SYNTHESIS OF A CLASS OF MULTIVARIABLE NETWORK FUNCTIONS AS CASCADE OF SINGLE-VARIABLE LOSSLESS TWO-PORTS

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ABSTRACT

SYNTHESIS OF A CLASS OF MULTIVARIABLE NETWORK FUNCTIONS AS CASCADE OF SINGLE-VARIABLE LOSSLESS TWO-PORTS

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This study is concerned with the problem of realization of a multivariable positive real function (MPRF) of arbitrary degree in each variable as a terminated cascade structure of single-variable lossless two-port networks. The proposed technique of realization is based on cascade-separability of MPRFs into single-variable functions, and on the applicability of the method of single-variable Darlington synthesis to multivariable functions.

Conditions for the realization of an m-variable positive real function (PRF) as the driving-point impedance of a single-variable lossless two-port network terminated by an impedance function of the remaining (m-1)-variables are developed. It is shown that augmentation as in the case of single-variable Darlington synthesis is not possible for multivariable functions. Consequently, the extracted lossless two-port is non-reciprocal unless certain even part condition is satisfied. It is established that when the extraction of a single-variable lossless two-port network is possible in more than one variable, the choice of any one variable over the others is not to be preferred.

Using the result of cascade extraction of a single-variable lossless two-port, realization techniques for a class of ladder networks are proposed. In particular, realizability conditions for a multivariable ladder structure with or without a resistive termination which is cascade of several single-variable lossless ladder networks with all of their transmission zeros at the origin or at infinity are derived. Also, using a real part condition, an explicit solution is provided for the realization of a class of multivariable resistively-terminated lossless ladder networks with all of their transmission zeros at the origin or at infinity. The reactive elements of these ladder networks do not follow a sequence which is dependent on the types of the elements.

Some properties of a cascade structure of single-variable lossless two-ports, each in a distinct variable, and terminated by a single-variable impedance function are investigated from the partial derivative point of view. These properties are studied in greater detail when any one of the lossless two-ports takes the form of a ladder network. In particular, conditions involving partial derivatives for the extraction of a single-variable lowpass or highpass ladder network from a multivariable function are derived.

Conditions are developed for the realization of the voltage transfer function of a resistively-terminated cascade of p_1- and p_2-variable lowpass or highpass ladder networks. The solution is provided by converting the problem of a transfer function realization to the problem of two-variable impedance function realization. It is shown that these ladder networks can be transformed into wave digital filters realizing a class of digital transfer functions.

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LIST OF ABBREVIATIONS AND SYMBOLS

HPN	: Hurwitz Polynomial in the Narrow Sense
SHP	: Strictly Hurwitz Polynomial
PRF	: Positive Real Function
SPRF	: Single-Variable Positive Real Function
MPRF	: Multivariable Positive Real Function
RF	: Reactance Function
TRF	: Two-Variable Reactance Function
UE	: Unit Element
$s, p_1, p_2, \ldots, p_m$	: Complex Frequency Variables
$p_{1,\ldots,m}$	: $\{p_1, p_2, \ldots, p_m\}$
Ω_i	: $\{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m\}$
Ω_{ij}	: $\{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_m\}$
$\Omega_{i,\ldots,k}$	: $\{p_1, p_2, \ldots, p_{i-2}, p_{i-1}, p_{k+1}, p_{k+2}, \ldots, p_m\}$
Φ_i	: $\{j\omega_1, j\omega_2, \ldots, j\omega_{i-1}, j\omega_{i+1}, \ldots, j\omega_m\}$
$\delta_{p_i}(Z)$	: Degree of the Variable p_i in an MPRF Z
$Z_{p_i}(p_{1,\ldots,m})$	: $\dfrac{\partial Z(p_{1,\ldots,m})}{\partial p_i}$
$Ev(\cdot)$	: Even Part of a Rational Function
$Nu(\cdot)$	: Numerator of a Rational Function
$m_k(p_i)$, $m_k^{(i)}(p_i)$	: Single-Variable Even Polynomials
$n_k(p_i)$, $n_k^{(i)}(p_i)$	: Single-Variable Odd Polynomials

$M_k(p_{1,\ldots,m})$, $M_k^{(i)}(p_{i,\ldots,m})$	: Multivariable Even Polynomials
$N_k(p_{1,\ldots,m})$, $N_k^{(i)}(p_{i,\ldots,m})$	: Multivariable Odd Polynomials
$\lvert\cdot\rvert$	: Absolute Value of a Function
$\mathrm{Re}\,(\cdot)$	: Real Part of a Complex Argument
$\triangleq$	: Equal by Definition

CHAPTER I
INTRODUCTION

1.1 GENERAL

The theory of rational functions of several complex variables, though not new, has only recently found extensive applications in many complicated systems [1]. The behaviour of space-invariant multi-dimensional optical processing systems has been characterized by rational transfer functions of several variables [2], [3]. Multi-dimensional digital filters have used the principles and mathematical tools of multivariable theory [4]-[8]. It has been shown that the theory of multivariable positive real functions (MPRFs) provides an effective means of dealing with the problems of variable parameter networks [9], [10]. For networks consisting of lumped reactances, in addition to commensurate or non-commensurate lengths of transmission lines, the network functions become irrational. In such cases, the theory of single-variable lumped networks is not directly applicable due to the transcendental nature of the network functions. However, the realization problem of these mixed lumped-distributed networks is investigated by the conversion of the transcendental functions of $s = \sigma + j\omega$ into polynomial functions of several variables, and as a result the network functions become rational functions of several variables. With this approach, the system functions of mixed lumped-distributed networks can conveniently be compared with those of lumped networks. Thus, it has been possible to extend many single-variable concepts to multivariable functions, and to determine where the techniques of the lumped network synthesis is applicable directly or indirectly to multivariable networks. A review on the similarities and dissimilarities between single-variable and two-variable reactance functions may be found in [11].

Although a great deal of work has already been done in the area of multidimensional systems [1], it is not established that positive realness of a multivariable impedance function, like that of a single-variable function, is in general a necessary and sufficient condition for its realization as a linear, passive network. For instance, it is always possible to synthesize a given two-variable reactance function or matrix [12]-[14], but in the present state of art, it has not been possible to realize a given MPRF [15], [16].

Restricting the discussion to multivariable synthesis with constrained topology, it has been shown that even though an arbitrary two-variable reactance matrix can be synthesized, the same cannot be realized with a prescribed topology. In this case, the function requires some conditions in addition to the two-variable reactance property. The realization of a cascade of commensurate transmission lines and lumped reactances terminated in a resistance requires a set of conditions on the input impedance [17]. It is also known that if the even part of an MPRF is prescribed, it is not always possible to generate the function [18]. This means that if a multivariable voltage transfer function is given, it may not be possible to realize it as a reactance network terminated in a resistance.

It is clear from the foregoing discussion that special studies have to be carried out whenever realizations of multivariable network functions with constrained topologies are to be investigated.

1.2 REALIZATION OF MULTIVARIABLE CASCADE STRUCTURES

Extensive work has been done on the problem of multivariable synthesis of resistively-terminated cascade of commensurate or non-commensurate uniform lossless transmission lines and lumped passive lossless two-port networks. With growing interest in integrated circuits, these kinds of structures have become particularly useful in the design of microwave filters using TEM mode with or without lumped discontinuities, and networks containing semiconductor elements and commensurate transmission lines. Applications of multivariable cascade structures are also found in the design of multidimensional digital filters and acoustic filters. Various authors have studied these cascade structures, and each has given necessary and sufficient conditions for the structure with a presumed topology. A brief review of their work follows in the subsequent paragraphs of this section.

Ansell [19] has established the realizability conditions for a symmetrical two-port consisting of a cascade of commensurate unit elements (UEs) with a lumped shunt capacitor at its center. Saito [20] has, via an extension of Richards' transformation, derived the necessary and sufficient conditions for the realization of three different structures: (i) cascaded commensurate UEs terminated by a lumped reactance, (ii) cascaded non-commensurate UEs terminated by a resistance, and (iii) cascaded commensurate UEs terminated by a resistance and shunted by lumped reactances. Scanlan and Rhodes [21] have investigated the problem when a multivariable impedance function may be realized by means of a cascade of passive, lumped, lossless, two-port networks connected by means of a non-commensurate transmission lines and terminated in a resistor. Shirakawa, Takahashi and Ozaki [22] have considered the synthesis of cascaded transmission-line networks of non-commensurate UEs with the following structures:

(i) A cascade connection of m transmission lines $N_i (i=1,2,\ldots,m)$ in a prescribed order, each N_i being a transmission line with open ended stubs composed of UEs of single-variable p_i

(ii) A transmission line with open ended stubs composed of UEs of p_2, connected in cascade between two transmission lines of UEs of p_1

(iii) A cascade connection of m transmission lines $N_i (i=1,2,\ldots,m)$ in an arbitrary order, each N_i being a UE of variable p_i, and

(iv) A transmission line composed of commensurate UEs with open ended stubs, partitioned from a multi-variable network.

Kamp and Neirynck [23] have derived the conditions under which a multivariable transfer matrix can be synthesized as a cascade connection of lossless non-commensurate transmission lines. Later, Kamp [24] generalized these results for a cascade structure of non-commensurate transmission lines with parallel or series stubs, consisting of open-circuited or short-circuited single lines.

Youla and Ott [25] gave the realizability conditions for a cascade structure containing at most two commensurate UEs and three capacitors. The method, however, is difficult to extend to cascade structures with more number of elements. Uruski and Piekarski [26] have discussed the conditions on the driving-point admittance of a resistively-terminated structure which is a cascade of commensurate UEs separated by shunt lumped capacitors or series lumped inductors. They have obtained these results by making use of the theory of bigradient arrays.

Premoli [27] derived the conditions for the synthesis of cascaded non-commensurate UEs terminated by a resistor from the study of the impedance function in special set of points without using the concept of positive real function. Subsequently [28], he extended this method to structures considered by Scanlan and Rhodes [21], namely, the networks of cascaded non-commensurate UEs shunted by lumped capacitances. He has also shown that Saito's realizability condition [20] of real positivity for the input admittance of a network composed of cascaded non-commensurate UEs, closed on a resistance is partially redundant and can be replaced by a much simpler condition [29].

Kamp [30]-[32] has presented the necessary and sufficient conditions under which an MPRF of arbitrary degree in each variable can be realized as the driving-point impedance of a cascade of non-commensurate UEs terminated in a finite positive resistance. He has, by transforming the multivariable impedance into a single-variable transcendental function, shown that these conditions are equivalent to those of Kinariwala's [33]. He has also derived the conditions for the realizability of an MPRF of the first degree in all variables except one, as the input impedance of a resistively-terminated cascade of lossless two-ports separated by non-commensurate series and shunt stubs [34]. Furthermore, unlike Saito [20], he gave the realizability conditions for a class of multivariable reactance functions.

Koga [35] has presented a general solution to the problem of synthesizing a passive two-port consisting of a cascade of commensurate or non-commensurate UEs, and lumped passive lossless two-ports in an arbitrary sequence with a resistive termination at the receiving end. However, Rhodes and Martson [36] have, through an example, shown that Koga's conditions are not sufficient to guarantee a canonic, passive network in a cascade configuration.

Rao and Ramachandran [37] have given the necessary and sufficient conditions for an MPRF of arbitrary degree in each variable to be realizable by a cascade of non-commensurate UEs separated by lumped lossless two-ports (all of the same single variable) terminated in 1Ω resistor in terms of multivariable reactance functions generated from the even and odd polynomials of the given function. In a subsequent paper [38], they have presented the conditions for the realizability of a two-variable function by a resistively-terminated cascade of commensurate UEs separated by series lumped inductors on one side and shunt lumped capacitors on the other side. They achieved this by showing an equivalence relation between such a structure and a two-variable resistively-terminated lowpass ladder network.

Youla, Rhodes and Marston [39] have presented an explicit solution with the aid of two-variable positive-real concept for the realizability of resistively-terminated cascade of commensurate UEs separated by lumped, passive, lossless, two-port networks. Later these authors advanced a complete and compact solution for the problem [17], [40]. A significant point of their later solution is that the difficulty of testing a two-variable polynomial for positive realness is replaced by relatively simpler tests.

Fujimoto and Ishii [41] have established the realizability conditions for a resistively-terminated cascade structure of passive, lossless, lumped two-ports separated by commensurate UEs. In addition to the test of the positive-reality of the two-variable impedance function, their realizability conditions require certain tests on single-variable functions.

Phan [42] has considered the synthesis problem of a class of networks made up of non-commensurate UEs separated by passive, lumped, lossless two-ports, and terminated in a passive lumped network by employing a direct and explicit approach which eliminates the prerequisite of multivariable positive-reality condition in favour of some simpler one-variable type conditions.

Some work has also been done on the realization of MPRFs as cascade structures of lumped building blocks involving single-variable functions [42], [43]. Phan [42] has derived the realizability conditions from the chain parameter characterization of passive, lumped, lossless two-ports. In [43] the development of the realizability conditions is based on the single-variable Darlington theory and is restricted to reciprocal realizations.

In the above cited literature, most of the synthesis is carried out in terms of cascade of lumped, lossless two-ports and transmission lines, terminated in positive resistances. Naturally, there could be many other cascade structures, each of them having different interconnection of lumped and distributed elements. It may be pointed out that in the cascade synthesis, the problem of realizing an MPRF, where the degree of each variable is arbitrary, as a tandem connection of resistively-terminated commensurate or non-commensurate UEs separated by lossless two-ports has been solved only for the case of two-variable functions. Also, most of the realization techniques make use of Richard's transformation on the driving-point function or the reflection coefficient. Finally, it is noted that the realizability conditions are frequently given in terms of the even parts of impedance functions, and therefore, they are restricted to non-reactance functions.

1.3 SCOPE OF THE THESIS

This thesis aims at the study of multivariable structures of cascade of single-variable lumped lossless two-ports terminated by positive real impedances. Realization technique for these structures is given without taking recourse to Richard's transformation. The synthesis is based on cascade-separability of MPRFs of arbitrary degree in each variable into single-variable functions and on the utilization of single-variable Darlington theory of resistively-terminated lossless networks.

In Chapter II, necessary and sufficient conditions are developed for the realization of an MPRF as the driving-point function of an extracted single-variable lossless two-port terminated by a driving-point function which is an MPRF in the rest of the variables. Conditions are also derived for the case when the extraction of a lossless two-port is possible in either of the two variables p_i or p_j. Some special cases where both the p_i- and p_j-variable two-port networks reduce to simple series or simple shunt branches are discussed.

Chapter III studies some special cases of the cascade structures of Chapter II. Realizability conditions for a resistively-terminated cascade of m single-variable lossless two-ports, where each two-port is either a ladder network with all of its transmission zeros at the origin or at infinity or a Fujisawa-type lowpass ladder network are derived. Also, for a class of MPRFs, an even part condition is developed for the realization of a resistively-terminated ladder network with all of its transmission zeros either at the origin or at infinity, where the reactive elements are not ordered according to their types.

In Chapter IV, some partial derivative properties of the impedance function of a cascade structure of $(m-1)$ lossless two-ports of variables p_1 to p_{m-1} terminated by a network of variable p_m are studied. Those cases where some of the lossless two-ports are ladder networks with all of their transmission zeros at the origin or at infinity are examined. Necessary and sufficient conditions involving partial derivatives under which an m-variable reactance or positive real function can be realized as the impedance function of a p_i-variable ladder network with all of its transmission zeros at $p_i=0$ or at $p_i=\infty$, and terminated in a reactance or positive real impedance function in the remaining (m-1) variables are derived.

In Chapter V, necessary and sufficient conditions are established for the realization of the voltage transfer function of a resistively-terminated cascade of p_1-and p_2-variable ladder network with all of its transmission zeros at $p_i=0$ or at $p_i=\infty$ $(i=1,2)$. It is shown that these analog networks can be used to obtain two-dimensional wave digital filters.

CHAPTER II

CASCADE EXTRACTION OF A LOSSLESS TWO-PORT FROM A MULTIVARIABLE POSITIVE REAL FUNCTION

2.1 INTRODUCTION

As stated earlier, conditions in addition to positive realness are required to ensure that a given multivariable rational function is realizable as a cascade structure of lumped lossless two-ports terminated by a resistance. Skirakawa et al. [22] have given necessary and sufficient conditions on an MPRF whereby the extraction of a single-variable lowpass ladder network with all of its transmission zeros at infinity is possible. More recently [42], [43], some work has been reported on cascade extraction of a single-variable lumped lossless two-port from a given MPRF.

In this Chapter, some results on cascade realization of MPRFs are obtained [44], [45]. In particular, necessary and sufficient condition for an m-variable positive real function (PRF) to be realizable as the driving-point impedance of a lumped lossless two-port in one of the variables, with a termination of an MPRF in the remaining (m-1) variables is given. It is shown that augmentation with surplus factors as in the single-variable Darlington synthesis is not possible here. Consequently, the lossless two-port cannot always be realized with reciprocal elements only. In such a case, however, it can be realized using ideal gyrators. It is shown that, by repeated application of the condition for the cascade extraction, an MPRF is realizable as a cascade of single-variable lossless two-ports closed on a single-variable impedance.

Conditions are also found for the case when an m-variable PRF is realizable both as a p_i-variable lossless two-port network with a driving-point impedance termination $Z_{01}(\Omega_i)$*, and as a p_j-variable lossless two-port network with a driving-point impedance termination $Z_{02}(\Omega_j)$. Some special cases are also discussed where both the p_i- and p_j-variable two-ports reduce to either simple series or simple shunt branches.

2.2 THEOREMS ON CASCADE EXTRACTION

In this section, necessary and sufficient conditions are established such that an MPRF can be realized as a lossless two-port in one of the variables, with a termination whose driving-point impedance is an MPRF in the other variables. The synthesis procedure is based on the two theorems given below.

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*The following notations have been used throughout the thesis

$$p_{1,\ldots,m} = \{p_1, p_2, \ldots, p_m\}$$
$$\Omega_i = \{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m\}$$
$$\Omega_{ij} = \{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_m\}$$
$$\Omega_{i,\ldots,k} = \{p_1, p_2, \ldots, p_{i-2}, p_{i-1}, p_{k+1}, p_{k+2}, \ldots, p_m\}$$
$$\Phi_i = \{j\omega_1, j\omega_2, \ldots, j\omega_{i-1}, j\omega_{i+1}, \ldots, j\omega_m\}$$

Theorem 2.2.1

The necessary and sufficient conditions for a reduced* m-variable rational function $Z(p_{1,\ldots,m})$ expressible in the form

$$Z(p_{1,\ldots,m}) = \frac{N(p_{1,\ldots,m})}{D(p_{1,\ldots,m})} = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)} \quad (2.2.1)$$

where $m_1(p_i)$ and $m_2(p_i)$ are even and $n_1(p_i)$ and $n_2(p_i)$ are odd polynomials of the single variable p_i, and $P(\Omega_i)$ and $Q(\Omega_i)$ are polynomials in the variables Ω_i, to be an MPRF are that

(i) $\dfrac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ is a single-variable positive real function (SPRF) in p_i, and

(ii) $\dfrac{P(\Omega_i)}{Q(\Omega_i)}$ is an MPRF in Ω_i

Proof

Necessity: Let each $p_j (j=1,\ldots,m; j\neq i)$ assume an arbitrary positive real value a_j, then $Z(a_1,\ldots,a_{i-1},p_i,a_{i+1},\ldots,a_m) = \dfrac{k_1m_1(p_i)+k_2n_1(p_i)}{k_2m_2(p_i)+k_1n_2(p_i)}$ is an SPRF, where k_1 and k_2 are positive real values of $P(\Omega_i)$ and $Q(\Omega_i)$ for these p_j's. This implies that $\dfrac{k_1m_1(p_i)+k_1n_2(p_i)}{k_2m_2(p_i)+k_2n_1(p_i)}$ is an SPRF. Since k_1/k_2 is a positive constant, $\dfrac{m_1(p_i)+n_2(p_i)}{m_2(p_i)+n_1(p_i)}$ is also

*A function is said to be reduced if its numerator and denominator polynomials are relatively prime.

an SPRF. Hence, $\frac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ is an SPRF. To prove (ii), proceed as follows: If Z does not have a critical frequency at $p_i=0$, then $Z(p_1,\ldots,p_{i-1},0,p_{i+1},\ldots,p_m) = \frac{m_1(0)}{m_2(0)} \cdot \frac{P(\Omega_i)}{Q(\Omega_i)}$, and since $Z(p_1,\ldots,p_{i-1},0,$ $p_{i+1},\ldots,p_m)$ is an MPRF, $\frac{P}{Q}$ is also an MPRF. On the other hand, if Z has a critical frequency at $p_i=0$, then it must be either a simple zero or a simple pole, but not both, because the given function is reduced. If it has a zero at $p_i=0$ (the proof is similar in the case when Z has a pole at $p_i=0$), generate an MPRF as follows:

$$F(p_{1,\ldots,m}) = \frac{\frac{\partial}{\partial p_i}[m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)]}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)} \tag{2.2.2}$$

$$= \frac{m_2'(p_i)Q(\Omega_i)+n_2'(p_i)P(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)}$$

where $m_2'(p_i) = \frac{dm_2(p_i)}{dp_i}$ and $n_2'(p_i) = \frac{dn_2(p_i)}{dp_i}$ are respectively odd and even polynomials in p_i. Since $F(p_{1,\ldots,m})$ does not have a pole or a zero at $p_i=0$, then $F(p_1,\ldots,p_{i-1},0,p_{i+1},\ldots,p_m) = \frac{n_2'(0)P(\Omega_i)}{m_2(0)Q(\Omega_i)}$ must also be an MPRF. Since $\frac{n_2'(0)}{m_2(0)}$ is a positive constant, $\frac{P(\Omega_i)}{Q(\Omega_i)}$ is an MPRF in the variables Ω_i. Thus, Condition (ii) is also satisfied.

Sufficiency: Since $\frac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ is an SPRF, $\frac{m_2(p_i)+n_1(p_i)}{m_1(p_i)+n_2(p_i)}$ is also an SPRF. Further, since $\frac{P(\Omega_i)}{Q(\Omega_i)}$ is an MPRF, $\frac{m_2(p_i)+n_1(p_i)}{m_1(p_i)+n_2(p_i)} + \frac{P(\Omega_i)}{Q(\Omega_i)} =$

$\frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)+m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)}{[m_1(p_i)+n_2(p_i)]Q(\Omega_i)}$ is an MPRF. This implies that $[m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)+m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)]$ is a Hurwitz polynomial in the narrow sense (HPN). Now, let $Z_1(\Omega_i)$ denote $\frac{P(\Omega_i)}{Q(\Omega_i)}$, then

$$\operatorname{Re} Z(j\omega_1,\ldots,j\omega_m) = \frac{\operatorname{Re} Z_1(\phi_i)[m_1(j\omega_1)m_2(j\omega_1)-n_1(j\omega_1)n_2(j\omega_1)]}{|m_2(j\omega_i)+n_2(j\omega_i)Z_1(\phi_i)|^2} \quad (2.2.3)$$

Since $Z_1(\Omega_i)$ is an MPRF, $\operatorname{Re} Z_1(\phi_i) \geq 0$ for all real $\omega_j (j=1,\ldots,i-1, i+1,\ldots,m)$, and since $\frac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ is an SPRF, $[m_1(j\omega_i)m_2(j\omega_i) -n_1(j\omega_i)n_2(j\omega_i)] \geq 0$ for all real ω_i. Hence, $\operatorname{Re} Z(j\omega_1,\ldots,j\omega_m) \geq 0$ for all real $\omega_k (k=1,\ldots,m)$. Thus, $Z(p_{1,\ldots,m})$ satisfies the real part condition, and the sum of its numerator and denominator polynomials is an HPN, and therefore, it is an MPRF.

Theorem 2.2.2*

The necessary and sufficient condition for a reduced MPRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a p_i-variable lumped lossless two-port network with a driving-point impedance termination $Z_0(\Omega_i) = \frac{S(\Omega_i)}{T(\Omega_i)}$ (Fig. 2.1) is that Z can be written as:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)} \quad (2.2.4)$$

*This theorem is also proved in [42], but the proof here is along the lines of single-variable Darlington synthesis.

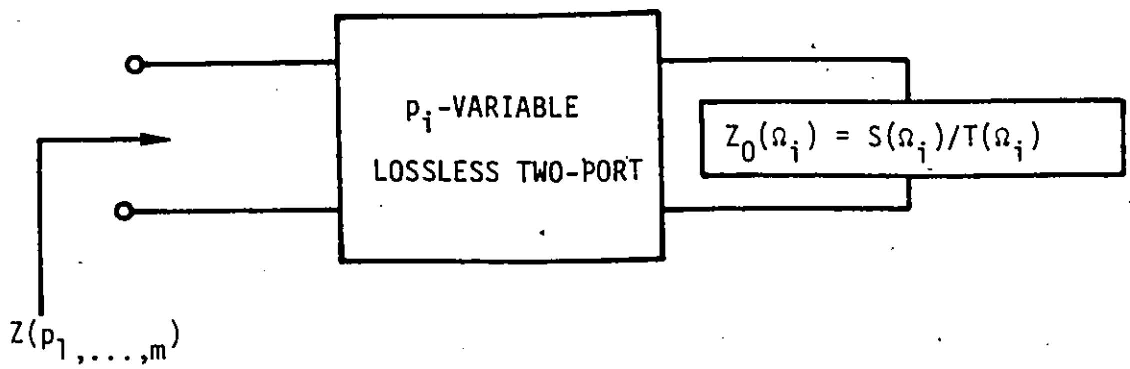

FIG. 2.1. Cascade Extraction of a p_i-Variable Lossless Two-Port from an MPRF $Z(p_{1,\ldots,m})$.

where $m_1(p_i)$ and $m_2(p_i)$ are even and $n_1(p_i)$ and $n_2(p_i)$ are odd polynomials in p_i, and $P(\Omega_i)$ and $Q(\Omega_i)$ are polynomials in the variables Ω_i.

Proof

Necessity: If $Z(p_{1,\ldots,m})$ is realizable as the impedance function of the network shown in Fig. 2.1, it can always be written in the form given by (2.2.4), and necessity follows.

Sufficiency: Eqn. (2.2.4) can be rewritten in the following two forms:

Case A:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)}{n_2(p_i)} \frac{\dfrac{n_1(p_i)}{m_1(p_i)} + \dfrac{P(\Omega_i)}{Q(\Omega_i)}}{\dfrac{m_2(p_i)}{n_2(p_i)} + \dfrac{P(\Omega_i)}{Q(\Omega_i)}} \tag{2.2.5a}$$

Case B:

$$Z(p_{1,\ldots,m}) = \frac{n_1(p_i)}{m_2(p_i)} \frac{\dfrac{m_1(p_i)}{n_1(p_i)} + \dfrac{Q(\Omega_i)}{P(\Omega_i)}}{\dfrac{n_2(p_i)}{m_2(p_i)} + \dfrac{Q(\Omega_i)}{P(\Omega_i)}} \tag{2.2.5b}$$

From (2.2.5a) and (2.2.5b), the following identifications can be made for the two cases:

Case A	Case B
$z_{11}(p_i) = \dfrac{m_1(p_i)}{n_2(p_i)}$	$z_{11}(p_i) = \dfrac{n_1(p_i)}{m_2(p_i)}$
$z_{22}(p_i) = \dfrac{m_2(p_i)}{n_2(p_i)}$	$z_{22}(p_i) = \dfrac{n_2(p_i)}{m_2(p_i)}$

$$z_{12}(p_i)z_{21}(p_i) = \frac{m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)}{n_2^2(p_i)} \qquad z_{12}(p_i)z_{21}(p_i) = \frac{n_1(p_i)n_2(p_i)-m_1(p_i)m_2(p_i)}{m_2^2(p_i)}$$

$$Z_0(\Omega_i) = \frac{S(\Omega_i)}{T(\Omega_i)} = \frac{P(\Omega_i)}{Q(\Omega_i)} \qquad Z_0(\Omega_i) = \frac{S(\Omega_i)}{T(\Omega_i)} = \frac{Q(\Omega_i)}{P(\Omega_i)}$$

Since $F(p_i) = \frac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ is a PRF, the zeros of $[m_1(p_i)m_2(p_i) - n_1(p_i)n_2(p_i)]$ have quadrantal symmetry in the p_i-plane, and hence, the numerator of $z_{12}(p_i)z_{21}(p_i)$ for the two cases can always be expressed as $[m_0^2(p_i)-n_0^2(p_i)]$, where $m_0(p_i)$ and $n_0(p_i)$ are respectively even and odd polynomials in p_i. Hence, further identifications as given below are possible:

<u>Case A</u>

$$z_{12}(p_i) = \frac{m_0(p_i)+n_0(p_i)}{n_2(p_i)}$$

$$z_{21}(p_i) = \frac{m_0(p_i)-n_0(p_i)}{n_2(p_i)}$$

<u>Case B</u>

$$z_{12}(p_i) = \frac{n_0(p_i)+m_0(p_i)}{m_2(p_i)}$$

$$z_{21}(p_i) = \frac{n_0(p_i)-m_0(p_i)}{m_2(p_i)}$$

In view of Theorem 2.2.1, $F(p_i)$ is an SPRF in p_i and $Z_0(\Omega_i)$ is an MPRF in Ω_i, and hence, by Darlington synthesis, the open-circuit two-port parameters as given by the two cases are realizable using inductors, capacitors and ideal gyrators [46]. Realization of $Z(p_{1,\ldots,m})$ is obtained by terminating the two-port with the MPRF $Z_0(\Omega_i)$. This proves the sufficiency, and the theorem is established.

As seen from the two-port parameters for the two cases, the extracted lossless two-port is non-reciprocal. However, if $[m_1(p_i)m_2(p_i) - n_1(p_i)n_2(p_i)]$ is plus or minus a perfect square, then a reciprocal realization can be obtained [46] because, in such a case, the following choices are possible:

Case A:

$$z_{12}(p_i) = z_{21}(p_i) = \sqrt{\frac{m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)}{n_2(p_i)}} \tag{2.2.6a}$$

Case B:

$$z_{12}(p_i) = z_{21}(p_i) = \sqrt{\frac{n_1(p_i)n_2(p_i)-m_1(p_i)m_2(p_i)}{m_2(p_i)}} \tag{2.2.6b}$$

In the case when $[m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)]$ is not a perfect square, neither $z_{12}(p_i)$ nor $z_{21}(p_i)$ as given by (2.2.6) is a rational function. These parameters are rational functions only if the numerator of the even part of $F(p_i)$ is a perfect square. The procedure in the single-variable Darlington synthesis is to augment the numerator and the denominator of $F(p_i)$ by the surplus factor $[m_0(p_i)+n_0(p_i)]$, a polynomial obtained by taking the left half p_i-plane zeros of $[m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)]$. This procedure, however, cannot be followed in the cascade realization of $Z(p_{1,\ldots,m})$, because when the p_i-variable lossless two-port is realized as a reciprocal structure made possible by proper augmentation of $F(p_i)$, and terminated by $Z_0(\Omega_i)$, the resulting MPRF will be

$$Z_1(p_{1,\ldots,m}) = \frac{\begin{array}{c}[m_1(p_i)m_0(p_i)+n_1(p_i)n_0(p_i)]P(\Omega_i)\\ +[n_1(p_i)m_0(p_i)+m_1(p_i)n_0(p_i)]Q(\Omega_i)\end{array}}{\begin{array}{c}[m_2(p_i)m_0(p_i)+n_2(p_i)n_0(p_i)]Q(\Omega_i)\\ +[n_2(p_i)m_0(p_i)+m_2(p_i)n_0(p_i)]P(\Omega_i)\end{array}}$$

instead of that given by (2.2.4). On the other hand, if $Z(p_{1,\ldots,m})$ is augmented by the surplus factor $m_0(p_i)+n_0(p_i)$, then

$$Z(p_1,\Omega_i) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)} \times \frac{m_0(p_i)+n_0(p_i)}{m_0(p_i)+n_0(p_i)}$$

$$= \frac{\begin{array}{c}[m_1(p_i)m_0(p_i)+m_1(p_i)n_0(p_i)]P(\Omega_i)\\ +[n_1(p_i)m_0(p_i)+n_1(p_i)n_0(p_i)]Q(\Omega_i)\end{array}}{\begin{array}{c}[m_2(p_i)m_0(p_i)+m_2(p_i)n_0(p_i)]Q(\Omega_i)\\ +[n_2(p_i)m_0(p_i)+n_2(p_i)n_0(p_i)]P(\Omega_i)\end{array}}$$

is not expressible in the form of (2.2.4), and thus, not cascade-realizable. Hence, if $[m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)]$ is neither a perfect square nor the negative of a perfect square, a reciprocal realization of the lossless two-port is not possible, even though a non-reciprocal realization exists.

Theorems 2.2.1 and 2.2.2 suggest the following definition:

<u>Definition</u>: A reduced MPRF $Z(p_{1,\ldots,m})$ which can be written in the form:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)}$$

is said to be <u>cascade-expressible in the variable p_i</u> .

According to Theorem 2.2.1, if $Z(p_{1,\ldots,m})$ is cascade-expressible in the variable p_i, then $F(p_i)$ is an SPRF and $\frac{P(\Omega_i)}{Q(\Omega_i)}$ is an MPRF, and by Theorem 2.2.2, it can always be realized as the impedance function of a p_i-variable lumped lossless two-port terminated by a driving-point impedance function $Z_0(\Omega_i)$. Hence, it is worth pointing out that for the extraction of a lumped lossless two-port from a cascade-expressible MPRF $Z(p_{1,\ldots,m})$ it is not necessary to test the positive realness of $F(p_i)$ and $\frac{P(\Omega_i)}{Q(\Omega_i)}$, and thus, the necessity of the three conditions of a theorem given in [43] seems redundant.

2.3 ALTERNATIVE CONDITIONS FOR CASCADE EXTRACTION

According to Theorem 2.2.2, the necessary and sufficient condition for a lossless two-port in the variable p_i to be completely extractable from an MPRF is that the function be cascade-expressible in the variable p_i. The following theorems provide alternative methods to test whether a given MPRF is cascade-expressible in the variable p_i.

Theorem 2.3.1

The necessary and sufficient conditions for a reduced MPRF $Z(p_{1,\ldots,m}) = \frac{N(p_{1,\ldots,m})}{D(p_{1,\ldots,m})}$ to be cascade-expressible in the variable p_i are that

$$\text{(i)} \quad N(p_i,\Omega_i)+N(-p_i,\Omega_i) = 2m_1(p_i)P(\Omega_i) \tag{2.3.1a}$$

$$\text{(ii)} \quad N(p_i,\Omega_i)-N(-p_i,\Omega_i) = 2n_1(p_i)Q(\Omega_i) \tag{2.3.1b}$$

$$\text{(iii)} \quad D(p_i,\Omega_i)+D(-p_i,\Omega_i) = 2m_2(p_i)Q(\Omega_i) \tag{2.3.1c}$$

$$\text{(iv)} \quad D(p_i,\Omega_i)-D(-p_i,\Omega_i) = 2n_2(p_i)P(\Omega_i) \tag{2.3.1d}$$

where $m_1(p_i)$ and $m_2(p_i)$ are even and $n_1(p_i)$ and $n_2(p_i)$ are odd polynomials in p_i, and $P(\Omega_i)$ and $Q(\Omega_i)$ are polynomials in Ω_i.

Proof

Necessity: Since $Z(p_{1,\ldots,m})$ is cascade-expressible, it is readily seen from (2.2.1) that conditions given by (2.3.1) are satisfied.

Sufficiency: Adding (2.3.1a) with (2.3.1b) and (2.3.1c) with (2.3.1d), yields $2N(p_{1,\ldots,m}) = 2m_1(p_i)P(\Omega_i)+2n_1(p_i)Q(\Omega_i)$ and $2D(p_{1,\ldots,m}) = 2m_2(p_i)Q(\Omega_i)+2n_2(p_i)P(\Omega_i)$ respectively, and hence, $Z(p_{1,\ldots,m})$ is cascade-expressible in p_i.

Theorem 2.3.2

The necessary and sufficient condition for a reduced MPRF $Z(p_{1,\ldots,m})$ to be cascade-expressible in the variable p_i is that when the function is written in the form:

$$Z(p_{1,\ldots,m}) = \frac{A_n(\Omega_i)p_i^n+B_{n-1}(\Omega_i)p_i^{n-1}+A_{n-2}(\Omega_i)p_i^{n-2}+\ldots+B_1(\Omega_i)p_i+A_0(\Omega_i)^*}{B_n(\Omega_i)p_i^n+A_{n-1}(\Omega_i)p_i^{n-1}+B_{n-2}(\Omega_i)p_i^{n-2}+\ldots+A_1(\Omega_i)p_i+B_0(\Omega_i)} \tag{2.3.2}$$

then all A_i's must be polynomials in Ω_i which are constant multiples of each other, and all B_i's must be polynomials in Ω_i which are also constant multiples of each other.

Proof

Since all $A_i(\Omega_i)$'s are constant multiples of each other, and all $B_i(\Omega_i)$'s are also constant multiples of each other, the function

*Without loss of generality, n is assumed to be even.

given by (2.3.2) is obviously cascade-expressible in p_i. Conversely, a cascade-expressible function in the variable p_i can always be expressed in the form of (2.3.2).

Example 2.3.1

Consider the two-variable reactance function (TRF) given as an example in [43].

$$Z(p_1,p_2) = \frac{N(p_1,p_2)}{D(p_1,p_2)} = \frac{6p_1^2p_2^2+3p_1^2+6p_2^2+10p_1p_2+3}{5p_1^2p_2+6p_1p_2^2+3p_1+5p_2} .$$

The following relations are obtained by applying the conditions given by (2.3.1) on $Z(p_1,p_2)$:

$$\begin{aligned}
N(p_1,p_2)+N(-p_1,p_2) &= 2(p_1^2+1)(6p_2^2+3) &&= 2m_1(p_1)P(p_2)\\
N(p_1,p_2)-N(-p_1,p_2) &= 2(2p_1)(5p_2) &&= 2n_1(p_1)Q(p_2)\\
D(p_1,p_2)+D(-p_1,p_2) &= 2(p_1^2+1)(5p_2) &&= 2m_2(p_1)Q(p_2)\\
D(p_1,p_2)-D(-p_1,p_2) &= 2(p_1)(6p_2^2+3) &&= 2n_2(p_1)P(p_2)
\end{aligned}$$

Thus, the given TRF is cascade-expressible in the variable p_i, and it can be written as:

$$Z(p_1,p_2) = \frac{(p_1^2+1)[6p_2^2+3]+(2p_1)[5p_2]}{(p_1^2+1)[5p_2]+(p_1)[6p_2^2+3]} = \frac{m_1(p_1)P(p_2)+n_1(p_1)Q(p_2)}{m_2(p_1)Q(p_2)+n_2(p_1)P(p_2)}$$

Since $[m_1(p_1)m_2(p_1)-n_1(p_1)n_2(p_1)] = (p_1^2+1)^2-2p_1^2 = p_1^4+1$ is not a perfect square, realization of the lossless two-port will require the use of an ideal gyrator. Let $[m_0(p_1)^2-n_0(p_1)^2] = (m_1m_2-n_1n_2) = [(p_1^2+1)^2 - (\sqrt{2}p_1)^2]$, i.e. $m_0(p_1) = (p_1^2+1)$ and $n_0(p_1) = \sqrt{2}p_1$, and make the following identifications:

$$z_{11}(p_1) = \frac{m_1(p_1)}{n_2(p_1)} = \frac{p_1^2+1}{p_1}$$

$$z_{22}(p_1) = \frac{m_2(p_1)}{n_2(p_1)} = \frac{p_1^2+1}{p_1}$$

$$z_{12}(p_1) = \frac{m_0(p_1)+n_0(p_1)}{n_2(p_1)} = \frac{p_1^2+\sqrt{2}p_1+1}{p_1}$$

$$z_{21}(p_1) = \frac{m_0(p_1)-n_0(p_1)}{n_2(p_1)} = \frac{p_1^2-\sqrt{2}p_1+1}{p_1}$$

$$Z_0(p_2) = \frac{P(p_2)}{Q(p_2)} = \frac{6p_2^2+3}{5p_2}$$

A realization of $Z(p_1;p_2)$ is shown in Fig. 2.2. The termination $Z_0(p_2)$ is a single-variable reactance function in p_2, and it is always realizable as a lossless network. Note that since $Z(p_1,p_2)$ is not cascade-expressible in the variable p_2, a realization of Z as a lossless two-port in the p_2-variable, with a termination of p_i-variable reactance function, is not possible.

2.4 THEOREMS ON CASCADE SYNTHESIS

The conditions discussed for the cascade extraction in the preceding two sections will now be generalized for a realization of an MPRF as a resistively-terminated cascade of single-variable lossless two-ports.

<u>Theorem 2.4.1</u>

The necessary and sufficient condition for a reduced m-variable

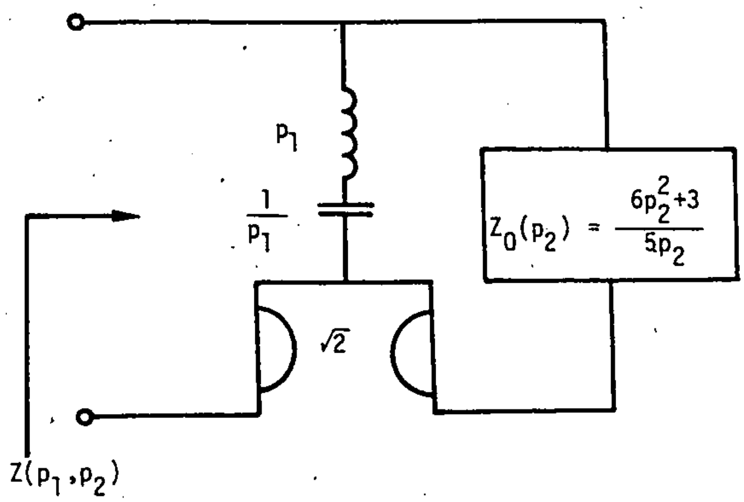

FIG. 2.2. A Cascade Realization of $Z(p_1,p_2)$ of Example 2.3.1.

PRF $Z_1(p_{1,\ldots,m}) = \dfrac{P_1(p_{1,\ldots,m})}{Q_1(p_{1,\ldots,m})}$ to be realizable as the input impedance of a resistively-terminated cascade of single-variable lumped lossless two-ports of variables p_1 to p_m (Fig. 2.3) is that the function $Z_1(p_{1,\ldots,m})$ can be decomposed as:

$$Z_i(p_{i,\ldots,m}) = \frac{m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})+n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})}{m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})+n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})},$$

$$i=1,2,\ldots,(m-1) \qquad (2.4.1)$$

where $m_1^{(i)}(p_i)$ and $m_2^{(i)}(p_i)$ are even and $n_1^{(i)}(p_i)$ and $n_2^{(i)}(p_i)$ are odd polynomials of p_i, and $P_{i+1}(\Omega_{1,\ldots,i})$ and $Q_{i+1}(\Omega_{1,\ldots,i})$ are multivariable polynomials of $\Omega_{1,\ldots,i}$.

<u>Proof</u>

The theorem can readily be proved by repeated application of Theorem 2.2.2.

<u>Theorem 2.4.2</u>

The necessary and sufficient condition for a reduced m-variable PRF $Z_1(p_{1,\ldots,m}) = \dfrac{P_1(p_{1,\ldots,m})}{Q_1(p_{1,\ldots,m})}$ to be realizable as the input impedance of a resistively-terminated cascade of single-variable lumped lossless two-ports of variables p_1 to p_m are that, for $i=1,2,\ldots,(m-1)$,

$$P_i(p_i,\Omega_{1,\ldots,i})+P_i(-p_i,\Omega_{1,\ldots,i}) = 2m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})$$

$$P_i(p_i,\Omega_{1,\ldots,i})-P_i(-p_i,\Omega_{1,\ldots,i}) = 2n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})$$

$$Q_i(p_i,\Omega_{1,\ldots,i})+Q_i(-p_i,\Omega_{1,\ldots,i}) = 2m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})$$

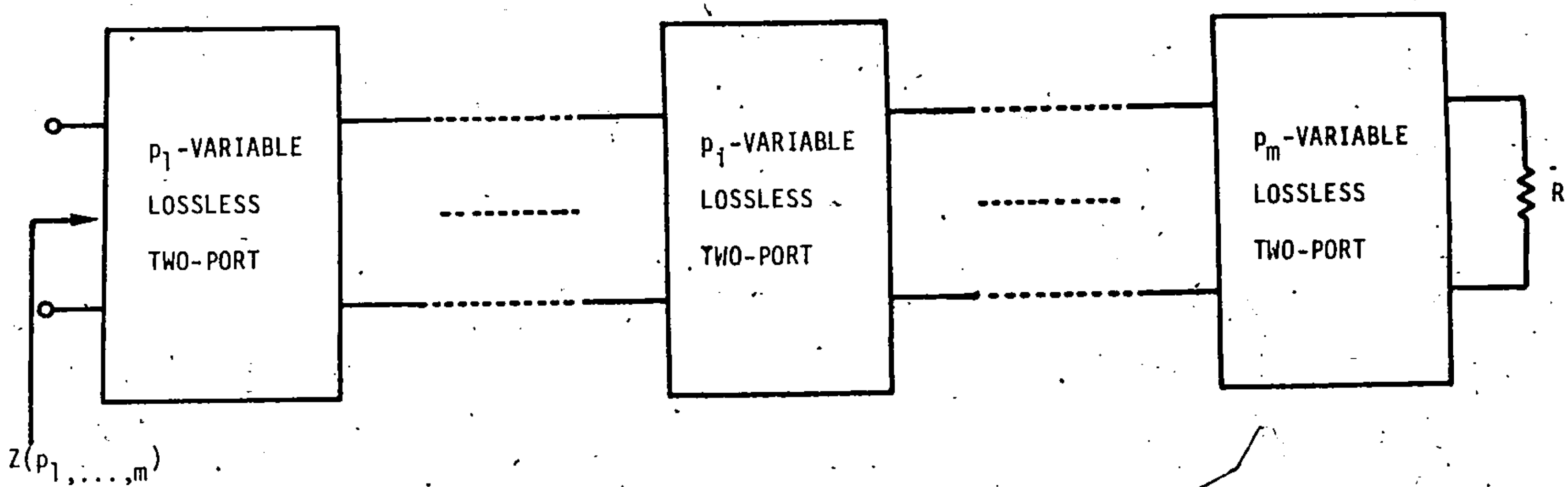

FIG. 2.3 A Realization of an MPRF as a Resistively-Terminated Cascade of Single-Variable Lossless Two-Ports,

$$Q_i(p_i,\Omega_{1,\ldots,i}) - Q_i(-p_i,\Omega_{1,\ldots,i}) = 2n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})$$

where the symbols have the same meanings as in Theorem 2.4.1.

Proof

The proof follows directly from Theorem 2.3.1.

2.5 CHOICE OF THE FIRST LOSSLESS TWO-PORT NETWORK

If an MPRF Z is cascade-expressible in only one of the variables p_i, then by Theorem 2.2.2 a p_i-variable lossless two-port can be extracted from it, and the position of this two-port is unique in the overall realization of Z. However, if the function is cascade-expressible in more than one variable, the first lossless two-port could be in any one of these variables. In such a case, one may expect that the choice of one two-port over the other possible two-ports will lead to a larger number of extractions of cascade two-ports. Necessary and sufficient conditions for the first lossless two-port to be extractable in either of the two variables p_i or p_j are now developed.

Theorem 2.5.1

The necessary and sufficient condition for a reduced m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedances of both a p_i-variable lossless two-port terminated by a driving-point impedance $Z_{01}(\Omega_i)$ and as a p_j-variable lossless two-port terminated by a driving-point impedance $Z_{02}(\Omega_j)$, is that the function Z be expressible as:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)[m_2(p_j)A(\Omega_{ij})+n_2(p_j)B(\Omega_{ij})] + n_1(p_i)[\gamma m_2(p_j)B(\Omega_{ij})+\delta n_2(p_j)A(\Omega_{ij})]}{m_1(p_i)[\gamma m_2(p_j)B(\Omega_{ij})+\delta n_2(p_j)A(\Omega_{ij})] + (\gamma\delta n_1(p_i))[m_2(p_j)A(\Omega_{ij})+n_2(p_j)B(\Omega_{ij})]} \qquad (2.5.1)$$

where $m_1(p_i)$ and $n_1(p_i)$ are respectively even and odd polynomials in the variable p_i, $m_2(p_j)$ and $n_2(p_j)$ are the corresponding polynomials in the variable p_j, $A(\Omega_{ij})$ and $B(\Omega_{ij})$ are multivariable polynomials in the variables Ω_{ij}; and γ and δ are nonnegative constants.

Proof

Necessity: If $Z(p_{1,\ldots,m})$ is realizable as the input impedance of a p_i-variable lossless two-port terminated by an MPRF $Z_{01}(\Omega_i)$, then by Theorem 2.2.2 it is expressible as:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_3(p_i)Q(\Omega_i)+n_3(p_i)P(\Omega_i)} \qquad (2.5.2)$$

where $m_1(p_i)$ and $m_3(p_i)$ are even and $n_1(p_i)$ and $n_3(p_i)$ are odd polynomials of the variable p_i, and $P(\Omega_i)$ and $Q(\Omega_i)$ are polynomials of the variables Ω_i. Without loss of generality, the polynomials P and Q can be expressed as:

$$P(\Omega_i) = m_2(p_j)A(\Omega_{ij}) + n_2(p_j)B(\Omega_{ij}) \qquad (2.5.3a)$$

$$Q(\Omega_i) = m_4(p_j)C(\Omega_{ij}) + n_4(p_j)D(\Omega_{ij}) \qquad (2.5.3b)$$

where $m_2(p_j)$ and $m_4(p_j)$ are even and $n_2(p_j)$ and $n_4(p_j)$ are odd polynomials of the variable p_j, and $A(\Omega_{ij})$, $B(\Omega_{ij})$, $C(\Omega_{ij})$ and

$D(\Omega_{ij})$ are polynomials of the variables Ω_{ij}. Thus, (2.5.2) can be rewritten as:

$$Z(p_{1,\ldots,m}) = \frac{\begin{array}{c} m_2(p_j)[m_1(p_i)A(\Omega_{ij})]+m_4(p_j)[n_1(p_i)C(\Omega_{ij})] \\ +n_2(p_j)[m_1(p_i)B(\Omega_{ij})]+n_4(p_j)[n_1(p_i)D(\Omega_{ij})] \end{array}}{\begin{array}{c} m_2(p_j)[n_3(p_i)A(\Omega_{ij})]+m_4(p_j)[m_3(p_i)C(\Omega_{ij})] \\ +n_2(p_j)[n_3(p_i)B(\Omega_{ij})]+n_4(p_j)[m_3(p_i)D(\Omega_{ij})] \end{array}} \quad (2.5.4)$$

If the function given by (2.5.4) is to be realizable as the input impedance of a p_j-variable lossless two-port terminated by an MPRF $Z_{02}(\Omega_j)$, then by Theorem 2.2.2 $Z(p_{1,\ldots,m})$ must be cascade-expressible in the variable p_j. This requires that $m_4(p_j) = \alpha m_2(p_j)$ and $n_4(p_j) = \beta n_2(p_j)$, where α and β are constants. Eqn. (2.5.4) then becomes:

$$Z(p_{1,\ldots,m}) = \frac{\begin{array}{c} m_2(p_j)[m_1(p_i)A(\Omega_{ij})+n_1(p_i)(\alpha C(\Omega_{ij}))] \\ +n_2(p_j)[m_1(p_i)B(\Omega_{ij})+n_1(p_i)(\beta D(\Omega_{ij}))] \end{array}}{\begin{array}{c} m_2(p_j)[m_3(p_i)(\alpha C(\Omega_{ij}))+n_3(p_i)A(\Omega_{ij})] \\ +n_2(p_j)[m_3(p_i)(\beta D(\Omega_{ij}))+n_3(p_i)B(\Omega_{ij})] \end{array}} \quad (2.5.5)$$

The cascade-expressibility of $Z(p_{1,\ldots,m})$ in the variable p_j further requires that the bracketed expressions of (2.5.5) be related to each other by the equations given below:

$$m_3(p_i)(\alpha C(\Omega_{ij}))+n_3(p_i)A(\Omega_{ij}) = \gamma[m_1(p_i)B(\Omega_{ij})+n_1(p_i)(\beta D(\Omega_{ij}))] \quad (2.5.6a)$$

$$m_3(p_i)(\beta D(\Omega_{ij}))+n_3(p_i)B(\Omega_{ij}) = \delta[m_1(p_i)A(\Omega_{ij})+n_1(p_i)(\alpha C(\Omega_{ij}))] \quad (2.5.6b)$$

where γ and δ are constants. Equations (2.5.6a) and (2.5.6b) can be decomposed as follows:

$$\alpha m_3(p_i)C(\Omega_{ij}) = \gamma m_1(p_i)B(\Omega_{ij}) \ , \ n_3(p_i)A(\Omega_{ij}) = \gamma\beta n_1(p_i)D(\Omega_{ij})$$

$$\beta m_3(p_i)D(\Omega_{ij}) = \delta m_1(p_i)A(\Omega_{ij}) \ , \ n_3(p_i)B(\Omega_{ij}) = \delta\alpha n_1(p_i)C(\Omega_{ij})$$

From these relations, the following can be obtained:

$$m_3(p_i) = k_1 m_1(p_i) \tag{2.5.7a}$$

$$C(\Omega_{ij}) = k_2 B(\Omega_{ij}) \tag{2.5.7b}$$

$$D(\Omega_{ij}) = k_3 A(\Omega_{ij}) \tag{2.5.7c}$$

$$n_3(p_i) = k_4 n_1(p_i) \tag{2.5.7d}$$

where k_1, k_2, k_3 and k_4 are constants related to α, β, γ and δ as follows:

$$k_1 \ k_2 = \frac{\gamma}{\alpha} \tag{2.5.8a}$$

$$k_1 \ k_3 = \frac{\delta}{\beta} \tag{2.5.8b}$$

$$\frac{k_4}{k_3} = \beta\gamma \tag{2.5.8c}$$

$$\frac{k_4}{k_2} = \alpha\delta \tag{2.5.8d}$$

Three of the four equations given by (2.5.8) are independent. Solving for k_2, k_3 and k_4 in terms of k_1, α, β, γ and δ gives:

$$k_2 = \frac{\gamma}{\alpha k_1} \tag{2.5.9a}$$

$$k_3 = \frac{\delta}{\beta k_1} \tag{2.5.9b}$$

$$k_4 = \frac{\gamma\delta}{k_1} \tag{2.5.9c}$$

Thus, substituting from (2.5.9a) and (2.5.9b) into (2.5.7b) and (2.5.7c) gives:

$$C(\Omega_{ij}) = \frac{\gamma}{\alpha k_1} B(\Omega_{ij}) \tag{2.5.10a}$$

$$D(\Omega_{ij}) = \frac{\delta}{\beta k_1} A(\Omega_{ij}) \tag{2.5.10b}$$

Using (2.5.6) and (2.5.10) into (2.5.5), and without loss of generality letting $k_1=1$, yields (2.5.1). Thus, $Z(p_{1,\ldots,m})$ satisfies the necessity of the condition.

Sufficiency: The function given by (2.5.1) is in the cascade-expressible form in the variable p_i. Moreover, it can also be expressed in the cascade-expressible form in the variable p_2 as:

$$Z(p_{1,\ldots,m}) = \frac{m_2(p_j)[m_1(p_i)A(\Omega_{ij})+\gamma n_1(p_i)B(\Omega_{ij})] + n_2(p_j)[m_1(p_i)B(\Omega_{ij})+\delta n_1(p_i)A(\Omega_{ij})]}{(\gamma m_2(p_j))[m_1(p_i)B(\Omega_{ij})+\delta n_1(p_i)A(\Omega_{ij})] + (\delta n_2(p_j))[m_1(p_i)A(\Omega_{ij})+\gamma n_1(p_i)B(\Omega_{ij})]} \tag{2.5.11}$$

Hence, by Theorem 2.2.2, $Z(p_{1,\ldots,m})$ can be realized both as a p_i-variable lossless two-port terminated in an impedance $Z_{01}(\Omega_i)$, and as a p_j-variable lossless two-port terminated in an impedance $Z_{02}(\Omega_j)$, and the theorem is proved.

The two realizations of the function $Z(p_{1,\ldots,m})$ of the previous theorem are shown in Fig. 2.4. It may be noted that the terminations after the extraction of the p_i-variable and the p_j-variable networks in both the cases are the same, namely, $A(\Omega_{ij})/B(\Omega_{ij})$. Hence, it is immaterial in which variable a lossless two-port is extracted first.

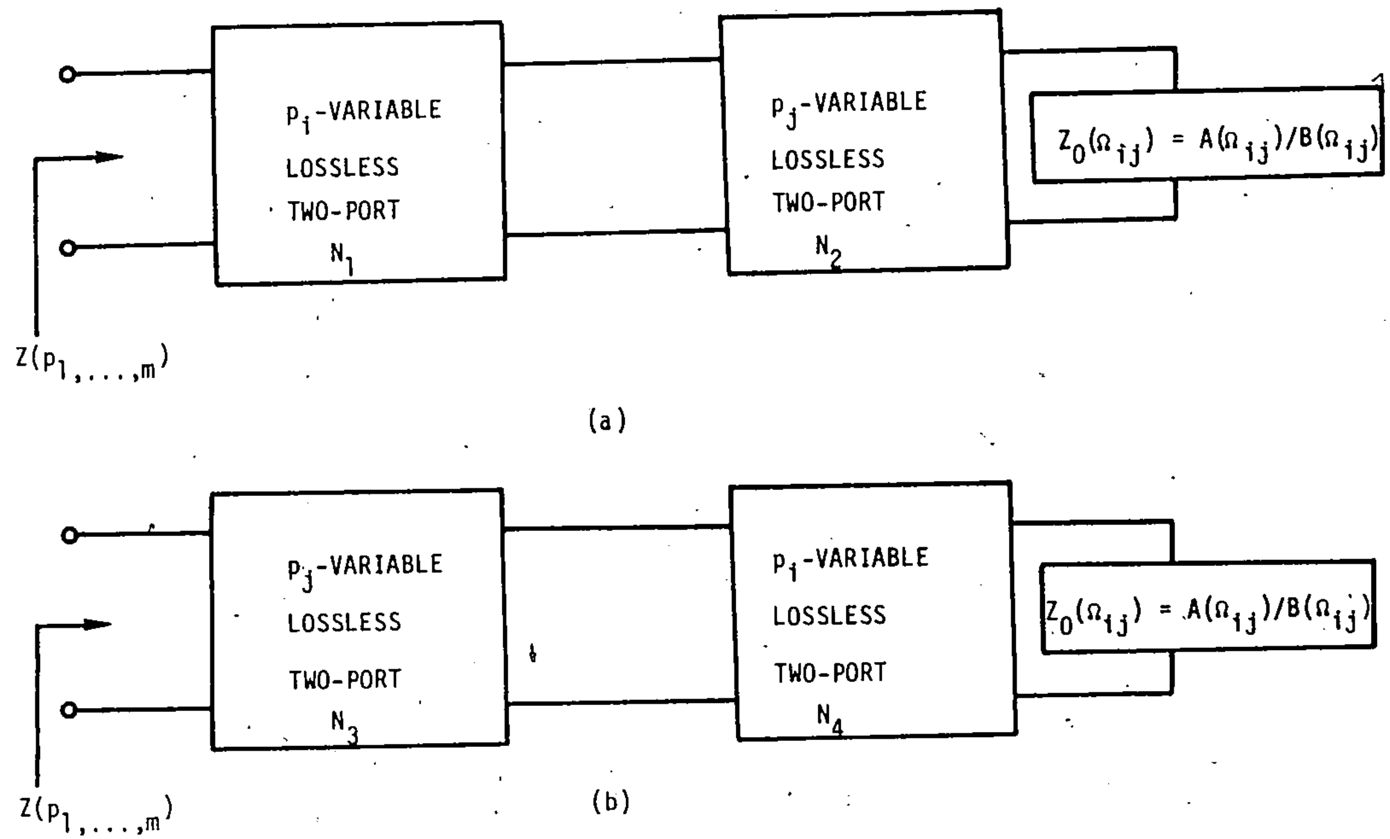

FIG. 2.4. Cascade Extraction of p_i- and p_j-Variable Lossless Two-Ports from an MPRF $Z(p_{1,\dots,m})$.

Special Cases

The function considered in Theorem 2.5.1 is now studied, when any one of the constants γ and δ becomes zero or infinity, and the forms of realizations of the function for these cases are determined.

Case (i): γ or δ is zero

When $\gamma=0$, the function given by (2.5.1) reduces to

$$Z(p_{1,\ldots,m}) = \frac{n_1(p_i)}{m_1(p_i)} + \frac{m_2(p_j)}{\delta n_2(p_j)} + \frac{B(\Omega_{ij})}{\delta A(\Omega_{ij})} \qquad (2.5.12)$$

Hence, $Z(p_{1,\ldots,m})$ can be realized as the input impedance of a series combination of three impedances: $Z_1(p_i) = \frac{n_1(p_i)}{m_1(p_i)}$, $Z_2(p_j) = \frac{m_2(p_j)}{\delta n_2(p_j)}$ and $Z_3(\Omega_{ij}) = \frac{B(\Omega_{ij})}{\delta A(\Omega_{ij})}$.

Case (ii): γ or δ is infinity

When $\gamma=\infty$, the function $Z(p_{1,\ldots,m})$ is reduced to

$$Z(p_{1,\ldots,m}) = \frac{1}{\frac{m_1(p_i)}{n_1(p_i)} + \frac{\delta n_2(p_j)}{m_2(p_j)} + \frac{\delta A(\Omega_{ij})}{B(\Omega_{ij})}} \qquad (2.5.13)$$

Hence, $Z(p_{1,\ldots,m})$ can be realized as the input impedance of a parallel combination of three admittances: $Y_1(p_i) = \frac{m_1(p_i)}{n_1(p_i)}$, $Y_2(p_j) = \frac{\delta n_2(p_j)}{m_2(p_j)}$ and $Y_3(\Omega_{ij}) = \frac{\delta A(\Omega_{ij})}{B(\Omega_{ij})}$.

The function reduces to a similar form as given by (2.5.12) or (2.5.13) when the constant δ becomes zero or infinity. Thus, when any one of the constants γ and δ becomes zero or infinity, then both the p_i- and the p_j-variable networks reduce to either simple series or simple shunt branches where the positions of the p_i- and p_j-variable networks can be interchanged without altering the structures of individual networks.

<u>Example 2.5.1</u>

Consider the two-variable PRF given below:

$$Z(p_1,p_2) = \frac{p_1^2p_2^2+2p_1^2p_2+3p_1p_2^2+p_1^2+12p_1p_2+p_2^2+3p_1+2p_2+1}{2p_1^2p_2^2+8p_1^2p_2+12p_1p_2^2+2p_1^2+24p_1p_2+2p_2^2+12p_1+8p_2+2}$$

This function can be rewritten as:

$$Z(p_1,p_2) = \frac{\begin{array}{c}(p_1^2+1)[(p_2^2+1)\cdot(1)+(2p_2)\cdot(1)]+(\frac{3}{2}\,p_1)\\ [(2)\cdot(p_2^2+1)\cdot(1)+(4)\cdot(2p_2)\cdot(1)]\end{array}}{\begin{array}{c}(p_1^2+1)[(2)\cdot(p_2^2+1)\cdot(1)+(4)\cdot(2p_2)\cdot(1)]+(2)\cdot(4)\cdot(\frac{3}{2}\,p_1)\\ [(p_2^2+1)\cdot(1)+(2p_2)\cdot(1)]\end{array}}$$

Thus, $Z(p_1,p_2)$ satisfies the condition of Theorem 2.5.1, with $A=B=1$ and $\gamma=2$ and $\delta=4$, and it is realizable both as the input impedance of a lossless two-port in p_1 terminated by a driving-point function in p_2, and as the input impedance of a lossless two-port in p_2 terminated by a driving-point function in p_1. As a check, note that $Z(p_1,p_2)$ is cascade-expressible in both the variables p_1 and p_2 as:

$$Z(p_1,p_2)=\frac{(p_1^2+1)[p_2^2+2p_2+1]+(\frac{3}{2}\,p_1)[2p_2^2+8p_2+2]}{(p_1^2+1)[2p_2^2+8p_2+2]+(12p_1)[p_2^2+2p_2+1]}$$

$$=\frac{(p_2^2+1)[p_1^2+3p_1+1]+(p_2)[2p_1^2+12p_1+2]}{(p_2^2+1)[2p_1^2+12p_1+2]+(8p_2)[p_1^2+3p_1+1]}.$$

Realizations for both the cases are shown in Fig. 2.5.

2.6 SUMMARY AND DISCUSSION

In this chapter, a necessary and sufficient condition for the realization of an m-variable PRF as the input impedance of a single-variable lossless two-port network terminated by a driving-point impedance function in the rest of the variables has been obtained. The condition is that the function be cascade-expressible in one of the variables.

It is shown that the single-variable lossless two-port cannot always be realized by reciprocal elements only, but in general it requires ideal gyrators for realization. Alternative conditions which are suitable for testing the cascade-expressibility of a given MPRF in the variable p_i are also given. It may be noted that the results on cascade extraction hold also for multivariable reactance functions. Since the termination of the extracted lossless two-port is also an MPRF, it has been shown that a realization of the given function as a cascade of single-variable lossless two-ports closed on a single-variable impedance function is possible, if the function is successively decomposable in the cascade-expressible forms.

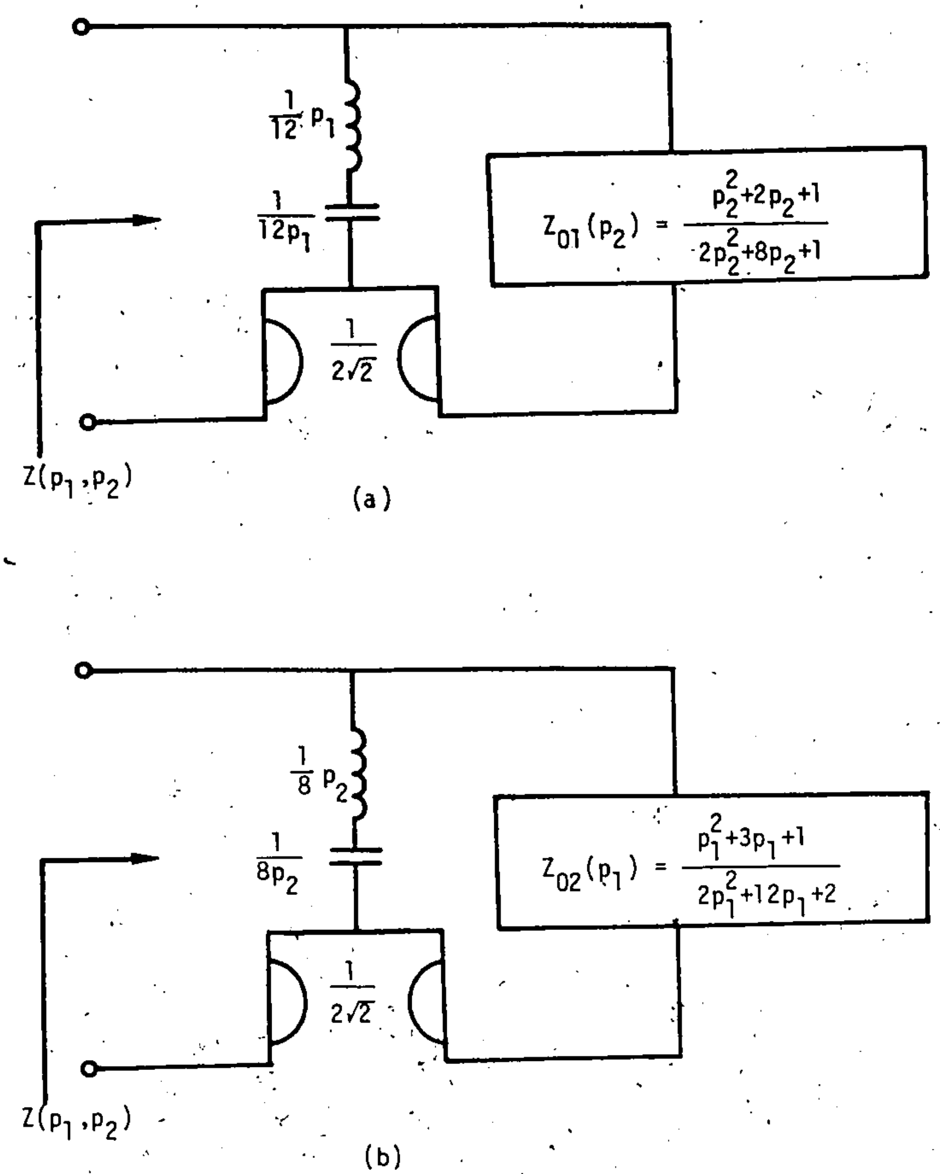

FIG. 2.5. Cascade Realizations of $Z(p_1,p_2)$ of Example 2.5.1.

In some cases the given MPRF may be cascade-expressible in more than one variable. A condition has been obtained so that a given MPRF can be realized by both of the following possibilities: (i) as the input impedance of a p_i-variable lossless two-port terminated by a driving-point impedance function of variables Ω_i and (ii) as the input impedance of a p_j-variable lossless two-port terminated by a driving-point impedance function of variables Ω_j. This condition has clearly established that the choice of one variable over the other is not to be preferred because in both cases the terminations, after extracting the p_i- and p_j-variable lossless two-ports, are the same.

CHAPTER III

LADDER REALIZATION OF MULTIVARIABLE REACTANCE AND POSITIVE REAL FUNCTIONS

3.1 INTRODUCTION

In the previous chapter, results were obtained regarding the extraction of a single-variable lumped lossless two-port from a given m-variable PRF, the termination being an MPRF in the remaining (m-1) variables. The lossless nature has been the only constraint put on the extracted network. However, it would be desirable that the extracted network may have certain characteristics as required by its applications. For instance, one may be interested in obtaining a resistively-terminated lossless ladder realization of an MPRF because of its application in the design of microwave and multidimensional digital filters.

It is known that while a single-variable reactance function can always be synthesized as a lowpass ladder network by a continued-fraction expansion, not every SPRF can be realized by this technique. Naturally, conditions in addition to positive realness are required to realize multivariable functions by continued fraction expansion. However, these conditions and those for the realizability of general multivariable ladder networks are not available in the literature.

This chapter is concerned with the ladder realization of MPRFs. In Section 3.2, necessary and sufficient conditions are established for the realization of resistively-terminated Fujisawa-type lowpass ladder networks and ladder networks whose transmission zeros are either at the origin or at infinity in the various p_i-planes [47]. In Section 3.3, using the even part of an MPRF which is of the first degree in all variables except one, realizability condition for a resistively-terminated lowpass ladder network with all of its transmission zeros at infinity is derived [48]. Using this realization, condition for a resistively-terminated highpass ladder network with all of its transmission zeros at the origin is also obtained.

The symbols which shall frequently be used in this chapter are listed below:

$\delta_{p_i}(Z)$: Degree of the variable p_i in an MPRF Z.

$m_1(p_i)$, $m_2(p_i)$, $m_1^{(i)}(p_i)$, $m_2^{(i)}(p_i)$: even polynomials in the variable p_i.

$n_1(p_i)$, $n_2(p_i)$, $n_1^{(i)}(p_i)$, $n_2^{(i)}(p_i)$: odd polynomials in the variable p_i

$P(\Omega_i)$, $Q(\Omega_i)$: multivariable polynomials in the variables Ω_i

$P_{i+1}(\Omega_{i,\ldots,k})$, $Q_{i+1}(\Omega_{i,\ldots,k})$: multivariable polynomials in the variables $\Omega_{i,\ldots,k}$

$A_1^{(i)}(p_{i,\ldots,m})$, $A_0^{(i)}(p_{i,\ldots,m})$, $B_1^{(i)}(p_{i,\ldots,m})$, $B_0^{(i)}(p_{i,\ldots,m})$: multivariable polynomials in the variables $p_{i,\ldots,m}$

$M_1^{(i)}(p_{i,\ldots,m})$, $M_2^{(i)}(p_{i,\ldots,m})$: multivariable even polynomials in the variables $p_{i,\ldots,m}$

$N_1^{(i)}(p_{i,\ldots,m})$, $N_2^{(i)}(p_{i,\ldots,m})$: multivariable odd polynomials in the variables $p_{i,\ldots,m}$.

3.2 RESULTS ON LADDER REALIZATION

Lemmas 3.2.1 and 3.2.2 of this section establish necessary and sufficient conditions for a single-variable ladder extraction from a given MPRF while the remaining theorems and corollaries derive conditions for a complete resistively-terminated ladder realization of an MPRF.

Lemma 3.2.1

The necessary and sufficient conditions for a reduced m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a p_i-variable lossless ladder network with all of its transmission zeros (a) at $p_i=\infty$, (b) at $p_i=0$ or (c) both at $p_i=0$ and $p_i=\infty$, and terminated by a multivariable positive real impedance function $Z_0(\Omega_i)$, are that

(i) $Z(p_{1,\ldots,m})$ is cascade-expressible in the variable p_i , that is, it can be written as:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)}$$

(ii) $m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i) = Rp_i^{2k}$ where k is non-negative integer and R is non-zero real constant, and for the three cases:

(a) $k=0$ and $R>0$, when all the transmission zeros of the lossless ladder network are at $p_i=\infty$;

(b) $k=n$, when all the transmission zeros of the lossless ladder network are at $p_i=0$;

(c) $0<k<n$, when the transmission zeros of the lossless ladder network are at $p_i=0$ or $p_i=\infty$;

where $n=\delta_{p_i}(Z)$.

Proof

The necessity is obvious. Hence, only the sufficiency is proved. Because of Condition (i), the open-circuit parameters of the lossless two-port can be chosen from either Case A or Case B, as mentioned in Section 2.2.

(a) Since $k=0$ and $R>0$, the choice of the parameters of Case A with $z_{12}(p_i) = z_{21}(p_i) = \dfrac{\sqrt{R}}{n_2(p_i)}$ is possible. Since $z_{12}(p_i)$ has all of its zeros at $p_i=\infty$, realization of the open-circuit parameters will give a lowpass ladder network with p_i-inductors in the series arms and p_i-capacitors in the shunt arms. Realization of $Z(p_1,\ldots,_m)$ is obtained by terminating the lowpass ladder network by the multivariable positive real impedance function $Z_0(\Omega_i) = P(\Omega_i)/Q(\Omega_i)$.

(b) For a positive R , choose the parameters of Case A, and in that case $z_{12}(p_i) = z_{21}(p_i) = \dfrac{\sqrt{R}p_i^n}{n_2(p_i)}$, whereas for a negative R, choose the parameters of Case B, and in that case $z_{12}(p_i) = z_{21}(p_i) = \frac{\sqrt{-R}p_i^n}{m_2(p_i)}$. Since $z_{12}(p_i)$ has all of its zeros at $p_i=0$, realization of the open-circuit parameters will give a highpass ladder network with p_i-capacitors in the series arms and p_i-inductors in the shunt arms. This ladder network when terminated by an appropriate multivariable positive real impedance of $P(\Omega_i)/Q(\Omega_i)$ or $Q(\Omega_i)/P(\Omega_i)$, gives a realization of $Z(p_{1,\ldots,m})$.

(c) Again, depending on a positive or a negative R, choose Case A or Case B, and then $z_{12}(p_i) = z_{21}(p_i) = \frac{\sqrt{R}p_i^k}{n_2(p_i)}$ or $\frac{\sqrt{-R}p_i^k}{m_2(p_i)}$ $(0<k<n)$. Thus, $z_{12}(p_i)$ has zeros both at $p_i=0$ and $p_i=\infty$, and hence, realization of the open-circuit parameters will give a lossless ladder network with its transmission zeros both at the origin and at infinity of the p_i-plane. A realization of $Z(p_{1,\ldots,m})$ is obtained by terminating this bandpass ladder network with the impedance function $Z_0(\Omega_i)$.

<u>Theorem 3.2.1</u>

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a resistively-terminated lossless ladder network which is a cascade of m two-ports of variables p_1 to p_m, in that order, and each two-port having all of its transmission zeros at the origin or at infinity, are that,

(i) For $i=1,2,\ldots,(m-1)$, $Z(c_1,c_2,\ldots,c_{i-1},p_i,\ldots,p_m)$ is cascade-expressible in the variable p_i, that is,

$$Z(c_1,c_2,\ldots,c_{i-1},p_i,\ldots,p_m)=\frac{m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})+n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})}{m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})+n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})}$$

(ii) $m_1^{(i)}(p_i)m_2^{(i)}(p_i)-n_1^{(i)}(p_i)n_2^{(i)}(p_i) = R_i p_i^{2k_i}$, $i=1,2,\ldots,m$

where R_i is a non-zero real constant, $k_i=0$ or $\delta_{p_i}(Z)$ and $c_j=0$ or ∞ $(j=1,\ldots,i-1)$ depending on whether $k_j=0$ or $\delta_{p_j}(Z)$.

Proof

Only the sufficiency will be proved, as the necessity immediately follows from the analysis of such a network. For $i=1$, (i) and (ii) are the conditions of Lemma 3.2.1. Hence, from the MPRF $Z(p_{1,\ldots,m})$, a p_1-variable lossless ladder network with all of its transmission zeros either at $p_1=\infty$ or at $p_1=0$, depending on whether k_1 is zero or $\delta_{p_1}(Z)$, can be extracted. Termination $Z_1(\Omega_1)$ of this ladder network is given by $Z(c_1,p_2,p_3,\ldots,p_m)$, where $c_1=0$ or ∞ depending on whether the extracted lossless two-port is a lowpass or a highpass ladder network. The driving-point MPRF $Z_1(\Omega_1)$, with (i) and (ii) for $i=2$, again satisfies the realizability conditions of Lemma 3.2.1, and a p_2-variable lossless lowpass or highpass ladder network can be extracted from it. Thus, at this stage of synthesis, the cascade structure consists of p_1- and p_2-variable lossless two-ports terminated by the driving-point MPRF $Z_2(\Omega_{1,2}) = Z(c_1, c_2, p_3, p_4, \ldots, p_m)$, where $c_2=0$ or ∞ depending on whether the p_2-variable lossless two-port is a lowpass or a highpass ladder network. This process is repeated until a single-variable PRF $Z_{m-1}(p_m) = Z(c_1, c_2, \ldots, c_{m-1}, p_m)$, with the condition that $m_1^{(m)}(p_m)m_2^{(m)}(p_m)-n_1^{(m)}(p_m)n_2^{(m)}(p_m) = R_m p_m^{2k_m}$, is obtained. This is realized by a resistively-terminated lowpass or highpass ladder network with all of its transmission zeros either at $p_m=\infty$ or at $p_m=0$. Thus, $Z(p_{1,\ldots,m})$ is realized as the input impedance of a resistively-terminated lossless ladder network which is a cascade of m lossless two-ports of variables p_1 to p_m , each two-port having all of its transmission zeros at the origin or at infinity.

Corollary 3.2.1

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of the resistively-terminated lowpass ladder network of Fig. 3.1 are that,

(i) For $i=1,2,\ldots,(m-1)$, $Z(0,0,\ldots,0,p_i,\ldots,p_m)$ is cascade-expressible in the variable p_i , that is,

$$Z(0,0,\ldots,0,p_i,\ldots,p_m) = \frac{m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})+n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})}{m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})+n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})}$$

(ii) $m_1^{(i)}(p_i)m_2^{(i)}(p_i)-n_1^{(i)}(p_i)n_2^{(i)}(p_i) = R_i$, $i=1,2,\ldots,m$

where R_i is a non-zero positive constant.

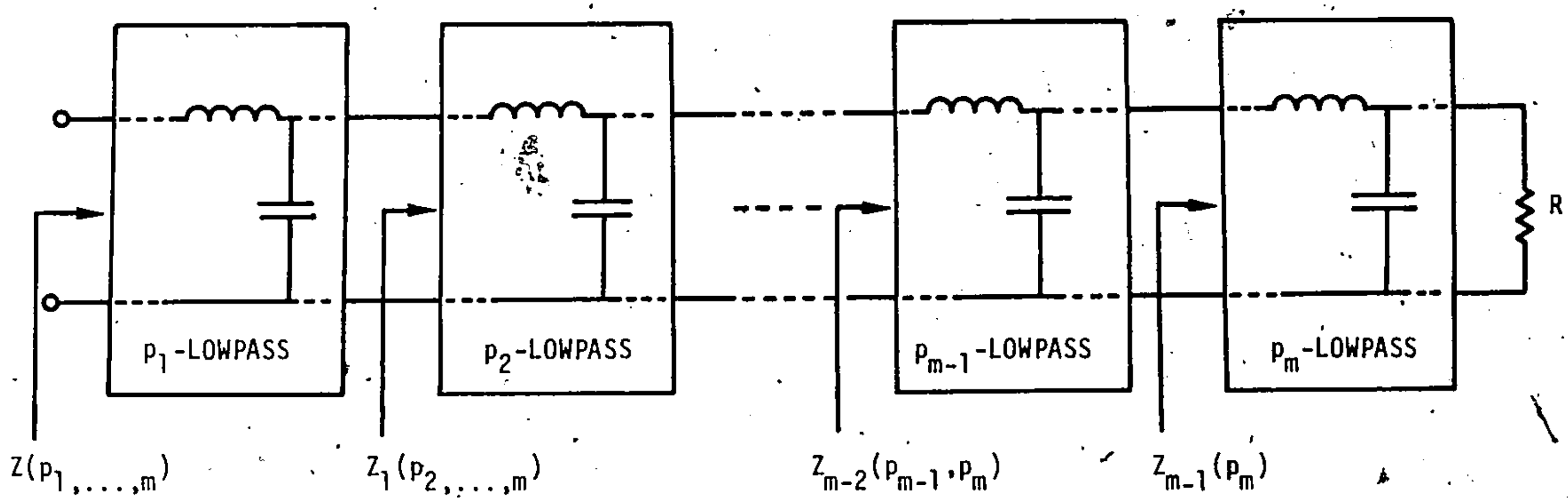

FIG. 3.1. A Resistively-Terminated Lowpass Ladder Realization of an m-Variable PRF $Z(p_{1,\ldots,m})$.

Corollary 3.2.2

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of the resistively-terminated highpass ladder network of Fig. 3.2 are that,

(i) For $i=1,2,\ldots,(m-1)$, $Z(\infty,\ldots,\infty,p_i,\ldots,p_m)$ is cascade-expressible in the variable p_i, that is,

$$Z(\infty,\ldots,\infty,p_i,\ldots,p_m) = \frac{m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})+n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})}{m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\ldots,i})+n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\ldots,i})}$$

(ii) $m_1^{(i)}(p_i)m_2^{(i)}(p_i)-n_1^{(i)}(p_i)n_2^{(i)}(p_i) = R_i p_i^{2n_i}$, $i=1,2,\ldots,m$

where R_i is a non-zero real constant and $n_i=\delta_{p_i}(Z)$.

Corollaries 3.2.1 and 3.2.2 are special cases of Theorem 3.2.1.

Lemma 3.2.2

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a p_i-variable mid-series and mid-shunt lowpass ladder network terminated by a multivariable positive real impedance function $Z_0(\Omega_i)$ are that

(i) $Z(p_{1,\ldots,m})$ is cascade-expressible in the variable p_i, that is, it can be written as:

$$Z(p_{1,\ldots,m}) = \frac{m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)}$$

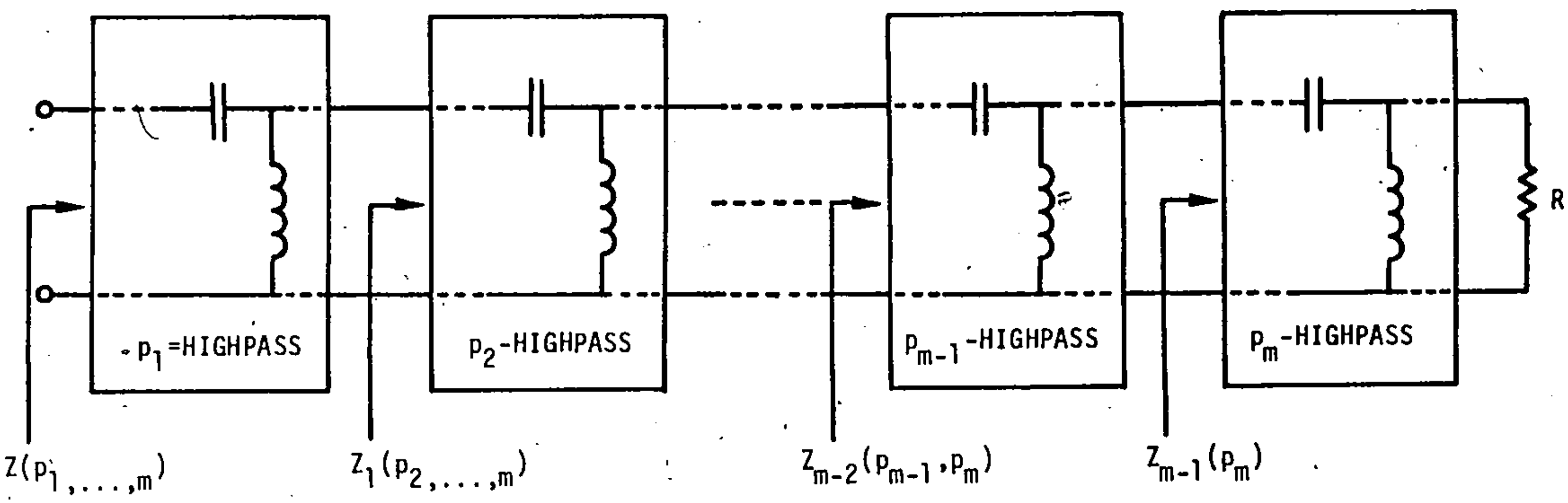

FIG. 3.2. A Resistively-Terminated Highpass Ladder Realization of an m-Variable PRF $Z(p_{1,\ldots,m})$.

(ii) The single-variable PRF, $F(p_i) = \frac{m_1(p_i)+n_1(p_i)}{m_2(p_i)+n_2(p_i)}$ satisfies the Fujisawa conditions [49] given below:

(a) Zeros of the numerator of the even part of $F(p_i)$ are restricted to the imaginary axis of the p_i-plane.

(b) $F(p_i)$ has a pole or zero at $p_i=\infty$.

(c) $n_1(p_i)$ or $n_2(p_i)$ (or both) have at least one more zero than there are zeros of the numerator of the even part of $F(p_i)$.

(d) If $\omega_1 < \omega_2 < \cdots < \omega_k$ are the imaginary axis zeros mentioned in (a), then if $n_1(p_i)$ satisfies Condition (c), any value ω_j has at least j zeros of $m_1(p_i)$ between $p_i=0$ and itself, and if $n_2(p_i)$ satisfies Condition (c), ω_j has at least j zeros of $m_2(p_i)$ between $p_i=0$ and itself.

Proof.

Sufficiency: Since $F(p_i)$ is a single-variable PRF satisfying the Fujisawa conditions, it can always be realized as the impedance function of a one-ohm terminated lossless mid-series and mid-shunt lowpass ladder network in the variable p_i [49]. This network, with the resistive termination replaced by the MPRF $P(\Omega_i)/Q(\Omega_i)$, gives a realization of $Z(p_{1,\ldots,m})$.

Theorem 3.2.2

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\dots,m})$ to be realizable as the input impedance of a resistively-terminated lossless ladder network which is a cascade of m Fujisawa-type lowpass ladder networks of variables p_1 to p_m, are that,

(i) For $i=1,2,\dots,(m-1)$, $Z(0,0,\dots,0,p_i,\dots,p_m)$ is cascade-expressible in the variable p_i, that is,

$$Z(0,0,\dots,p_i,\dots,p_m) = \frac{m_1^{(i)}(p_i)P_{i+1}(\Omega_{1,\dots,i}) + n_1^{(i)}(p_i)Q_{i+1}(\Omega_{1,\dots,i})}{m_2^{(i)}(p_i)Q_{i+1}(\Omega_{1,\dots,i}) + n_2^{(i)}(p_i)P_{i+1}(\Omega_{1,\dots,i})}$$

(ii) $F_i(p_i) = \dfrac{m_1^{(i)}(p_i)+n_1^{(i)}(p_i)}{m_2^{(i)}(p_i)+n_2^{(i)}(p_i)}$, $i=1,2,\dots,m$

satisfies the Fujisawa conditions.

Proof

The driving-point function of the termination of a p_i-variable Fujisawa-type lowpass ladder network can be obtained by letting $p_i=0$ in the overall driving-point function of the network. The proof of this theorem can be obtained by using this fact and repeatedly applying Lemma 3.2.2.

It was shown in Chapter II that from a given MPRF Z, a lossless two-port in the variable p_i can be extracted if and only if the function is cascade-expressible in the variable p_i. Thus, an m-variable PRF may require as many as m trials to test if the function is cascade-expressible in any one of the variables. However, in the case of extraction of a ladder network whose transmission zeros are all either at the origin or at infinity, the test of cascade expressibility of a function is much simpler than in the general case. If a p_i-variable ladder network with all of its transmission zeros at $p_i=\infty$ has to be extracted from Z, then the highest degrees of the numerator and the denominator polynomials of Z in the variable p_i must differ by one. Similarly, if a p_i-variable ladder network with all of its transmission zeros at $p_i=0$ has to be extracted from Z, then the lowest degrees of the numerator and the denominator polynomials of Z in the variable p_i must also differ by one. Hence, for the extraction of these types of ladder networks one ought to check Condition (i) of Lemma 3.2.1 by testing if the given MPRF is cascade-expressible in a variable in which the highest or the lowest degrees of the numerator and the denominator polynomials differ by one. In general, if the highest or the lowest degrees of the numerator and the denominator polynomials of an MPRF in a variable differ by one, then that particular variable cannot be contained in the load.

The following examples are considered to illustrate the techniques of ladder realizations based on the results obtained in this section.

Example 3.2.1

Realize the PRF of three variables given below:

$$Z(p_1,p_2,p_3)=\frac{\begin{array}{l}4p_1^2p_2^2p_3+4p_1^2p_2p_3^2+2p_1p_2^2p_3^2+4p_1^2p_2^2+2p_1p_2^2p_3+4p_1^2p_2p_3\\+4p_1^2p_2+2p_1p_2p_3+4p_1^2p_3+2p_1p_2^2+p_2^2p_3+p_2p_3^2+4p_1^2+2p_1p_2\\+p_2^2+p_2p_3+p_2+p_3+1\end{array}}{\begin{array}{l}2p_1p_2^2p_3+2p_1p_2p_3^2+p_2^2p_3^2+2p_1p_2^2+p_2^2p_3+2p_1p_2p_3+2p_1^2+p_2^2\\+p_2p_3+2p_1p_3+2p_1+p_2\end{array}}$$

Since the highest degrees of the numerator and the denominator polynomials in the variable p_1 differ by one, an attempt is made to rewrite Z in the cascade-expressible form in this variable.

$$Z(p_1,p_2,p_3) = \frac{(4p_1^2+1)[p_2^2p_3+p_2p_3^2+p_2^2+p_2p_3+p_2+p_3+1]+2p_1[p_2^2p_3^2+p_2^2p_3+p_2^2+p_2p_3+p_2]}{[p_2^2p_3^2+p_2^2p_3+p_2^2+p_2p_3+p_2]+2p_1[p_2^2p_3+p_2p_3^2+p_2^2+p_2p_3+p_2+p_3+1]}$$

The real part condition of the p_1-variable network can be checked by

$$G_1(p_1) = m_1^{(1)}(p_1)m_2^{(1)}(p_1)-n_1^{(1)}(p_1)n_2^{(1)}(p_1)$$

$$= (4p_1^2+1)(1)-(2p_1)(2p_1) = 1$$

The termination of the p_1-variable lowpass ladder network is given by

$$Z(0,p_2,p_3) = \frac{p_2^2p_3+p_2p_3^2+p_2^2+p_2p_3+p_2+p_3+1}{p_2^2p_3^2+p_2^2p_3+p_2^2+p_2p_3+p_2}$$

Since the lowest degrees of the numerator and the denominator polynomials of $Z(0,p_2,p_3)$ in the variable p_2 differ by one, this function is rewritten in the cascade-expressible form in the variable p_2 as given below:

$$Z(0,p_2,p_3) = \frac{(p_2^2+1)[p_3+1]+p_2[p_3^2+p_3+1]}{p_2^2[p_3^2+p_3+1]+p_2[p_3+1]}$$

Now, the numerator of the even part of the p_2-variable network and the driving-point impedance of its termination are given as

$$G_2(p_2) = m_1^{(2)}(p_2)m_2^{(2)}(p_2)-n_1^{(2)}(p_2)n_2^{(2)}(p_2) \qquad [2\delta_{p_2}(Z)]$$

$$= (p_2^2+1)(p_2^2)-(p_2)(p_2) = p_2^4 = p_2$$

$$Z(0,\infty,p_3) = \frac{p_3+1}{p_3^2+p_3+1}$$

Finally, the numerator of the even part of $Z(0,\infty,p_3)$ is given by

$$G_3(p_3) = m_1^{(3)}(p_3)m_2^{(3)}(p_3)-n_1^{(3)}(p_3)n_2^{(3)}(p_3)$$

$$= (1)(p_3^2+1)-(p_3)(p_3) = 1$$

Thus, $Z(p_1,p_2,p_3)$ satisfies the conditions of Theorem 3.2.1, and it can be realized as the input impedance of a resistively-terminated ladder network. The necessary parameters of realization are as follows:

$$z_{11}^{(1)}(p_1) = \frac{m_1^{(1)}(p_1)}{n_2^{(1)}(p_1)} = \frac{4p_1^2+1}{2p_1} \qquad z_{12}^{(1)}(p_1) = \frac{\sqrt{G_1(p_1)}}{n_2^{(1)}(p_1)} = \frac{1}{2p_1}$$

$$z_{11}^{(2)}(p_2) = \frac{m_1^{(2)}(p_2)}{n_2^{(2)}(p_2)} = \frac{p_2^2+1}{p_2} \qquad z_{12}^{(2)}(p_2) = \frac{\sqrt{G_2(p_2)}}{n_2^{(2)}(p_2)} = \frac{p_2^2}{p_2} = p_2$$

A complete realization of Z is shown in Fig. 3.3.

<u>Example 3.2.2</u>

Consider a two-variable PRF given by

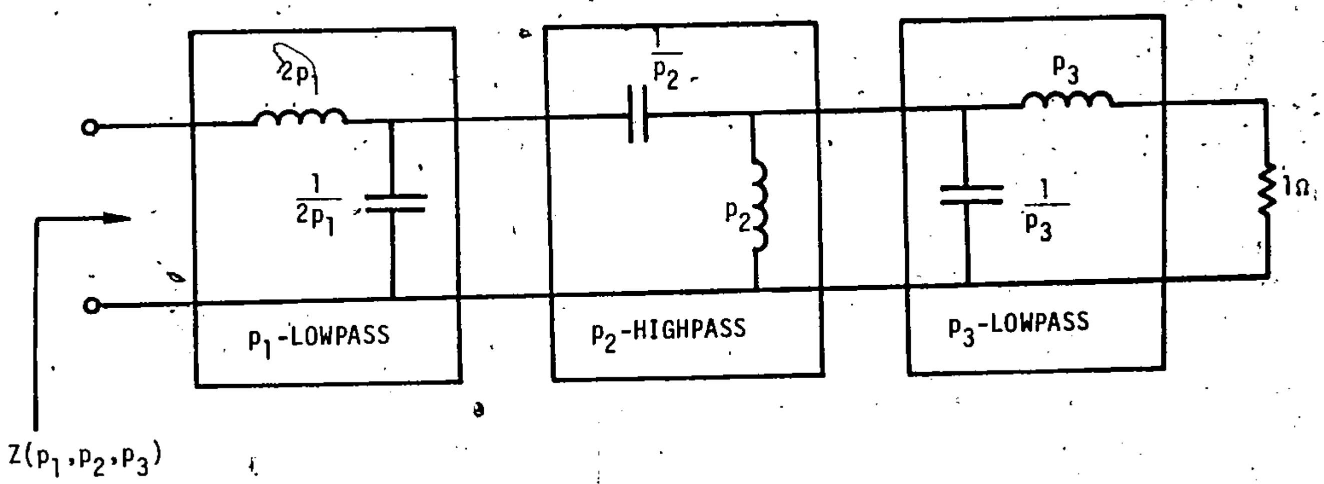

FIG. 3.3. A Cascade Realization of $Z(p_1,p_2,p_3)$ of Example 3.2.1 Using Theorem 3.2.1.

$$Z(p_1,p_2) = \frac{\begin{array}{l} p_1^6p_2^2+2p_1^5p_2^3+p_1^6p_2+4p_1^5p_2^2+p_1^6+2p_1^5p_2+5p_1^4p_2^2+4p_1^3p_2^3+2p_1^5+5p_1^4p_2 \\ +8p_1^3p_2^2+5p_1^4+4p_1^3p_2+5p_1^2p_2^2+2p_1p_2^3+4p_1^3+5p_1^2p_2+4p_1p_2^2+5p_1^2 \\ +2p_1p_2+p_2^2+2p_1+p_2+1 \end{array}}{\begin{array}{l} p_1^5p_2^2+2p_1^4p_2^3+p_1^5p_2+4p_1^4p_2^2+p_1^5+2p_1^4p_2+4p_1^3p_2^2+3p_1^2p_2^3+2p_1^4+4p_1^3p_2 \\ +6p_1^2p_2^2+4p_1^3+3p_1^2p_2+2p_1p_2^2+p_2^3+3p_1^2+2p_1p_2+2p_2^2+2p_1+p_2+1 \end{array}}$$

Here, the highest degrees of numerator and denominator polynomials of Z in the variable p_2 are the same, whereas those in the variable p_1 differ by one. Hence, the variable p_1 cannot be a load variable of a cascade realization of Z, if one exists. Therefore, an attempt is made to check if Z is in the cascade-expressible form in the variable p_1:

$$Z(p_1,p_2) = \frac{\begin{array}{l} (p_1^6p_2^2+p_1^6p_2+p_1^6+5p_1^4p_2^2+5p_1^4p_2+5p_1^4+5p_1^2p_2^2+5p_1^2p_2+5p_1^2+p_2^2+p_2+1) \\ +(2p_1^5p_2^3+4p_1^5p_2^2+2p_1^5p_2+4p_1^3p_2^3+2p_1^5+8p_1^3p_2^2+4p_1^3p_2+2p_1p_2^3+4p_1^3 \\ +4p_1p_2^2+2p_1p_2+2p_1) \end{array}}{\begin{array}{l} (2p_1^4p_2^3+4p_1^4p_2^2+2p_1^4p_2+3p_1^2p_2^3+2p_1^4+6p_1^2p_2^2+3p_1^2p_2+p_2^3+3p_1^2+2p_2^2+p_2+1) \\ +(p_1^5p_2^2+p_1^5p_2+p_1^5+4p_1^3p_2^2+4p_1^3p_2+4p_1^3+2p_1p_2^2+2p_1p_2+2p_1) \end{array}}$$

$$Z(p_1,p_2) = \frac{\begin{array}{l} [(p_1^6+5p_1^4+5p_1^2+1)p_2^2+(p_1^6+5p_1^4+5p_1^2+1)p_2+(p_1^6+5p_1^4+5p_1^2+1)] \\ +[(2p_1^5+4p_1^3+2p_1)p_2^3+(4p_1^5+8p_1^3+4p_1)p_2^2+(2p_1^5+4p_1^3+2p_1)p_2 \\ +(2p_1^5+4p_1^3+2p_1)] \end{array}}{\begin{array}{l} [(2p_1^4+3p_1^2+1)p_2^3+(4p_1^4+6p_1^2+2)p_2^2+(2p_1^4+3p_1^2+1)p_2+(2p_1^4+3p_1^2+1)] \\ +[(p_1^5+4p_1^3+2p_1)p_2^2+(p_1^5+4p_1^3+2p_1)p_2+(p_1^5+4p_1^3+2p_1)] \end{array}}$$

$$Z(p_1,p_2) = \frac{(p_1^6+5p_1^4+5p_1^2+1)[p_2^2+p_2+1]+(2p_1^5+4p_1^3+2p_1)[p_2^3+2p_2^2+p_2+1]}{(2p_1^4+3p_1^2+1)[p_2^3+2p_2^2+p_2+1]+(p_1^5+4p_1^3+2p_1)[p_2^2+p_2+1]}$$

Thus, Z is cascade-expressible in the variable p_1. Obtain the single-variable PRFs given below:

$$\frac{m_1^{(1)}(p_1)+n_1^{(1)}(p_1)}{m_2^{(1)}(p_1)+n_2^{(1)}(p_1)} = \frac{(p_1^6+5p_1^4+5p_1^2+1)+(2p_1^5+4p_1^3+2p_1)}{(2p_1^4+3p_1^2+1)+(p_1^5+4p_1^3+2p_1)} \qquad (3.2.1)$$

and

$$\frac{P(p_2)}{Q(p_2)} = Z(0,p_2) = \frac{(p_2^2+1)+(p_2)}{(2p_2^2+1)+(p_2^3+p_2)} \qquad (3.2.2)$$

The single-variable functions given by (3.2.1) and (3.2.2) satisfy the realizability conditions of Fujisawa lowpass ladder networks. First, the function given by (3.2.1) is realized with a one-ohm resistive termination, and then this one-ohm resistance is replaced by a realization of the function given by (3.2.2). A complete Fujisawa-type lowpass ladder realization of $Z(p_1,p_2)$ is shown in Fig. 3.4.

3.3 LADDER REALIZATION USING REAL PART CONDITION

So far, the discussion has been on the realization of a lowpass or a highpass ladder network where the cascade-expressibility is one of the conditions that the given input impedance has to satisfy. In this section, the realizability conditions are derived involving the numerator of the even part. In particular, necessary and sufficient conditions are derived under which an m-variable PRF of the first degree in all variables except one can be realized as a resistively-terminated ladder network with all of its transmission zeros at the origin or at infinity. First, a theorem on a resistively-terminated lowpass ladder realization is

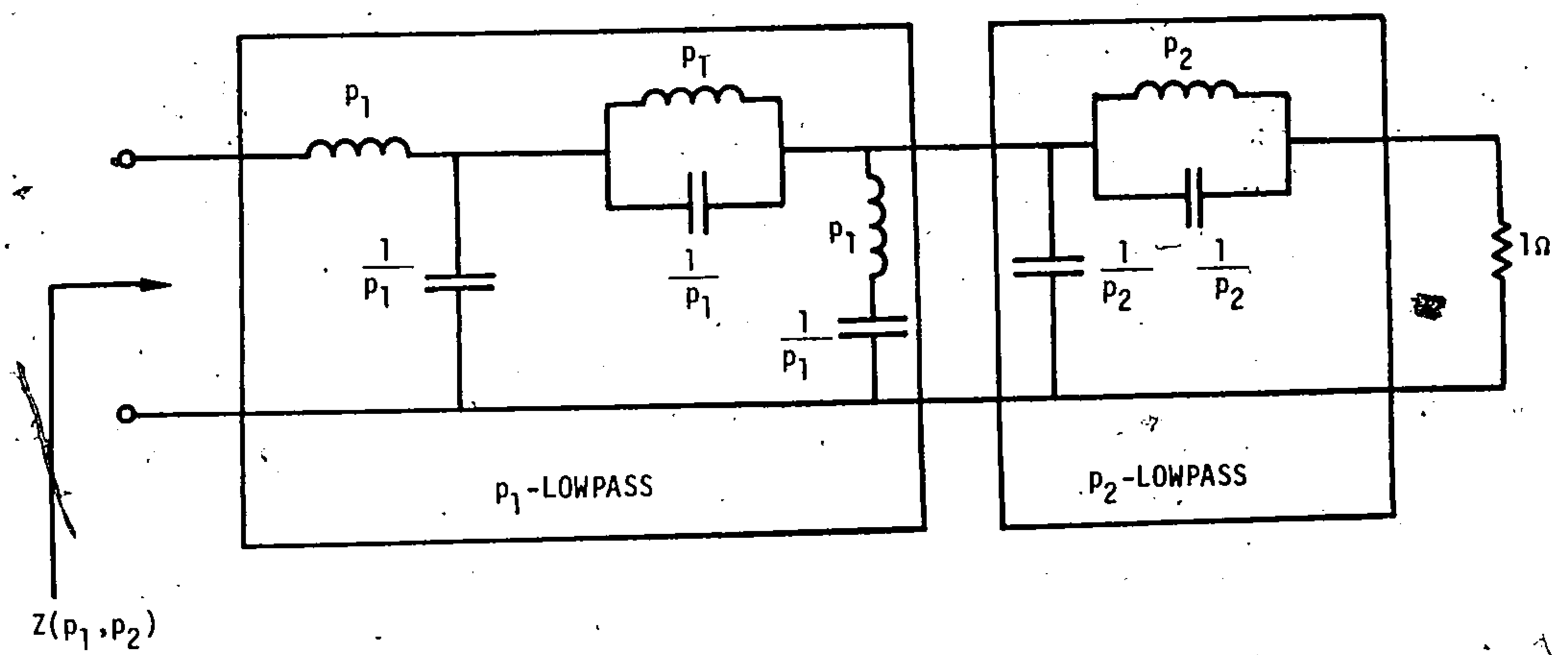

FIG. 3.4. A Cascade Realization of $Z(p_1,p_2)$ of Example 3.2.2 Using Theorem 3.2.2.

established, then using this theorem and the transformation $p_i \to \frac{1}{p_i}$, the condition for the realizability of a resistively-terminated highpass ladder network is derived.

Theorem 3.3.1

The necessary and sufficient condition for an m-variable PRF $F^{(1)}(p_{1,\ldots,m}) = \frac{M_1^{(1)}(p_{1,\ldots,m})+N_1^{(1)}(p_{1,\ldots,m})}{M_2^{(1)}(p_{1,\ldots,m})+N_2^{(1)}(p_{1,\ldots,m})}$, which is of the first degree in all variables except one, to be realizable as the input impedance of a resistively-terminated lowpass ladder network, with inductors in the series arms and capacitors in the shunt arms, is that

$$M_1^{(1)}(p_{1,\ldots,m})M_2^{(1)}(p_{1,\ldots,m})-N_1^{(1)}(p_{1,\ldots,m})N_2^{(1)}(p_{1,\ldots,m}) = R$$

where R is a non-zero positive real constant.

Proof

The necessity follows immediately since, for such a realization, all the transmission zeros are at $p_k=\infty$ $(k=1,\ldots,m)$ independent of other variables.

Sufficiency: Assume that the degrees in none of the variables of the numerator and the denominator polynomials of $F_1^{(1)}(p_{1,\ldots,m})$ differ by one. Also, without loss of generality, assume that $\delta_{p_k}[F^{(1)}(p_{1,\ldots,m})] = 1$ $(k=1,\ldots,m-1)$ and $\delta_{p_m}[F^{(1)}(p_{1,\ldots,m})] = n$. Now, $F^{(1)}(p_{1,\ldots,m})$ can be rewritten as:

$$F^{(1)}(p_{1,\ldots,m}) = \frac{M_1^{(1)}(p_{1,\ldots,m})+N_1^{(1)}(p_{1,\ldots,m})}{M_2^{(1)}(p_{1,\ldots,m})+N_2^{(1)}(p_{1,\ldots,m})}$$

$$= \frac{A_1^{(2)}(p_{2,\ldots,m})p_1+A_0^{(2)}(p_{2,\ldots,m})}{B_1^{(2)}(p_{2,\ldots,m})p_1+B_0^{(2)}(p_{2,\ldots,m})} \qquad (3.3.1a)$$

where

$$M_1^{(1)}(p_{1,\ldots,m})M_2^{(1)}(p_{1,\ldots,m})-N_1^{(1)}(p_{1,\ldots,m})N_2^{(1)}(p_{1,\ldots,m}) = R \qquad (3.3.1b)$$

The (m-1)-variable PRF $F^{(2)}(p_{2,\ldots,m}) = \frac{A_0^{(2)}(p_{2,\ldots,m})}{B_0^{(2)}(p_{2,\ldots,m})}$ can be expressed as

$$F^{(2)}(p_{2,\ldots,m}) = \frac{A_0^{(2)}(p_{2,\ldots,m})}{B_0^{(2)}(p_{2,\ldots,m})} = \frac{M_1^{(2)}(p_{2,\ldots,m})+N_1^{(2)}(p_{2,\ldots,m})}{M_2^{(2)}(p_{2,\ldots,m})+N_2^{(2)}(p_{2,\ldots,m})}$$

$$= \frac{A_1^{(3)}(p_{3,\ldots,m})p_2+A_0^{(3)}(p_{3,\ldots,m})}{B_1^{(3)}(p_{3,\ldots,m})p_2+B_0^{(3)}(p_{3,\ldots,m})} \qquad (3.3.2a)$$

Condition (3.3.1b) implies that the PRF given by (3.3.2a) must satisfy the condition:

$$M_1^{(2)}(p_{2,\ldots,m})M_2^{(2)}(p_{2,\ldots,m})-N_1^{(2)}(p_{2,\ldots,m})N_2^{(2)}(p_{2,\ldots,m}) = R \qquad (3.3.2b)$$

Continuing this way, obtain the following PRFs with the corresponding real part conditions:

$$F^{(3)}(p_{3,\ldots,m}) = \frac{A_0^{(3)}(p_{3,\ldots,m})}{B_0^{(3)}(p_{3,\ldots,m})} = \frac{M_1^{(3)}(p_{3,\ldots,m})+N_1^{(3)}(p_{3,\ldots,m})}{M_2^{(3)}(p_{3,\ldots,m})+N_2^{(3)}(p_{3,\ldots,m})}$$

$$= \frac{A_1^{(4)}(p_{4,\ldots,m})p_3+A_0^{(4)}(p_{4,\ldots,m})}{B_1^{(4)}(p_{4,\ldots,m})p_3+B_0^{(4)}(p_{4,\ldots,m})} \quad (3.3.3a)$$

$$M_1^{(3)}(p_{3,\ldots,m})M_2^{(3)}(p_{3,\ldots,m})-N_1^{(3)}(p_{3,\ldots,m})N_2^{(3)}(p_{3,\ldots,m}) = R \quad (3.3.3b)$$

$$\vdots$$

$$F^{(m-2)}(p_{m-2,\ldots,m}) = \frac{A_0^{(m-2)}(p_{m-2,\ldots,m})}{B_0^{(m-2)}(p_{m-2,\ldots,m})}$$

$$= \frac{M_1^{(m-2)}(p_{m-2,\ldots,m})+N_1^{(m-2)}(p_{m-2,\ldots,m})}{M_2^{(m-2)}(p_{m-2,\ldots,m})+N_2^{(m-2)}(p_{m-2,\ldots,m})}$$

$$= \frac{A_1^{(m-1)}(p_{m-1,m})p_{m-2}+A_0^{(m-1)}(p_{m-1,m})}{B_1^{(m-1)}(p_{m-1,m})p_{m-2}+B_0^{(m-1)}(p_{m-1,m})} \quad (3.3.4a)$$

$$M_1^{(m-2)}(p_{m-2,\ldots,m})M_2^{(m-2)}(p_{m-2,\ldots,m})-N_1^{(m-2)}(p_{m-2,\ldots,m})N_2^{(m-2)}(p_{m-2,\ldots,m})=R \quad (3.3.4b)$$

$$F^{(m-1)}(p_{m-1,m}) = \frac{A_0^{(m-1)}(p_{m-1,m})}{B_0^{(m-1)}(p_{m-1,m})} =$$

$$= \frac{M_1^{(m-1)}(p_{m-1,m})+N_1^{(m-1)}(p_{m-1,m})}{M_2^{(m-1)}(p_{m-1,m})+N_2^{(m-1)}(p_{m-1,m})}$$

$$= \frac{A_1^{(m)}(p_m)p_{m-1}+A_0^{(m)}(p_m)}{B_1^{(m)}(p_m)p_{m-1}+B_0^{(m)}(p_m)} \quad (3.3.5a)$$

$$M_1^{(m-1)}(p_{m-1,m})M_2^{(m-1)}(p_{m-1,m})-N_1^{(m-1)}(p_{m-1,m})N_2^{(m-1)}(p_{m-1,m}) = R \qquad (3.3.5b)$$

$$F^{(m)}(p_m) = \frac{A_0^{(m)}(p_m)}{B_0^{(m)}(p_m)} = \frac{M_1^{(m)}(p_m)+N_1^{(m)}(p_m)}{M_2^{(m)}(p_m)+N_2^{(m)}(p_m)} \qquad (3.3.6a)$$

$$M_1^{(m)}(p_m)M_2^{(m)}(p_m)-N_1^{(m)}(p_m)N_2^{(m)}(p_m) = R \qquad (3.3.6b)$$

Note that none of the polynomials $A_0^{(i)}$'s and $B_0^{(i)}$'s $(i=2,\ldots,m)$ can be identically zero, zince in that case the real part conditions will not be satisfied. Also, none of the polynomials $A_1^{(i)}$'s and $B_1^{(i)}$'s $(i=2,\ldots,m)$ can be identically zero, since in that case a degree difference in the variable p_i will be reflected between the numerator and the denominator polynomials of the given PRF $F^{(1)}(p_{1,\ldots,m})$. For instance, if in (3.3.5a), $B_1^{(m)}(p_m) \equiv 0$, then the degrees in the variable p_{m-1} of the numerator and the denominator polynomials of $F^{(m-1)}(p_{m-1,m})$ differ by one, that is, the degrees in the variable p_{m-1} of the polynomials $A_0^{(m-1)}(p_{m-1,m})$ and $B_0^{(m-1)}(p_{m-1,m})$ of (3.3.4a) differ by one. The function $A_1^{(m-1)}(p_{m-1,m})/B_1^{(m-1)}(p_{m-1,m})$ is a ratio of even and odd polynomials because of Condition (3.3.4b). Assume that $A_1^{(m-1)}$ is even and $B_1^{(m-1)}$ is odd. Since $A_1^{(m-1)}(p_{m-1,m})/A_0^{(m-1)}(p_{m-1,m})$ and $B_1^{(m-1)}(p_{m-1,m})/B_0^{(m-1)}(p_{m-1,m})$ are PRFs, $A_1^{(m-1)}(p_{m-1,m}) = \alpha M_1^{(m-1)}(p_{m-1,m})$ and $B_1^{(m-1)}(p_{m-1,m}) = \beta N_2^{(m-1)}(p_{m-1,m})$, where α and β

are positive constants. Thus, $F^{(m-2)}(p_{m-2,\ldots,m})$ as given by (3.3.4a) can be rewritten as:

$$F^{(m-2)}(p_{m-2,\ldots,m}) = \frac{\alpha M_1^{(m-1)}(p_{m-1,m})\,p_{m-2}+M_1^{(m-1)}(p_{m-1,m})+N_1^{(m-1)}(p_{m-1,m})}{\beta N_2^{(m-1)}(p_{m-1,m})\,p_{m-2}+M_2^{(m-1)}(p_{m-1,m})+N_2^{(m-1)}(p_{m-1,m})} \qquad (3.3.7)$$

From (3.3.7), it is clear that the degrees in the variable p_{m-1} of the numerator and the denominator polynomials of $F^{(m-2)}(p_{m-2,\ldots,m})$ also differ by one. Continuing in this way, it can be concluded that the degrees in the variable p_{m-1} of the numerator and the denominator polynomials of the given function $F^{(1)}(p_{1,\ldots,m})$ differ by one. Since this is not possible, the polynomial $B_1^{(m)}(p_m)$ cannot be identically zero.

The single-variable PRF $F^{(m)}(p_m)$ as given by (3.3.6a) satisfies Condition (3.3.6b). Hence, the degrees in the variable p_m of its numerator and denominator polynomials $A_0^{(m)}(p_m)$ and $B_0^{(m)}(p_m)$ differ by one. Since the PRF as given by (3.3.5a) satisfies the real part condition (3.3.5b), $A_1^{(m)}(p_m)/B_1^{(m)}(p_m)$ is a reactance function. Assume that it is a ratio of even by odd polynomials. Moreover, since $A_1^{(m)}(p_m)/A_0^{(m)}(p_m)$ and $B_1^{(m)}(p_m)/B_0^{(m)}(p_m)$ are PRFs, $A_1^{(m)}(p_m) = \gamma M_1^{(m)}(p_m)$ and $B_1^{(m)}(p_m) = \delta N_2^{(m)}(p_m)$, where γ and δ are positive constants. Hence, the function $F^{(m-1)}(p_{m-1,m})$ can be rewritten as

$$F^{(m-1)}(p_{m-1,m}) = \frac{\gamma M_1^{(m)}(p_m)p_{m-1}+M_1^{(m)}(p_m)+N_1^{(m)}(p_m)}{\delta N_2^{(m)}(p_m)p_{m-1}+M_2^{(m)}(p_m)+N_2^{(m)}(p_m)} \qquad (3.3.8)$$

It is obvious from (3.3.8) that the degrees in the variable p_m of the numerator and denominator polynomials of $F^{(m-1)}(p_{m-1,m})$ differ by one. Thus, using the PRFs and the corresponding real part conditions successively from (3.3.6) to (3.3.1), one arrives at the conclusion that the degrees in the variable p_m of the numerator and the denominator polynomials of the given function $F^{(1)}(p_{1,\ldots,m})$ differ by one. This contradicts the original assumption of no degree difference. Hence, if the PRF given by (3.3.1a) has to satisfy the real part condition given by (3.3.1b), the degrees in at least one of the variables of the numerator and the denominator polynomials of the given function must differ by one. Consequently, by Ozaki and Kasami's theorem [50], removal of a pole at infinity from $F^{(1)}(p_{1,\ldots,m})$ or $1/F^{(1)}(p_{1,\ldots,m})$ is possible. If the numerator polynomial is one degree higher in a particular variable than the denominator polynomial, then the function has a pole at infinity in that variable independent of the other variables which can be extracted as a series inductor. On the other hand, if the numerator polynomial is one degree lower than the denominator polynomial, then from the inverted function a shunt capacitor can be extracted. In either case, the remaining function is positive real whose degree in one of the variables is reduced by one from that of $F^{(1)}(p_{1,\ldots,m})$. Moreover, the numerator of the even part of the remaining function is still R, and thus, the process is repeatable. Hence, by repeated removal of poles or zeros, a zero degree function is achieved, which can be realized as a resistor. This proves the sufficiency, and the theorem is established.

Theorem 3.3.2

The necessary and sufficient condition for an m-variable PRF $F(p_{1,\ldots,m}) = \frac{M_1(p_{1,\ldots,m})+N_1(p_{1,\ldots,m})}{M_2(p_{1,\ldots,m})+N_2(p_{1,\ldots,m})}$, which is of the first degree in all variables except one, to be realizable as the input impedance of a resistively-terminated highpass ladder network, with capacitors in the series arms and inductors in the shunt arms, is that

$$M_1(p_{1,\ldots,m})M_2(p_{1,\ldots,m})-N_1(p_{1,\ldots,m})N_2(p_{1,\ldots,m}) = Hp_1^{2k_1}p_2^{2k_2}\cdots p_m^{2k_m} \tag{3.3.9}$$

where H is a non-zero real constant and $k_i=\delta_{p_i}(F)$ $(i=1,2,\ldots,m)$.

Proof

Let $F^{(1)}(p_{1,\ldots,m}) = \frac{M_1^{(1)}(p_{1,\ldots,m})+N_1^{(1)}(p_{1,\ldots,m})}{M_2^{(1)}(p_{1,\ldots,m})+N_2^{(1)}(p_{1,\ldots,m})}$ be an m-variable PRF obtained from $F(p_{1,\ldots,m})$ after making the transformation $p_i \to \frac{1}{p_i}$ $(i=1,\ldots,m)$. Since $F(p_{1,\ldots,m})$ satisfies the condition given by (3.3.9), it can be shown that $F^{(1)}(p_{1,\ldots,m})$ will satisfy the condition given below:

$$M_1^{(1)}(p_{1,\ldots,m})M_2^{(1)}(p_{1,\ldots,m})-N_1^{(1)}(p_{1,\ldots,m})N_2^{(1)}(p_{1,\ldots,m}) = R \tag{3.3.10}$$

where R is a non-zero positive real constant. Thus, by Theorem 3.3.1, $F^{(1)}(p_{1,\ldots,m})$ can be realized as the driving-point impedance function of a resistively-terminated lowpass ladder network. From this realization by making the retransformation $p_i \to \frac{1}{p_i}$ $(i=1,\ldots,m)$ on it, a realization of F as the input impedance of a resistively-terminated highpass ladder network with capacitors in the series arms and inductors in the shunt arms are obtained.

Example 3.3.1

Consider the following three-variable PRF:

$$Z(p_1,p_2,p_3) = \frac{M_1(p_1,p_2,p_3)+N_1(p_1,p_2,p_3)}{M_2(p_1,p_2,p_3)+N_2(p_1,p_2,p_3)}$$

$$= \frac{(24p_1p_2^4p_3+18p_1p_2^3+12p_1p_2^2p_3+4p_2^4+8p_2^3p_3+6p_1p_2+4p_2^2) + (6p_1p_2^4+24p_1p_2^3p_3+15p_1p_2^2+6p_1p_2p_3+2p_2^3+4p_2^2p_3+3p_1+2p_2)}{(12p_1p_2^4p_3+30p_1p_2^2p_3+4p_2^3p_3+6p_1p_3+4p_2p_3) + (36p_1p_2^3p_3+8p_2^4p_3+12p_1p_2p_3+8p_2^2p_3)}$$

Here $M_1(p_1,p_2,p_3)M_2(p_1,p_2,p_3)-N_1(p_1,p_2,p_3)N_2(p_1,p_2,p_3)$ is evaluated to be $288p_1^2p_2^8p_3^2$. Thus, $Z(p_1,p_2,p_3)$ satisfies the condition of Theorem 3.3.2, and it can be realized as the impedance function of a resistively-terminated highpass ladder network. A complete realization is shown in Fig. 3.5.

3.4 SUMMARY AND DISCUSSION

This chapter has established several necessary and sufficient conditions for the realization of resistively-terminated multivariable lossless ladder networks. First, conditions have been derived for the extraction of a lowpass, a highpass or a bandpass ladder network, with

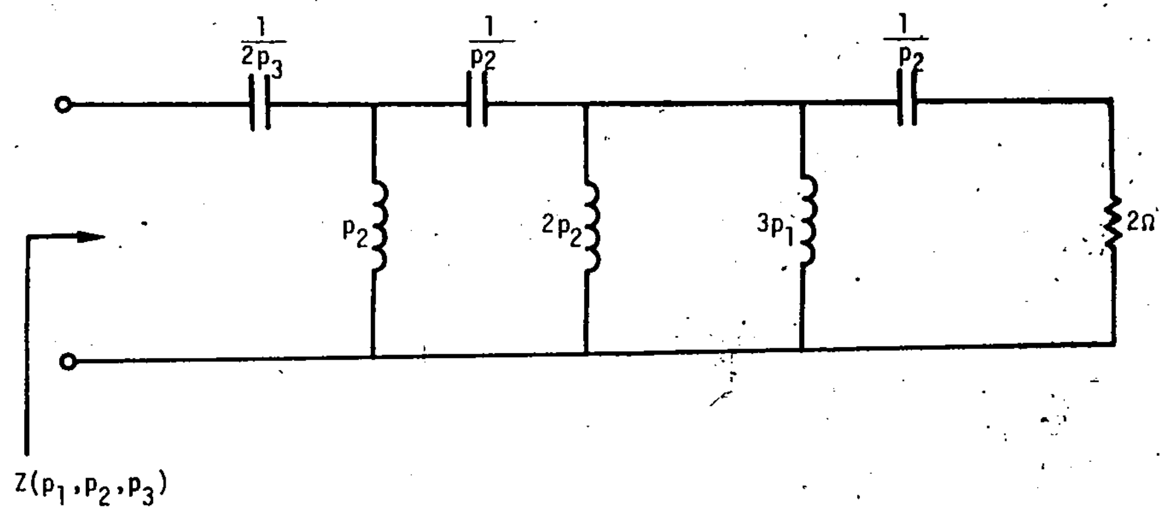

FIG. 3.5. A Resistively-Terminated Highpass Ladder Realization of $Z(p_1,p_2,p_3)$ of Example 3.3.1.

all of its transmission zeros either at $p_i=0$ or at $p_i=\infty$, or a Fujisawa-type lowpass ladder network. Next, using the fact for the extracted lowpass and highpass ladder network, the impedance function of the terminating load can be obtained by respectively letting $p_i=0$ and $p_i=\infty$ in the given MPRF, conditions for a complete ladder realization have been derived. These conditions allow a realization of an MPRF as a resistively-terminated cascade of m lossless two-ports such that each two-port is either a lowpass or a highpass ladder network. It has been shown that the highest or the lowest degree difference of a particular variable between the numerator and the denominator polynomials determines the choice of the variable in which the conditions for a ladder extraction with all of its transmission zeros at the origin or at infinity ought to be tested. It may be noted that Theorem 3.2.1 also gives the conditions for a multivariable reactance function to be realizable as the input impedance of a lossless ladder network if Condition (ii) holds only for $i=1,\ldots,(m-1)$.

Condition using the numerator of the even part has been derived for the realization of a class of resistively-terminated lowpass ladder networks. Using this realization, condition for a resistively-terminated highpass ladder network with all of its transmission zeros at the origin has also been derived. These realizations do not assume a specific structure of the ladder network such as the one where each of the series and shunt arms contains reactive elements in all of the variables [38], or the ladder network of Section 3.2 where the structure consists of a cascade of distinct single-variable lossless two-ports in each of the variables.

CHAPTER IV

PARTIAL DERIVATIVE PROPERTIES OF MULTIVARIABLE CASCADED NETWORKS

4.1 INTRODUCTION

Some properties and techniques of generation of multivariable reactance and positive real functions, from the partial derivative point of view, may be found in the literature [15], [19], [51]. However, no such study seems to have been done for multivariable cascade structures. In this chapter [52], [53], some partial derivative properties of a multivariable structure which is a cascade of single-variable lossless two-ports terminated by a single-variable impedance function are established. These properties are then utilized to derive conditions for the extraction of single-variable lossless ladder network with all of its transmission zeros at the origin or at infinity. Finally, some synthesis procedures are outlined for the extraction of these kinds of ladder networks.

The following symbols shall frequently be used in this chapter:

$Z_{p_i}(p_{1,\ldots,m})$: partial derivative of the multivariable function $Z(p_{1,\ldots,m})$ with respect to the variable p_i .

Ev $F(p_i)$: even part of the positive real function $F(p_i)$.

Nu $G(p_{1,\ldots,m})$: numerator of the rational function $G(p_{1,\ldots,m})$.

4.2 PARTIAL DERIVATIVE PROPERTIES OF A MULTIVARIABLE CASCADE OF SINGLE-VARIABLE LOSSLESS TWO-PORTS WITH A REACTIVE OR POSITIVE REAL LOAD

This section will first establish some general partial derivative properties of a network which is a cascade of $(m-1)$ lossless two-ports of variables p_1 to p_{m-1}, and terminated in a reactance or positive real impedance function of variable p_m. Those cases where some of the lossless two-ports are lowpass or highpass ladder networks with all of their transmission zeros either at the origin or at infinity will then be studied. Using these partial derivative properties, necessary and sufficient conditions under which an m-variable PRF or reactance function (RF) can be realized as the input impedance of a p_i-variable lowpass or highpass ladder network, with all of its transmission zeros at $p_i=\infty$ or $p_i=0$, and terminated in a reactance or positive real impedance function of variables Ω_i will be derived.

<u>Theorem 4.2.1</u>

The necessary condition for an m-variable PRF $Z_1(p_1,\ldots,_m)$ to be the input impedance of a one-ohm resistively-terminated network (Fig. 4.1) which is a cascade of m lossless two-ports of variables p_1 to p_m, is that

$$Z_{1p_m}(p_1,\ldots,_m) = \frac{\left[\prod_{i=1}^{m-1}\{\mathrm{NuEv}\ F_i(p_i)\}\right]\cdot \mathrm{Nu}\,\dfrac{dF_m(p_m)}{dp_m}}{\prod_{j=1}^{m-1}\left[m_2^{(j)}(p_j)Q_{j+1}(\Omega_1,\ldots,_j)+n_2^{(j)}(p_j)P_{j+1}(\Omega_1,\ldots,_j)\right]^2} \qquad (4.2.1)$$

where $F_i(p_i) = \dfrac{m_1^{(i)}(p_i)+n_1^{(i)}(p_i)}{m_2^{(i)}(p_i)+n_2^{(i)}(p_i)}$ is the driving-point function of the ith two-port when it is terminated by a one-ohm resistance, and

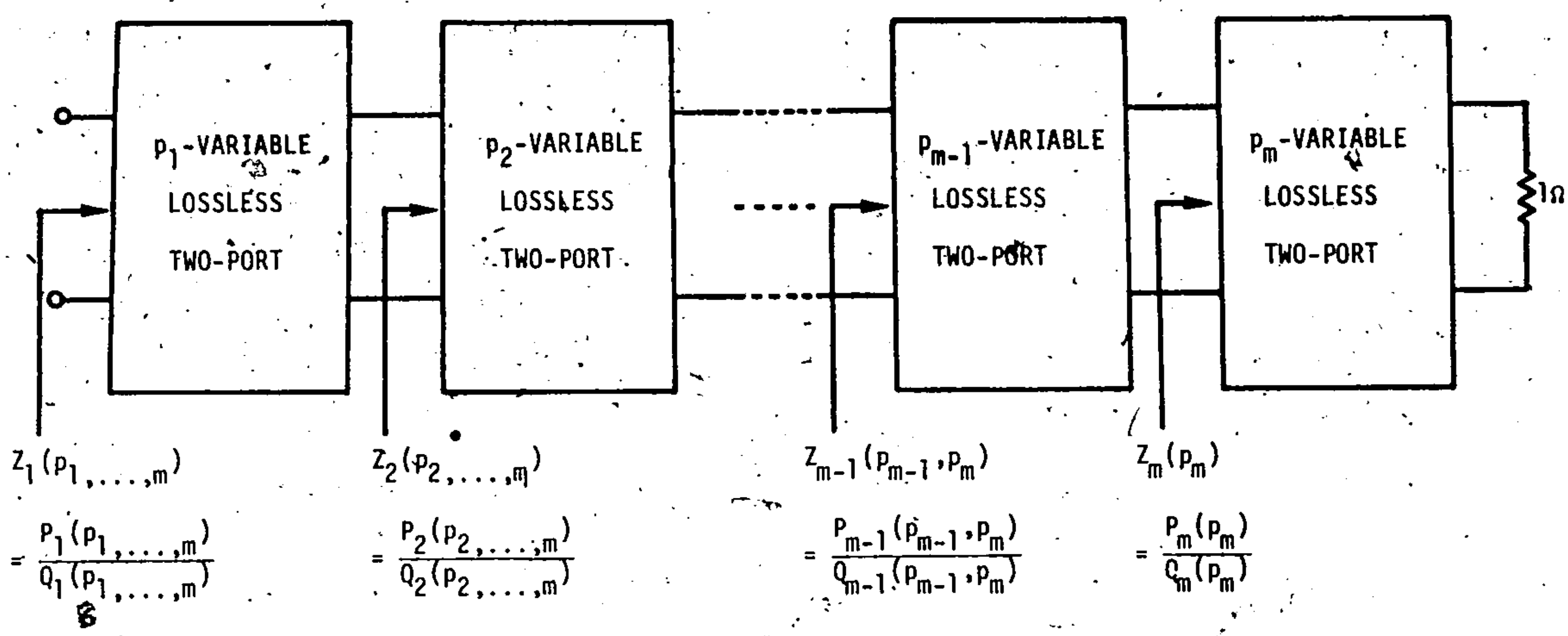

FIG. 4.1. A Resistively-Terminated Multivariable Network as a Cascade of m Lossless Two-Ports of Variables p_1 to p_m.

$\frac{P_{j+1}(\Omega_{1,\ldots,j})}{Q_{j+1}(\Omega_{1,\ldots,j})} = Z_{j+1}(p_{j+1,\ldots,m})$ is the driving-point function of the termination of the jth two-port $(i=1,\ldots,m\ ,\ \ j=1,\ldots,m-1)$.

Proof

The driving-point function of the given network (Fig. 4.1) can be written as:

$$Z_1(p_{1,\ldots,m}) = \frac{m_1^{(1)}(p_1)P_2(\Omega_1)+n_1^{(1)}(p_1)Q_2(\Omega_1)}{m_2^{(1)}(p_1)Q_2(\Omega_1)+n_2^{(1)}(p_1)P_2(\Omega_1)} \tag{4.2.2}$$

Taking partial derivative of (4.2.2) with respect to p_m gives:

$$Z_{1p_m}(p_{1,\ldots,m}) = \frac{[m_1^{(1)}(p_1)m_2^{(1)}(p_1)-n_1^{(1)}(p_1)n_2^{(1)}(p_1)]\ \text{Nu}\ Z_{2p_m}(p_{2,\ldots,m})}{[m_2^{(1)}(p_1)Q_2(\Omega_1)+n_2^{(1)}(p_1)P_2(\Omega_1)]^2} \tag{4.2.3}$$

But Z_2 can be written as:

$$Z_2(p_{2,\ldots,m}) = \frac{m_1^{(2)}(p_2)P_3(\Omega_{1,2})+n_1^{(2)}(p_2)Q_3(\Omega_{1,2})}{m_2^{(2)}(p_2)Q_3(\Omega_{1,2})+n_2^{(2)}(p_2)P_3(\Omega_{1,2})} \tag{4.2.4}$$

and its partial derivative with respect to p_m is given by

$$Z_{2p_m}(p_{1,\ldots,m}) = \frac{[m_1^{(2)}(p_2)m_2^{(2)}(p_2)-n_1^{(2)}(p_2)n_2^{(2)}(p_2)]\ \text{Nu}\ Z_{3p_m}(p_{3,\ldots,m})}{[m_2^{(2)}(p_2)Q_3(\Omega_{1,2})+n_2^{(2)}(p_2)P_3(\Omega_{1,2})]^2} \tag{4.2.5}$$

This process is continued until Z_{m-1p_m} as given below is obtained:

$$Z_{m-1p_m}(p_{m-1},p_m) = \frac{[m_1^{(m-1)}(p_{m-1})m_2^{(m-1)}(p_{m-1})-n_1^{(m-1)}(p_{m-1})n_2^{(m-1)}(p_{m-1})]\mathrm{Nu}Z_{mp_m}(p_m)}{[m_2^{(m-1)}(p_{m-1})Q_m(p_m)+n_2^{(m-1)}(p_{m-1})P_m(p_m)]^2} \tag{4.2.6}$$

Hence, Z_{1p_m} can be rewritten as:

$$Z_{1p_m}(p_{1,\ldots,m}) = \frac{[\prod_{i=1}^{m-1}(m_1^{(i)}(p_i)m_2^{(i)}(p_i)-n_1^{(i)}(p_i)n_2^{(i)}(p_i)]\ \mathrm{Nu}\ Z_{mp_m}(p_m)}{\prod_{j=1}^{m-1}[(m_2^{(j)}(p_j)Q_{j+1}(\Omega_{1,\ldots,j})+n_2^{(j)}(p_j)P_{j+1}(\Omega_{1,\ldots,j})]^2} \tag{4.2.7}$$

But $m_1^{(i)}(p_i)m_2^{(i)}(p_i)-n_1^{(i)}(p_i)n_2^{(i)}(p_i) = \mathrm{NuEv}\ F_i(p_i)$, and thus, (4.2.7) yields (4.2.1).

<u>Theorem 4.2.2</u>

The necessary condition for an m-variable PRF (RF) $Z_1(p_{1,\ldots,m})$ to be the input impedance of a network (Fig. 4.2) which is a cascade of k lossless two-ports of variable p_1 to p_k , and terminated by an (m-k)-variable driving-point PRF (RF) of variables $\Omega_{1,\ldots,k}$ is that

$$Z_{1p_\ell}(p_{1,\ldots,m}) = \frac{[\prod_{i=1}^{k}\{\mathrm{NuEv}\ F_i(p_i)\}].\mathrm{Nu}\frac{\partial}{\partial p_\ell}\ Z_{k+1}(\Omega_{1,\ldots,k})}{\prod_{j=1}^{k}[m_2^{(j)}(p_j)Q_{j+1}(\Omega_{1,\ldots,j})+n_2^{(j)}(p_j)P_{j+1}(\Omega_{1,\ldots,j})]^2} \tag{4.2.8}$$

where $p_\ell \in \Omega_{1,\ldots,k}$, $F_i(p_i) = \dfrac{m_1^{(i)}(p_i)+n_1^{(i)}(p_i)}{m_2^{(i)}(p_i)+n_2^{(i)}(p_i)}$ $(i=1,\ldots,k)$

is the driving-point function of the ith two-port when it is terminated

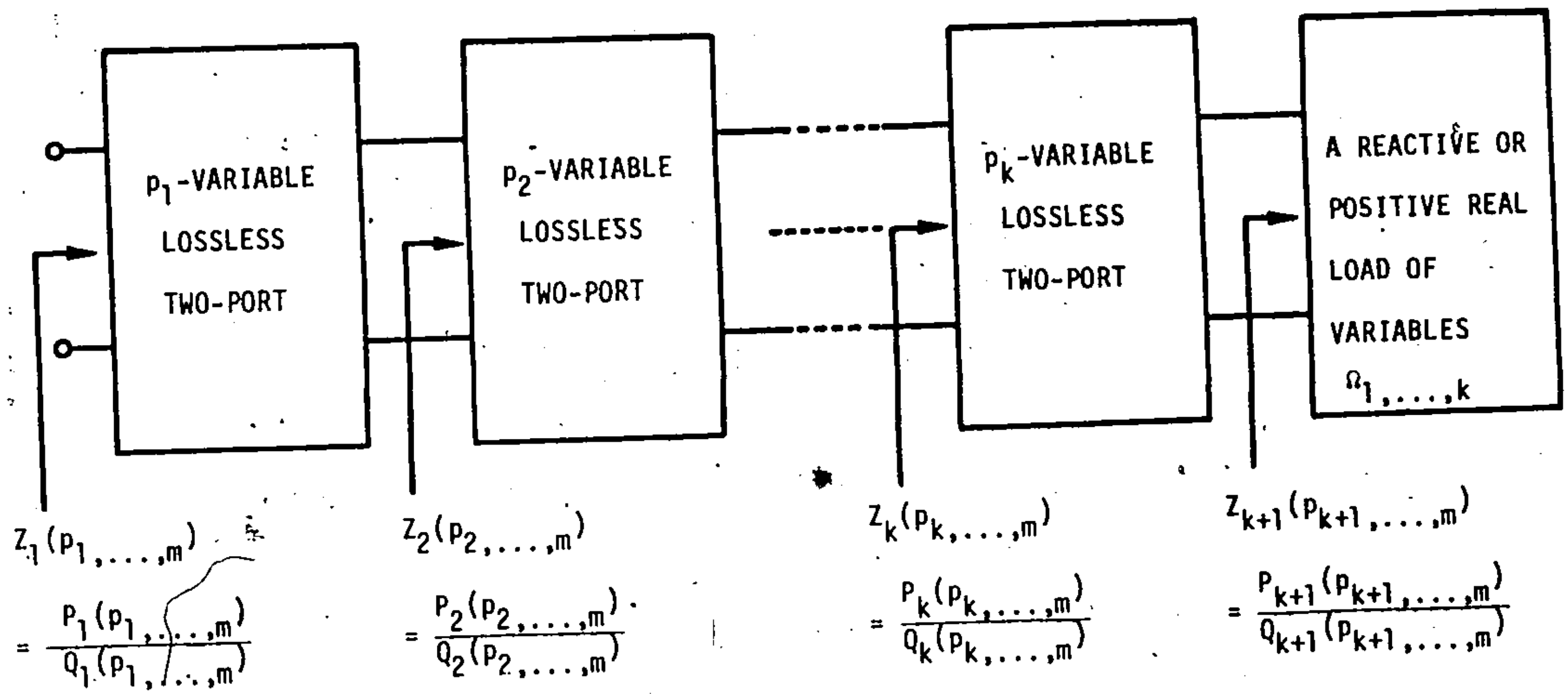

FIG. 4.2. A Multivariable Network as a Cascade of k Lossless Two-Ports of Variables p_1 to p_k with a Reactive or Positive Real Load of Variables $p_{k+1},\ldots,p_m$.

by a one-ohm resistance, and $\frac{P_{j+1}(\Omega_{1,\ldots,j})}{Q_{j+1}(\Omega_{1,\ldots,j})} = Z_{j+1}(\Omega_{1,\ldots,j})$ $(j=1,\ldots,k)$ is the driving-point function of the termination of the jth two-port.

The proof of this theorem is along the same lines as those of Theorem 4.2.1.

Corollary 4.2.1

If in the network of Fig. 4.2, any one of the lossless two-ports is a lowpass ladder network with inductors in the series arms and capacitors in the shunt arms, then $NuZ_{1p_\ell}(p_{1,\ldots,m})$ $(p_\ell \in \Omega_{1,\ldots,k})$ is independent of the variable of that lossless two-port.

Proof

Theorem 4.2.2 gives

$$NuZ_{1p_\ell}(p_{1,\ldots,m}) = [\prod_{i=1}^{k}\{NuEv\ F_i(p_i)\}]\cdot Nu\ \frac{\partial}{\partial p_\ell}\ Z_{k+1}(\Omega_{1,\ldots,k}) \quad (4.2.9)$$

If the jth lossless two-port, where $j \in \{1,\ldots,k\}$, is a lowpass ladder network as stated in the corollary, then $NuEv\ F_j(p_j)$ is identically a positive constant. Hence, from (4.2.9), $Nu\ Z_{1p_\ell}(p_{1,\ldots,m})$ is independent of p_j.

Corollary 4.2.2

If in the network of Fig. 4.2, any one of the lossless two-ports, say the one in the variable p_j, is a highpass ladder network with capacitors in the series arms and inductors in the shunt arms, then

Nu $Z_{1p_\ell}(p_{1,\ldots,m})$ $(p_\ell \in \Omega_{1,\ldots,k})$ has $2(n_j-1)$th order zero at $p_j=0$, where $n_j=\delta_{p_j}(Z_1)$.

Proof

If the jth lossless two-port, where $j \in \{1,\ldots,k\}$, is a highpass ladder network as stated in the corollary, then NuEv $F_j(p_j) = H_j p_j^{2n_j}$, where H_j is a real constant. Moreover, the corresponding portion of the denominator of (4.2.8), that is, the polynomial $[m_2^{(j)}(p_j)Q_{j+1}(\Omega_{1,\ldots,j})+n_2^{(j)}(p_j)P_{j+1}(\Omega_{1,\ldots,j})]^2$ will have p_j^2 as a factor. Hence, from (4.2.8) Nu $Z_{1p_\ell}(p_{1,\ldots,m})$ has $2(n_j-1)$th order zero at $p_j=0$.

Theorem 4.2.3

If an m-variable PRF (RF) ,

$$Z(p_{1,\ldots,m}) = \frac{N(p_{1,\ldots,m})}{D(p_{1,\ldots,m})} = \frac{N(p_{1,\ldots,m})}{m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)} \tag{4.2.10}$$

where $m_2(p_i)$ and $n_2(p_i)$ are respectively even and odd polynomials of p_i , and $P(\Omega_i)$ and $Q(\Omega_i)$ are multivariable polynomials in Ω_i , satisfies the condition:

$$\begin{aligned} \text{Nu } Z_{p_\ell}(p_{1,\ldots,m}) = {} & [m_1(p_i)m_2(p_i)-n_1(p_i)n_2(p_i)] \times \\ & [P_{p_\ell}(\Omega_i)Q(\Omega_i)-P(\Omega_i)Q_{p_\ell}(\Omega_i)] \end{aligned} \tag{4.2.11}$$

where $p_\ell \in \Omega_i$, and $m_1(p_i)$ and $n_1(p_i)$ are respectively even and odd polynomials of p_i , then $N(p_{1,\dots,m})$ is expressible as:

$$N(p_{1,\dots,m}) = m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i) \tag{4.2.12}$$

Proof

Taking partial derivative of $Z(p_{1,\dots,m})$ with respect to the variable $p_\ell \in \Omega_i$, gives:

$$Z_{p_\ell}(p_{1,\dots,m}) = \frac{N_{p_\ell}(p_{1,\dots,m})[m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)] - N(p_{1,\dots,m})[m_2(p_i)Q_{p_\ell}(\Omega_i)+n_2(p_i)P_{p_\ell}(\Omega_i)]}{[m_2(p_i)Q(\Omega_i)+n_2(p_i)P(\Omega_i)]^2} \tag{4.2.13}$$

Now, if the condition given by (4.2.11) has to be satisfied, then from (4.2.13) the following two relationships must hold:

$$N_{p_\ell}(p_{1,\dots,m})Q(\Omega_i)-N(p_{1,\dots,m})Q_{p_\ell}(\Omega_i) = m_1(p_i)[P_{p_\ell}(\Omega_i)Q(\Omega_i)-P(\Omega_i)Q_{p_\ell}(\Omega_i)] \tag{4.2.14}$$

$$N_{p_\ell}(p_{1,\dots,m})P(\Omega_i)-N(p_{1,\dots,m})P_{p_\ell}(\Omega_i) = -n_1(p_i)[P_{p_\ell}(\Omega_i)Q(\Omega_i)-P(\Omega_i)Q_{p_\ell}(\Omega_i)] \tag{4.2.15}$$

If $P_{p_\ell}(\Omega_i)Q(\Omega_i)-P(\Omega_i)Q_{p_\ell}(\Omega_i) \neq 0$, then (4.2.14) and (4.2.15) yield:

$$N(p_{1,\dots,m}) = m_1(p_i)P(\Omega_i)+n_1(p_i)Q(\Omega_i)$$

Thus, the theorem is proved.

Theorem 4.2.4

The necessary and sufficient conditions for an m-variable RF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a lowpass ladder network with p_i-inductors in the series arms and p_i-capacitors in the shunt arms, beginning with an inductor, and terminated by an (m-1)-variable RF $Z_0(\Omega_i)$ are that,

(i) The function Z is of the form

$$Z(p_{1,\ldots,m}) = \frac{A_n(\Omega_i)p_i^n + A_{n-1}(\Omega_i)p_i^{n-1} + \cdots + A_1(\Omega_i)p_i + A_0(\Omega_i)}{B_{n-1}(\Omega_i)p_i^{n-1} + B_{n-2}(\Omega_i)p_i^{n-2} + \cdots + B_1(\Omega_i)p_i + B_0(\Omega_i)} \tag{4.2.16}$$

(ii) The polynomial $\text{Nu}\, Z_{p_j}(p_{1,\ldots,m})$ is independent of p_i for all $p_j \in \Omega_i$.

Proof

Necessity: If the function Z is realizable as stated in the theorem, it is obviously expressible in the form of (4.2.16). The necessity of Condition (ii) follows directly from Corollary 4.2.1.

Sufficiency: Without loss of generality, assume that n is even, and the reactance function Z is a ratio of even by odd polynomials. Thus, the polynomial coefficients $A_k (k=0,\ldots,n)$ is even or odd depending on whether k is even or odd, whereas the polynomial coefficient $B_\ell (\ell=0,\ldots,n-1)$ is even or odd depending on whether ℓ is odd or even. Now, for the function given by (4.2.16), obtain

$$\text{Nu } Z_{p_j}(p_{1,\ldots,m}) = \sum_{k=0}^{2n-1} a_k(\Omega_i)p_i^k \quad , \quad p_j \in \Omega_i \tag{4.2.17}$$

where $a_k(\Omega_i)$ is given by

$$a_k(\Omega_i) = \sum_{q+r=k} A_q'(\Omega_i)B_r(\Omega_i) - \sum_{q+r=k} A_q(\Omega_i)B_r'(\Omega_i) \tag{4.2.18}$$

with prime representing the partial derivative with respect to the variable p_j . By Condition (ii), $a_{2n-1} = 0$, that is,

$$A_n' B_{n-1} - A_n B_{n-1}' = 0 \tag{4.2.19}$$

Because of the given form of Z , $A_n(\Omega_i)$ or $B_{n-1}(\Omega_i)$ cannot be identically zeros. Thus, from (4.2.19), $\frac{\partial}{\partial p_j}(\frac{A_n}{B_{n-1}}) = 0$. Since this is true for all $p_j \in \Omega_i$, $A_n/B_{n-1} = k_n$ is a positive constant. Hence, a pole at $p_i=\infty$ independent of the other variables can be removed from Z , that is, $Z(p_{1,\ldots,m}) = k_n p_i + Z_1(p_{1,\ldots,m})$, such that,

$$Z_1(p_{1,\ldots,m}) = \frac{C_{n-1}(\Omega_i)p_i^{n-1}+C_{n-2}(\Omega_i)p_i^{n-2}+\cdots+C_1(\Omega_i)p_i+C_0(\Omega_i)}{B_{n-1}(\Omega_i)p_i^{n-1}+B_{n-2}(\Omega_i)p_i^{n-2}+\cdots+B_1(\Omega_i)p_i+B_0(\Omega_i)} \tag{4.2.20}$$

where $C_0(\Omega_i) = A_0(\Omega_i)$ and $C_\ell(\Omega_i) = A_\ell(\Omega_i) - k_n B_{\ell-1}(\Omega_i)$ $(\ell=1,2,\ldots,n-1)$. Z_1 is also an RF and the polynomial coefficient C_ℓ is even or odd depending on whether ℓ is even or odd. Again, because of Condition (ii), a_{2n-2} is identically zero, that is,

$$A_n' B_{n-2} + A_{n-1}' B_{n-1} - A_n B_{n-2}' - A_{n-1} B_{n-1}' = 0 \tag{4.2.21}$$

Substituting $A_n = k_n B_{n-1}$ in (4.2.21) yields:

$$(A_{n-1}-k_n B_{n-2})' B_{n-1}-(A_{n-1}-k_n B_{n-2}) B'_{n-1} = 0 \qquad (4.2.22)$$

Since $A_{n-1}-k_n B_{n-2} = C_{n-1}$, (4.2.22) can be rewritten as:

$$C'_{n-1} B_{n-1}-C_{n-1} B'_{n-1} = 0 \qquad (4.2.23)$$

The above relationship holds only when either C_{n-1} is identically zero or it is a constant multiple of B_{n-1} . Since B_{n-1} is even polynomial of variables Ω_i and C_{n-1} is odd, the second possibility is ruled out, and $C_{n-1} \equiv 0$. From (4.2.20), obtain

$$Y_1(p_{1,\ldots,m}) = \frac{1}{Z_1(p_{1,\ldots,m})}$$

$$= \frac{B_{n-1}(\Omega_i)p_i^{n-1}+B_{n-2}(\Omega_i)p_i^{n-2}+\cdots+B_1(\Omega_i)p_i+B_0(\Omega_i)}{C_{n-2}(\Omega_i)p_i^{n-2}+C_{n-3}(\Omega_i)p_i^{n-3}+\cdots+C_1(\Omega_i)p_i+C_0(\Omega_i)} \qquad (4.2.24)$$

Note that $C_{n-2}(\Omega_i) \not\equiv 0$, because in that case Y_1 would not be an RF. The reactance function Y_1 has the same form as Z except that its degree in the variable p_i is reduced from n to $(n-1)$. Since $Z(p_{1,\ldots,m}) = k_n p_i + \frac{1}{Y_1(p_{1,\ldots,m})}$, $\text{Nu } Z_{p_j}(p_{1,\ldots,m}) = -\text{Nu } Y_{1p_j}(p_{1,\ldots,m})$ for all $p_j \in \Omega_i$. Moreover, since $\text{Nu } Z_{p_j}(p_{1,\ldots,m})$ is independent of p_i , $\text{Nu } Y(p_{1,\ldots,m})$ is also independent of p_i . Thus, Y_1 too satisfies the two conditions of the theorem, and its degree in the variable p_i is one less than that of Z . Hence, a pole at $p_i=\infty$ independent of the other variables can be removed from Y_1 . This process of pole removal at $p_i=\infty$ can successively be repeated until the degree in the variable p_i of the remaining function reduces to zero. Thus, the reactance function Z is realized as the input impedance of a lowpass ladder network with p_i-inductors in the series arms and p_i-capacitors in the shunt arms, beginning with an inductor, and terminated by an RF of variables Ω_i .

Theorem 4.2.5

The necessary and sufficient conditions for an m-variable RF $Z(p_{1,\ldots,m})$ to be realizable as the input impedance of a highpass ladder network with p_i-capacitors in the series arms and p_i-inductors in the shunt arms, beginning with a capacitor, and terminated by an (m-1)-variable RF $Z_0(\Omega_i)$ are that,

(i) The function Z is of the form

$$Z(p_{1,\ldots,m}) = \frac{A_n(\Omega_i)p_i^n+A_{n-1}(\Omega_i)p_i^{n-1}+\ldots+A_1(\Omega_i)p_i+A_0(\Omega_i)}{B_n(\Omega_i)p_i^n+B_{n-1}(\Omega_i)p_i^{n-1}+\ldots+B_2(\Omega_i)p_i^2+B_1(\Omega_i)p_i} \quad (4.2.25)$$

(ii) For all $p_j \in \Omega_i$, $Z_{p_j}(p_{1,\ldots,m})$ has $2(n-1)$th order zero at $p_i=0$, where $n=\delta_{p_i}(Z)$.

Proof

If the function Z satisfies the two conditions, $Z_{p_j}(p_{1,\ldots,m})$ must be of the form,

$$Z_{p_j}(p_{1,\ldots,m}) = \frac{H(\Omega_i)p_i^{2(n-1)}}{[B_n(\Omega_i)p_i^{n-1}+B_{n-1}(\Omega_i)p_i^{n-2}+\ldots+B_2(\Omega_i)p_i+B_1(\Omega_i)]^2} \quad (4.2.26)$$

where $H(\Omega_i)$ is a polynomial in Ω_i. If $Z_1(p_{1,\ldots,m})$ is a function obtained from $Z(p_{1,\ldots,m})$ by making the transformation $p_i \to \frac{1}{p_i}$ on it, then it has the same form as (4.2.16), and the partial derivative of $Z_1(p_{1,\ldots,m})$ with respect to $p_j \in \Omega_i$ is given by

$$Z_{1p_j}(p_{1,\ldots,m}) = \frac{H(\Omega_i)}{[B_1(\Omega_i)p_i^{n-1}+B_2(\Omega_i)p_i^{n-2}+\cdots+B_{n-1}(\Omega_i)p_i+B_n(\Omega_i)]^2} . \quad (4.2.27)$$

From (4.2.27), it is obvious that the Nu $Z_{1p_j}(p_{1,\ldots,m})$ is independent of p_i. Thus, $Z_1(p_{1,\ldots,m})$ satisfies the two conditions of Theorem 4.2.4, and as a result it is realizable as the input impedance of a p_i-variable lowpass ladder network terminated by an RF of variables Ω_i. From this realization, by making the retransformation $p_i \to \frac{1}{p_i}$ on it, a realization of Z is obtained as the input impedance of a highpass ladder network with p_i-capacitors in the series arms and p_i-inductors in the shunt arms, beginning with a capacitor, and terminated by an (m-1)-variable RF of variables Ω_i.

Lemma 4.2.1

If $F_1(p_{1,\ldots,m}) = \frac{M_1(p_{1,\ldots,m})+N_1(p_{1,\ldots,m})}{M_2(p_{1,\ldots,m})+N_2(p_{1,\ldots,m})}$ is an m-variable PRF of variables $p_{1,\ldots,m}$, then the functions given by

(i) $$F_2(p_{1\ldots,m+1}) = \frac{M_1(p_{1,\ldots,m})p_{m+1}+N_1(p_{1,\ldots,m})}{M_2(p_{1,\ldots,m})+N_2(p_{1,\ldots,m})p_{m+1}}$$

(ii) $$F_3(p_{1,\ldots,m+1}) = \frac{M_1(p_{1,\ldots,m})+N_1(p_{1,\ldots,m})p_{m+1}}{M_2(p_{1,\ldots,m})p_{m+1}+N_2(p_{1,\ldots,m})}$$

are (m+1)-variable RFs of variables $p_{1,\dots,m+1}$.

Proof

If $F_1 = \frac{M_1+N_1}{M_2+N_2}$ is an MPRF, then $\frac{M_1+N_2}{M_2+N_1}$ is also an MPRF. Since the sum of two positive real functions is also a positive real function, $\frac{M_1+N_2}{M_2+N_1} + \frac{1}{p_{m+1}} = \frac{M_1p_{m+1}+N_1+M_2+N_2p_{m+1}}{p_{m+1}(M_2+N_1)}$ is an MPRF. This implies that the numerator polynomial of this function, $M_1p_{m+1}+N_1+M_2+N_2p_{m+1}$ is HPN. Since the ratio of the even part to the odd part of such a polynomial is an RF, $\frac{M_1p_{m+1}+N_1}{M_2+N_2p_{m+1}} = F_2$ is an (m+1)-variable RF. In a similar way, it can be proved that F_3 is also an (m+1)-variable RF.

Theorem 4.2.6

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\dots,m}) = \frac{M_1(p_{1,\dots,m})+N_1(p_{1,\dots,m})}{M_2(p_{1,\dots,m})+N_2(p_{1,\dots,m})}$ to be realizable as the input impedance of a lowpass ladder network with p_i-inductors in the series arms and p_i-capacitors in the shunt arms, beginning with an inductor, and terminated by an (m-1)-variable PRF $Z_0(\Omega_1)$ are that the (m+1)-variable function $Z_1(p_{1,\dots,m+1}) = \frac{M_1(p_{1,\dots,m})p_{m+1}+N_1(p_{1,\dots,m})}{M_2(p_{1,\dots,m})+N_2(p_{1,\dots,m})p_{m+1}}$ satisfies the conditions of Theorem 4.2.4.

Proof

Necessity: If Z is realizable as the impedance function of a p_i-variable lowpass ladder network with all of its transmission zeros at $p_i=\infty$, and terminated by an (m-1)-variable PRF $Z_0(\Omega_i) = \frac{M_a(\Omega_i)+N_a(\Omega_i)}{M_b(\Omega_i)+N_b(\Omega_i)}$, then in this realization replacing the termination by $\frac{M_a(\Omega_i)p_{m+1}+N_a(\Omega_i)}{M_b(\Omega_i)+N_b(\Omega_i)p_{m+1}}$ will give a realization of Z_1. The driving-point function of this structure, by Theorem 4.2.4, satisfies the two conditions of that theorem.

Sufficiency: The (m+1)-variable function Z_1, by Lemma 4.2.1, is an RF function. Since this function satisfies the two conditions of Theorem 4.2.4, it can be realized as the input impedance of a p_i-variable lowpass ladder network with all of its transmission zeros at $p_i=\infty$, and terminated by an m-variable RF of variables $\{\Omega_i, p_{m+1}\}$. In this realization, letting $p_{m+1} = 1$ gives a realization of Z.

Theorem 4.2.7

The necessary and sufficient conditions for an m-variable PRF $Z(p_{1,\ldots,m}) = \frac{M_1(p_{1,\ldots,m})+N_1(p_{1,\ldots,m})}{M_2(p_{1,\ldots,m})+N_2(p_{1,\ldots,m})}$ to be realizable as the input impedance of a highpass ladder network with p_i-capacitors in the series arms and p_i-inductors in the shunt arms, beginning with a capacitor, and terminated by an (m-1)-variable PRF $Z_0(\Omega_i)$ are that the (m+1)-variable function $Z_1(p_{1,\ldots,m+1}) = \frac{M_1(p_{1,\ldots,m})p_{m+1}+N_1(p_{1,\ldots,m})}{M_2(p_{1,\ldots,m})+N_2(p_{1,\ldots,m})p_{m+1}}$ or $Z_2(p_{1,\ldots,m+1}) = \frac{M_1(p_{1,\ldots,m})+N_1(p_{1,\ldots,m})p_{m+1}}{M_2(p_{1,\ldots,m})p_{m+1}+N_2(p_{1,\ldots,m})}$ satisfies the conditions of Theorem 4.2.5.

Proof.

Necessity: If Z is realizable as the impedance function of a p_i-variable highpass ladder network, with all of its transmission zeros at $p_i=0$, and terminated by an (m-1)-variable PRF $Z_0(\Omega_i) = \frac{M_a(\Omega_i)+N_a(\Omega_i)}{M_b(\Omega_i)+N_b(\Omega_i)}$, then in this realization replacing the termination by $\frac{M_a(\Omega_i)p_{m+1}+N_a(\Omega_i)}{M_b(\Omega_i)+N_b(\Omega_i)p_{m+1}}$ will give a realization of Z_1 or Z_2 depending on whether the last element of the p_i-variable ladder network is a shunt or a series branch element. The driving-point function of this structure, by Theorem 4.2.5, satisfies the two conditions of that theorem.

Sufficiency: The (m+1)-variable function Z_1 or Z_2, by Lemma 4.2.1, is a reactance function. If any one of these functions satisfies the two conditions of Theorem 4.2.5, then it can be realized as the input impedance of a p_i-variable highpass ladder network with all of its transmission zeros at $p_i=0$, and terminated by an m-variable RF of variables $\{\Omega_i, p_{m+1}\}$. In this realization, letting $p_{m+1}=1$ gives a realization of Z.

4.3 SYNTHESIS PROCEDURES FOR LADDER EXTRACTION

In this section, some techniques of lowpass and highpass ladder extraction from an RF or a PRF are discussed. Once a multivariable function satisfies the conditions of any one of the theorems 4.2.4 through 4.2.7, a ladder network with all its transmission zeros either at $p_i=0$ or at $p_i=\infty$ can be extracted from the function. In other words, the given function can be expressed in a continued fraction expansion form as given below:

$$Z^*(p_{1,\dots,m}) = z_1(p_i) + \cfrac{1}{y_2(p_i) + \cfrac{1}{z_3(p_i) + \cfrac{1}{y_4(p_i) + \cfrac{}{\ddots \cfrac{1}{z_{n-1}(p_i) + \cfrac{1}{y_n(p_i) + \cfrac{Q(\Omega_i)}{P(\Omega_i)}}}}}}} \qquad (4.3.1)$$

where $z_j(p_j)$ and $y_k(p_i)$ $(j=1,3,\dots,n-1\ ;\ k=2,4,\dots,n)$ are of the form $K_\ell p_i$ or $\frac{1}{K_\ell p_i}$ $(\ell=1,2,\dots,n)$, K_ℓ being a positive constant.

From (4.3.1), it is obvious that the p_i-variable lossless two-port is independent of the terminating load of variables Ω_i, if the load is non-zero and finite. Since the load is reactance or positive real function, it can never become zero or infinity for all $p_k \in \Omega_i$ with $\text{Re}\, p_k > 0$. Hence, for any point in this polydomain, the lossless two-port is independent of the load. Consequently, for the extraction of the p_i-variable lossless ladder network, it would be much simpler to work with the single-variable PRF $Z(k_1,\dots,k_{i-1}, p_i, k_{i+1},\dots,k_m)$; where all k_j's are non-zero positive real constants.

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*Without loss of generality, the degree of $Z(p_{1,\dots,m})$ in the variable p_i is assumed to be even.

In this case the terminating load becomes purely resistive and must be replaced by an appropriate RF or PRF of variables Ω_i in order to obtain a realization of $Z(p_{1,\ldots,m})$. Synthesis procedures for the extraction of a lowpass or a highpass ladder network, with all of its transmission zeros at $p_i=0$ or at $p_i=\infty$, are now outlined.

Extraction of a Lowpass Ladder Network

In order that a p_i-variable lowpass ladder network with all of its transmission zeros at $p_i=\infty$ can be extracted from an m-variable RF or PRF $Z(p_{1,\ldots,m})$, the function must satisfy the conditions of Theorem 4.2.4 or Theorem 4.2.6. Once these conditions are satisfied, synthesis can be carried out by following the steps given below:

(i) Obtain a single-variable PRF $Z_1(p_i) = Z(k_1,\ldots,k_{i-1},p_i,k_{i+1},\ldots,k_m)$, where k_j's $(j=1,\ldots,m\ ;\ j\neq i)$ are non-zero positive constants.

(ii) Realize $Z_1(p_i)$ as a lowpass ladder network with all of its transmission zeros at $p_i=\infty$, and terminated in a resistance R.

(iii) Replace the resistive termination R by the (m-1)-variable RF or PRF $Z(p_1,\ldots,p_{i-1},0,p_{i+1},\ldots,p_m)$. This gives a driving-point impedance realization of $Z(p_{1,\ldots,m})$.

Extraction of a Highpass Ladder Network

In order that a p_i-variable highpass ladder network with all of its transmission zeros at $p_i=0$ can be extracted from an m-variable

RF and PRF $Z(p_{1,\dots,m})$, the function must satisfy the conditions of Theorem 4.2.5 or Theorem 4.2.7. Once these conditions are satisfied, synthesis can be carried out by following the steps given below:

(i) Obtain a single-variable PRF $Z_1(p_i) = Z(k_1,\dots,k_{i-1},p_i,k_{i+1},\dots,k_m)$, where k_j's ($j=1,\dots,m$, $j\neq i$) are non-zero positive constants.

(ii) Realize $Z_1(p_i)$ as a highpass ladder network with all of its transmission zeros at $p_i=0$, and terminated in a resistance R .

(iii) Replace the resistive termination R by the $(m-1)$-variable RF or PRF $Z(p_1,\dots,p_{i-1},\infty,p_{i+1},\dots,p_m)$. This gives a driving-point impedance realization of $Z(p_{1,\dots,m})$.

Example 4.3.1

Consider the TRF given below:

$$Z(p_1,p_2) = \frac{N(p_1,p_2)}{D(p_1,p_2)} = \frac{\begin{matrix}108p_1^5p_2^3+54p_1^5p_2+324p_1^4p_2^2+540p_1^3p_2^3+54p_1^4 \\ +270p_1^3p_2+540p_1^2p_2^2+324p_1p_2^3+90p_1^2 \\ +162p_1p_2+108p_2^2+18\end{matrix}}{\begin{matrix}78p_1^4p_2^3+39p_1^4p_2+234p_1^3p_2^2+360p_1^2p_2^3+39p_1^3+180p_1^2p_2 \\ +300p_1p_2^2+108p_2^3+50p_1+54p_2\end{matrix}}$$

Degree in the variable p_1 of the numerator polynomial of $Z(p_1,p_2)$ is one higher than that of the denominator, and the function can be expressed in the form of (4.2.16), where $p_i=p_2$. The numerator of the partial derivative of $Z(p_1,p_2)$ with respect to p_2 as given by

$$Nu\ Z_{p_2}(p_1,p_2) = 108p_2^4+72p_2^3+36p_2^2+12p_2+3$$

is independent of p_1. Thus, $Z(p_1,p_2)$ satisfies the two conditions of Theorem 4.2.4, and it can be realized as the input impedance of a p_1-variable lowpass ladder network with all of its transmission zeros at $p_1=\infty$, and terminated in a p_2-variable RF. First, the function $Z_1(p_1) = Z(p_1,1)$ as given by

$$Z_1(p_1) = \frac{162p_1^5+378p_1^4+810p_1^3+630p_1^2+486p_1+126}{117p_1^4+273p_1^3+540p_1^2+350p_1+162}$$

is realized as a resistively-terminated lowpass ladder network. Next, in order to obtain a realization of $Z(p_1,p_2)$, the resistive termination of $R = \frac{7}{9}$ ohm is replaced by the RF $Z(0,p_2) = \dfrac{6p_2^2+1}{6p_2^3+3p_2}$ (Fig. 4.3).

4.4 SUMMARY

In this chapter, some partial derivative properties of a class of driving-point functions have been discussed. First, a general result concerning the partial derivative of the impedance function of a network which is a cascade of $(m-1)$ lossless two-ports of variables p_1 to p_{m-1}, and terminated by a reactance or positive real function of variable p_m is obtained. It has been shown that for such networks, there exists a relationship between the numerators of the even part and the partial derivative of the impedance function. The partial derivative

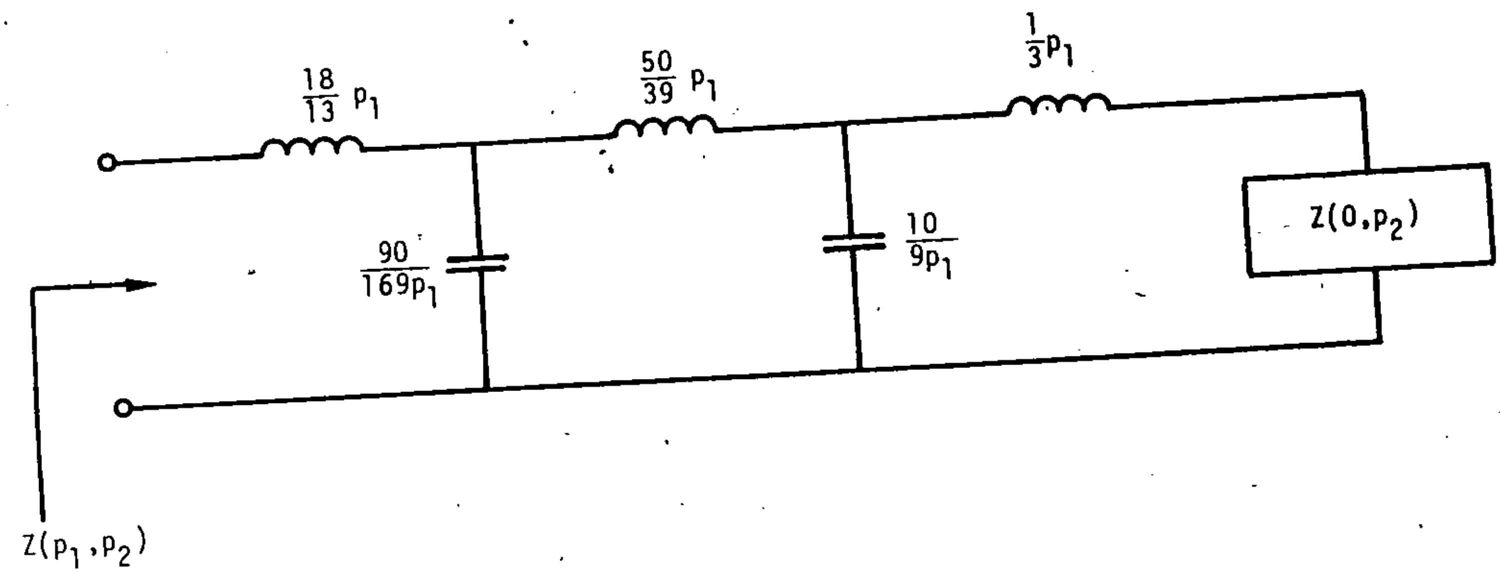


FIG. 4.3. A Realization of the Two-Variable Reactance Function $Z(p_1,p_2)$ of Example 4.3.1.

result is then examined for the cases where one or more lossless two-ports may be lowpass or highpass ladder networks with all of their transmission zeros at the origin or at infinity. Next, using these partial derivative results, necessary and sufficient conditions under which an m-variable RF or PRF can be realized as the impedance function of a p_i-variable lowpass or highpass ladder network, with all of its transmission zero at $p_i=0$ or at $p_i=\infty$, and terminated by a reactance or a positive real function of Ω_i are established. Some synthesis procedures are also discussed for the extraction of these kinds of ladder networks.

CHAPTER V

LADDER REALIZATION OF A CLASS OF TWO-DIMENSIONAL VOLTAGE TRANSFER FUNCTIONS WITH APPLICATION TO WAVE DIGITAL FILTERS

5.1 INTRODUCTION

In recent years, a great deal of attention has been paid to the development of two-dimensional (2-D) digital filtering because of its application in the fields of picture processing and geophysics [54], [55]. Design techniques of such filters, however, encounter the problems of stability and realization.

Instability arises in the case of an infinite impulse response type of 2-D digital filter which is usually implemented in a recursive fashion. The transfer function of such a filter is a rational function of two variables z_1 and z_2. The stability, in the bounded-input bound-output sense, of the 2-D digital filter is guaranteed if and only if the transfer function has no poles in $\{(z_1,z_2) : |z_1| \geq 1 , |z_2| \geq 1\}$, and no non-essential singularities of the second kind in the same region except possibily on $\{(z_1,z_2) : |z_1| = 1 , |z_2| = 1\}$ [56]. This introduces difficulties in testing for the stability and the existing techniques [57], [58] are often tedious to use. It has been reported [59], [60] that a stable 2-D digital lowpass transfer function can be obtained to approximate a circularly symmetric response starting with a two-variable passive analog network and then using double bilinear z-transformation.

The other problem in 2-D digital filtering is the realization techniques. Implementation of 2-D filters in the direct form is not desirable because of the poor sensitivity associated with the finite word-lengths of multiplier coefficients. More recently [60], a technique has been developed whereby starting with a doubly terminated lossless ladder network in two-variables, each series and shunt arm element is replaced by the corresponding wave digital two-port and the individual two-ports are cascaded to obtain the overall digital realization. This technique ensures the sensitivity properties of the analog domain.

It has been reported [59] that in the case of a doubly-terminated lossless network a better approximation of a circularly symmetric lowpass response is obtained when p_1- and p_2-variable networks are cascaded than in the case of any other combination of these variables.

This chapter [61], [62] deals with the realization of the voltage transfer function of a resistively-terminated two-port which is a cascade of p_1- and p_2-variable lossless two-ports (Fig. 5.1), such that each two-port has all of its transmission zeros either at $p_i=0$ or at $p_i=\infty$ $(i=1,2)$. This analog network can then be used to obtain a digital realization by using the technique described in [60].

5.2 REALIZATION OF A TWO-VARIABLE RESISTIVELY-TERMINATED LOSSLESS LADDER NETWORK WITH ITS TRANSMISSION ZEROS AT THE ORIGIN OR AT INFINITY

In this section conditions are derived for the realization of a rational function as the voltage transfer function of a resistively-terminated lowpass, highpass or bandpass ladder network which is a cascade

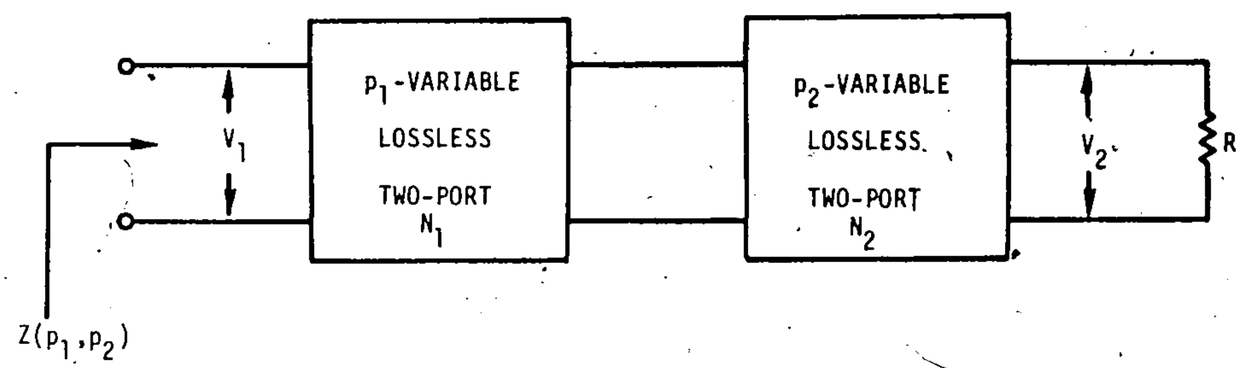


FIG. 5.1. A Resistively-Terminated Cascade Structure of p_1- and p_2-Variable Lossless Two-Ports.

of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros at $p_i=0$ or $p_i=\infty$ $(i=1,2)$. The technique is based on transforming the problem of a transfer function realization to that of a driving-point function realization.

Lemma 5.2.1

If $D(p_1,p_2) = m_1(p_1)P(p_2)+n_1(p_1)Q(p_2)$, where $m_1(p_1)$ and $n_1(p_1)$ are respectively even and odd polynomials is p_1, and $P(p_2)$ and $Q(p_2)$ are polynomials in p_2, is a strictly Hurwitz polynomial (SHP) in the two-variables p_1 and p_2, then

(i) $m_1(p_1)+n_1(p_1)$ is a single-variable SHP in p_1.

(ii) $\dfrac{P(p_2)}{Q(p_2)}$ is an SPRF in p_2.

Proof

Let $P(p_2)$ and $Q(p_2)$ be expressed as $[M_a(p_2)+N_a(p_2)]$ and $[M_b(p_2)+N_b(p_2)]$ respectively, where $M_a(p_2)$ and $M_b(p_2)$ are even, and $N_a(p_2)$ and $N_b(p_2)$ are odd polynomials in p_2. The polynomial $D(p_1,p_2)$ can now be expressed as:

$$D(p_1,p_2) = m_1(p_1)M_a(p_2)+n_1(p_1)N_b(p_2)+m_1(p_1)N_a(p_2)+n_1(p_1)M_b(p_2) \quad (5.2.1)$$

From (5.2.1), a TRF can be formed as:

$$F(p_1,p_2) = \frac{m_1(p_1)M_a(p_2)+n_1(p_1)N_b(p_2)}{m_1(p_1)N_a(p_2)+n_1(p_1)M_b(p_2)} \quad (5.2.2)$$

(i) If $p_2=c_2$ is a positive constant, then $F(p_1,c_2)$ is an SPRF. Thus, from (5.2.2) $[m_1(p_1)M_a(c_2)+n_1(p_1)N_b(c_2)]$ is a Hurwitz polynomial, that is, $\frac{M_a(c_2)}{N_b(c_2)} \cdot \frac{m_1(p_1)}{n_1(p_1)}$ or $\frac{m_1(p_1)}{n_1(p_1)}$ is a reactance function. Now the polynomials $m_1(p_1)$ and $n_1(p_1)$ can have only the imaginary axis zeros. But they cannot have common factors because in that case $D(p_1,p_2)$ will not be an SHP. Hence $m_1(p_1)+n_1(p_1)$ is a single-variable SHP in p_1.

(ii) If $p_1=c_1$ is a positive constant, then the function given by

$$F(c_1,p_2) = \frac{m_1(c_1)M_a(p_2)+n_1(c_1)N_b(p_2)}{m_1(c_1)N_a(p_2)+n_1(c_1)M_b(p_2)}$$

is an SPRF. This implies that function given by

$$F_1(c_1,p_2) = \frac{m_1(c_1)[M_a(p_2)+N_a(p_2)]}{n_1(c_1)[M_b(p_2)+N_b(p_2)]}$$

$$= \frac{m_1(c_1)}{n_1(c_1)} \cdot \frac{P(p_2)}{Q(p_2)}$$

is also an SPRF. Hence, $\frac{P(p_2)}{Q(p_2)}$ is an SPRF.

<u>Theorem 5.2.1</u>

The necessary and sufficient conditions for a two-variable rational function $T(p_1,p_2)$ to be the voltage transfer function of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros at infinity, are that,

(i) the function is expressible in the form:

$$T(p_1,p_2) = \frac{K}{D(p_1,p_2)} = \frac{K}{m_1(p_1)P(p_2)+n_1(p_1)Q(p_2)} \quad (5.2.3)$$

where K is a positive constant and $D(p_1,p_2)$ is a two-variable SHP, and

(ii) the numerator of the even part of $P(p_2)/Q(p_2)$ is a positive constant.

<u>Proof</u>

The necessity is obvious. Hence, only the sufficiency is proved here. Since $D(p_1,p_2) = m_1(p_1)P(p_2)+n_1(p_1)Q(p_2)$ is a two-variable SHP, by Lemma 5.2.1, $m_1(p_1)+n_1(p_1)$ is a single-variable SHP. Construct a single-variable PRF, $\frac{m_1(p_1)+n_1(p_1)}{m_2(p_1)+n_2(p_1)}$, such that $m_1(p_1)m_2(p_1)$ $-n_1(p_1)n_2(p_1) = K_1$, where K_1 is a positive real constant. Note that the construction of such a function is always possible. Next, construct a two-variable function given below:

$$Z(p_1,p_2) = \frac{m_1(p_1)P(p_2)+n_1(p_1)Q(p_2)}{m_2(p_1)Q(p_2)+n_2(p_1)P(p_2)}$$

Now, $\frac{m_1(p_1)+n_1(p_1)}{m_2(p_1)+n_2(p_1)}$ is an SPRF in p_1, and by Lemma 5.2.1, $\frac{P(p_2)}{Q(p_2)}$ is also an SPRF. Hence, by Theorem 2.2.1, $Z(p_1,p_2)$ is a two-variable PRF. Moreover, $Z(p_1,p_2)$ satisfies the two conditions of Corollary 3.2.1, and therefore, it can be realized as the input impedance of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros at infinity. This network realizes the transfer function given by (5.2.3) within a multiplicative constant.

Theorem 5.2.2

The necessary and sufficient condition for a two-variable rational function $T(p_1,p_2)$ to be the voltage transfer function of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros at the origin, is that after making the transformation $p_i \to \frac{1}{p_i}$ $(i=1,2)$ in the given function, the resulting function satisfies the conditions of Theorem 5.2.1.

Theorem 5.2.3

The necessary and sufficient condition for a two-variable rational function $T(p_1,p_2)$ to be the voltage transfer function of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, and has all of its transmission zeros at $p_1=\infty$ and $p_2=0$, is that after making the transformation $p_2 \to \frac{1}{p_2}$ in the given function, the resulting function satisfies the conditions of Theorem 5.2.1.

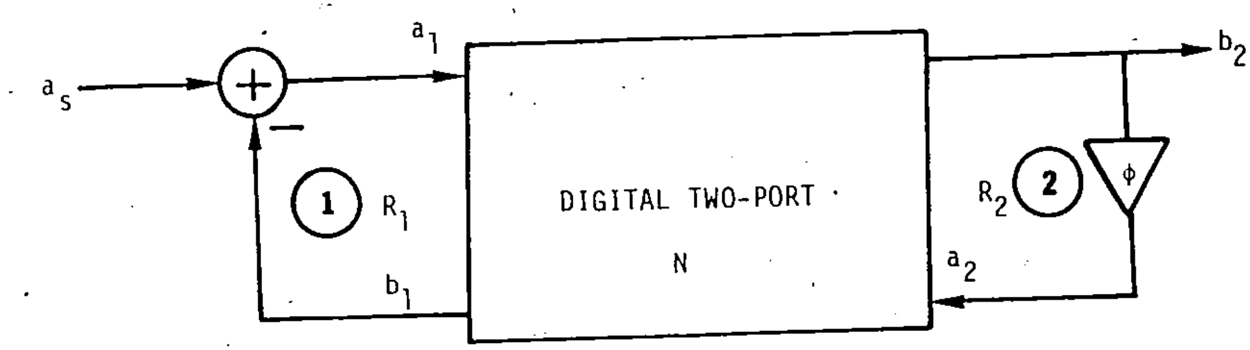
Theorems 5.2.2 and 5.2.3 can be proved readily. Note that Theorem 5.2.3 covers only a special class of resistively-terminated bandpass ladder networks, that is, ladder networks having all of their transmission zeros at $p_1=\infty$ and $p_2=0$.

5.3 APPLICATION TO TWO-DIMENSIONAL WAVE DIGITAL FILTERS

The class of digital transfer functions for which the corresponding analog transfer functions satisfy the conditions of Theorem 5.2.1, 5.2.2 or 5.2.3 may now be realized by using the approach given in [60]. Starting with the analog realization of $T(p_1,p_2) = \frac{V_2}{V_1}$ as a resistively-terminated lossless two-port, obtain the corresponding digital realization of $H(z_1,z_2) \triangleq \frac{V_2}{V_1}$ with $p_i = \frac{z_i-1}{z_i+1}$ $(i=1,2)$ by using Fig. 5.2, where the digital two-port N is obtained from the corresponding analog network as follows: each of the series and shunt arm elements is replaced by the corresponding wave digital two-port, and the individual two-ports are then cascaded to obtain the overall digital two-port N. The class of analog ladder networks realized in Section 5.2 contains series and shunt inductors or capacitors. The digital two-ports corresponding to these series and shunt elements are summarized in Table 5.1. The technique of realization is illustrated by considering the following example.

Example 5.3.1

Let an analog transfer function in two variables be given as:

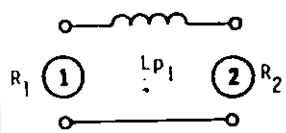
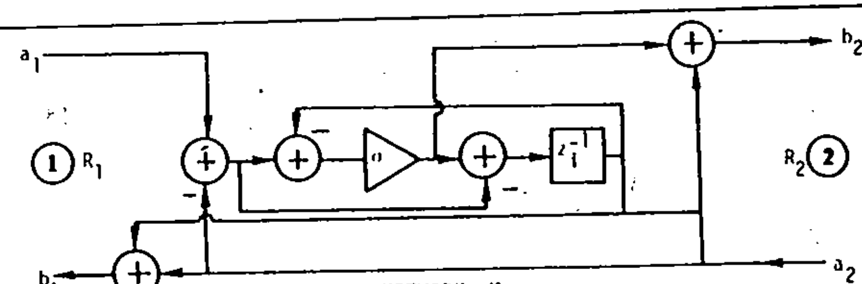
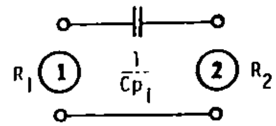
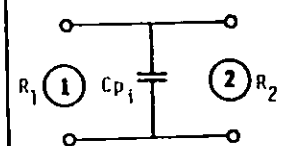
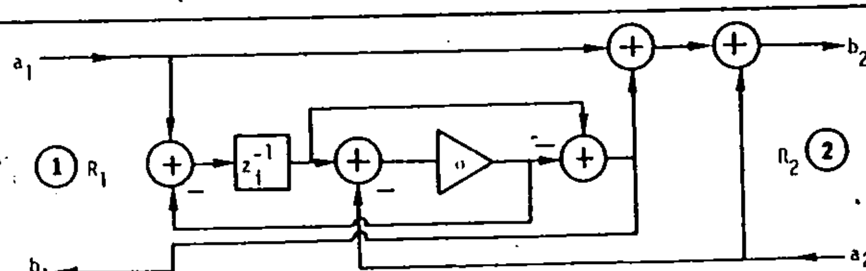
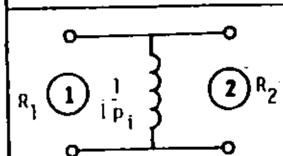


$$\phi = \frac{R-R_2}{R+R_2} \quad , \quad \frac{b_2}{a_s} = \frac{1}{1+\phi} H(z_1,z_2)$$

$$H(z_1,z_2) \triangleq \frac{V_2}{V_1} \text{ with } p_i = \frac{z_i-1}{z_i+1} \quad , \ i=1,2$$

FIG. 5.2. A Wave Digital Realization of $\frac{b_2}{a_s} = \frac{1}{1+\phi} H(z_1,z_2)$ Corresponding to an Analog Realization of $\frac{V_2}{V_1} = T(p_1,p_2)$.

TABLE 5.1 - A Set of Digital Two-Port Realizations Corresponding to the Series and the Shunt Elements of a Two-Variable Analog Ladder Network

ELEMENT	CORRESPONDING WAVE DIGITAL TWO-PORT	RELATIONS
	 NETWORK N_a	$\alpha = R_2/R_1$ $R_1 = R_2+L$
	Same as N_a with z_i^{-1} replaced by $-z_i^{-1}$ NETWORK N_b	$\alpha = R_2/R_1$ $R_1 = R_2+1/C$
	 NETWORK N_c	$\alpha = G_2/G_1$ $G_1 = G_2+C$
	Same as N_c with z_i^{-1} replaced by $-z_i^{-1}$ NETWORK N_d	$\alpha = G_2/G_1$ $G_1 = G_2+\frac{1}{L}$

$$T(p_1,p_2) = \frac{p_2^2}{p_1^2p_2^2+\sqrt{2}p_1^2p_2+\sqrt{2}p_1p_2^2+p_1^2+p_1p_2+p_2^2+\sqrt{2}p_2+1} \tag{5.3.1}$$

The 2-D digital transfer function is obtained from the above transfer function by using double bilinear z-transformation, $p_i = \frac{z_i-1}{z_i+1}$, $i=1,2$.

$$H(z_1,z_2) = \frac{(z_1+1)^2(z_2-1)^2}{\begin{array}{l}(z_1-1)^2(z_2-1)^2+\sqrt{2}(z_1-1)^2(z_2-1)(z_2+1)\\ +\sqrt{2}(z_1-1)(z_1+1)(z_2-1)^2+(z_1-1)^2(z_2+1)^2\\ +(z_1-1)(z_1+1)(z_2-1)(z_2+1)+(z_1+1)^2(z_2-1)^2\\ +\sqrt{2}(z_1+1)^2(z_2-1)(z_2+1)+(z_1+1)^2(z_2+1)^2\end{array}} \tag{5.3.2}$$

The function given by (5.3.1) cannot be a lowpass or a highpass function with all of its transmission zeros either at $p_i=\infty$ or $p_i=0$ $(i=1,2)$. However, after making the transformation $p_2 \to \frac{1}{p_2}$, the resulting function satisfies the conditions of Theorem 5.2.1. Hence, the given function $T(p_1,p_2)$ can be realized as the transfer function of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, and has all of its transmission zeros at $p_1=\infty$ and $p_2=0$. The analog realization is shown in Fig. 5.3. Now, starting with this singly-terminated lossless ladder network in two variables, each of the series and shunt arm elements is replaced by the corresponding wave digital two-port of Table 5.1, and the individual two-ports are then cascaded to obtain a digital realization of the transfer function $H(z_1,z_2)$ as given by 5.3.2. Fig. 5.4 shows this digital realization.

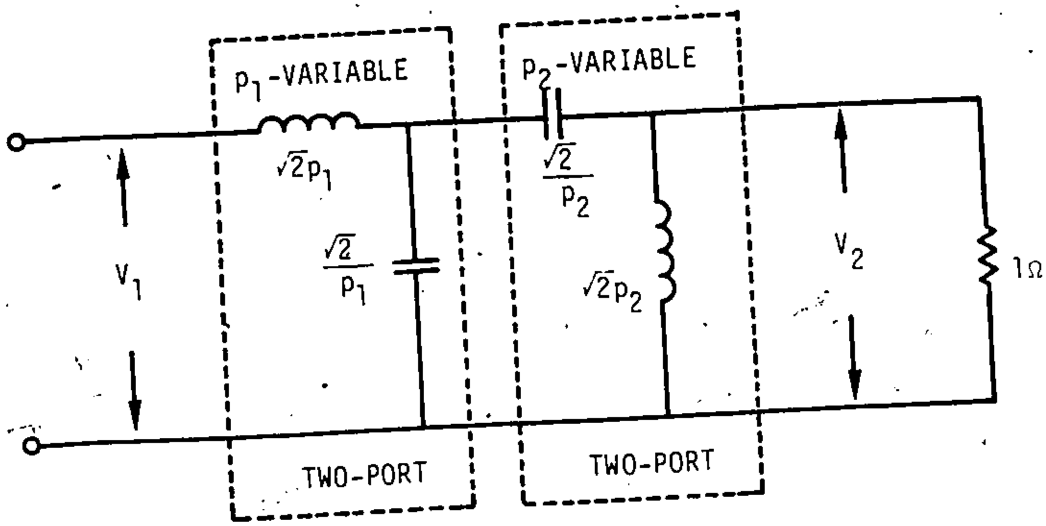
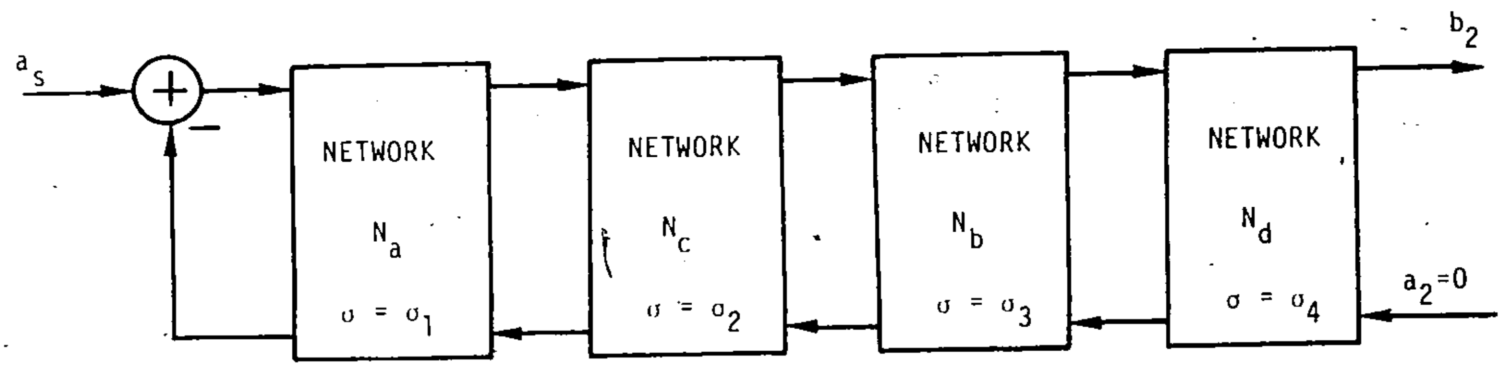


FIG. 5.3. A Voltage Transfer Function Realization of $T(p_1,p_2)$ of Example 5.3.1.



$$\sigma_1 = \frac{\sqrt{2}}{2\sqrt{2}+1} \qquad \sigma_2 = \frac{1}{\sqrt{2}+1}$$

$$\sigma_3 = \frac{1}{\sqrt{2}+2} \qquad \sigma_4 = \frac{\sqrt{2}}{\sqrt{2}+1}$$

FIG. 5.4. A Wave Digital Realization of the Transfer Function $H(z_1,z_2)$ of Example 5.3.1.

5.4 SUMMARY

Necessary and sufficient conditions have been derived for the realization of a two-variable rational function as the voltage transfer function of a resistively-terminated lowpass, highpass or bandpass ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros either at the origin or at infinity. The process of realization has been carried out by transforming the problem of a transfer function realization, under the given conditions, to the problem of a two-variable positive real impedance function realization. It has been shown that from these analog realizations, 2-D digital realizations can be obtained by using the wave digital technique proposed in [60].

CHAPTER VI

CONCLUSIONS

6.1 SUMMARY

The thesis has investigated the realization problem of multivariable structures which are cascade of single-variable lumped lossless two-port networks terminated by positive real impedances. The technique of realization has been based on the decomposability of multivariable functions into simpler functions and on the z-parameter characterization of passive lumped lossless two-ports.

Condition for the realization of an m-variable PRF of arbitrary degree in each variable as the input impedance of a p_i-variable lossless two-port terminated by an impedance function of the remaining (m-1) variables has been derived using the technique of single-variable Darlington synthesis. Since the terminating load of the lossless two-port is also an MPRF, it has been shown that by repeatedly using the result of single-variable cascade-extraction, the given function can be tested for a complete cascade realization. Each lossless two-port of the cascade structure is non-reciprocal unless the numerator of the even part of the impedance function has a product separable perfect square factor in the variable of the lossless two-port. Some alternative conditions have also been proposed for the cascade realization. In some cases, an MPRF may be cascade-expressible in more than one variable. It has been shown that in such cases the choice of one variable over the others is not to be preferred because after the extraction of lossless two-ports in all such variables, the terminating loads are the same within constant multiples regardless of the order in which the lossless two-ports are extracted.

The result of cascade extraction of single-variable lossless two-port from a multivariable impedance function has then been applied for the realization of a cascade of single-variable ladder networks, each in a distinct variable, and having all of its transmission zeros either at the origin or at infinity. For such a network the choice of the variable in which the first lossless two-port ought to be extracted is determined by the difference of the highest or the lowest degree of a variable between the numerator and denominator polynomials of the impedance function. Realizability condition has also been derived for a resistively-terminated ladder structure where each lossless two-port is a Fujisawa-type lowpass ladder network.

Using an even part condition, realization for a class of MPRF as the impedance functions of resistively-terminated lowpass or highpass ladder networks with all of their transmission zeros either at the origin or at infinity have been obtained. The reactance elements of these ladder networks are not grouped together to follow some predetermined sequence.

Some general properties of the impedance function of a multi-variable cascade structure of single-variable lossless two-ports with or without a resistive termination have been examined from the partial derivative point of view. It has been shown that for such a network there exists a relationship between the partial derivative of the impedance function and the even part of individual two-ports when they are terminated by positive resistors. A more detailed study is made when any one of the lossless two-ports assumes the structure of a ladder network. Specifically, conditions making use of partial derivatives have been derived for the cascade extraction of a single-variable lossless ladder network with all of its transmission zeros either at the origin or at infinity.

Finally, utilizing the conditions for the impedance function realization, necessary and sufficient conditions have been established for the realization of a two-variable rational function as the voltage transfer function of a resistively-terminated ladder network which is a cascade of p_1- and p_2-variable lossless two-ports, each two-port having all of its transmission zeros either at the origin or at infinity. It has been shown that the realization of a class of digital transfer functions can be obtained from these analog ladder networks by employing a technique of wave digital filter design.

6.2 SUGGESTIONS FOR FURTHER INVESTIGATION

The work of this thesis leads to the following problems for further investigation.

(i) The thesis has been mainly concerned with the synthesis of singly-terminated multivariable structures which are cascade of single-variable lossless two-ports. Another problem of practical significance which can be studied is the realization of doubly-terminated multivariable structures.

(ii) In Chapter II, it has been shown that the condition for cascade extraction of the p_i-variable lumped lossless two-port from an m-variable PRF is that the function be cascade-expressible in the varaible p_i. However, this would require as many as m trials to determine whether the given function is cascade-expressible in any one of the m variables. It would be desirable to construct a simple testing procedure to determine in which variable the function is cascade-expressible.

(iii) A problem suited for the synthesis of a cascade structure of lumped lossless two-ports and UEs would be the realization of an m-variable PRF as the input impedance of the p_i-variable lumped lossless two-port terminated by an impedance function in which the degree of the variable p_i is lower than that in the given function.

(iv) In Chapter III, using an even part condition, an MPRF of the first degree in all variables except one was realized as the impedance function of a resistively-terminated lowpass or highpass ladder network. The possibility of extending this realization to impedance functions of arbitrary degree in each variable can be explored.

(v) In Chapter V, the conditions for the realization of the transfer function of a resistively-terminated cascade of p_1- and p_2-variable ladder networks, each network having all of its transmission zeros either at the origin or at infinity, were obtained. The problem can further be investigated to include ladder networks with other finite imaginary axis transmission zeros.

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