Tests of Homogeneity of Variances
of Normal Linear Models

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ABSTRACT

TESTS OF HOMOGENEITY OF VARIANCES
OF NORMAL LINEAR MODELS

A.B.M. Nur Enayet Talukder

Many test procedures have been proposed in the literature to detect the heteroscedasticity of statistical populations. Each of the tests has its own assumptions regarding the population distributions. Some of the tests were developed with a view to detect heteroscedasticity in linear models. These tests can also be used to test the equality of variances of normal populations with common mean. Since the classical methods of 'optimum' test constructions did not work well in this situation, researchers developed these tests through some ad hoc approach. In this thesis attempt is made to find a Bayesian solution to test equality of variances of \( k(\geq 2) \) normal populations with common location parameters and to extend it for detecting heteroscedasticity in \( k(\geq 2) \) linear models with common regression parameters.

In Chapter II of this thesis we consider the Bayesian solution for testing equality of variances of two normal populations with common means. This method is compared with the classical F-test in terms of highest posterior density intervals and is found to have shorter widths. The Bayesian solution is extended for testing equality of variances in two linear models with common regression parameters in Chapter III, the method is applied to a real life problem (Booth and DeWitt's (1960) data) and is observed to give better result than F-test. The extension of
the Bayesian method discussed in Chapter II and III, for the case of more than two normal populations is difficult. Finally, a 'likelihood ratio type' test, proposed by Chaubey (1980) is considered for the detection of heteroscedasticity in \( k \geq 2 \) linear models. This test is studied in detail for a particular case to investigate its properties. A chi-square approximation, which seems to be reasonable is proposed for the test statistic.
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CHAPTER I

INTRODUCTION

1.1 Introduction

Many statistical theories have been developed on the basis of some assumptions on the nature of the random variation present in a set of data or among similar statistical processes, e.g. the theory of linear models, time series analysis, analysis of variance. For an efficient statistical analysis it is necessary to test the assumptions before applying the method of analysis relying on them.

Sometimes it may be of direct interest to test the equality of variances of different populations. Many statistical methods have been developed to test such a hypothesis under specified distributional structure, such as F-test for testing equality of variances of two normal populations, Bartlett’s test and Hartley’s test for testing equality of variances of $k$ normal populations. All these tests assume that the data have come from normal populations with different location parameters. However, there are situations where data may come from populations with the same location parameter, e.g. in making inference about the precision of two measuring instruments (see Grubbs, 1948). Under such situations how can we test the equality of the variances ‘efficiently’? The word ‘efficiently’ means the optimality of the test. Many researchers have worked on this problem such as Chaubey (1980,1981), Harrison and McCabe (1979), Theil (1971) and Goldfeld and Quandt (1965),
but most of these consider a general linear model set-up. Most of the
tests are suggested on an ad hoc basis. Some of the tests are very
complicated in terms of their distributions and/or their computations
while others may not use all the available information.

In this thesis an attempt is made to find tests which use all
the information present in the sample data and also have some theoretical basis. With this spirit we investigate a Bayesian solution of
the problem. Bayesian solution is difficult for k(>2) population
case. For such a situation we consider the proposal of Chaubey (1980)
of a 'likelihood ratio type test' which is computationally simple and
uses all the sample data. In the following section (see 1.2) we dis-
cuss some preliminaries of the Bayesian Inference and Section 1.3
presents the detailed outline of the thesis.

1.2 Some Preliminaries of Bayesian Inference

We here provide some preliminaries of Bayesian Inference. How-
ever for a detailed treatment the reader is referred to the excellent
text by Box and Tiao (1973).

In Bayesian inference, the parameter \( \theta \) about which the inference
is to be made, is considered as a random variable with a specified
distribution. This distribution is called the prior distribution of
\( \theta \). Given the prior distribution \( p(\theta) \), the corresponding posterior
distribution \( p(\theta|Y) \) of \( \theta \) given the observed data \( (Y) \) is obtained
as

\[
p(\theta|Y) \propto L(\theta|Y) \, p(\theta)
\]

(1.2.1)

where \( L(\theta|Y) \) is the likelihood function of \( \theta \) given the data.
\[ Y = (y_1, y_2, \ldots, y_n) \]. The inference about \( \theta \) is then based on \( p(\theta | Y) \).

From (1.2.1) it is clear that the selection of prior for \( \theta \) is important in Bayesian analysis. Some guidelines for the choice of a prior distribution have been put forward by Raiffa and Schlaifer (1961).

However, Sir Harold Jeffreys (1961) first gave a rule for the choice of 'non-informative' prior. According to Jeffreys, the prior distribution for a parameter \( \theta \) is approximately non-informative if it is taken proportional to the square root of Fisher's information measure. He justified this rule on the grounds of its invariance under parameter transformation.

In Bayesian analysis usually interval estimation about the parameter \( \theta \) is done through the highest posterior density (H.P.D.) intervals of a specified probability content. In the next section we give the definition of H.P.D. interval and an algorithm to construct such an interval.

1.2.1 Highest Posterior Density (HPD) Intervals

**Definition:** The interval \((\theta_L, \theta_U)\) is said to be highest posterior density interval of content \((1 - \alpha)\) for \( \theta \) if

i) \( \Pr(\theta \in (\theta_L, \theta_U)) > \Pr(\theta \in (\theta_L, \theta_U)) \)

ii) \( (\theta_U - \theta_L) \) is minimum.

\( \theta_U \)

and iii) \( \int_{\theta_L}^{\theta_U} p(\theta | Y) d\theta = 1 - \alpha \).

The above conditions lead us to minimize \((\theta_U - \theta_L)\) subject to the constraint

\[ \int_{\theta_L}^{\theta_U} p(\theta | Y) d\theta = 1 - \alpha \]
To get the required interval we minimize the following function with respect to \( \theta_L, \theta_U \):

\[
G(\theta_L, \theta_U, \lambda) = (\theta_U - \theta_L)^2 + \lambda \left( \int_{\theta_L}^{\theta_U} p(\theta|Y) d\theta - (1 - \alpha) \right) \tag{1.2.2}
\]

Taking the derivative of (1.2.2) and setting to zero we get,

\[
\frac{\partial G}{\partial \theta_U} = 1 + \lambda \frac{\partial}{\partial \theta_U} p(\theta_U|Y) = 0,
\]

and

\[
\frac{\partial G}{\partial \theta_L} = -1 - \lambda \frac{\partial}{\partial \theta_L} p(\theta_L|Y) = 0, \tag{1.2.3}
\]

where \( p(\theta|Y) \) is the posterior distribution function, which imply,

\[
p(\theta_U|Y) = p(\theta_L|Y). \tag{1.2.4}
\]

Thus, \((\theta_L, \theta_U)\) is an HPD interval of content \((1 - \alpha)\) for \( \theta \) if

\[
\int_{\theta_L}^{\theta_U} p(\theta|Y) d\theta = 1 - \alpha \tag{1.2.5}
\]

and

\[
p(\theta_L|Y) = p(\theta_U|Y). \tag{1.2.6}
\]

The condition (1.2.6) implies that if we have a unimodal posterior pdf we choose the interval around the mode. Choosing the interval \((\theta_L, \theta_U)\) in this way, satisfies the conditions (i), (ii) and (iii), hence \((\theta_L, \theta_U)\) satisfying (1.2.5) and (1.2.6) is a HPD interval.

HPD interval is not invariant under parameter transformations.

For this reason we consider HPD interval for a function of \( \theta \) with locally uniform non-informative prior. The interval for \( \theta \) based on this HPD interval is called the "standardized" HPD interval.
1.2.2 An Algorithm To Find The HPD-Intervals

Let \((\theta_L, \theta_U)\) be the HPD interval of content \((1 - \alpha)\) for \(\theta\) based on \(p(\theta|Y)\). That is \((\theta_L, \theta_U)\) is such that

\[
\int_{\theta_L}^{\theta_U} p(\theta|Y) d\theta = 1 - \alpha
\]

and the value of \(p(\theta|Y)\) is equal at \(\theta = \theta_L\) and \(\theta = \theta_U\). However, to get the standardized HPD intervals (in our case) we have to consider the posterior pdf of \(u = \log \theta\) given \(Y\). The standardized HPD interval is computed from \((\theta_{LS}, \theta_{US})\) where \((\theta_{LS}, \theta_{US})\) are such that

\[
\begin{align*}
\int_{\ln\theta_{US}}^{\ln\theta_{US}} p(u|Y) du &= 1 - \alpha \\
\int_{\ln\theta_{LS}}^{\ln\theta_{LS}} p(u|Y) du &= 1 - \alpha
\end{align*}
\]

and

\[
p(u|Y)_{u=\log\theta_{LS}} = p(u|Y)_{u=\log\theta_{US}}
\]

The following iterative procedure can be adopted to find \((\theta_L, \theta_U)\).

Let \(\hat{\theta}\) be the mode of \(p(\theta|Y)\) then the equation

\[
p(\theta|Y) = cp(\theta|Y); \quad 0 < c < 1
\]

has two roots (Fig.1.2) for every \(c\) in the interval \((0,1)\). Let us call them \(\hat{\theta}_{1c}, \hat{\theta}_{2c}\).

(Fig. 1.2)
We vary $c$ over the range $(0, 1)$ and choose that value of $c$ for which the two solutions $\theta_{1c}, \theta_{2c}$ of the equation (1.2.7) are such that

\[
\int_{\theta_{1c}}^{\theta_{2c}} p(\theta|y) \, d\theta = 1 - \alpha.
\]

The interval $(\theta_{1c}, \theta_{2c})$ is the required HPD interval of content $(1 - \alpha)$ for $\theta$.

1.3 Outline Of The Thesis

As discussed in the beginning of this chapter that our interest is in testing equality of the variances of normal populations, we consider the problem of testing equality of variances of two normal populations with same mean $\mu$, in Chapter II, where some earlier proposed tests are briefly described and a Bayesian solution is proposed for the problem. In Chapter III we consider the problem of testing equality of variances of two populations where the means of the populations depend on some common factors in a linear set up. We describe the tests proposed by Chaubey (1981) and Theil (1971) in this chapter. A Bayesian solution for this case is proposed and compared with $F$-test. We consider the problem of testing equality of $k \geq 2$ linear models in Chapter IV. In this chapter we describe the tests proposed by Goldfeld and Quandt, Harrison and McCabe (1979). A proposal is made to use Bayesian analysis using the method described in Chapter III by grouping the data by some criteria as was done by Goldfeld and Quandt (1965), and Harrison and McCabe (1979). But for $k(> 2)$ populations, a straightforward Bayesian solution is difficult and we propose a 'likelihood ratio type' test for this case.
approximation for the distribution of the test criterion in terms of a chi-square distribution is proposed. A special case is investigated in detail to study the properties of the test.
CHAPTER II

TESTING EQUALITY OF VARIANCES OF TWO
NORMAL POPULATIONS WITH EQUAL MEANS

2.1 Introduction

Testing equality of variances of two populations with equal means commonly arises in a situation described as follows. Let \( \mu \) be the true length of a rod which we wish to measure using two measuring instruments. We want to compare the precision of the instruments i.e. we want to test the hypothesis \( H_0: \sigma_1^2 = \sigma_2^2 \), where \( \sigma_1^2 \) and \( \sigma_2^2 \) are the variances of the two instruments, using the sample data (see also Grubbs, 1948). The more general situation, where the observations depend on other factors, is a particular case of testing homoscedasticity in a general linear model (see Theil, 1971, pp. 214-218). This general case will be discussed in Chapter III. However, because of the importance of the two sample problem we discuss it in detail in this chapter.

There are situations where we have two independent samples from two normal populations with different means \( \mu_1 \) and \( \mu_2 \) respectively, e.g. in analysis of variance, when the experimental error appears to vary considerably from treatment to treatment, so that a preliminary test for the homogeneity of the error variances is required before we proceed to investigate the treatment means. It is well known that the best test of \( H_0 \) in this situation is provided by the F-test. The Bayesian solution for testing \( H_0 \) in this case, with the assumption
of noninformative reference prior also gives the F-test (see Box and Tiao, 1975, pp. 110). Our focus is mainly on the equal mean situation discussed at the beginning of this chapter. A considerable amount of work has been done in the literature on this problem (see Chaubey, 1981 and reference therein), most of which discuss the problem in a general linear model set-up. These tests can be adapted in the situation discussed here, as a particular case. Some of the tests have been critically reviewed by Chaubey (loc. cit.). In the same paper he proposed a test based on the average of the squared residuals and named it average squared residual test (ASR-test). ASR-test is a modification of the test proposed by C.R. Rao (1970) using the MINQUE of the variances, which in some cases may produce negative estimates for the variances. ASR-test depends on the least squares residuals like the test proposed by C.R. Rao, but it removes the difficulty of using negative estimates of the variances produced by the method of MINQUE.

Some of these tests are described in section 2.2 and a Bayesian solution for the problem is presented in section 2.3. Section 2.4 presents an empirical comparison of the different test procedures.

2.2 Some Tests of $H_0$

Let $y_{11}, y_{12}, \ldots, y_{n_1}$ and $y_{21}, y_{22}, \ldots, y_{n_2}$ be two independent random samples of size $n_1$ and $n_2$ from two normal populations with equal mean $\mu$ and variances $\sigma_1^2$ and $\sigma_2^2$ respectively. We wish to test $H_0: \sigma_1^2 = \sigma_2^2$. Below we present some common tests for $H_0$.

F-Test

The following test statistic is used to test $H_0$,
\[ R_1 = \frac{s_1^2}{s_2^2} \]  

(2.2.1)

where \( s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \), \( \bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \), for \( i = 1, 2 \), \( j = 1, 2, \ldots, n_i \). The statistic \( R_1 \) has the F-distribution with \((n_1 - 1, n_2 - 1)\) degrees of freedom. This test does not take into consideration the equality of the population means.

**ASR-Test**

The ASR-test procedure leads to the following test statistic for the particular problem we are discussing in this chapter,

\[ R_2 = \frac{\tilde{\sigma}_1^2}{\tilde{\sigma}_2^2} \]  

(2.2.2)

where \( \tilde{\sigma}_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 \), \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n_i} y_{ij} \) and \( n = n_1 + n_2 \). The test statistic \( R_2 \) is based on the MINQ estimates of the variances \( \tilde{\sigma}_i^2 \) (see Rao and Chaubey, 1978 for MINQE).

ASR-test is empirically found (Chaubey 1981) to be locally more powerful than the F-test. But the exact distribution of the ASR-test statistic is complicated. Chaubey (1981) also gave an approximation of the distribution for the test statistic \( R_2 \).

**BLUS Test**

The adaptation of Theil's (1971) procedure leads to the following test statistic
\[
R_3 = \frac{\sum_{i=1}^{v_1} e_{1i}^2 / v_1}{v_2 \sum_{j=1}^{v_2} e_{2j}^2 / v_2}
\]

(2.2.3)

where \( e_{ij} \) \( i = 1,2 \) are BLU (best linear unbiased with scalar covariance) residuals obtained from the least squares residuals obtained from the least squares residuals \( e_{ij} = y_{ij} - \bar{y} \) \( i = 1,2; j = 1,2, \ldots, n_i \), \( v_1 \) and \( v_2 \) are respective degrees of freedom such that \( v_1 + v_2 = n - 1 \).

Under \( H_0 \), \( R_3 \) has the F-distribution with \( (v_1, v_2) \) degrees of freedom. This test is compared in terms of power in Chaubey (1981) and is found to be less powerful than the tests based on the test statistics \( R_1 \) and \( R_2 \). Though the distribution of \( R_3 \), under \( H_0 \) is simple, its construction is arbitrary and complicated. Goldfeld and Quandt (1965) considered the test statistic \( R_1 \) for testing homogeneity of variances by dividing the total sample in two halves.

There is no clear statistical theory behind the derivation of the test statistics discussed so far. In the next section we are presenting a Bayesian method for testing \( H_0 \).

2.3 A Bayesian Analysis

The null hypothesis \( H_0: \sigma_1^2 = \sigma_2^2 \) is equivalent to \( H_0: (\sigma_1^2 / \sigma_2^2) = 1 \).

We shall consider Bayesian inference on the ratio \( \sigma_1^2 / \sigma_2^2 = w \), on the basis of the posterior probability density function of \( w \) given the knowledge of the data, \( Y \) i.e. \( p(w|Y) \). When there is a posterior evidence that the ratio \( w = \sigma_1^2 / \sigma_2^2 \) is close to one, we may accept \( H_0 \) and if this ratio significantly differs from one, we reject \( H_0 \). Inference about \( w \) is made by constructing Highest Posterior Density (HPD) intervals of \( w \). First we derive the posterior distribution of \( w \).
2.3.1 Posterior Distribution of \( w \)

The likelihood function of the observations \( Y = \{y_{ij}, i=1,2; j=1,2,...,n_i\} \) is

\[
L(\mu, \sigma_1^2, \sigma_2^2 | Y) \propto \left( \frac{n_1}{\sigma_1^2} \right)^{\frac{n_1}{2}} \left( \frac{n_2}{\sigma_2^2} \right)^{\frac{n_2}{2}} \exp \left( -\frac{1}{2} \left( \frac{n_1}{\sigma_1^2} \sum_{j=1}^{n_1} (y_{1j} - \mu)^2 + \frac{n_2}{\sigma_2^2} \sum_{j=1}^{n_2} (y_{2j} - \mu)^2 \right) \right)
\]

subject to \( 0 < \mu < \infty \), \( 0 < \sigma_1^2 < \infty \), \( 0 < \sigma_2^2 < \infty \) \hspace{1cm} (2.3.1)

We consider the non-informative reference prior on \( \mu, \sigma_1^2, \sigma_2^2 \) as

\[
p(\mu, \sigma_1^2, \sigma_2^2) \propto \frac{1}{\sigma_1^2 \sigma_2^2}
\]

The joint posterior pdf of \( (\mu, \sigma_1^2, \sigma_2^2) \) given \( Y \) with respect to the above prior is given by

\[
p(\mu, \sigma_1^2, \sigma_2^2 | Y) \propto L(\mu, \sigma_1^2, \sigma_2^2 | Y) p(\mu, \sigma_1^2, \sigma_2^2)
\]

\[
\propto \left( \frac{n_1}{\sigma_1^2} \right)^{\frac{n_1}{2} - 1} \left( \frac{n_2}{\sigma_2^2} \right)^{\frac{n_2}{2} - 1} \exp \left( -\frac{1}{2} \left( \frac{n_1}{\sigma_1^2} \sum_{j=1}^{n_1} (y_{1j} - \mu)^2 + \frac{n_2}{\sigma_2^2} \sum_{j=1}^{n_2} (y_{2j} - \mu)^2 \right) \right)
\]

\[
\propto a(\sigma_1^2)^{-\frac{n_1}{2} - 1} (\sigma_2^2)^{-\frac{n_2}{2} - 1} \exp \left( -\frac{1}{2} \sigma_1^2 \sum_{j=1}^{n_1} (y_{1j} - \mu)^2 \sum_{j=1}^{n_2} (y_{2j} - \mu)^2 \right)
\]

\[
\times \exp \left( \frac{1}{2\sigma_1^2} \sum_{j=1}^{n_1} \left( \frac{\bar{y}_1 - \mu}{\sigma_1} \right)^2 + \lambda(\bar{y}_2 - \mu)^2 \right), \quad (2.3.3)
\]

where

\[
s_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} (y_{1j} - \bar{y}_1)^2
\]

\[
\bar{y}_1 = \frac{\sum_{j=1}^{n_1} y_{1j}}{n_1}
\]

\[
v_1 = (n_1 - 1)
\]
\[ w = \frac{\sigma_1^2}{\sigma_2^2} \]

and
\[ \lambda = \frac{w \cdot \sigma_2^2}{\mu_1} \]

Writing
\[ (\bar{y}_1 - \mu)^2 + \lambda(\bar{y}_2 - \mu)^2 = (\bar{y}_1 - \bar{y}_2)^2 \frac{\lambda}{1 + \lambda} + (\mu - \mu)^2 (1 + \lambda) \]

where
\[ \mu = \frac{\bar{y}_1 + \lambda \bar{y}_2}{1 + \lambda} \]

the posterior pdf of \((\mu, \sigma_1^2, \sigma_2^2)\) given \(Y\) becomes
\[
p(\mu, \sigma_1^2, \sigma_2^2 | Y) \propto (\sigma_1^2)^{-\frac{n_1}{2} - 1} (\sigma_2^2)^{-\frac{n_2}{2} - 1} \exp\left(- \frac{1}{2\sigma_1^2} (\nu_1 s_1^2 + \nu_2 s_2^2 + \frac{n_1 \lambda}{1 + \lambda} (\bar{y}_1 - \bar{y}_2)^2)\right) \times \exp\left(- \frac{\mu^2}{2\sigma_2^2}\right). \tag{2.3.4}
\]

The marginal posterior pdf of \((\sigma_1^2, \sigma_2^2)\) is obtained by integrating out \(\mu\) from (2.3.4) over its range \(-\infty < \mu < \infty\). Thus
\[
p(\sigma_1^2, \sigma_2^2 | Y) \propto \int_{-\infty}^{\infty} p(\mu, \sigma_1^2, \sigma_2^2 | Y) \, d\mu
\]
\[
\propto (\sigma_1^2)^{-\frac{n_1}{2} + 1} (\sigma_2^2)^{-\frac{n_2}{2} + 1} \exp\left(- \frac{1}{2\sigma_1^2} (s^2(w))\right) \tag{2.3.5}
\]

where \(s^2(w) = [\nu_1 s_1^2 + \nu_2 s_2^2 + \frac{n_1 \lambda}{1 + \lambda} (\bar{y}_1 - \bar{y}_2)^2]\).

Making the following transformation
\[
(\sigma_1^2, \sigma_2^2) + (\sigma_1^2, w = \sigma_1^2/\sigma_2^2)
\]

in (2.3.5) and noting that the Jacobian of the transformation is \(\sigma_1^2/\sigma_2^2\), we get the posterior pdf of \((\sigma_1^2, w)\) as
p(\sigma_1^2, w|Y) \propto (\sigma_1^2)^{-\frac{n_1+n_2}{2}} \left(1+\lambda \right)^{-\frac{1}{2}} \exp \left(-\frac{1}{2\sigma_1^2} (s^2(w)) \right) \tag{2.3.6}

; 0 < w < \infty , \sigma_1^2 > 0 .

Hence the marginal posterior pdf of w is

\[ p(w|Y) \propto \int_0^\infty p(\sigma_1^2, w|Y) d\sigma_1^2 \]

\[ = \frac{n_2}{\alpha(1+\lambda)^{-\frac{1}{2}}} \frac{1}{w^2} \left[ s^2(w) \right]^{-\frac{1}{2}} \frac{n_1+n_2}{2} \]

\[ ; 0 < w < \infty . \tag{2.3.7} \]

The final form of the posterior pdf of w given Y is

\[ p(w|Y) \propto \left(1+\frac{w}{n_1} \right)^{-\frac{n_2}{2}-1} \frac{w}{2} \left[ s^2(w) + \frac{n_1}{n_1+n_2} y_1^2 + \frac{n_2}{n_1+n_2} (y_1-y_2)^2 \right]^{-\frac{n_1+n_2}{2}-1} \]

; 0 < w < \infty . \tag{2.3.8}

Therefore,

\[ p(w|Y) = K \cdot g(w) , \tag{2.3.9} \]

where

\[ g(w) = \left(1+\frac{w}{n_1} \right)^{-\frac{n_2}{2}-1} \frac{w}{2} \left[ s^2(w) \right]^{-\frac{1}{2}} \frac{n_1+n_2}{2} \]

and

\[ K^{-1} = \int_0^\infty g(w) \, dw . \tag{2.3.10} \]

2.3.2 A Transformation for the Computation of \( K^{-1} \) and the Distribution of \( w \)

The following transformation is used to replace the improper integral in (2.3.10) which helps to get rid of the truncation error involved in a numerical integration of an improper integral.
The transformation used is
\[ z = \frac{w}{1+w} \quad ; \quad 0 < z < 1 \]
which implies that \( w = z/(1-z) \) and \( dw = 1/(1-z)^2 \, dz \). Hence \( K \) in (2.3.10) can be written as
\[ K^{-1} = \int_0^1 [g(w)]_{w=z/(1-z)} (1-z)^{-2} \, dz \quad . \quad (2.3.11) \]
The integral in (2.3.11) may be conveniently evaluated using IMSL subroutine DCADRE or any other comparable subroutine.

The posterior pdf of \( w \) and \( \log w \) are obtained through the distribution of \( z \) and those distributions are used to find HPD intervals and standardized HPD intervals for \( w \) respectively.

The posterior pdf of \( w \) given \( Y \) is found using the posterior distribution of \( z \) as follows:
\[ p(w|Y) = [p(z|Y) \cdot (1-z)^2]_{z=w/(1+w)} \quad . \quad (2.3.12) \]
and the posterior distribution of \( u = \log w \) given \( Y \) is obtained as
\[ p(u|Y) = [p(z|Y) \cdot z(1-z)]_{z=e^u/(1+e^u)} \quad . \quad (2.3.13) \]
The modes of \( p(w|Y) \) and \( p(u=\log w|Y) \) are obtained iteratively (or otherwise) by solving respectively, \( h(w) = 0 \) and \( h^*(w) = 0 \),
where
\[ h(w) = \frac{n_2}{2} - 1 \cdot w^{-1} - \frac{n_2}{2(n_1+n_2 \cdot w)} - \frac{(n_1+n_2-1)}{2s^2(w)} \left[ v_2 \cdot s_2^2 + \frac{n_1^2 n_2 (\bar{y}_1 - \bar{y}_2)^2}{(n_1+n_2 \cdot w)^2} \right] \]
and
\[ h^*(w) = \frac{n_2}{2w} - \frac{n_2}{2(n_1+n_2 \cdot w)} - \frac{(n_1+n_2-1)}{2s^2(w)} \left[ v_2 \cdot s_2^2 + \frac{n_1^2 n_2 (\bar{y}_1 - \bar{y}_2)^2}{(n_1+n_2 \cdot w)^2} \right] \]

FORTRAN source listing for this purpose is provided in Appendix A2.1 (Subroutine SOLMOD and SOLMOD1).
Figure 2.1a POSTERIOR DISTRIBUTION OF $z$ WHEN $n_1 = 5$, $n_2 = 10$,

$s_1^2 = .67$, $s_2^2 = 1.03$, $D = .31$. 
Figure 2.1b POSTERIOR DISTRIBUTION OF $u$ WHEN $n_1 = 9$, $n_2 = 10$,

$s_1^2 = .67$, $s_2^2 = 1.03$, $D = .31$
Figure 2.1c  POSTERIOR DISTRIBUTION OF $w$ WHEN $n_1 = 5$, $n_2 = 10$,

$$s_1^2 = .67, \quad s_2^2 = 1.03, \quad D = .31$$
Figure 2.2a POSTERIOR DISTRIBUTION OF $z$ WHEN $n_1 = 5$, $n_2 = 5$,

$$s_1^2 = 1.71, \ s_2^2 = 4.79, \ \text{and} \ D = -0.81.$$
Figure 2.2b POSTERIOR DISTRIBUTION OF \( u \) WHEN \( n_1 = 5 \), \( n_2 = 5 \), 
\[ s^2_1 = 1.71 \text{, } s^2_2 = 4.79 \text{ and } D = -.81 \]
Figure 2.2c  POSTERIOR DISTRIBUTION OF $w$ WHEN $n_1 = 5$, $n_2 = 5$,

$s_1^2 = 1.71$, $s_2^2 = 4.79$ and $D = -0.81$
Figure 2.3a  POSTERIOR DISTRIBUTION OF \( z \) WHEN \( n_1 = 10 \), \( n_2 = 5 \),

\[
\sigma_1^2 = 2.25, \quad \sigma_2^2 = 1.10 \quad \text{and} \quad D = .15
\]
Figure 2.3b POSTERIOR DISTRIBUTION OF $u$ WHEN $n_1 = 10$, $n_2 = 5$,

\[ s_1^2 = 2.25, \quad s_2^2 = 1.10 \text{ and } D = .15 \]
Figure 2.3c. POSTERIOR DISTRIBUTION OF $w$ WHEN $n_1 = 10, \ n_2 = 5,$

$s_1^2 = 2.25, \ s_2^2 = 1.10$ \text{and} \ $D = .15$
Figure 2.4a  POSTERIOR DISTRIBUTION OF z WHEN $n_1 = 5$, $n_2 = 5$,

$s_1^2 = .87$, $s_2^2 = 14.59$, $D = -.65$
Figure 2.4b POSTERIOR DISTRIBUTION OF $u$ WHEN $n_1 = 5$, $n_2 = 5$,

$a_1^2 = .87$, $a_2^2 = 14.59$, $D = -.65$
Figure 2.4c  POSTERIOR DISTRIBUTION OF $w$ WHEN $n_1 = 5, n_2 = 5$,

\[ s_1^2 = .87 , \quad s_2^2 = 14.59 , \quad D = -.65 \]
The posterior distributions $p(z|Y)$, $p(w|Y)$ and $p(u|Y)$ are computed using the IMSL integrating subroutine DCADRE and are plotted for different values of $n_1$, $n_2$, $s_1$, $s_2$ and $D = \overline{y}_1 - \overline{y}_2$ (see Figs. 2.1 to 2.4). The fortran program listing is provided in Appendix A2.1. The values of $s_1$, $s_2$ and $D$ were randomly generated using GCCHS and GGNOF subroutines of IMSL for generating chi-square and normal random variables. The distribution $p(u|Y)$ looks symmetric in each case, hence inferences based on $\log w$ may be quite appropriate. In the next section we present an empirical comparison of Bayes' procedures based on $w$ and $u$ and the $F$-statistic.

2.4 An Empirical Comparison of Bayes-Procedure with Other Analysis

We generate $s_1^2$, $s_2^2$ and $D$ such that $(n_1 - 1)s_1^2 / \chi_{n_1 - 1}^2$, $(n_2 - 1)s_2^2 / \chi_{n_2 - 1}^2$ and $D \sim N(0, \frac{1}{n_1} + \frac{1}{wn_2})$ for given $n_1$, $n_2$ and $w$, 100 times. We confine our attention to the small samples because of the computational limitations. (Even for values of $(n_1, n_2)$ as small as $(20, 20)$, computer time needed to generate two samples, compute HPD and standardized HPD intervals 100 times was more than 3000 CPU seconds). On the basis of those generated samples we computed the HPD intervals of content .95, .99 based on $p(w|Y)$ and standardized HPD intervals which are based on $p(\log w|Y)$ using the algorithm described in Chapter I. HPD intervals are also calculated using $F$-distribution (see Box and Tiao, 1973, Eq. 2.6.2). We compare these intervals in terms of the average number of times they contain the specified ratio $w$ (coverage frequency), which is presented in Table 2.1. We did not compare BLUS test since it is found less powerful than $F$-test (Chaubey, 1981). Also, we could not compare ASR because
### TABLE 2.1  COMPARISON OF THE BAYES-PROCEDURE WITH CLASSICAL F-TEST

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the procedure for finding the confidence intervals using the distribution of ASR-test statistic under $H_0$ is not clear. The source listing of the FORTRAN program which generates the samples and finds the HPD intervals mentioned above is given in the Appendix A2.2. This program uses IMSL subroutine ZFALSE and DCADRE to compute HPD intervals. It is observed that HPD-interval based on $p(\log w|Y)$ is almost always better than HPD-interval based on F-distribution on comparing the coverage frequency. The HPD-intervals based on $p(w|Y)$ and F-statistic may have smaller widths but in such cases they loose in terms of coverage frequency.
CHAPTER III

TESTING EQUALITY OF VARIANCES OF TWO LINEAR
MODELS WITH COMMON REGRESSION PARAMETERS

3.1 Introduction

It is quite natural in practice that the observations from two sources depend on some common as well as some uncommon factors. But we are treating the case where the observations from two sources linearly depend only on common factors. For example, in analyzing an investment time series data relating to two corporations Boot and DeWitt (1960) assumed that the annual investment is a linear function of expected profitability and real capital at the beginning of the year (see Box and Tiao, 1973, Chapter 9). In this example the data from two sources i.e. the corporations, depend only on two common factors. There are situations where data may depend on \( p(>2) \) common factors. Such a situation is represented by the model,

\[
Y_i = X_i \beta + \varepsilon_i ; \quad i = 1, 2 \quad (3.1.1)
\]

where \( Y_i \) is an \((n_1 \times 1)\) observation vector, \( X_i \) is an \((n_1 \times p)\) known matrix (matrix of observations on the \(p\) factors), \( \beta \) is \((p \times 1)\) unknown parameter vector and \( \varepsilon_i \) is the \((n_1 \times 1)\) disturbance vector, which is assumed to be normally distributed with zero mean and dispersion matrix \( \sigma_1^2 I_{n_1} \), \( \varepsilon_1 \) being independent of \( \varepsilon_2 \).

Because of the independence of the two sources it is quite natural to expect that the variances in two sources will be different. If they, for some reason, could be assumed to be equal then the data from the two
different sources could be pooled under one linear model and the analysis of a single linear model would be applied. Hence to detect the differences in the variances associated with the two linear models is of interest. Thus, we are interested in testing the hypothesis \( H_0 : \sigma_1^2 = \sigma_2^2 \)
or equivalently \( H_0 : \sigma_1^2 / \sigma_2^2 = 1 \). This is a general case of the problem discussed in Chapter II. As pointed out before, because of the generality of this problem, it has attracted the attention of many researchers.

Goldfeld and Quandt (1968) considered the problem of testing the homogeneity of variances of a linear model against the alternative that the variances are not equal. They divided the observations into two groups by means of some criterion and then computed the sum of squares of residuals from the regressions fitted separately to the two groups by least squares method and used the ratio of resultant residual sum of squares as a test statistic. In this approach, the test statistic has \( F \)-distribution under the null hypothesis, even if the regression parameters of two linear models are different. They have also proposed a non-parametric test for testing \( H_0 \).

We already mentioned in Chapter II that Theil (1971, section 5.2) proposed a test for testing \( H_0 \) using what he called BLUS (Best linear unbiased with scalar covariance) residuals instead of least squares residuals. Theil's BLUS-test statistic also has an \( F \)-distribution under \( H_0 \), which is its only simplicity (see Chaubey 1981).

Chaubey (1981) proposed a test for testing \( H_0 \) using the average of the squared least squares residuals. He named the test as ASR-test. ASR-test is found to be invariant under some important transformations and the power function of the ASR-test depends on the ratio \( w = \sigma_1^2 / \sigma_2^2 \).
ASR-test is locally more powerful though F-test is found to be reasonable for large samples.

If we consider k(>2) sources instead of two then the problem in general is to compare the variances of k(>2) linear models. Harrison and McCabe (1979) suggested a test to compare k-variances and called it "Bounds test". The test statistic used by Harrison and McCabe is similar to that of ASR-test statistic though in a more general context. The problem of k(>2) linear models may not be easy to deal with, however, we discuss this problem in Chapter IV.

A Bayesian analysis for testing $H_0$ for the model (3.1.1) is presented in section 3.3. In section 3.4 we present an example using the Boot and DeWitt's (1960) data to which different methods have been applied and compare the F-test and Bayes-procedure. As it has not been possible to compute the confidence interval for ASR-test statistic, we could not compare ASR-test with Bayes-procedure. Since BLUS-test and F-test were compared in Chaubey (1981) and F-test showed better results than BLUS-test, we compared only F-test with the Bayes procedure. Section 3.2 describes various tests discussed above.

3.2 Some Common Tests of $H_0$

One simple method to test $H_0$ is to consider the F-statistic, computed in the following way,

$$R_1 = \frac{\tilde{e}_1^T \tilde{e}_1 / (n_1 - p)}{\tilde{e}_2^T \tilde{e}_2 / (n_2 - p)}$$  \hspace{1cm} (3.2.1)

where $$\tilde{e}_1 = [I_{n_1} - X_1(X_1'X_1)^{-1}X_1']y_1$$.
\((i = 1,2)\) i.e. \(\hat{\varepsilon}_i's\) are the least-squares residuals for the \(i\)th model. Obviously, this F-statistic ignores the equality of the regression parameters of the two linear models in (3.1.1). Under the null hypothesis, the test statistic in (3.2.1) has F-distribution with \((n_1 - p, n_2 - p)\) degrees of freedom.

**BLUS-Test**

Theil (1971) observed that the least squares residuals are not convenient for testing \(H_0\), since the least square residuals \(\varepsilon\), in the combined model

\[ Y = X\beta + \varepsilon, \quad (3.2.2) \]

where \(Y' = (Y_1' : Y_2')\), \(X' = (X_1' : X_2')\), \(\varepsilon' = (\varepsilon_1' : \varepsilon_2')\), have non-scalar covariance matrix under \(H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2\), viz.

\[ D(\varepsilon) = \sigma^2Q \quad (3.2.3) \]

where \(Q = (I - X(X'X)^{-1}X')\) is an idempotent matrix. Theil suggested to obtain that estimator of \(\varepsilon\) which is linear in \(Y\), unbiased and has the minimum variance among the class of other estimators with the scalar covariance matrix \(\sigma^2I\). Such an estimator of the residual vector is obtained by choosing the estimator \(\hat{\varepsilon} = CY\), where \(C\) is such that

\[ CX = 0 \quad \text{and} \quad CC' = I \quad (3.2.4) \]

\(C\) has at most \((n - r)\) rows where \(r = \text{rank}(X)\) and the order of \(I\) in (3.2.4) is \((n - r)\). Therefore, \(\hat{\varepsilon}\) represents at most \((n - r)\) components of the disturbance vector \(\varepsilon\).

The model (3.2.2) can be partitioned as

\[
\begin{bmatrix}
Y_o \\
\vdots \\
Y'_o
\end{bmatrix} = 
\begin{bmatrix}
X_o \\
\vdots \\
X'_o
\end{bmatrix} \beta + 
\begin{bmatrix}
\varepsilon_o \\
\vdots \\
\varepsilon'_o
\end{bmatrix} 
\quad (3.2.5)
\]
and equivalently,

\[
\begin{bmatrix}
Y_o^c \\
\vdots \\
Y_o^c
\end{bmatrix}
= \begin{bmatrix}
X_o^c \\
\vdots \\
X_o^c
\end{bmatrix} \hat{\beta} + \begin{bmatrix}
\epsilon_o^c \\
\vdots \\
\epsilon_o^c
\end{bmatrix}
\]  

(3.2.6)

where \( \hat{\beta} \) is the least-squares estimator of \( \beta \), \( \epsilon \)'s are the least-squares residuals and \( X_o \) is non-singular. Assuming that \( X_o \) corresponds to the first \( r \) observations, it can be shown that

\[
\epsilon_o^c = (X_o^c X_o^{-1})^c \epsilon_o^c.
\]  

(3.2.7)

From (3.2.7) we conclude that the 1st \( r \) residuals and last \((n-r)\) residuals are equally informative about \( \epsilon \). So the \((n-r)\) residuals having scalar covariance matrix could be made to correspond to the last \((n-r)\) disturbances. Let \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \) denote the BLUS-residuals for the first and second models respectively. The dimensions of \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \) depends on the selection of \( X_o \). If \( X_o \) has \( r_1 \) rows of \( X_1 \) and \( r_2 \) rows of \( X_2 \), then \( r = r_1 + r_2 \). The statistic

\[
R_2 = \frac{\hat{\epsilon}_1^T \hat{\epsilon}_1 / (n_1 - r_1)}{\hat{\epsilon}_2^T \hat{\epsilon}_2 / (n_2 - r_2)}
\]  

(3.2.8)

which has the \( F \)-distribution with \((n_1 - r_1, n_2 - r_2)\) degrees of freedom, can be used to test \( H_0 \).

The ASR-Test

The least-squares residual vector \( \epsilon \) of (3.2.2) can be written as \( \epsilon' = (\epsilon_1', \epsilon_2') \) where

\[
\epsilon_1' = (I_{n_1} - X_1(X'X)^{-1}X_1')Y_1 - X_1(X'X)^{-1}X_1'Y_2,
\]

and

\[
\epsilon_2' = (I_{n_2} - X_2(X'X)^{-1}X_2')Y_2 - X_2(X'X)^{-1}X_2'Y_1.
\]

The ASR-test statistic for testing \( H_0 \) is constructed on the basis
of the above residual vectors as follows.

\[ R_3 = \frac{E_1 e_1 / n_1}{e_2 e_2 / n_2} \]  \hspace{1cm} (3.2.9)

The acceptance region for \( H_0 : \sigma_1^2 = \sigma_2^2 \) against \( H_1 : \sigma_1^2 > \sigma_2^2 \) based on \( R_3 \) is given as

\[ R_3 \leq C \]  \hspace{1cm} (3.2.10)

where \( C \) is such that

\[ \Pr(R_3 \leq C | H_o^c) = 1 - \alpha. \]  \hspace{1cm} (3.2.11)

3.3 The Bayesian Solution

A Bayesian approach for testing \( H_0 \) can be given through the Bayesian inference on the ratio \( w = \sigma_1^2 / \sigma_2^2 \). When there is a posteriori evidence that this ratio is significantly close to one, we may accept \( H_0 \). Thus, we first derive the posterior distribution of \( w \).

3.3.1 Posterior Distribution of \( w \)

The likelihood function of the model (3.1.1) is given by

\[ L(\beta, \sigma_1^2, \sigma_2^2 | Y) \propto \left( \sigma_1^2 \right)^{-n_1/2} \left( \sigma_2^2 \right)^{-n_2/2} \exp\left( -\frac{1}{2} \frac{(Y - X \beta)'(Y - X \beta)}{\sigma_1^2} \right). \]  \hspace{1cm} (3.3.1)

We consider the non-informative reference prior for \( (\beta, \sigma_1^2, \sigma_2^2) \)

\[ p(\beta, \sigma_1^2, \sigma_2^2) \propto \frac{1}{\sigma_1^2 \sigma_2^2} \]  \hspace{1cm} (3.3.2)

The posterior distribution with respect to the above prior is given by

\[ p(\beta, \sigma_1^2, \sigma_2^2 | Y) \propto \left( \sigma_1^2 \right)^{-n_1-1} \left( \sigma_2^2 \right)^{-n_2-1} \exp\left( -\frac{v_1 s_1^2 + v_2 s_2^2}{\sigma_1^2} + \frac{v_1 s_1^2 + v_2 s_2^2}{\sigma_2^2} \right). \]  \hspace{1cm} (3.3.3)
where $v_i = n_i - p$, $s_i^2 = v_i^{-1}(Y_i - \bar{X}_i\bar{\beta}_i)^\top(Y_i - \bar{X}_i\bar{\beta}_i)$,

$\bar{\beta}_i = (X_i'X_i)^{-1}X_i'y_i$, $s_i(\beta) = (\beta - \bar{\beta}_i)'X_iX_i(\beta - \bar{\beta}_i)$, $i = 1, 2$.

Integrating out $\beta$ from (3.3.3) gives the posterior distribution of $\sigma_1^2, \sigma_2^2$, namely,

$$
p(\sigma_1^2, \sigma_2^2|Y) \propto (\sigma_1^2)^{-\frac{n_1}{2} - 1} (\sigma_2^2)^{-\frac{n_2}{2} - 1} \exp\left[-\frac{v_1 s_1^2 + v_2 s_2^2}{\sigma_1^2} + \frac{v_1 s_1^2 + v_2 s_2^2}{\sigma_2^2}\right] \int_{-\infty}^{\infty} \exp\left(-\frac{\frac{1}{\sigma_1^2} S_1(\beta) + \frac{1}{\sigma_2^2} S_2(\beta)}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right) d\beta \tag{3.3.4}
$$

Using the following identity to simplify (3.3.4),

$$
S_1(\beta) + wS_2(\beta) = S(\beta|w) + D(w) \tag{3.3.5}
$$

where

$$
S(\beta|w) = (\beta - \bar{\beta}_w)'(X_1'X_1 + wx_2'x_2)(\beta - \bar{\beta}_w),
$$

$$
D(w) = w(\bar{\beta}_1 - \bar{\beta}_2)'X_1 X_1 (x_1'x_1 + wx_2'x_2)^{-1}x_2 x_2 (\bar{\beta}_1 - \bar{\beta}_2)
$$

and

$$
\bar{\beta}_w = (X_1'X_1 + wx_2'x_2)^{-1}(x_1'x_1\bar{\beta}_1 + wx_2'x_2\bar{\beta}_2).
$$

Using (3.3.5) in (3.3.4) gives,

$$
p(\sigma_1^2, \sigma_2^2|Y) \propto (\sigma_1^2)^{-\frac{n_1}{2} - 1} (\sigma_2^2)^{-\frac{n_2}{2} - 1} \exp\left[-\frac{v_1 s_1^2 + v_2 s_2^2 + D(w)}{\sigma_1^2} \right] \int_{-\infty}^{\infty} \exp\left(-\frac{\frac{1}{\sigma_1^2} S(\beta|w)}{\sigma_1^2}\right) d\beta,
$$

$$
\propto (\sigma_1^2)^{-\frac{n_1}{2} - 1} (\sigma_2^2)^{-\frac{n_2}{2} - 1} \left|X_1'X_1 + wx_2'x_2\right|^{-\frac{1}{2}} \exp\left[-\frac{v_1 s_1^2 + v_2 s_2^2 + D(w)}{\sigma_1^2} \right]. \tag{3.3.6}
$$
Making the transformation from \((\sigma_1^2, \sigma_2^2)\) to \((\sigma_1^2, w)\) as in section 2.3, and integrating out \(\sigma_1^2\) from (3.3.6) gives the posterior distribution of \(w\) as
\[
p(w|Y) = Kw \frac{n_2 - 1}{2} |x_1'x_1 + wx_2'x_2|^{-\frac{1}{2}} [s^2(w)]^{-\frac{n_1 + n_2 - p}{2}}
\]
where the constant \(K\) is determined from
\[
K^{-1} = \int_{-\infty}^{\infty} \frac{n_2 - 1}{2} |x_1'x_1 + wx_2'x_2|^{-\frac{1}{2}} [s^2(w)]^{-\frac{n_1 + n_2 - p}{2}} \, dw \quad (3.3.7)
\]
and
\[
s^2(w) = (v_1s_1^2 + vw^2s_2^2 + D(w))/(n_1 + n_2 - p).
\]
This distribution is derived in Box and Tiao (1973) to be used in computation of the posterior distribution of any subset of \(\beta\) and is simplified as
\[
p(w|Y) = K|x_1'x_1|^{-\frac{1}{2}} \left[ \frac{p}{j\lambda_j(1+w\lambda_j)} \right]^{-\frac{1}{2}} w \frac{n_2 - 1}{2} [s^2(w)]^{-\frac{n_1 + n_2 - p}{2}},
\]
for \(0 < w < \infty\), \( (3.3.8)\)
where \(\lambda_1, \lambda_2, \ldots, \lambda_p\) are diagonal elements of a matrix \(\Lambda\) given by
\[
\Lambda = P'X_2'X_2P
\]
where \(P\) is such that \(P'X_1'X_1P = I\). Computation of \(\hat{\beta}_w\) and \(s^2(w)\) now simplifies to
\[
\hat{\beta}_w = P(I+w\Lambda)^{-1}(\hat{\phi}_1 + w\hat{\phi}_2), \quad \hat{\phi}_1 = P^{-1}\hat{\beta}_1, \quad i = 1, 2
\]
and
\[
s^2(w) = (n_1 + n_2 - p)^{-1}[v_1s_1^2 + vw^2s_2^2 + \frac{p}{j=1} \frac{\lambda_j}{1+w\lambda_j} (\hat{\phi}_{1j} - \hat{\phi}_{2j})^2] .
\]
To replace the improper integral in (3.3.7), we make the transformation \(z = \frac{w}{1+w}\) and the constant \(K\) is given by
\[ K^{-1} = \left| \mathbf{X}' \mathbf{X} \right|^{-\frac{1}{2}} \int_0^1 \left[ \left( \prod_{j=1}^p (1 + w \lambda_j) \right)^{-\frac{1}{2}} w^{-\frac{n_2}{2}} \times \left\{ \int_0^{n_1 + n_2 - p} \right\} \frac{1}{z} (1 - z)^{-2} dz \right] \frac{z}{1-z} \]

where \([ \cdot ] \) expresses the fact that the expression in square brackets is evaluated for \( w = \frac{z}{1-z} \). Computational aspects of this is addressed in section 3.4. HPD intervals based on \( p(z|Y), p(w|Y) \) and \( p(u = \log w|Y) \) for Boot and DeWitt's data are obtained using the algorithm described in Chapter I and are presented in the next section.

The posterior densities \( p(z|Y), p(w|Y) \) and \( p(u = \log w|Y) \) are obtained through the procedures discussed in section 2.3.2. When the regression parameter \( \beta \) is not known to be common \textit{a priori}, the HPD intervals can be obtained using F-distribution (see Box and Tiao (1973), pp. 141).

3.4 An example

We consider the time series data reported in Boot and DeWitt (1960), relating to two large companies, General Electric (GE) and Westinghouse (W)*. In this model, price deflated gross investment (I) is assumed to be a linear function of expected profitability (P) and real capital stock at the beginning of the year (S). As in Grunfeld (1958) the value of outstanding shares at the beginning of the year is taken as a measure of a company's expected profitability. The two investment relations are:

* This example is taken from Box and Tiao (1973).
\[ Y_{1j} = \beta_{01} + \beta_1 x_{11j} + \beta_2 x_{12j} + \epsilon_{1j}, \]
\[ Y_{2j} = \beta_{02} + \beta_1 x_{21j} + \beta_2 x_{22j} + \epsilon_{2j}, \]

where \( j \) denotes the index of a year, \( j = 1, 2, \ldots, 20 \)

and variables are identified as,

<table>
<thead>
<tr>
<th>variable</th>
<th>GE</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( Y_{1j} )</td>
<td>( Y_{2j} )</td>
</tr>
<tr>
<td>P</td>
<td>( x_{11j} )</td>
<td>( x_{21j} )</td>
</tr>
<tr>
<td>S</td>
<td>( x_{12j} )</td>
<td>( x_{22j} )</td>
</tr>
<tr>
<td>Error</td>
<td>( \epsilon_{1j} )</td>
<td>( \epsilon_{2j} )</td>
</tr>
</tbody>
</table>

Performing similar analysis as in section (3.3), we can eliminate \( \beta_{01}, \beta_{02} \) and represent the posterior of \( (\beta_1, \beta_2, \sigma_1^2, \sigma_2^2) \) in the form (3.3.3) (see Box and Tiao (1973), pp. 496-497) where,

\[
\begin{bmatrix}
3.254 & 0.233 \\
0.233 & 1.193
\end{bmatrix} = X_1^T X_1 = 10^6, \quad X_2^T X_2 = 10^6, \quad
\begin{bmatrix}
0.940 & 0.195 \\
0.195 & 0.074
\end{bmatrix}
\]

\[
\hat{\beta}_1 = \begin{bmatrix}
0.02655 \\
0.15170
\end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix}
0.05289 \\
0.09241
\end{bmatrix}
\]

\( s_1^2 = 0.777 \times 10^3, \quad s_2^2 = 0.104 \times 10^3, \quad n_1 = n_2 = 19, \quad p = 2, \)
\( v_1 = v_2 = 17 \). The matrices \( P, \Lambda \) and vectors \( \phi_1 \) and \( \phi_2 \) are
\[
\begin{bmatrix}
-1.9814 & .52196 \\
.99469 & .22306
\end{bmatrix}, \quad \Lambda = 
\begin{bmatrix}
.02703 & 0 \\
0 & .30513
\end{bmatrix}
\]
\[
\hat{\phi}_1 = P^{-1}\bar{\phi}_1 = 10^3 \begin{bmatrix}
.14331 \\
.10527
\end{bmatrix}, \quad \hat{\phi}_2 = P^{-1}\bar{\phi}_2 = 10^3 \begin{bmatrix}
.07128 \\
.12839
\end{bmatrix}
\]

The modes of \( p(w|Y) \) and \( p(u = \log w|Y) \) are obtained by solving \( h(w) = 0 \) and \( h^*(w) = 0 \) iteratively (or otherwise) respectively, where

\[
h(w) = (n_2 - 2)w^{-1} - \sum \frac{1}{\lambda_j (1 + w\lambda_j)^{-1}} - \frac{1}{s^2(w)}
\]
\[
\times \left[ \frac{\nu_2 s_2^2}{\nu_2} + \sum \frac{1}{\lambda_j (1 + w\lambda_j)^{-1}} - \frac{1}{s^2(w)} \left( \hat{\phi}_1 - \hat{\phi}_2 \right)^2 \right]
\]
\[
h^*(w) = n_2 w^{-1} - \sum \frac{1}{\lambda_j (1 + w\lambda_j)^{-1}} - \frac{1}{s^2(w)}
\]
\[
\times \left[ \frac{\nu_2 s_2^2}{\nu_2} + \sum \frac{1}{\lambda_j (1 + w\lambda_j)^{-1}} - \frac{1}{s^2(w)} \left( \hat{\phi}_1 - \hat{\phi}_2 \right)^2 \right]
\]

FORTRAN source listing for this purpose is provided in Appendix A3.1 (subroutines SOLMOD and SOLMOD1). For this data the mode of \( p(w|Y) \) and \( p(u|Y) \) are found to be at \( \hat{w} = 5.8586 \) and \( \hat{w}^* = 7.349 \) respectively. The subroutine DCADRE from IMSL Library was used to compute the constant (3.3.8). The value of \( K \) is obtained to be

\[
K^{-1} = 1385105765137285 \times 10^{-43}
\]

The posterior pdf of \( z \) is given in figure 3.1 and that of \( w \) and \( u \) are given in figures 3.2 and 3.3 respectively.

HPD - intervals were obtained using the algorithm given in Chapter I. These are given in Table 3.1. For each \( c \) the roots \( u_{1c}, u_{2c} \) were obtained using the subroutine ZFALSE from IMSL
Figure 3.1 POSTERIOR DISTRIBUTION OF $z$ GIVEN $Y$. 
Figure 3.2 POSTERIOR DISTRIBUTION OF $w$ GIVEN $Y$. 
Figure 3.3 POSTERIOR DISTRIBUTION OF \( u \) GIVEN \( Y \).
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Based on $w$</th>
<th>Based on $u$</th>
<th>Based on $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limit</td>
<td>Limit</td>
<td>Width</td>
<td>Limit</td>
</tr>
<tr>
<td>0.01</td>
<td>1.38</td>
<td>23.69</td>
<td>22.31</td>
</tr>
<tr>
<td>0.05</td>
<td>1.98</td>
<td>16.83</td>
<td>14.83</td>
</tr>
<tr>
<td>0.10</td>
<td>2.37</td>
<td>14.18</td>
<td>11.81</td>
</tr>
<tr>
<td>0.25</td>
<td>3.13</td>
<td>10.88</td>
<td>7.75</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$u$ Based</td>
<td>$F$ Based</td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>------------</td>
<td>------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lower Limit</td>
<td>Upper Limit</td>
<td>Width</td>
</tr>
<tr>
<td>.01</td>
<td>2.07</td>
<td>26.60</td>
<td>24.54</td>
</tr>
<tr>
<td>.05</td>
<td>2.83</td>
<td>19.32</td>
<td>16.49</td>
</tr>
<tr>
<td>.10</td>
<td>3.31</td>
<td>16.47</td>
<td>13.15</td>
</tr>
</tbody>
</table>
Library. The program is given in Appendix A3.1. We also evaluate equal tail Bayesian intervals and these are given in table 3.2, which seem not to be too far off from the HPD - intervals.

Two points become evident in the analysis of the given data:

i) HPD - intervals based on w and u are shorter than those based on F.

ii) The ratio of two variances is concentrated far away from one.
CHAPTER IV

TESTING EQUALITY OF VARIANCES

IN k(≥ 2) LINEAR MODELS

4.1. Introduction

Let us consider k(≥ 2) linear models with common β parameter

\[ Y_1 = X_1 \beta + \varepsilon_1, \quad i = 1, 2, 3, \ldots, k \]  \hspace{1cm} (4.1.1)

where \( Y_i \) is an \((n_i \times 1)\) vector of observations, \( X_i \) is a known \((n_i \times p)\) matrix with rank \( p \) and \( p \leq n_i \), \( \beta \) is a \( p \times 1 \) vector of parameters, \( \varepsilon_i \) is disturbance vector of order \((n_i \times 1)\) for the \( i \)th model with

\[ E(\varepsilon_i) = 0, \quad \text{var}(\varepsilon_i) = \sigma_i^2 I_{n_i}, \quad \text{cov}(\varepsilon_i, \varepsilon_j) = 0; \quad i \neq j \]

and assumed to be normally distributed. The situations where such a case of linear models occurs is discussed in previous chapters (also see Box and Tiao (1973), pp. 478). We are interested in testing the hypothesis \( H_0: \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 \). The problem of testing \( H_0 \) has been considered by many researchers (see Chaubey 1981, and Harrison and McCabe (1979) and references therein). The test proposed by Harrison and McCabe (1979) divides the observations into two groups following the suggestion of Goldfeld and Quandt (1965), though the test statistic is different from the test statistic proposed by Goldfeld and Quandt. The Bayesian analysis for this \( k \)-linear model case is difficult. However, a simple way to get a Bayesian solution for this case is to divide the observations into two groups and apply the procedures discussed in section 3.3. The other way is to generalize the likelihood ratio statistic for testing the homogeneity of variances in normal populations, which
has some Bayesian support (see Box and Tiao 1973, pp. 134). In this
spirit we consider the proposal of Chaubey (1980) to use a test similar
to likelihood ratio test, which is based on ordinary least squares
residuals. The test statistic proposed here is computationally simpler
than that given by the standard likelihood ratio test. We have studied
the properties of the test for two normal populations with common mean
as a special case of (4.1.1) and give results for general case \( k > 2 \).
It is observed that the proposed test statistic has an approximate chi-
square distribution with \((k-1)\) degrees of freedom. The present test
is also computationally simpler than some of the other tests proposed
in literature. The next section (4.2) gives the tests proposed by
Goldfeld and Quandt and Harrison and McCabe and the new test is described
in section 4.3. Section 4.4 presents an approximation to the distribu-
tion of the new test statistic in terms of its moments which are evalu-
ated for the special case with \( k = 2 \) and studied in detail for com-
parison purposes.

4.2 Goldfeld and Quandt's Parametric Test

Goldfeld and Quandt's parametric test for testing \( H_0 \) can be des-
dcribed for the combined model,

\[
Y = X\beta + \varepsilon \tag{4.2.1}
\]

where \( Y = [Y_1' \ldots Y_k']' \), \( X = [X_1' \ldots X_k']' \), \( \varepsilon = [\varepsilon_1' \ldots \varepsilon_k']' \)
as follows.

First the observations are arranged in ascending order by the values
of the variable \( X_t \), where \( X_t \) is the potential deflator in econo-
metric problem. The ordering is given in terms of the second subscript
of \( X_t \) indexed so that \( x_{ti} \preceq x_{tj} \) if and only if \( i < j \), and the
remaining variables are indexed so that the indexed values correspond with those of $x_{t1}$. This ordering is based on the presumption that variance of the observations increase with $x_t$. Ignoring $m$ central observations, two separate least-squares regressions are fitted to the first $(n-m)/2$ and last $(n-m)/2$ observations, provided that $(n-m)/2 > p$, the number of parameters to be estimated, and that the $(n-m)/2$ observations be distributed over at least $p$ distinct points in $X$ space, i.e. the rank of matrix with the $(n-m)/2$ observations must be at least equal to $p$. Denoting $S_1$ and $S_2$, the sum of squares of the residuals from the regressions based on the relatively small and relatively large values of $x_t$ respectively, the test statistic is formed as

$$R_1 = S_2 / S_1$$  (4.2.2)

which has F-distribution with $(n-m-2p-2)$, $(n-m-2p-2)$ degrees of freedom under the null hypothesis.

**Harrison and McCabe's Test**

Harrison and McCabe (1979) proposed the following test statistic for testing $H_0$,

$$b = \frac{e'\Lambda e}{e'e}$$  (4.2.3)

where $\Lambda$ is an appropriate selector matrix of order $(n \times n)$ with $m(0 < m < n)$ ones and $(n-m)$ zeros on its principal diagonal and zeros elsewhere, and $e$ is the least squares residuals obtained from the combined model (4.2.1).

This test uses the F-distribution for finding the critical values of $b$, and one sided $H_0$ would be rejected if $b < b_L^*$ and accepted if $b > b_U^*$ and the test is inconclusive if $b_L^* < b < b_U^*$; where $b_L^*$ and $b_U^*$ are the critical values determined from the F-distribution.
and $b^*_U$ are critical values of test statistic for a required level of
significance and obtained using the $\alpha$ critical value of $F$ as

$$b^*_L = \left[ 1 + \frac{(n-m)F_\alpha(n-m,m-k)}{m-k} \right]^{-1},$$

$$b^*_U = \left[ 1 + \frac{(n-m-k)F_\alpha(n-m-k,m)}{m} \right]^{-1},$$

where $F_\alpha(\nu_1,\nu_2)$ denotes the upper $\alpha$-percentile of $F$-distribution
with degrees of freedom $(\nu_1,\nu_2)$. When the test is inconclusive, use
of an exact test using an exact or an approximate distribution of $b$
is proposed.

4.3 A Likelihood Ratio Type Test for $H_0$

Chaubey (1980) proposed the following likelihood ratio type test
for testing $H_0$. This test is motivated by the likelihood ratio test
criterion (A) for testing equality of variances of $k$ normal popu-
lations, which has the form

$$\Lambda = \prod_{i=1}^{k} \left( \frac{\hat{\sigma}_i^2}{\hat{s}^2} \right)^{n_i/2},$$

(4.3.1)

where $\hat{\sigma}_i^2$ is the maximum likelihood estimate of $\sigma_i^2$ and $\hat{s}^2$ is the
maximum likelihood estimate of $\sigma^2$ (the common variance under the
null hypothesis). Since the computation of $\hat{\sigma}_i^2$'s are difficult in
the present case it is replaced by

$$\hat{\sigma}_i^2 = \bar{e}'_i e_i/n_i,$$

where $e_i = Y_i - X_i \hat{\beta}$ and $\hat{\beta}$ is the ordinary least-squares estimate of
$\beta$ for the combined model given in (4.2.1) and substituting the maximum
likelihood estimator ($\hat{\sigma}^2 = e'e/n$) in place of $s^2$ in (4.3.1) we
obtain

$$T = -2\ln \Lambda = - \sum_{i=1}^{k} n_i \ln(e'_i e_i/e'e) + \sum_{i=1}^{k} n_i \ln(n_i/n).$$

(4.3.2)
The test statistic $T$ for large samples has approximately a chi-square distribution with $(k-1)$ degrees of freedom, even though $\sigma_1^2$ are not individual MLE's of $\sigma_1^2$. The exact distribution of $T$ is very difficult to obtain and we shall instead obtain an approximation to its distribution as follows.

4.4 Approximation to the Distribution of $T$ Under $H_0$

We approximate $T$ by a constant multiple of chi-square variable, because $T$ is a linear function involving the variables $(e_1e_1/e'e)$ which behaves like a beta variable and negative logarithm of beta variable behaves like a chi-square variable (see Wise (1950)). Thus letting $T = \chi^2_v$ and equating the first two moments we get,

$$a = \mu_1/2\mu_2^1, \quad v = 2\mu_1/\mu_2,$$

(4.4.1)

where $\mu_1$ and $\mu_2$ represent the mean and variance of $T$ respectively. These are not explicitly obtainable. However the approximate moments can be obtained as in the following section.

4.4.1 Approximate Moments of $T$

Note from (4.3.2) that

$$T = g(b) + \sum_{i=1}^{k} n_i \ln(n_i/n),$$

where $b = (b_1, b_2, \ldots, b_k)$, $b_1 = e_1e_1/e'e$

and

$$g(b) = -\sum_{i=1}^{k} n_i \ln(b_i) - n_k \ln(1 - \sum_{i=1}^{k} b_i).$$

(4.4.2)

We find the approximate moments of $T$ by expanding $g(b)$ in Taylor's series about $\theta = E(b)$,

$$g(b) = g(\theta) + \sum_{i=1}^{k-1} (b_i - \theta_i) \left. \frac{\partial g(b)}{\partial b_i} \right|_{b=\theta} + \frac{1}{2!} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (b_i - \theta_i) (b_j - \theta_j) \left. \frac{\partial^2 g(b)}{\partial b_i \partial b_j} \right|_{b=\theta}.$$

(4.4.3)
Taking expectation on both sides of (4.4.3) we get,

\[ \delta_1 = E(g(b)) - g(\theta) = g(\theta) + \frac{1}{2} \sum_{i \neq j} \frac{\partial^2 g(b)}{\partial b_i \partial b_j} \bigg|_{b=\theta} \]

where

\[ \mu_{ij} = E(b_i - \theta_i)(b_j - \theta_j). \]

Also from (4.4.3),

\[ \delta_2 = E(g(b) - g(\theta))^2 = \sum_{i \neq j} E(b_i - \theta_i)(b_j - \theta_j) \left( \frac{\partial g(b)}{\partial b_i} \cdot \frac{\partial g(b)}{\partial b_j} \right) \bigg|_{b=\theta} + \sum_{i} E(b_i - \theta_i)^2 \left( \frac{\partial g(b)}{\partial b_i} \right)^2 \bigg|_{b=\theta} \]

\[ + \sum_{i \neq j} E(b_i - \theta_i)(b_j - \theta_j) \left( \frac{\partial^2 g(b)}{\partial b_i^2} + \frac{\partial^2 g(b)}{\partial b_j^2} \right) \bigg|_{b=\theta} \]

\[ + \frac{1}{4} \sum_{i \neq j} E(b_i - \theta_i)^2(b_j - \theta_j) \left( \frac{\partial^2 g(b)}{\partial b_i \partial b_j} \right)^2 \bigg|_{b=\theta} + 2 \cdot \frac{\partial^2 g(b)}{\partial b_i^2} \cdot \frac{\partial^2 g(b)}{\partial b_j^2} \bigg|_{b=\theta} \]

\[ + \frac{1}{4} \sum_{i \neq j} E(b_i - \theta_i)^2(b_j - \theta_j)^2 \left( \frac{\partial^2 g(b)}{\partial b_i^2} \cdot \frac{\partial^2 g(b)}{\partial b_j^2} + \frac{\partial^2 g(b)}{\partial b_i \partial b_j} \right)^2 \bigg|_{b=\theta} \]

\[ + \sum_{i} E(b_i - \theta_i)^3 \left( \frac{\partial g(b)}{\partial b_i} \right) \cdot \frac{\partial^2 g(b)}{\partial b_i^2} \bigg|_{b=\theta} \]
\[
\frac{1}{2} \sum_{j} \sum_{j} E(b_i - \theta_i)^3 (b_j - \theta_j) \left( \frac{\partial^2 g(b)}{\partial b_i^2} \right) \left( \frac{\partial^2 g(b)}{\partial b_j \partial b_i} \right) \bigg|_{b=\hat{b}}
\]
\[
\frac{1}{4} \sum_{i} E(b_i - \theta_i)^4 \left( \frac{\partial^2 g(b)}{\partial b_i^2} \right)^2 \bigg|_{b=\hat{b}}
\]

 ordinaria, approximations for \( \delta_1 \) and \( \delta_2 \) may be appropriate by considering only the terms with first derivative and ignoring the terms with second and higher order derivatives, but at least in one situation it is not so, as we will see in section 4.4.3. That is why we retained terms with second derivatives in (4.4.3) and (4.4.5). Using these approximations we can obtain \( \mu_1' \) and \( \mu_2' \) to be used in (4.4.1) as

\[
\mu_1' = \delta_1 + \varepsilon n_1 \ln(n_1/n),
\]

\[
\mu_2' = \delta_2 - \delta_1^2.
\]

Computation of \( \delta_1 \) and \( \delta_2 \) in (4.4.4) and (4.4.5) involve the moments, cross-moments and cross-moments of higher powers of \( b_i, b_j, b_1 \) which are obtained in the following section.

### 4.4.2 Computation of Moments, Cross-Moments and Cross-Moments of Higher Powers of \( b_i, b_j, b_1 \)

Since \( b_i \)'s are independent of \( e'e \) by Geary's Theorem (1933), the moments of \( b_i \)'s are easily obtained as

\[
\mu_r' = E(b_i^r) = \frac{E(e_i'e_i)^r}{E(e'e)^r}
\]

The raw moments in the numerator and the denominator of (4.4.8) are obtained from the following formula for the rth cumulant \( k_r \) of a quadratic form in normal variables with zero mean and covariance matrix \( \Sigma \) (see Searle, 1971, pp. 56, Corollary 1.1),
\[ k_r = (r - 1)! \cdot 2^{r-1} \text{tr}(\Sigma^r) \quad (4.4.9) \]

and using the following relation (see Chaubey and Talukder, 1983, Lemma 1, pp. 676-677) successively between the raw moments \( m_r \) and the cumulants \( k_r \),

\[ m_{r+1} = \sum_{i=0}^{r-1} \binom{r}{i} m_i k_{r-i} \quad \text{for } r = 0, 1, 2, \ldots \quad (4.4.10) \]

The cross moments between \( b_i b_j b_l \) and other cross moments of higher order can be obtained using Geary's result as

\[ E(b_i^r b_j^s b_l^t) = \frac{E(e_i e_j e_l)^r (e_i e_j e_l)^s (e_i e_j e_l)^t}{E(e_i e_j e_l)^{r+s}} \quad (4.4.11) \]

where \( r, s, t \) take values from the set \( 0, 1, 2, 3 \). Noting that \( e = Q \epsilon, e_i e_j = e_i A_i e_j \), where \( Q = I - X'X'^{-1}X' \), and \( A_i \) is block-diagonal matrix with \( i \)-th Block being \( I_{n_i} \) and zero elsewhere, the denominator of (4.4.11) can be evaluated using (4.4.9) and the expectations in the numerator can be evaluated using the results of Magnus (1978) quoted below:

**Theorem:**

Let \( A, B \) and \( C \) be symmetric matrices of order \( n \) and \( \epsilon \sim N(0, \Sigma) \) where \( \Sigma \) is positive definite, then

i) \[ E(e' A \epsilon \cdot e' B \epsilon) = (\text{tr} A \Sigma)(\text{tr} B \Sigma) + 2 \text{tr} \Sigma B \Sigma \]

ii) \[ E(e' A \epsilon \cdot e' B \epsilon \cdot e' C \epsilon) = (\text{tr} A \Sigma)(\text{tr} B \Sigma)(\text{tr} C \Sigma) \\
+ 2(\text{tr} A \Sigma)(\text{tr} B \Sigma C \Sigma) + (\text{tr} B \Sigma)(\text{tr} A \Sigma C \Sigma) + (\text{tr} C \Sigma)(\text{tr} A \Sigma B \Sigma) \\
+ 8 \text{tr} A \Sigma B \Sigma C \Sigma \]

iii) \[ E(e' A \epsilon)^2 (e' B \epsilon) = (\text{tr} A \Sigma)^2 (\text{tr} B \Sigma) + 4(\text{tr} A \Sigma)(\text{tr} A \Sigma B \Sigma) \\
+ 2(\text{tr} B \Sigma)(\text{tr} A \Sigma)^2 + 8 \text{tr}(A \Sigma)^2 B \Sigma \]
\[ \text{iv) } E(e'Ae)^2 \cdot (e'Bc)^2 = (\text{tr } AE)^2(\text{tr } BE)^2 + 16[(\text{tr } AE)(\text{tr } AE(\text{tr } BE)^2)] \\
+ (\text{tr } BE)(\text{tr } (AE)^2BE)] \\
+ 4[(\text{tr } (AE)^2)(\text{tr } (BE)^2) + 2(\text{tr } AEBE)^2] \\
+ 2(\text{tr } AE)^2 \cdot (\text{tr } (BE)^2) + 4(\text{tr } AE)(\text{tr } BE)(\text{tr } AEBE) \\
+ (\text{tr } BE)^2(\text{tr } (AE)^2)[ \\
+ 16[\text{tr } (AEBE)^2 + 2\text{tr } (AE)^2(\text{BE}^2)] \]

\[ \text{v) } E(e'Ae)^2 \cdot (e'Bc) \cdot (e'Cc) = (\text{tr } AE)^2(\text{tr } BE)(\text{tr } CE) \\
+ 8[(\text{tr } BE)(\text{tr } (AE)^2CE) + (\text{tr } CE)(\text{tr } (AE)^2BE) + 2(\text{tr } AE)(\text{tr } AEBECE)] \\
+ 4[(\text{tr } (AE)^2)(\text{tr } BECE) + (\text{tr } AEBE)(\text{tr } AECE) + (\text{tr } AEC)(\text{tr } AEBE)] \\
+ 2[(\text{tr } AE)^2(\text{tr } BECE) + 2(\text{tr } AE)(\text{tr } BE)(\text{tr } AEC)] \\
+ 2(\text{tr } AE)(\text{tr } CE)(\text{tr } AEBE) + (\text{tr } BE)(\text{tr } CE)(\text{tr } (AE)^2)] \\
+ 16[\text{tr } (AE)^2BECE + \text{tr } (AE)^2CEBE + \text{tr } (AEBEAC)] \]

\[ \text{vi) } E(e'Ae)^3 \cdot e'Bc = (\text{tr } AE)^3(\text{tr } BE) \\
+ 24(\text{tr } AE)(\text{tr } (AE)^2BE) + 8(\text{tr } BE)(\text{tr } (AE)^3) \\
+ 12(\text{tr } (AE)^2)(\text{tr } AEBE) + 6(\text{tr } AE)^2(\text{tr } AEBE) \\
+ (\text{tr } AE)(\text{tr } BE)(\text{tr } (AE)^2)] \\
+ 16[2\text{tr } (AE)^3BE + \text{tr } (AE)^2BEAE] \]

\subsection*{4.4.3 A Special Case}

In this section we consider the special case of (4.1.1) where,

\[ k = 2, X_1 = \frac{1}{n_1}, X_2 = \frac{1}{n_2}, Q = (I - \frac{1}{n} E_{n\times n}), A_1 = \begin{bmatrix} I_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \\
A_2 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n_2} \end{bmatrix} \text{ and } E_{n\times n} \text{ is } n\times n \text{ matrix all of whose elements are unity, } I_{n_1} \text{ is the identity matrix of order } n_1 \text{ and } \frac{1}{n_i}, \text{ is a } (n_1 \times 1), (i = 1, 2) \text{ vector of elements unity. Partitioning } Q \text{ as} \]
\[
Q = \begin{bmatrix}
I_{n_1} - \frac{1}{n} E_{n_1}^{\times n_1} & -\frac{1}{n} E_{n_1}^{\times n_2} \\
\frac{1}{n} E_{n_2}^{\times n_1} & I_{n_2} - \frac{1}{n} E_{n_2}^{\times n_2}
\end{bmatrix}
\]

and multiplying by \( A_1 \) and \( A_2 \) respectively, we get.

\[
A_1Q = \begin{bmatrix}
I_{n_1} - \frac{1}{n} E_{n_1}^{\times n_1} & -\frac{1}{n} E_{n_1}^{\times n_2} \\
0 & 0
\end{bmatrix}
\quad (4.4.12)
\]

\[
A_2Q = \begin{bmatrix}
0 & 0 \\
-\frac{1}{n} E_{n_2}^{\times n_1} & I_{n_2} - \frac{1}{n} E_{n_2}^{\times n_2}
\end{bmatrix}
\quad (4.4.13)
\]

In this case \( g(b_1) = -n_1 \ln b_1 - n_2 \ln(1-b_1) \). The Taylor's series expansion of \( g(b_1) \) about \( \theta_1 = E(b_1) = n_1/n \) gives as a special case of (4.4.3), since

\[
g''(\theta_1) = -\frac{n_1}{\theta_1} + \frac{n_2}{1-\theta_1} = 0
\]

\[
\delta_1 = g(\theta_1) + \frac{1}{2} m_2 g''(\theta_1), \quad (4.4.14)
\]

\[
\delta_2 = \frac{1}{4} m_4 \left[ g''(\theta_1) \right]^2. \quad (4.4.15)
\]

Applying the method discussed in section 4.2 we find the central moments \( m_i (i = 1, 2, 3, 4) \) of \( b_1 \) which are,

\[
m_1 = \frac{n_1}{n},
\]

\[
m_2 = \frac{2n_1(n-2)(n-n_1)}{n^2(n^2-1)},
\]
\[
\hat{m}_3 = \frac{8n_1(n-3)(n'-2n')(n-p_1)}{n^2(n-1)(n+3)}
\]

and
\[
\hat{m}_4 = \frac{12n_1(n-n')(n_1(n-n')(n-6)(n-10)+4n^2(n-1)(n+3)(n+5)}{n^4(n-1)(n+3)(n+5)}
\]

Using (4.4.6) and (4.4.7) we get
\[
\mu'_1 = \frac{n(n-2)}{(n^2-1)}
\]

and
\[
\mu'_2 = \frac{12n^4(n-4)}{n_1n_2(n^2-1)(n+3)(n+5)} + \frac{3n^2(n-6)(n-10)}{(n^2-1)(n+3)(n+5)}
\]

(4.4.17)

It is observed (see table 4.1) that as the sample size \( n \) increases the mean \( \mu'_1 \) and the variance \( \mu'_2 \) of \( T \) approaches to 1 and 2 respectively, and also the constants of approximation \( a \) and \( v \) in (4.4.1) approach to unity which shows the justification of the chi-square approximation to \( T \).

Table - 4.1: Constants of Approximation (4.4.1) for different \( n_1 \) and \( n_2 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( v )</th>
<th>( a )</th>
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<tbody>
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<td>.912</td>
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<td>.553</td>
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<td>15</td>
<td>.902</td>
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</table>

Since \( \mu'_1 \) and \( \mu'_2 \) in (4.4.16) and (4.4.17) are symmetric in \( n_1 \) and \( n_2 \) we get the same value for \( \mu'_1, \mu'_2, v \) and \( a \) as in table 4.1 if we increase \( n_1 \) keeping \( n_2 \) fixed. In the next section we investigate the accuracy of this approximation.
4.4.4 Adequacy of the Approximation

In order to check the accuracy of the chi-square approximation to the distribution of $T$ under $H_0$, a computer simulation is performed. Approximate $100(1-\alpha)$% points of $T$ denoted by $T^*_{1-\alpha} = \chi^2_{\nu;\alpha}$, where $\chi^2_{\nu;\alpha}$ denotes the upper $\alpha$ percentile of chi-square distribution with $\nu$ degrees of freedom are obtained for different $n_1$ and $n_2$ by evaluating $\chi^2_{\nu;\alpha}$ through the IMSL subroutine MDCHI. We generate 5000 random values of $T$ using the fact that

$$e_1'\hat{a}_1 = U_1 + \frac{n_2}{(n_1+n_2)} U_3$$
$$e_2'\hat{a}_2 = U_2 + \frac{n_1}{(n_1+n_2)} U_3$$

where $U_1, U_2, U_3$ are independent chi-square random variables with $n_1 - 1, n_2 - 1$, and 1 degrees of freedom respectively when $\sigma^2_1 = \sigma^2_2 = 1$ (see Chaubey (1981) for the proof). The relative frequency of the event $T \geq T^*_{1-\alpha}$ denoted by $f_{\alpha}$ is observed and presented in table 4.2. If the chi-square approximation to the distribution of $T$ is reasonable one should expect $f_{\alpha} = \alpha$. It is evident from table 4.2 that the approximation is in close agreement with the true values of $T^*_{1-\alpha}$ even for a small value of $n = 15$. It is also observed that for $n < 15$, the approximation is not very good. Numerical calculations also indicate that as $n$ grows large, $f_{\alpha}$ approaches closer to $\alpha$. 
### Table 4.2: Approximate \((1 - \alpha)100\%\) Points of \(T^*\) and a Comparison For Their Accuracy

<table>
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<th>(n_1)</th>
<th>(n_2)</th>
<th>(T^*_{.90})</th>
<th>(T^*_{.95})</th>
<th>(T^*_{.99})</th>
<th>(f_{.10})</th>
<th>(f_{.05})</th>
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REFERENCES


PROGRAM MAIN(INPUT, OUTPUT, GRAPHD, TAPEx=GRAPHD)
THIS PROGRAM COMPUTES P(Z/Y), P(U/Y) AND P(W/Y)
FOR DIFFERENT SAMPLE SIZES M, N, SAMPLE
VARIANCES V1, V2 AND DIFFERENCE BETWEEN THE
SAMPLE MEANS D
DIMENSION X(99), Y(99), Z(99), U(99), W(99), FU(99), FW(99)
COMMON/B/L/M, N, V1, V2, D
PRINT 5
PRINT 15
FORMAT("ENTER VALUES FOR M, N, V1, V2, D")
READ*, M, N, V1, V2, D
1, ", D = ", F8.4)
CALL CMEMAN(X, Y, Z, U, W, FU, FW)
DO 10 I=1, 99
WRITE(6, 100) X(I), Y(I), Z(I), U(I), W(I), FU(I), FW(I)
100 CONTINUE
STOP
END
SUBROUTINE CMEMAN (X, Y, Z, U, W, FU, FW).
COMPUTES THE POSTERIOR DISTRIBUTION OF
Z, U AND W
DIMENSION X(99), Y(99), Z(99), U(99), W(99), FU(99), FW(99)
COMMON /B/L/M, N, V1, V2, D
EXTERNAI F
AL=0. OEO
AU=1. OEO
AUX=0. OI
CONS=DCADRE(F, AL, AU, 1. OE-25, 1. OE-1, ERR, IER)
I=1
10 FX=DCADRE(F, AL, AUX, 1. OE-25, 1. OE-1, ERR, IER)
X(I)=AUX
Y(I)=FX/CONS
Z(I)=F(AUX)/CONS
I=I+1
AUX=AUX+0.01EO
IF(AUX.LE. 99) GO TO 10
DO 11 J=1, 99
W(J)=X(J)/(1.-X(J))
U(J)=ALOG(W(J))
FW(J)=Z(J)**(1.-X(J))**2.*
FU(J)=FW(J)*W(J)
11 CONTINUE
RETURN
END
FUNCTION F(X)
COMMON /B/L/M, N, V1, V2, D
IF(X.EQ. 0.0EO .OR. X.EQ. 1.0EO) GO TO 1
W=X/(1.-X)
N1=M-1
N2=N-1
TI=(N1*V1+N2*W*V2)
T2=M*N*W*W/D/(M+N*W)
FP=(T1+T2)/W**((N/2.0)-1.0)*FP**(-0.5)*((1.0-X)**(-2.0))
F=FP/(T1+T2)**((M+N-1.0)/2.0)
GO TO 2
1 F=0. OEO
2 RETURN
END
PROGRAM PLOTM (INPUT, OUTPUT, TAPE99, TAPE5=INPUT, 
1TAPE6=OUTPUT)

C
C
This program plots \( P(Z/Y) \), \( P(U/Y) \) and \( P(W/Y) \)
using the data generated by the preceding
program.

C
C
DIMENSION X(101), Y(101), Z(101), U(101), W(101), 
1FU(101), FW(101)

DO 10 I=1,100
READ(5,100)X(I),Y(I),Z(I),U(I),W(I),FU(I),FW(I)
10 CONTINUE

CALL SCALE(X, 6, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', -1.6, 0., X(100), X(101))
CALL SCALE(Z, 6, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', 1.6, 90., Z(100), Z(101))
CALL NEWPEN(1)
CALL PLOT(1, 5, 1.5, -3)
CALL LINE(X, Z, 99, 1, 0, 0)
CALL SYMBOL(- 29, 2.7, 14, 102, 90., 0)
CALL SYMBOL(- 29, 2.9, 14, 40, 90., 0)
CALL SYMBOL(- 29, 3.1, 14, 120, 90., 0)
CALL SYMBOL(- 29, 3.3, 14, 124, 90., 0)
CALL SYMBOL(- 29, 3.5, 14, 121, 90., 0)
CALL SYMBOL(- 29, 3.7, 14, 41, 90., 0)
CALL SYMBOL(2. 95, - 6, 14, 120, 0., 0)
CALL PLOT(0, 0, 999)
CALL SCALE(U, 9, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', -1.6, 0., U(100), U(101))
CALL SCALE(FU, 6, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', 1.6, 90., FU(100), FU(101))
CALL NEWPEN(1)
CALL PLOT(1, 5, 1.5, -3)
CALL LINE(U, FU, 99, 1, 0, 0)
CALL SYMBOL(- 29, 2.9, 14, 40, 90., 0)
CALL SYMBOL(- 29, 3.1, 14, 117, 90., 0)
CALL SYMBOL(- 29, 3.3, 14, 124, 90., 0)
CALL SYMBOL(- 29, 3.5, 14, 121, 90., 0)
CALL SYMBOL(- 29, 3.7, 14, 41, 90., 0)
CALL SYMBOL(2. 95, - 6, 14, 117, 0., 0)
CALL PLOT(0, 0, 999)
CALL SCALE(W, 48, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', -1.6, 0., W(100), W(101))
CALL SCALE(FW, 3, 99, 1)
CALL AXIS(1, 5, 1.5, ' ', 1.6, 90., FW(100), FW(101))
CALL NEWPEN(1)
CALL PLOT(1, 5, 1.5, -3)
CALL LINE(W, FW, 99, 1, 0, 0)
CALL SYMBOL(- 29, 2.7, 14, 102, 90., 0)
CALL SYMBOL(- 29, 2.9, 14, 40, 90., 0)
CALL SYMBOL(- 29, 3.1, 14, 119, 90., 0)
CALL SYMBOL(- 29, 3.3, 14, 124, 90., 0)
CALL SYMBOL(- 29, 3.5, 14, 121, 90., 0)
CALL SYMBOL(- 29, 3.7, 14, 41, 90., 0)
CALL SYMBOL(2. 95, - 6, 14, 119, 0., 0)
CALL PLOT(0, 0, 999)
CALL PLOT(0, 0, 9999)
STOP
END
PROGRAM QDATA2(INPUT, OUTPUT, RES, TAPE1=RES)
THIS PROGRAM GENERATES THE SAMPLE VARIANCES
AND THE DIFFERENCE BETWEEN THE SAMPLE MEANS
FOR TWO INDEPENDENT NORMAL POPULATIONS FOR
GIVEN SAMPLE SIZES AND THE RATIO W USING
THE FACT THAT THE SAMPLE VARIANCES ARE DISTRI-
BUTED AS CHISQUARES AND DIFFERENCE BETWEEN
THE MEANS IS NORMAL AND COMPUTES THE
HPD INTERVALS BASED ON W AND LOG(W) FOR THE
CORRESPONDING SAMPLE SIZES AND SAMPLE VARIANCES
IT ALSO OBSERVES THE NUMBER OF TIMES THE HPD-
INTERVALS INCLUDE THE SPECIFIED W AND THE
AVERAGE WIDTHS OF THE INTERVALS
COMMON /BL1/M,N,V1,V2,D/UBL/W1C,W2C/BL1/ALPHA
1/UBL1/W1CU,W2CU
DOUBLE PRECISION DSEED
DIMENSION N1(6),W(5),Z(10)
DATA ALPHAS/.09/,CMIN/.0001/,CMAX/.9999/
DATA FMIN/.220/,FMAX/.6.086/
IFW=0
I=0
WTHI=0.0
WTHI=0.0
DATA N1/5,10,15,20,30,50/,W/.1,.5,1.0,1.5,2.0/
DO 10 I=2,2
N=N1(I)
COND=1,1
DDI=12 K=3,3
L=0
DSEE=34567365.1.D0
HI=W(K)
M1=M-1
NU2=N-1
WRITE(1,101)W1,ALPHA
101 FORMAT(5X,*W1=*,@10.4,*,ALPHA=*,@10.4)
15 R=0.G05Q(DSEE)
D=R*SQRT(1./M1+1./W1/N)
CALL GCHS(DSEE,NU1,Z,U1)
CALL GCHS(DSEE,NU2,Z,U2)
V1=U1/NU1
V2=U2/(NU2*W1)
CALL HPDW(M,N,V1,V2,D,ALPHA,CMIN,CMAX,W1C,W2C)
W1=FMIN+V1/V2
W2=FMAX+V1/V2
CALL HPDW(M,N,V1,V2,D,ALPHA,CMIN,CMAX,W1C,W2C)
IF((W1.LT.W2C),AND. (W1.GT.W1C))IFW=IFW+1
IF((W1.LT.W2C),AND. (W1.GT.W1C))IFU=IFU+1
IF((W1.LT.W2C),AND. (W1.GT.W1C))IFF=IFF+1
WTHI=WTHI+(W2C-W1C)
WTHI=WTHI+(W2C-W1C)
WTHI=WTHI+(W2C-W1C)
WRITE(1,100)M,N,V1,V2,D
100 FORMAT(5X,*M=*,@13,*N=*,@13,*,V1=*,F8.4,*,V2=*,F8.4)
1, *D=*, FB. 4, /)
WRITE(1, 102) W1C, W2C, W1CU, W2CU, W1F, W2F
L=L+1
IF(L L.E. 99) GO TO 15
WTHW=WTHW/100. 0
WTHU=WTHU/100. 0
WTHF=WTHF/100. 0
WRITE(1, 103)
103 FORMAT(/, 5X, *AVERAGE WIDTHS=* )
WRITE(1, 104) WTHW, WTHU, WTHF
WRITE(1, 105)
105 FORMAT(/, 5X, *COVERAGE FREQUENCY=* )
WRITE(1, 106) IFW, IFU, IFF
12 CONTINUE
11 CONTINUE
10 CONTINUE
STOP
SUBROUTINE HPDW(M, N, V1, V2, D, ALPHA, CMIN, CMAX, W1C, W2C)
C COMPUES HPD INTERVAL BASED ON W
COMMON /BL/CONST
EXTERNAL F, HPDC
CONST D (DGADAR, F, 0.0EO, 1.0EO, 1.0E-25, 1.0E-10, ERR, IER)
ITMAX =100
EPS1 =1.0E-10
XL1 =CMIN
XR1 =CMAX
CALL ZEROHPD(HPDC, XL1, XR1, EPS1, ITMAX1, CSOL)
RETURN
END
SUBROUTINE SOLMOD(W)
C COMPUTES THE MODE OF P(W/Y)
COMMON /BLI/M, N, V1, V2, D
EXTERNAL H
EPS=1.0E-10
NSIG =10
XL =1.0E-20
XR=(V1/V2)+20.0
ITMAX =100
CALL ZFMLE(H, EPS, NSIG, XL, XR, W, ITMAX, IER)
RETURN
END
FUNCTION H(W)
COMMON /BLI/M, N, V1, V2, D
T1=(N/2, 0)-1.0)/W
T2 = N*(2.0*(M+N*W))
T3= M+N-1.0
T4 = (N-1.0)*V2+M*M*D*D /(M+N*W)**2.0
T6 = 2.0*((M-1.0)*V1+(N-1.0)*V2*W)
T7 = M*N*M*D*D/(M+N*W)
FP = T3*T4
FUNCTION EXPD2C(X1C, X2C, ITMAX, IER)

C THIS FUNCTION F2(X) FINDS THE EXPONENTIAL EXP2D = EXPD(X) SUCH THAT X2D = 100 * EXP2D = EXPD(X) * 100, WHERE X IS THE VALUE OF THE VARIABLE X1C.

CALL ZFALSE(F2(1, 1.0E-10, 10, XM, XR, X2C, ITMAX, IER))

RETURN

END

FUNCTION EXPD2C(X1C, X2C, ITMAX, IER)

C THIS FUNCTION F2(X) FINDS THE EXPONENTIAL EXP2D = EXPD(X) SUCH THAT X2D = 100 * EXP2D = EXPD(X) * 100, WHERE X IS THE VALUE OF THE VARIABLE X1C.

CALL ZFALSE(F2(1, 1.0E-10, 10, XM, XR, X2C, ITMAX, IER))

RETURN

END

FUNCTION EXPD2C(X1C, X2C, ITMAX, IER)

C THIS FUNCTION F2(X) FINDS THE EXPONENTIAL EXP2D = EXPD(X) SUCH THAT X2D = 100 * EXP2D = EXPD(X) * 100, WHERE X IS THE VALUE OF THE VARIABLE X1C.

CALL ZFALSE(F2(1, 1.0E-10, 10, XM, XR, X2C, ITMAX, IER))

RETURN

END
FUNCTION F1(X)
COMMON /BL/CONST
F1 = F(X)/CONST
RETURN
END

SUBROUTINE ZEYROHPD(HPD, XMIN, XMAX, ERROR, ITM, XSOL)
C FINDS THE SOLUTION SUCH HPD IS ZERO
C SOLUTION XSOL LIES BETWEEN XMIN AND XMAX
C ITM = MAX NO. OF ITERATIONS
C ERROR = ERROR BOUND
FMIN = HPD(XMIN)
FMAX = HPD(XMAX)
ITMX = 1

IF( (FMIN.GT.0.0 .AND. FMAX.GT.0.0) .OR.
1.(FMIN.LT.0.0 .AND. FMAX.LT.0.0)) GO TO 2
4 X = (XMIN+XMAX)/2.0
F = HPD(X)
IF(ABS(F).LE.ERROR) GO TO 6
ITMX = ITMX + 1
IF( ITMX.GT.ITM) GO TO 5
IF( F.GT.0.0) GO TO 3
XMAX = X
GO TO 4
3 XMIN = X
GO TO 4
5 PRINT*, "NO. OF ITERATIONS HAVE REACHED MAXIMUM, VALUE OF"
1 HPD="",F
GO TO 1
2 PRINT*, "SOLUTION DOES NOT LIE IN THE GIVEN INTERVAL"
GO TO 1
6 XSOL = X
1 RETURN
END

SUBROUTINE HPD11(M, N, V1, V2, D, ALPHA, CMIN, CMAX, W1CU, W2CU)
C COMPUTES HPD INTERVAL BASED ON U
COMMON /BL/CONST
EXTERNAL HPDC1
ITMAX2=100
EPS2=1.0E-10
XL2=CMIN
XR2 = CMAX
CALL ZEYROHPD(HPDC1, XL2, XR2, EPS2, ITMAX2, CSOL1)
RETURN
END

SUBROUTINE SOLMODI(W)
C COMPUTES THE MODE OF P(LOGW/Y)
COMMON /BL/M, N, V1, V2, D
EXTERNAL H1
EPS= 1.0E-10
NSIQ =10
XL = 1.0E-10
XR = (V1/V2)+20.0
ITMAX =100
CALL ZFALSE(H1, EPS, NSIQ, XL, XR, W, ITMAX, IER)
RETURN
FUNCTION H1(W)
COMON /BLI/M, N, V1, V2, D
T1 = N/(2*W)
T2 = N/(2.0*(M+N*W))
T3 = M+N-1.0
T4 = (N-1.0)*V2+N*M*W*D*D/(M+N*W)**2.0
T6 = 2.0*(M-1.0)*V1+(N-1.0)*V2*W
T7 = M*N*W*D*D/(M+N*W)
FP = T3*T4
FP = FP/(T6+2.0*T7)
H1 = T1-T2-FP
RETURN
END

FUNCTION HPDC1(C)
COMMON /BLI/M, N, V1, V2, D
EXTERNAL F1, F2U
IF THIS FUNCTION FINDS THE VALUES W1C, W2C SUCH THAT
THE INTERVAL (W1C, W2C) HAVE THE SAME POSTERIOR
ORDINATES FOR A GIVEN C, THE VALUE OF ORDISTATE OF Log(W)
BEING THE PRODUCT OF C AND THE POSTERIOR ORDISTATE
AT THE MODE.
AND THEN GIVES THE PROBABILITY CONTENT OF THIS
INTERVAL
CALL SOLMOD1(W)
X = W / (1.+W)
X1 = 1.0E-10
X2 = 1.0E-10
F1X = F1(X)
F2UX = F2U(X)
PRW = PRX*(1.-X)**2.0
PRU = PRW+W
PRC1 = C*PRU
ITMAX = 100
XM = X
CALL ZFALSN(F2U, 1.0E-10, 6, XL, XM, X1C, ITMAX, IER)
X1M = X
ITMAX = 100
CALL ZFALSN(F2U, 1.0E-10, 6, XM, X1M, X2C, ITMAX, IER)
U1C = ALOG(X1C/(1.-X1C))
U2C = ALOG(X2C/(1.-X2C))
W1CU = EXP(U1C)
W2CU = EXP(U2C)
PROBC = DCAKE(F1, X1C, X2C, 1.0E-100, 1.0E-50, ERR, IER)
HPDC1 = PROBC-(1.-ALPHA)
RETURN
END

FUNCTION F2U(X)
COMMON /UBL3/PRC1
F2U = F1(X)*X*(1.-X)-PRC1
RETURN
END
APPENDIX A3.1

PROGRAM HPDINT(OUTPUT, TAPE4)
 This program computes the HPD-intervals based on W and U for boot and De Witt's Data
 COMMON /UBL/W1C, W2C/BL1/ALPHA/UBL1/W1CU, W2CU
 DATA ALPHA/, 05/, CMIN/, 0001/, CMAX/, 9999/
 WRITE(1, 101) ALPHA
 101 FORMAT(5X, *ALPHA=* F10.4)
 CALL HPDM(ALPHA, CMIN, CMAX, W1C, W2C)
 CALL HPDM1(ALPHA, CMIN, CMAX, W1C, W2C)
 WRITE(1,102) W1C, W2C, W1CU, W2CU
 F10.4, *W2CU=*, F10.4, /)
 STOP
 END

SUBROUTINE HPDW(ALPHA, CMIN, CMAX, W1C, W2C)
 Computes HPD-interval based on W
 COMMON /BL/CONST
 - EXTERNAL F, HPDC
 CONST = DCADRE(F, O, OEO, 1. OEO, 1. OEO-25, 1. OEO-10, ERR, IER).
 ITMAX = 100
 EPS1 = 1. OEO-10
 XL = CMIN
 XR = CMAX
 CALL ZEROHPD(HPDC, XL, XR, EPS1, ITMAX1, CSOL)
 RETURN
 END

SUBROUTINE SOLMOD(W)
 Computes the mode of P(W/Y)
 EXTERNAL H
 EPS = 1. OEO-10
 NSIQ =10
 XL = 1. OEO-20
 XR = (V1*V2)+20.0
 ITMAX = 100
 CALL ZFALSE(H, EPS, NSIQ, XL, XR, W, ITMAX, IER)
 RETURN
 END

FUNCTION H(W)
 T1 = 17.0/W
 T2 = 0.2703/(1.0+ 0.2703*W)
 T3 = 30513/(1.0+ 30513*W)
 T4 = 17*(777.0+104.0*W)
 SSW = (T4+W*(T2+5188.3209+T3*534.5344))/36.0
 T4 = 17.0*104.0
 T3 = 0.2703*5188.3209/(1.0+ 0.2703*W)*2.0
 T6 = 30513*534.5344/(1.0+ 30513*W)*2.0
 H = T1-T2-T3-(T4+T5+T6)/SSW
 RETURN
 END

STOP
 HPDC(C)
 COMMON /BL/CONST/BL1/ALPHA/BL3/PRC/UBL/W1C, W2C
 EXTERNAL F1, F2
 This function finds the values W1C, W2C such that
 the interval (W1C, W2C) have the same posterior
 ordinates for a given C, the value of ordinate
 being the product of C and the posterior ordinate
AT THE MODE.

AND THEN GIVES THE PROBABILITY CONTENT OF THIS

INTERVAL.

CALL SOLMOD(W)
X = W / (1.+W)
XL = 1.0E-20
XR = 1.0-(1.0E-20)
PRX = F1(X)
PRW = PRX*PRX
PRC = C*PRW
ITMAX = 100
XM = X
CALL ZFALSE(F2, 1.0E-10, 10, XL, XM, X1C, ITMAX, IER)
ITMAX = 100
XM = X
CALL ZFALSE(F2, 1.0E-10, 10, XM, XR, X2C, ITMAX, IER)
U1C = ALGO(X1C/(1.-X1C))
U2C = ALGO(X2C/(1.-X2C))
W1C = EXP(U1C)
W2C = EXP(U2C)
PROBC = DCADRE(F1, X1C, X2C, 1.0E-100, 1.0E-50, ERR, IER)
HPDC = PROBC-(1.-ALPHA)
RETURN.

END

FUNCTION F2(X)
COMMON /BL3/PRC
F2 = F1(X)**(1.0.0)**2.0-PRC
RETURN

END

FUNCTION F(X)
IF(X EQ 0.0.E0 .OR. X EQ 1.0.E0) GO TO 1
W = X/(1.-X)
T1 = W**0.02703/(1.0+.02703*W)
T2 = 30513/(1.0+.30513*W)
T3 = 17*(777.0+104.9*W)
SSGW = (T3+T1*3188.3209+T2*534.5344)/36.0
FP = (1.0+.02703*W)**(1.0+.30513*W)
F = W**8.5*((1.0-X)**(-2.0))*SSGW**(-18.0)*FP*(-0.5)
GO TO 2
F = 0.0.E0
RETURN

END

FUNCTION F1(X)
COMMON /BL/CONST
F1 = F(X)/CONST
RETURN

END

SUBROUTINE ZEROHPD(HPD, XMIN, XMAX, ERROR, ITM, XSOL)
FINDS THE SOLUTION SUCH HPD IS ZERO
SOLUTION XSOL LIES BETWEEN XMIN AND XMAX
ITM = MAX NO OF ITERATIONS
ERROR = ERROR BOUND
FMIN = HPD(XMIN)
FMAX = HPD(XMAX)
ITMX = 1
IF((FMIN GT 0.0 .AND. FMAX GT 0.0) .OR.
1(FMIN.LT.0.0.AND.FMAX.LT.0.0)) GO TO 2
4 X = (XMIN+XMAX)/2.0
F = HPD(X)
IF(ABS(F).LE.ERROR) GO TO 6
ITMX = ITMX + 1
IF( ITMX.GT.ITM) GO TO 5
IF(F.GT.0.0) GO TO 3
XMAX = X
GO TO 4
3 XMIN = X
GO TO 4
5 PRINT*:"NO. OF ITERATIONS HAVE REACHED MAXIMUM. VALUE OF
1 HPD=",F
GO TO 1
2 PRINT*:"SOLUTION DOES NOT LIE IN THE GIVEN INTERVAL"
GO TO 1
6 PRINT*:"XSOl = X"
1 RETURN
END
C SUBROUTINE HPD1(ALPHA, CMIN, CMAX, W1CU, W2CU)
C COMPUTES HPD INTERVAL BASED ON U,
COMMON /BL/CONST
EXTERNAL HPDC1
ITMAX=100
EPS=1.0E-10
XL=CMIN
XR=CMAX
CALL ZEROPD(HPDC1, XL, XR, EPS2, ITMAX2, CSOL1)
RETURN
END
C SUBROUTINE SOLMOD1(W)
C COMPUTES THE MODE OF P(LOGW/Y)
COMMON /BL/BL1,ALPHA/UBL3/PRC1/UBL1/W1CU, W2CU
C EXTERNAL H1
EPS=1.0E-10
NSIG =10
XL = 1.0E-10
XR = (V1/V2)+20.0
ITMAX =100
CALL ZFALSE(H1, EPS, NSIG, XL, XR, W, ITMAX, IER)
RETURN
END
FUNCTION H1(W)
T1=19/W
T2 = .02703/(1.0+.02703*W)
T3 = 30513/(1.0+30513*W)
T4 = .02703*5188.3209/(1.0+.02703*W)**2.0
T5 = 30513*534.5344/(1.0+30513*W)**2.0
T6=17*1777.0+104.0*W
SSGW=(T6*W*(T2*5188.3209+T3*534.5344))/36.0
T7=17*104.0
H1 = T1-T2-T3-(T7+T4+T5)/SSGW
RETURN
END
FUNCTION HPDC1(C)
COMMON /BL/BL1/ALPHA/UBL3/PRC1/UBL1/W1CU, W2CU
EXTERNAL F1, F2U

This function finds the values W1C, W2C such that
the interval (W1C, W2C) have the same posterior
ordinates for a given c, the value of ordinate of log(W)
being the product of c and the posterior ordinate
at the mode.
And then gives the probability content of this
interval.

CALL ZFAIL1(W)
X = W / (1. + W)
XL = 1.0E-10
XR = 1.0E-10
PRX = F1(X)
PRW = PRX * (1. - X)**2.0
PRU = PRW * W
PRC1 = C * PRU
ITMAX = 100
XM = X
CALL ZFAIL(F2U, 1.0E-10, 6, XL, XM, X1C, ITMAX, IER)
XM = X
ITMAX = 100*
CALL ZFAIL(F2U, 1.0E-10, 6, XM, XR, X2C, ITMAX, IER)
U1C = ALOG(X1C/(1. - X1C))
U2C = ALOG(X2C/(1. - X2C))
W1CU = EXP(U1C)
W2CU = EXP(U2C)
PROBC = DCADRE(F1, X1C, X2C, 1.0E-100, 1.0E-50, ERR, IER)
HPDC1 = PROBC - (1. - ALPHA)
RETURN
END

FUNCTION F2U(X)
COMMON /UBL3/PRC1
F2U = F1(X) * X * (1. - X) - PRC1
RETURN
END