpha (\theta^4 - \beta^4) + \alpha^4 (\theta - \beta)] \\ + [\delta^4 + \eta^4 - 6\delta^2\eta^2] [\theta \beta (\theta^2 - \beta^2) - \alpha (\theta^3 - \beta^3) + \alpha^3 (\theta - \beta)] \}$$

$$A_{141} = \frac{1}{\Delta} \{ 4\delta\eta\beta [(\delta^2 - \eta^2 - \theta^2)^2 + 4\delta^2\eta^2] - 4\delta\eta\theta [(\delta^2 - \eta^2 - \beta^2)^2 + 4\delta^2\eta^2] \\ - 2\eta(\theta^2 - \beta^2) [(\delta^2 - \eta^2 - \theta^2)(\delta^2 - \eta^2 - \beta^2) - 4\delta^2\eta^2] + \\ 4\delta^2\eta(\theta^2 - \beta^2) (2\delta^2 - 2\eta^2 - \theta^2 - \beta^2) \}$$

$$A_{142} = \frac{1}{\Delta} \{ -4\delta\eta\alpha [(\delta^2 - \eta^2 - \theta^2)^2 + 4\delta^2\eta^2] + 4\delta\eta\theta [(\delta^2 - \eta^2 - \alpha^2)^2 + 4\delta^2\eta^2] \\ + 2\eta(\theta^2 - \alpha^2) [(\delta^2 - \eta^2 - \theta^2)(\delta^2 - \eta^2 - \alpha^2) - 4\delta^2\eta^2] - \\ 4\delta^2\eta(\theta^2 - \alpha^2) (2\delta^2 - 2\eta^2 - \theta^2 - \alpha^2) \}$$

$$A_{143} = \frac{1}{\Delta} \{ 4\delta\eta\alpha [(\delta^2 - \eta^2 - \beta^2)^2 + 4\delta^2\eta^2] - 4\delta\eta\beta [(\delta^2 - \eta^2 - \alpha^2)^2 + 4\delta^2\eta^2] \\ - 2\eta(\beta^2 - \alpha^2) [(\delta^2 - \eta^2 - \beta^2)(\delta^2 - \eta^2 - \alpha^2) + \\ 4\delta^2\eta(\beta^2 - \alpha^2) (2\delta^2 - 2\eta^2 - \alpha^2 - \beta^2) \}$$

$$A_{144} = \frac{2}{\Delta} \{ -2\alpha\delta\eta(\theta^2 - \beta^2) (2\delta^2 - 2\eta^2 - \theta^2 - \beta^2) + 2\delta\eta\beta(\theta^2 - \alpha^2) (2\delta^2 - 2\eta^2 - \theta^2 - \alpha^2) \\ - 2\delta\eta\theta(\beta^2 - \alpha^2) (2\delta^2 - 2\eta^2 - \alpha^2 - \beta^2) + \eta(\beta^2 - \alpha^2)(\theta^2 - \beta^2)(\theta^2 - \alpha^2) \}$$

E.10

$$A_{145} = \frac{2}{\Delta} \{ \alpha(\theta^2 - \beta^2)[(\delta^2 - \eta^2 - \theta^2)(\delta^2 - \eta^2 - \beta^2) - 4\delta^2\eta^2] \\ - \beta(\theta^2 - \alpha^2)[(\delta^2 - \eta^2 - \theta^2)(\delta^2 - \eta^2 - \alpha^2) - 4\delta^2\eta^2] \\ + \theta(\beta^2 - \alpha^2)[(\delta^2 - \eta^2 - \beta^2)(\delta^2 - \eta^2 - \alpha^2) - 4\delta^2\eta^2] \\ - \delta(\beta^2 - \alpha^2)(\theta^2 - \beta^2)(\theta^2 - \alpha^2) \}$$

$$A_{151} = \frac{2\eta}{\Delta} (\theta - \beta)[(\delta - \beta)^2 + \eta^2][(\delta - \theta)^2 + \eta^2]$$

$$A_{152} = -\frac{2\eta}{\Delta} (\theta - \alpha)[(\delta - \alpha)^2 + \eta^2][(\delta - \theta)^2 + \eta^2]$$

$$A_{153} = \frac{2\eta}{\Delta} (\beta - \alpha)[(\delta - \beta)^2 + \eta^2][(\delta - \alpha)^2 + \eta^2]$$

$$A_{154} = \frac{2}{\Delta} \{ -\eta(\beta - \alpha)(\theta - \beta)(\theta - \alpha)[(\delta - \beta)(\delta - \alpha) - \eta^2 + (\delta - \theta)(2\delta - \alpha - \beta)] \}$$

$$A_{155} = \frac{2}{\Delta} \{ (\beta - \alpha)(\theta - \beta)(\theta - \alpha)[(\delta - \theta)[(\delta - \beta)(\delta - \alpha) - \eta^2] - \eta^2(2\delta - \alpha - \beta)] \}$$

So, the elements of the 5×5 fundamental matrix are written:

$$y_{ij} = \sum_{k=1}^3 A_{ijk} (\lambda_k)^{i-1} \exp(\lambda_k x) + (A_{1j4} + iA_{1j5}) (\lambda_4)^{i-1} \exp(\lambda_4 x) \\ + (\lambda_{j4} - iA_{1j5}) (\lambda_5)^{i-1} \exp(\lambda_5 x) \quad (E.9)$$

E.2.1.3 Case of Five Real Roots

In this case the eigenvalues are of the form:

$$\lambda_1 = \alpha$$

$$\lambda_2 = \beta$$

$$\lambda_3 = \theta$$

$$\lambda_4 = \delta$$

$$\lambda_5 = \eta$$

(E.10)

So the following parameters are defined:

$$\Delta = (\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\delta - \theta)(\delta - \beta)(\delta - \alpha)(\eta - \delta)(\eta - \theta)(\eta - \beta)(\eta - \alpha)$$

$$A_{111} = \beta\theta\delta\eta (\theta - \beta)(\delta - \theta)(\delta - \beta)(\eta - \delta)(\eta - \theta)(\eta - \beta) \frac{1}{\Delta}$$

$$A_{112} = -\frac{\alpha\theta\delta\eta}{\Delta} (\theta - \alpha)(\delta - \theta)(\delta - \alpha)(\eta - \delta)(\eta - \theta)(\eta - \alpha)$$

$$A_{113} = \frac{\alpha\beta\delta\eta}{\Delta} (\beta - \alpha)(\delta - \beta)(\delta - \alpha)(\eta - \delta)(\eta - \beta)(\eta - \alpha)$$

$$A_{114} = -\frac{\alpha\beta\theta\eta}{\Delta} (\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\eta - \theta)(\eta - \beta)(\eta - \alpha)$$

$$A_{115} = \frac{\alpha\beta\theta\delta}{\Delta} (\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\delta - \theta)(\delta - \beta)(\delta - \alpha)$$

$$A_{121} = \frac{1}{\Delta} \{ -\theta^2\delta^2\eta^2(\delta - \theta)(\eta - \delta)(\eta - \theta) + \beta^2\delta^2\eta^2(\delta - \beta)(\eta - \delta)(\eta - \beta) \\ - \beta^2\theta^2\eta^2(\theta - \beta)(\eta - \theta)(\eta - \beta) + \beta^2\theta^2\delta^2(\theta - \beta)(\delta - \theta)(\delta - \beta) \}$$

$$A_{122} = \frac{1}{\Delta} \{ \theta^2\delta^2\eta^2(\delta - \theta)(\eta - \delta)(\eta - \theta) - \alpha^2\delta^2\eta^2(\delta - \alpha)(\eta - \delta)(\eta - \alpha) \\ + \alpha^2\theta^2\eta^2(\theta - \alpha)(\eta - \theta)(\eta - \alpha) - \alpha^2\theta^2\delta^2(\theta - \alpha)(\delta - \theta)(\delta - \alpha) \}$$

$$A_{123} = \frac{1}{\Delta} \{ -\beta^2\delta^2\eta^2(\delta - \beta)(\eta - \delta)(\eta - \beta) + \alpha^2\delta^2\eta^2(\delta - \alpha)(\eta - \delta)(\eta - \alpha) \\ - \alpha^2\beta^2\eta^2(\beta - \alpha)(\eta - \beta)(\eta - \alpha) + \alpha^2\beta^2\delta^2(\beta - \alpha)(\delta - \beta)(\delta - \alpha) \}$$

$$A_{124} = \frac{1}{\Delta} \{ \beta^2\theta^2\eta^2(\theta - \beta)(\eta - \theta)(\eta - \beta) - \alpha^2\theta^2\eta^2(\theta - \alpha)(\eta - \theta)(\eta - \alpha) \\ + \alpha^2\beta^2\eta^2(\beta - \alpha)(\eta - \beta)(\eta - \alpha) - \alpha^2\beta^2\theta^2(\beta - \alpha)(\theta - \beta)(\theta - \alpha) \}$$

$$A_{125} = \frac{1}{\Delta} \{ -\beta^2\theta^2\delta^2(\theta - \beta)(\delta - \theta)(\delta - \beta) + \alpha^2\theta^2\delta^2(\theta - \alpha)(\delta - \theta)(\delta - \alpha) \\ - \alpha^2\beta^2\delta^2(\beta - \alpha)(\delta - \beta)(\delta - \alpha) + \alpha^2\beta^2\theta^2(\beta - \alpha)(\theta - \beta)(\theta - \alpha) \}$$

$$A_{131} = \frac{1}{\Delta} \{ \delta^3\eta^3(\eta - \delta)(\theta - \beta) + \theta^3\eta^3(\eta - \theta)(\beta - \delta) + \theta^3\delta^3(\delta - \theta)(\eta - \beta) \\ + \beta^3\eta^3(\eta - \delta)(\delta - \theta) + \beta^3\delta^3(\delta - \beta)(\theta - \eta) + \beta^3\theta^3(\theta - \beta)(\eta - \delta) \}$$

$$A_{132} = \frac{1}{\Delta} \{ \delta^3 \eta^3 (\eta - \delta) (\alpha - \theta) + \theta^3 \eta^3 (\eta - \theta) (\delta - \alpha) + \theta^3 \delta^3 (\delta - \theta) (\alpha - \eta)$$

$$+ \alpha^3 \eta^3 (\eta - \alpha) (\theta - \delta) + \delta^3 \alpha^3 (\delta - \alpha) (\eta - \theta) + \theta^3 \alpha^3 (\theta - \alpha) (\delta - \eta) \}$$

$$A_{133} = \frac{1}{\Delta} \{ \delta^3 \eta^3 (\eta - \delta) (\beta - \alpha) + \beta^3 \eta^3 (\eta - \beta) (\alpha - \delta) + \beta^3 \delta^3 (\delta - \beta) (\eta - \alpha)$$

$$+ \alpha^3 \eta^3 (\eta - \alpha) (\delta - \beta) + \alpha^3 \delta^3 (\delta - \alpha) (\beta - \eta) + \alpha^3 \beta^3 (\beta - \alpha) (\eta - \delta) \}$$

$$A_{134} = \frac{1}{\Delta} \{ \theta^3 \delta^3 (\delta - \theta) (\alpha - \beta) + \beta^3 \delta^3 (\delta - \beta) (\alpha - \theta) + \theta^3 \beta^3 (\theta - \beta) (\alpha - \delta)$$

$$+ \alpha^3 \delta^3 (\delta - \alpha) (\beta - \theta) + \alpha^3 \theta^3 (\theta - \alpha) (\delta - \beta) + \alpha^3 \beta^3 (\beta - \alpha) (\theta - \delta) \}$$

$$A_{135} = \frac{1}{\Delta} \{ \theta^3 \eta^3 (\eta - \theta) (\beta - \alpha) + \beta^3 \eta^3 (\eta - \beta) (\alpha - \theta) + \theta^3 \beta^3 (\theta - \beta) (\eta - \alpha)$$

$$+ \alpha^3 \eta^3 (\eta - \alpha) (\theta - \beta) + \alpha^3 \theta^3 (\theta - \alpha) (\beta - \eta) + \alpha^3 \beta^3 (\beta - \alpha) (\eta - \theta) \}$$

$$A_{141} = \frac{1}{\Delta} \{ \theta (\eta^2 - \delta^2) (\eta^2 - \theta^2) (\delta^2 - \theta^2) - \beta (\delta^2 - \beta^2) (\eta^2 - \delta^2) (\eta^2 - \beta^2)$$

$$+ \delta (\theta^2 - \beta^2) (\eta^2 - \theta^2) (\eta^2 - \beta^2) - \eta (\theta^2 - \beta^2) (\delta^2 - \theta^2) (\delta^2 - \beta^2) \}$$

$$A_{142} = \frac{1}{\Delta} \{ -\theta (\eta^2 - \delta^2) (\eta^2 - \theta^2) (\delta^2 - \theta^2) + \alpha (\delta^2 - \alpha^2) (\eta^2 - \delta^2) (\eta^2 - \alpha^2)$$

$$- \delta (\theta^2 - \alpha^2) (\eta^2 - \theta^2) (\eta^2 - \alpha^2) + \eta (\theta^2 - \alpha^2) (\delta^2 - \theta^2) (\delta^2 - \alpha^2) \}$$

$$A_{143} = \frac{1}{\Delta} \{ \beta (\eta^2 - \delta^2) (\eta^2 - \beta^2) (\delta^2 - \beta^2) - \alpha (\delta^2 - \alpha^2) (\eta^2 - \delta^2) (\eta^2 - \alpha^2)$$

$$+ \delta (\beta^2 - \alpha^2) (\eta^2 - \beta^2) (\eta^2 - \alpha^2) - \eta (\beta^2 - \alpha^2) (\delta^2 - \beta^2) (\delta^2 - \alpha^2) \}$$

$$A_{144} = \frac{1}{\Delta} \{ -\alpha (\theta^2 - \beta^2) (\eta^2 - \theta^2) (\eta^2 - \beta^2) + \beta (\theta^2 - \alpha^2) (\eta^2 - \theta^2) (\eta^2 - \alpha^2)$$

$$- \theta (\beta^2 - \alpha^2) (\eta^2 - \beta^2) (\eta^2 - \alpha^2) + \eta (\beta^2 - \alpha^2) (\theta^2 - \beta^2) (\theta^2 - \alpha^2) \}$$

$$A_{145} = \frac{1}{\Delta} \{ \alpha (\theta^2 - \beta^2) (\delta^2 - \theta^2) (\delta^2 - \beta^2) - \beta (\theta^2 - \alpha^2) (\delta^2 - \theta^2) (\delta^2 - \alpha^2)$$

$$+ \theta (\beta^2 - \alpha^2) (\delta^2 - \beta^2) (\delta^2 - \alpha^2) - \delta (\beta^2 - \alpha^2) (\theta^2 - \beta^2) (\theta^2 - \alpha^2) \}$$

E.13

$$A_{151} = \frac{1}{\Delta} \{(\theta - \beta)(\delta - \theta)(\delta - \beta)(\eta - \delta)(\eta - \theta)(\eta - \beta)\}$$

$$A_{152} = -\frac{1}{\Delta} \{(\theta - \alpha)(\delta - \theta)(\delta - \alpha)(\eta - \delta)(\eta - \theta)(\eta - \alpha)\}$$

$$A_{153} = \frac{1}{\Delta} \{(\beta - \alpha)(\delta - \beta)(\delta - \alpha)(\eta - \delta)(\eta - \beta)(\eta - \alpha)\}$$

$$A_{154} = \frac{1}{\Delta} \{(\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\eta - \theta)(\eta - \beta)(\eta - \alpha)\}$$

$$A_{155} = \frac{1}{\Delta} \{(\beta - \alpha)(\theta - \beta)(\theta - \alpha)(\delta - \theta)(\delta - \beta)(\delta - \alpha)\}$$

So, the elements of the 5×5 fundamental matrix are written:

$$y_{ij} = \sum_{k=1}^5 A_{ijk} (\lambda_k)^{i-1} \exp(\lambda_k x) \quad (E.11)$$

E.3 General Solution to Equation (6.5)

The general solution to equation (6.5) will be the sum of the homogeneous solution and a particular solution. One considers (E.2) and defines by $T(t)$ the 2×2 fundamental matrix of this equation. According to the hypothesis of paragraph (6.1), the motion is assumed to be harmonic. So, $y(x,t)$ and the exciting force will be the product of two independent functions. One performs a series expansion of the term containing the exciting force and, applies the method used in solving the non-homogeneous partial differential equation of the second order.

If one considers the boundary values with respect to t and defines the displacement;

$$y(x,t) = y(x) \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad (E.12)$$

where the array column designates the value of $T(t)$ and its derivatives at $t=0$, then $y(x,t)$ can be expressed in the form:

E.14

$$y(x,t) = Y(x,t) y(0) + \int_0^x \int_0^t Y(x-\mu, t-\tau) B(\mu, \tau) y(\mu, \tau) d\mu d\tau \\ + \int_0^x \int_0^t Y(x-\mu, t-\tau) f(\mu, \tau) d\mu d\tau \quad (6.25)$$

In this formula, $B(\mu, \tau)$ is assumed to be a 5×5 matrix of the perturbator terms and $f(\mu, \tau)$ the array column of terms including $p(\mu, t)$.

APPENDIX F

FORCED VIBRATION AND DAMPED MOTION OF THE
STACK WITH VARIABLE INERTIA

F.1

APPENDIX F

FORCED VIBRATION AND DAMPED MOTION OF THE
STACK WITH VARIABLE INERTIA

F.1 Damped Motion

Let us consider the partial differential equation (6.21) of the sixth order.

One performs an expansion in series of analytic functions inside the disk of radius convergence $\inf \left(\frac{1}{a}, \frac{1}{b} \right)$.

So the following parameters are defined:

$$\theta_1(x, m, n) = -(9a\phi_1(x, m, n) + 3b\phi_2(x, m, n))$$

$$\theta_2(x, m, n) = -(18a^2\phi_3(m, n, x) + 18ab\phi_4(m, n, x))$$

$$\theta_3(x, m, n) = -(6a^3\phi_6(x, m, n) + 12a^2b\phi_5(x, m, n))$$

$$\begin{aligned} \theta_4(x, m, n) = -\frac{1}{Kg} [P - q_0 \{-6\ell + [6 - 3\ell(a+b)]x + \\ [3(a+b) - 2ab\ell]x^2 + 2abx^3\}] \phi_7(x, m, n) \end{aligned}$$

$$\begin{aligned} \theta_5(x, m, n) = \frac{q_0}{Kg} [12 - 9\ell(a+b) + \{15(a+b) - 12ab\ell\}x + \\ 16abx^2] \phi_7(x, m, n) \end{aligned}$$

$$\theta_6(x, m, n) = \frac{q_0}{Kg} [9(a+b) - 12ab\ell + 24abx] \phi_7(x, m, n)$$

$$\theta_7(x, m, n) = -\frac{q_0}{Kg} [6 + 3(a+b)x + 2abx^2] \phi_7(x, m, n)$$

$$\theta_8(x, m, n) = -\frac{q_0}{Kg} [3(a+b) + 4abx] \phi_7(x, m, n)$$

$$\theta_9(x, m, n) = -\frac{C}{K} \phi_7(x, m, n) \quad (F.1)$$

F.2

The following inhomogeneous partial differential equation is written:

$$c_s \frac{\partial^6 y(x,t)}{\partial x^5 \partial t} + \frac{\partial^5 y(x,t)}{\partial x^5} + (9a + 3b) \left(c_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right)$$

$$+ (18a^2 + 18ab) \left(c_s \frac{\partial^4 y(x,t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x,t)}{\partial x^3} \right) + (6a^3 + 12a^2b) (c_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t})$$

$$+ \left(\frac{\partial^2 y(x,t)}{\partial x^2} \right) + \frac{P + 6q_0 \ell}{Kg} \frac{\partial^3 y(x,t)}{\partial x^3} - \frac{q_0}{Kg} (12 - 9\ell(a+b)) \frac{\partial^2 y(x,t)}{\partial x^2}$$

$$- \frac{q_0}{Kg} (9(a+b) - 12ab\ell) \frac{\partial y(x,t)}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^3 y(x,t)}{\partial x \partial t^2} +$$

$$\frac{3(a+b)q_0}{Kg} \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{c}{K} \frac{\partial^2 y(x,t)}{\partial x \partial t} =$$

$$\frac{1}{K(1+ax)^3(1+bx)} \frac{\partial}{\partial x} p(x,t) + \phi(x, \frac{\partial^k y(x,t)}{\partial x^5 \partial t^u}, t) \quad (6.22)$$

where:

$$\phi(x, \frac{\partial^k y(x,t)}{\partial x^5 \partial t^u}, t) = \theta_1(x, m, n) \left(c_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right) +$$

$$\theta_2(x, m, n) \left(c_s \frac{\partial^4 y(x,t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x,t)}{\partial x^3} \right) + \theta_3(x, m, n) \left(c_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x,t)}{\partial x^2} \right)$$

$$\theta_4(x, m, n) \frac{\partial^3 y(x,t)}{\partial x^3} + \theta_5(x, m, n) \frac{\partial^2 y(x,t)}{\partial x^2} + \theta_6(x, m, n) \frac{\partial y(x,t)}{\partial x}$$

F.3

$$+ \theta_6(x, m, n) \frac{\partial^3 y(x, t)}{\partial x \partial t^2} + \theta_7(x, m, n) \frac{\partial^2 y(x, t)}{\partial t^2} + \theta_8(x, m, n) \frac{\partial^2 y(x, t)}{\partial x \partial t}$$

$$k = 1,5 \quad s = 0,4 \quad u = 0,2 \quad s + u = k$$

As the proposed solution, one applies the d'Alembert's method and a derivation with respect to the time variable t to the homogeneous equation in (6.22).

Consequently, equations (6.23) and (6.24) are obtained:

$$\frac{d^5 y(x)}{dx^5} + a_1 \frac{d^4 y(x)}{dx^4} + a_2 \frac{d^3 y(x)}{dx^3} + a_3 \frac{d^2 y(x)}{dx^2} + a_4 \frac{dy(x)}{dx} + a_5 y(x) = 0 \quad (6.23)$$

$$\ddot{T} + \omega^2 \dot{T} - \lambda' T = 0 \quad (6.24)$$

where:

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18ab$$

$$a_3 = 6a^3 + 12a^2b$$

$$a_4 = \frac{C'}{KC_S} - \frac{6q_0\omega^2}{C'_S Kg}$$

$$a_5 = -\frac{3\omega^2(a+b)q_0}{C'_S Kg}$$

ω is the frequency and λ' a constant of integration. One can observe that the coefficient of the term T in equation (6.22) is the left-hand side of the static buckling equation (4.8). By applying the proposed method, this coefficient disappears.

So, the solution to (6.22) is given by (6.25).

APPENDIX G

CHIMNEY WITH CONSTANT DIAMETER

G.1

APPENDIX G

CHIMNEY WITH CONSTANT DIAMETERG.1 Undamped Motion

One considers equation (6.26) and the method developed in Appendix E. With the modification introduced by the parameter d , one will define:

$$\theta_1(x, m, n) = - (9a\phi_1(x, m, n) + 3b\phi_2(x, m, n))$$

$$\theta_2(x, m, n) = - (18a^2\phi_3(x, m, n) + 18ab\phi_4(x, m, n) + \frac{1}{Kg} [P -$$

$$q_0 \{ -6\omega + [6 - 6b(1+d)\omega]x + [3b(1+d) - 12d\omega b^2]x^2 +$$

$$2db^3x^3 \})\phi_1(x, m, n)$$

$$\theta_3(x, m, n) = -(6a^3\phi_5(x, m, n) + 18a^2b\phi_6(x, m, n) - \frac{q_0}{Kg} \{ 12 -$$

$$9b\omega(1+d) + [15b(1+d) - 12d\omega b^2]x + 16db^2x^2 \})\phi_2(x, m, n))$$

$$\theta_4(x, m, n) = \frac{q_0}{Kg} [9b(1+d) - 12d\omega b^2 + 24db^2x]\phi_3(x, m, n)$$

$$\theta_5(x, m, n) = - \frac{q_0}{Kg} [6 + 3b(1+d)x + 2db^2x^2]\phi_4(x, m, n)$$

$$\theta_6(x, m, n) = - \frac{q_0}{Kg} [3b(1+d) + 4db^2x]\phi_5(x, m, n)$$

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18a^2b + (P + 6\omega q_0)/Kg$$

$$a_3 = 6a^3 + 18a^2b - \frac{q_0}{Kg} [12 - 9b\omega(1+d)]$$

$$a_4 = - \frac{q_0}{Kg} [9b(1+d) - 12d\omega b^2 - 6\omega^2]$$

$$a_5 = - \frac{3q_0}{Kg} \omega^2 b(1+d)$$

G.2

$$\begin{aligned}
 \phi(x, \frac{\partial^k y(x, t)}{\partial x^m \partial t^n}, t) = & \theta_1(x, m, n) \frac{\partial^4 y(x, t)}{\partial x^4} + \theta_2(x, m, n) \frac{\partial^3 y(x, t)}{\partial x^3} \\
 & + \theta_3(x, m, n) \frac{\partial^2 y(x, t)}{\partial x^2} + \theta_4(x, m, n) \frac{\partial y(x, t)}{\partial x} + \theta_5(x, m, n) \frac{\partial^3 y(x, t)}{\partial x \partial t^2} \\
 & + \theta_6(x, m, n) \frac{\partial^2 y(x, t)}{\partial t^2}
 \end{aligned} \tag{G.1}$$

So, the solution to equation (6.26) is given by (6.25).

APPENDIX H

CHIMNEY OF CONSTANT INSIDE DIAMETER
FORCED VIBRATION AND DAMPED MOTION.

H.1

APPENDIX H

CHIMNEY OF CONSTANT INSIDE DIAMETERFORCED VIBRATION AND DAMPED MOTIONH.1 Equation of Motion

As in Appendix F, one considers the damped motion of the stack of constant inside diameter. With the modification introduced by the parameter d , the expansion in series leads to the following parameters:

$$\theta_1(x, m, n) = -(9a\phi_1(x, m, n) + 3b\phi_2(x, m, n))$$

$$\theta_2(x, m, n) = -(18a^2\phi_3(x, m, n) + 18ab\phi_4(x, m, n))$$

$$\theta_3(x, m, n) = -(6a^3\phi_5(x, m, n) + 12a^2b\phi_6(x, m, n))$$

$$\theta_4(x, m, n) = -\frac{1}{Kg} [P - q_0 \{-6\ell + [6 - 6b(1+d)\ell]x + [3b(1+d) - 2d\ell b^2]x^2 + 2dbx^3\}] \phi_7(x, m, n)$$

$$\theta_5(x, m, n) = \frac{q_0}{Kg} [12 - 9b\ell(1+d) + \{15b(1+d) - 12d\ell b^2\}x + 16db^2x^2] \phi_7(x, m, n)$$

$$\theta_6(x, m, n) = \frac{q_0}{Kg} [9b(1+d) - 12d\ell b^2 + 24db^2x] \phi_7(x, m, n)$$

$$\theta_7(x, m, n) = -\frac{q_0}{Kg} [6 + 3(1+d)bx + 2db^2x^2] \phi_7(x, m, n)$$

$$\theta_8(x, m, n) = -\frac{q_0}{Kg} [3b(1+d) + 4db^2x] \phi_7(x, m, n)$$

$$\theta_9(x, m, n) = -\frac{C}{K} \phi_7(x, m, n)$$

$$a_1 = 9a + 3b$$

$$a_2 = 18a^2 + 18ab$$

$$a_3 = 6a^3 + 12a^2b$$

$$a_4 = \frac{C_s}{KC_s} - \frac{6q_0\omega^2}{C_s \text{ Kg}}$$

$$a_5 = -\frac{3\omega^2 b(1+d)q_0}{C_s \text{ Kg}}$$

$$\phi(x, \frac{\partial^k y(x,t)}{\partial x^s \partial t^u}, t) = \theta_1(x, m, n) \left(C_s \frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 y(x,t)}{\partial x^4} \right) +$$

$$\theta_2(x, m, n) \left(C_s \frac{\partial^4 y(x,t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x,t)}{\partial x^3} \right) + \theta_3(x, m, n) \left(C_s \frac{\partial^3 y(x,t)}{\partial x^2 \partial t} \right.$$

$$\left. \frac{\partial^2 y(x,t)}{\partial x^2} \right) + \theta_4(x, m, n) \frac{\partial^3 y(x,t)}{\partial x^3} + \theta_5(x, m, n) \frac{\partial^2 y(x,t)}{\partial x^2} +$$

$$\theta_6(x, m, n) \frac{\partial y(x,t)}{\partial x} + \theta_8(x, m, n) \frac{\partial^3 y(x,t)}{\partial x \partial t^2} + \theta_7(x, m, n) \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$+ \theta_9(x, m, n) \frac{\partial^2 y(x,t)}{\partial x \partial t} \quad (H.1)$$

So, equation (6.28) is obtained. The general solution has the same form as that of (6.22).

APPENDIX I

CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETER

APPENDIX I

CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETERI.1 Undamped MotionI.1.1 Equation of Motion

Let us consider the partial differential equation (6.30). One will perform the expansion in series of the coefficients inside the circle of radius of convergence $1/a$.

So, the following parameters are defined:

$$\theta_1(x, m, n) = -6a\phi_1(x, m, n)$$

$$\theta_2(x, m, n) = -[6a^2\phi_3(x, m, n) + \frac{1}{Kg}(P + 3q_0\{2\ell - (a\ell - 2)x - ax^2\})\phi_6(x, m, n)]$$

$$\theta_3(x, m, n) = -\frac{3q_0}{Kg}(2a\ell - 2 - 3ax)\phi_6(x, m, n)$$

$$\theta_4(x, m, n) = -\frac{3q_0}{Kg}(2 + ax)\phi_6(x, m, n)$$

$$\begin{aligned} \phi(x, \frac{\partial^k y}{\partial x^5 \partial t^4}, t) = & \theta_1(x, m, n) \frac{\partial^3 y(x, t)}{\partial x^3} + \theta_2(x, m, n) \frac{\partial^2 y(x, t)}{\partial x^2} \\ & + \theta_3(x, m, n) \frac{\partial y(x, t)}{\partial x} + \theta_4(x, m, n) \frac{\partial^2 y(x, t)}{\partial t^2} \end{aligned}$$

So, one obtains equation (I.1) below

$$\begin{aligned} \frac{\partial^4 y(x, t)}{\partial x^4} + 6a \frac{\partial^3 y(x, t)}{\partial x^3} + [6a + \frac{1}{Kg}(P + 6\ell q_0)] \frac{\partial^2 y(x, t)}{\partial x^2} \\ - \frac{3q_0}{Kg}(-2 + 2a\ell) \frac{\partial y(x, t)}{\partial x} + \frac{6q_0}{Kg} \frac{\partial^2 y(x, t)}{\partial t^2} = \\ \frac{p(x, t)}{K[1 + ax]^3} + \phi \left(x, \frac{\partial^k y(x, t)}{\partial x^5 \partial t^4}, t \right) \end{aligned} \quad (I.1)$$

If the d'Alembert's method is applied to the homogeneous part, the following parameters will be defined:

$$a_1 = 6a$$

$$a_2 = \frac{P + 6\ell q_0}{Kg}$$

$$a_3 = \frac{6q_0(\alpha\ell - 1)}{Kg}$$

$$a_4 = \frac{6\omega^2 q_0}{Kg}$$

where ω is the frequency.

The characteristic polynomial is an equation of the fourth degree. So, the roots are of the same form as those in Appendix B. The general solution is given by (6.33).

APPENDIX J

CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETER

APPENDIX J

CHIMNEY OF CONSTANT THICKNESS AND VARIABLE DIAMETER, DAMPED MOTIONJ.1 Equation of Motion

Let us consider the partial differential equation (6.34). One will perform an expansion in series of the coefficients. Then the following parameters are defined

$$\theta_1(x, m, n) = -6a\phi_1(x, m, n)$$

$$\theta_2(x, m, n) = -6a^2\phi_3(x, m, n)$$

$$\theta_3(x, m, n) = -\frac{1}{Kg} [P + 3q_0 \{2\ell - (a\ell - 2)x - ax^2\}] \phi_6(x, m, n)$$

$$\theta_4(x, m, n) = -\frac{3q_0}{Kg} [-2 + 2a\ell - 3ax] \phi_6(x, m, n)$$

$$\theta_5(x, m, n) = -\frac{3q_0}{Kg} (2 + ax) \phi_6(x, m, n)$$

$$\theta_6(x, m, n) = \frac{C'}{K} \phi_6(x, m, n)$$

$$\phi(x, \frac{\partial^k y}{\partial x^s \partial t^u}, t) = \theta_1(x, m, n) \left(C_s \frac{\partial^4 y(x, t)}{\partial x^3 \partial t} + \frac{\partial^3 y(x, t)}{\partial x^3} \right) + \theta_2(x, m, n) \cdot$$

$$\left(C_s \frac{\partial^3 y(x, t)}{\partial x^2 \partial t} + \frac{\partial^2 y(x, t)}{\partial x^2} \right) + \theta_3(x, m, n) \frac{\partial^2 y(x, t)}{\partial x^2} + \theta_4(x, m, n) \frac{\partial y(x, t)}{\partial x} +$$

$$\theta_5(x, m, n) \frac{\partial^2 y(x, t)}{\partial t^2} + \theta_6(x, m, n) \frac{\partial y(x, t)}{\partial t} \quad (J.1)$$

So one obtains equation (6.35). If the method used in solving (6.22) is applied to (6.35), the following parameters will be defined:

$$a_1 = 6a$$

$$a_2 = 6a^2$$

$$a_3 = 0$$

$$a_4 = \frac{T}{KC^4} s \left(C^4 - \frac{6\omega^2 q_0}{g} \right)$$

The characteristic polynomial is an equation of the fourth degree. So, the roots are of the same form as those in Appendix B. The general solution is given by (6.33).

APPENDIX K

INFLUENCE OF THE AXIAL FORCE ON THE
STATIC FLEXURAL BENDING MOMENT

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K.1 Differential Equation of the Elastic Line

Let us consider the bar with variable inertia acted upon by the set of vertical and horizontal loads at ends 1 and 2. The parameters having been defined in Appendix A, the differential equation of the elastic line is written:

$$y^{(2)}(x) = \frac{1}{K(1+ax)^3(1+bx)} (M_2 + V_2x - Py - px^2/2) \quad (K.1)$$

Inside the disk of radius of convergence $\inf(1/a, 1/b)$, one will perform an expansion in series, as before, of the denominator. So, the elementary differential equation of the second order is obtained.

$$y^{(2)}(x) + \omega^2 y = \frac{1}{K} (M_2 + V_2x - \frac{px}{2}) + (M_2 + V_2x - Py - \frac{px^2}{2}) \phi_7(x, m, n) \quad (K.2)$$

where:

$$\omega^2 = \frac{P}{K}$$

So, the influence of the axial load is taken into account in the static flexural bending moment of the stack.