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A BLOCKING MODEL
AND ITS APPLICATIONS IN COMMUNICATION NETWORKS

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Abstract

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Performance analysis of resource-sharing systems such as packet radio and circuit-switching networks is particularly difficult, as contention introduces statistical dependencies among the elements of the system. Much of the work in this area has focussed on networks with simple topologies and symmetrically-loaded networks. In the more general context of multihop packet radio networks, blocking and hidden-terminal interference among tranceivers are prevalent. Blocking is a nondestructive interaction, whereby a transmission is inhibited to resolve contention. On the other hand, hidden-terminal interference is a destructive interaction, resulting in transmission collisions. Some means of incorporating the topology of the network into a performance model is required in order to properly characterize these interactions.

In this thesis, a model is described, extended and applied to determine throughput, blocking and interference in broadcast networks and circuit-switched interconnection networks. The notions of blocking and interference are formally described as relations on the set of links of a network. Symmetry of the blocking relation results in a product-form Markov model for the dynamics of the system. A transmission interaction model is given to systematically categorize all possible interactions for a class of channel-access protocols. Nonpersistent CSMA, BTMA and Directional CSMA are

modeled within this framework. In addition, two nonrealizable protocols, CFMA and TCAP are introduced as capacity measures for networks with omnidirectional and directional transmission capabilities, respectively. A comprehensive comparative study of CSMA, BTMA and D-CSMA is conducted for various networks, using CFMA and TCAP to determine relative capacity utilizations.

The computational aspects of the model are greatly simplified for special cases such as symmetric loading, interference-free systems or regular networks. For these cases, the model is developed further to obtain efficient recursive methods to compute performance measures directly. These methods are then applied to study throughput and blocking in a variety of circuit-switched interconnection networks, where centralized control precludes the need to deal with interference. This approach captures a range of switch architectures and allows perturbation analysis and more general asymmetric-loading situations to be handled.

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Chapter 1

INTRODUCTION

1.1 Communication Networks

Communication networks play a vital role in our present-day hign-tech society with its growing dependency on information exchange. They can be used to carry information between distant network stations, as demonstrated by public telephone networks. On the other hand, they can be used to deliver information within a local environment. Typical examples are Local Area Networks (LANs) and Private Branch exchanges (PBXs). When hosts computers and terminals are involved in data transmission and reception, communication networks are often referred to as computer communication networks. The most important characteristic of a computer communication network is that it can permit remote sites to use each other's facilities, hardware, software or data. Currently, as more and more advanced technologies are being adopted into communication environments, applications in communication networks increase very rapidly. In fact, each year, there are many new product lines and services being developed and introduced into applications. At present time, facsimile, electronic mail, and video service have been widely used in industries, educational institutes, and government agencies. Therefore, it is not an overstatement to say that communication networks have become an essential part of our modern society.

Concerning communication network problems, one fundamental question often posed is: how should a communication network be designed for data transmission?

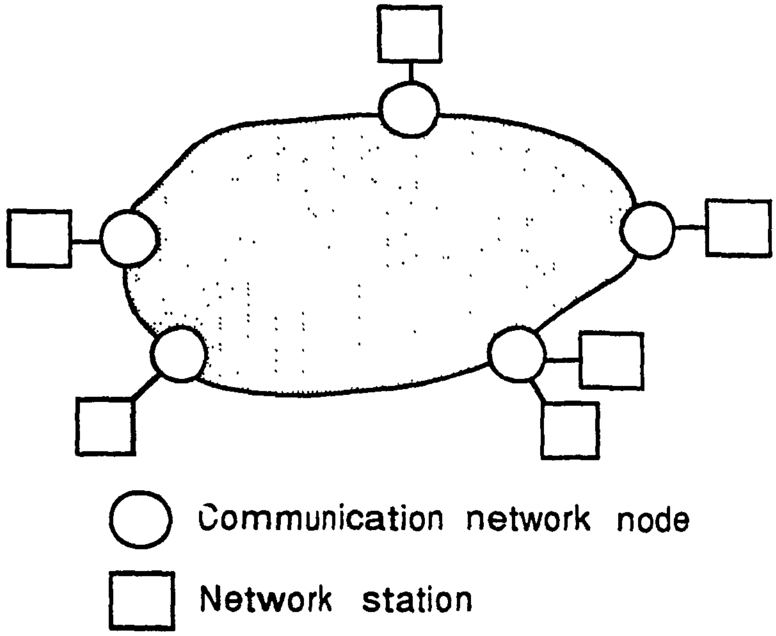


Figure 1.1: Interconnection via a communication network

One simple but very expensive solution is to directly connect every pair of devices with a direct point-to-point communication link. A more feasible solution is to attach the communication devices to a shared communication network. Figure 1.1 illustrates this concept in a general way. We refer to the device wishing to communicate as a station, and each station attaches to a network node. The set of nodes to which station is attached is the boundary of the communication network, which is capable of transferring data between pairs of attached stations.

Based on the architecture and techniques used to transfer data, communication networks can be categorized as (1) Switched Communication Networks (Circuit-Switched Networks and Packet-Switched Networks) and (2) Broadcast Communication Networks (Packet Radio Networks, Satellite Networks, and Local Networks). A switched communication network consists of an interconnection of nodes, in which data are transmitted from source to destination by being routed through the network nodes. Data entering the network from a station are routed to their final destination by being switched along the transmission path from node to node. In the case of circuit switching, a dedicated path from transmitter to receiver is first established, then a stream of data from the transmitting station follows this path to its final destination without stopping at the intermediate nodes. During the course of its operations, three phases of actions take place, namely, (1) circuit establishment, (2) data transfer, and (3) circuit disconnection. In broadcasting networks, each station is attached to a transmitter/receiver that communicates over a medium shared by other stations. In its simplest form (one hop transmission), a transmission from any one station is broadcast to and received by all other stations. In this case, the transmission facility is shared and only a certain stations can successfully transmit at a time. This leads to the need for some mechanism for controlling access to the shared medium.

The key questions in any medium access control technique can be abbreviated into two simple words: "where" and "how" [Sta85]. "Where" refers to whether control is exercised in a centralized or in a distributed fashion. In a centralized case, a controller is designated who has the authority to grant access to the network. A station wishing to transmit must wait until it receives permission from the controller. Typically examples are circuit switching networks. In a decentralized (distributed) network, the stations collectively perform a medium access control function to dynamically determine the order in which stations transmit. Packet switching networks and most broadcasting networks are the often encountered applications of this scheme.

The second question "how" refers to how the channel access control technique operates. It is constrained by network topology and determined by a trade off among competing factors: cost, performance, and complexity. In general, access control techniques can be categorized as being synchronous or asynchronous. With synchronous techniques, a specific capacity is dedicated to a connection. Circuit switching is a typical application of this technique. Synchronous techniques are inappropriate to handle bursty traffic. On the other hand, asychronous transmission techniques are applied to handle this problem. With asychnronous technique, the capacity resources are allocated in a dynamic fashion, more or less in response to immediate needs.

The asynchronous transmission schemes include Round-robin, Reservation, and Contention. With Round-robin, each station in turn is provided with an opportunity to transmit a message. It is efficient when many stations have data to transmit over an extended period of time. However, when only a few stations have data to transmit at any given time, it has low efficiency. In this case, reservation and contention techniques perform much better. Usually, resveration is employed for stream traffic. On the other hand, contention is applied for bursty traffic. Stream traffic is characterized by lengthy and continuous transmissions (i.e., voice communication, telemetry, and bulk file transfer). Burst traffic is characterized by short sporadic transmissions (i.e., interactive terminal-host traffic). These techniques are of necessity distributed in nature, they are efficient under light to moderate load. However, under heavy load, performance tends to degrade [Sta85].

1.2 Blocking and Interference in Communication Networks

In computer communication networks, we often observe the following characteristics:

- Input traffic required to be transmitted in the network is often bursty. In this situation, contention for the transmission medium is the access technique often applied.
- Communication links are often simplex or half-duplex. That is, communication is either unidirectional or a station can only transmit message to its neighboring stations, or receive message destined to it, but not both at the same time.
- The transmitting station generally knows only limited amount of information about the intended receiver or the intermediate nodes along its way to a final destination.
- Distributed control techniques are often applied to the networks. Each station collects its local information and decides what to do during the course of data communication.

Based on the above observations, we define two terminologies to describe communication actions. They are transmission blocking and transmission interference. Both are closely relate to the media access protocols. Transmission blocking is defined to be the event in which the transmission scheduled to take place is not allowed to actually happen due to the fact that the other transmitting facilities involved in the course of data communications are not available. This transmission blocking is the result of the transmitter's actions after it collects its local information and knows that this scheduled transmission action would not be successful. Transmission blocking often occurs in the following situations:

1. The transmission medium is occupied by the other entities of the system, so that a scheduled transmission would not be able to access to the medium exclusively.

2. The transmitter detects the situation that the receiving station is unable to accept the scheduled transmission. This could be the case when the receiver is transmitting or receiving other data.

3. The transmitter is busy at the time of its scheduled transmission. This happens when the transmitter is busy receiving data from or transmitting data to other stations.

4. If the buffer capacity in the network is not very large, a transmission could also be blocked whenever the receiver cannot store the incoming data in the buffer.

Therefore, blocking often occurs when the controller of the data transmission detects the status of the system and withdraws its scheduled transmission. In this case, the other parts of the system are not affected by this blocking. They can still be involved in communication with other parts of the system.

On the other hand, transmission interference is defined as the event in which the intended receiver of a transmission is receiving other transmissions and is, therefore, incapable of responding. Thus, transmission collision is the result at the receiver. Under this situation, it is up to the receiver to decide what to do next. Two collision capabilities are considered: *perfect capture* and *zero capture*. With the perfect capture, the receiver will accept one transmission and abandon the other transmissions. It is often the transmission which first reaches the receiver which is accepted. Under the zero capture scheme, all transmissions are abandoned at the receiver. In our performance studies, we will assume the system with the scheme of perfect capture only. Transmission Interference is a commonly-encountered phenomenon for communication systems with distributed control, where transmitters do not have complete information concerning the state of neighboring nodes. It is highly dependent on the traffic rate and it determines the saturation point of the system. For example, when the system is operating under heavy-load conditions, transmission interference will cause the system performance to deteriorate to the point where no transmission can successfully pass through the channel.

1.3 Contents of the Thesis

The material in this thesis consists of four major parts, and they will be presented in the following chapters:

In Chapter 2, we describe and formalize the blocking model for its applications in broadcast networks. Basically, we provide some prerequisites for applying the blocking model to communication networks. We study the reversibility condition for element activation blockings, consequently, we establish the criterion for applicability of the blocking model in communication network systems. In addition, we create the Transmission Blocking Graph (TBG) and the Reception Interference Graph (RIG) to generally describe blocking and interference in the given network. With the information that the TBG and the RIG provide, performance measures can be computed in a systematic way.

In Chapter 3, we elaborate the analytical results from the blocking model. First of all, we study some computational techniques for performance measures. Based on that, we develop a general iterative-estimation technique for networks of arbitrary topologies. Secondly, we categorize the operation of different channel access protocols by considering the possible interactions between two transmissions in the network. Thus, the TBG and the RIG can be systematically generated. Two ideal protocols, Collision-Free Multiple Access (CFMA) and Totoal Capacity Allocation Protocol (TCAP), are created to provide relative channel capacity measurements for systems with omnidirectional transmission and directional transmission, respectively. Thirdly, we consider interference-free systems. With these systems, the partition function alone can be used to measure the performance of the system. Lastly, we establish recursive relations for partition functions of some regular networks operating under TCAP with uniform-normalized transmission rates.

In Chapter 4, we apply the blocking model to study the performance of some packet radio networks. We consider the uniform link traffic and the uniform end-to-end traffic cases, individually. Further, we compare the channel throughput for networks operating in these two cases. The iterative-estimation method is employed for performance measures.

In Chapter 5, we apply the blocking model to circuit switching networks in which their operation can be characterized by TCAP. By using the recursive expressions among partition functions for networks of regular topologies, we generate the performance measures in recursive forms as well. Typically, we study the completely-connected networks, the crossbar networks, and the recirculating shift network under symmetric loading. Similarly, we study crossbar networks operating under skewed loading where the input ports are partitioned into many different groups.

1.4 Major Contributions

The major contributions of this research work can be sumarized into the following areas

1. The formalization of the blocking model to broadcast network applications is established by the TBG and the RIG of the given system. With the TBG and RIG generated, system performance measures can be determined.

2. The high flexibility of the blocking model is well demonstrated in the examples by referring to nodes and to directed links as basic elements of the system with omnidirectional transmission operations. Both approach can quantify the same performance measures.

3. The transmission interactions model provides a systematic mean to describe the operations of different channel access protocols. By using this model, the TBG and the RIG can be generated for system performance studies.

4. Two non-realizable channel access protocols, CFMA and TCAP, are created to provide relative channel capacity measurements for systems with omnidirectional transmission and directional transmission, respectively. By comparing the operating protocol (i.e., CSMA, BTMA, or D-CSMA) with CFMA or TCAP, the true channel capacity utilization of the system is revealed.

5. Interference-free systems provide a direct, easily-used method to quantify system performance measures.

6. The generation of recursive expressions for system performance measures provide us with a simple method to analyse network systems of large size under symmetric loading. This certainly removes the limitation imposed on analyzing networks of large size.

7. The general recursive expressions for performance measures of the crossbar network under skewed loading provides us with a highly-efficient computational method to study network performance of these kinds.

Chapter 2

A General Model of Blocking in a Resource Sharing Environment

2.1 Introduction

In this chapter, we describe a blocking model for systems in which individual elements contend to use a distributively-shared resource. The word "distributively" refers to those situations where each element is allowed to use only certain portions of the resource that are allocated to it. The model is quite general; it can be applied to many systems that share a resource distributively. However, we are interested in employing this model to the area of communication networks, particularly packet radio networks (PRNs) and circuit-switching networks (CSNs). Circuit switching networks which use a centrally-shared resource are considered as special cases of the distributed resource-sharing model.

The model described in this chapter was originally developed by Boorstyn, Kershenbaum and Maglaris in [KB84], [BKM87] to study the throughput behavior of nonpersistent carrier sense multiple access (CSMA) in multihop packet radio networks with perfectcapture capabilities. It was later extended by Tobagi and Brazio [TB83] to include other protocols. The model has its mathematical foundations in the theory of reversible Markov fields [Kel79] and special cases of the model are well known in the theory of statistical thermodynamics [Yem83].

Brazio and Tobagi [BT84] investigated some theoretical aspects of the model. They derived necessary and sufficient conditions for reversibility of the underlying

Markov process and considered throughput computation in zero capture. In describing this model, we establish necessary and sufficient conditions for reversibility in a much simpler manner, using Kolmogorov's criterion and some well-known notions in graph theory. For reasons of tractability, we restrict our attention to the perfect-capture case, when studying packet radio networks. However, the notion of capture is not needed in our application of the model to switching networks.

For our purposes, we represent the topology of a communication network by a graph $G = (N, L)$, where $N = \{1, 2, \ldots, m\}$ is a set of m nodes representing the communication stations in the network and $L = \{l_1, l_2, \ldots, l_n\}$ is a set of n directed links representing the possible physical data-communication channels in the network. Figure 2.1 illustrates a 4-node chain, where $N = \{1, 2, 3, 4\}$ and $L = \{l_1, l_2, l_3, l_4, l_5, l_6\} = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}$. Figure 2.1 will be repeatedly referred to in examples throughout this thesis.

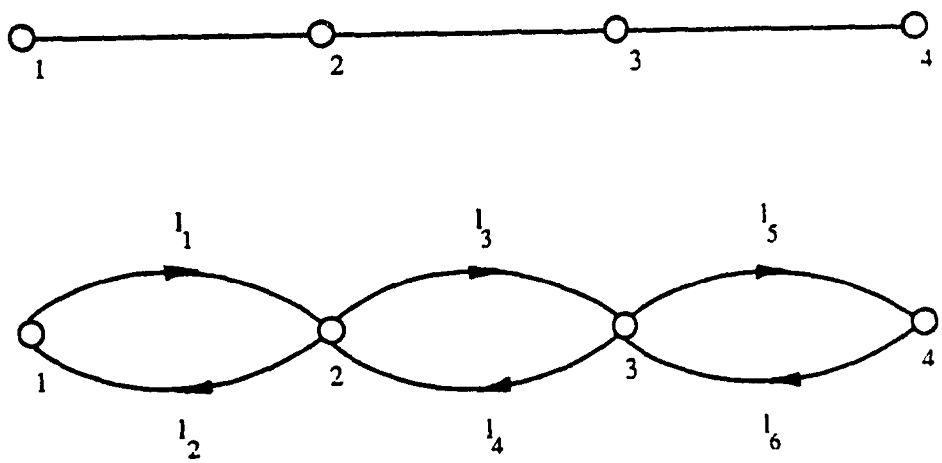


Figure 2.1: The graphical representation of a 4-node chain network

In a similar way, a network topology can also be represented by its adjacency matrix. Using the terminology of broadcast networks, we say that node j can *hear* node i if link $(i, j) \in L$. The *adjacency* or *hearing matrix* $H = [h_{ij}]_{m \times m}$ of the network is the matrix whose elements are given by

$$h_{ij} = \begin{cases} 1 & \text{if node } j \text{ can hear node } i; \\ 0 & \text{otherwise.} \end{cases}$$

Each nonzero entry h_{ij} in the hearing matrix corresponds to a directed link (i, j) in the network from node i to node j. We call node i the source and node j the destination of directed link (i, j). In broadcast media, a message initiated by a given node will reach nodes other than its intended receiver, thus possibly colliding with messages destinated to these nodes. We say the directed link is *active* whenever the source node is transmitting a message intended for the destination node on that link, and *idle* otherwise. For a given link $l_i \in L$, we denote its source node by s_i and its destination node by d_i. Thus a directed link l_i is the ordered pair (s_i, d_i).

In analyzing a communication network for the performance measures of interest in this thesis, namely, throughput, blocking and interference, we need only consider two states (*active* or *idle*) for the nodes or edges. For the purpose of generality, we will use the term *elements* to refer to either nodes or directed links. Generally speaking, activity on a directed link implies activity of the nodes associated with it. For instance, given a directed link $l_i = (s_i, d_i)$, its activation signifies the fact that node s_i is transmitting while node d_i may or may not be attempting to receive the transmitted message.

2.2 A Markov Model of Blocking

Consider a set $V = \{1, 2, \cdots, n\}$ of n elements, where at each instant of time t, each element $i \in V$ can be in one of two states. We refer to the states of an element as *idle* and *active*. Collectively, the elements of V assume an n-dimensional binary state called the *system state*. The system state may be represented by the subset $S(t) \subseteq V$ consisting of all active elements of V at time t or, equivalently, by an incidence vector $I(S(t)) = b_1(t), b_2(t), \cdots, b_n(t)$, where

$$b_i(t) = \begin{cases} 1 & \text{if } i \in S(t) \text{ (element } i \text{ is active at time } t\text{);} \\ 0 & \text{otherwise (element } i \text{ is idle at time } t\text{).} \end{cases}$$

For notational simplicity, we usually drop the dependency of $S(t)$ on t by writing S or $b_1 b_2 \cdots b_n$ to describe the system state and assume it is understood that S is a function of time.

At points in time governed by an independent Poisson process of rate λ_i, element

$i \in V$ attempts an *activation.* An interference structure in the form of a *blocking relation* among the elements of V determines whether element i can become active. We say that element $j \in V$ *blocks* element $i \in V$ if element i's activation attempts are not permitted whenever element j is active. Blocked activation attempts are discarded by the system. If element i is active at the time of an attempt to activate it, we assume the attempt is also blocked and, hence, the relation is reflexive. Then, element i may become active at time t if and only if all elements which block i are idle at time t. Once activated, element i remains active for an independent, exponentially-distributed duration with mean $1/\mu_i$ and then *deactivates,* returning to the idle state. Under the assumptions stated so far, the system state $S(t)$ is a continuous-time finite Markov chain. The restriction to exponential active times can be relaxed in favor of more general distributions. This will be discussed in Section 2.3.

We refer to the directed graph (V, E) of the blocking relation on V as the *blocking graph* of the system, where E is the set of edges $\{(i,j) : i, j \in V,\ i \text{ blocks } j\}$. Given the blocking graph of the elements of V, we can generate the states and transitions of this chain. To investigate further, we define the following sets:

1. $B^*(i)$ is the collection of all elements $j \in V - \{i\}$, which are blocked by i, or

   $$B^*(i) = \{j : j \in V - \{i\},\ j \text{ is blocked by } i\}.$$

   Since element i blocks itself, the *blocked set* of i or the set of elements blocked by element i is $B(i) = B^*(i) \cup \{i\}$.

2. Inversely, the *blocking set* of i is the collection $\tilde{B}(i)$ of all elements $j \in V$ which block i from becoming active, or

   $$\tilde{B}(i) = \{j : j \in V,\ j \text{ blocks } i\}.$$

   In general, $\tilde{B}(i) \neq B(i)$.

3. The state S blocks element $j \in V$ if there exist some element $i \in S$ which blocks j. Therefore, $B(S)$ is the collection of all elements $j \in V$, which are blocked by state S, or

   $$B(S) = \{j : j \in V,\ S \text{ blocks } j\}.$$

Clearly,

$$B(S) = \bigcup_{i \in S} B(i).$$

4. $U(S) = V - B(S)$ is the set of all elements $j \in V - S$ which are not blocked by state S.

The assertions and definitions that follow characterize the states of the system in a constructive fashion, providing a means of generating them systematically.

Since deactivations are not blocked, if $S \subseteq V$ is a state, then all elements in S can deactivate in any sequence, leading to the idle state ϕ. Hence, we have the following assertion:

Assertion 2.1 *ϕ is a state and ϕ is reachable from every state S.*

On the other hand, activations are blocked according to the given blocking relation and the current state of the system. Not every sequence of element activations is permitted.

Definition 2.1 *A sequence $(i_1, i_2, \cdots, i_m)$ of element activations, starting from state ϕ, where $i_j \in V$, $j = 1, \cdots, m$, is* **permissible** *if and only if i_j is not blocked by state $\{i_1, i_2, \cdots, i_{j-1}\}$, $j = 2, \cdots, m$. Note: No element is blocked by state ϕ.*

Assertion 2.2 *A non-empty subset $S \subseteq V$ is a state if and only if there exists some permissible sequence of activations of all and only the elements in S.*

The sufficiency of this condition is obvious. That only the elements in S need to be activated can be established by noting that activation of an element not in S can only block more elements from becoming active. It never introduces new candidate elements for activation. Any element not in S which is activated must be deactivated prior to reaching S. Hence, only the elements in S need to be activated in reaching S.

Definition 2.2 *The state space Ω is the set of all subsets $S \subseteq V$ satisfying Assertion 2.2 together with the empty set ϕ.*

In the remainder of this section, we clarify the definitions presented above by demonstrating how the model may be applied to characterize the operational behaviour of a channel access protocol in a broadcast network, under certain assumptions. We consider the solution to this model and the calculation of performance measures in subsequent sections.

Given is a multihop broadcast network with topology $G = (N, L)$ and hearing matrix H. Each communication link of the network is an element of the blocking model, i.e., $V = L$. Messages are scheduled for transmission on link $l_i = (s_i, d_i)$ at node s_i according to a Poisson process of rate λ_i. These scheduled transmissions may be considered to include both newly-arrived messages and backlogged messages which were previously transmitted unsuccessfully, as long as the rescheduling of backlogged traffic is randomized in such a way as to render the total traffic offered Poisson in character. Link l_i is active whenever node s_i is transmitting a message to node d_i and idle otherwise. A fundamental assumption required is that state-transitions are made instantaneously, based on the state at the transition instants. Thus, we assume that propagation delays over the links of the network are negligible, as propagation delay would destroy the memoryless property of the process. Each message arrival on link $l_i = (s_i, d_i)$ is an activation attempt of l_i. Depending on the state of the network at the time and the channel access protocol in use, node s_i will either accept the activation attempt and activate l_i or it will block the attempt, leaving the state of l_i unchanged. Further assumptions will be presented when we consider computation of performance measures in Section 2.5.

To illustrate, we consider two common channel access protocols: (1): NP-CSMA (Nonpersistent Carrier Sense Multiple Access) and (2): RI-BTMA (Receiver-Initiated Busy tone Multiple Access). The operations of these two protocols are first explained below:

NP-CSMA: Under NP-CSMA, transmission attempts on a link are blocked if its source node is currently transmitting or senses carrier from any node it can hear. Formally, link $l_i = (s_i, d_i)$ is blocked by $l_j = (s_j, d_j)$ whenever $l_j = (s_j, d_j)$ is active and either $h_{s_j s_i} = 1$ or $s_i = s_j$.

RI-BTMA: The destination of a link emits a busy tone on a separate channel whenever that link is active. Transmission attempts on a link are blocked when its source node is currently transmitting or senses either carrier or busy tone from any node it can hear. Formally, link $l_i = (s_i, d_i)$ is blocked by $l_j = (s_j, d_j)$ if $l_j = (s_j, d_j)$ is active and either $h_{s_j s_i} = 1$, $h_{d_j s_i} = 1$ or $s_i = s_j$.

Thus, the elements of the blocking model in each case are the links of the network and the blocking relation is a function of the topology of the network. The following examples clarify the definition of each protocol. In this context, we will refer to the blocking graph as the Transmission Blocking Graph (TBG). In the TBG, a directed edge from l_i to l_j signifies that transmissions on link l_i blocks transmission attempts on link l_j.

Example 2.1 The TBG for the 4-node chain PRN of Figure 2.1 operating under NP-CSMA is given in Figure 2.2.

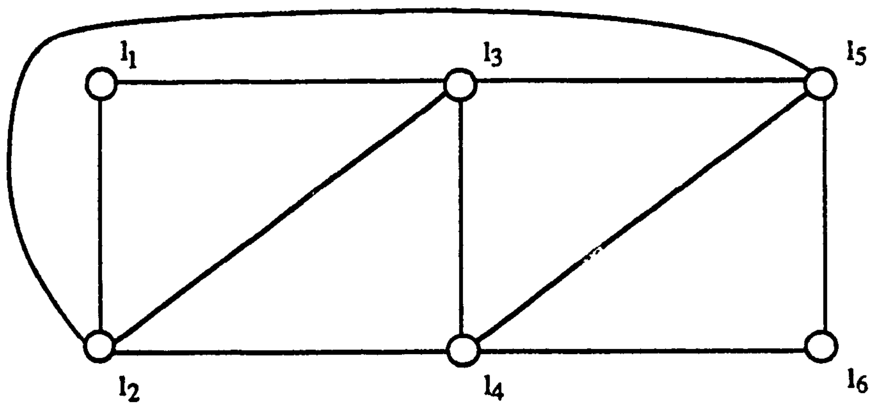


Figure 2.2: The TBG for the 4-node chain PRN operating under NP-CSMA

For this network $G = (N, L)$, we have,

$$N = \{1,2,3,4\}; \qquad L = \{l_1, l_2, l_3, l_4, l_5, l_6\} = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}.$$

Inspecting the TBG, the blocked set for each element $l_i \in L$ is given below:

$$\begin{aligned} B(l_1) &= \{l_1, l_2, l_3\}; & B(l_2) &= \{l_1, l_2, l_3, l_4, l_5\}; \\ B(l_3) &= \{l_1, l_2, l_3, l_4, l_5\}; & B(l_4) &= \{l_2, l_3, l_4, l_5, l_6\}; \\ B(l_5) &= \{l_2, l_3, l_4, l_5, l_6\}; & B(l_6) &= \{l_4, l_5, l_6\}. \end{aligned}$$

Note that $\tilde{B(l_i)} = B(l_i)$ for all $l_i \in L$.

The state space is shown in Figure 2.3, and the state-transition diagram is given in Figure 2.4.

The state space of the system is:

$$\Omega = \{\phi, \{l_1\}, \{l_2\}, \{l_3\}, \{l_4\}, \{l_5\}, \{l_6\}, \{l_1, l_4\}, \{l_1, l_5\}, \{l_1, l_6\}, \{l_2, l_6\}, \{l_3, l_6\}\}.$$

The blocked set for each state $S \in \Omega$ is listed below:

$$\begin{array}{ll} B(\{l_1\}) = \{l_1, l_2, l_3\}; & B(\{l_2\}) = \{l_1, l_2, l_3, l_4, l_5\}; \\ B(\{l_3\}) = \{l_1, l_2, l_3, l_4, l_5\}; & B(\{l_4\}) = \{l_2, l_3, l_4, l_5, l_6\}; \\ B(\{l_5\}) = \{l_2, l_3, l_4, l_5, l_6\}; & B(\{l_6\}) = \{l_4, l_5, l_6\}; \\ B(\{l_1, l_4\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; & B(\{l_1, l_5\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; \\ B(\{l_1, l_6\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; & B(\{l_2, l_6\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; \\ B(\{l_3, l_6\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\}. & \end{array}$$

Correspondingly, the unblocked sets are listed as follows:

$$\begin{array}{ll} U(\{l_1\}) = \{l_4, l_5, l_6\}; & U(\{l_2\}) = \{l_6\}; \\ U(\{l_3\}) = \{l_6\}; & U(\{l_4\}) = \{l_1\}; \\ U(\{l_5\}) = \{l_1\}; & U(\{l_6\}) = \{l_1, l_2, l_3\}; \\ U(\{l_1, l_4\}) = \phi; & U(\{l_1, l_5\}) = \phi; \\ U(\{l_1, l_6\}) = \phi; & U(\{l_2, l_6\}) = \phi; \\ U(\{l_3, l_6\}) = \phi. & \end{array}$$

□

Example 2.2: The TBG for the 4-node chain PRN of Figure 2.1 operating under RI-BTMA is given in Figure 2.5. The graphical representation of the state space is shown in Figure 2.6, while the state-transition diagram is presented in Figure 2.7.

The blocked and blocking set are:

$$\begin{array}{ll} B(l_1) = \{l_1, l_2, l_3, l_4, l_5\}; & \tilde{B}(l_1) = \{l_1, l_2, l_3, l_4\}; \\ B(l_2) = \{l_1, l_2, l_3, l_4, l_5\}; & \tilde{B}(l_2) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; \\ B(l_3) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; & \tilde{B}(l_3) = B(l_3); \\ B(l_4) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; & \tilde{B}(l_4) = B(l_4); \\ B(l_5) = \{l_2, l_3, l_4, l_5, l_6\}; & \tilde{B}(l_5) = \{l_1, l_2, l_3, l_4, l_5, l_6\}; \\ B(l_6) = \{l_2, l_3, l_4, l_5, l_6\}; & \tilde{B}(l_6) = \{l_3, l_4, l_5, l_6\}. \end{array}$$

In this case, $B(l_i) \neq \tilde{B}(l_i)$ for $l_i = l_1, l_2, l_5, l_6$.

The state space of the system is

$$\Omega = \{\phi, \{l_1\}, \{l_2\}, \{l_3\}, \{l_4\}, \{l_5\}, \{l_6\}, \{l_1, l_6\}, \{l_2, l_6\}^d, \{l_5, l_1\}^d\}.$$

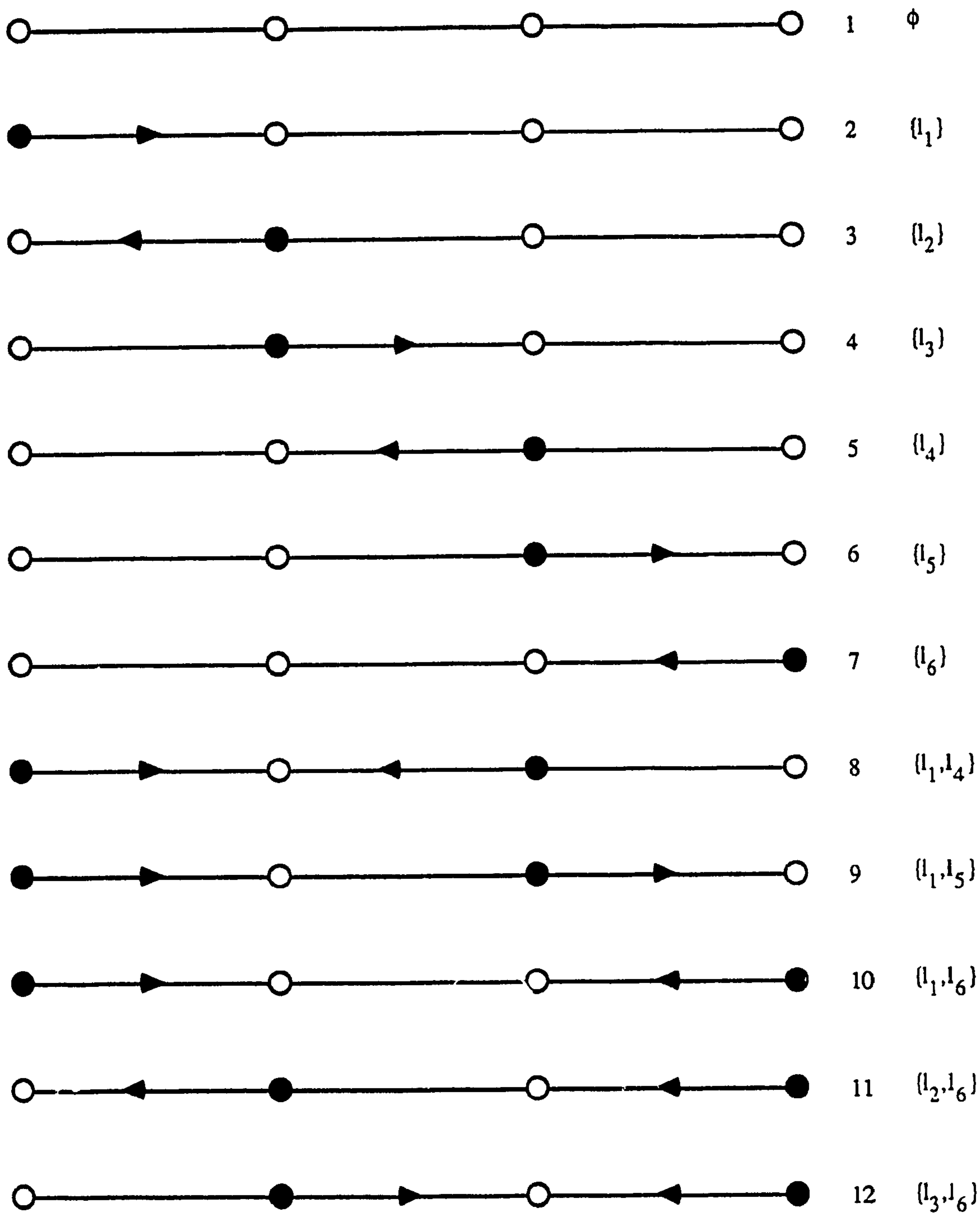


Figure 2.3: The state space for the 4-node chain PRN operating under NP-CSMA

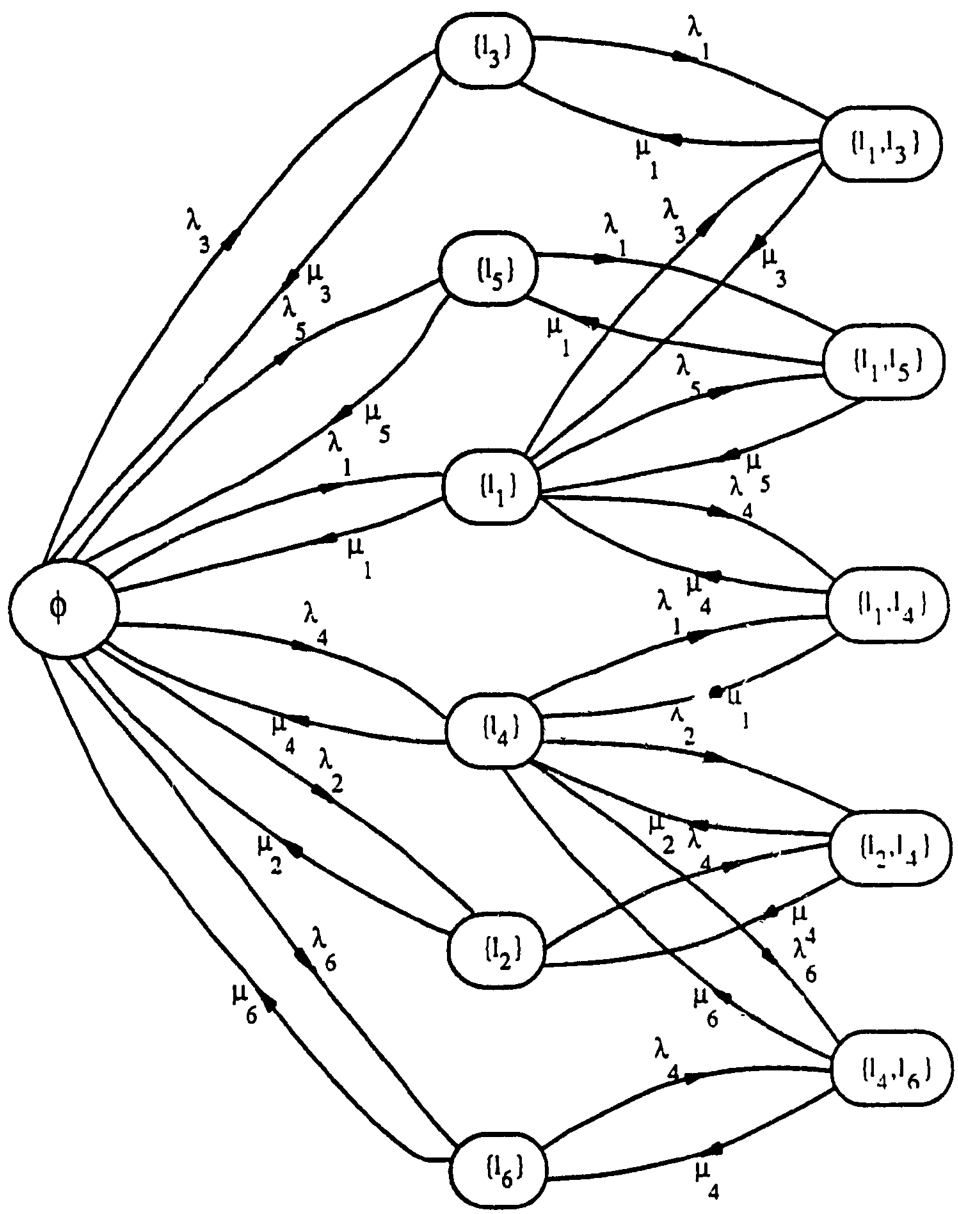


Figure 2.4: The state-transition diagram for the 4-node chain PRN operating under NP-CSMA

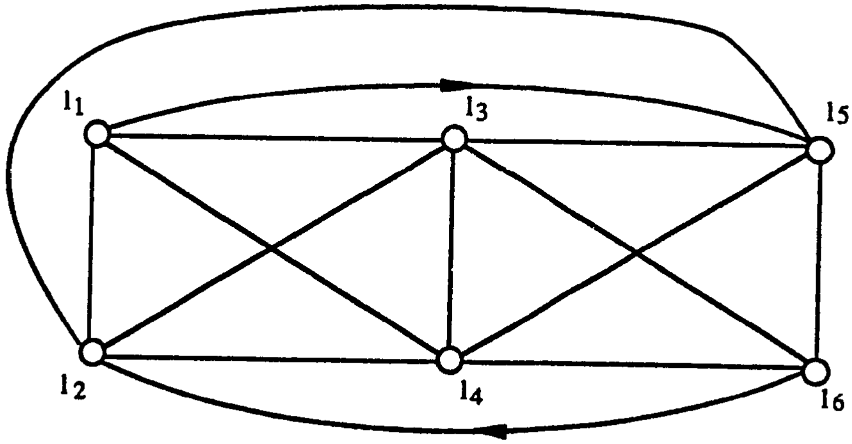


Figure 2.5: The T˜G for the 4-node chain network operating under RI-BTMA

The notation $\{l_i, l_j\}^d$ is used to denote an ordered state of the system, it can arise only if the activation of l_i precedes the activation of l_j. The blocked and unblocked sets of the states are listed as follows:

$B(\{l_1\}) = \{l_1, l_2, l_3, l_4, l_5\};$ $B(\{l_2\}) = \{l_1, l_2, l_3, l_4, l_5\};$
$B(\{l_3\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\};$ $B(\{l_4\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\};$
$B(\{l_5\}) = \{l_2, l_3, l_4, l_5, l_6\};$ $B(\{l_6\}) = \{l_2, l_3, l_4, l_5, l_6\};$
$B(\{l_1, l_6\}) = \{l_1, l_2, l_3, l_4, l_5, l_6\};$ $B(\{l_2, l_6\}^d) = \{l_1, l_2, l_3, l_4, l_5, l_6\};$
$B(\{l_5, l_1\}^d) = \{l_1, l_2, l_3, l_4, l_5, l_6\}.$

$U(\{l_1\}) = \{l_6\};$ $U(\{l_2\}) = \{l_6\};$
$U(\{l_3\}) = \phi;$ $U(\{l_4\}) = \phi;$
$U(\{l_5\}) = \{l_1\};$ $U(\{l_6\}) = \{l_1\};$
$U(\{l_1, l_6\}) = \phi;$ $U(\{l_2, l_6\}^d) = \phi;$
$U(\{l_5, l_1\}^d) = \phi.$

$\{l_2, l_6\}^d$ and $\{l_5, l_1\}^d$ are the states for which the order of activation is relevant. State $\{l_2, l_6\}^d$ is reached by the activation sequence (l_2, l_6) only. The activation sequence (l_6, l_2) is not permissible since the activation of l_6 blocks the subsequent activation of l_2. Similarly, state $\{l_5, l_1\}^d$ is reached only through the activation sequence (l_5, l_1). Note that in Fig 2.6, the states marked with a $*$ on their state numbers are the ordered states of the given system. The numbers assigned on the graph represent the order of activations.

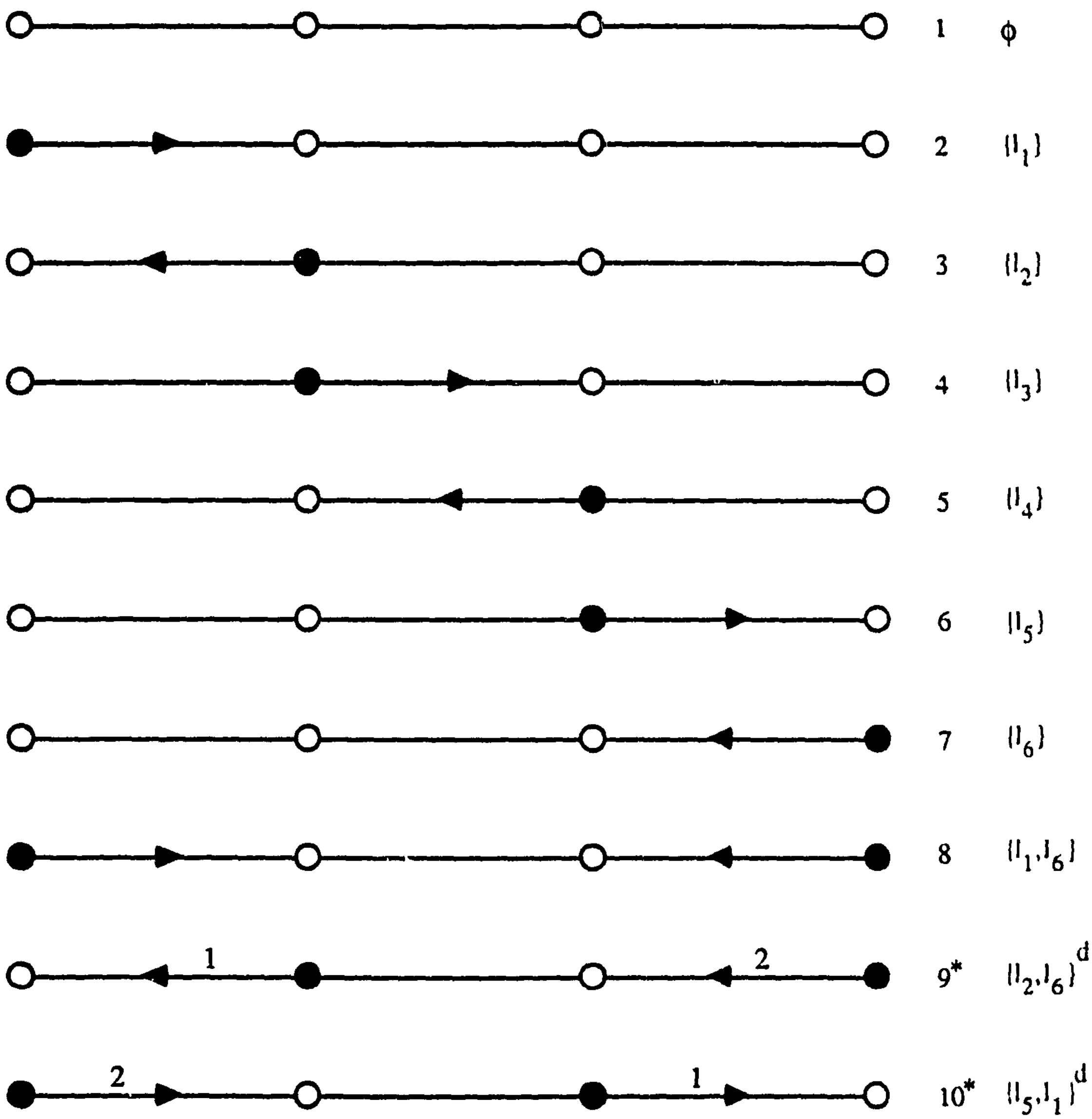


Figure 2.6: The State Space for the 4-node chain PRN operating under RI-BTMA

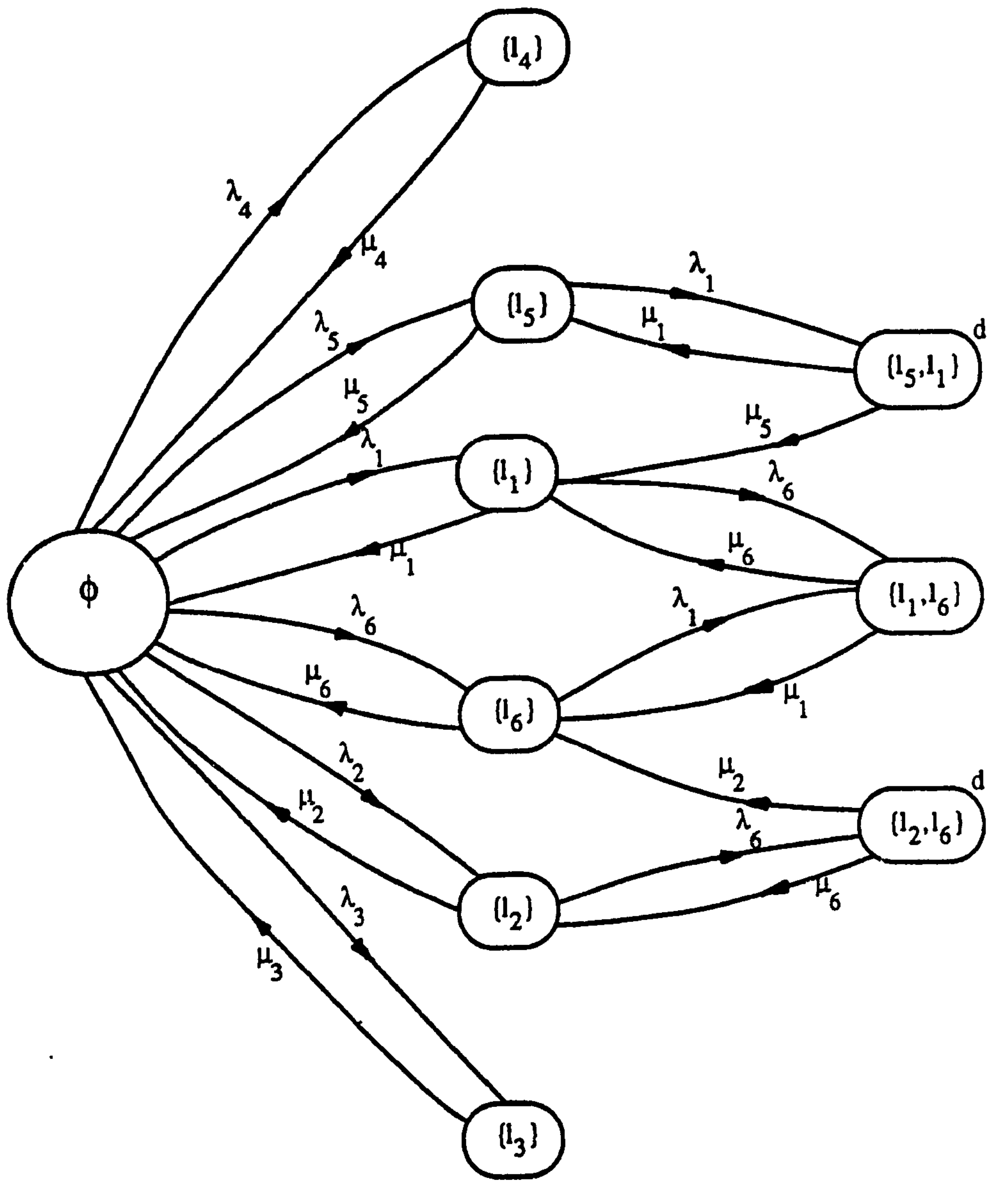


Figure 2.7: The state-transition diagram for the 4-node chain PRN operating under RI-BTMA

2.3 Reversibility Conditions and the Product-Form Solution

In this section, necessary and sufficient conditions are given for reversibility of the Markov process of Section 2.2 and the associated product-form solution for the equilibrium probabilities of the states is presented. The reader is referred to [BT84] for the construction of the global balance equations of the model. We restrict our investigation to reversible processes for the computational simplicity offered by a product-form solution.

Time reversibility of a stochastic process is the property which renders its sample functions statistically indistinguishable from the sample functions of the process running in reverse time. Stationarity is easily seen to be a necessary condition for reversibility. For instance, if the mean of the process increases in forward time over some interval then the mean of the process decreases in reverse time over the same interval. Hence, sample functions from the forward and the reversed processes would exhibit these behaviors, making them distinguishable with respect to the arrow of time.

The reversibility conditions for a stationary continuous-time Markov chain are characterized in terms of the state-transition rates and the steady-state distribution with the detailed-balance equations[Kel79].

Theorem 2.1 *A stationary continuous-time Markov chain is reversible if and only if there exist a collection of positive numbers $\{Q(S), S \in \Omega\}$, summing to unity, such that*

$$Q(S_1)q(S_1, S_2) = Q(S_2)q(S_2, S_1) \tag{2.1}$$

for all $S_1, S_2 \in \Omega$, where $q(S_i, S_j)$ is the transition rate from S_i to S_j. When such a collection exists, it is the stationary probability distribution.

Equations (2.1) are called the detailed-balance equations of the reversible continuous-time Markov chain. A necessary structural property of a reversible Markov chain is apparent from the detailed-balance conditions. That is, since $Q(S)$ is a strictly positive distribution, the transitions of a reversible process are symmetric ($q(S_j, S_i) > 0$ if $q(S_i, S_j) > 0$). This property is intuitively obvious, since if the process makes transitions from S_i to S_j but not from S_j to S_i then the reversed process makes transitions from S_j to S_i but not from S_i to S_j, distinguishing forward and reversed transition sequences.

An equivalent necessary and sufficient condition for reversibility in terms of the transition rates only is Kolmogorov's condition.

Theorem 2.2 *A stationary continuous-time Markov chain is reversible if and only if for any finite sequence of states $S_1, S_2, \ldots, S_l \in \Omega$, the transition rates satisfy:*

$$q(S_1, S_2)q(S_2, S_3)\ldots q(S_l, S_1) = q(S_1, S_l)q(S_l, S_{l-1})\ldots q(S_2, S_1) \tag{2.2}$$

From Kolmogorov's condition, it is clear that the reverse of every state-transition sequence of a reversible Markov process is also a state-transition sequence of the process. This is also an intuitive notion since the reverse of every state-transition sequence of the process is a state-transition sequence of the reversed process. Note that the reverse of every state-transition sequence is a transition sequence if and only if the transitions are symmetric.

Suppose $S_0, S_1, \ldots, S_l$ is any state-transition sequence of the process. Then, by repeatedly applying Equation (2.1), it is easily seen that the steady-state probability distribution for this Markov chain satisfies

$$Q(S_l) = Q(S_0) \prod_{k=1}^{l} \frac{q(S_{k-1}, S_k)}{q(S_k, S_{k-1})}. \tag{2.3}$$

Equation (2.3) is called the product-form solution of the system.

Theorem 2.3 *A stationary continuous-time Markov process possesses the product-form distribution for the steady-state probabilities given in Equation 2.3 if and only if it is reversible.*

Having stated the relevant theorems of reversibility, we now present necessary and sufficient conditions for reversibility of the blocking model.

Definition 2.3 *An independent set of a graph $G = (V, E)$ is a set of vertices of G such that no pair of vertices in the set are connected by an edge of G.*

Reversibility of the blocking model can be characterized in terms of the blocking relation as follows:

Theorem 2.4 *The Markov process associated with the blocking model is reversible if and only if the states of the process are independent sets of the blocking graph.*

Proof. Necessity: If a state $S \in \Omega$ is not an independent set of the blocking graph then some element $i \in S$ blocks some other element $j \in S$. Hence, not all of the $|S|!$ activation sequences of S are permissible. However, since deactivations are not blocked, all $|S|!$ deactivation sequences of S are permissible, implying there are state-transition sequences of the process whose reversals are not state-transition sequences of the process. Thus, the process cannot be reversible.

Sufficiency: Consider an arbitrary sequence $S_0, S_1, \cdots, S_l$ of state transitions of the process leading from an arbitrary initial state $S_0 = S$ back to itself $S_l = S$. Each transition (S_j, S_{j+1}), $j = 0, 1, \cdots, l-1$ is either an activation, in which case $S_{j+1} = S_j \cup \{i\}$ for some $i \in V - S_j$, or a deactivation, in which case $S_{j+1} = S_j - \{i\}$ for some $i \in S_j$. If (S_j, S_{j+1}) is an activation of element i then $q(S_j, S_{j+1}) = \lambda_i$ and since $S_j = S_{j+1} - \{i\}$ then $q(S_{j+1}, S_j) = \mu_i$. On the other hand, if (S_j, S_{j+1}) is a deactivation of some element i then $q(S_j, S_{j+1}) = \mu_i$ and since $S_j = S_{j+1} \cup \{i\}$ is an independent set then i is not blocked by S_{j+1}. Hence $q(S_{j+1}, S_j) = \lambda_i$. Let a_i and d_i be the number of times element $i \in V$ is activated and deactivated, respectively, in the sequence $S_0, S_1, \cdots, S_l$. Since $S_l = S_0$, each element must be activated and deactivated the same number of times, i.e., $a_i = d_i$, for all $i \in V$. Forming the product of the transition rates in each direction, we have

$$\prod_{j=0}^{l-1} q(S_j, S_{j+1}) = \prod_{i \in V} \lambda_i^{a_i} \mu_i^{d_i} = \prod_{i \in V} (\lambda_i \mu_i)^{a_i} = \prod_{j=0}^{l-1} q(S_{j+1}, S_j).$$

Hence, Kolmogorov's condition is satisfied; the process is reversible.

Theorem 2.5 *The states of the process are independent sets of the blocking graph if and only if the blocking relation is symmetric.*

Proof. Necessity: If there exists a pair of elements $i, j \in V$ such that i blocks j but j does not block i then the activation sequence (j, i) is permissible and leads from state ϕ to the state $\{i, j\}$ which is not an independent set.

Sufficiency: If j blocks i whenever i blocks j then i and j cannot be active simultaneously since one must be activated first and activation of either blocks the other from becoming active. Thus, in each state S, there is no pair of elements that block each other. Since two elements either block each other or neither blocks the other then no element in S blocks another element of S. In other words, S is an independent set.

Putting Theorems 2.4 and 2.5 together, we have

Theorem 2.6 *The Markov process is reversible if and only if the blocking relation is symmetric.*

Theorem 2.6 means that a blocking system has a product-form solution if and only if $B(i) = \tilde{B}(i)$ for all $i \in V$. Therefore, for the network of Examples 2.1 and 2.2, the NP-CSMA protocol yields a product-form solution while the RI-BTMA protocol does not.

Furthermore, for any state S in the reversible Markov chain, any order of activation of the elements in S allows S to be reached from the empty state ϕ. Under this situation, the state-transition diagram for state S is shown in Figure 2.8.

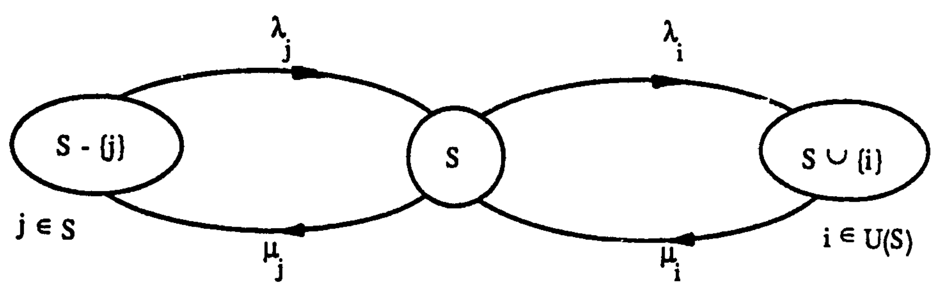


Figure 2.8: A portion of the state-transition diagram for the reversible Markov chain

The product-form solution takes the form

$$Q(S) = Q(\phi)\left(\prod_{i\in S}\rho_i\right) \tag{2.4}$$

where $\rho_i = \lambda_i/\mu_i$ and $Q(\phi)$ is the normalization constant, which is given by

$$Q(\phi) = \left(\sum_{S\in\Omega}\left(\prod_{i\in S}\rho_i\right)\right)^{-1}, \tag{2.5}$$

and equals the probability of finding all the elements idle. For convenience, we use the symbol $SP\{X\}$ to denote the sum of products over the set of states X. Further, we follow the notation used in [BKM87], designating $SP(A)$ as the sum of the products over the set of independent subsets of A. Thus, in general, we write

$$SP\{X\} = \sum_{S\in X}\left(\prod_{i\in S}\rho_i\right), \tag{2.6}$$

$$SP(A) = \sum_{S\subseteq A}\left(\prod_{i\in S}\rho_i\right), \tag{2.7}$$

where, in Equation 2.7, it is to be understood that S ranges over the set of independent subsets of A, not the power set of A. With these two notations, we can write $Q(\phi) = (SP\{\Omega\})^{-1} = (SP(V))^{-1}$.

In Section 2.2, we specified the state of the elements $i \in V$ as being either active or inactive, based on the assumption that both an element's activation scheduling and its active duration were memoryless (exponentially distributed). However, having established a product-form solution for the equilibrium probabilities under a symmetric blocking relation, the assumption that an element's active duration is exponentially distributed can be relaxed by the use of the method of stages, while keeping the product-form solution for the steady-state probability distribution.

The method of stages, which was originally introduced by Erlang [Kle75] and further developed by Cox [CM65], can approximate arbitrary service-length distributions with a combination of exponential services. Boorstyn and Kershenbaum [BKM87] have shown this to be true for the CSMA model. The proof for the general case follows exactly the proof given in [BKM87]. Instead of using the network graph, we simply substitute the blocking graph. Hence, we mention here that the equilibrium results are insensitive to the form of the distribution of active durations. The reader is referred to [BKM87] for further details.

Many other extensions of the model are possible in light of these theorems. For instance, state-dependent transition rates can be incorporated while maintaining reversibility. A simple example would be to treat each element as a binary source. The underlying model for the on-off sources has the same form as the Poisson source model. Only the blocking and the throughput computations change. It is possible to break the symmetry of the blocking relation and maintain a product form if we introduce a blocking relation governing deactivations of elements. A multiset extension of the model is also possible, where a blocking relation is defined over a multiset of elements instead of over a set of elements. Multiple copies of an element may be active at the same time, but copies of elements which block each other may not. These extensions will not be considered in this work. We mention them here only to motivate further research. The approach taken in this thesis will be to examine the structure of the blocking model more closely and exploit its computational simplicity to develop efficient methods for computing throughput and blocking in packet radio networks and circuit-switching networks.

2.4 Vector Representation of the State Probability Distributions

As was described in Section 2.2, given blocking system with n elements represented by the set V, we could represent the states S of the system by the incidence vector $I(S)$. Denoting the incidence vector of state S as $I(S) = b_1 b_2 \cdots b_n$, the steady-state distribution given in Equation (2.4) can be expressed as

$$Q(S) = Q(\phi)\rho_1^{b_1}\rho_2^{b_2}\ldots\rho_n^{b_n}. \tag{2.8}$$

When $\rho_i = \rho$ for all $i \in V$, as in a symmetric-loading or uniform-activation situation, we have

$$Q(S) = Q(\phi)\rho^{\sum_i b_i} = Q(\phi)\rho^{W(b_1 b_2 \ldots b_n)} = Q(\phi)\rho^{|S|},$$

where $W(b_1b_2\cdots b_n)$ is the weight of the state S defining the number of 1's in the incidence vector $I(S)$, or simply the number of elements in S.

Example 2.3. Given a reversible stochastic system of n elements, $V=\{1,2,\cdots,n\}$, we consider the following different blocking situations, their state-transition diagrams for $n=3$, and their steady-state probability distributions $Q(S)$:

(a) **No blocking** (i.e., Figure 2.9).

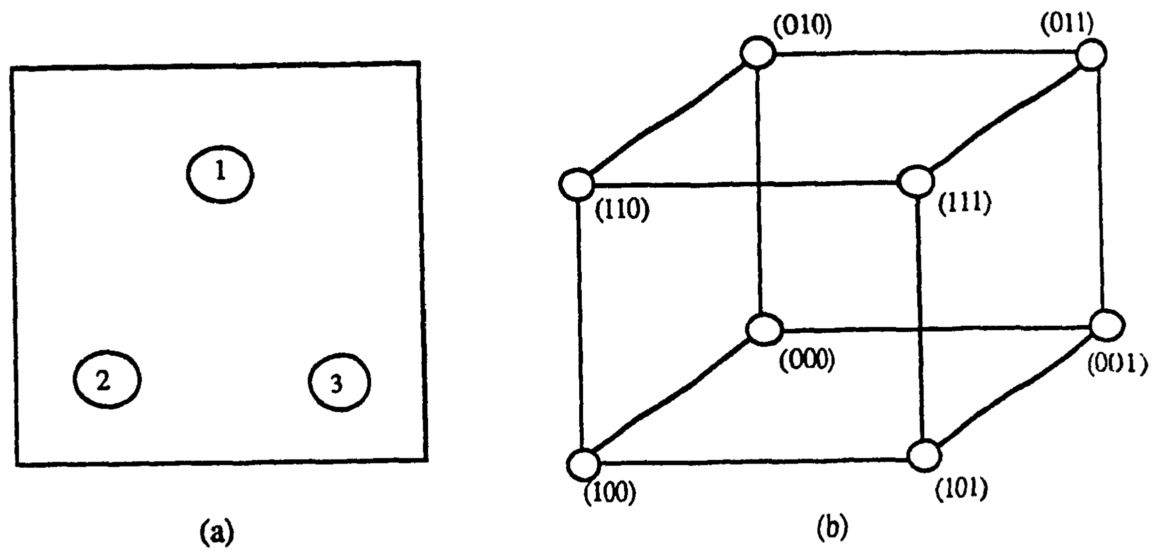


Figure 2.9: (a) The blocking graph and (b) the state-transition diagram of the no blocking system.

In this case,

$$Q(S)=Q(\phi)\rho_1^{b_1}\rho_2^{b_2}\ldots\rho_n^{b_n},$$

where $Q(\phi)=(SP\{\Omega\})^{-1}, \Omega=2^V$, and

$$Q^{-1}(\phi)=\sum_{S\in 2^V}\prod_{i\in S}\rho_i=\prod_{i\in V}(1+\rho_i).$$

We have,

$$Q(S)=\frac{\prod_{i\in S}\rho_i}{\prod_{i\in V}(1+\rho_i)}=\frac{\prod_{i=1}^{n}\rho_i^{b_i}}{\prod_{i=1}^{n}(1+\rho_i)}=\prod_{i=1}^{n}\left(\frac{\rho_i^{b_i}}{1+\rho_i}\right)=\prod_{i\in S}Q(\{i\})\prod_{i\in V-S}(1-Q(\{i\})).$$

In this non-blocking situation, elements are independent of each other. For each element i in the system, the probability of it becoming active is thus

$$Q(\{i\}) = \frac{\rho_i}{1+\rho_i}.$$

In the uniform activation case, $\rho_i = \rho$, for all $i \in V$, and

$$Q(S) = Q(\phi)\rho^{W(b_1 b_2 \ldots b_n)} = Q(\phi)\rho^{|S|},$$

with

$$Q^{-1}(\phi) = \sum_{S\in\Omega} \rho^{|S|} = \sum_{k=0}^{n} \binom{n}{k} \rho^k = (1+\rho)^n.$$

Thus,

$$Q(S) = \rho^{|S|}/(1+\rho)^n.$$

(b) **Complete Blocking** (i.e., Figure 2.10).

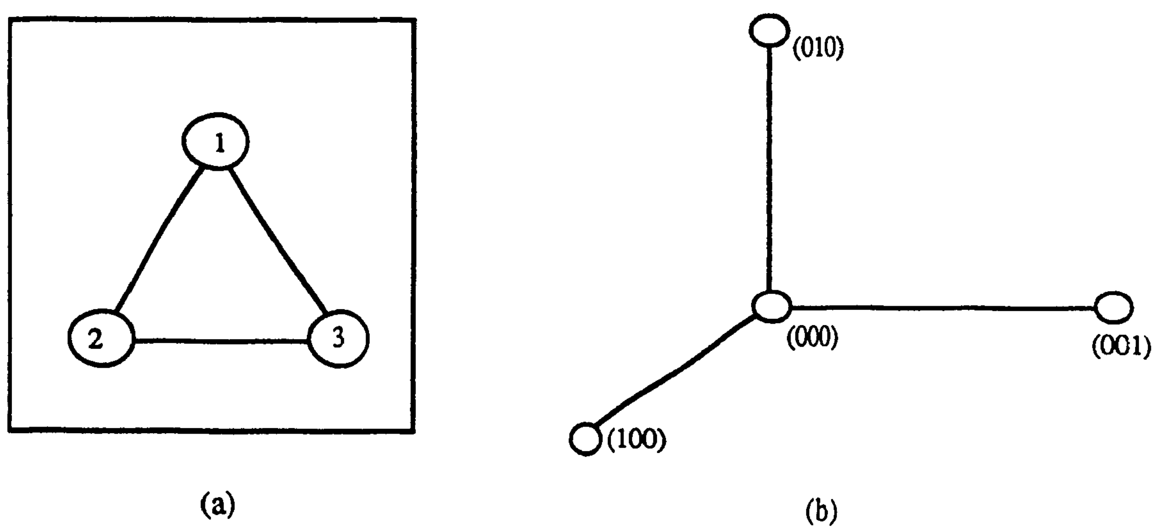


Figure 2.10: (a) The blocking graph and (b) the state-transition diagram of the complete blocking system.

In this case,

$$Q(S) = Q(\phi)\rho_1^{b_1}\rho_2^{b_2}\ldots\rho_n^{b_n},$$

where $Q(\phi) = (SP\{\Omega\})^{-1}$, $\Omega = \{\phi, \{1\}, \{2\}, \cdots, \{n\}\}$, thus $Q^{-1}(\phi) = 1 + \sum_{i=1}^{n} \rho_i$.

Then,

$$Q(S) = \frac{\prod_{i\in S} \rho_i}{1 + \sum_{i\in V} \rho_i} = \frac{\prod_{i=1}^{n} \rho_i^{b_i}}{1 + \sum_{i=1}^{n} \rho_i}.$$

When $\rho_i = \rho, i = 1, 2, \ldots, n$, we have

$$Q(S) = Q(\phi)\rho^{\sum_i b_i} = Q(\phi)\rho^{|S|},$$

where $Q^{-1}(\phi) = 1 + n\rho$. Thus,

$$Q(S) = \begin{cases} \frac{\rho}{1+n\rho} & \text{for } S \neq \phi; \\ \frac{1}{1+n\rho} & \text{for } S = \phi. \end{cases}$$

(c) **Partial Blocking** (i.e., Figure 2.11).

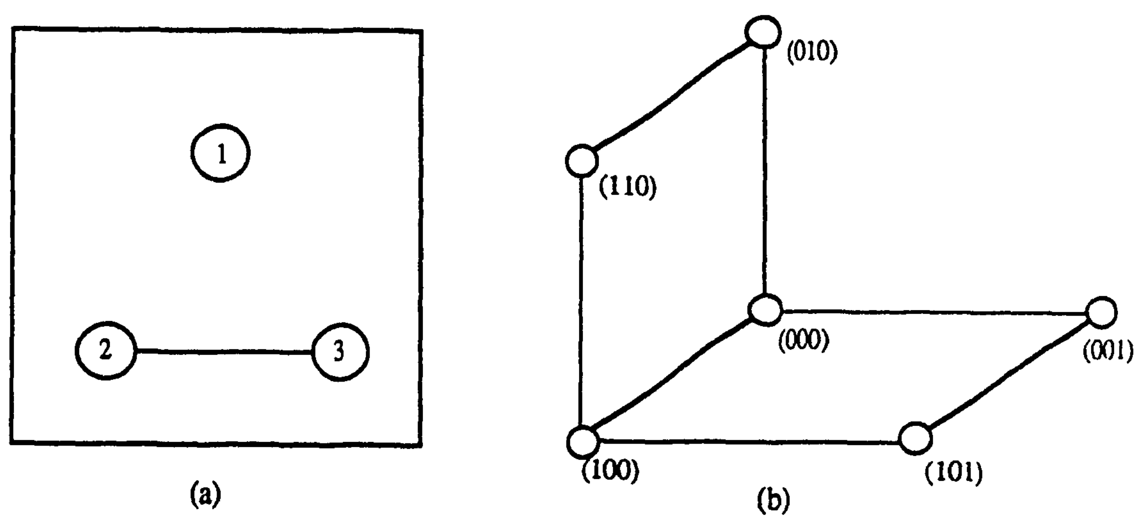


Figure 2.11: (a) The blocking graph and (b) the state-transition diagram of the partial blocking system.

For the case illustrated in Figure 2.11, $V = \{1, 2, 3\}$, we have

$$Q(S) = Q(\phi)\rho_1^{b_1}\rho_2^{b_2}\rho_3^{b_3},$$

where $Q(\phi) = (SP\{\Omega\})^{-1}$, $\Omega = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$, thus,

$$Q^{-1}(\phi) = 1 + \rho_1 + \rho_2 + \rho_3 + \rho_1\rho_2 + \rho_1\rho_3 = (1 + \rho_1)(1 + \rho_2 + \rho_3).$$

Note that $\{1\}$ and $\{2,3\}$ are independent blocking subsystems and that $\{2,3\}$ is a complete blocking subsystem. When $\rho_1 = \rho_2 = \rho_3$, we have,

$$Q(S) = Q(\phi)\rho^{\sum_i b_i} = Q(\phi)\rho^{|S|},$$

where

$$Q^{-1}(\phi) = 1 + 3\rho + 2\rho^2 = (1+\rho)(1+2\rho).$$

2.5 Analysis of System Performance

The reversible stochastic model described in the previous section can provide measures of throughput, blocking probability and interference probability. In this section, we demonstrate how to quantify these measures of performance in terms of the underlying model. The characterization of these quantities is general enough to be applicable to a number of different systems which are subject to some form of contention and interference.

Thus far, the model described in Section 2.2 uses a blocking relation, among otherwise independent elements, to determine whether a scheduled element activation can take place or not. Hence, blocking probabilities for the elements can be directly computed in terms of the equilibrium state probabilities. In order to quantify throughput, we consider two cases: interference-free systems and systems susceptible to interference. By an *interference-free* system, we mean one in which every element activation is considered to be successful and contributes to its throughput. The throughput of an element i in this interference-free model is then just its activation rate $Q(\{i\})$. In an *interference-susceptible* system, an element activation may or may not lead to a success. An active element can be interfered with during its active life by another element (an interfering element) becoming active. Interference takes the form of transmission collisions when modeling broadcast networks. In this case, the throughput of an element is much more difficult to compute and further information must be provided concerning the nature of the interference. In the following, we describe a simple model for interference and the associated throughput calculation and state the underlying assumptions. Although the former model may be considered a special case of the latter, it is worth considering separately since the interference-free case is computationally much simpler and it is applicable to a number of systems where centralized control precludes interference among active elements, such as in a switch. We will consider the application of these models to measure throughput and blocking in packet radio networks and communication switches in the chapters to follow.

Transmission collisions are an inherent problem in multihop broadcast networks due to the distributed nature of the access protocol. Each node must operate its transmitter based on limited knowledge of the state of the network. A transmission from one node may interfere with a transmission from another node or they may interfere with each other if, while trying to receive one transmission, a receiver hears another (a collision). Depending on the capabilities of the nodes and on the topology of the network, a receiver may be able to continue to correctly receive the first transmission and reject the other as noise, provided the interfering power is not too great. With channel sensing, a node can exercise precaution by blocking transmissions when it is known that they may interfere with ongoing transmissions.

A simple and unified approach to account for this type of interference in the stochastic model is to represent it as another relation on the elements. For example, we may say that element i *interferes with* element j if element i must be idle at the time element j is activated and while element j is active, in order for element j's activity to be successful. Under different assumptions, we may say that element i interferes with element j if element i must be idle at the time element j is activated, implying that element j can tolerate the subsequent activation of element i. The distinction between these definitions of interference has to do with the notion of *capture*. By employing orthogonal codes, a transceiver may have the ability to lock on to or capture the first transmission it hears and reject subsequently-initiated transmissions as wide-band noise[Nel84]. The first definition of interference assumes no such capability. It represents one extreme of the spectrum of capture capabilities and it is referred to as the *zero-capture* assumption. The second definition of interference assumes the receiver can successfully receive a transmission as long as all interferers are quiet at the instant the transmission is first heard by the receiver. This is a sophisticated capability and represents the other extreme of the spectrum, the *perfect-capture* assumption. In general, the capture capabilities of a receiver lie somewhere between these extremes. For instance, a receiver may not be able to reject a transmission if it is much closer, hence stronger, than the one it is trying to receive or if too many interfering transmissions occur while receiving one. The assumption of perfect-capture is an optimistic one which yields a simple quantification of throughput since the determination of success is made instantaneously. We will assume perfect capture for the computational simplicity it provides. Hence, our results can be considered as upperbounds for typical capture capabilities.

Having introduced the notion of an interference relation on the elements of the model, we may now quantify the throughput. Further, we assume error-free reception in the absence of a collision and that acknowledgements are instantaneous. Transmissions which are blocked or result in collision are thus known to the transmitter and rescheduled. The blocking model does not incorporate the backlog of messages at each node, for each link. Thus, it does not provide a significant quantification of message delay beyond what can be expected of the no-buffering bound [BG87]. We define $\rho_i = \lambda_i/\mu_i$ as the *normalized* scheduling rate for element i. If we denote the rate of successful activations of element i as r_i, then r_i/ρ_i is the probability $P_S(i)$ that a scheduled activation of element i is successful. In general,

$$P_S(i) = \frac{r_i}{\rho_i} = Pr\left\{ \begin{array}{l} \text{element } i \text{ is not blocked at its scheduling point and} \\ \text{it is not interfered with throughout its active duration.} \end{array} \right\}.$$

In modeling a communication network, r_i and ρ_i denote the success rate and attempt rate, respectively, for transmissions on link l_i. With zero capture capabilities, we would write

$$\frac{r_i}{\rho_i} = Pr\left\{ \begin{array}{l} \text{scheduling point for transmission over link } l_i = (s_i, d_i) \text{ is not blocked} \\ \text{at node } s_i \text{ and the transmission does not collide with another transmission} \\ \text{while being received at } d_i. \end{array} \right\}.$$

In perfect capture, we may write

$$\frac{r_i}{\rho_i} = Pr\left\{ \begin{array}{l} \text{a transmission attempt over link } l_i \text{ is not blocked} \\ \text{or interfered with at its scheduling point.} \end{array} \right\}.$$

The TBG defines the possibility of blocking for each element i at its scheduling point. Similarly, let the Reception Interference Graph (RIG) define the collision relation on the elements, at the receiving end of a transmission. Then the corresponding set of interference states is denoted as

$$\Omega_I = \{S : \text{some element in } S \text{ interferes with an other element in } S\}.$$

Referring back to Example 2.1 of the 4-node chain with the NP-CSMA protocol, states $\{l_1, l_4\}$ and $\{l_3, l_6\}$ are within the state space Ω of the system, yet the pair of elements inside each of these two states interfere with each other respectively at the receiving nodes 2 and 3. Therefore, $\{l_1, l_4\}, \{l_3, l_6\} \in \Omega_I$. Further, we use the notation $I(i)$ to represent the collection of all elements j which do not block i, but interfere with i,

$$I(i) = \{j : j \text{ does not block } i, j \text{ interferes with } i\}.$$

The information $I(i)$ can be directly obtained from the RIG of the system. By subtracting the interfering sets generated from the RIG from the state space Ω, generated from the TBG, we obtain a set of successful states Ω_S. These are the states of the system that guarantee successful transmission, and thus contribute to the throughput. Ω_S can be represented by the Successful Transmission Graph (STG) which is obtained by subtracting the RIG from the complement of TBG. In addition, let

$$S(i) = \{j : j \text{ does not block or interfere with } i\}.$$

and

$$\Omega_S(i) = \{S : S \in \Omega_S, S \text{ contains elements in } S(i) \text{ only}\} .$$

Then, the probability that the element i is active successfully is given by

$$\frac{r_i}{\rho_i} = \sum_{S \in \Omega_S(i)} Q(S) = \frac{\sum_{S \in \Omega_S(i)} \prod_{j \in S} \rho_j}{\sum_{S \in \Omega} \prod_{j \in S} \rho_j},$$

so

$$\frac{r_i}{\rho_i} = \frac{SP\{\Omega_S(i)\}}{SP\{\Omega\}}. \quad (2.9)$$

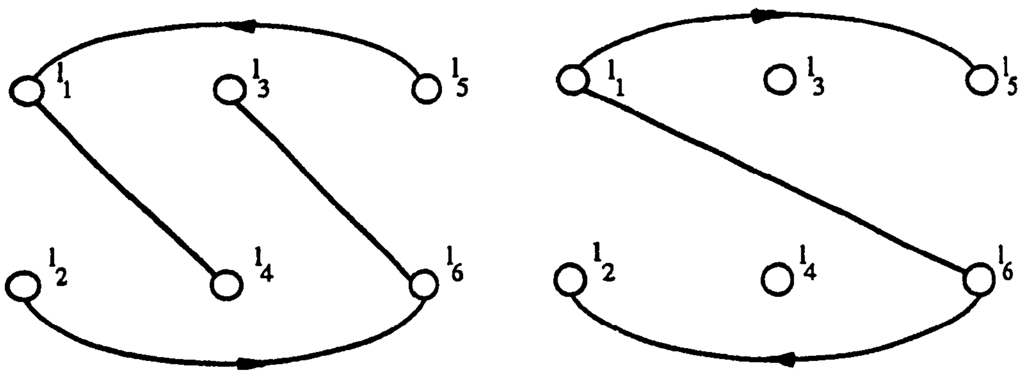
It is also true that

$$\frac{r_i}{\rho_i} = \frac{\sum_{S \subseteq S(i)} \prod_{j \in S} \rho_j}{\sum_{S \subseteq V} \prod_{j \in S} \rho_j},$$

therefore

$$\frac{r_i}{\rho_i} = \frac{SP(S(i))}{SP(V)}. \tag{2.10}$$

Example 2.4(a): Referring back to Example 2.1 (NP-CSMA), the corresponding TBG is shown in Figure 2.2, the RIG and the STG are shown in Figure 2.12.



(a) (b)

Figure 2.12: (a) The RIG and (b) the STG for the system in Example 2.4(a)

From the RIG, the interference sets are:

$$\Omega_I = \{\{l_1, l_4\}, \{l_2, l_6^d\}, \{l_3, l_6\}\}, \{l_5, l_1\}^d\},$$

where the directional state $\{l_i, l_j\}^d$ only exists when the activation of link l_i precedes the activation of link l_j. We have

$$\begin{array}{lll} I(l_1) = \{l_4, l_5\}; & I(l_2) = \phi; & I(l_3) = \{l_6\}; \\ I(l_4) = \{l_1\}; & I(l_5) = \phi; & I(l_6) = \{l_2, l_3\}. \end{array}$$

Considering the STG, the set of the successful transmission states are:

$$\Omega_S = \{\{l_1\}, \{l_2\}, \{l_3\}, \{l_4\}, \{l_5\}, \{l_6\}, \{l_1, l_6\}, \{l_1, l_5\}^d, \{l_6, l_2\}^d\}.$$

Thus,

$$\begin{array}{lll} S(l_1) = \{l_6\}; & S(l_2) = \{l_6\}; & S(l_3) = \phi; \\ S(l_4) = \phi; & S(l_5) = \{l_1\}; & S(l_6) = \{l_1\}, \end{array}$$

$$\Omega_S(l_1) = \{\phi, \{l_6\}\}; \quad \Omega_S(l_2) = \{\phi, \{l_6\}\}; \quad \Omega_S(l_3) = \phi;$$
$$\Omega_S(l_4) = \phi; \quad \Omega_S(l_5) = \{\phi, \{l_1\}\}; \quad \Omega_S(l_6) = \{\phi, \{l_1\}\},$$

and

$$\begin{aligned} SP\{\Omega\} = SP(V) &= 1 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 \\ &\quad + \rho_2\rho_6 + \rho_3\rho_6. \end{aligned}$$

We have

$$\begin{aligned} P_S(1) = P_S(2) &= \frac{r_1}{\rho_1} = \frac{r_2}{\rho_2} = \frac{1}{SP(V)}(1+\rho_6); \\ P_S(3) = P_S(4) &= \frac{r_3}{\rho_3} = \frac{r_4}{\rho_4} = \frac{1}{SP(V)}, \\ P_S(5) = P_S(6) &= \frac{r_5}{\rho_5} = \frac{r_6}{\rho_6} = \frac{1}{SP(V)}(1+\rho_1). \end{aligned}$$

□

Similarly, let $\Omega_B(i)$ be the set of all states S in Ω that contain at least one element in $B(i)$. Any of these states will block the activation of element i.

$$\Omega_B(i) = \{S : S \in \Omega, S \text{ contains at least one element in } B(i)\}.$$

The probability that element i is blocked is

$$\begin{aligned} P_B(i) &= Pr\{\text{element } i \text{ is blocked}\} \\ &= \textstyle\sum_{S \in \Omega_B(i)} Pr(S) \\ &= \frac{\sum_{S \in \Omega_B(i)} \prod_{j \in S} \rho_j}{\sum_{S \in \Omega} \prod_{j \in S} \rho_j}, \end{aligned}$$

or

$$P_B(i) = \frac{SP\{\Omega_B(i)\}}{SP\{\Omega\}} = \frac{SP\{\Omega_B(i)\}}{SP(V)}. \tag{2.11}$$

Lastly, let $\Omega_I(i)$ be the set of all states $S \in \Omega$, where $S \notin \Omega_B(i)$ and S contains at least one element in $I(i)$, then

$$\Omega_I(i) = \{S : S \in \Omega - \Omega_B(i), S \text{ contains at least one element in } I(i)\}.$$

The probability that element i is interfered with can be expressed as:

$$\begin{aligned} P_I(i) &= Pr\{\text{element } i \text{ is interfered with}\} \\ &= \textstyle\sum_{S \in \Omega_I(i)} Pr(S) \\ &= \frac{\sum_{S \in \Omega_I(i)} \prod_{j \in S} \rho_j}{\sum_{S \in \Omega} \prod_{j \in S} \rho_j}, \end{aligned}$$

or

$$P_I(i) = \frac{SP\{\Omega_I(i)\}}{SP\{\Omega\}} = \frac{SP\{\Omega_I(i)\}}{SP(V)}. \tag{2.12}$$

In general, we have the following relations:

$$P_I(i) + P_B(i) + P_S(i) = 1; \tag{2.13}$$

$$\Omega_I(i) + \Omega_B(i) + \Omega_S(i) = \Omega,$$

$$S(i) \cup B(i) \cup I(i) = V.$$

Thus

$$\frac{r_i}{\rho_i} = \frac{SP(S(i))}{SP(V)} = \frac{SP(V - B(i) - I(i))}{SP(V)}. \tag{2.14}$$

Example 2.4(b): Calculating (1) $P_B(i)$, (2) $P_I(i)$.

(1)

$$\begin{array}{lll} B(l_1) = \{l_1, l_2, l_3\}; & B(l_2) = \{l_1, l_2, l_3, l_4, l_5\}; & B(l_3) = B(l_2); \\ B(l_4) = \{l_2, l_3, l_4, l_5, l_6\}; & B(l_5) = B(l_4); & B(l_6) = \{l_4, l_5, l_6\}. \end{array}$$

$$\begin{aligned} \Omega_B(l_1) &= \{\{l_1\}, \{l_2\}, \{l_3\}, \{l_1, l_4\}, \{l_1, l_5\}, \{l_1, l_6\}, \{l_2, l_6\}, \{l_3, l_6\}\}; \\ \Omega_B(l_2) &= \{\{l_1\}, \{l_2\}, \{l_3\}, \{l_4\}, \{l_5\}, \{l_1, l_4\}, \{l_1, l_5\}, \{l_1, l_6\}, \{l_2, l_6\}, \{l_3, l_6\}\}; \\ \Omega_B(l_4) &= \{\{l_2\}, \{l_3\}, \{l_4\}, \{l_5\}, \{l_6\}, \{l_1, l_4\}, \{l_1, l_5\}, \{l_1, l_6\}, \{l_2, l_6\}, \{l_3, l_6\}\}; \\ \Omega_B(l_6) &= \{\{l_4\}, \{l_5\}, \{l_6\}, \{l_1, l_4\}, \{l_1, l_5\}, \{l_1, l_6\}, \{l_2, l_6\}, \{l_3, l_6\}\}, \end{aligned}$$

$$\Omega_B(l_3) = \Omega_B(l_2); \qquad \Omega_B(l_5) = \Omega_B(l_4).$$

$$\begin{aligned} P_B(1) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 + \rho_3\rho_6); \\ P_B(2) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 \\ &\quad + \rho_2\rho_6 + \rho_3\rho_6); \\ P_B(4) &= \frac{1}{SP(V)}(\rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 \\ &\quad + \rho_2\rho_6 + \rho_3\rho_6); \\ P_B(6) &= \frac{1}{SP(V)}(\rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 + \rho_3\rho_6); \end{aligned}$$

$$P_B(3) = P_B(2); \qquad P_B(5) = P_B(4).$$

(2)

$$\Omega_I(l_1) = \{\{l_4\}, \{l_5\}\}; \quad \Omega_I(l_2) = \phi; \quad \Omega_I(l_3) = \{\{l_6\}\};$$

$$\Omega_I(l_4) = \{\{l_1\}\}; \quad \Omega_I(l_5) = \phi; \quad \Omega_I(l_6) = \{\{l_2, l_3\}\}.$$

$$P_I(1) = \frac{1}{SP(V)}(\rho_4 + \rho_5); \quad P_I(2) = 0; \quad P_I(3) = \frac{\rho_6}{SP(V)};$$

$$P_I(4) = \frac{\rho_1}{SP(V)}; \quad P_I(5) = 0; \quad P_I(6) = \frac{1}{SP(V)}(\rho_2 + \rho_3).$$

□

Example 2.4(c) In communication network systems, transmissions over each channel are directly related to transmission and reception opertions in its corresponding node pair. Thus, the original system state-transition diagram refering to link activations of the 4-node chain network operating under CSMA, which was shown in Figure 2.4, can be redrawn, as shown in Figure 2.13, to describe the system nodal operations. If we let $r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = r$, $\rho_2 = \rho_3 = \rho_4 = \rho_5$, $\rho_1 = \rho_6$, and denote the normalized scheduling transmission rate at each node i as ρ_i', then, $\rho_1' = \rho_1$, $\rho_2' = \rho_2 + \rho_3$, $\rho_3' = \rho_4 + \rho_5$, $\rho_4' = \rho_6$, and

$$SP(V') = 1 + \rho_1' + \rho_2' + \rho_3' + \rho_4' + \rho_1'\rho_3' + \rho_1'\rho_4' + \rho_2'\rho_4'.$$

This $SP(\cdot)$ can be obtained by directly considering nodes as active elements of the system. Thus, in this case, we can measure the system performances by refering to either transmission operations over the channels or broadcasting operations at the stations.

By letting $\rho_1' = \rho_4'$, $\rho_2' = \rho_3'$, we have $\rho_2' = \rho_1'(2 + \rho_1')$, and $r = Q(\phi)(\rho_1'(1 + \rho_1'))$ with

$$Q(\phi) = \left(SP(V')\right)^{-1} = \left(1 + 6\rho_1' + 7\rho_1'^2 + 2\rho_1'^3\right)^{-1}.$$

Further, we have

$$\begin{aligned} P_B(1)' &= P_B(4)' = P_B(1) = P_B(4) = Q(\phi)\rho_1'(3 + 6\rho_1' + 2\rho_1'^2); \\ P_B(2)' &= P_B(2) + P_B(6) = Q(\phi)\rho_1'(5 + 7\rho_1 + 2\rho_1'^2); \end{aligned}$$

$$
\begin{aligned}
P_B(3)' &= P_B(3)+P_B(5)=Q(\phi)\rho_1'(5+7\rho_1'+2\rho_1'^2),\\
P_I(1)' &= P_I(4)'=P_I(1)=P_I(4)=Q(\phi)\rho_1'(2+\rho_1');\\
P_I(2)' &= P_I(2)+P_I(6)=Q(\phi)\rho_1';\\
P_I(3)' &= P_I(3)+P_I(5)=Q(\phi)\rho_1'.
\end{aligned}
$$

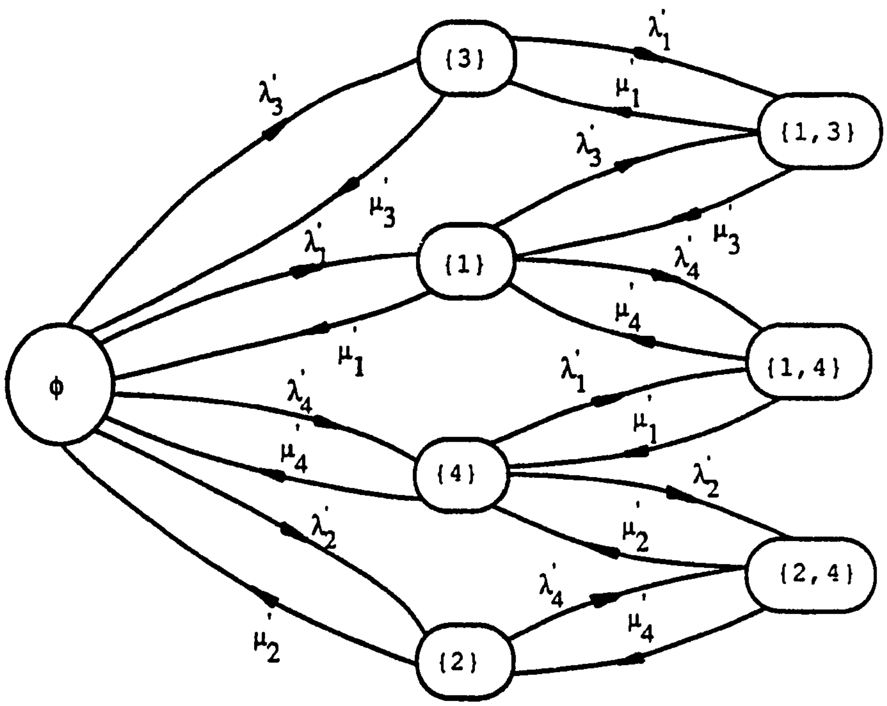


Figure 2.13: The revised state-transition diagram of Example 2.4

Chapter 3

Computational Techniques and Special Cases

3.1 Introduction

The mathematical model presented in Chapter 2 is quite general. In applying the model to analyze a packet radio network, the elements can represent the nodes (communication stations) or the directed links (communication channels) of the given network, and both approaches can quantify similar performance measures. Some of the assumptions in the model may be relaxed, as mentioned in Section 2.3 of Chapter 2, thus extending the range of applicability of the model. On the other hand, more restrictive assumptions can be made, introducing some very special cases of blocking systems. In this chapter, we will investigate some of these possibilities. First, we discuss some computational aspects of the model which provides the motivation for considering special cases.

3.2 Computational Techniques

The computation of performance measures becomes complicated as the number of elements in the system increases. These difficulties can be alleviated to some degree by using special techniques applicable to the blocking model.

3.2.1 Computation of SP(A)

As presented in Equations (2.10) and (2.14), the element-activation success probability $P_S(i) = r_i/\rho_i$ can, in general, be expressed as a ratio of sum-of-products expressions of the form $SP(A)/SP(V)$, where

$$SP(A) = \sum_{S \subseteq A} \prod_{i \in S} \rho_i;$$
$$SP(V) = \sum_{S \subseteq V} \prod_{i \in S} \rho_i,$$

with $A \subseteq V$ and $V = \{1, 2, \ldots, n\}$ is the set of all elements in the system.

A straightforward algorithm to evaluate $SP(A)$ is to generate all independent subsets of A. In the worst case, all $2^{|A|}$ subsets of A would be generated and this approach is thus exponentially hard. However, we can apply some properties of the $SP(\cdot)$ function to streamline the computation for large networks. Kershenbaum and Boorstyn [KB84] developed an algorithm for computing the sum of products expressions which they called the SP algorithm. The technique is based on the theorem below.

Theorem 3.1 *(a) If two subsets of elements X and Y are isolated from each other, then*

$$SP(X \cup Y) = SP(X)SP(Y) \tag{3.1}$$

(b) For any element i in the set X, it is true that

$$SP(X) = SP(X - i) + \rho_i SP(X - B(i)) \qquad i \in X, \tag{3.2}$$

where $B(i)$ is the set of elements blocked by i, or equivalently the set of elements blocking i.

The proof of this theorem is given in Appendix A.

In Kershenbaum and Boorstyn's studies, they considered networks with omnidirectional transceivers. With omnidirectional tranceivers, a node with carrier sensing capabilities detects carrier if any of its neighbors are transmitting a message. Thus, for this special case, the elements of the model are the nodes of the network and the blocking set $B(i)$ for node i is simply its set of neighbors N_i. The network graph and the blocking graph are one in the same; the blocking relation is the hearing relation. Hence, the CSMA model is reversible if and only if the hearing relation is symmetric, i.e., $h_{ij} = h_{ji}$. In general, however, the blocking relation is a relation on the link set of the network, derived from the hearing relation and the protocol. In this more general setting, the SP algorithm is simply applied to the blocking graph rather than the network graph. Our approach will be to systematically categorize all possible blocking and interference relations among the links of a given network in order to construct these relations for a given protocol. We then apply the SP algorithm with the relations.

By applying Equation (3.2) repeatedly to an arbitrary subset C of set V, expanding over C's possible subsets S, $S \subseteq C$, it can be shown that

$$SP(V) = SP(V-C) + \sum_{S \subseteq C} \{\textstyle\prod_{i \in S} \rho_i SP(V - C - B(S))\}, \tag{3.3}$$

where $B(S)$ denotes the set of blocked elements of state S (which includes all elements in S). Finally, if C is a cut of the blocking graph, i.e., a set of elements which when removed decomposes the network into two isolated subsets X and Y, then for all independent sets $S \subseteq C$,

$$SP(V - C - B(S)) = SP(X - B(S))SP(Y - B(S)),$$

and

$$SP(V - C) = SP(X)SP(Y).$$

Example 3.1. Consider the blocking graph of Figure 3.1, where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. By choosing $C = \{5, 6, 7\}$, the graph is decomposed into three subsets X, Y and C, with $X = \{1, 2, 3, 4\}$, $Y = \{8, 9, 10, 11\}$ being isolated from each other.

By repeatedly applying Equation (3.2), we have,

$$\begin{aligned} SP(V) &= SP(1,2,3,4,5,6,7,8,9,10,11) \\ &= SP(1,2,3,4,6,7,8,9,10,11) + \rho_5 SP(1,4,7,10,11) \end{aligned}$$

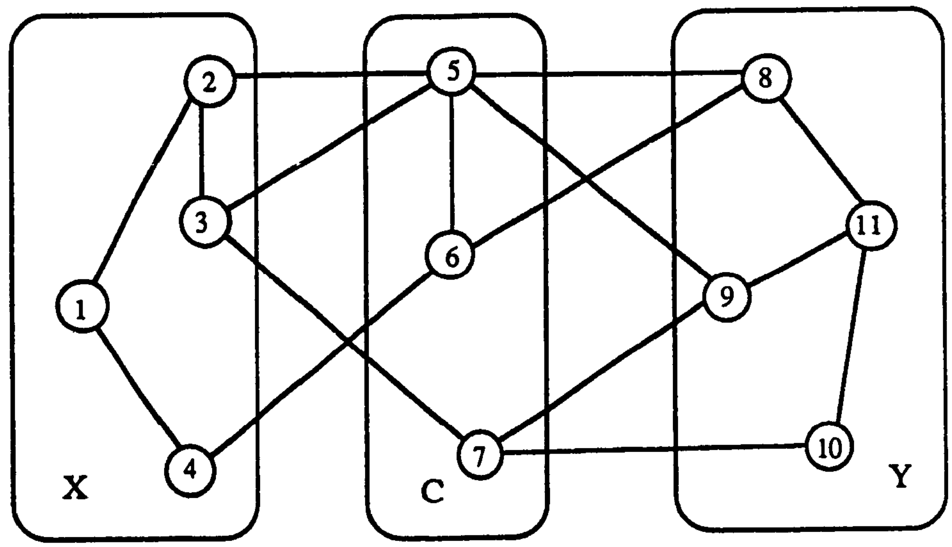


Figure 3.1: Decomposition of the blocking graph

$$
\begin{aligned}
&= SP(1,2,3,4,7,8,9,10,11) + \rho_6 SP(1,2,3,7,9,10,11) \\
&\quad + \rho_5 SP(1,4,10,11) + \rho_5\rho_7 SP(1,11) \\
&= SP(1,2,3,4,8,9,10,11) + \rho_7 SP(1,2,8,11) \\
&\quad + \rho_6 SP(1,2,3,9,10,11) + \rho_6\rho_7 SP(1,2,11) \\
&\quad + \rho_5 SP(1,4,10,11) + \rho_5\rho_7 SP(1,11) \\
&= SP(1,2,3,4)SP(8,9,10,11) + \rho_5 SP(1,4)SP(10,11) \\
&\quad + \rho_6 SP(1,2,3)SP(9,10,11) + \rho_7 SP(1,2)SP(8,11) \\
&\quad + \rho_5\rho_7 SP(1)SP(11) + \rho_6\rho_7 SP(1,2)SP(11)
\end{aligned}
$$

This result can be directly obtained by the application of Equation (3.3). Thus by applying these simplifications, we evaluate the sum-of-products for subsets of at most 4 elements instead of the original 11 elements.

The SP algorithm is based on the observation that the $SP(\cdot)$ expression for a set containing n mutually independent elements, can be written as

$$SP(V) = \prod_{i=1}^{n}(1+\rho_i). \tag{3.4}$$

If this expression is expanded directly into a sum-of-products form, it would contain 2^n terms. However, only n additions and $(n-1)$ multiplications are required to evaluate

Equation (3.4). In a similar fashion, $SP(X)$ can be evaluated by grouping together common subexpressions and using the distribution of multiplication over addition to reduce the number of arithmetic operations. Equation (3.2) is the starting point.

By using this recursive method, the expansion of $SP(X)$ can generate up to $2^{|X|}$ terms, but fewer terms will be generated in general, since $X - B(i)$ will quickly become ϕ and terminate the recursion. Furthermore, a dramatic reduction in the number of generated terms is possible, if we recognize sets D which have already appeared and, consequently, for which $SP(D)$ is already known. Under this situation, the $SP(D)$ value is substituted directly without recursive expansion required.

Example 3.2 Refering back to the 4-node chain PRN of Figure 2.1 with CSMA protocol, and with six directed link elements, $V = \{l_1, l_2, l_3, l_4, l_5, l_6\}$, its TBG is given in Figure 2.2 while its recursive expansion is depicted in the tree diagram of Figure 3.2.

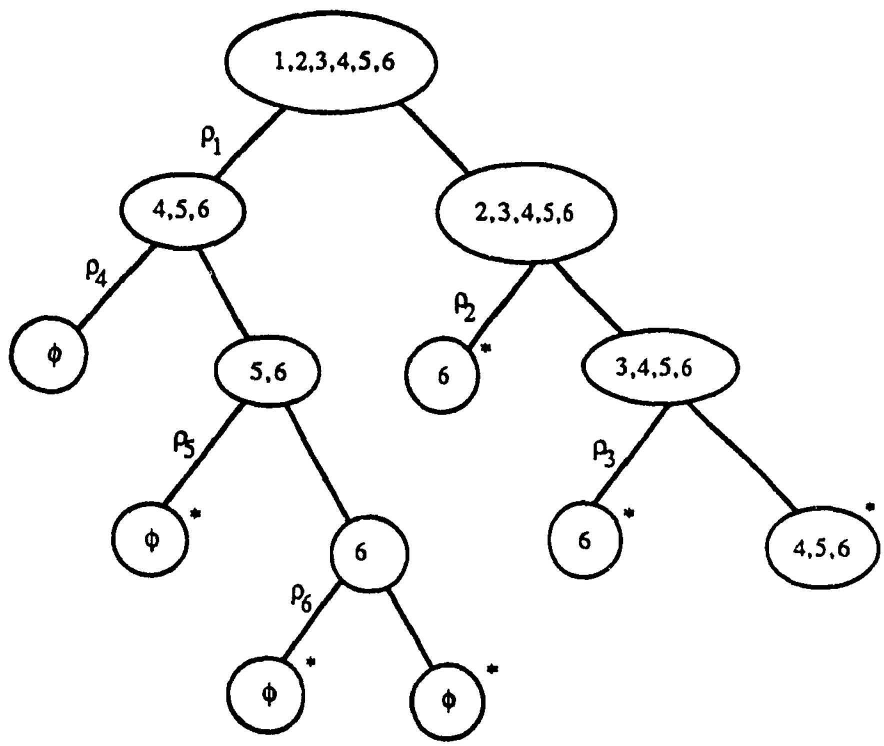


Figure 3.2: Recursive expansion of $SP(V)$ for the 4-node chain PRN operating under CSMA

Table 3.1: Iterative calculation for the 4-node chain PRN under CSMA protocol

Row	Set(X)	$SP(X-B(i))$	$SP(X-i)$	ρ_i	$SP(X)$
1	$\{l_1,l_2,l_3,l_4,l_5,l_6\}$	$\{l_4,l_5,l_6\}$	$\{l_2,l_3,l_4,l_5,l_6\}$	ρ_1	-
2	$\{l_4,l_5,l_6\}$	ϕ	$\{l_5,l_6\}$	ρ_4	-
3	ϕ	-	-	ρ_0	1
4	$\{l_5,l_6\}$	ϕ	$\{l_6\}$	ρ_5	-
5	$\{l_6\}$	ϕ	ϕ	ρ_6	-
6	$\{l_2,l_3,l_4,l_5,l_6\}$	$\{l_6\}$	$\{l_3,l_4,l_5,l_6\}$	ρ_2	-
7	$\{l_3,l_4,l_5,l_6\}$	$\{l_6\}$	$\{l_4,l_5,l_6\}$	ρ_3	-
8	$\{l_1\}$	ϕ	ϕ	ρ_1	-

The nodes in this tree diagram are the result of applying Equation (3.2). For example,

$$\begin{aligned} SP(\{l_1,l_2,l_3,l_4,l_5,l_6\}) &= \rho_1 SP(\{l_1,l_2,l_3,l_4,l_5,l_6\} - B(l_1)) \\ &\quad + SP(\{l_1,l_2,l_3,l_4,l_5\} - \{l_1\}) \\ &= \rho_1 SP(\{l_4,l_5,l_6\}) + SP(\{l_2,l_3,l_4,l_5\}) \end{aligned}$$

where $B(l_1) = \{l_1,l_2,l_3\}$.

For the sake of simplicity, in each case, $SP(X)$ is expanded on the lowest-numbered element in set X. Further, an asterisk marked on a node of the tree diagram signifies that the terms corresponding to the node have already been generated and, thus, need not be expanded. In total, we have seven distinct nodes in this figure. Table 3.1 is constructed to store all the relevant information. The order of entries in this table follows the traversal order of the tree in Figure 3.2. The entries in the $SP(X-B(i))$ and $SP(X-i)$ fields are obtained from the subordinate branches of the given node X. The $SP(X)$ value is generated by starting at the leafs of the tree with $SP(\phi)=1$ and working back towards the root, combining subordinate nodes of internal nodes according to Equation (3.2).

We generate the $SP(\cdot)$ expression for subsets introduced in the $SP(V)$ expansion. However, these subsets are not all that is needed. On the other hand, For performance measures, we not only require $SP(V)$ but also $SP(V-B(i)-I(i))$ for all i. Subsets not generated in this tree are introduced by expanding the sets $V - B(i) - I(i)$, for $i \in V$, for the required numerators. Here, we include $SP(V - (B(l_6) \cup I(l_6))) = SP(\{l_1\})$, which is the entry in row 8.

3.2.2 The iterative-estimation algorithm

The iterative-estimation calculating method for per-link throughput calculation of a given communication network can be directly developed from Equation (2.14), formally

$$\frac{r_i}{\rho_i} = P_S(i) = \frac{SP(V - B(i) - I(i))}{SP(V)} = F_i(\overline{\rho}), \tag{3.5}$$

where $\overline{\rho} = (\rho_1, \rho_2 \ldots \rho_n)$. Therefore, given an expected successful transmission rate r_i, the tra..mission scheduling rate Bρ_i can be iteratively estimated by using

$$\rho_i^{(k+1)} = \frac{r_i}{F_i(\overline{\rho}^{(k)})} \tag{3.6}$$

with

$$F_i(\overline{\rho}^{(k)}) = \frac{SP^{(k)}(V - B(i) - I(i))}{SP^{(k)}(V)},$$

where $\overline{\rho}^{(k)}$ is the value of $\overline{\rho}$ after the k^{th} iteration and $SP^{(k)}(\cdot)$ is the corresponding $SP(\cdot)$ value for the estimated $\overline{G}^{(k)}$ values. According to this expression, the ρ_i values can be iteratively computed for their new estimates given their current estimates. It is reported [BK80] that if the system can support the given r_i, then the iteration will converge to a fixed point if we start with $\rho_i = r_i$. Otherwise, it will diverge. It is clear that $\rho_i > r_i$.

To obtain the maximum throughput a link l_i can support, we assume an initial value for r_i and apply the iterative-estimation method to determine the required attempt rates. If the iteration diverges (and it does so quickly) then r_i is too high, the link will not support that much throughput. Otherwise, the iteration converges to a set of attempt rates that will yield the desired success rate r_i. If r_i is too high, we reduce its value and try again. If r_i is too low, we increase its value and try again. In this way, we may proceed in a binary search (or one using iterpolation) for the maximum throughput supported by each link of the network. The blocking probability $P_B(i)$ and interference probability $P_I(i)$ of every element i can be obtained from the generated $\overline{\rho}$ values, as given by Equations (2.11) and (2.12), respectively.

3.3 The Channel Access Protocols

In this section, we consider some channel access protocols used in packet radio networks. PRNs operating under these protocols satisfy the reversibility condition of the blocking model. As the result, we can employ the computational-techniques developed from the blocking model to measure their system performance.

3.3.1 The transmission interactions model

It is desired to provide a modeling framework which is general enough to describe a number of channel access protocols yet sufficiently restricted to maintain reversibility of the underlying Markov chain. To this end, we use the links of a network graph G with a symmetric hearing matrix $H = [h_{ij}]$ as the elements of the blocking system. The blocking and interference relations on the links are determined by taking the topology of G and the access protocol into consideration. At the outset, we consider distributed protocols in which a transmission may influence another transmission only if they are *topologically near* one another. Specifically, a transmission may *interact with* (block or interfere with) another transmission only if they involve a common node or neighboring nodes. Transmissions not involving the same node or neighbors are considered to be too distant from one another to influence each other in any way. However, the blocking model does not preclude such interactions from consideration.

Let $e_{ij} = (i, j)$ and $e_{uv} = (u, v)$ denote links of G connecting node i to node j and node u to node v, respectively. Under the limited range-of-influence assumption mentioned above, there are a total of eight topological situations in which a transmission on link e_{uv} may block or interfere with a transmission on link e_{ij}. Figure 3.3 organizes these eight possible cases into four classes with two levels within each class. Level L1 consists of transmission pairs which may *directly interact* by involving a common node, i.e., either both transmissions originate at the same source or the destination of one is the source of the other or they have the same destination. Level L2 describes

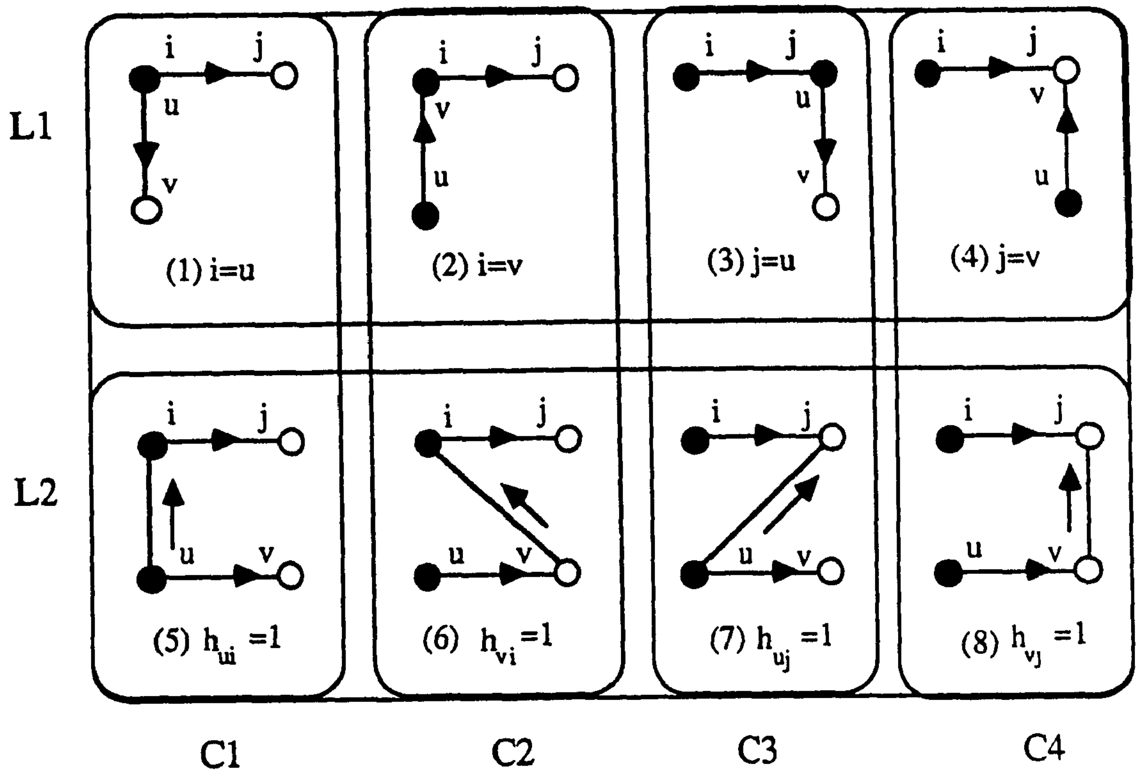


Figure 3.3: Possibility of transmission interactions in a symmetric hearing situation

transmission pairs which may interact *indirectly* via neighboring nodes. In the class context, transmission interactions are described by relating a transmission on e_{uv} to transmitting node i and receiving node j of a transmission on e_{ij}. In class C1, e_{ij} is effected by e_{uv} through the sources i and u; either $i = u$ or u is a neighbor of i. In class $C2$, e_{ij} is effected by e_{uv} by coupling between the source i and the destination v; either $i = v$ or v is a neighbor of i. Similarly, class C3 describes interactions where coupling is between the source u and destination j; either $j = u$ or u is a neighbor of j. Finally, class $C4$ describes interactions that may occur through the destinations j and v; either $j = v$ or j and v are neighbors.

Thus, all transmission interactions are systematically categorized. This categories provide a general framework within which different channel access protocols may be described by identifying possible blocking and interference relations for each. We point out that interaction case C4(L2) is not encountered in applying the model in this thesis, because this situation does not correspond to any blocking or interference relation in the systems we consider. However, we include it here for completeness. [B

As was mentioned in Section 2.3, under the blocking model, every pair of elements in the blocking relation must be symmetrically related in order to [Bgenerate a product-form solution. In this transmission-interactions model, the basic interactions we consider are those in which transmissions on e_{ij} are effected by transmissions on e_{uv}. To satisfy reversibility, the blocking relation must include the reverse interactions. The reverse of each interaction is one of the given cases. For example, case C1 (L1) is its own reverse; e_{ij} and e_{uv} exchange roles. On the other hand, the reverse of interaction case C2(L1) is C3(L1). Thus, C2(L1) and C3(L1) form a pair of reversible interactions. If one is included, the other must also be included to maintain reversibility. Each of the following sets describes an interaction case and its reverse: {C1(L1)}; {C2(L1), C3(L1)}; {C4 (L1)}; {C1(L2)}; {C2(L2), C3(L2)}; {C4(L2)}. A set containing a single case is an interaction whose reverse has the same topology. For a set containing a pair of cases, the reverse of one interaction case has the topology of the other.

Using the information provided in Figure 3.3, we will characterize the transmission interactions for different channel access protocols. We consider two common protocols for networks with omnidirectional line-of-sight transceivers: Carrier-Sense Multiple Access (CSMA) and conservative Busy-Tone Multiple Access (BTMA), both of the nonpersistent variety. In addition, we introduce a nonrealizable protocol which we dub Collision-Free Multiple Access (CFMA) and which provides an upperbound on the achievable throughput capacity for any nonpersistent protocol operating in a network with omnidirectional transceivers. CFMA is nonrealizable in the sense that it requires information which is generally unavailable to a node operating in a distributed environment. We also apply the model to line-of-sight networks with directional transmission capabilities. For these networks, we study Directional-CSMA (D-CSMA) and a nonrealizable protocol which we call the Total Capacity Allocation Protocol (TCAP). TCAP is nonrealizable as a distributed protocol for the same reasons cited for CFMA. It is used to provide an upperbound on the throughput capacity of any nonpersistent protocol operating in a network with directional transmission capabilities. CFMA and TCAP are considered to measure the capacity of the network for omnidirectional and directional transmission networks, respectively.

The operation of each protocol is described below:

CSMA Under CSMA, transmission attempts on a link are blocked if its source node is currently transmitting or senses carrier from any node it can hear. In this case, transmissions are omnidirectional. The transmission from a node will be heard by all its neighbors. Therefore, all of its neighbors' transmission attempts will be blocked.

D-CSMA For D-CSMA, transmission attempts on a link are blocked if its source node is currently transmitting or senses carrier from its destination node. Unlike CSMA, transmissions in D-CSMA are directional, blocking and interference may happen only when transmission interactions are directly invloved. D-CSMA offers higher throughput values than that of CSMA.

BTMA For the conservative scheme of BTMA, the transmissions are omnidirectional and collision free. A busy tone is broadcasted by all neighbors of the transmitting node. Transmission attempts on a link will be blocked when its source node is currently transmitting or a busy tone is heared.

CFMA CFMA is a nonrealizable ideal protocol for networks with omnidirectional transmission capability. The transmitter is assumed to know whether its transmission will collide with an ongoing one or whether an ongoing transmission will collide with it. A transmission attempt is blocked in either case. CFMA provides an upper bound on the throughput for systems operating with omnidirectional transceivers.

TCAP TCAP is a nonrealizable ideal protocol for network with directional-transmission capability. The transmitter is assumed to know whether the intended receiver is busy, either transmitting or receiving. A transmission attempt on a link is blocked when either the source node or the destination node is currently involved in another transmission action. TCAP provides an upper bound on the throughput for systems operating with directional transceivers.

Based on these descriptions, we can define the blocking and interference sets of each link e_{ij}, for each channel access protocol, in a systematic way. This representation is shown in Figure 3.4. These definitions are presented below. The numbers in the set descriptions correspond to the cases depicted in Figure 3.4.

CSMA:

$$\begin{aligned} B(e_{ij}) &= \{e_{uv} : (1)\, i=u, (2)\, i=v, (3)\, j=u, (5)\, h_{ui}=1\}, \\ I(e_{ij}) &= \{e_{uv} : (4)\, j=v, (7)\, h_{uj}=1\}. \end{aligned}$$

D-CSMA:

$$\begin{aligned} B(e_{ij}) &= \{e_{uv} : (1)\, i=u, (2)\, i=v, (3)\, j=u\}, \\ I(e_{ij}) &= \{e_{uv} : (4)\, j=v\}. \end{aligned}$$

BTMA:

$$\begin{aligned} B(e_{ij}) &= \left\{ e_{uv}\text{:} \quad \begin{array}{l} (1)\ i=u,\ (2)\ i=v,\ (3)\ j=u,\ (4)\ j=v, \\ (5)\ h_{ui}=1,\ (6)\ h_{vi}=1,\ (7)\ h_{uj}=1 \end{array} \right\}, \\ I(e_{ij}) &= \phi. \end{aligned}$$

CFMA:

$$\begin{aligned} B(e_{ij}) &= \left\{ e_{uv}\text{:} \quad \begin{array}{l} (1)\ i=u,\ (2)\ i=v,\ (3)\ j=u,\ (4)\ j=v, \\ (6)\ h_{vi}=1,\ (7)\ h_{uj}=1 \end{array} \right\}, \\ I(e_{ij}) &= \phi. \end{aligned}$$

TCAP:

$$\begin{aligned} B(e_{ij}) &= \{e_{uv} : (1)\, i=u, (2)\, i=v, (3)\, j=u, (4)\, j=v\}, \\ I(e_{ij}) &= \phi. \end{aligned}$$

With these blocking and interference sets provided for each protocol, the TBG and the RIG for the 4-node chain PRN of Figure 2.1 can be generated as depicted in Figure 3.5. Based on the transmission interaction model of Figure 3.4 and its application in Figure 3.5, we have the following observations:

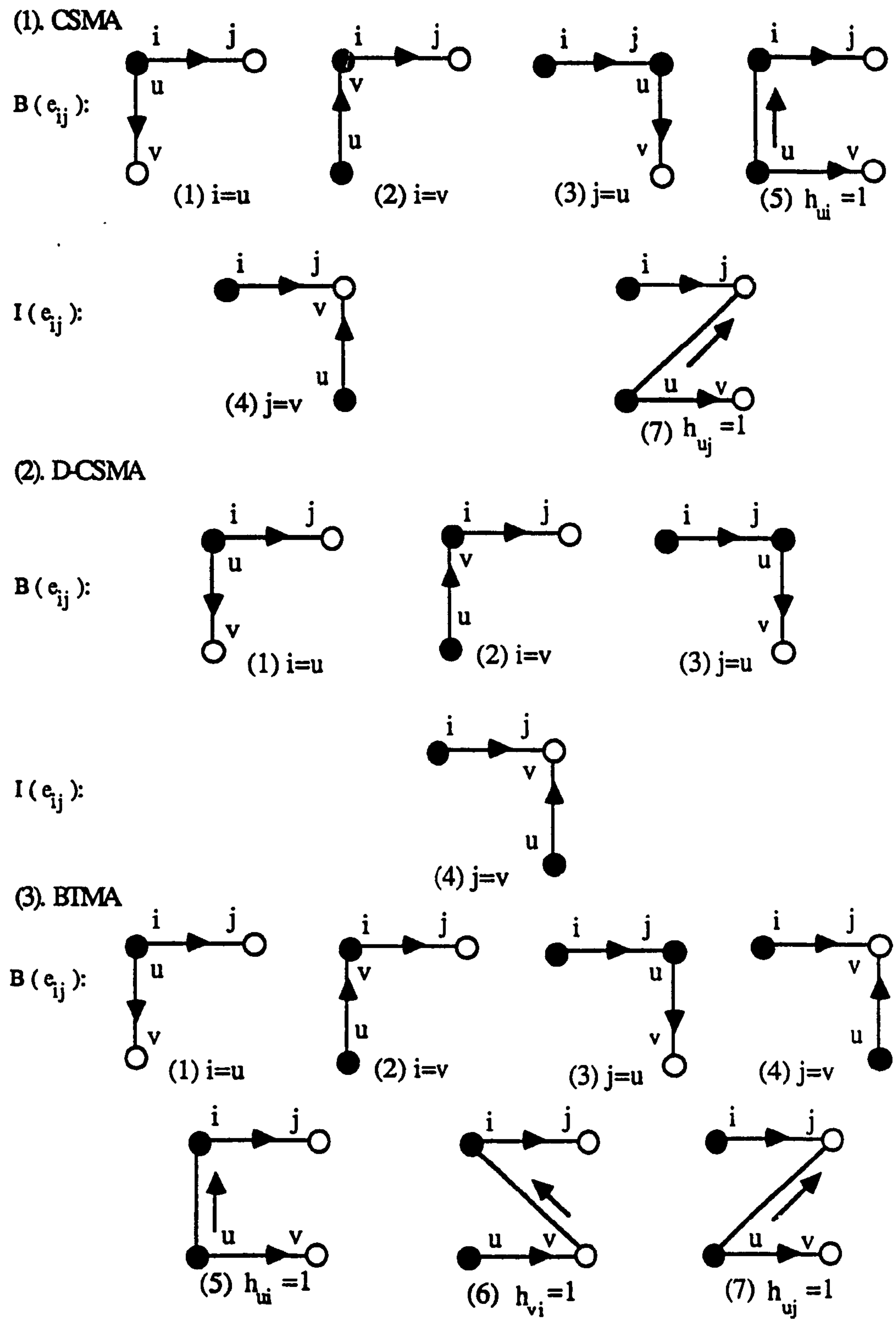


Figure 3.4: Blocking and interference for: (a) CSMA, (b) D-CSMA, (c) BTMA, (d) CFMA, (e) TCAP

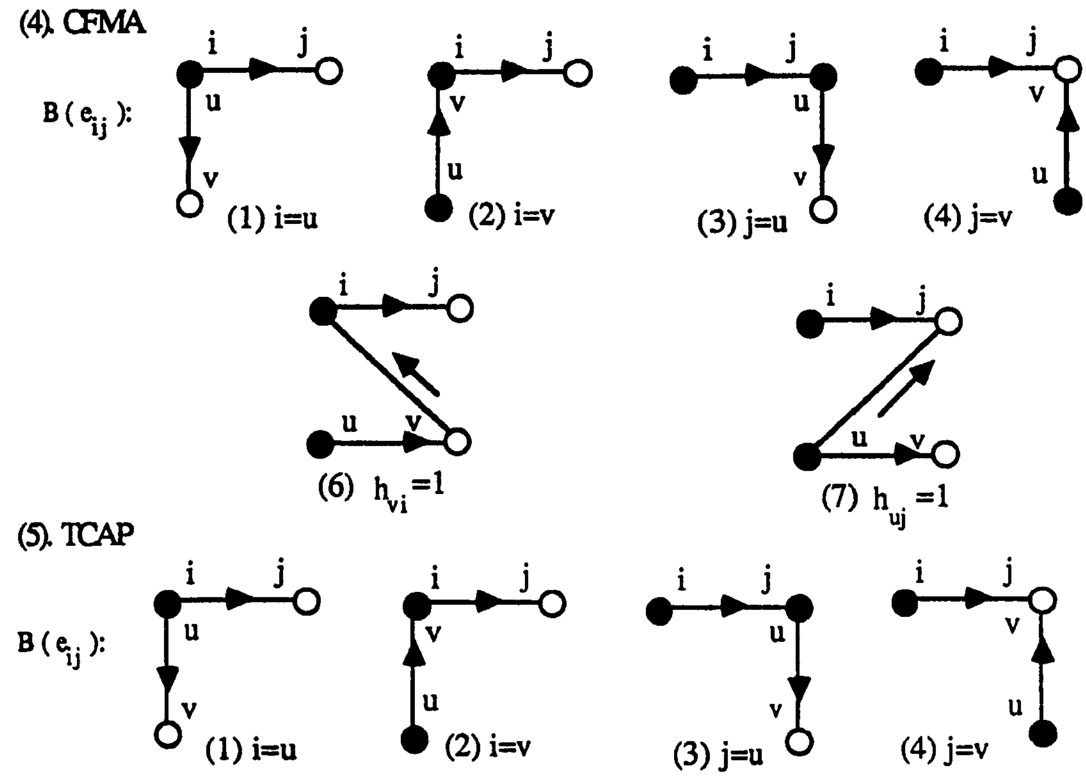


Figure 3.4: (Continued) Blocking and interference for: (a) CSMA, (b) D-CSMA, (c) BTMA, (d) CFMA, (e) TCAP

- BTMA, CFMA, and TCAP are collision-free as they are characterized by $I(e_{ij}) = \phi$. On the other hand, CSMA and D-CSMA are interference-susceptible since $I(e_{ij}) \neq \phi$. These results can be verified by observing the RIG of each protocol in Figure 3.5.

- All transmissions interactions of a directional-transmission system (with D-CSMA or TCAP) belong to the first level of Figure 3.3. In other words, the transmission operations are determined completely by direct interactions.

- For an omnidirectional-communication system (with CSMA, or BTMA, or CFMA), some transmission interactions are indirect while others are direct.

- For an omnidirectional-transmission system, CSMA *underprotects* the system since it causes some interference at the receiver; on the other hand, the protocol BTMA *overprotects* the system since it eliminates the interferences by the expenses of inhibiting some successful transmissions; CFMA *ideally protects* the system since it eliminates the interference problem without the cost of inhibiting any successful transmission. Similarly, for the systems with directional-transmission, D-CSMA *underprotects* the system while CFMA *ideally protects* the system.

- Systems using TCAP have directional-transmission capability and they are interference-free. TCAP can be thought of as as directional-BTMA or interference-free D-CSMA.

- Any pair of elements in a reversible transmission-interaction set must be simultaneousely present in the blocking model, as they are reflected in the TBG of the system. However, symmetry is not required for the interference relation, as is apparent from the RIG. The CSMA protocol demonstrates this in Figure 3.5.

Example 2.5(a). For the 4-node chain PRN of Figure 2.1, the rules for generating the blocking and interference sets of each channel access protocol result in the TBG and the RIG shown in Figure 3.5.

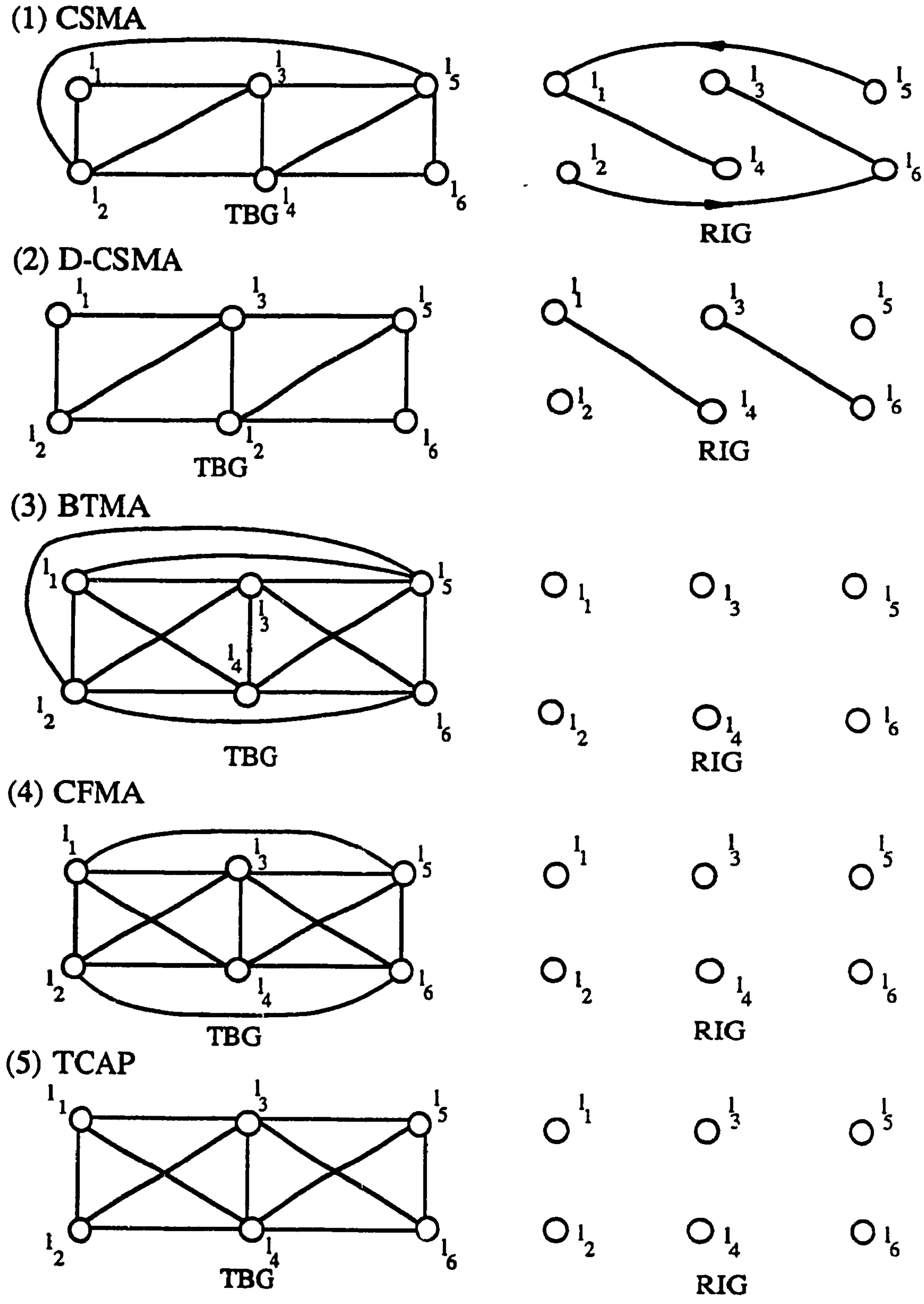


Figure 3.5: The TBG and the RIG for different channel access protocols

With element set $V = \{l_1, l_2, l_3, l_4, l_5, l_6\}$, we have

(a) CSMA.

$$B(l_1) = \{l_1, l_2, l_3\}; \qquad B(l_2) = \{l_1, l_2, l_3, l_4, l_5\};$$
$$B(l_3) = \{l_1, l_2, l_3, l_4, l_5\}; \qquad B(l_4) = \{l_2, l_3, l_4, l_5, l_6\};$$
$$B(l_5) = \{l_2, l_3, l_4, l_5, l_6\}; \qquad B(l_6) = \{l_4, l_5, l_6\}.$$

$$I(l_1) = \{l_4, l_5\}; \qquad I(l_2) = \phi; \qquad I(l_3) = \{l_6\};$$
$$I(l_4) = \{l_1\}; \qquad I(l_5) = \phi; \qquad I(l_6) = \{l_2, l_3\}.$$

$$SP(V) = 1 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 + \rho_3\rho_6.$$

$$\frac{r_1}{\rho_1} = \frac{(1+\rho_6)}{SP(V)}; \qquad \frac{r_2}{\rho_2} = \frac{(1+\rho_6)}{SP(V)}; \qquad \frac{r_3}{\rho_3} = \frac{1}{SP(V)};$$
$$\frac{r_4}{\rho_4} = \frac{1}{SP(V)}; \qquad \frac{r_5}{\rho_5} = \frac{1+\rho_1}{SP(V)}; \qquad \frac{r_6}{\rho_6} = \frac{1+\rho_1}{SP(V)}.$$

$$\begin{aligned}
P_B(1) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 + \rho_3\rho_6); \\
P_B(2) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 \\
&\quad + \rho_3\rho_6); \\
P_B(4) &= \frac{1}{SP(V)}(\rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 \\
&\quad + \rho_3\rho_6); \\
P_B(6) &= \frac{1}{SP(V)}(\rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_6 + \rho_3\rho_6),
\end{aligned}$$

$$P_B(3) = P_B(2), \qquad P_B(5) = P_B(4),$$

and

$$P_I(1) = \tfrac{1}{SP(V)}(\rho_4 + \rho_5); \qquad P_I(2) = 0; \qquad P_I(3) = \tfrac{1}{SP(V)}(\rho_6);$$
$$P_I(4) = \tfrac{1}{SP(V)}(\rho_1); \qquad P_I(5) = 0; \qquad P_I(6) = \tfrac{1}{SP(V)}(\rho_2 + \rho_3).$$

(b) D-CSMA:

$$B(l_1) = \{l_1, l_2, l_3\}; \qquad B(l_2) = \{l_1, l_2, l_3, l_4\}; \qquad B(l_3) = \{l_1, l_2, l_3, l_4, l_5\};$$
$$B(l_4) = \{l_2, l_3, l_4, l_5, l_6\}; \qquad B(l_5) = \{l_3, l_4, l_5, l_6\}; \qquad B(l_6) = \{l_4, l_5, l_6\}.$$
$$I(l_1) = \{l_4\}; \qquad I(l_2) = \phi; \qquad I(l_3) = \{l_6\};$$
$$I(l_4) = \{l_1\}; \qquad I(l_5) = \phi; \qquad I(l_6) = \{l_3\}.$$

$SP(V) = 1 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6 + \rho_3\rho_6.$

$\frac{r_1}{\rho_1} = \frac{1}{SP(V)}(1 + \rho_5 + \rho_6);$ $\quad \frac{r_2}{\rho_2} = \frac{1}{SP(V)}(1 + \rho_5 + \rho_6);$ $\quad \frac{r_3}{\rho_3} = \frac{1}{SP(V)};$

$\frac{r_4}{\rho_4} = \frac{1}{SP(V)};$ $\quad \frac{r_5}{\rho_5} = \frac{1}{SP(V)}(1 + \rho_1 + \rho_2);$ $\quad \frac{r_6}{\rho_6} = \frac{1}{SP(V)}(1 + \rho_1 + \rho_2).$

$$
\begin{aligned}
P_B(1) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6 + \rho_3\rho_6); \\
P_B(2) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6 \\
&\quad + \rho_3\rho_6); \\
P_B(3) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 \\
&\quad + \rho_2\rho_6 + \rho_3\rho_6); \\
P_B(4) &= \frac{1}{SP(V)}(\rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 \\
&\quad + \rho_2\rho_6 + \rho_3\rho_6); \\
P_B(5) &= \frac{1}{SP(V)}(\rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6 \\
&\quad + \rho_3\rho_6); \\
P_B(6) &= \frac{1}{SP(V)}(\rho_4 + \rho_5 + \rho_6 + \rho_1\rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6 + \rho_3\rho_6).
\end{aligned}
$$

$P_I(1) = \frac{1}{SP(V)}(\rho_4);$ $\quad P_I(2) = 0;$ $\quad P_I(3) = \frac{1}{SP(V)}(\rho_6);$

$P_I(4) = \frac{1}{SP(V)}(\rho_1);$ $\quad P_I(5) = 0;$ $\quad P_I(6) = \frac{1}{SP(V)}(\rho_3).$

(c) BTMA.

$B(l_1) = \{l_1, l_2, l_3, l_4, l_5\};$ $\quad B(l_2) = V;$ $\quad B(l_3) = V;$

$B(l_4) = V;$ $\quad B(l_5) = V;$ $\quad B(l_6) = \{l_2, l_3, l_4, l_5, l_6\},$

and $I(l_i) = \phi$, for all $i \in V$.

$$
\begin{aligned}
SP(V) &= 1 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_6. \\
\frac{r_1}{\rho_1} &= \frac{(1+\rho_6)}{SP(V)}; \qquad \frac{r_6}{\rho_6} = \frac{(1+\rho_1)}{SP(V)}; \\
\frac{r_2}{\rho_2} &= \frac{r_3}{\rho_3} = \frac{r_4}{\rho_4} = \frac{r_5}{\rho_5} = \frac{1}{SP(V)}. \\
P_B(1) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_1\rho_6);
\end{aligned}
$$

$$
\begin{aligned}
P_B(6) &= \frac{1}{SP(V)}(\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6); \\
P_B(2) &= P_B(3) = P_B(4) = P_B(5) \\
&= \frac{1}{SP(V)}(\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6),
\end{aligned}
$$

and

$$P_I(i) = 0, \qquad \text{for all } i \in V.$$

(d) CFMA.

$$
\begin{array}{lll}
B(l_1) = \{l_1, l_2, l_3, l_4, l_5\}; & B(l_2) = \{l_1, l_2, l_3, l_4, l_6\}; & B(l_3) = V; \\
B(l_4) = V; & B(l_5) = \{l_1, l_3, l_4, l_5, l_6\}; & B(l_6) = \{l_2, l_3, l_4, l_5, l_6\},
\end{array}
$$

and $I(l_i) = \phi$, for all $i \in V$.

$$SP(V) = 1+\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6+\rho_2\rho_5.$$

$$
\frac{r_1}{\rho_1} = \frac{(1+\rho_6)}{SP(V)}; \qquad \frac{r_2}{\rho_2} = \frac{(1+\rho_5)}{SP(V)}; \qquad \frac{r_3}{\rho_3} = \frac{1}{SP(V)};
$$

$$
\frac{r_4}{\rho_4} = \frac{1}{SP(V)}; \qquad \frac{r_5}{\rho_5} = \frac{1+\rho_2}{SP(V)}; \qquad \frac{r_6}{\rho_6} = \frac{1+\rho_1}{SP(V)}.
$$

$$
\begin{aligned}
P_B(1) &= \frac{1}{SP(V)}(\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_1\rho_6+\rho_2\rho_5); \\
P_B(3) &= \frac{1}{SP(V)}(\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6+\rho_2\rho_5); \\
P_B(5) &= \frac{1}{SP(V)}(\rho_1+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6+\rho_2\rho_5); \\
P_B(6) &= \frac{1}{SP(V)}(\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_1\rho_6+\rho_2\rho_5),
\end{aligned}
$$

$$P_B(2) = P_B(1) \qquad\qquad P_B(4) = P_B(3),$$

and

$$P_I(i) = 0 \qquad \text{for all } i \in V.$$

(e) TCAP.

$$
\begin{array}{lll}
B(l_1) = \{l_1, l_2, l_3, l_4\}; & B(l_2) = \{l_1, l_2, l_3, l_4\}; & B(l_3) = V; \\
B(l_4) = V; & B(l_5) = \{l_3, l_4, l_5, l_6\}; & B(l_6) = \{l_3, l_4, l_5, l_6\},
\end{array}
$$

and $I(l_i) = \phi$, for all $i \in V$.

$$SP(V) = 1 + \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6.$$

$$\begin{aligned}
\frac{r_1}{\rho_1} &= \frac{r_2}{\rho_2} = \frac{(1+\rho_5+\rho_6)}{SP(V)}; \qquad \frac{r_3}{\rho_3} = \frac{r_4}{\rho_4} = \frac{1}{SP(V)}; \\
\frac{r_5}{\rho_5} &= \frac{r_6}{\rho_6} = \frac{1+\rho_1+\rho_2}{SP(V)}. \\
P_B(1) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6); \\
P_B(3) &= \frac{1}{SP(V)}(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6); \\
P_B(5) &= \frac{1}{SP(V)}(\rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_1\rho_5 + \rho_1\rho_6 + \rho_2\rho_5 + \rho_2\rho_6),
\end{aligned}$$

$$P_B(2) = P_B(1); \qquad P_B(4) = P_B(3); \qquad P_B(6) = P_B(5),$$

and

$$P_I(i) = 0 \qquad \text{for all } i \in V.$$

3.3.2 Relative measurements of channel capacity utilizations

Channel capacity utilizations can be obtained for each practical protocol (i.e., CSMA, D-CSMA and BTMA) by using CFMA and TCAP as channel capacity measures for omnidirectional and directional transmission networks, respectively. We define the utilization of a link's capacity as the ratio of the maximum thrBoughput value achievable under a given protocol to the corresponding throughput value achieved in the same network under CFMA or TCAP, for the same transmission attempt rate. Since CFMA and TCAP generate the maximum possible throughput values, these relative measurements give the percentage of each link's total capacity that can be utilized with the given channel access protocol in an omnidirectional or a directional transmission environment, respectively. We refer to these as the relative channel capacity measurement for omnidirectional transmission (RCCMOT) and the relative channel capacity measurement for directional transmission (RCCMDT), respectively. Note that D-CSMA is a protocol for directional transmission systems.

D-CSMA should, therefore, be measured relative to TCAP only. However, communication networks with the omnidirectional transmission protocols (i.e., CSMA and BTMA) can be measured relative to either CFMA or TCAP. TCAP measures the total capacity of the network since it achieves as many successful transmissions as the topology of the network will support.

In general, given a system with channel access protocol X, we define

$$T_1^{(X)}(l_i) = \frac{r_{max}^{(X)}(l_i)}{r_{\rho_i^*}^{(CFMA)}(l_i)}; \tag{3.7}$$

$$T_2^{(X)}(l_i) = \frac{r_{max}^{(X)}(l_i)}{r_{\rho_i^*}^{(TCAP)}(l_i)}, \tag{3.8}$$

respectively, as the RCCMOT and the RCCMDT measurements for directed link l_i, where $r_{max}^{(X)}(l_i)$ is the maximum throughput value over link l_i for channel access protocol X, and $r_{\rho_i^*}^{CFMA}(l_i)$ and $r_{\rho_i^*}^{TCAP}(l_i)$ are the throughput values over link l_i evaluated, respectively, for CFMA and TCAP at the attempt rate ρ_i^* where $r_{max}^{(X)}(l_i)$ is achieved.

3.4 Partition Functions of Blocking Systems

In Section 2.3 of Chapter 2, we concluded that the steady-state probability distribution of the reversible stochastic blocking system with n elements $V = \{1, 2, \cdots, n\}$ is

$$Q(S) = Q(\phi) \prod_{i \in S} \rho_i = Q(\phi) \rho_1^{b_1} \rho_2^{b_2} \cdots \rho_n^{b_n},$$

where $b(S) = b_1 b_2 \cdots b_n$ is the bit pattern of the state S, and $Q^{-1}(\phi) = SP(V)$ is called the *partition function* of the given system. The partition function will also be denoted as $\beta(G, \overline{\rho})$ signifying that it is a function of the topology of the network G and the scheduling-rate vector $\overline{\rho}$, where $\overline{\rho} = (\rho_1, \rho_2, \cdots, \rho_n)$. $\beta(G, \overline{\rho})$ describes the dependencies among the elements of the system and it is defined by the network topology and the channel access protocol. $\beta(G, \overline{\rho})$ is completely described by the TBG, more specifically, by blocking interactions at the transmitting end of transmissions only. It is independent of the interference at the receiving end, which is described by the RIG. Special considerations can be placed upon $\beta(G, \overline{\rho})$ and they will be presented in the following subsections.

3.4.1 The partition function of an interference-free system

For a communication system operating under an interference-free protocol, $I(i) = \phi$, for all $i \in V$. In this case, Equation (2.14) is modified as

$$\frac{r_i}{\rho_i} = P(B(i) \text{ is idle}) = \frac{SP(V - B(i))}{SP(V)}.$$

With an interference-free communication system, transmissions never experience interference at the receivers and hence, every transmission is successful. Thus, transmission attempts result in successful transmissions if and only if they are not blocked by the protocol in effect at the transmitters. As usual, the states of the system are the independent sets of the blocking graph. The interference probability is zero over each channel. In general, for each element i in the system, the state space implied by the partition function $SP(V)$ can be separated into two parts: (1) states containing elements $j \in B(i)$ block element i; and (2) states contain only elements $j \in V - B(i)$ do not effect element i's activation, allowing i to be successful. Thus, we write

$$P_B(i) = \frac{SP(V) - SP(V - B(i))}{SP(V)},$$
$$P_I(i) = 0.$$

In addition, given a partition function $SP(V)$ of a general communication system (which is not necessarily interference-free), we observe the following result:

Theorem 3.2 *The partition function $SP(V)$ of the blocking model satifies*

$$\frac{\partial(SP(V))}{\partial\rho_i} = SP(V - B(i)).$$

The proof of this Theorem is given in Appendix B.

Therefore, for the interference-free system, the probability of successful transmission over link l_i can be expressed as the probability of successful transmission over link i can be expressed as

$$\frac{r_i}{\rho_i} = \frac{\partial(ln(SP(V)))}{\partial\rho_i} = \frac{1}{SP(V)}\frac{\partial(SP(V))}{\partial\rho_i} = \frac{N(\overline{G})}{D(\overline{G})},$$

while its blocking probability can be written as

$$P_B(i) = \frac{SP(V) - SP(V - B(i))}{SP(V)} = \frac{D(\overline{G}) - N(\overline{G})}{D(\overline{G})},$$

where $N(\overline{G})$ and $D(\overline{G})$ are respectively, the numerator and denominator polynomials in the expression for r_i/ρ_i.

Typical interference-free systems are communication networks with BTMA, CFMA, or TCAP. However, common networks such as a complete network and a ring network are also interference-free systems under CSMA or D-CSMA, for the given assumptions of the model.

Example 2.5(b) (continuation of Ex.2.5(a))

(a) CSMA.

By letting $\hat{\rho}_1 = \rho_1 = \rho_2 = \rho_5 = \rho_6$, $\hat{\rho}_2 = \rho_3 = \rho_4$, we have

$$\beta(\vec{\rho}) = SP(V) = 1 + 4\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1\hat{\rho}_2 + 3\hat{\rho}_1^2.$$

Let $\hat{r}_1 = r_1 = r_2 = r_5 = r_6 = \frac{\hat{\rho}_1(1+\hat{\rho}_1)}{\beta(\vec{\rho})}$, $\hat{r}_2 = r_3 = r_4 = \frac{\hat{\rho}_2}{\beta(\vec{\rho})}$. When $r = \hat{r}_1 = \hat{r}_2$, we have $\hat{\rho}_2 = \hat{\rho}_1(1+\hat{\rho}_1)$. Then

$$r = \frac{\hat{\rho}_1(1+\hat{\rho}_1)}{\beta(\vec{\rho})},$$

with $\beta(\vec{\rho}) = 1 + 6\hat{\rho}_1 + 7\hat{\rho}_1^2 + 2\hat{\rho}_1^3$. Further,

$$\begin{aligned}
\hat{P}_B(1) &= P_B(1) = P_B(6) = \frac{1}{\beta(\vec{\rho})}(2\hat{\rho}_1 + \hat{\rho}_2 + 3\hat{\rho}_1^2 + 2\hat{\rho}_1\hat{\rho}_2) = \frac{1}{\beta(\vec{\rho})}(3\hat{\rho}_1 + 6\hat{\rho}_1^2 + 2\hat{\rho}_1^3);\\
\hat{P}_B(2) &= P_B(2) = P_B(3) = P_B(4) = P_B(5) = \frac{1}{\beta(\vec{\rho})}(3\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1\hat{\rho}_2 + 3\hat{\rho}_1^2)\\
&= \frac{1}{\beta(\vec{\rho})}(5\hat{\rho}_1 + 7\hat{\rho}_1^2 + 2\hat{\rho}_1^3).\\
\hat{P}_I(1) &= P_I(1) = P_I(6) = \frac{1}{\beta(\vec{\rho})}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{\hat{\rho}_1(2+\hat{\rho}_1)}{\beta(\vec{\rho})};\\
\hat{P}_I(2) &= P_I(2) = P_I(5) = 0;\\
\hat{P}_I(3) &= P_I(3) = P_I(4) = \frac{\hat{\rho}_1}{\beta(\vec{\rho})}.
\end{aligned}$$

(b) D-CSMA.

By letting $\hat{\rho}_1 = \rho_1 = \rho_2 = \rho_5 = \rho_6$, and $\hat{\rho}_2 = \rho_3 = \rho_4$, we get

$$\beta(\vec{\rho}) = SP(V) = 1 + 4\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1\hat{\rho}_2 + 4\hat{\rho}_1^2.$$

Let $\hat{r}_1 = r_1 = r_2 = r_5 = r_6 = \frac{\hat{\rho}_1(1+2\hat{\rho}_1)}{\beta(\bar{\rho})}$ and $\hat{r}_2 = r_3 = r_4 = \frac{\hat{\rho}_2}{\beta(\bar{\rho})}$. When $r = \hat{r}_1 = \hat{r}_2$, we have $\hat{\rho}_2 = \hat{\rho}_1(1+2\hat{\rho}_1)$. Then

$$r = \frac{\hat{\rho}_1(1+2\hat{\rho}_1)}{\beta(\bar{\hat{\rho}})},$$

with $\beta(\bar{\hat{\rho}}) = 1 + 6\hat{\rho}_1 + 10\hat{\rho}_1^2 + 4\hat{\rho}_1^3$. In addition, we have

$$\hat{P}_B(1) = P_B(1) = P_B(6) = \frac{1}{\beta(\bar{\hat{\rho}})}(2\hat{\rho}_1 + \hat{\rho}_2 + 4\hat{\rho}_1^2 + 2\hat{\rho}_1\hat{\rho}_2) = \frac{1}{\beta(\bar{\hat{\rho}})}(3\hat{\rho}_1 + 8\hat{\rho}_1^2 + 4\hat{\rho}_1^3);$$

$$\hat{P}_B(2) = P_B(2) = P_B(5) = \frac{1}{\beta(\bar{\hat{\rho}})}(2\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1\hat{\rho}_2 + 4\hat{\rho}_1^2) = \frac{1}{\beta(\bar{\hat{\rho}})}(4\hat{\rho}_1 + 10\hat{\rho}_1^2 + 4\hat{\rho}_1^3);$$

$$\hat{P}_B(3) = P_B(3) = P_B(4) = \frac{1}{\beta(\bar{\hat{\rho}})}(3\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1\hat{\rho}_2 + 4\hat{\rho}_1^2) = \frac{1}{\beta(\bar{\hat{\rho}})}(5\hat{\rho}_1 + 10\hat{\rho}_1^2 + 4\hat{\rho}_1^3).$$

$$\begin{aligned}
\hat{P}_I(1) &= P_I(1) = P_I(6) = \frac{\hat{\rho}_2}{\beta(\bar{\hat{\rho}})} = \frac{\hat{\rho}_1(1+2\hat{\rho}_1)}{\beta(\bar{\hat{\rho}})};\\
\hat{P}_I(2) &= P_I(2) = P_I(5) = 0;\\
\hat{P}_I(3) &= P_I(3) = P_I(4) = \frac{\hat{\rho}_1}{\beta(\bar{\hat{\rho}})}.
\end{aligned}$$

(c) BTMA.

By letting $\hat{\rho}_1 = \rho_1 = \rho_6$, $\hat{\rho}_2 = \rho_2 = \rho_3 = \rho_4 = \rho_5$, we obtain

$$\beta(\bar{\hat{\rho}}) = SP(V) = 1 + 2\hat{\rho}_1 + 4\hat{\rho}_2 + \hat{\rho}_1^2.$$

$$\frac{\hat{r}_1}{\hat{\rho}_1} = \frac{1}{2}\frac{\partial}{\partial\hat{\rho}_1}(ln(\beta(\bar{\hat{\rho}}))) = \frac{(1+\hat{\rho}_1)}{\beta(\bar{\hat{\rho}})}; \qquad P_B(1) = \frac{1}{\beta(\bar{\hat{\rho}})}(\hat{\rho}_1 + 4\hat{\rho}_2 + \hat{\rho}_1^2)$$

$$\frac{\hat{r}_2}{\hat{\rho}_2} = \frac{1}{4}\frac{\partial}{\partial\hat{\rho}_2}(ln(\beta(\bar{\hat{\rho}}))) = \frac{1}{\beta(\bar{\hat{\rho}})}; \qquad P_B(2) = \frac{1}{\beta(\bar{\hat{\rho}})}(2\hat{\rho}_1 + 4\hat{\rho}_2 + \hat{\rho}_1^2).$$

Let $r = \hat{r}_1 = \hat{r}_2$, we get $\hat{\rho}_2 = \hat{\rho}_1(1+\hat{\rho}_1)$ and $\beta(\bar{\hat{\rho}}) = 1 + 6\hat{\rho}_1 + 5\hat{\rho}_1^2$. Therefore,

$$r = \frac{\hat{\rho}_1(1+\hat{\rho}_1)}{1 + 6\hat{\rho}_1 + 5\hat{\rho}_1^2}, \qquad r_{max} = r|_{\hat{\rho}_1 \to \infty} = 0.2.$$

$$\begin{aligned}
\hat{P}_B(1) &= P_B(1) = P_B(6) = \frac{5\hat{\rho}_1 + 5\hat{\rho}_1^2}{1 + 6\hat{\rho}_1 + 5\hat{\rho}_1^2};\\
\hat{P}_B(2) &= P_B(2) = P_B(3) = P_B(4) = P_B(5) = \frac{6\hat{\rho}_1 + 5\hat{\rho}_1^2}{1 + 6\hat{\rho}_1 + 5\hat{\rho}_1^2}.
\end{aligned}$$

(d) CFMA.

By letting $\hat{\rho}_1 = \rho_1 = \rho_2 = \rho_5 = \rho_6$, $\hat{\rho}_2 = \rho_3 = \rho_4$, we have

$$\beta(\vec{\rho}) = SP(V) = 1 + 4\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1^2.$$

$$\frac{\hat{r}_1}{\hat{\rho}_1} = \frac{1}{4}\frac{\partial}{\partial\hat{\rho}_1}(ln\beta(\vec{\rho})) = \frac{(1+\hat{\rho}_1)}{\beta(\vec{\rho})}; \qquad \hat{P}_B(1) = \frac{1}{\beta(\vec{\rho})}(3\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1^2)$$

$$\frac{\hat{r}_2}{\hat{\rho}_2} = \frac{1}{2}\frac{\partial}{\partial\hat{\rho}_2}(ln\beta(\vec{\rho})) = \frac{1}{\beta(\vec{\rho})}; \qquad \hat{P}_B(2) = \frac{1}{\beta(\vec{\rho})}(4\hat{\rho}_1 + 2\hat{\rho}_2 + 2\hat{\rho}_1^2).$$

Let $r = \hat{r}_1 = \hat{r}_2$, we get $\hat{\rho}_2 = \hat{\rho}_1(1+\hat{\rho}_1)$ and $\beta(\vec{\rho}) = 1 + 6\hat{\rho}_1 + 4\hat{\rho}_1^2$. Thus,

$$r = \frac{\hat{\rho}_1(1+\hat{\rho}_1)}{1+6\hat{\rho}_1+4\hat{\rho}_1^2}, \qquad r_{max} = r|_{\hat{\rho}_1\to\infty} = 0.25.$$

$$\begin{aligned}\hat{P}_B(1) &= P_B(1) = P_B(2) = P_B(5) = P_B(6) = \frac{5\hat{\rho}_1 + 4\hat{\rho}_1^2}{1+6\hat{\rho}_1+4\hat{\rho}_1^2};\\ \hat{P}_B(2) &= P_B(3) = P_B(4) = \frac{6\hat{\rho}_1 + 4\hat{\rho}_1^2}{1+6\hat{\rho}_1+4\hat{\rho}_1^2}.\end{aligned}$$

(e) TCAP.

By letting $\hat{\rho}_1 = \rho_1 = \rho_2 = \rho_5 = \rho_6$, and $\hat{\rho}_2 = \rho_3 = \rho_4$, we have

$$\beta(\vec{\rho}) = SP(V) = 1 + 4\hat{\rho}_1 + 2\hat{\rho}_2 + 4\hat{\rho}_1^2.$$

$$\frac{\hat{r}_1}{\hat{\rho}_1} = \frac{1}{4}\frac{\partial}{\partial\hat{\rho}_1}(ln\beta(\vec{\rho})) = \frac{(1+2\hat{\rho}_1)}{\beta(\vec{\rho})}; \qquad \hat{P}_B(1) = \frac{1}{\beta(\vec{\rho})}(2\hat{\rho}_1 + 2\hat{\rho}_2 + 4\hat{\rho}_1^2);$$

$$\frac{\hat{r}_2}{\hat{\rho}_2} = \frac{1}{2}\frac{\partial}{\partial\hat{\rho}_2}(ln\beta(\vec{\rho})) = \frac{1}{\beta(\vec{\rho})}; \qquad \hat{P}_B(2) = \frac{1}{\beta(\vec{\rho})}(4\hat{\rho}_1 + 2\hat{\rho}_2 + 4\hat{\rho}_1^2).$$

Let $r = \hat{r}_1 = \hat{r}_2$, we get $\hat{\rho}_2 = \hat{\rho}_1(1+2\hat{\rho}_1)$ and $\beta(\vec{\rho}) = 1 + 6\hat{\rho}_1 + 8\hat{\rho}_1^2$. Therefore,

$$r = \frac{\hat{\rho}_1(1+2\hat{\rho}_1)}{1+6\hat{\rho}_1+8\hat{\rho}_1^2}, \qquad r_{max} = r|_{\hat{\rho}_1\to\infty} = 0.25.$$

$$\begin{aligned}\hat{P}_B(1) &= P_B(1) = P_B(2) = P_B(5) = P_B(6) = \frac{4\hat{\rho}_1 + 8\hat{\rho}_1^2}{1+6\hat{\rho}_1+8\hat{\rho}_1^2};\\ \hat{P}_B(2) &= P_B(3) = P_B(4) = \frac{6\hat{\rho}_1 + 8\hat{\rho}_1^2}{1+6\hat{\rho}_1+8\hat{\rho}_1^2}.\end{aligned}$$

□

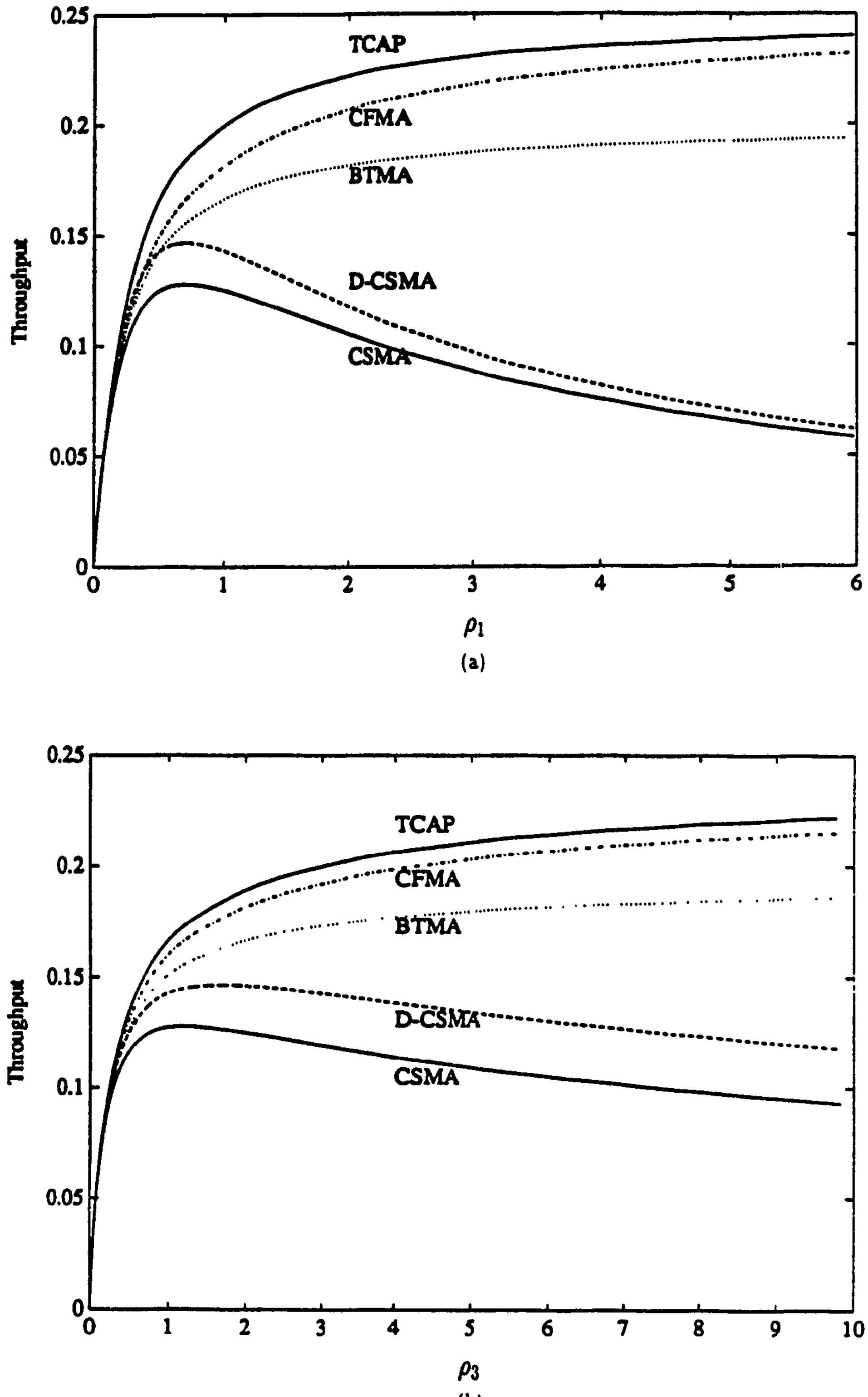


Figure 3.6: Effects of channel access protocol on throughput: (a) Throughput versus ρ_1, (b) Throughput versus ρ_3

The throughput comparisons of the given 4-node chain PRN with different channel access protocols are presented in Figure 3.6.

The RCCMOT and the RCCMDT over links l_1 and l_3 of this 4-node chain PRN are shown in Table 3.2 and Table 3.3, respectively. In this 4-node chain network, $T_1^{(X)}(l_1) = T_1^{(X)}(l_3)$ for all protocols, whereas $T_2^{(X)}(l_1) = T_2^{(X)}(l_3)$ for D-CSMA and BTMA, and $T_2^{(X)}(l_3) > T_1^{(X)}(l_1)$ for CSMA.

Table 3.2: The RCCMOT and the RCCMDT for link l_1 of the 4-node chain PRN

Protocol (X)	ρ_1^*	$r_{max}^{(X)}(l_1)$	$r_{\rho_1^*}^{(CFMA)}(l_1)$	$r_{\rho_1^*}^{(TCAP)}(l_1)$	$T_1^{(X)}(l_1)$	$T_2^{(X)}(l_1)$
CSMA	0.7070	0.1277	0.1667	0.1847	0.7661	0.6914
D-CSMA	0.7072	0.1464	-	0.1847	-	0.7926
BTMA	∞	0.2	0.25	0.25	0.80	0.80

Table 3.3: The RCCMOT and the RCCMDT for link l_3 of the 4-node chain PRN

Protocol (X)	ρ_3^*	$r_{max}^{(X)}(l_3)$	$r_{\rho_3^*}^{(CFMA)}(l_3)$	$r_{\rho_3^*}^{(TCAP)}(l_3)$	$T_1^{(X)}(l_3)$	$T_2^{(X)}(l_3)$
CSMA	1.2068	0.1277	0.1667	0.1734	0.7661	0.7364
D-CSMA	1.7067	0.1464	-	0.1847	-	0.7926
BTMA	∞	0.2	0.25	0.25	0.80	0.80

3.4.2 The partition function for systems with uniform normalized activation rates

For a given system with uniform normalized activation rates, we have $\rho_i = \rho$ for all $i \in V$, where $V = \{1, 2, \ldots, n\}$ is the set of n elements in the system. Then for any state $S \in \Omega$, the steady-state probability distribution is

$$Q(S) = Q(\phi)\rho^{|S|}. \tag{3.9}$$

The partition function is,

$$\beta(G,\rho) = (Q(\phi))^{-1} = \sum_{S \in \Omega} \rho^{|S|} = \sum_{k} P(G,k)\rho^{k},$$

where $P(G, k)$ is the number of distinct independent subsets of V having k elements. Under this uniform situation, every element behaves identically. Then, for an interference-free system,

$$\frac{r}{\rho} = \frac{1}{n}\frac{\partial ln\beta(\rho)}{\partial \rho} = \frac{N(\rho)}{D(\rho)}; \tag{3.10}$$

$$Pb = \frac{D(\rho) - N(\rho)}{D(\rho)}, \tag{3.11}$$

where n is the total number of elements, and Pb is the blocking probability for every element in the system.

3.5 The Partition Functions for Systems with TCAP

As demonstrated in Subsection 3.4.1, a communication system is collision-free for BTMA, CFMA, or TCAP. In this section, our attention is focussed on TCAP for its special relation to network topology.

Based on the rules to generate the TBG for TCAP, a scheduled transmission over directed link l_i from source s_i to destination d_i, is blocked by the activation of any directed link originating from, or destined to, node s_i or d_i. Therefore, given a directed network G, a given link will be blocked by its adjacent links. Using this rule, the partition function is directly obtained from the network. For TCAP, the partition function assumes a special form, which is related to the matching polynomial. The *matching polynomial* [GG81], [Far79], [God79] has been studied in graph theory. It occurs in various forms, none of which is exactly the form we require. In the following, we borrow some of the results known in the literature and reshape them for our problem. The proofs of these properties are presented in the appendices for completness. In this context, we refer to our form of the matching polynomial as the matching function $\alpha(\tilde{G})$ of the matching network $\tilde{G}$. $\tilde{G}$ is defined as an undirected graph which has the same topological connection as the directed graph G.

3.5.1 The matching function of the matching network

In order to obtain some computational results that $\alpha(\tilde{G})$ can provide to $\beta(G)$, we first need to describe the matching function $\alpha(\tilde{G})$ of the matching network $\tilde{G}$, then, we establish the relations between $\alpha(\tilde{G})$ and the partition function $\beta(G,\rho)$ of the directed network G. For simplicity of analysis, we assume uniform normalized activation scheduling rate $\tilde{\rho}$ over each edge in $\tilde{G}$.

Definition 3.1 *Two edges (i,j) and (u,v) of an undirected graph $\tilde{G}$ are* **independent** *if they have no node in common, i.e., $i \neq u$, $i \neq v$, $j \neq u$, and $j \neq v$.*

Definition 3.2 *A* **matching** *of an undirected graph $\tilde{G}$ is a set of pairwise independent edges of $\tilde{G}$.*

By a *matching network*, we mean a system in which interference takes place only for adjacent (non-independent) edges.

Definition 3.3 *Let $\tilde{G}$ be an matching network of n vertices and $P(\tilde{G},k)$ be the number of ways in which one can select k independent edges in $\tilde{G}$, $k = 1,2,\ldots,\lfloor n/2 \rfloor$, where $P(\tilde{G},0) \triangleq 1$ for all $\tilde{G}$. Then the matching function $\alpha(\tilde{G})$ of the graph $\tilde{G}$ is given by*

$$\alpha(\tilde{G}) = \alpha(\tilde{G},\tilde{\rho}) = \sum_{k=0}^{\lfloor n/2 \rfloor} P(\tilde{G},k)\tilde{\rho}^k. \tag{3.12}$$

Let e be an edge incident to the vertices v and w. Among the $P(\tilde{G},k)$ selections of k independent edges, there are $P(\tilde{G}-e,k)$ selections which do not contain e and $P(\tilde{G}-v-w,k-1)$ selections which contain e. Thus,

$$P(\tilde{G},k) = P(\tilde{G}-e,k) + P(\tilde{G}-v-w,k-1). \tag{3.13}$$

From this property, it is easy to establish the following.

Theorem 3.3 *The matching function of the matching graph $\tilde{G}$ satisfies*

$$\alpha(\tilde{G},\tilde{\rho}) = \alpha(\tilde{G}-e,\tilde{\rho}) + \tilde{\rho}\alpha(\tilde{G}-v-w,\tilde{\rho}). \tag{3.14}$$

The proof of Theorem 3.2 is given in Appendix C. Note the similarity of Equation (3.14) to Equation (3.2). Equation (3.14) simply expresses the recursion of Equation (3.2) in terms of edge and node deletions in $\tilde{G}$, when the elements of the blocking model happen to be the edges of $\tilde{G}$ and the blocking relation is the dependent-edge relation on the edges of $\tilde{G}$. In this model, the independent sets of the blocking graph are the matchings of the graph $\tilde{G}$.

Corollary 3.1 *Let the vertex v be adjacent to the vertices $w_1, w_2, \ldots, w_d$. Let further $\tilde{H} = \tilde{G} - v$, then*

$$\alpha(\tilde{G}, \tilde{\rho}) = \alpha(\tilde{H}, \tilde{\rho}) + \tilde{\rho} \sum_{j=1}^{d} \alpha(\tilde{H} - w_j, \tilde{\rho}). \tag{3.15}$$

The proof of this Corollary is given in Appendix D.

Theorem 3.4 *For two matching networks $\tilde{G}1$ and $\tilde{G}2$, it is true that*

$$\alpha(\tilde{G}1 \cup \tilde{G}2, \tilde{\rho}) = \alpha(\tilde{G}1, \tilde{\rho})\alpha(\tilde{G}2, \tilde{\rho}). \tag{3.16}$$

This theorem is just the application of Equation (3.1). Its proof is omitted.

3.5.2 Relationship between partition functions and matching functions

The matching function $\alpha(\tilde{G}, \tilde{\rho})$ of the given network $\tilde{G}$ refers to sets of possible successful activations of edges in the system. In order to use the partition function to analyze the communication network with the directed graph G, we need to establish a general relation between $\alpha(\tilde{G}, \tilde{\rho})$ and $\beta(G, \rho)$. A communication network operating with simplex data links, such as a switch, can be represented by a directed graph G in which every pair of nodes is connected by a single directed link. In this case, an edge in $\tilde{G}$ corresponds to a directed link in G. Therefore, $\beta(G, \rho) = \alpha(\tilde{G}, \tilde{\rho})$. However, for a communication system operating with half-duplex data links, an edge connecting a pair of nodes in the network represents two possible transmissions, one in each direction. In other words, activation of an edge in the matching network represents the activation of one of the two directed links corresponding to this edge. In this case, $\tilde{\rho} = 2\rho$, and $\beta(G, \rho) = \alpha(\tilde{G}, \tilde{\rho})|_{\tilde{\rho}=2\rho}$.

Therefore, for a network G operating with simplex data links, we have,

$$\beta(G,\rho) = \beta(G-e,\rho)+\rho\beta(G-v-w,\rho); \quad (3.17)$$

$$\beta(G,\rho) = \beta(H,\rho)+\rho\sum_{j=1}^{d}\beta(H-w_j,\rho), \quad (3.18)$$

$$\beta(G1\cup G2,\rho) = \beta(G1,\rho)\beta(G2,\rho). \quad (3.19)$$

Similarly, for a network G operating with half-duplex data links, the relations between its partition functions are

$$\beta(G,\rho) = \beta(G-e,\rho)+2\rho\beta(G-v-w,\rho); \quad (3.20)$$

$$\beta(G,\rho) = \beta(H,\rho)+2\rho\sum_{j=1}^{d}\beta(H-w_j,\rho), \quad (3.21)$$

$$\beta(G1\cup G2,\rho) = \beta(G1,\rho)\beta(G2,\rho). \quad (3.22)$$

Example 3.7. Consider a circuit-switching 3-cube network H operating with half-duplex data links. Assuming its transmissions are one-hop only, then its performance can be measured by relating to the partition functions.

For the development of its throughput expression, we refer to Figure 3.7. Let H_2 be obtained from H by removing 2 adjacent nodes. Then given a normalized attempted transmission rate of ρ_{ij} from node i to node j, the successful-transmission rate can be expressed as

$$r_{ij} = \rho_{ij}Pr[\text{no blocking at } i \text{ and } j] = \rho_{ij}P(i,j) = \rho_{ij}\sum_{S\subseteq H_2} Q(S).$$

Under symmetric loading, $r_{ij}=r$, and $\rho_{ij}=\rho$. Therefore, the per-link throughput is

$$r=\rho\sum_{S\subseteq H_2} Q(S) = \rho\beta^{-1}(H,\rho)\sum_{s\subseteq H_2}\rho^{|S|},$$

where $\beta(H,\rho)=\sum_{S\subseteq H}\rho^{|S|}$. Since $\beta(H_2,\rho)=\sum_{S\subseteq H_2}\rho^{|S|}$, the per-link throughput is

$$r=\rho\frac{\beta(H_2,\rho)}{\beta(H,\rho)}.$$

Since there are 24 directed links in H, the network throughput is $= 24\cdot r$. Similarly, by using Equation (3.10), we also have

$$r=\frac{\rho}{24}\frac{\frac{\partial}{\partial\rho}\beta(H,\rho)}{\beta(H,\rho)}.$$

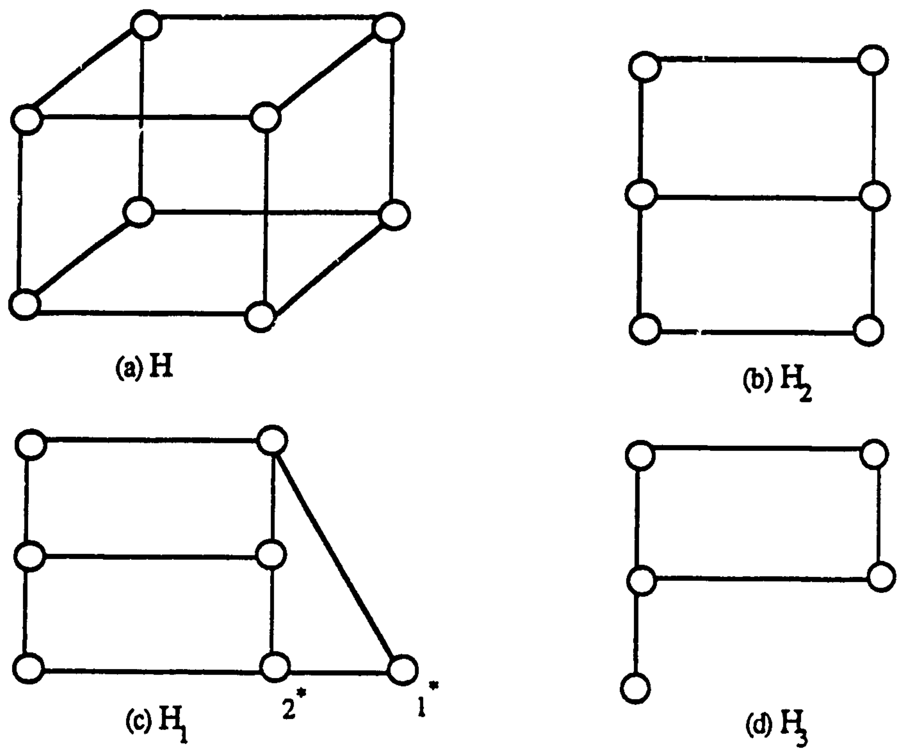


Figure 3.7: Decomposition of the hypercube H

Therefore, we can write

$$\beta(H_2,\rho) = \frac{\rho}{24}\frac{\partial}{\partial\rho}\beta(H,\rho).$$

According to Equations (3.21) and (3.22), we obtain

$$\begin{aligned}
\beta(H_2,\rho) &= \beta(CH_5,\rho) + 2\rho\,(2\beta(CH_4,\rho) + \beta(CH_2 \cup CH_2,\rho));\\
\beta(CH_5,\rho) &= 1 + 8\rho + 12\rho^2;\\
\beta(CH_4,\rho) &= 1 + 6\rho + 4\rho^2;\\
\beta(CH_2 \cup CH_2,\rho) &= \beta(CH_2,\rho)\beta(CH_2,\rho) = (1+2\rho)(1+2\rho) = 1 + 4\rho + 4\rho^2;\\
\beta(H_2,\rho) &= 1 + 14\rho + 44\rho^2 + 24\rho^3,
\end{aligned}$$

where CH_n is the chain network with n nodes. Let H_1 be obtained from H by deleting one node, then

$$\begin{aligned}
\beta(H,\rho) &= \beta(H_1,\rho) + 6\rho\beta(H_2,\rho);\\
\beta(H_1,\rho) &= \beta(H_1 - 1^*,\rho) + 4\rho\beta(H_1 - 1^* - 2^*,\rho)
\end{aligned}$$

$$= \beta(H_2,\rho)+4\rho\beta(H_3,\rho)$$
$$= \beta(H_2,\rho)+4\rho[\beta(L_4,\rho)+2\rho\beta(CH_3,\rho)],$$

where L_n is the Loop with n nodes. Since $\beta(L_4,\rho)=1+8\rho+8\rho^2$ and $\beta(CH_3,\rho)=1+4\rho$, then

$$\beta(H_1,\rho) = 1+18\rho+84\rho^2+88\rho^3;$$
$$\beta(H,\rho) = 1+24\rho+168\rho^2+352\rho^3+144\rho^4.$$

Therefore,

$$r=\rho\frac{\beta(H_2,\rho)}{\beta(H,\rho)}=\frac{\rho}{24}\frac{\frac{\partial}{\partial\rho}\beta(H,\rho)}{\beta(H,\rho)}=\frac{\rho+14\rho^2+44\rho^3+24\rho^4}{1+24\rho+168\rho^2+352\rho^3+144\rho^4},$$

and

$$Pb=\frac{10\rho+124\rho^2+328\rho^3+144\rho^4}{1+24\rho+168\rho^2+352\rho^3+144\rho^4}.$$

So, $r_{\max}=\lim_{\rho\to\infty} r=\frac{1}{6}$. The performance measures are demonstrated in Figure 3.8.

3.5.3 The partition functions of some regular graphs

Regular graphs or graphs with symmetric topologies lead to simplifications in the sum-of-products expression for $\beta(G,\rho)$ and hence to recursions or even closed forms. In this section, we generate recursive expressions for partition functions of networks with regular topologies. Specifically, we consider the partition function for the path network P_n, the cycle network C_n, the completely-connected network K_n, the crossbar switching network $K_{n,m}$, and the recirculating shift network RS_n, respectively.

The results are directly presented here. Their detailed developments are given in Appendix E. We assume unidirectional transmissions in the networks P_n, C_n, $K_{n,m}$ and RS_n and bidirectional transmissions in the network K_n.

(a) **Cycle network C_n and path network P_n.**

A cycle network is a directed network. For every node in the network, there are two directed links associated with it: one originates from it and the other terminates

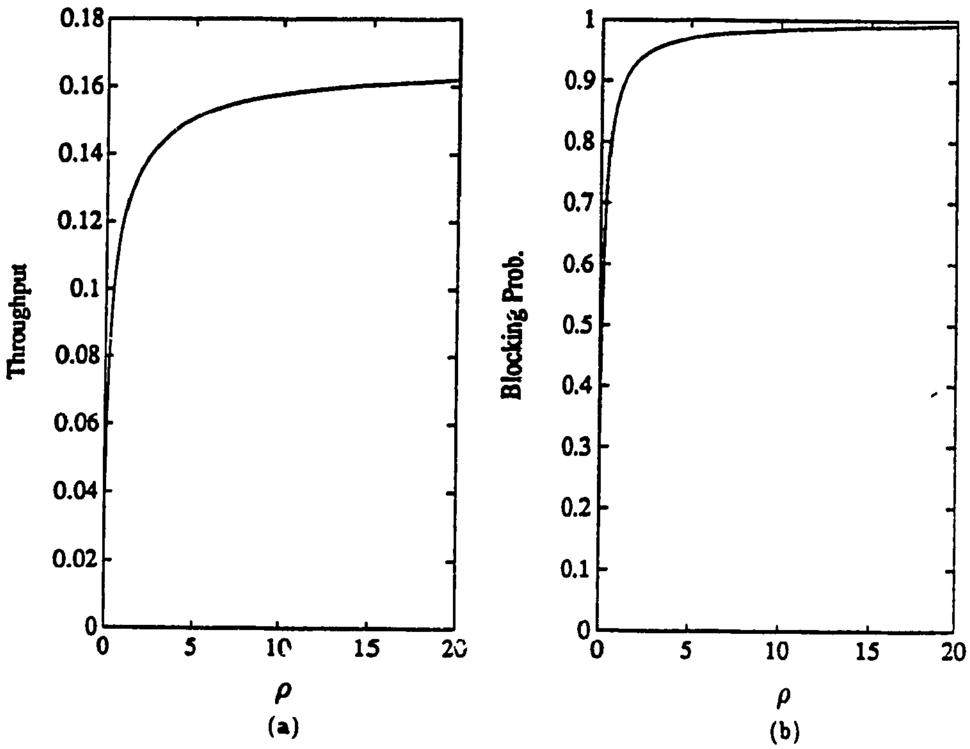


Figure 3.8: Performance measures of the hypercube network: (a) Throughput, (b) Blocking probability

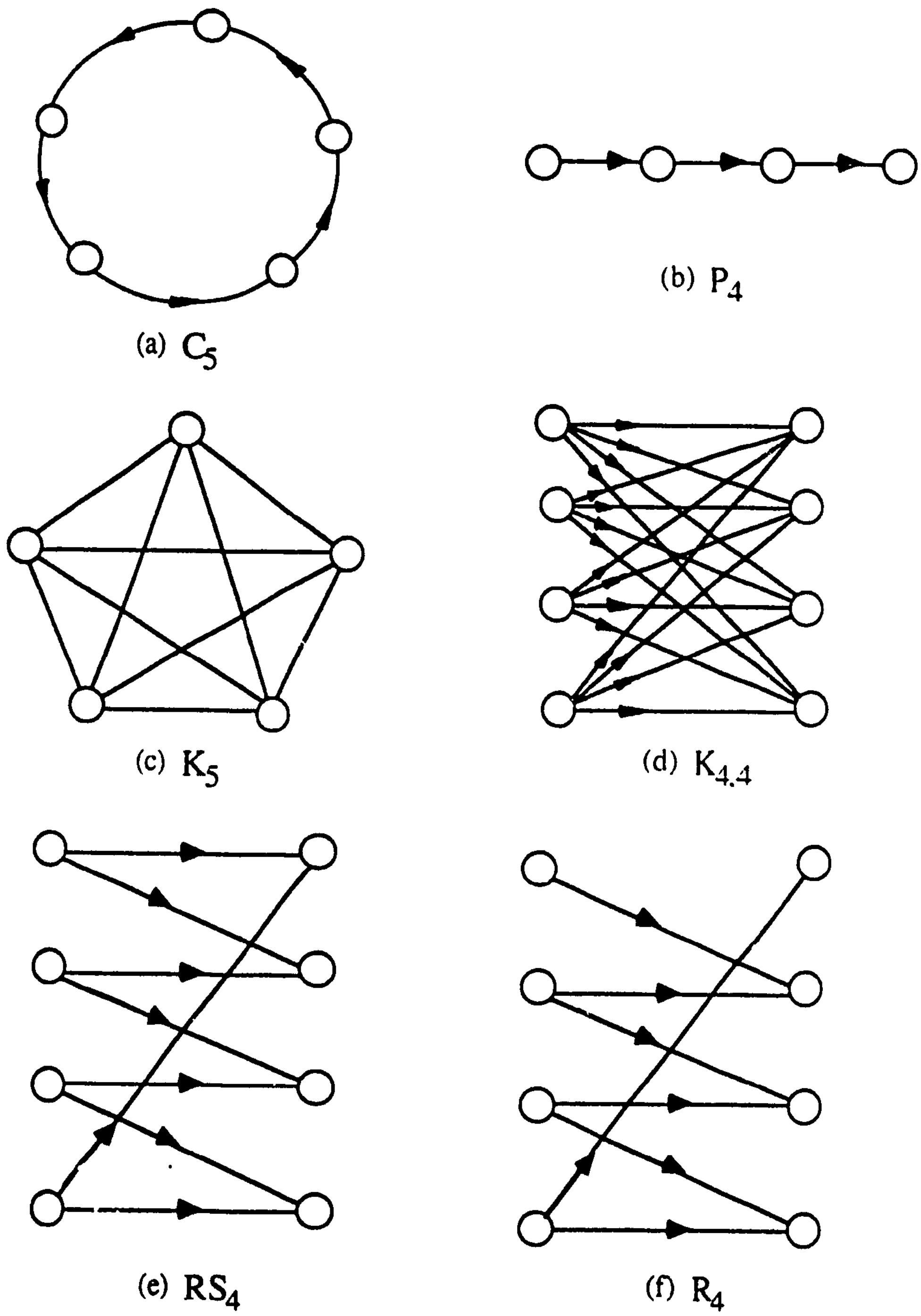


Figure 3.9: Some typical regular graphs

at destinated to it. Figure 3.9(a) shows the 5-node cycle C_5. The path network results from the cycle network when one directed link is removed from it. Figure 3.9(b) is the 4-node path P_4.

The partition functions of a cycle network C_n and a path network P_n satisfy the following relations:

$$\beta(C_n,\rho) = \beta(P_n,\rho)+\rho\beta(P_{n-2},\rho), \tag{3.23}$$

$$\beta(C_n,\rho) = \beta(P_{n-1},\rho)+2\rho\beta(P_{n-2},\rho), \tag{3.24}$$

$$\beta(P_n,\rho) = \beta(P_{n-1},\rho)+\rho\beta(P_{n-2},\rho), \tag{3.25}$$

$$\beta(C_n,\rho) = 2\beta(P_n,\rho)-\beta(P_{n-1},\rho), \tag{3.26}$$

$$\beta(P_n,\rho) = \frac{1}{1+4\rho}\left(\beta(C_{n+1},\rho)+2\rho\beta(C_n,\rho)\right), \tag{3.27}$$

$$\beta(C_n,\rho) = \beta(C_{n-1},\rho)+\rho\beta(C_{n-2},\rho), \tag{3.28}$$

with

$$\beta(P_1,\rho)=1; \qquad \beta(P_2,\rho)=1+\rho,$$
$$\beta(C_1,\rho)=1; \qquad \beta(C_2,\rho)=1+2\rho.$$

(b) **Complete Graph K_n.**

The complete network K_n is formed by connecting each possible pair of the nodes in the given node set N of the graph $G=(N,L)$. Figure 3.9(c) shows K_5.

By applying Equation (3.21), we have

$$\beta(K_n,\rho) = \beta(K_{n-1},\rho)+2(n-1)\rho\beta(K_{n-2},\rho); \tag{3.29}$$

$$\beta(K_n,\rho) = [1+2(n-1)\rho]\beta(K_{n-2},\rho)+\sum_{k=1}^{(n-4)/2}\prod_{j=1}^{k}(n-2j)(2\rho)^k\beta(K_{n-2-2k},\rho);$$

$$\beta(K_n,\rho) = [1+2(2n-3)\rho]\beta(K_{n-2},\rho)-4(n-2)(n-3)\rho^2\beta(K_{n-4},\rho); \tag{3.30}$$

$$\beta(K_n,\rho) = \beta(K_{n-1},\rho)+\sum_{k=1}^{(n-2)/2}\prod_{j=1}^{k}(n-2j+1)(2\rho)^k\beta(K_{n-1-2k},\rho);$$

$$\beta(K_n,\rho) = [1+2(n-1)\rho]\beta(K_{n-1},\rho)-4(n-1)(n-2)\rho^2\beta(K_{n-3},\rho), \tag{3.31}$$

with $\beta(K_1,\rho)=1, \quad \beta(K_2,\rho)=1+2\rho$.

(c) **Bipartite Graph $K_{n,m}$**

In this type of graph, the node set is divided into two parts: n input nodes and m output nodes. The graph is formed by connecting each input node to each output node. Figure 3.9(d) is the graph $K_{4,4}$.

By applying Equation (3.18) to the $K_{n,m}$ graph, we have

$$\beta(K_{n,m},\rho) = \sum_{k=0}^{m} \binom{m}{k} \frac{n!}{(n-k)!}\rho^k;$$

$$\beta(K_{n,m},\rho) = \beta(K_{n,m-1},\rho) + n\rho\beta(K_{n-1,m-1},\rho); \tag{3.32}$$

$$\beta(k_{n,m},\rho) = \frac{n!}{(n-m)!}\rho^m + \sum_{k=0}^{m-1} \frac{n!}{(n-k)!}\rho^k \beta(K_{n-k,m-k-1},\rho);$$

$$\beta(K_{n,m},\rho) = (1+n\rho)\beta(K_{n-1,m-1},\rho) + \sum_{k=2}^{m} \frac{(m-1)!}{(m-k)!}\rho^{(k-1)}\beta(K_{n-k,m-k},\rho);$$

$$\begin{aligned}\beta(K_{n,m},\rho) &= [1+(m+n-1)\rho]\beta(K_{n-1,m-1},\rho) \\ &\quad -(n-1)(m-1)\rho^2\beta(K_{n-2,m-2},\rho);\end{aligned} \tag{3.33}$$

$$\beta(K_{n,m},\rho) = (1+n\rho)\beta(K_{n,m-1},\rho) - n(m-1)\rho^2\beta(K_{n-1,m-2},\rho), \tag{3.34}$$

with $\beta(K_{n,0},\rho)=1$, $\beta(K_{n,1},\rho)=1+n\rho$.

(d) **Recirculating shift network RS_n**

The structure of a recirculating shift network RS_n is like the bipartite graph $K_{n,n}$, however, $K_{n,n}$ has n links connecting each input port to all output ports while each input port of RS_n is connected to only two output ports. If we remove one edge from RS_n, we obtain the network R_n which is required in deriving the recursive expression for computing the RS_n network throughput. Figure 3.9(e) and Figure 3.9(f) show the network RS_4 and the network R_4, respectively.

By applying Equations (3.18) and (3.19), we have

$$\beta(RS_n,\rho) = (1+2\rho)\beta(RS_{n-1},\rho) - \rho^2\beta(RS_{n-2},\rho); \quad (3.35)$$

$$\beta(R_n,\rho) = (1+2\rho)\beta(R_{n-1},\rho) - \rho^2\beta(R_{n-2},\rho), \quad (3.36)$$

for $n > 3$, given

$$\begin{aligned} \beta(RS_1,\rho) &= 1+\rho; & \beta(R_1,\rho) &= 1+\rho; \\ \beta(RS_2,\rho) &= 1+4\rho+2\rho^2; & \beta(R_2,\rho) &= 1+3\rho+\rho^2; \\ \beta(RS_3,\rho) &= 1+6\rho+9\rho^2+2\rho^3, & \beta(R_3,\rho) &= 1+5\rho+6\rho^2+\rho^3. \end{aligned}$$

In addition, the following expressions can be generated:

$$\beta(RS_n,\rho) = (1+3\rho)\beta(RS_{n-1},\rho) + \sum_{i=1}^{n-2}(-1)^i\rho^i(1+4\rho)\beta(RS_{n-1-i},\rho);$$

$$\beta(RS_n,\rho) = \beta(R_n,\rho) + \rho\beta(R_{n-1},\rho); \quad (3.37)$$

$$\beta(R_n,\rho) = \frac{1+3\rho}{1+4\rho}\beta(RS_n,\rho) - \frac{\rho^2}{1+4\rho}\beta(RS_{n-1},\rho);$$

$$\beta(R_n,\rho) = \frac{1}{1+4\rho}\beta(RS_{n+1},\rho) + \frac{\rho}{1+4\rho}\beta(RS_n,\rho). \quad (3.38)$$

Chapter 4

Applications of the Blocking Model in Packet Radio Networks

In Chapters 2 and 3, we described and extended the blocking model for communication network systems. This model defines a pair of relations in which information concerning blocking and interference are represented. We can carry out performance studies for many network systems with the computational techniques developed for this model. In this chapter, we will apply the model to study the performance of packet radio networks (PRNs).

4.1 Some Basic Considerations

In applying the blocking model to a communication network with topology $G = (N, L)$, it is preferable to refer to directed links $l_i \in L$ rather than nodes $j \in N$ as the basic elements of the system since this allows us to solve some problems that would otherwise be precluded. For example:

(1) For communication networks with directional-transmission characteristics, such as a network using D-CSMA, transmission operations can only be described by activations of directed links in the network graph.

(2) In employing this blocking model to a multihop broadcast network with virtual-circuits, where many channels may co-exist in each data link, we assign a weight $w(l_i)$ to each directed link l_i of G to designate the number of channels within that link. It is usually true that the weights of directed links originating at the same

node are different. In this case, performance can be measured much easily when we consider directed links as basic elements.

With transmission interactions among the directed links l_i in a PRN governed by the TBG and the RIG, we measure the performance of networks for each of the following channel access protocols: CSMA, BTMA, D-CSMA, CFMA and TCAP. In applying this blocking model to a PRN with arbitrary topology, the computational techniques explained in Chapter 3 are directly applied. However, efficient implementation of these techniques requires appropriate data structures and programming procedures. Considering this, we integrate the procedures for computing performance measures with some additional algorithms, so that performance studies can be systematically carried out for any system which can be described with the blocking model.

4.2 Additional Algorithms Required for Performance Measures

Basically, we include the following two preprocessing algorthms to calculate performance measures for a variety of PRNs: (1) Generation of the TBG and the RIG, and (2) calculation of the link weights. These are explained in the following subsections.

4.2.1 Generation of the TBG and the RIG

As was presented in Subsection 3.3.1, there are, four classes of interactions that e_{ij} can have with e_{uv}, namely,

1. C_1 (e_{ij} is effected by e_{uv} through the sources i and u):
   - (L_1) The sources of e_{ij} and e_{uv} are the same ($i = u$),
   - (L_2) u is a neighbor of i ($h_{ui} = 1$).
2. C_2 (e_{ij} is effected by e_{uv} by coupling between the source i and the destination v):

- (L_1) The source of e_{ij} is the destination of e_{uv} ($i = v$),
- (L_2) v is a neighbor of i ($h_{vi} = 1$).

3. C_3 (e_{ij} is effected by e_{uv} by coupling between the destination j and the source u):
   - (L_1) The destination of e_{ij} is the source of e_{uv} ($j = u$),
   - (L_2) u is a neighbor of j ($h_{uj} = 1$).
4. C_4 (e_{ij} is effected by e_{uv} through the destinations j and v):
   - (L_1) The destinations of e_{ij} and e_{uv} are the same ($v = j$),
   - (L_2) j is a neighbor of j ($h_{vj} = 1$).

Therefore, by identifying all these interactions between e_{ij} and e_{uv} for each protocol, we can construct the corresponding TBG and RIG. Note that the numbers C_i and L_j, $i = 1, 2, 3, 4$, $j = 1, 2$, in the equations represent, respectively, the class and the level of the transmission interaction in the model.

- For CSMA:

$$B(e_{ij}) = \left\{ e_{uv} : \begin{array}{c} 1(C_1(L_1)) : i = u \\ 2(C_2(L_1)) : i = v \\ 3(C_3(L_1)) : j = u \\ 5.(C_1(L_2)) : h_{ui} = 1 \end{array} \right\};$$

$$I(e_{ij}) = \left\{ e_{uv} : \begin{array}{c} 4(C_4(L_1)) : j = v \\ 7(C_3(L_2)) : h_{uj} = 1 \end{array} \right\}.$$

- For D-CSMA:

$$B(e_{ij}) = \left\{ e_{uv} : \begin{array}{c} 1(C_1(L_1)) : i = u \\ 2(C_2(L_1)) : i = v \\ 3(C_3(L_1)) : j = u \end{array} \right\};$$

$$I(e_{ij}) = \{ e_{uv} : 4(C_4(L_1)) : j = v \}.$$

- For BTMA:

$$
\begin{aligned}
B(e_{ij}) &= \left\{ e_{uv} : \begin{array}{r} 1(C_1(L_1)) : i = u \\ 2(C_2(L_1)) : i = v \\ 3(C_3(L_1)) : j = u \\ 4(C_4(L_1)) : j = v \\ 5(C_1(L_2)) : h_{ui} = 1 \\ 6(C_2(L_2)) : h_{vi} = 1 \\ 7(C_3(L_2)) : h_{uj} = 1 \end{array} \right\}; \\
I(e_{ij}) &= \emptyset.
\end{aligned}
$$

- For CFMA:

$$
\begin{aligned}
B(e_{ij}) &= \left\{ e_{uv} : \begin{array}{r} 1(C_1(L_1)) : i = u \\ 2(C_2(L_1)) : i = v \\ 3(C_3(L_1)) : j = u \\ 4(C_4(L_1)) : j = v \\ 6(C_2(L_2)) : h_{vi} = 1 \\ 7(C_3(L_2)) : h_{uj} = 1 \end{array} \right\}; \\
I(e_{ij}) &= \emptyset.
\end{aligned}
$$

- For TCAP:

$$
\begin{aligned}
B(e_{ij}) &= \left\{ e_{uv} : \begin{array}{r} 1(C_1(L_1)) : i = u \\ 2(C_2(L_1)) : i = v \\ 3(C_3(L_1)) : j = u \\ 4(C_4(L_1)) : j = v \end{array} \right\}; \\
I(e_{ij}) &= \emptyset.
\end{aligned}
$$

Therefore, the blocking set $B(e_{ij})$ and interference set $I(e_{ij})$ for every link e_{ij} in the system can be systematically generated. The state space of the system can be generated by using Equation (3.2). By applying the computational-technique in Subsection 3.2.2, the throughput over each directed link l_i can then be calculated. The blocking probability $P_B(i)$ and the interference probability $P_I(i)$ are obtained by direct applications of Equation (2.11) and Equation (2.12), respectively.

4.2.2 Generating a routing for the network

In a multihop communication network operating virtual-circuits, routing has to be established for transmissions from each station to the other stations. We consider static routing, where the transmissions between pairs of stations follow pre-established paths. Further, we assume each virtual circuit is routed along a single path (a shortest path) connecting the end nodes. Thus, each directed link l_i in the network may carry traffic for more than one virtual circuit. The number of channels $w(l_i)$ using link l_i is called the *weight* of directed link l_i and it depends on how traffic is routed through the network.

To explain this concept further, we once again refer to the 4-node chain PRN of Figure 2.1. Let σ_{ij}, $i \neq j$, be the required successful-transmission rate from node i to node j. The *traffic matrix* $R = [\sigma_{ij}]$ gives the required throughput for each possible channel in the network. For the four node network,

$$R = \begin{bmatrix} 0 & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} \\ \sigma_{2,1} & 0 & \sigma_{2,3} & \sigma_{2,4} \\ \sigma_{3,1} & \sigma_{3,2} & 0 & \sigma_{3,4} \\ \sigma_{4,1} & \sigma_{4,2} & \sigma_{4,3} & 0 \end{bmatrix}.$$

To simplify, we assume uniform end-to-end traffic requirements, where the required successful-transmission rates between all pairs of stations are equal, $\sigma_{ij} = \sigma$ for all $i, j \in N$. The R matrix has the form

$$R = \begin{bmatrix} 0 & \sigma & \sigma & \sigma \\ \sigma & 0 & \sigma & \sigma \\ \sigma & \sigma & 0 & \sigma \\ \sigma & \sigma & \sigma & 0 \end{bmatrix}.$$

In order to achieve an equal successful-transmission rate σ over each channel, the successful transmission rate over each link l_i will have to be $r_i = w(l_i)\sigma$. Note: Links which carry no traffic have $w(l_i) = 0$, and, hence $r_i = \rho_i = 0$ for all such links. They are not considered as elements of the model but they must be used to determine possible interactions between links which carry nonzero traffic.

In order to calculate the link weights, we need to find transmission paths between every pair of nodes in the network G. Floyd's algorithm [ST81] provides a solution to this problem. We briefly explain it here.

Let d_{ij} denote the length of directed link e_{ij} in graph G, and $D = [d_{ij}]$ denote the $m \times m$ matrix of directed lengths in G. As a simple approximation, we will use the hop count as a measure of path distance. Thus

$$d_{ij} = \begin{cases} 1 & j \in NBR(i); \\ 0 & j = i; \\ \infty & j \neq i \text{ and } j \notin NBR(i), \end{cases}$$

where $NBR(i)$ is the set of neighbors of i.

Starting with the matrix $D^{(o)} = D$, Floyd's algorithm constructs a sequence $D^{(1)}, D^{(2)}, \cdots, D^{(m)}$ of $m \times m$ matrices so that the entry $d_{ij}^{(m)}$ in $D^{(m)}$ gives the distance from node i to node j in G. The matrix $D^{(k)} = [d_{ij}^{(k)}]$ is constructed from the matrix $D^{(k-1)} = [d_{ij}^{(k-1)}]$ according to the following rule:

$$d_{ij}^{(k)} = min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}.$$

If we let $p_{ij}^{(k)}$ denote a path of minimum length among all the directed i to j paths which pass through nodes from the set $\{1, 2, \cdots, k\}$ only, then

for $0 \leq k \leq n$, $d_{ij}^{(k)}$ is equal to the length of $p_{ij}^{(k)}$.

The weight of a directed link is generated by counting the number of paths using the link. Not only do we need to know the shortest distance between every pair of nodes in the network, we also require a path which has this length so that the weight of each directed link in the network can be determined. As a result, in addition to constructing the sequence $D^{(0)}, D^{(1)}, \cdots, D^{(m)}$, we construct another sequence $Z^{(0)}, Z^{(1)}, \cdots, Z^{(m)}$ of matrices such that the entry $z_{ij}^{(k)}$ of $Z^{(k)}$ gives the node immediately following node i in $p_{ij}^{(k)}$. Starting with $Z^{(0)}$, where

$$z_{ij}^{(0)} = \begin{cases} j & \text{if } d_{ij} \neq \infty \text{ ;} \\ 0 & \text{if } d_{ij} = \infty \text{ ,} \end{cases}$$

$Z^{(k)} = [z_{ij}^{(k)}]$ can be obtained from $Z^{(k-1)} = [z_{ij}^{(k-1)}]$ according to the following rule: Let

$$M = min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}, \tag{4.1}$$

then

$$z_{ij}^{(k)} = \begin{cases} z_{ij}^{(k-1)} & \text{if } M = d_{ij}^{(k-1)}; \\ z_{ik}^{(k-1)} & \text{if } M < d_{ij}^{(k-1)}. \end{cases} \tag{4.2}$$

Therefore, if $M = d_{ij}^{(k-1)}$, then

$$\text{length of } p_{ij}^{(k)} = \text{length of } p_{ij}^{(k-1)},$$

and

$$z_{ij}^{(k)} = z_{ij}^{(k-1)}.$$

On the other hand, if $M < d_{ij}^{(k-1)}$, then

$$p_{ij}^{(k)} = p_{ik}^{(k-1)} \text{ concatenated with } p_{kj}^{(k-1)},$$

and

$$z_{ij}^{(k)} = z_{ik}^{(k-1)}.$$

The shortest i to j path is given by the sequence $i, i_1, i_2, \cdots, i_p, j$ of nodes: $i_1 = z_{ij}^{(m)}$, $i_2 = z_{i_1 j}^{(m)}$, $i_3 = z_{i_2 j}^{(m)}$, $\cdots$, and, $j = z_{i_p j}^{(m)}$.

Example 4.1 Given the 4-node chain PRN of Figure 2.1, Floyd's algorithm generates the z_{ij} entries given in Table 4.1. By following the entries in the matrix from any row i, for any column j, we construct the shortest path selected by the algorithm from node i to node j. For instance, messages from node 1 to node 4 follow the path $(1,2), (2,3), (3,4)$. The order of node activations is $(1,2,3)$. Table 4.2 gives the order of directed-link activations for each source-destination pair, as well as the weight $w(l_i)$ of each link l_i in the network.

The throughput that a link l_i must support for uniform end-to-end traffic is simply $r_i = w(l_i)\sigma$. Note that the corresponding network under uniform link traffic can be considered as the special case with $w(l_i) = 1$ for all $l_i \in L$. We consider both of these cases in our analysis.

4.3 Performance Analysis

In this section, we analyze the performance of different PRN systems by employing the iterative-estimation algorithm presented in Chapter 3. The analysis will be

Table 4.1: Nodal activations in constructing the shortest paths for the 4-node chain

	1	2	3	4
1	-	2	2	2
2	1	-	3	3
3	2	2	-	4
4	3	3	3	-

Table 4.2: Generating the routing for the 4-node chain network

Transmission from node i to node j	Activation sequence of links l_i in the network										
$1 \rightarrow 2$	l_1										
$1 \rightarrow 3$	l_1	$\rightarrow$			l_3						
$1 \rightarrow 4$	l_1	$\rightarrow$			l_3	$\rightarrow$			l_5		
$2 \rightarrow 1$			l_2								
$2 \rightarrow 3$					l_3						
$2 \rightarrow 4$					l_3	$\rightarrow$			l_5		
$3 \rightarrow 1$			l_2	$\leftarrow$			l_4				
$3 \rightarrow 2$							l_4				
$3 \rightarrow 4$									l_5		
$4 \rightarrow 1$			l_2	$\leftarrow$			l_4	$\leftarrow$			l_6
$4 \rightarrow 2$							l_4	$\leftarrow$			l_6
$4 \rightarrow 3$											l_6
	$w(l_1)=3$		$w(l_2)=3$		$w(l_3)=4$		$w(l_4)=4$		$w(l_5)=3$		$w(l_6)=3$

carried out by focussing on two directed links in the communication network under investigation. It should be noted that performance measurement information for every directed link in the network is simultaneously generated. Therefore, we can study the system performance in many different directions or levels of detail by appropriately referring to the generated information.

4.3.1 Comparing the system performance for different channel access protocols

In this section, we study the performance of different channel access protocols. Typically, we consider the 4-node chain PRN which was shown in Figure 2.1, operating under: (1) CSMA; (2) D-CSMA; (3) BTMA; (4) CFMA; and (5) TCAP. In this section, links l_1 and l_3 are chosen to examine the performance of the network and the performance over link l_3 will be plotted in the figures. Further, we study the performance for this network with virtual-circuits carrying uniform end-to-end traffic and under uniform link traffic.

The channel throughputs for this 4-node chain PRN operating under different channel access protocols are shown in Figure 4.1. For the network with virtual-circuit operation, the successful transmission rate over link l_i is obtained by scaling its channel throughput value with the associated weight $w(l_i)$. Figure 4.2 shows the throughputs over l_1 and l_3.

Notice Figure 4.1(a) represents only the increasing portion of curves in Figure 3.7(b). The decreasing region of the curves has been truncated. This is a limitation of the iterative-estimation method used for throughput calculation, since convergence to a fixed point is guaranteed only in the stable operating region.

Table 4.3 presents the maximum throughputs of this 4-node chain with other associated measures.

The relative throughput measurements of the network with CSMA, D-CSMA and BTMA, with the respect to that of CFMA and TCAP are demonstrated in Figures 4.3(a), 4.4(a), and 4.5(a) respectively. These figures show comparison results for the 4-node chain network. The RCCMOT and the RCCMDT for data link l_3 of

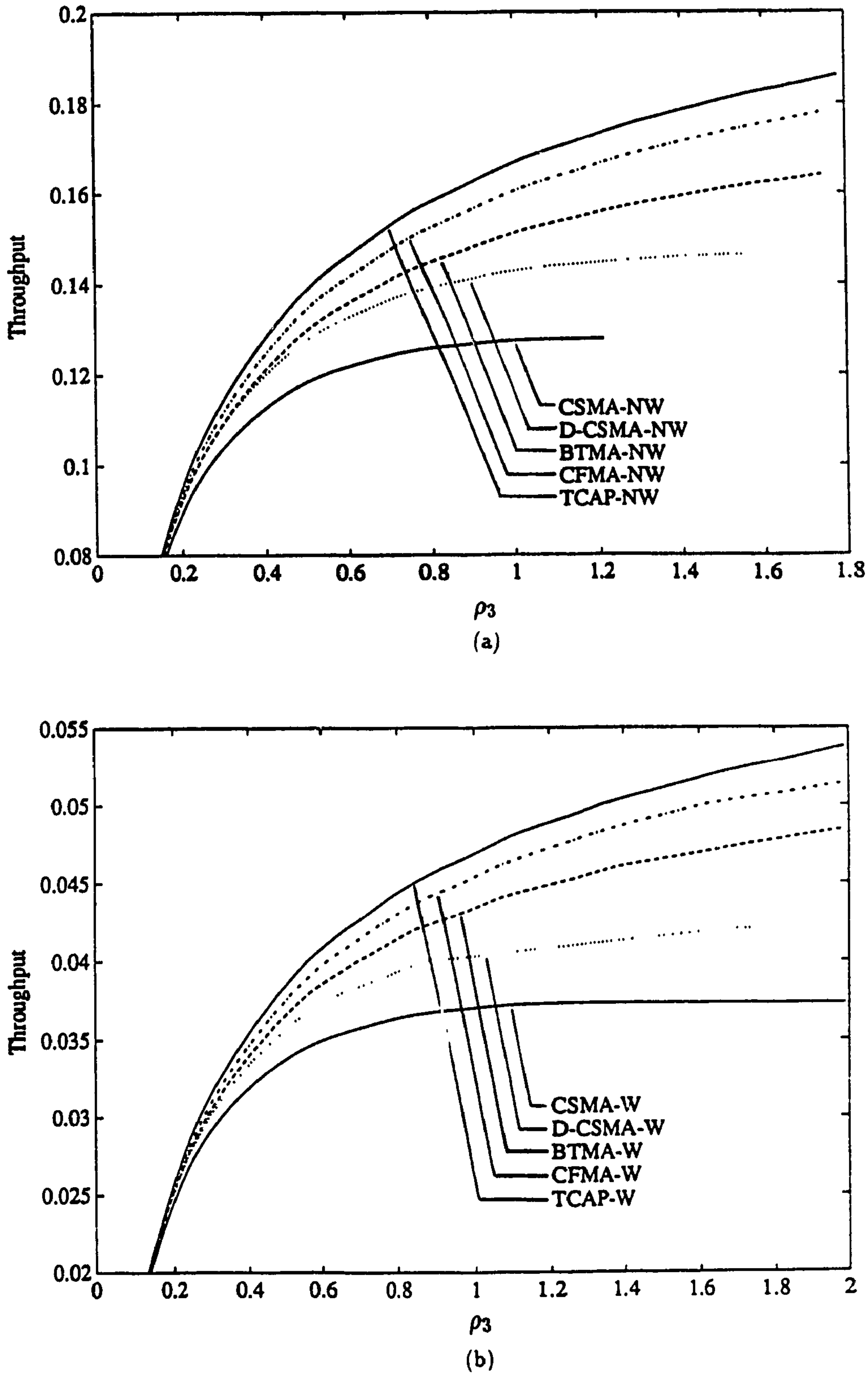


Figure 4.1: Throughput of the systems with different channel access protocols (a) uniform link traffic, (b) uniform end-to-end traffic.

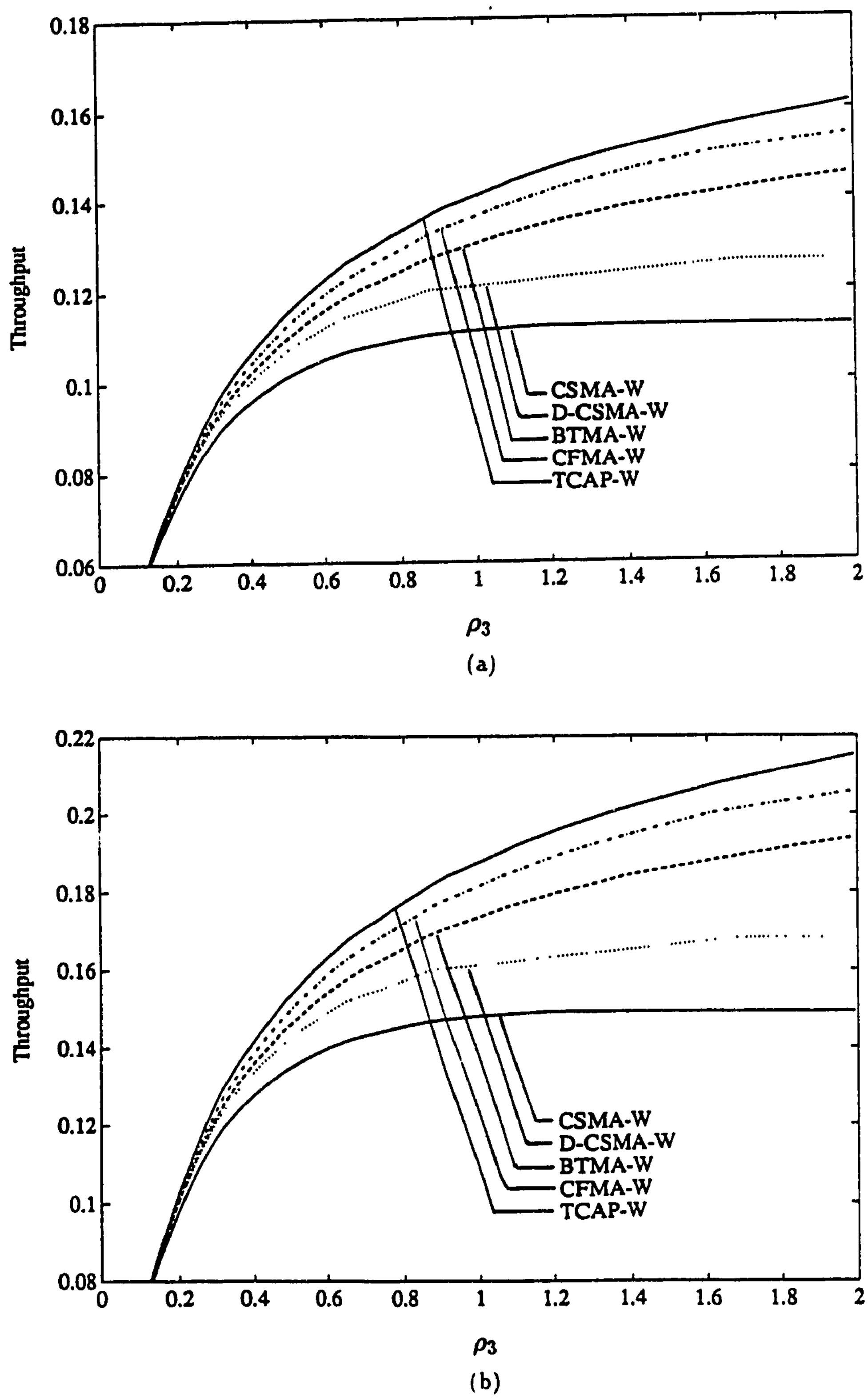


Figure 4.2: Throughputs over data links (a) l_1 and (b) l_3 of the virtual-circuit communication network.

this network are shown in Table 4.4.

The channel throughput comparsions for both traffic cases under CSMA, D-CSMA and BTMA can also be observed from Figures 4.3(a), 4.4(a) and 4.5(a). Blocking probabilities are given in Figures 4.3(b), 4.4(b) and 4.5(b). Interference probabilities are shown in Figure 4.6. Of course, only the curves corresponding to the interference-susceptible protocols CSMA and D-CSMA have non-zero interference probability values.

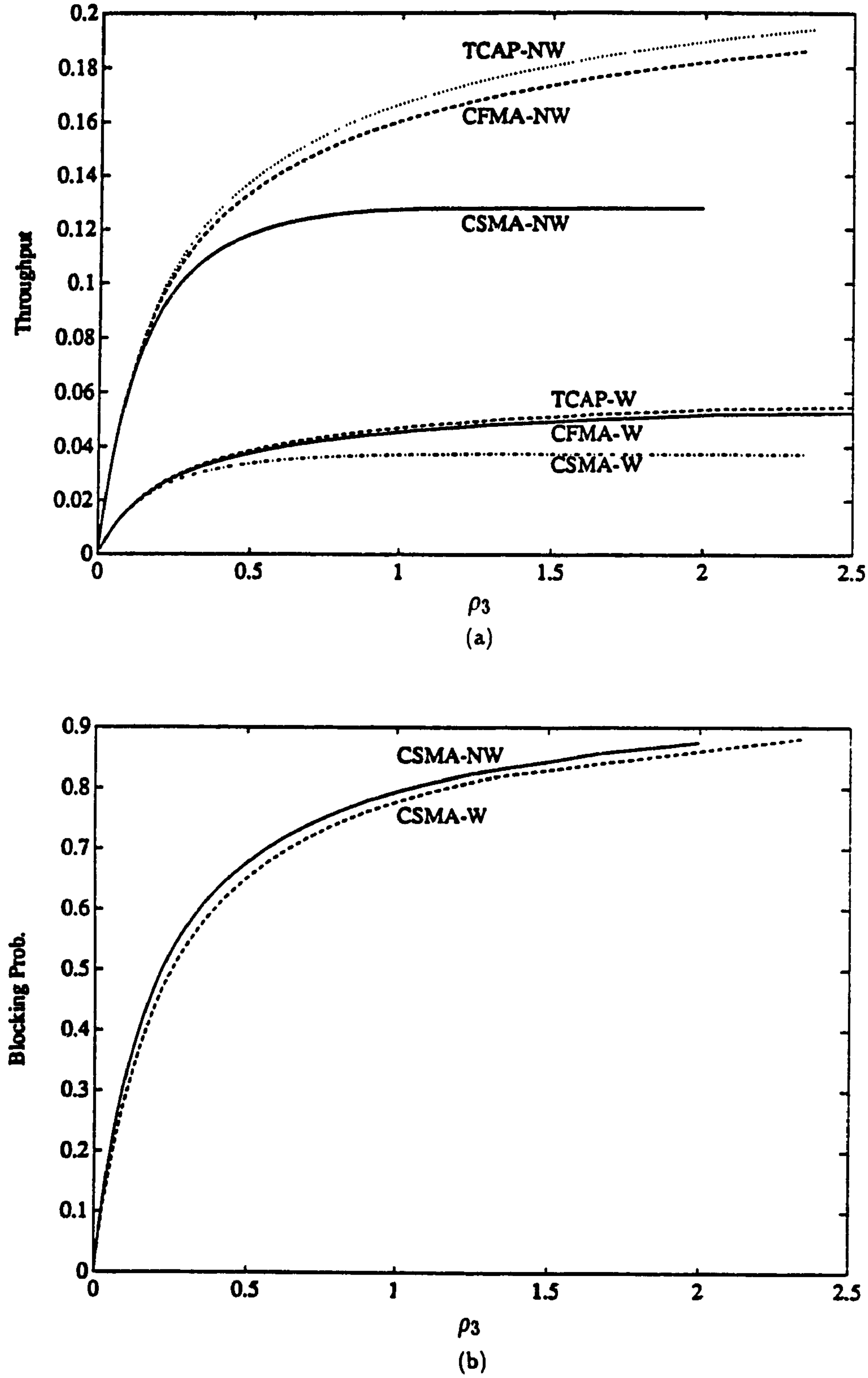
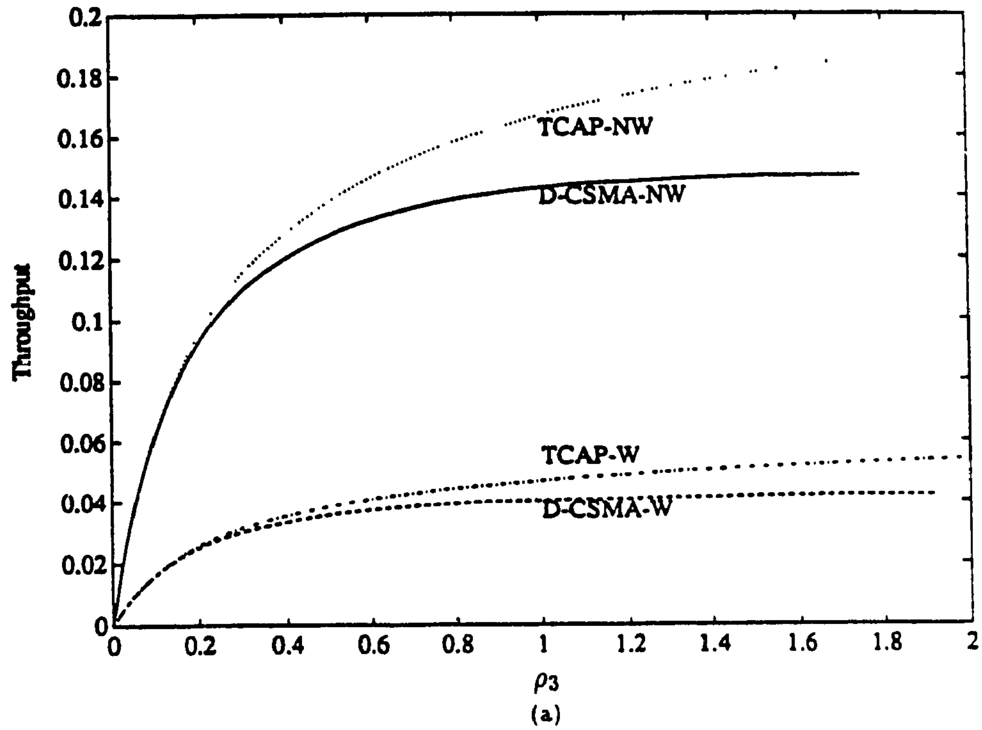


Figure 4.3: Performance measures for the 4-node chain operating under CSMA: (a) Throughput, (b) Blocking probability



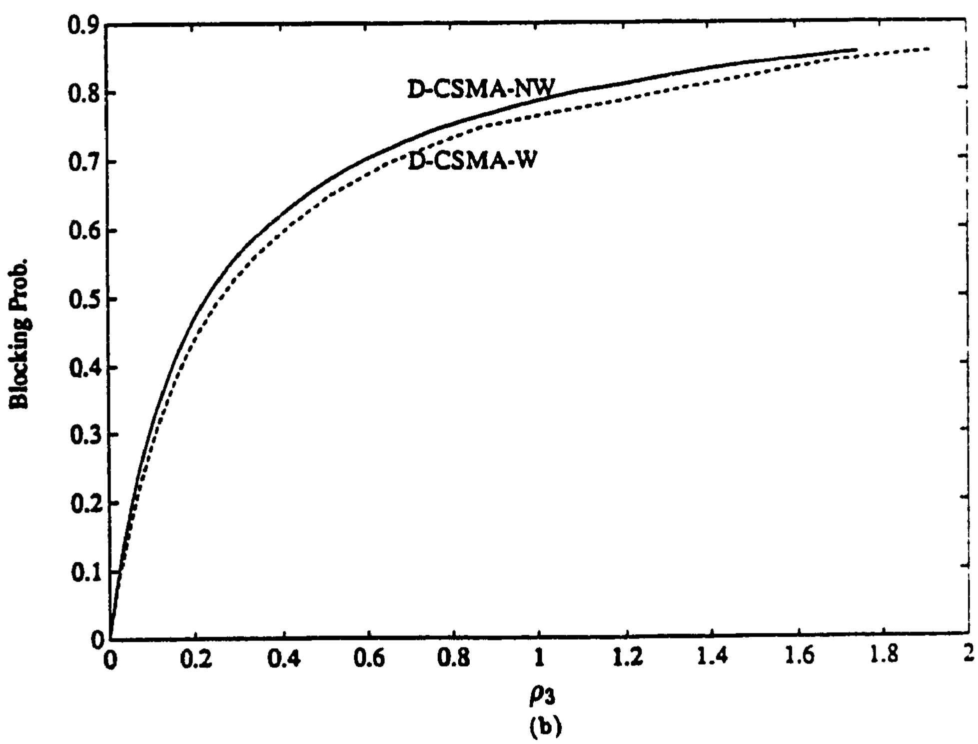


Figure 4.4: Performance measures for the 4-node chain operating under D-CSMA: (a) Throughput, (b) Blocking probability

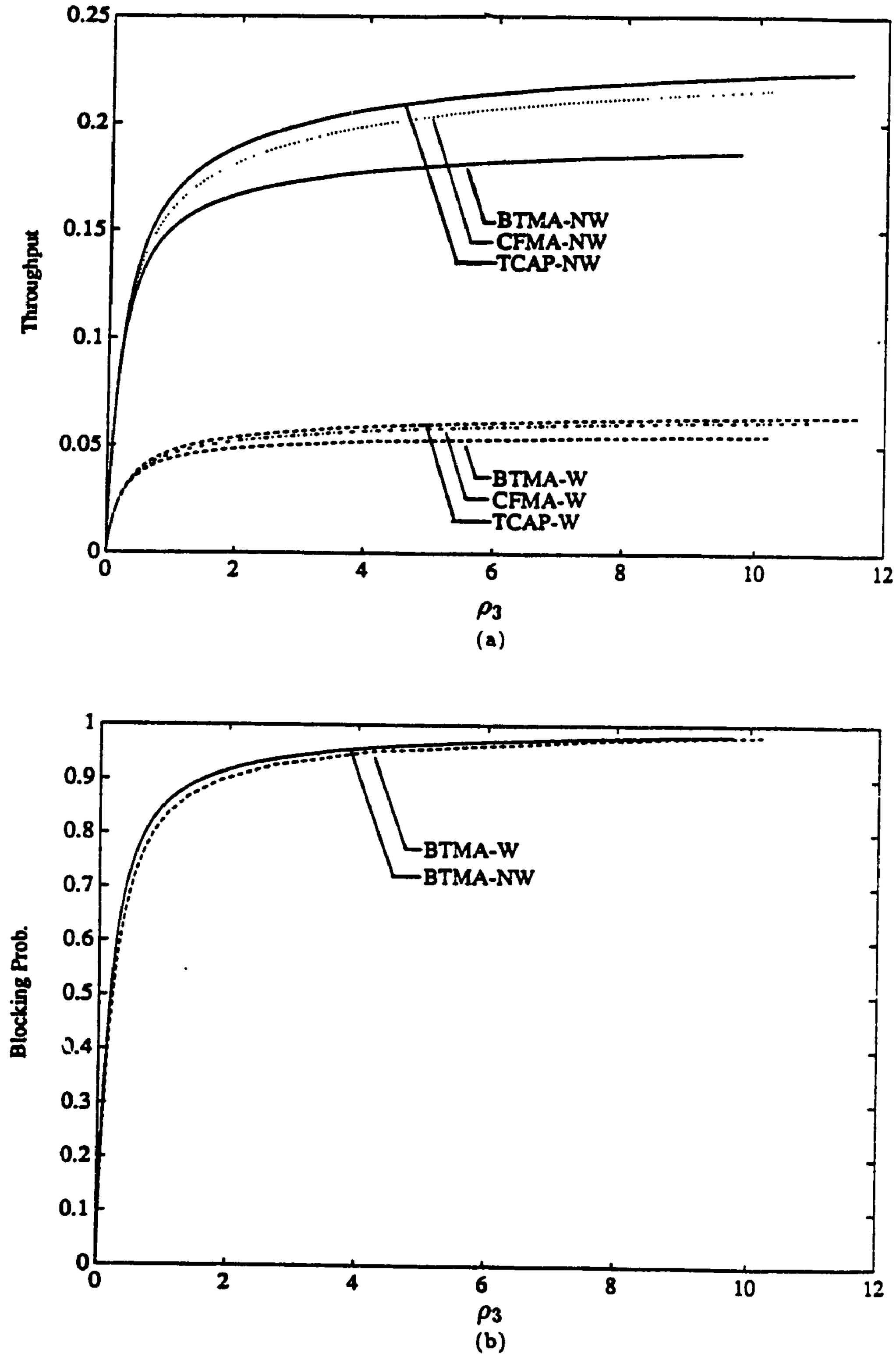


Figure 4.5: Performance measures for the 4-node chain operating under BTMA: (a) Throughput, (b) Blocking probability

Table 4.3: Maximum throughput of the 4-node chain PRN

Protocol	Wht	r_{max}	ρ_1	ρ_3	$P_B(1)$	$P_B(3)$	$P_I(1)$	$P_I(3)$
CSMA	NW	0.1277	0.7081	1.2094	0.6171	0.8196	0.2025	0.0748
	W	0.0373	0.922	2.337	0.6806	0.8815	0.1980	0.0546
D-CSMA	NW	0.1464	0.7057	1.7018	0.6460	0.8532	0.1464	0.0607
	W	0.0420	0.634	1.915	0.6330	0.8567	0.1680	0.0556
BTMA	NW	0.2	118.2	14089	0.9983	1.0	0.0	0.0
	W	0.0560	3.500	21.00	0.9520	0.9893	0.0	0.0
CFMA	NW	0.25	185.1	34410	0.9998	1.0	0.0	0.0
	W	0.0700	20.51	588.00	0.9898	0.9995	0.0	0.0
TCAP	NW	0.25	130.7	344252	0.9981	1.0	0.0	0.0
	W	0.0700	10.50	307.86	0.9800	0.9991	0.0	0.0

Table 4.4: The RCCMOT and the RCCMDT for l_3 of the 4-node chain PRN

protocol	Wht	ρ_3^*	$r_{max}^{(X)}(l_3)$	$r_{\rho_3^*}^{(CFMA)}(l_3)$	$r_{\rho_3^*}^{(TCAP)}(l_3)$	$T_1^{(X)}(l_3)$	$T_2^{(X)}(l_3)$
CSMA	NW	1.2094	0.1277	0.1668	0.1734	0.7655	0.7364
CSMA	W	2.337	0.0373	0.0532	0.055	0.7011	0.6782
D-CSMA	NW	1.7018	0.1464	-	0.1847	-	0.7926
D-CSMA	W	1.915	0.042	-	0.0537	-	0.7821
BTMA	NW	34410	0.20	0.25	0.25	0.8	0.8
BTMA	W	21	0.056	0.070	0.070	0.8	0.8

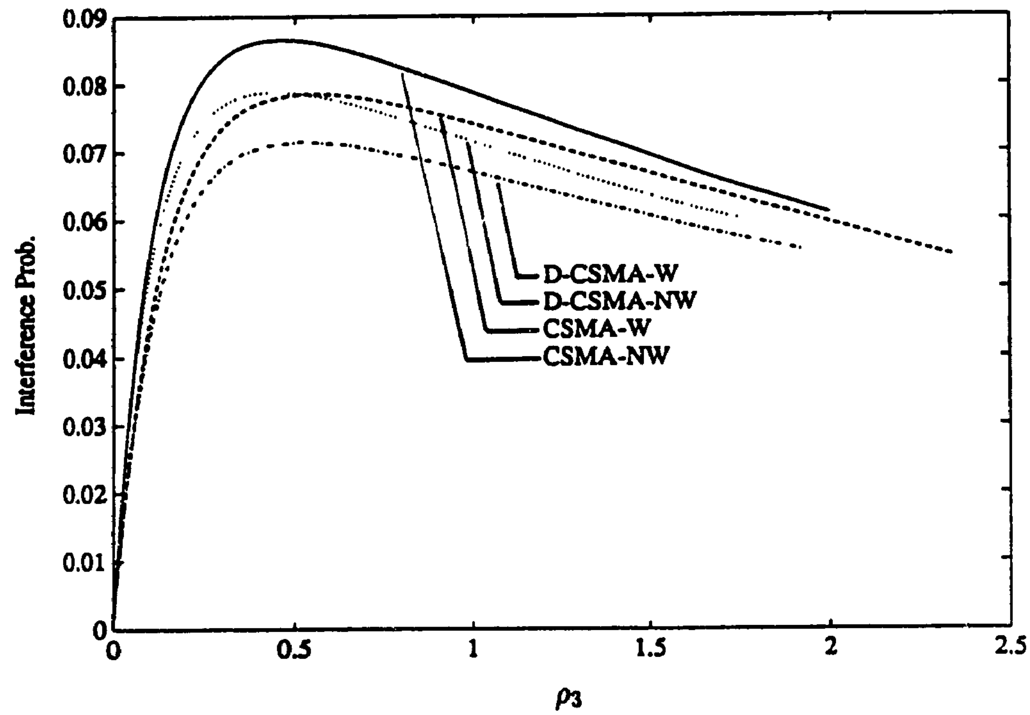


Figure 4.6: Interference Probability for the 4-node chain operating under different protocols

The blocking probability and interference probability presented in these figures describe the attempted and the actual transmissions over the data link l_3 of the given system. They are determined by $\Omega_B(3)$, $\Omega_I(3)$ and Ω as given by Equations (2.11) and (2.12). Since in the uniform end-to-end traffic applications, the original relation between any two attempt rates ρ_i and ρ_j for the uniform link traffic application have been scaled by the ratio $w(l_i)/w(l_j)$, the ρ_i values over those heavily-loaded links $l_i \in L$ need to be scaled up, while in those lightly-loaded links $l_j \in L$ must be scaled down to meet the necessary uniform end-to-end traffic requirements.

In general, the blocking probability over each data link l_i is proportional to the transmission-attempt rates over directed links l_j belonging to states in $\Omega_B(i)$. The interference probability over link l_i, on the other hand, is an increasing function of the attempt rate ρ_i only when ρ_i is relatively small. It decreases after ρ_i passes a certain value. The relation presented in Equation (2.13) holds in general. Transmission blocking only disables scheduled transmissions, it does not effect other transmissions taking place in the network. Interference, on the other hand, damages transmissions in the system. As more and more transmissions over link $l_j \in I(i)$ take place, the chance of any of the transmissions over link l_i being damaged also increases. These damaged transmissions block or interfere with other scheduled or actual transmissions. Thus, the system eventually reaches the point where almost all attempted transmissions are blocked or result in collision, beyond which the throughput of the system deteriorates.

4.3.2 Performance measures of some PRNs of different topologies

In this subsection, we conduct a similar study to that in Subsection 4.3.1 for different PRNs. We are interested in the effects of topology on performance. Specifically, given networks of different topologies, we will analyse their performance under CSMA with respect to TCAP. The RCCMDT measurements will also be considered. Note that similar studies can be carried out for other channel access protocols (i.e., BTMA, D-CSMA).

The networks to be studied are shown in Figure 4.7. These networks are selected to represent some of the frequently encountered local area networks. For each communication network considered in this section, we marked the numbers 1 and 2 with an arrow on it to represent the two directed links being focussed on (see Figure 4.8). The directed link marked with * is the one for which performance is plotted in the figures. As in the last section, the performance studies will be undertaken for the uniform end-to-end and uniform link traffic cases.

For some networks, there are multiple paths with the same number of hops between two nodes. For example, in the hypercube network G2, there are 6 possible shortest paths of 3-hops from node 1 to node 8. In these cases, application of Floyd's algorithm to calculate the routing produces a severe imbalance in the utilization of link capacities; many links are not utilized at all, while others are heavily loaded. As was implied by its decision making rule in Equations (4.1) and (4.2), Floyd's algorithm is biased to choose a path connecting nodes with small index numbers. For

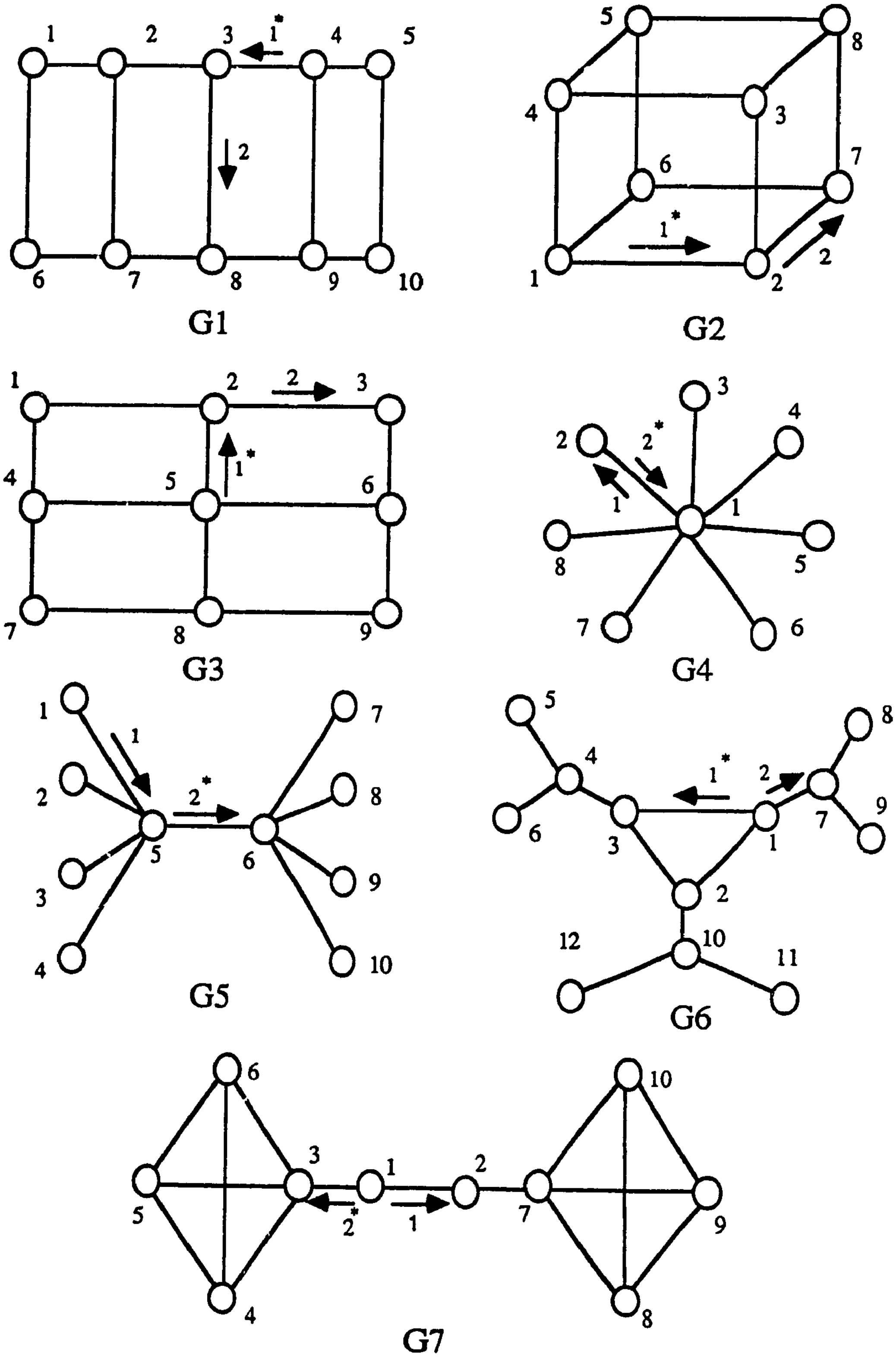


Figure 4.7: Typical network topologies

example, in selecting the path from node 1 to node 8 in network G2, it will choose the path (1, 2, 3, 8) rather than the path (1, 6, 7, 8). When we consider the fact that path (1, 2, 7) rather than path (1, 6, 7), and path (1, 2, 3) rather than path (1, 4, 3) have been similarly chosen, we see the links (1, 2) and (2, 3) belong to many paths, whereas the links (1, 6) and (6, 7) do not. Similar problems can be observed for networks G1 and G3. In order to equalize link capacity utilization, some kind of modification to Floyd's algorithm is required to remove its bias.

One of the modifications we have tried is to add a small value ϵ to a directed link to stretch its length once it has been included in a path. A selected link is considered to be a little longer than an unselected link. It thus has a lower priority of being selected again. This method alleviates the problem to some degree, but it still does not solve the problem completely. The routing decisions are made based on the path selection information provided in the last iteration only, it does not consider path selection information generated during the same iteration. Consequently, it responses too slowly to the change of weights in the links. What is required in a *good* routing is a set of, preferably edge-disjoint, paths between each pair of nodes, along which the end-to-end traffic can be allocated, so as to efficiently utilize the global capacity of the network. The general solution to this routing problem is to select sets of candidate paths for each virtual circuit and formulate an optimization problem. A flow-deviation approach [Hay84] can be used to solve this problem, but we do not pursue this further.

However, for any network with only one shortest path between each pair of stations, such as the networks of G4, G5, G6, and G7, there is no problem. We will consider networks G4, G5, G6, and G7 under uniform end-to-end and uniform link traffic, whereas for networks G1, G2, and G3, we consider uniform link traffic only.

The weights for G4 are $w(l_i) = 7$, for all $l_i \in L$, whereas, the weights for G5, G6, and G7 are shown in Table 4.5, 4.6, and 4.7, respectively. Note that based on the fact that $w(e_{ij}) = w(e_{ji})$, the weights $w(e_{ij})$, for $i > j$, are not presented in these tables.

The performance measures of networks G1, G2, G3 under uniform link traffic are

Table 4.5: Assignment of weights to Network G5

i	j	$w(e_{ij})$	i	j	$w(e_{ij})$	i	j	$w(e_{ij})$
1	5	9	4	5	9	6	8	9
2	5	9	5	6	25	6	9	9
3	5	9	6	7	9	6	10	9

Table 4.6: Assignment of weights to network G6

i	j	$w(e_{ij})$	i	j	$w(e_{ij})$	i	j	$w(e_{ij})$
1	2	16	2	10	27	7	8	11
1	3	16	3	4	27	7	9	11
1	7	27	4	5	11	10	11	11
2	3	16	4	6	11	10	12	11

Table 4.7: Assignment of weights to network G7

i	j	$w(e_{ij})$	i	j	$w(e_{ij})$	i	j	$w(e_{ij})$
1	2	25	3	6	7	7	9	7
1	3	24	4	5	1	7	10	7
2	7	24	4	6	1	8	9	1
3	4	7	5	6	1	8	10	1
3	5	7	7	8	7	9	10	1

shown in Figure 4.8, Figure 4.9, and Figure 4.10, respectively. For these networks, their throughput values with respect to that of TCAP are compared, as well as their blocking probabilities and interference probabilities. Their channel capacities and related measures are presented in Table 4.8.

Similarly, the performance measures of networks G4, G5, G6, and G7 are shown in Figure 4.11, Figure 4.12, Figure 4.13, and Figure 4.14, respectively. For these networks, throughput values with respect to that of TCAP are compared for both traffic situations. Blocking probability and interference probability are also compared for these cases.

Consider the 8-node star network G4. Because of its symmetric topology, the weights are evenly distributed among all links in the network. The analysis is identical for the uniform end-to-end and uniform link traffic cases, with the exception that the throughput value in the channel is scaled down by $w(l_i) = 7$ times. The per-link successful transmission rate remains unchanged.

The channel capacities and other related measures of these networks are presented in Table 4.9. From this table, it can be seen that a heavily-loaded data link does not necessarily introduce more blocking and interference, or become saturated more easily than a less heavily-loaded data link. For example, consider the network G6, where $w(e_{1,3}) > w(e_{1,7})$. Under TCAP, data link $e_{1,3}$ has larger blocking probability and can become saturated more easily than link $e_{1,7}$, even though $e_{1,3}$ carries less traffic.

The RCCMDT measurements of these networks are presented in Table 4.10.

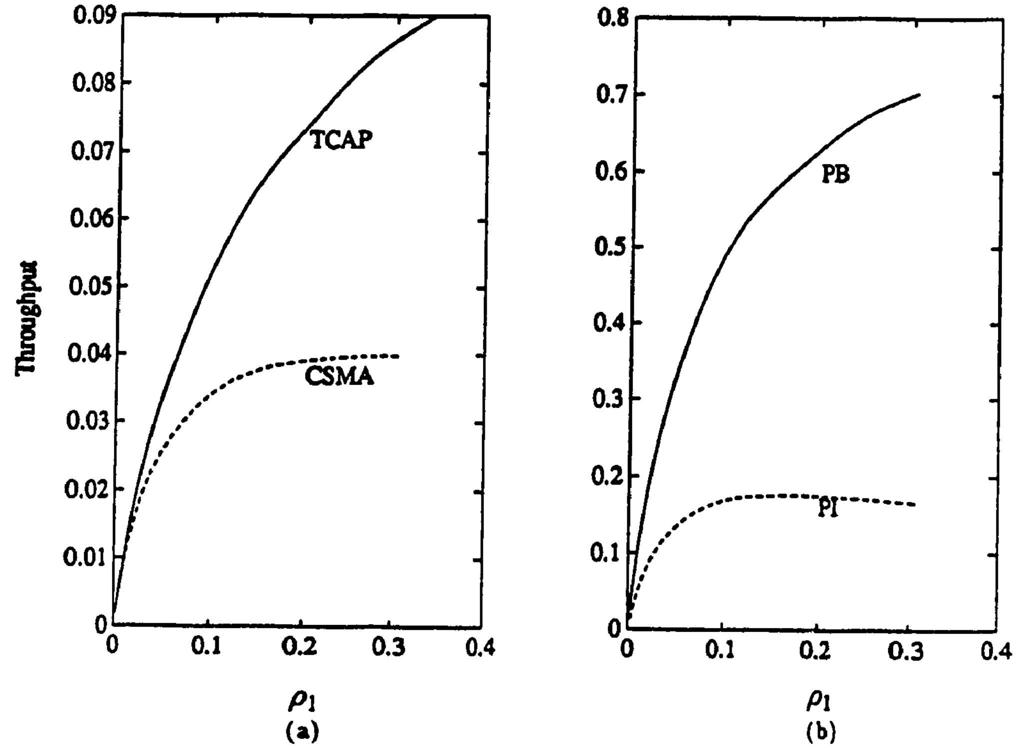


Figure 4.8: Performance measures for G1: (a) Throughput, (b) $P_B(1)$ and $P_I(1)$

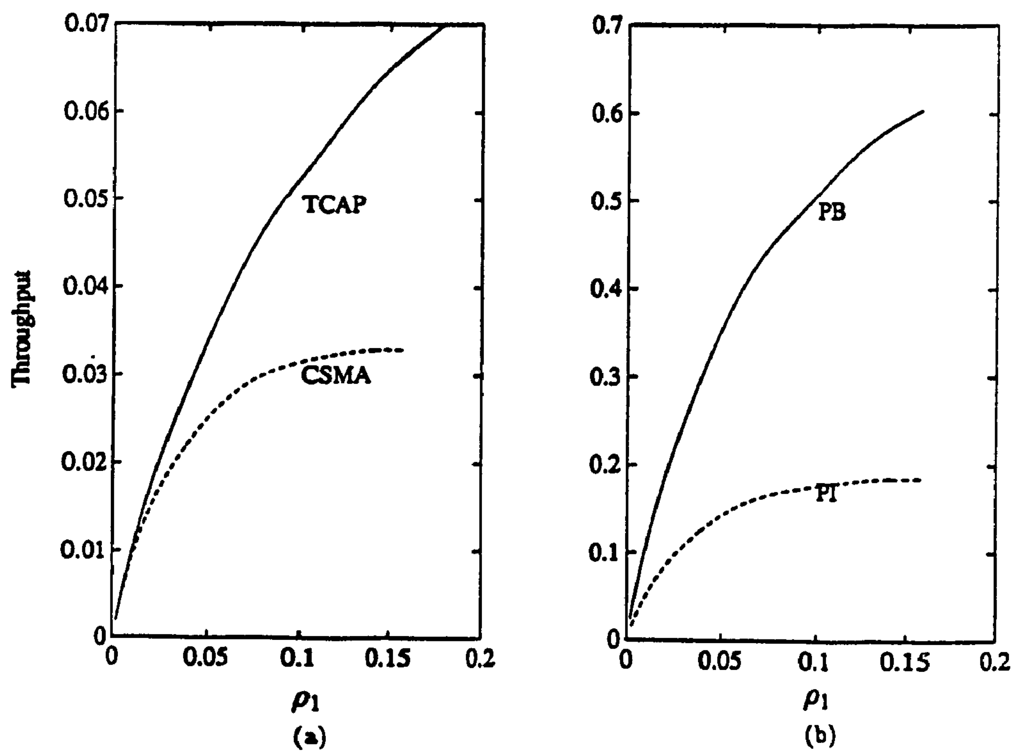


Figure 4.9: Performance measures for G2: (a) Throughput, (b) $P_B(1)$ and $P_I(1)$

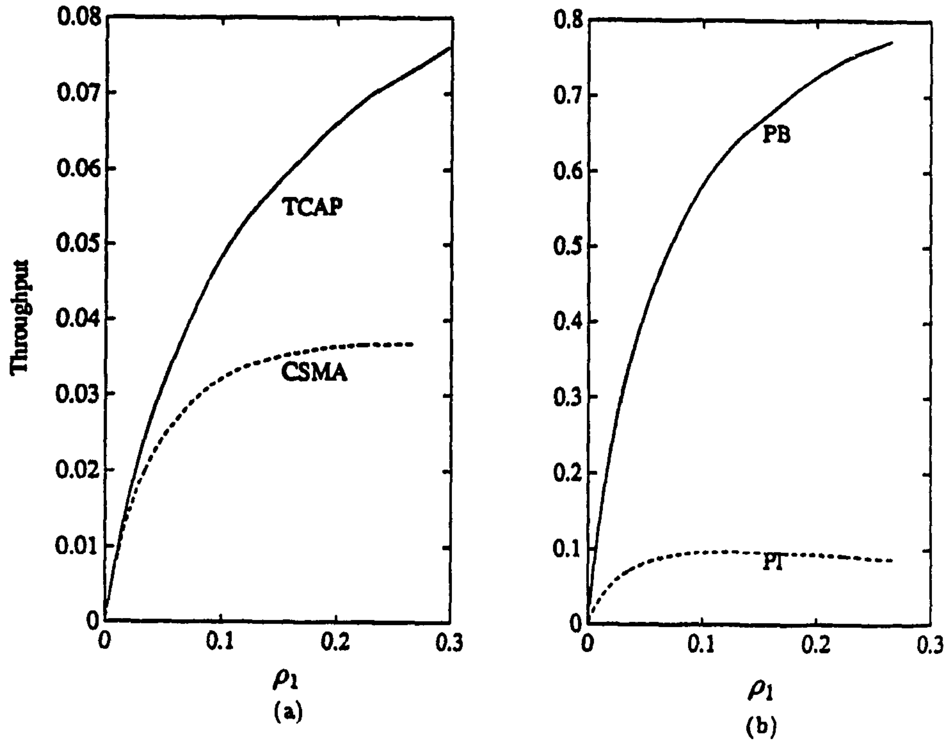


Figure 4.10: Performance measures for G3: (a) Throughput, (b) $P_B(1)$ and $P_I(1)$

Table 4.8: Performance for different PRNs under uniform link traffic

Graph	Protocol	r_{max}	ρ_1	ρ_2	$P_B(1)$	$P_B(2)$	$P_I(1)$	$P_I(2)$
G1	TCAP	0.1520	8.229	9.708	0.9815	0.9843	0.0	0.0
	CSMA	0.0398	0.305	0.242	0.7027	0.7027	0.1668	0.1328
G2	TCAP	0.1660	146.51	146.51	0.9989	0.9989	0.0	0.0
	CSMA	0.0330	0.157	0.157	0.6045	0.6045	0.1853	0.1853
G3	TCAP	0.1240	38.91	0.661	0.9968	0.8124	0.0	0.0
	CSMA	0.0368	0.263	0.136	0.7734	0.6086	0.0876	0.1208

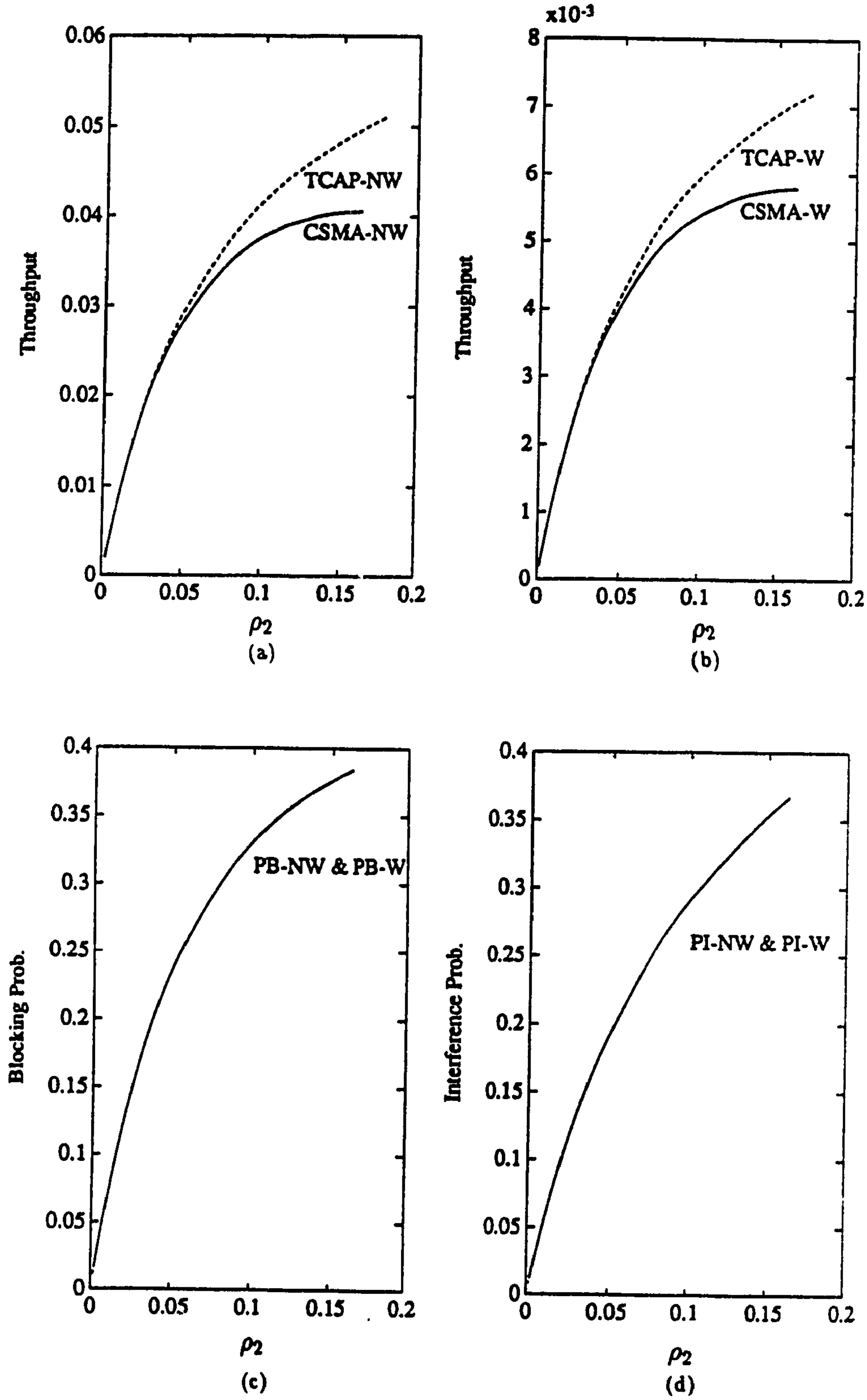


Figure 4.11: Performance measures for network G4: (a) Throughput for uniform link traffic, (b) Throughput for uniform end-to-end traffic, (c) Blocking Probability, and (d) Interference Probability.

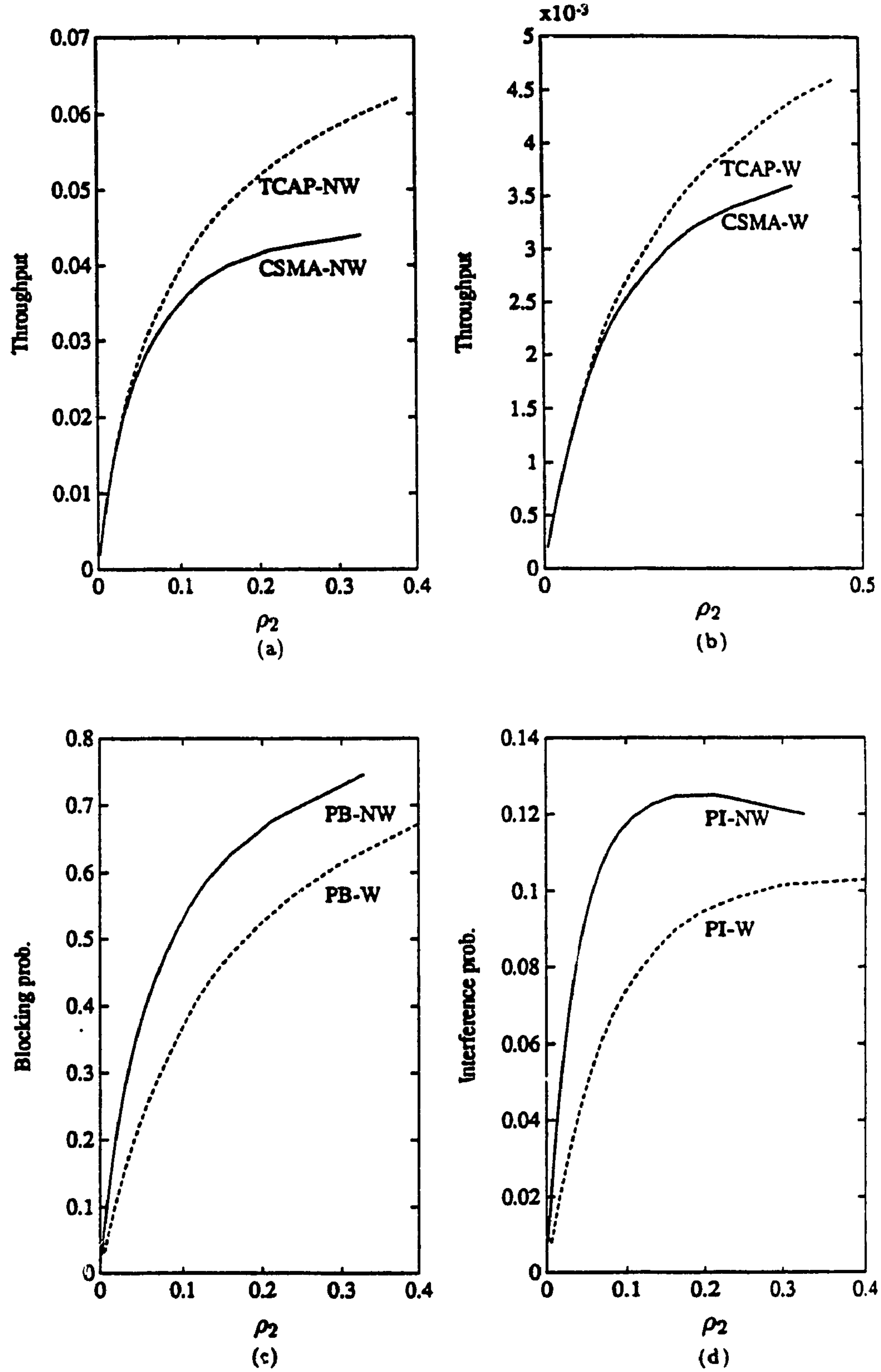


Figure 4.12: Performance measures for network G5: (a) Throughput for uniform link traffic, (b) Throughput for uniform end-to-end traffic, (c) Blocking Probability, and (d) Interference Probability.

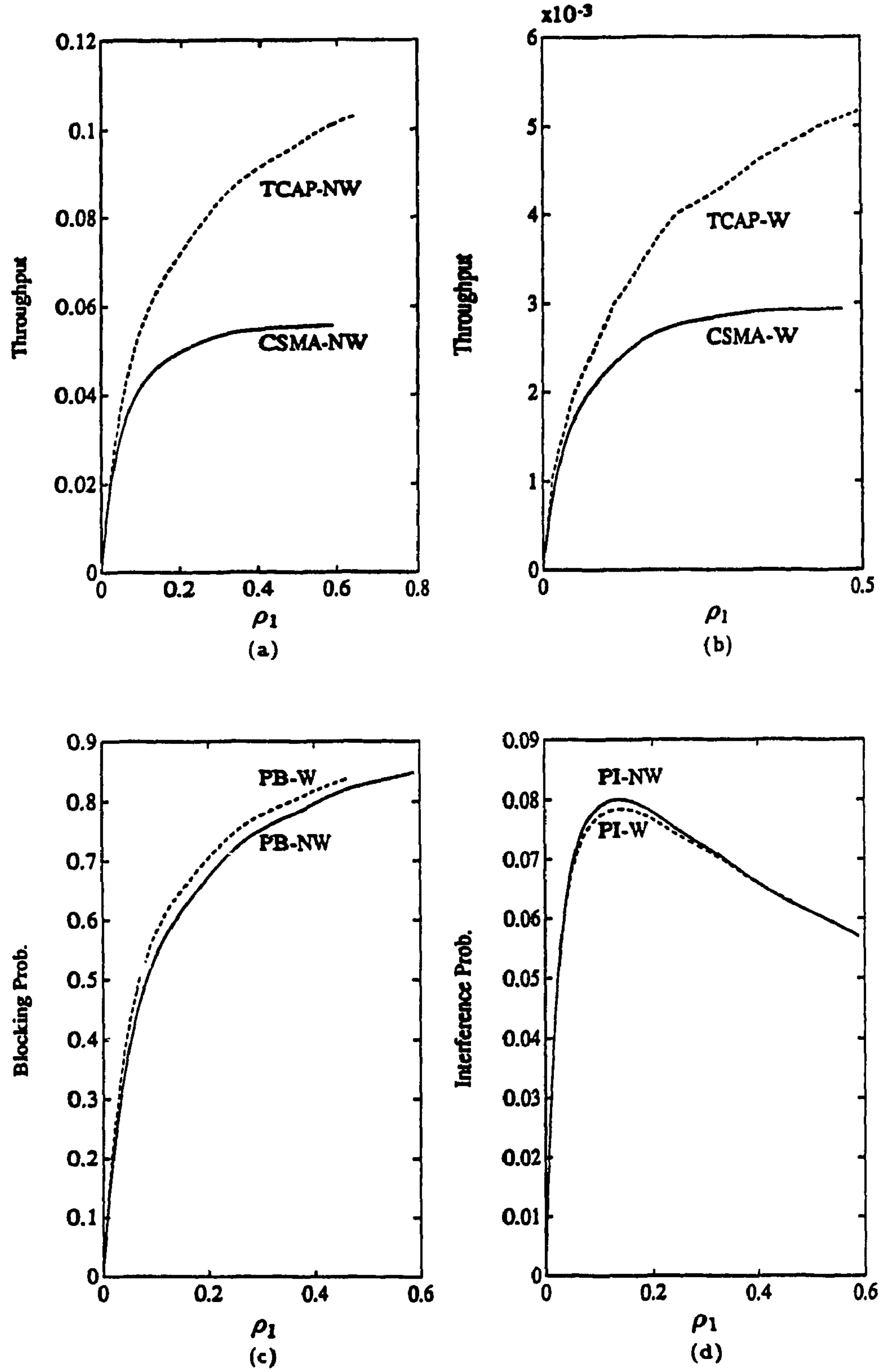


Figure 4.13: Performance measures for network G6: (a) Throughput for uniform link traffic, (b) Throughput for uniform end-to-end traffic, (c) Blocking Probability, and (d) Interference Probability.

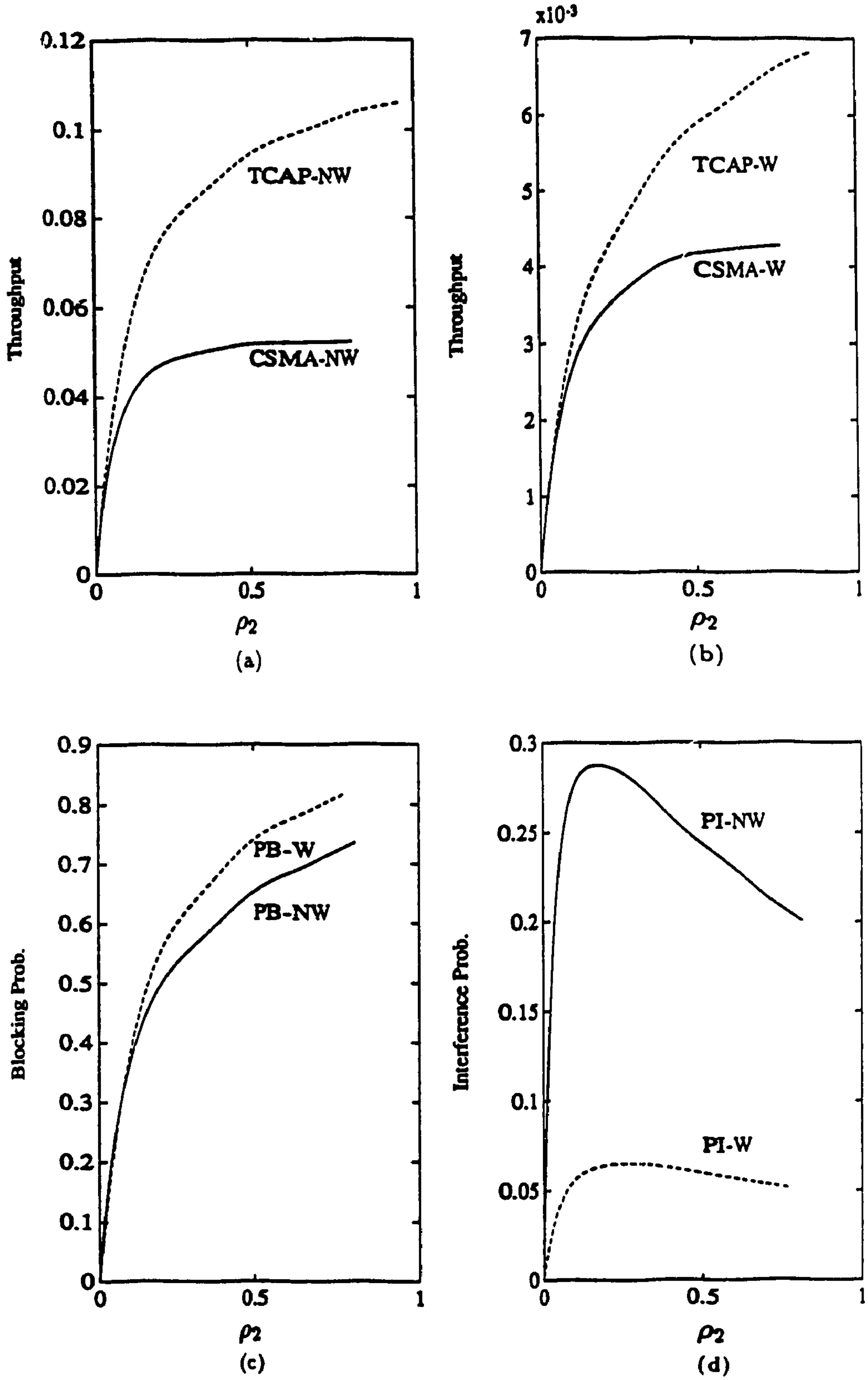


Figure 4.14: Performance measures for network G7: (a) Throughput for uniform link traffic, (b) Throughput for uniform end-to-end traffic, (c) Blocking Probability, and (d) Interference Probability.

Table 4.9: System performance for different network topologies

Graph	Protocol		r_{max}	ρ_1	ρ_2	$P_B(1)$	$P_B(2)$	$P_I(1)$	$P_I(2)$
G4	TCAP	(NW)	0.0710	11.77	11.77	0.9940	0.9940	0.0	0.0
		(W)	0.0102	51.43	51.43	0.9986	0.9986	0.0	0.0
	CSMA	(NW)	0.0406	0.163	0.163	0.7513	0.3843	0.0	0.367
		(W)	0.0058	0.163	0.163	0.7505	0.3840	0.0	0.366
G5	TCAP	(NW)	0.0960	2.400	48.48	0.9600	0.9980	0.0	0.0
		(W)	0.0076	0.940	22.23	0.9272	0.9915	0.0	0.0
	CSMA	(NW)	0.0440	0.173	0.328	0.3687	0.7459	0.377	0.120
		(W)	0.0036	0.099	0.399	0.3271	0.6715	0.344	0.103
G6	TCAP	(NW)	0.1520	34.28	13.66	0.9956	0.9889	0.0	0.0
		(W)	0.0083	338.41	32.17	0.9996	0.9930	0.0	0.0
	CSMA	(NW)	0.0554	0.588	0.493	0.8488	0.8488	0.057	0.0388
		(W)	0.0029	0.465	0.565	0.8363	0.8363	0.064	0.025
G7	TCAP	(NW)	0.1240	0.367	23.10	0.6622	0.9946	0.0	0.0
		(W)	0.0091	10.99	6.464	0.9793	0.9661	0.0	0.0
	CSMA	(NW)	0.0522	0.304	0.815	0.7353	0.7353	0.093	0.20
		(W)	0.0043	1.043	0.772	0.8158	0.8158	0.081	0.0505

Table 4.10: The RCCMDT for different networks

Graph	Wht	l_i	ρ_i^*	$r_{max}^{(X)}(l_i)$	$r_{\rho_i^*}^{(TCAP)}(l_i)$	$T_2^{(X)}(l_i)$
G1	NW	1	0.305	0.0398	0.087	0.4575
G2	NW	1	0.157	0.033	0.0672	0.4911
G3	NW	1	0.263	0.0368	0.0731	0.5034
G4	NW	2	0.163	0.0406	0.050	0.8120
G4	W	2	0.163	0.0058	0.0071	0.8169
G5	NW	2	0.328	0.044	0.0573	0.7679
G5	W	2	0.399	0.0036	0.00439	0.8200
G6	NW	1	0.588	0.0554	0.1012	0.5474
G6	W	1	0.465	0.0029	0.0052	0.5577
G7	NW	2	0.815	0.0522	0.1036	0.504
G7	W	2	0.772	0.0043	0.00665	0.6466

Chapter 5

Applications of the Blocking Model in Circuit Switching Networks

5.1 Introduction

As discussed in Chapter 3, the operation of a circuit switching network can be described with TCAP.

In a switch, there are no transmission collisions at the receivers, every unblocked transmission becomes successful. Furthermore, the system ultilizes the maximum resources that can be offered by the given network.

As in the PRNs cases, the throughput and blocking in CSNs can be obtained by using the iterative-estimation method. This method is limited by its complexity to moderate size networks. The reason is that this method proceeds by first constructing the state space of the system, which consumes a lot of memory and many CPU cycles. In general, we cannot employ this calculating method to study the performance of networks of large size.

However, in analyzing the performance of a circuit-switching network, we realize the advantages offered by using the recursive properties of partition functions, as given in Equations (3.17) to (3.22). For some networks with regular topologies, the partition functions can be expressed in terms of partition functions of smaller networks of the same topology. In Subsection 3.5.3, we have demonstrated the recursive results of partition functions for a completely-connected network, a crossbar switching network, and a recirculating shift network. The recursive relations among partition functions can be used to derive expressions for performance measures. In this chapter, we examine these possibilities.

This chapter consists of two parts. In the first part, we will study the performance for some CSNs of regular topology under symmetrical loading. In the second part, we will study the performance for some switches under skewed loading conditions.

5.2 Performance Measures for Networks Under Symmetric Loading

In this section, we will analyze three regular network systems with symmetric loading. These networks are: (1) the completely-connected network, (2) the crossbar switching network, and (3) a recirculating shift network. We will analyze their performance individually in the following subsections.

For any circuit switching-network G, every unblocked scheduled transmission becomes successful. If we denote the scheduled transmission rate over any link e_{ij} as ρ_{ij}, and the successful-transmission rate over this link as r_{ij}, we have

$$r_{ij} = \rho_{ij}\Pr[\text{no blocking at node } i \text{ and node } j] = \rho_{ij}P(i,j). \tag{5.1}$$

Under symmetric traffic load, if we let $r(G,\rho)$ denote the successful-transmission rate over any link e_{ij} of network G, we then have $\rho_{ij} = \rho$, $r_{ij} = r(G,\rho)$ and

$$\frac{r(G,\rho)}{\rho} = P(i,j). \tag{5.2}$$

In addition, we also denote the throughput of the overall system as $S(G,\rho)$. It is the summation of all per-link throughputs of the network. Thus, if the network contains l directed links, we have, $S(G,\rho) = l \cdot r(G,\rho)$.

5.2.1 Analysis of completely connected networks K_n

Consider a completely-connected switching network K_n with n input-output ports. For this network, K_{n-2} is obtained from K_n by removing the input node i and output node j. The successful transmission rate over link e_{ij} is

$$r_{ij} = \rho_{ij} \sum_{S \subseteq K_{n-2}} Q(S).$$

Under symmetric loading, we have

$$r(K_n, \rho) = \rho \sum_{S \subseteq K_{n-2}} \beta^{-1}(K_n, \rho)\rho^{|S|} = \rho\beta^{-1}(K_n, \rho) \sum_{S \subseteq K_{n-2}} \rho^{|S|},$$

where $\beta(K_n, \rho) = \sum_{S \subseteq K_n} \rho^{|S|}$. Further, since $\beta(K_{n-2}, \rho) = \sum_{S \subseteq K_{n-2}} \rho^{|S|}$, then

$$r(K_n, \rho) = \rho \frac{\beta(K_{n-2}, \rho)}{\beta(K_n, \rho)}. \tag{5.3}$$

By using Equation (3.10), we get the overall throughput of the system

$$S(K_n, \rho) = n(n-1) \cdot r(K_n, \rho) = \frac{\frac{\partial}{\partial \rho}\beta(K_n, \rho)}{\beta(K_n, \rho)},$$

then

$$r(K_n, \rho) = \frac{\rho}{n(n-1)} \cdot \frac{\frac{\partial}{\partial \rho}\beta(K_n, \rho)}{\beta(K_n, \rho)}. \tag{5.4}$$

By relating this expression with Equation (5.3), we have

$$\beta(K_{n-2}, \rho) = \frac{1}{n(n-1)} \frac{\partial}{\partial \rho} \beta(K_n, \rho).$$

From Equations (5.3) and (3.11), the throughput and blocking probability for this interference-free network system are,

$$\begin{aligned} \frac{r(K_n, \rho)}{\rho} &= \frac{\beta(K_{n-2}, \rho)}{\beta(K_n, \rho)}; \\ Pb(K_n, \rho) &= \frac{\beta(K_n, \rho) - \beta(K_{n-2}, \rho)}{\beta(K_n, \rho)}. \end{aligned}$$

Using Equation (3.30), the partition function for the directed-links-activation model of the completely-connected network K_n can be expressed as,

$$\beta(K_n, \rho) = [1 + 2(2n-3)\rho]\beta(K_{n-2}, \rho) - 4(n-2)(n-3)\rho^2\beta(K_{n-4}, \rho).$$

Thus,

$$\begin{aligned} \frac{r(K_n, \rho)}{\rho} &= \frac{\beta(K_{n-2}, \rho)}{[1 + 2(2n-3)\rho]\beta(K_{n-2}, \rho) - 4(n-2)(n-3)\rho^2\beta(K_{n-4}, \rho)} \\ &= \frac{1}{[1 + 2(2n-3)\rho] - [4(n-2)(n-3)\rho^2]\frac{\beta(K_{n-4}, \rho)}{\beta(K_{n-2}, \rho)}}. \end{aligned}$$

As the result, we have,

$$r(K_n,\rho) = \frac{\rho}{1+2(2n-3)\rho-4(n-2)(n-3)\rho r(K_{n-2},\rho)}; \quad (5.5)$$

$$Pb(K_n,\rho) = \frac{2(2n-3)\rho-4(n-2)(n-3)\rho r(K_{n-2},\rho)}{1+2(2n-3)\rho-4(n-2)(n-3)\rho r(K_{n-2},\rho)}, \quad (5.6)$$

with,

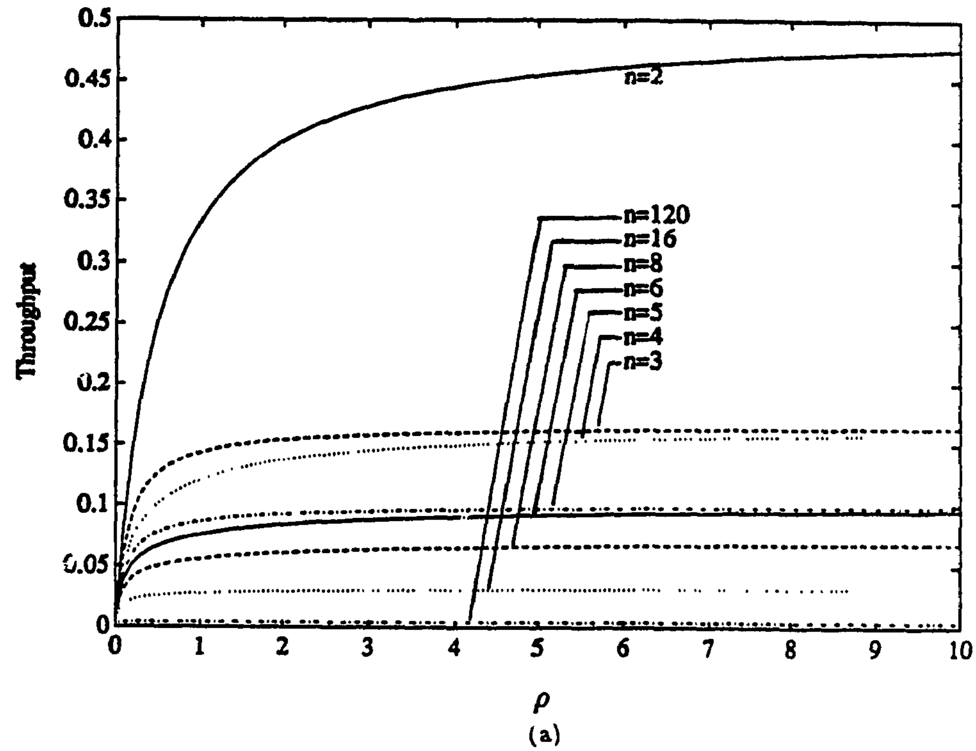
$$r(K_2,\rho) = \frac{\rho}{1+2\rho}; \qquad r(K_3,\rho) = \frac{\rho}{1+6\rho};$$
$$Pb(K_2,\rho) = \frac{2\rho}{1+2\rho}; \qquad Pb(K_3,\rho) = \frac{6\rho}{1+6\rho}.$$

The throughput for these completely-connected networks of various sizes are shown in Figure 5.1(a) while the blocking probability of the system is shown in Figure 5.1(b). It is seen that as the number of nodes in the network increases, the blocking probability of the system increases and the throughput of the system decreases correspondingly. When the n value is modestly large, (*i.e.*, 120), the throughput of the system is approximately zero, and the blocking probability approaches unity immediately. This can be explained by the following observations: For a complete graph K_n, each transmission over a directed link e_{ij} in the network can be blocked by $2(n-1)$ transmissions (including the transmission currently using the link e_{ij}) at the transmittion node i and $2(n-2)$ transmissions at the receiving node j. Therefore, each transmission in the network can be blocked by $2(2n-3)$ transmissions in the network. Consequently, as the number of nodes n increases, the number of transmissions possibly blocking the considered transmission increase as well. Therefore, when the number of nodes in the network is large, the chance that a scheduled transmission is blocked will be high, and the successful-transmission rate of the system will be small.

5.2.2 Analysis of the crossbar switching network $K_{n,m}$

The topological representation of the crossbar switch network with n input ports and m output ports is the bipartite graph $K_{n,m}$. The graph resulting from removing the input node i and the output node j from $K_{n,m}$ is $K_{n-1,m-1}$. We have

$$r_{ij} = \rho_{ij} \sum_{S \subseteq K_{n-1,m-1}} Q(S).$$



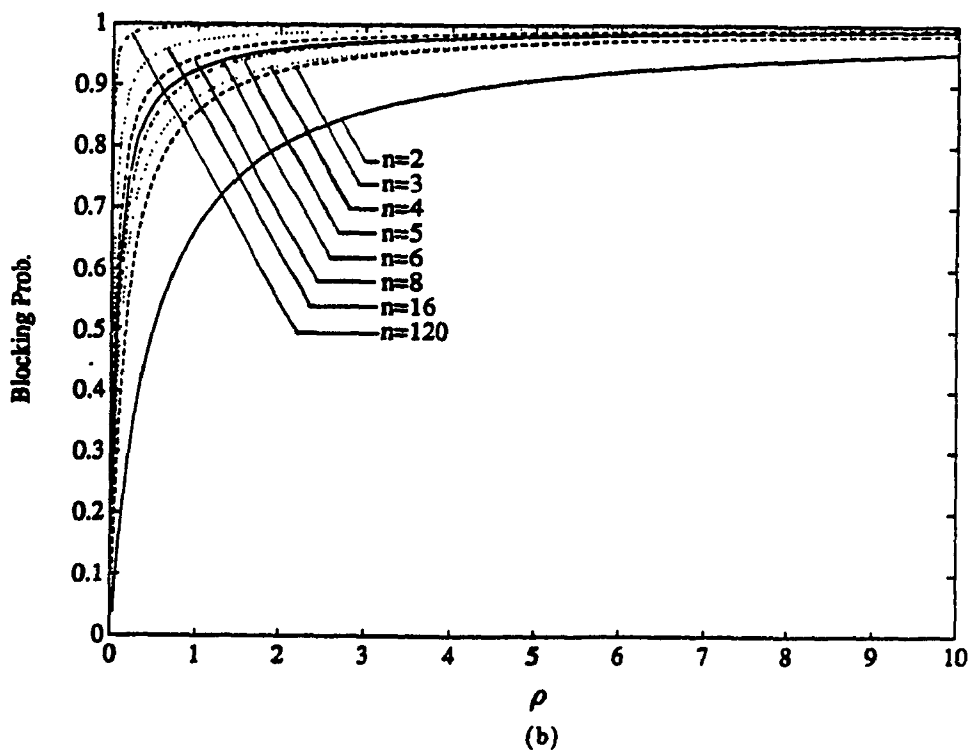


Figure 5.1: Performance for the K_n networks of different sizes: (a) Throughput, (b) Blocking probability.

Under symmetric loading, we have

$$r(K_{n,m},\rho) = \rho \sum_{S \subseteq K_{n-1,m-1}} \beta^{-1}(K_{n,m},\rho)\rho^{|S|} = \rho\beta^{-1}(K_{n,m},\rho) \sum_{S \subseteq K_{n-1,m-1}} \rho^{|S|},$$

where $\beta(K_{n,m},\rho) = \sum_{S \subseteq K_{n,m}} \rho^{|S|}$. Further, since $\beta(K_{n-1,m-1},\rho) = \sum_{S \subseteq K_{n-1,m-1}} \rho^{|S|}$, then

$$r(K_{n,m},\rho) = \rho \frac{\beta(K_{n-1,m-1},\rho)}{\beta(K_{n,m},\rho)}. \tag{5.7}$$

From Equation (3.10), the overall throughput of the system can be expressed as

$$S(K_{n,m},\rho) = (n \cdot m) \cdot r(K_{n,m},\rho) = \frac{\frac{\partial}{\partial\rho}\beta(K_{n,m},\rho)}{\beta(K_{n,m},\rho)}.$$

BThen

$$r(K_{n,m},\rho) = \frac{\rho}{n \cdot m} \frac{\frac{\partial}{\partial\rho}\beta(K_{n,m},\rho)}{\beta(K_{n,m},\rho)}. \tag{5.8}$$

By relating this expression to Equation (5.7), we obtain

$$\beta(K_{n-1,m-1},\rho) = \frac{1}{n \cdot m} \cdot \frac{\partial}{\partial\rho}\beta(K_n,\rho).$$

Equations (5.7) and (3.11) lead to

$$\begin{aligned} \frac{r(K_{n,m},\rho)}{\rho} &= \frac{\beta(K_{n-1,m-1},\rho)}{\beta(K_{n,m},\rho)}; \\ Pb(K_{n,m},\rho) &= \frac{\beta(K_{n,m},\rho) - \beta(K_{n-1,m-1},\rho)}{\beta(K_{n,m},\rho)}. \end{aligned}$$

Since

$$\beta(K_{n,m},\rho) = [1 + (m+n-1)\rho]\beta(K_{n-1,m-1},\rho) - (n-1)(m-1)\rho^2\beta(K_{n-2,m-2},\rho),$$

then,

$$\begin{aligned} r(K_{n,m},\rho) &= \frac{\rho\beta(K_{n-1,m-1},\rho)}{[1+(m+n-1)\rho]\beta(K_{n-1,m-1},\rho) - (n-1)(m-1)\rho^2\beta(K_{n-2,m-2},\rho)} \\ &= \frac{\rho}{[1+(m+n-1)\rho] - (n-1)(m-1)\rho^2\frac{\beta(K_{n-2,m-2},\rho)}{\beta(K_{n-1,m-1},\rho)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} r(K_{n,m},\rho) &= \frac{\rho}{1+(m+n-1)\rho - (n-1)(m-1)\rho r(K_{n-1,m-1},\rho)}; & (5.9) \\ Pb(K_{n,m},\rho) &= \frac{(m+n-1)\rho - (n-1)(m-1)\rho r(K_{n-1,m-1},\rho)}{1+(m+n-1)\rho - (n-1)(m-1)\rho r(K_{n-1,m-1},\rho)}, & (5.10) \end{aligned}$$

with

$$\begin{aligned} r(K_{n,1},\rho) &= \tfrac{\rho}{1+n\rho}; & r(K_{2,2},\rho) &= \tfrac{\rho(1+\rho)}{1+4\rho+2\rho^2}; \\ Pb(K_{n,1},\rho) &= \tfrac{n\rho}{1+n\rho}; & Pb(K_{2,2},\rho) &= \tfrac{3\rho+2\rho^2}{1+4\rho+2\rho^2}. \end{aligned}$$

For the crossbar switching network $K_{n,m}$, the numbers n and m have a direct effect on performance. Basically, there are m directed links connected to any input ports and n directed links connected to any output port of the switch. For a scheduled transmission over any link l_i, there are (m) other transmissions that can block it at the input port, and $(n-1)$ other transmissions that can block it at the output port. Therefore, as the numbers m and n increase, the probability of blocking for a transmission over link l_i, $P_B(i)$, will increase correspondingly. This causes the per-link throughput to decrease accordingly. Figure 5.2 demonstrates these results.

If we let $n = m$, we obtain the switch $K_{n,n}$, in which the number of input ports equals the number of output ports. In this case,

$$r(K_{n,n},\rho) = \frac{\rho}{1+(2n-1)\rho-(n-1)^2\rho r(K_{n-1,n-1},\rho)}; \tag{5.11}$$

$$Pb(K_{n,n},\rho) = \frac{(2n-1)\rho-(n-1)^2\rho r(K_{n-1,n-1},\rho)}{1+(2n-1)\rho-(n-1)^2\rho r(K_{n-1,n-1},\rho)}. \tag{5.12}$$

Figure 5.3 presents the performance of the $K_{n,n}$ switches.

With these recursive relations at hand, we can further develop some general recursive expressions to observe the effects on performance of varying number of the output ports while the number of input ports is fixed. A similar study can be undertaken by varying the number of input ports while the number of output ports remains unchanged. By observing the effects of these switch size variations, we can obtain the optimal operating parameters for these expander and concentrator networks. In the following, we develop recursive expressions for the performance measures. A Based on Equation (5.7), we have

$$\frac{r(K_{n,m},\rho)}{\rho} = \frac{\beta(K_{n-1,m-1},\rho)}{\beta(K_{n,m},\rho)},$$

and

$$\frac{r(K_{n,m-1},\rho)}{\rho} = \frac{\beta(K_{n-1,m-2},\rho)}{\beta(K_{n,m-1},\rho)}.$$

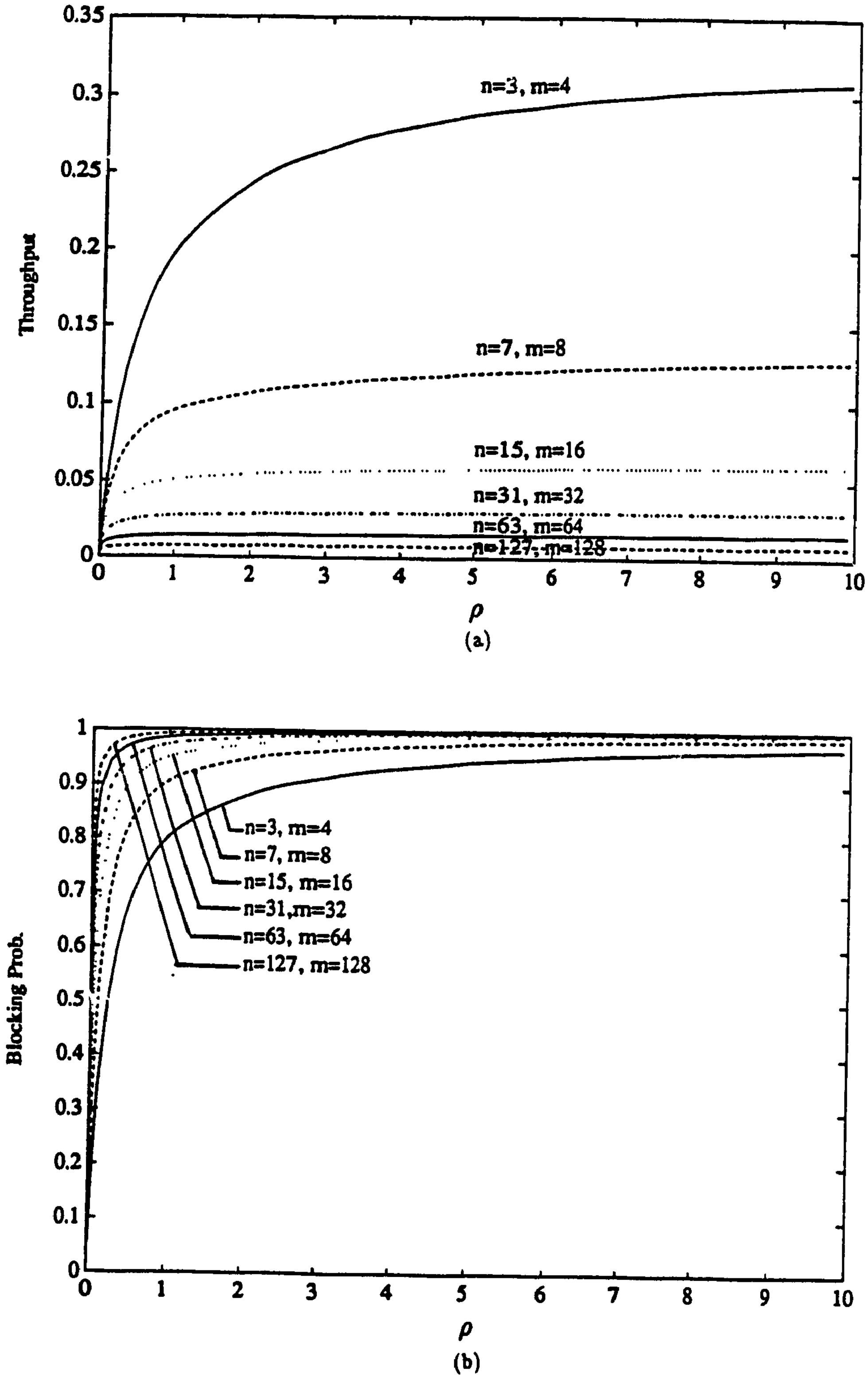


Figure 5.2: System performance for the switches of different sizes: (a) Throughput, (b) Blocking probability.

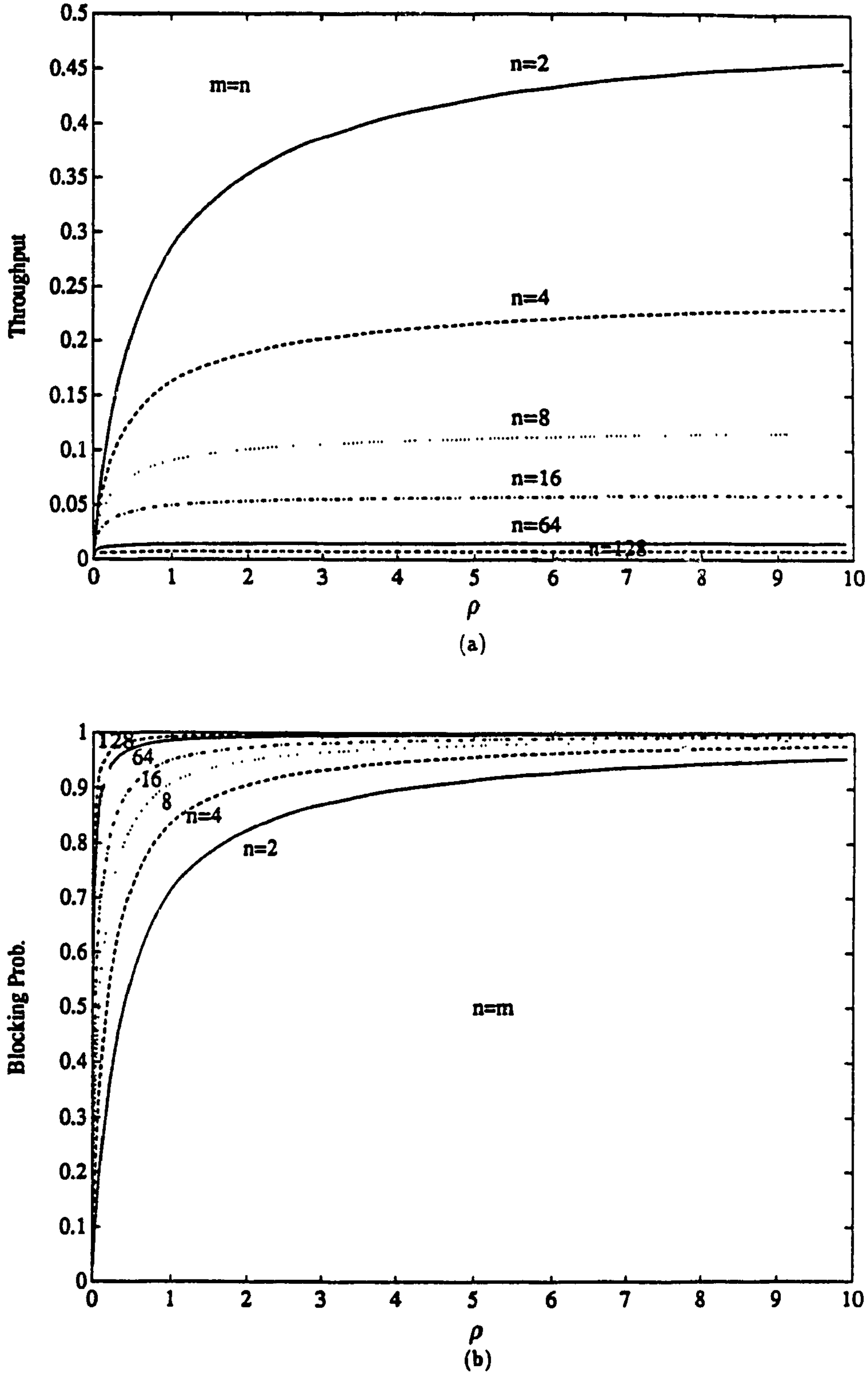


Figure 5.3: System performance for different sizes of the $K_{n,n}$ networks: (a) Throughput, (b) Blocking probability.

Since

$$\beta(K_{n,m},\rho) = \beta(K_{n,m-1},\rho) + n\rho\beta(K_{n-1,m-1},\rho),$$

then

$$\frac{\beta(K_{n-1,m-1},\rho)}{\beta(K_{n,m},\rho)} = \frac{1}{n\rho}\left(1 - \frac{\beta(K_{n,m-1},\rho)}{\beta(K_{n,m},\rho)}\right).$$

Thus,

$$r(K_{n,m},\rho) = \frac{1}{n}\left(1 - \frac{\beta(K_{n,m-1},\rho)}{\beta(K_{n,m},\rho)}\right). \tag{5.13}$$

Similarly, since

$$\beta(K_{n,m},\rho) = (1 + n\rho)\beta(K_{n,m-1},\rho) - n(m-1)\rho^2\beta(K_{n-1,m-2},\rho),$$

we have

$$\frac{\beta(K_{n,m-1},\rho)}{\beta(K_{n,m},\rho)} = \frac{1}{1+n\rho}\left(1 + n(m-1)\rho^2\frac{\beta(K_{n-1,m-2},\rho)}{\beta(K_{n,m},\rho)}\right).$$

Substituting this expression into Equation (5.13) leads to

$$r(K_{n,m},\rho) = \frac{1}{n}\left[1 - \frac{1}{1+n\rho}\left(1 + n(m-1)\rho^2\frac{\beta(K_{n-1,m-2},\rho)}{\beta(K_{n,m})}\right)\right]. \tag{5.14}$$

Also, since

$$\beta(K_{n,m},\rho) = (1 + n\rho)\beta(K_{n,m-1},\rho) - n(m-1)\rho^2\beta(K_{n-1,m-2},\rho),$$

then

$$\begin{aligned}
\frac{\beta(K_{n-1,m-2},\rho)}{\beta(K_{n,m},\rho)} &= \frac{\beta(K_{n-1,m-2},\rho)}{(1+n\rho)\beta(K_{n,m-1},\rho) - n(m-1)\rho^2\beta(K_{n-1,m-2},\rho)} \\
&= \frac{r(K_{n,m-1},\rho)}{\rho(1+n\rho) - n(m-1)\rho^2 r(K_{n,m-1},\rho)}.
\end{aligned}$$

Substituting this expression into Equation (5.14), we obtain

$$\begin{aligned}
r(K_{n,m},\rho) &= \frac{1}{n}\left[1 - \frac{1}{1+n\rho}\left(1 + \frac{n(m-1)\rho^2 r(K_{n,m-1},\rho)}{\rho(1+n\rho) - n(m-1)\rho^2 r(K_{n,m-1},\rho)}\right)\right] \\
&= \frac{1}{n}\left(1 - \frac{1}{(1+n\rho) - n(m-1)\rho r(K_{n,m-1},\rho)}\right).
\end{aligned}$$

Finally, we have

$$r(K_{n,m},\rho) = \frac{\rho(1-(m-1)r(K_{n,m-1},\rho))}{(1+n\rho) - n(m-1)\rho r(K_{n,m-1},\rho)}; \tag{5.15}$$

$$Pb(K_{n,m},\rho) = \frac{n\rho - (m-1)(n\rho-1)r(K_{n,m-1},\rho)}{(1+n\rho) - n(m-1)\rho r(K_{n,m-1},\rho)}, \tag{5.16}$$

with

$$\begin{aligned} r(K_{n,1},\rho) &= \tfrac{\rho}{1+n\rho}; & r(K_{n,2},\rho) &= \tfrac{\rho(1+(n-1)\rho)}{1+2n\rho+n(n-1)\rho^2}; \\ Pb(K_{n,1},\rho) &= \tfrac{n\rho}{1+n\rho}; & Pb(K_{n,2},\rho) &= \tfrac{(n+1)\rho+n(n-1)\rho^2}{1+2n\rho+n(n-1)\rho^2}. \end{aligned}$$

With these recursive relations, we study the performance behavior of the networks $K_{64,m}$, where the number of input ports is fixed at 64 while the number of outputs port varies. Their performances are shown in Figure 5.4 and Figure 5.5, respectively, for $m \leq 64$ and $m \geq 64$. In this case, every scheduled transmission over link e_{ij} can be blocked by m transmissions at the input port (including the transmission currently using link e_{ij} and up to 63 other transmissions destined to the same output port j. All together, it can be blocked by $(63 + m)$ possible transmissions in the switch. Therefore, as the number of output ports increases, the blocking probability increases, and the per-link throughput decreases accordingly. In addition, due to the high blocking probability introduced, the throughput of the switch with a large number of output ports will reach saturation relatively fast. This can be observed from Figure 5.4 and Figure 5.5.

Similarly, we can apply the equations

$$\begin{aligned} \beta(K_{n,m},\rho) &= \beta(K_{n-1,m},\rho) + m\rho\beta(K_{n-1,m-1},\rho); \\ \beta(K_{n,m},\rho) &= [1+m\rho]\beta(K_{n-1,m},\rho) - m(n-1)\rho^2\beta(K_{n-2,m-1},\rho), \end{aligned}$$

repeatedly to generate the relations between the throughput for switches of fixed output-port size with changing input-port sizes. In this case, we have

$$r(K_{n,m},\rho) = \frac{\rho[1-(n-1)r(K_{n-1,m},\rho)]}{(1+m\rho) - m(n-1)\rho r(K_{n-1,m},\rho)}; \tag{5.17}$$

$$Pb(K_{n,m},\rho) = \frac{m\rho - (n-1)(m\rho - 1)r(K_{n-1,m},\rho)}{(1+m\rho) - m(n-1)\rho r(K_{n-1,m},\rho)}, \tag{5.18}$$

with

$$\begin{aligned} r(K_{1,m},\rho) &= \tfrac{\rho}{1+m\rho}; & r(K_{2,m},\rho) &= \tfrac{\rho[1+(m-1)\rho]}{1+2m\rho+m(m-1)\rho^2} \\ Pb(K_{1,m},\rho) &= \tfrac{m\rho}{1+m\rho}; & Pb(K_{2,m},\rho) &= \tfrac{(m+1)\rho+m(m-1)\rho^2}{1+2m\rho+m(m-1)\rho^2} \end{aligned}$$

If we let $m = 64$, the system performance will be the same as in the case $K_{64,m}$.

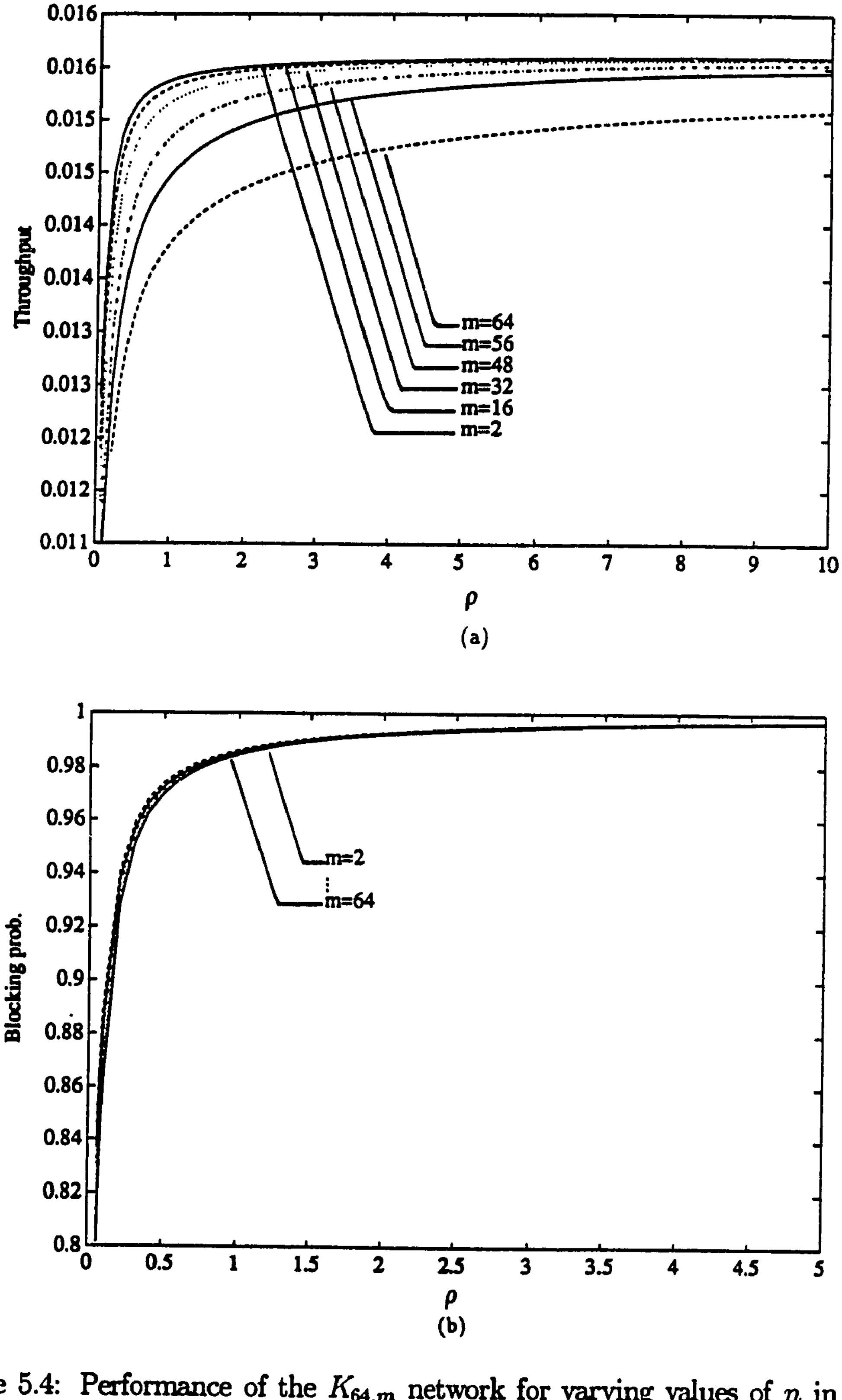


Figure 5.4: Performance of the $K_{64,m}$ network for varying values of n in the case $m \leq 64$: (a) Throughput, (b) Blocking probability.

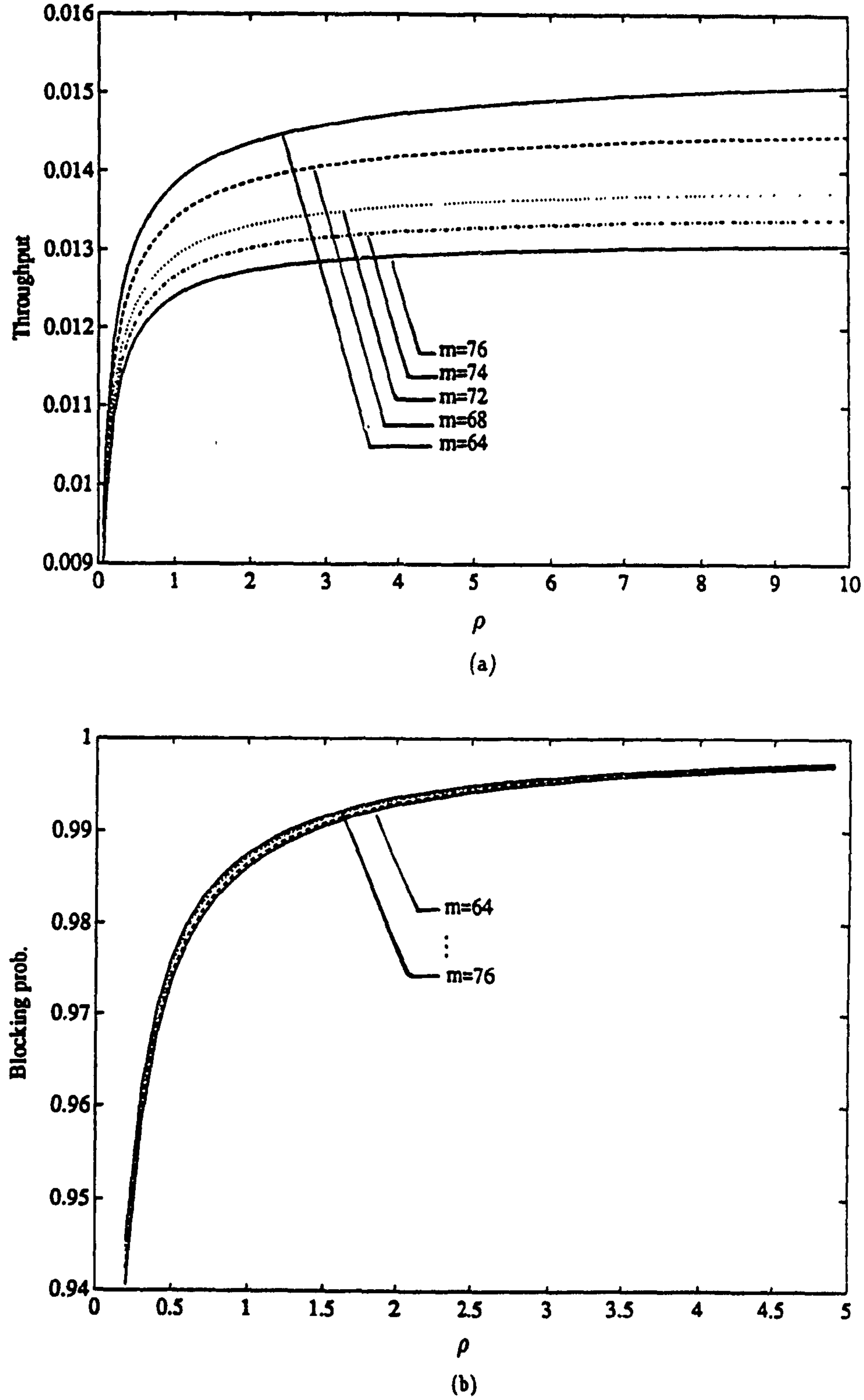


Figure 5.5: Performance of the $K_{64,m}$ network for varying values of n in the case $m \geq 64$: (a) Throughput, (b) Blocking probability.

5.2.3 Analysis of recirculating shift networks RS_n

For a recirculating shift network RS_n, if we assign R_n as the network generated from RS_n by removing one link from it, then we have the successful-transmission rate over a directed link e_{ij} connecting an input port i and an output port j as,

$$r_{ij} = \rho_{ij} \sum_{S \subseteq R_n} Q(S).$$

Under symmetric loading, we have

$$r(RS_n, \rho) = \rho \sum_{S \subseteq R_{n-1}} \beta^{-1}(RS_n, \rho)\rho^{|S|} = \rho\beta^{-1}(RS_n, \rho) \sum_{S \subseteq R_{n-1}} \rho^{|S|},$$

where $\beta(RS_n, \rho) = \sum_{S \subseteq RS_n} \rho^{|S|}$. Since $\beta(R_{n-1}, \rho) = \sum_{S \subseteq R_{n-1}} \rho^{|S|}$, then

$$\frac{r(RS_n, \rho)}{\rho} = \frac{\beta(R_{n-1}, \rho)}{\beta(RS_n, \rho)}. \tag{5.19}$$

The overall throughput of the system is

$$S = (2n) \cdot r(RS_n, \rho) = \frac{\frac{\partial}{\partial \rho}\beta(RS_n, \rho)}{\beta(RS_n, \rho)}.$$

So,

$$r(RS_n, \rho) = \frac{\rho}{2n} \frac{\frac{\partial}{\partial \rho}\beta(RS_n, \rho)}{\beta(RS_n, \rho)}, \tag{5.20}$$

and

$$\beta(R_{n-1}, \rho) = \frac{1}{2n} \cdot \frac{\partial}{\partial \rho}\beta(K_n, \rho).$$

By substituting the equations

$$\beta(RS_n, \rho) = (1 + 2\rho)\beta(RS_{n-1}, \rho) - \rho^2\beta(RS_{n-2}, \rho)$$

and

$$\beta(R_{n-1}, \rho) = \frac{1}{1+4\rho}\beta(RS_n, \rho) + \frac{\rho}{1+4\rho}\beta(RS_{n-1}, \rho)$$

into Equation (5.19), we have

$$\frac{r(RS_n, \rho)}{\rho} = \frac{\beta(R_{n-1}, \rho)}{\beta(RS_n, \rho)} = \frac{1}{1+4\rho}\left(1 + \rho\frac{\beta(RS_{n-1}, \rho)}{\beta(RS_n, \rho)}\right). \tag{5.21}$$

Further, since

$$\beta(RS_n,\rho) = (1+2\rho)\beta(RS_{n-1},\rho) - \rho^2\beta(RS_{n-2},\rho),$$

we have

$$\frac{\beta(RS_n,\rho)}{\beta(RS_{n-1},\rho)} = (1+2\rho) - \rho^2\frac{\beta(RS_{n-2},\rho)}{\beta(RS_{n-1},\rho)}. \tag{5.22}$$

From Equation (5.21)

$$r(RS_{n-1},\rho) = \frac{\rho}{1+4\rho} + \frac{\rho^2}{1+4\rho}\frac{\beta(RS_{n-2},\rho)}{\beta(RS_{n-1},\rho)}.$$

Rearranging this expression as

$$\rho^2\frac{\beta(RS_{n-2},\rho)}{\beta(RS_{n-1},\rho)} = (1+4\rho)r(RS_{n-1},\rho) - \rho,$$

and substituting it into Equation (5.22) yields

$$\frac{\beta(RS_n,\rho)}{\beta(RS_{n-1},\rho)} = (1+3\rho) - (1+4\rho)r(RS_{n-1},\rho).$$

Substituting this into Equation (5.21) leads to

$$r(RS_n,\rho) = \frac{\rho}{1+4\rho}\left(1 + \frac{\rho}{(1+3\rho)-(1+4\rho)r(RS_{n-1},\rho)}\right).$$

Therefore,

$$r(RS_n,\rho) = \frac{\rho(1-r(RS_{n-1},\rho))}{(1+3\rho)-(1+4\rho)r(RS_{n-1},\rho)}; \tag{5.23}$$

$$Pb(RS_n,\rho) = \frac{\rho(3-4r(RS_{n-1},\rho))}{(1+3\rho)-(1+4\rho)r(RS_{n-1},\rho)}, \tag{5.24}$$

with

$$r(RS_2,\rho) = \frac{\rho(1+\rho)}{1+4\rho+2\rho^2}; \qquad Pb(RS_2,\rho) = \frac{\rho(3+2\rho)}{1+4\rho+2\rho^2}.$$

The performance for recirculating shift networks RS_n of different sizes is shown in Figure 5.6.

In an RS_n network, every transmission over a link from input port i can be blocked by two transmissions at the input port and one transmission at the output port. Under this situation, the size of the network also has some effects on the activation of links. As the network size increases, the probability of the transmission being blocked by other transmissions also increases slightly, and the per-link throughput decreases with the increasing network size accordingly. However, the throughput will approach $\frac{1}{2}$ independent of the size of the switch.

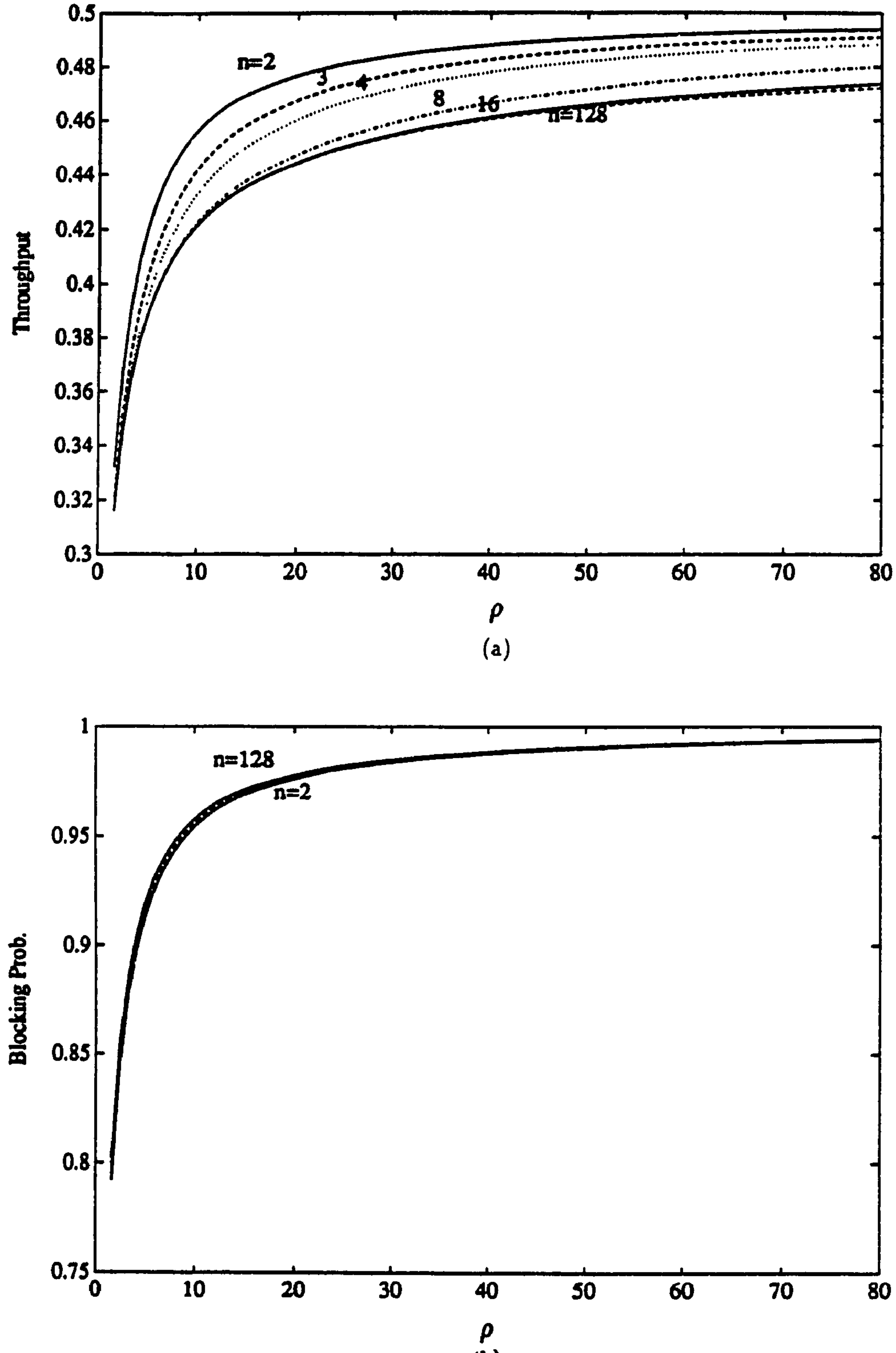


Figure 5.6: Performance for the RS_n network for varying values of n: (a) Throughput, (b) Blocking probability.

5.3 Analysis of Switches Under Asymmetric Loading Conditions

As we have demonstrated, the partition functions for networks with regular topology and symmetric message arrival rates can provide us with convenient recursive expressions to calculate the performance for large networks. However, for systems with asymmetric message arrival rates, the analysis becomes complicated. In general, the iterative-estimation method can be used to perform this task. However, recursive forms for the performance measures can be derived for asymmetrically-loaded switches if the load imbalance is localized to a single port or a group of ports. Such perturbations from symmetric loading can be handled easily with the blocking model. We will examine some of the possibilities in this section. The technique we employed is based on Equations (3.17) and (3.18).

The set of input ports $\{I_1, I_2, \cdots, I_n\}$ of a $K_{n,m}$ switch is assumed to be partitioned into N port groups. Each group may be allocated to serve a particular traffic type. The ports within each group are assumed to be uniformly loaded but different port groups may have different loads. Let p_i denote the number of input ports in group i and let ρ_i denote the load on any link originating at an input port belonging to group i. Then $\sum_{i=1}^{N} p_i = n$ and the partition function of the switch is denoted by $\beta_{p_1,p_2,\cdots,p_N}(K_{n,m}, \rho_1, \rho_2, \cdots, \rho_N)$. Further, $r^i_{p_1,p_2,\cdots,p_N}(K_{n,m}, \rho_1, \rho_2, \cdots, \rho_N)$. and $Pb^i_{p_1,p_2,\cdots,p_N}(K_{n,m}, \rho_1, \rho_2, \cdots, \rho_N)$ are the per-link throughput and blocking probability over any link belonging to group i, respectively.

We consider several situations separately.

5.3.1 Perturbing the load at a single input port on the symmetrically-loaded crossbar $K_{n,m}$

Consider a crossbar switch $K_{n,m}$ with n input ports $\{I_1, I_2, \cdots, I_n\}$ and m output ports $\{O_1, O_2, \cdots, O_n\}$. We assume $n-1$ input ports are symmetrically-loaded with load ρ_1 and the remaining port carries a load of ρ_2. By using Equations (3.17) and

(3.18), we derive the following recursive expressions for the partition function:

$$\beta(K_{n,m},\rho_1) = \beta(K_{n-1,m},\rho_1) + m\rho_1\beta(K_{n-1,m-1},\rho_1); \tag{5.25}$$

$$\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2) = \beta(K_{n-1,m},\rho_1) + m\rho_2\beta(K_{n-1,m-1},\rho_1); \tag{5.26}$$

$$\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2) = \beta(K_{n,m},\rho_1) + m(\rho_2-\rho_1)\beta(K_{n-1,m-1},\rho_1); \tag{5.27}$$

$$\begin{aligned}\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2) &= \beta_{n-2,1}(K_{n-1,m},\rho_1,\rho_2) \\ &\quad + m\rho_1\beta_{n-2,1}(K_{n-1,m-1},\rho_1,\rho_2).\end{aligned} \tag{5.28}$$

Then,

$$\begin{aligned}\frac{r^2_{n-1,1}(K_{n,m},\rho_2,\rho_2)}{\rho_2} &= \frac{\beta(K_{n-1,m-1},\rho_1)}{\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2)} \\ &= \frac{\beta(K_{n-1,m-1},\rho_1)}{\beta(K_{n,m},\rho_1)+m(\rho_2-\rho_1)\beta(K_{n-1,m-1},\rho_1)}.\end{aligned}$$

Therefore,

$$\frac{r^2_{n-1,1}(K_{n,m},\rho_1,\rho_2)}{\rho_2} = \frac{r(K_{n,m},\rho_1)}{\rho_1+m(\rho_2-\rho_1)r(K_{n,m},\rho_1)}.$$

Similarly, we define

$$\frac{r^1_{n-1,1}(K_{n,m},\rho_1,\rho_2)}{\rho_1} = \frac{\beta_{n-2,1}(K_{n-1,m-1},\rho_1,\rho_2)}{\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2)}.$$

Combining Equations (3.33) and (5.27) leads to

$$\begin{aligned}&\beta_{n-1,1}(K_{n,m},\rho_1,\rho_2) = \\ &\quad (1+m\rho_2+(n-1)\rho_1)\beta(K_{n-1,m-1},\rho_1) - (n-1)(m-1)\rho_1^2\beta(K_{n-2,m-2},\rho_1).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{r^1_{n-1,1}(K_{n,m},\rho_1,\rho_2)}{\rho_1} \\ &= \frac{\beta(K_{n-1,m-1},\rho_1)+(m-1)(\rho_2-\rho_1)\beta(K_{n-2,m-2},\rho_1)}{[1+m\rho_2+(n-1)\rho_1]\beta(K_{n-1,m-1},\rho_1)-(n-1)(m-1)\rho_1^2\beta(K_{n-2,m-2},\rho_1)} \\ &= \frac{\rho_1+(m-1)(\rho_2-\rho_1)r(K_{n-1,m-1},\rho_1)}{\rho_1[1+m\rho_2+(n-1)\rho_1]-(n-1)(m-1)\rho_1^2 r(K_{n-1,m-1},\rho_1)}.\end{aligned}$$

Therefore,

$$r^1_{n-1,1}(K_{n,m},\rho_1,\rho_2)$$
$$= \frac{\rho_1+(m-1)(\rho_2-\rho_1)r(K_{n-1,m-1},\rho_1)}{\rho_1[1+m\rho_2+(n-1)\rho_1]-(n-1)(m-1)\rho_1^2 r(K_{n-1,m-1},\rho_1)}; \tag{5.29}$$

$$Pb^1_{n-1,1}(K_{n,m},\rho_1,\rho_2)$$
$$= \frac{\rho_1[m\rho_2+(n-1)\rho_1]-(m-1)[(n-1)\rho_1^2+(\rho_2-\rho_1)]r(K_{n-1,m-1},\rho_1)}{\rho_1[1+m\rho_2+(n-1)\rho_1]-(n-1)(m-1)\rho_1^2 r(K_{n-1,m-1},\rho_1)} \tag{5.30}$$

and

$$r^2_{n-1,1}(K_{n,m},\rho_1,\rho_2) = \frac{\rho_2 r(K_{n,m},\rho_1)}{\rho_1+m(\rho_2-\rho_1)r(K_{n,m},\rho_1)}; \tag{5.31}$$
$$Pb^2_{n-1,1}(K_{n,m},\rho_1,\rho_2) = \frac{\rho_1+[m(\rho_2-\rho_1)-1]r(K_{n,m},\rho_1)}{\rho_1+m(\rho_2-\rho_1)r(K_{n,m},\rho_1)}, \tag{5.32}$$

with $r(K_{n,1},\rho_1)=\frac{\rho_1}{1+n\rho_1}$.

Using these expressions, we study the switch $K_{n,m}$ for links originating from ports in group 1 and group 2. We fix ρ_1 and allow ρ_2 to change. The throughput and blocking probability for the $K_{2,2}$ network are shown in Figure 5.7 and Figure 5.8, respectively. Figure 5.7(a) and Figure 5.8(a) presents performance results for a link in group 1, while Figure 5.7(b) and Figure 5.8(b) show performance results for a link in group 2. The ratio of blocking probabilities over links with ρ_1 and ρ_2 transmission attempt rates is shown in Figure 5.9.

To describe the performance of this kind of switch network, we first make the following observations: (1) The activation of a link in group 1 can be blocked by m links originated from the same input port and $(n-1)$ links destinated to the same output port(where $n-2$ links belong to group 1 and another link belongs group 2). Hence, it can be blocked by $(m+n-2)$ group 1 links and one group 2 link. (2) The activation of a group 2 link is blocked by m group 2 links at its input port and $(n-1)$ group 1 links at the output port.

For a $K_{2,2}$ network, which is depicted in Figure 5.10, the blocking of a transmission over link l_1 (or l_2) can be caused by two transmissions over links from I_1 (l_1 and l_2), and one transmission from I_2, (l_3 or l_4). We observe that the blocking occurring at the

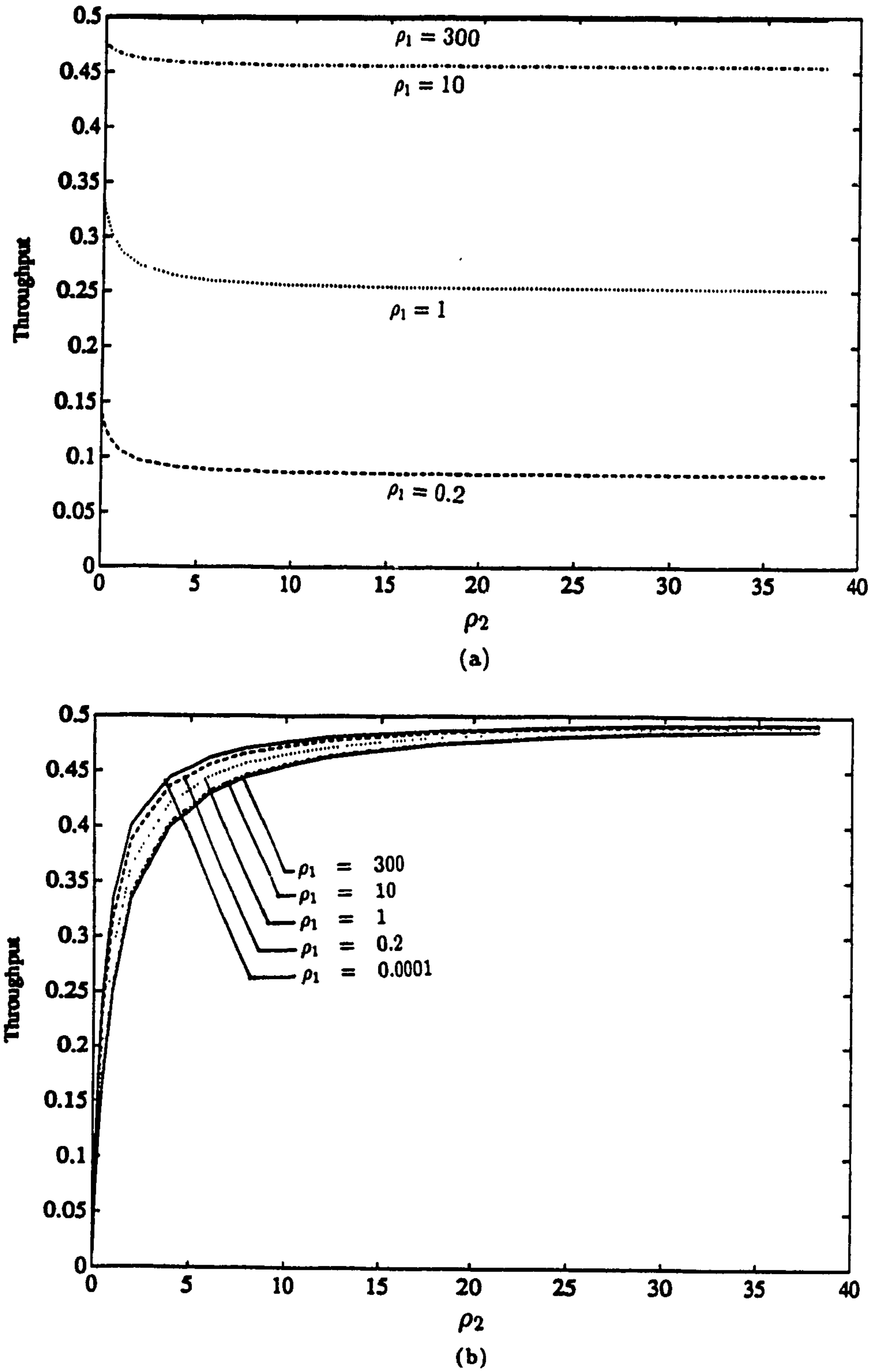


Figure 5.7: Throughput Analysis of a $K_{2,2}$ network: (a) for links in group 1, (b) for links in group 2.

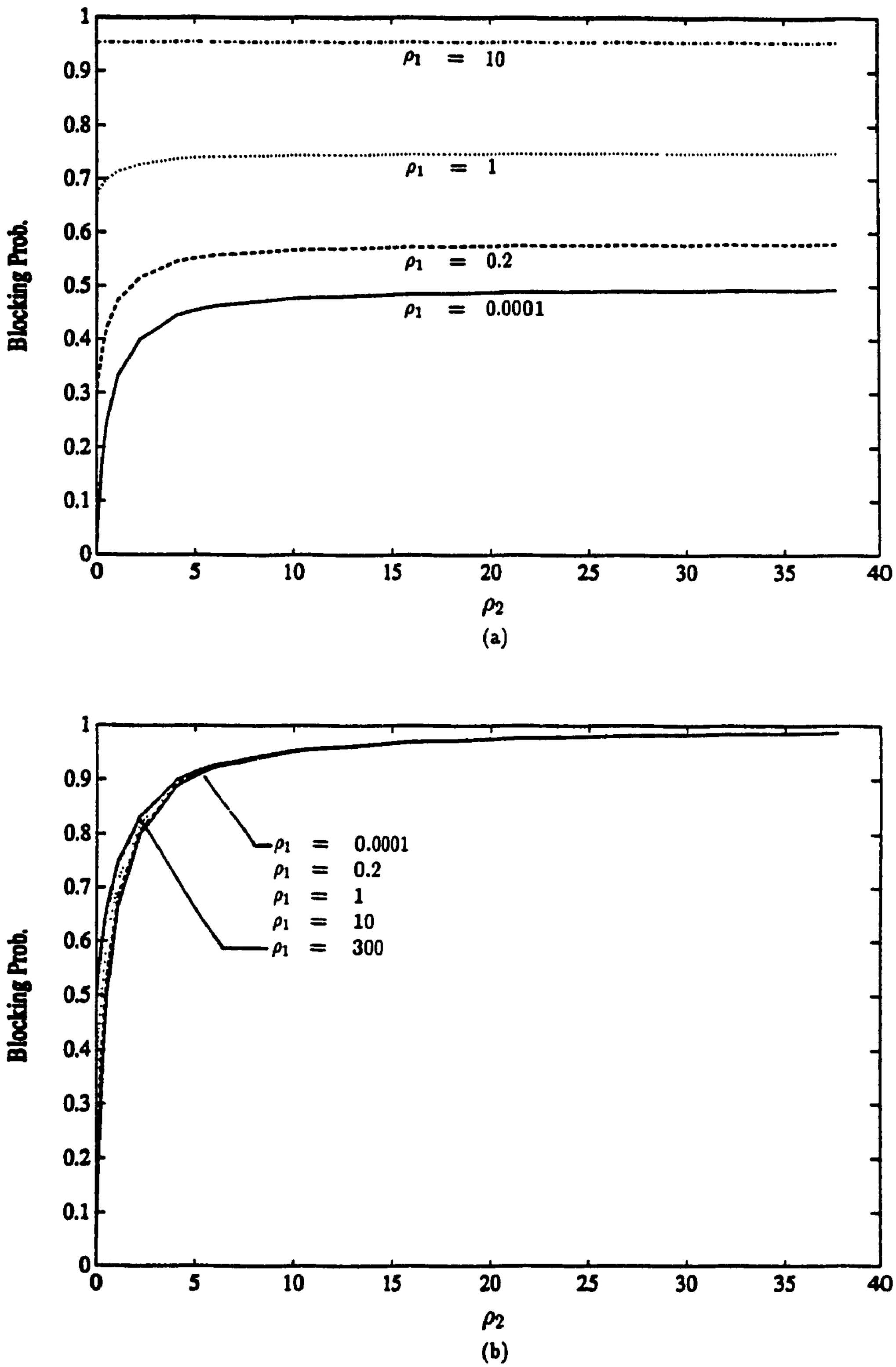


Figure 5.8: Blocking Probability of a $K_{2,2}$ network: (a) for links in group 1, (b) for links in group 2.

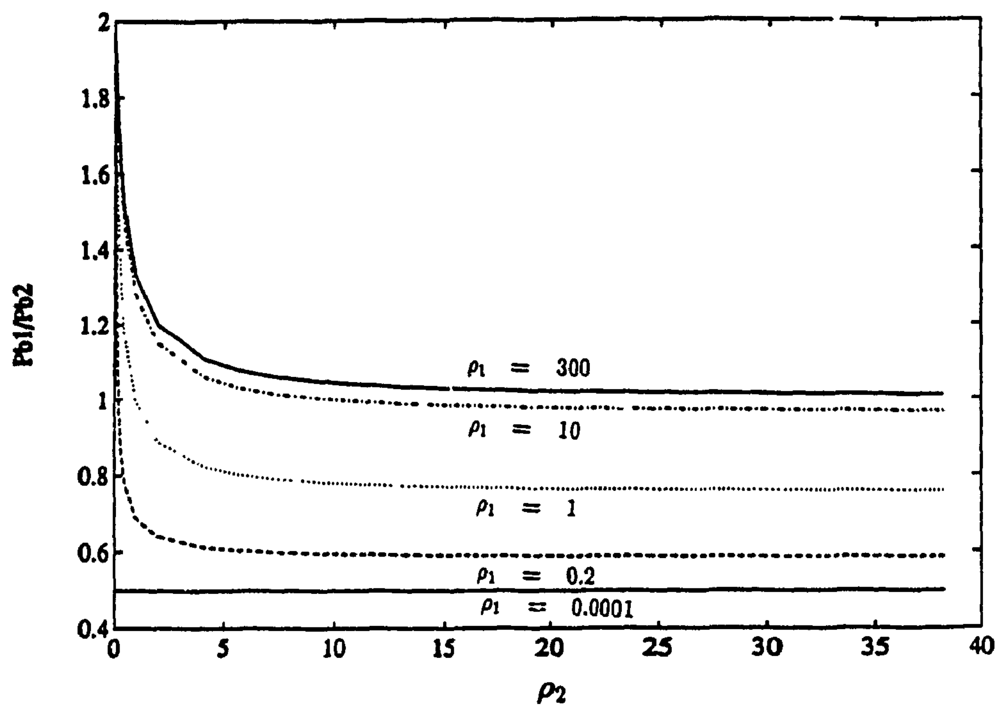


Figure 5.9: Comparision of blocking probability over links in group 1 and group 2 of a $K_{2,2}$ network.

input port has a more pronounced effect on system performance than that happening at the output port. Scheduled transmissions over any two links originating from the same input port block each other. The scheduled transmission over l_1 will be blocked at the input port I_1 if either l_1 or l_2 is active. The blocking of transmissions over l_1 at the output port O_1 can be caused by transmissions over l_3 only, and transmission over l_3 is only possible when l_4 is idle.

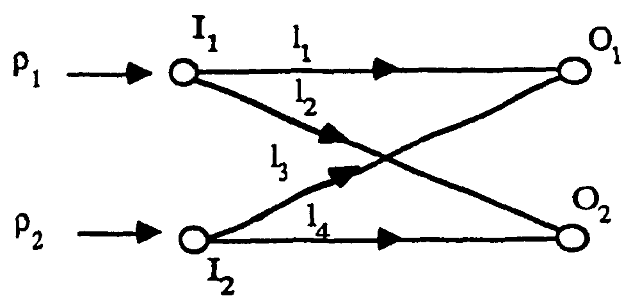


Figure 5.10: Illustration of a $K_{2,2}$ network

Therefore, when ρ_1 is large, it will cause $Pb^1(K_{n,m}, \rho_1, \rho_2)$ to reach 1 almost instantaneously no matter what the ρ_2 value is. However when ρ_1 is small, ρ_2 will have a strong influence over $Pb^1(K_{n,m}, \rho_1, \rho_2)$. Basically, $Pb^1(K_{n,m}, \rho_1, \rho_2)$ increases with increasing ρ_2, but it will never reach unity. $Pb^2(K_{n,m}, \rho_1, \rho_2)$ heavily depends on ρ_2, but it is relatively independent of ρ_1. It approaches unity when ρ_2 is large. Correspondingly, $r^1(K_{n,m}, \rho_1, \rho_2)$ is a decreasing function of ρ_2. However, it is insensitive to ρ_2 when ρ_2 is large. This is because as ρ_2 increases, there will be more unblocked transmissions over group 2 links which cause more blocking for transmissions over group 1 links. However, when ρ_2 is large enough, only a constant portion of attempted transmissions are not blocked, and thus block a certain number of transmissions from group 1 links. Furthermore, $r^2(K_{n,m}, \rho_1, \rho_2)$ is an increasing function of ρ_2, but a decreasing function of ρ_1. This is because the increment of ρ_1 causes more blocking at its output port, whereas an increment in ρ_2 increases its scheduled transmission rate, which can introduce more successful transmissions. Lastly, if we compare $Pb^1(K_{n,m}, \rho_1, \rho_2)$ and $Pb^2(K_{n,m}, \rho_1, \rho_2)$ as presented in Figure 5.9, we see that when $\rho_2 \leq \rho_1$, links from I_1 will have larger scheduled transmission rates than links from I_2, thus, transmissions over l_1 (or l_2) have more chance of being blocked than the transmissions over l_3 or l_4. Thus, $Pb^1(K_{n,m}, \rho_1, \rho_2) \geq Pb^2(K_{n,m}, \rho_1, \rho_2)$. On the other hand, when $\rho_2 \geq \rho_1$, $Pb^1(K_{n,m}, \rho_1, \rho_2) \leq Pb^2(K_{n,m}, \rho_1, \rho_2)$.

Similar studies are undertaken to observe the size effect of this kind of network, where two different situations are investigated: (1) $\rho_1 = 10$, (2) $\rho_1 = 0.2$. Under these two cases, the system performance behaviors are very much the same, therefore, we do not include every result for case (2). Figure 5.11, 5.12, and 5.13 shows, respectively, the throughput, blocking probability, and the blocking probability ratio $Pb^1(K_{n,m}, \rho_1, \rho_2)/Pb^2(K_{n,m}, \rho_1, \rho_2)$, for the case $\rho_1 = 10$. Since $Pb^1(K_{n,m}, \rho_1, \rho_2)$ is proportional to $(m + n - 2)\rho_1 + \rho_2$ and $Pb^2(K_{n,m}, \rho_1, \rho_2)$ is proportional to $(n - 1)\rho_1 + m\rho_2$, as the switch size increases, both $Pb^1(K_{n,m}, \rho_1, \rho_2)$ and $Pb^2(K_{n,m}, \rho_1, \rho_2)$ increase correspondingly. This causes their respective throughputs to decrease. Since ρ_1 is fixed, the increment in ρ_2 only causes more blocking at the output ports. Thus, $Pb^1(K_{n,m}, \rho_1, \rho_2)$ is an increasing function of ρ_2 while

$r^1(K_{n,m}, \rho_1, \rho_2)$ is a decreasing function of ρ_2. However, as ρ_2 increases, the chances of this attempted transmission being blocked also increases. On the other hand, it introduces more successful transmissions over group 2 links. Figure 5.11 and Figure 5.12 demonstrate these results.

The blocking probability ratio $Pb^1(K_{n,n}, \rho_1, \rho_2)/Pb^2(K_{n,n}, \rho_1, \rho_2)$ is shown in Figure 5.13 and Figure 5.14 for $\rho_1 = 10$ and $\rho_1 = 0.2$, respectively. We observe that the point $\rho_2 = \rho_1$ provides the boundary for the behavior of the relative blocking probabilities. More specifically, when $\rho_2 \leq \rho_1$, $Pb^1(K_{n,n}, \rho_1, \rho_2) \geq Pb^2(K_{n,n}, \rho_1, \rho_2)$, when $\rho_2 \geq \rho_1$, $Pb^1(K_{n,n}, \rho_1, \rho_2) \leq Pb^2(K_{n,n}, \rho_1, \rho_2)$. Furthermore, we observe that as the switch size increases, this blocking probability becomes insensitive to the transmission attempt rate over group 2 links, and the blocking probability over both group 1 and group 2 links becomes closer.

5.3.2 Performance measures for switches with arbitrary number of input-port groups

In this subsection, we intend to develop a general recursive formula for performance measures of a switch with an arbitrary number of input-port groups. We will develop expressions for some basic cases first.

(a) Switches partitioned in two input-port groups

In this case, p_1 and p_2 are any two positive integer numbers satisfying $p_1 + p_2 = n$. By applying Equation (3.18) on the group 1 input ports of the switch, we have

$$\beta_{p_1,p_2}(K_{n,m}, \rho_1, \rho_2) = \beta_{p_1-1,p_2}(K_{n-1,m}, \rho_1, \rho_2) + m\rho_1\beta_{p_1-1,p_2}(K_{n-1,m-1}, \rho_1, \rho_2). \quad (5.33)$$

Therefore, we can write

$$\frac{r^1_{p_1,p_2}(K_{n,m}, \rho_1, \rho_2)}{\rho_1} = \frac{\beta_{p_1-1,p_2}(K_{n-1,m-1}, \rho_1, \rho_2)}{\beta_{p_1,p_2}(K_{n,m}, \rho_1, \rho_2)} = \frac{1}{m\rho_1 + \frac{\beta_{p_1-1,p_2}(K_{n-1,m}, \rho_1, \rho_2)}{\beta_{p_1-1,p_2}(K_{n-1,m-1}, \rho_1, \rho_2)}}.$$

Furthermore, by using Equation (3.18) on one of its output ports, we have

$$\begin{aligned}\beta_{p_1-1,p_2}(K_{n-1,m}, \rho_1, \rho_2) &= \beta_{p_1-1,p_2}(K_{n-1,m-1}, \rho_1, \rho_2) \\ &\quad + (p_1 - 1)\rho_1\beta_{p_1-2,p_2}(K_{n-2,m-1}, \rho_1, \rho_2) \\ &\quad + p_2\rho_2\beta_{p_1-1,p_2-1}(K_{n-2,m-1}, \rho_1, \rho_2).\end{aligned}$$

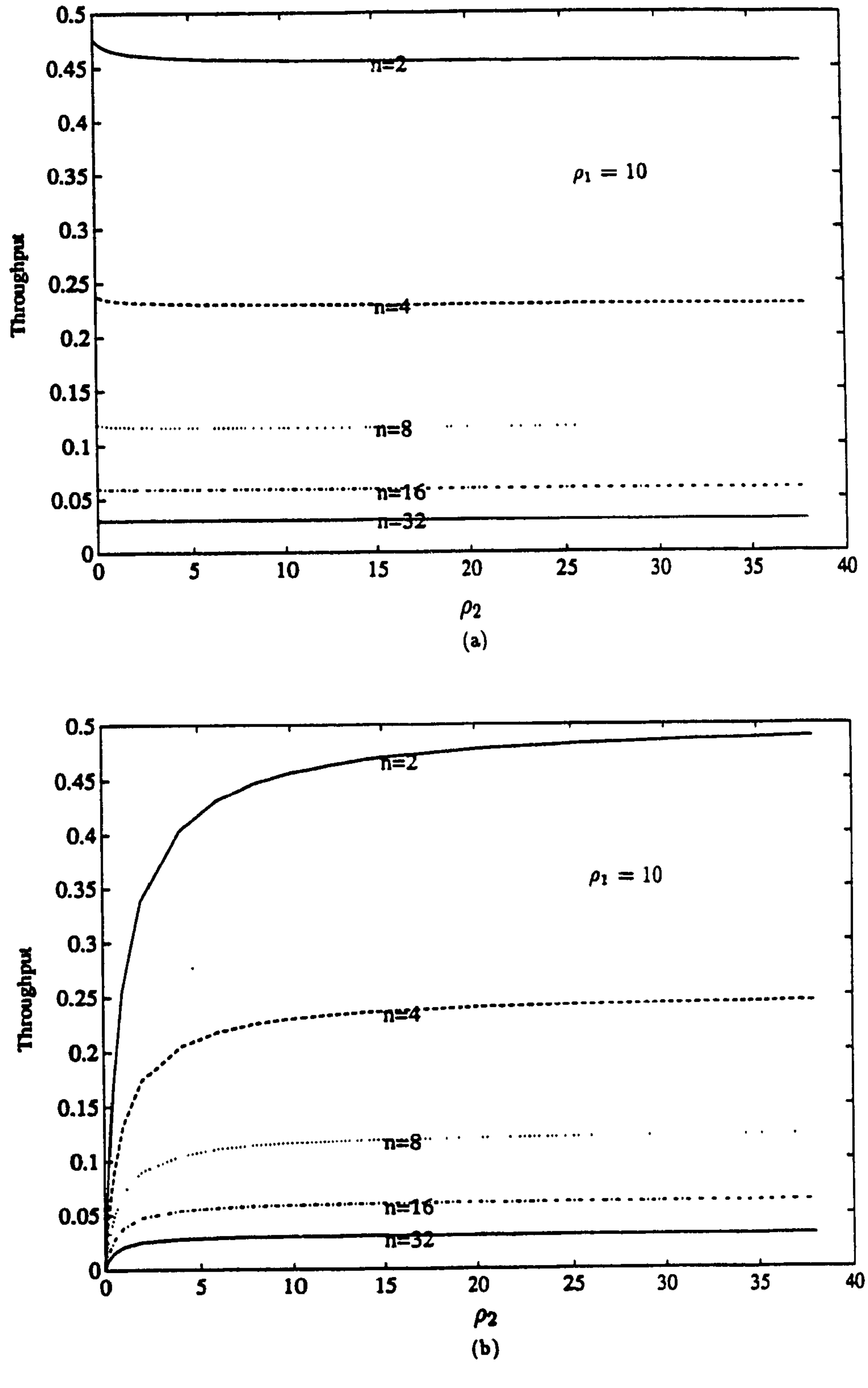


Figure 5.11: The throughput of the switch with $\rho_1 = 10$: (a) for links in group 1 and (b) for links in group 2, for various values of n.

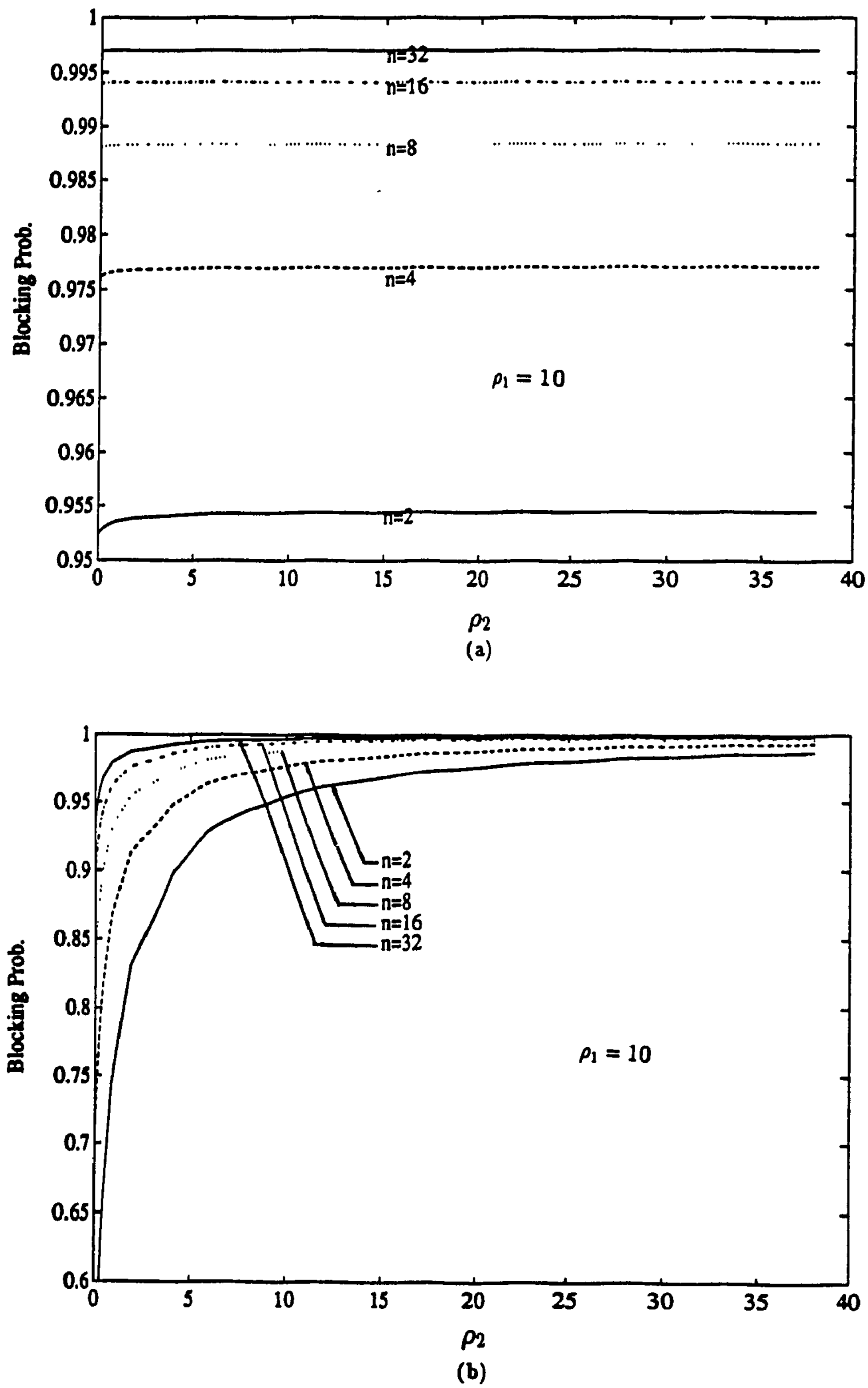


Figure 5.12: The blocking probability of the switch with $\rho_1 = 10$: (a) for links in group 1 and (b) for links in group 2, for various values on n.

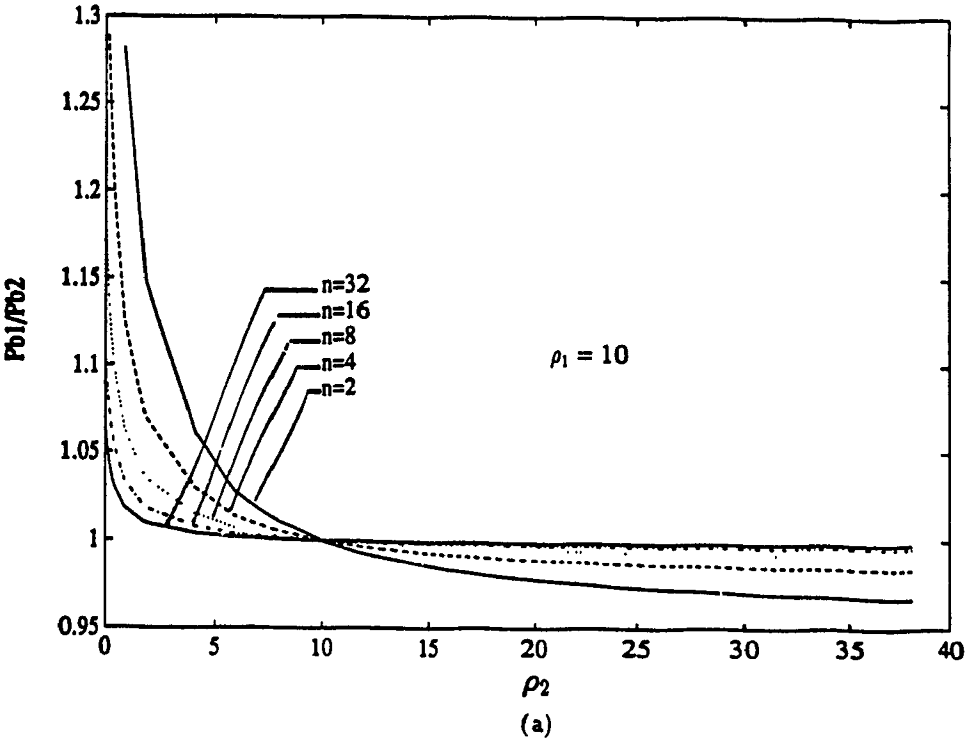


Figure 5.13: The switch size effects on blocking probability for the case $\rho_1 = 10$.

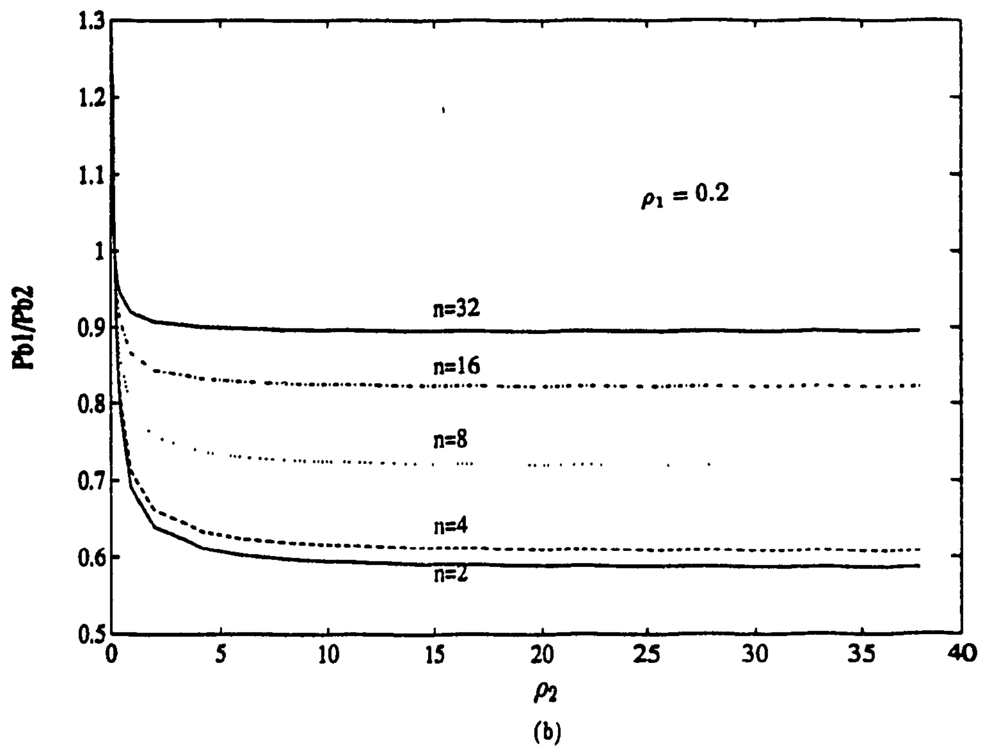


Figure 5.14: The switch size effects on blocking probability for the case $\rho_1 = 0.2$.

Hence,

$$\frac{r^1_{p_1,p_2}(K_{n,m},\rho_1,\rho_2)}{\rho_1} = \tag{5.34}$$
$$\frac{1}{1+m\rho_1+(p_1-1)\rho_1\frac{\beta_{p_1-2,p_2}(K_{n-2,m-1},\rho_1,\rho_2)}{\beta_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2)}+p_2\rho_2\frac{\beta_{p_1-1,p_2-1}(K_{n-2,m-1},\rho_1,\rho_2)}{\beta_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2)}}.$$

Further, applying Equation (3.18) on group 1 and group 2 input ports, respectively, yields

$$\begin{aligned}\beta_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2) &= \beta_{p_1-2,p_2}(K_{n-2,m-1},\rho_1,\rho_2)\\ &\quad+(m-1)\rho_1\beta_{p_1-2,p_2}(K_{n-2,m-2},\rho_1,\rho_2);\\ \beta_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2) &= \beta_{p_1-1,p_2-1}(K_{n-2,m-1},\rho_1,\rho_2)\\ &\quad+(m-1)\rho_2\beta_{p_1-1,p_2-1}(K_{n-2,m-2},\rho_1,\rho_2).\end{aligned}$$

Finally, substituting these expressions back to Equation (5.34), and letting

$$\begin{aligned}S_2(p_1,1) &= [1-(m-1)r^1_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2)];\\ S_2(p_1,2) &= [1-(m-1)r^2_{p_1-1,p_2}(K_{n-1,m-1},\rho_1,\rho_2)],\end{aligned}$$

we obtain

$$\frac{r^1_{p_1,p_2}(K_{n,m},\rho_1,\rho_2)}{\rho_1} = \frac{1}{1+m\rho_1+(p_1-1)\rho_1S_2(p_1,1)+p_2\rho_2S_2(p_1,2)}. \tag{5.35}$$

Similarly, by using the same method of development, and letting

$$\begin{aligned}S_2(p_2,1) &= [1-(m-1)r^1_{p_1,p_2-1}(K_{n-1,m-1},\rho_1,\rho_2)];\\ S_2(p_2,2) &= [1-(m-1)r^2_{p_1,p_2-1}(K_{n-1,m-1},\rho_1,\rho_2)],\end{aligned}$$

we have

$$\frac{r^2_{p_1,p_2}(K_{n,m},\rho_1,\rho_2)}{\rho_2} = \frac{1}{1+m\rho_2+p_1\rho_1S_2(p_2,1)+(p_2-1)\rho_2S_2(p_2,2)}. \tag{5.36}$$

(b) Switches partitioned into three input-port groups.

For the switches with three group of input ports, $p_1 + p_2 + p_3 = n$, Following the same method, we can generate similar results. Let

$$\begin{aligned}
S_3(p_1,1) &= [1-(m-1)r^1_{p_1-1,p_2,p_3}(K_{n-1,m-1})];\\
S_3(p_1,2) &= [1-(m-1)r^2_{p_1-1,p_2,p_3}(K_{n-1,m-1})];\\
S_3(p_1,3) &= [1-(m-1)r^3_{p_1-1,p_2,p_3}(K_{n-1,m-1})];\\
S_3(p_2,1) &= [1-(m-1)r^1_{p_1,p_2-1,p_3}(K_{n-1,m-1})];\\
S_3(p_2,2) &= [1-(m-1)r^2_{p_1,p_2-1,p_3}(K_{n-1,m-1})];\\
S_3(p_2,3) &= [1-(m-1)r^3_{p_1,p_2-1,p_3}(K_{n-1,m-1})];\\
S_3(p_3,1) &= [1-(m-1)r^1_{p_1,p_2,p_3-1}(K_{n-1,m-1})];\\
S_3(p_3,2) &= [1-(m-1)r^2_{p_1,p_2,p_3-1}(K_{n-1,m-1})];\\
S_3(p_3,3) &= [1-(m-1)r^3_{p_1,p_2,p_3-1}(K_{n-1,m-1})].
\end{aligned}$$

We have,

$$\frac{r^1_{p_1,p_2,p_3}(K_{n,m},\rho_1,\rho_2,\rho_3)}{\rho_1} = \frac{1}{1+m\rho_1+(p_1-1)\rho_1 S_3(p_1,1)+p_2\rho_2 S_3(p_1,2)+p_3\rho_3 S_3(p_1,3)}; \tag{5.37}$$

$$\frac{r^2_{p_1,p_2,p_3}(K_{n,m},\rho_1,\rho_2,\rho_3)}{\rho_2} = \frac{1}{1+m\rho_2+p_1\rho_1 S_3(p_2,1)+(p_2-1)\rho_2 S_3(p_2,2)+p_3\rho_3 S_3(p_2,3)}; \tag{5.38}$$

$$\frac{r^3_{p_1,p_2,p_3}(K_{n,m},\rho_1,\rho_2,\rho_3)}{\rho_3} = \frac{1}{1+m\rho_3+p_1\rho_1 S_3(p_3,1)+p_2\rho_2 S_3(p_3,2)+(p_3-1)\rho_3 S_3(p_3,3)}. \tag{5.39}$$

(c) Switches with N input-port groups.

Similarly, for switches partitioned in N input-port groups, $p_1 + p_2 + \cdots, p_N = n$, the same kind of development leads to the following results:

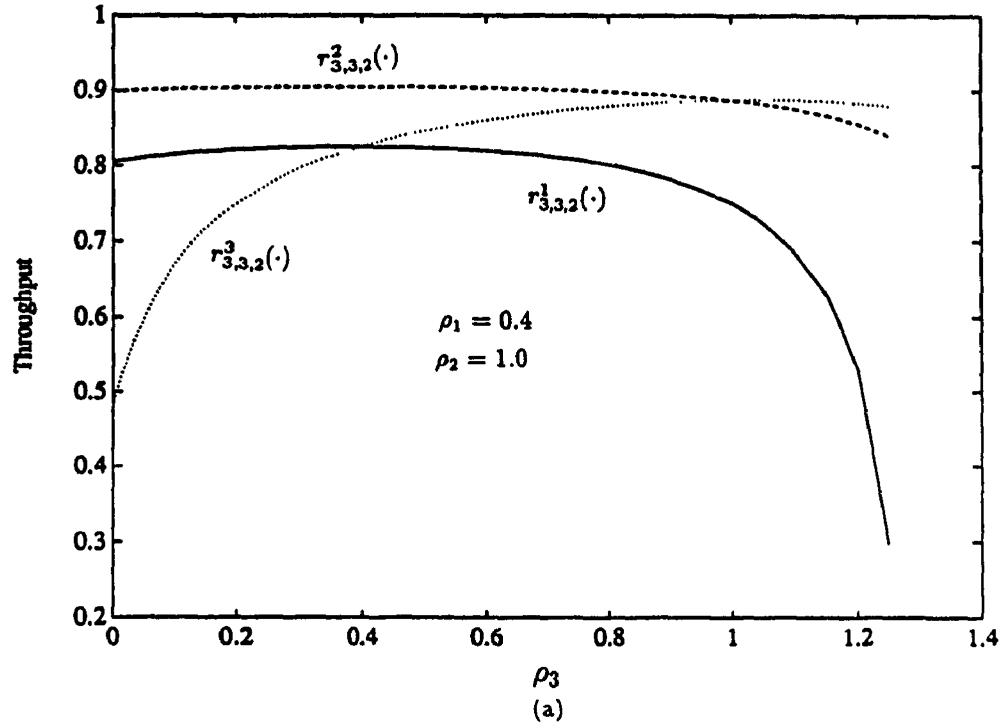
Defining the quantities $S_N(p_i, j)$, $1 \le i, j \le N$, as

$$S_N(p_i,j) = [1-(m-1)r^j_{p_1,p_2,\cdots,p_i-1,\cdots,p_N}(K_{n-1,m-1},\rho_1,\rho_2,\cdots,\rho_n)],$$

then

$$\frac{r^i_{p_1,p_2,\cdots,p_N}(K_{n,m},\rho_1,\rho_2,\cdots,\rho_N)}{\rho_i} = \frac{1}{1+m\rho_i+\sum_{\substack{j=1 \\ j\neq i}}^{N} p_j\rho_j S_N(p_i,j)+(p_i-1)\rho_i S_N(p_i,i)}. \tag{5.40}$$

To demonstrate, a switch $K_{8,8}$ with three input-port groups is considered. In this switch, $p_1 = 3$, $p_2 = 3$, $p_3 = 2$, and we let $\rho_1 = 0.4$, $\rho_2 = 1.0$, while ρ_3 changes. The performance of this switch is shown in Figure 5.15. In Figure 5.15(a), we present its throughput for links in groups 1, 2, and 3, respectively. Similarly, Figure 5.15(b) shows blocking probabilities for links in each group. It is interesting to see that the curve $r^3_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$ intersects with the curves $r^1_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$ and $r^2_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$, respectively at the operating point $\rho_3 = 0.4$ and $\rho_3 = 1.0$. However, the curve $Pb^3_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$ intersects with the curves $Pb^1_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$ at point $\rho_3 = 0.4$ also, but it does not intersect with the curve $Pb^2_{3,3,2}(K_{8,8},\rho_1,\rho_2,\rho_3)$ at the point $\rho_3 = 1.0$. The main emphasis of this example is to demonstrate the fact that we can use the developed technique to study switches under asymmetric loading. A detailed performance analysis of the switch is omitted.



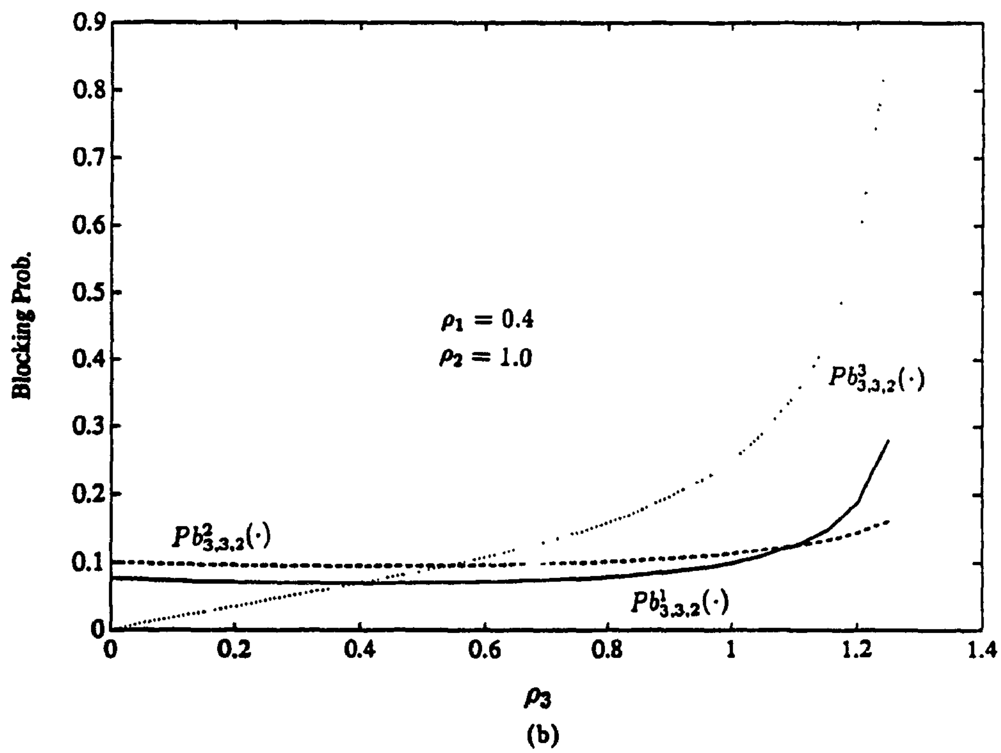


Figure 5.15: Performance measurements for a switch with three different message arrival rate values: (a) Throughput, (b) Blocking Probability.

Chapter 6

CONCLUSION

In this thesis, a blocking model is described and formalized for its applications in broadcast networks Under this model, the transmission and reception operations have been generally described by the TBG and the RIG. Therefore, system performance studies are basically processed by using information from the TBG and the RIG.

The blocking model can be extended or specialized for different applications. Typical extention is the relaxation of the service duration distribution so that it is not neccessary to be memoryless. On the other hand, the collision-free systems have been studied separately with the result that a much simplified procedure of obtaining its performance measures is possible.

The blocking model can be applied to the PRNs by using the iterative-estimation method. This computational method can be used in many different situations. However, it provides only the operating region of the throughput curve. Another disadvantage is that it consumes a large amount of memory and many CPU cycles. It is unable to deal with the networks of large size. However, for CSNs with regular topologies, the system performance can be calculated recursively. Therefore, the network size limitation imposed on the procedure, such as the case in PRNs, is less severe.

Extentions from this thesis are possible, we briefly introduce some of those possibilities.

- In this thesis, we consider the network systems with simplex data link or half-duplex data link. We can extent the transmission interaction model to deal with duplex data link operations. Under this case, the TBG and the RIG are generated according to the given system. The iterative-estimation method can still be directly applied for system performance measures.

- The recursive relations between partition functions are not only held for CSNs but also for PRNs with some regular topologies. For example, for the Cycle network with unidirectional transmission, which is the graphical model of the packet radio networek operating under D-CSMA, the partition function for this system is exactly the same as that of the TCAP, The recursive relation in its partition function, as well as in its throughput expression can be similarly generated.

- In Section 5.3, we analyzed the switches under asymmetric loading conditions. In that particular case, we partitioned the message arrival rates in the input ports of the switch into N groups, and we calculated the system performance measures recursively. Similarly, under some special applications, we can partition the message arrival rates at the output ports into N groups, and we use the same technique of development to recursively measure the system performance.

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Appendix A

Proof of Theorem 3.1

Proof. (a) Based on the $SP(\cdot)$ definition, we write

$$\begin{array}{lcl} SP(X) & = & \sum_{S\subseteq X}(\prod_{i\in S}\rho_i) \\ SP(Y) & = & \sum_{S\subseteq Y}(\prod_{i\in S}\rho_i) \\ SP(X\cup Y) & = & \sum_{S\subseteq X\cup Y}(\prod_{i\in S}\rho_i) \end{array}$$

For any two isolated subsets X and Y as depicted in the blocking graph of Figure A.1, it is true that

$$(S\subseteq X)\cup(S\subseteq Y) = S\subseteq(X\cup Y).$$

Thus, we have

$$\sum_{S\subseteq X\cup Y}(\prod_{i\in S}\rho_i) = \sum_{S\subseteq X}(\prod_{i\in S}\rho_i) + \sum_{S\subseteq Y}(\prod_{i\in S}\rho_i) + \sum_{\substack{S\subseteq(X\cup Y)\\ S\not\subseteq X\\ S\not\subseteq Y}} \prod_{i\in S}\rho_i.$$

Similarly,

$$S\subseteq(X\cup Y), S\not\subseteq X, S\not\subseteq Y \longrightarrow S = S_X\cup S_Y, S_X\subseteq X, S_Y\subseteq Y.$$

Hence

$$\begin{array}{lcl} \sum_{S\subseteq X\cup Y}(\prod_{i\in S}\rho_i) & = & \sum_{S\subseteq X}(\prod_{i\in S}\rho_i) + \sum_{S\subseteq Y}(\prod_{i\in S}\rho_i) + \sum_{S_X\subseteq X}\sum_{S_Y\subseteq Y}\prod_{\substack{i\in S_X\cup S_Y\\ S_X\neq\phi\\ S_Y\neq\phi}}\prod\rho_i \\ & = & \sum_{S\subseteq X}(\prod_{i\in S}\rho_i) + \sum_{S\subseteq Y}(\prod_{i\in S}\rho_i) + \sum_{\substack{S_X\subseteq X\\ S_X\neq\phi}}\sum_{\substack{S_Y\subseteq Y\\ S_Y\neq\phi}}\prod_{i\in S_Y}\prod_{i\in S_Y}\rho_i \\ & = & \sum_{S\subseteq X}(\prod_{i\in S}\rho_i) + \sum_{S\subseteq Y}(\prod_{i\in S}\rho_i) \end{array}$$

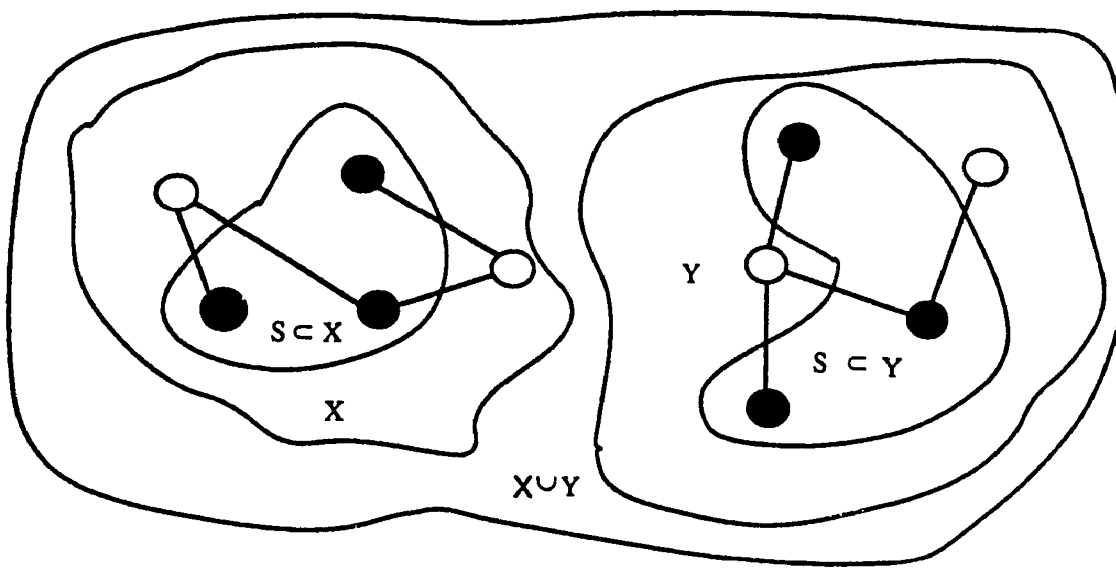

Figure A.1: Graph representation of two isolated sets

$$
\begin{aligned}
& +\Big(\sum_{\substack{S_X \subseteq X \\ S_X \neq \phi}} \textstyle\prod_{i \in S_X} \rho_i\Big)\Big(\sum_{\substack{S_Y \subseteq Y \\ S_Y \neq \phi}} \textstyle\prod_{i \in S_Y} \rho_i\Big) \\
= \; & \textstyle\sum_{S \subseteq X}(\prod_{i \in S} \rho_i) + \sum_{S \subseteq Y}(\prod_{i \in S} \rho_i) \\
& +\Big(\sum_{\substack{S \subseteq X \\ S \neq \phi}} \textstyle\prod_{i \in S} \rho_i\Big)\Big(\sum_{\substack{S \subseteq Y \\ S \neq \phi}} \textstyle\prod_{i \in S} \rho_i\Big) \\
= \; & \textstyle\sum_{S \subseteq X}(\prod_{i \in S} \rho_i) \sum_{S \subseteq Y}(\prod_{i \in S} \rho_i) \\
= \; & SP(X)SP(Y).
\end{aligned}
$$

(b) Equation (3.2) can be justified by the observation that each independent subset of X either contains element i and non of i's blocking neighbors or it does not contain element i. The elements containing i form $\rho_i SP(X-B(i))$ while the terms not containing i form $SP(X-i)$. Given an element i in an set X, $X = \{1, 2, \ldots, i, \ldots, n\}$, we denote $B^*(i)$ as the set of blocking for element i excluding element i. Since i blocks itself, then $B(i) = B^*(i) \cup \{i\}$ is the blocked set of element i. Consider a general blocking graph as depicted in Figure A.2. For a state $S \subseteq X$, we have:

$$\text{if } S \subseteq X - B(i), \text{ then, } S \cup \{i\} \subseteq X;$$

$$\text{if } i \in S, \text{ then } S \subseteq X - B^*(i);$$

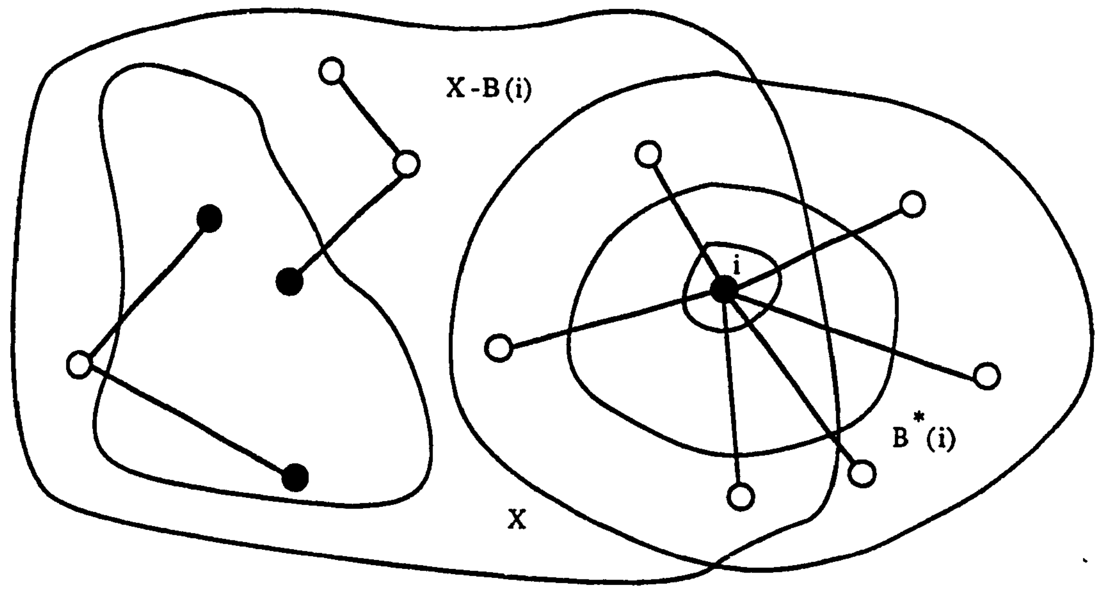

Figure A.2: Graph representation for the blocking set of element i

$$\text{if } i \notin S, \text{ then } S \subseteq X - \{i\}.$$

Therefore,

$$\textstyle\sum_{S \subseteq X}(\prod_{i \in S} \rho_i) = \sum_{S \subseteq X - \{i\}}(\prod_{i \in S} \rho_i) + \rho_i(\sum_{S \subseteq X - B(i)}(\prod_{i \in S} \rho_i)),$$

or

$$SP(A) = SP(X - \{i\}) + \rho_i SP(X - B(i)). \qquad q.e.d.$$

Appendix B

Proof of Theorem 3.2

Proof: We observe the fact that the $SP(V)$ expression can be separated into two components: (1) terms contain ρ_i and, (2) terms not containing ρ_i. The elements ρ_j, $j \neq i$ in the terms containing ρ_i must belong to the unblocked set $V - B(i)$. Thus, we can write

$$SP(V) = \rho_i \sum_{S \subseteq V - B(i)} (\prod_{j \in S} \rho_j) + \sum_{\substack{S \subseteq V \\ j \neq i}} (\prod_{j \in S} \rho_j).$$

Differentiating with respect to ρ_i, we obtain

$$\frac{\partial}{\partial \rho_i} SP(V) = \sum_{S \subseteq V - B(i)} (\prod_{j \in S} \rho_j) = SP(V - B(i)). \qquad q.e.d.$$

Appendix C

Proof of Theorem 3.3

Proof: Using $P(\tilde{G},k) = P(\tilde{G}-e,k) + P(\tilde{G}-v-w,k-1)$, we have

$$\begin{aligned}
\alpha(\tilde{G},\tilde{\rho}) &= \sum_{k=0}^{\lfloor n/2 \rfloor} P(\tilde{G},k)\tilde{\rho}^k \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (P(\tilde{G}-e,k) + P((\tilde{G}-v-w,k-1))\tilde{\rho}^k \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} P(\tilde{G}-e,k)\tilde{\rho}^k + \tilde{\rho} \sum_{k=0}^{\lfloor n/2 \rfloor} P(\tilde{G}-v-w,k-1)\tilde{\rho}^{k-1},
\end{aligned}$$

where $P(0,-1) \triangleq 0$. So

$$\alpha(\tilde{G},\tilde{\rho}) = \alpha(\tilde{G}-e,\tilde{\rho}) + \tilde{\rho} \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} P(\tilde{G}-v-w,j)\tilde{\rho}^j,$$

or

$$\alpha(\tilde{G},\tilde{\rho}) = \alpha(\tilde{G}-e,\tilde{\rho}) + \tilde{\rho}\alpha(\tilde{G}-v-w,\tilde{\rho}). \qquad q.e.d. \tag{C.1}$$

Appendix D

Proof of Corollary 3.1

Proof: By applying Equation (C.1) to network $\tilde{G}$ repeately, we have

$$\begin{aligned}
\alpha(\tilde{G}) &= \alpha(\tilde{G}-e_1)+\tilde{\rho}\alpha(\tilde{G}-v-w_1) \\
&= \alpha(\tilde{G}-e_1-e_2)+\tilde{\rho}\alpha(\tilde{G}-e_1-v-w_2)+\tilde{\rho}\alpha(\tilde{G}-v-w_1) \\
&= \alpha(\tilde{G}-e_1-e_2)+\tilde{\rho}\alpha(\tilde{G}-v-w_2)+\tilde{\rho}\alpha(\tilde{G}-v-w_1) \\
&\vdots \\
&= \alpha(\tilde{G}-e_1-e_2-\cdots-e_d)+\tilde{\rho}\sum_{j=1}^{d}\alpha(\tilde{G}-v-w_j) \\
&= \alpha(\tilde{G}-v)+\tilde{\rho}\sum_{j=1}^{d}\alpha(\tilde{G}-v-w_j)
\end{aligned}$$

Therefore,

$$\alpha(\tilde{G}) = \alpha(H)+\tilde{\rho}\sum_{j=1}^{d}\alpha(H-w_j). \qquad q.e.d. \tag{D.1}$$

Appendix E

The Partition Functions of Some Regular Networks

The partition functions of some regular networks will be analyzed individually in the following sections.

E.1 The partition functions of cycle networks and path networks

Given a cycle network C_n and a path network P_n of n nodes, we have

$$\beta(P_1,\rho) = 1; \qquad \beta(P_2,\rho) = 1+\rho;$$

$$\beta(C_1,\rho) = 1; \qquad \beta(C_2,\rho) = 1+2\rho.$$

By applying Equation (3.17) to C_n, we obtain the basic relationship

$$\beta(C_n,\rho) = \beta(P_n,\rho) + \rho\beta(P_{n-2},\rho). \tag{E.1}$$

Similarly, by applying Equation (3.18) to C_n, we get

$$\beta(C_n,\rho) = \beta(P_{n-1},\rho) + 2\rho\beta(P_{n-2},\rho). \tag{E.2}$$

Also, applying Equation (3.17) to the end node of P_n, we express P_n in terms of P_{n-1} and P_{n-2} as

$$\beta(P_n,\rho) = \beta(P_{n-1},\rho) + \rho\beta(P_{n-2},\rho). \tag{E.3}$$

Multiplying Equation (E.3) by 2, and subtracting it from Equation (E.2), we have

$$\beta(C_n,\rho) = 2\beta(P_n,\rho) - \beta(P_{n-1},\rho). \tag{E.4}$$

From Equation (E.2), we obtain

$$\beta(C_{n+1},\rho) = \beta(P_n,\rho) + 2\rho\beta(P_{n-1},\rho). \tag{E.5}$$

If we multiply by 2ρ on each side of Equation (E.4),

$$2\rho\beta(C_n,\rho) = 4\rho\beta(P_n,\rho) - 2\rho\beta(P_{n-1},\rho).$$

and combine this expression with Equation (E.5), then

$$\beta(P_n,\rho) = \frac{1}{1+4\rho}\left(\beta(C_{n+1},\rho) + 2\rho\beta(C_n,\rho)\right). \tag{E.6}$$

Finally, using Equation (E.4),

$$\rho\beta(C_{n-1},\rho) = 2\rho\beta(P_{n-1},\rho) - \rho\beta(P_{n-2},\rho).$$

and subtracting from Equation (E.5), we express the partition function of a cycle network on n nodes in terms of the partition functions of cycle networks on $n-1$ and $n-2$ nodes, respectively, namely

$$\beta(C_n,\rho) = \beta(C_{n-1},\rho) + \rho\beta(C_{n-2},\rho). \tag{E.7}$$

E.2 The Partition functions of completely connected networks

Let us momentarily drop the explict notational dependency of $\beta(K_n,\rho)$ on ρ by writing simply $\beta(K_n)$. In applying Equation (3.21) to a completely-connected network K_n, we have

$$\beta(K_n) = \beta(K_{n-1}) + 2(n-1)\rho\beta(K_{n-2}). \tag{E.8}$$

Thus, we have $\beta(K_1) = 1$, $\beta(K_2) = 1 + 2\rho$.

Recursively applying Equation (E.8) yields

$$\begin{aligned}
\beta(K_n) &= \beta(K_{n-1}) + 2(n-1)\rho\beta(K_{n-2}) \\
&= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-3}) \\
&= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) + 4(n-2)(n-4)\rho^2\beta(K_{n-5}) \\
&= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) + 4(n-2)(n-4)\rho^2\beta(K_{n-6}) \\
&\quad + 8(n-2)(n-4)(n-6)\rho^3\beta(K_{n-7}).
\end{aligned}$$

In general, we have

$$\beta(K_n) = [1+2(n-1)\rho]\beta(K_{n-2}) + \sum_{k=1}^{(n-4)/2}\prod_{j=1}^{k}(n-2j)(2\rho)^k\beta(K_{n-2-2k}). \tag{E.9}$$

Similarly,

$$\beta(K_{n-2}) = [1+2(n-3)\rho]\beta(K_{n-4}) + \sum_{k=1}^{(n-6)/2}\prod_{j=1}^{k}(n-2-2j)(2\rho)^k\beta(K_{n-4-2k});$$

$$\begin{aligned}
\beta(K_n) &= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) \\
&\quad + \sum_{k=2}^{(n-4)/2}\prod_{j=1}^{k}(n-2j)(2\rho)^k\beta(K_{n-2-2k}).
\end{aligned}$$

Let $l = k-1$, $p = j-1$. Then

$$\begin{aligned}
\beta(K_n) &= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) \\
&\quad + \sum_{l=1}^{(n-6)/2}\prod_{p=0}^{l}(n-2-2p)(2\rho)^{l+1}\beta(K_{n-4-2l}) \\
&= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) \\
&\quad + 2(n-2)\rho\sum_{l=1}^{(n-6)/2}\prod_{p=1}^{l}(n-2-2p)(2\rho)^{l}\beta(K_{n-4-2l}) \\
&= [1+2(n-1)\rho]\beta(K_{n-2}) + 2(n-2)\rho\beta(K_{n-4}) \\
&\quad + 2(n-2)\rho[\beta(K_{n-2}) - 2(1+(n-3)\rho)\beta(K_{n-4})].
\end{aligned}$$

Therefore, the partition function for K_n is expressed in terms of that for K_{n-2} and K_{n-4} as

$$\beta(K_n) = [1+2(2n-3)\rho]\beta(K_{n-2}) - 4(n-2)(n-3)\rho^2\beta(K_{n-4}). \tag{E.10}$$

By the same method of development, we obtain the alternative form

$$\beta(K_n) = \beta(K_{n-1}) + \sum_{k=1}^{(n-2)/2} \prod_{j=1}^{k} (n-2j+1)(2\rho)^k \beta(K_{n-1-2k}); \tag{E.11}$$

$$\beta(K_n) = [1+2(n-1)\rho]\beta(K_{n-1}) - 4(n-1)(n-2)\rho^2\beta(K_{n-3}). \tag{E.12}$$

E.3 The Partition functions of crossbar switching networks

Dropping the explicit dependency of $\beta(K_{n,m},\rho)$ on ρ by writing $\beta(K_{n,n})$ and applying Equation (3.18), we have

$$\beta(K_{n,0}) = 1; \quad \beta(K_{n,1}) = 1 + n\rho;$$

$$\begin{aligned}
\beta(K_{n,2}) &= \beta(K_{n,1}) + n\rho\beta(K_{n-1,1}) \\
&= 1 + 2n\rho + n(n-1)\rho^2; \\
\beta(K_{n,3}) &= \beta(K_{n,2}) + n\rho\beta(K_{n-1,2}) \\
&= 1 + 3n\rho + 3n(n-1)\rho^2 + n(n-1)(n-2)\rho^3 \\
&= \sum_{k=0}^{3} \binom{3}{k} \frac{n!}{(n-k)!}\rho^k; \\
\beta(K_{n,4}) &= \beta(K_{n,3}) + n\rho\beta(K_{n-1,3}) \\
&= 1 + 4n\rho + 6n(n-1)\rho^2 + 4n(n-1)(n-2)\rho^3 \\
&\quad + n(n-1)(n-2)(n-3)\rho^4 \\
&= \sum_{k=0}^{4} \binom{4}{k} \frac{n!}{(n-k)!}\rho^k.
\end{aligned}$$

In general,

$$\beta(K_{n,m}) = \sum_{k=0}^{m} \binom{m}{k} \frac{n!}{(n-k)!}\rho^k. \tag{E.13}$$

By using the relation, $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$, we obtain

$$\beta(K_{n,m}) = \beta(K_{n,m-1}) + n\rho\beta(K_{n-1,m-1}); \tag{E.14}$$

$$\beta(K_{n,m}) = \frac{n!}{(n-m)!}\rho^m + \sum_{k=0}^{m-1} \frac{n!}{(n-k)!}\rho^k \beta(K_{n-k,m-k-1}).$$

Applying Equation (3.18) directly on the network $K_{n,m}$ leads to

$$
\begin{aligned}
\beta(K_{n,m}) &= (1+n\rho)\beta(K_{n-1,m-1}) + \sum_{k=2}^{m} \frac{(m-1)!}{(m-k)!}\rho^{(k-1)}\beta(K_{n-k,m-k}); \\
\beta(K_{n,m}) &= [1+(m+n-1)\rho]\beta(K_{n-1,m-1}) \\
&\quad - (n-1)(m-1)\rho^2\beta(K_{n-2,m-2}); \qquad \text{(E.15)} \\
\beta(K_{n,m}) &= (1+n\rho)\beta(K_{n,m-1}) - n(m-1)\rho^2\beta(K_{n-1,m-2}). \qquad \text{(E.16)}
\end{aligned}
$$

E.4 The partition functions of recirculating shift networks

By applying Equation (3.18) to recirculating networks RS_n of different sizes, we have

$$
\begin{aligned}
\beta(RS_1,\rho) &= 1+\rho; \\
\beta(RS_2,\rho) &= 1+4\rho+2\rho^2; \\
\beta(RS_3,\rho) &= (1+3\rho)\beta(RS_2,\rho) - \rho(1+4\rho)\beta(RS_1,\rho) \\
&= 1+6\rho+9\rho^2+2\rho^3; \\
\beta(RS_4,\rho) &= (1+3\rho)\beta(RS_3,\rho) - \rho(1+4\rho)\beta(RS_2,\rho) + \rho^2(1+4\rho)\beta(RS_1,\rho) \\
&= (1+3\rho)\beta(RS_3,\rho) - \rho(1+4\rho)\beta(RS_2,\rho) + \rho^2(1+4\rho)\beta(RS_1,\rho) \\
&= (1+2\rho)\beta(RS_3,\rho) - \rho^2\beta(RS_2,\rho); \\
\beta(RS_5,\rho) &= (1+3\rho)\beta(RS_4,\rho) - \rho(1+4\rho)\beta(RS_3,\rho) + \rho^2(1+4\rho)\beta(RS_2,\rho) \\
&\quad - \rho^3(1+4\rho)\beta(RS_1,\rho) \\
&= (1+2\rho)\beta(RS_4,\rho) - \rho^2\beta(RS_3,\rho).
\end{aligned}
$$

Thus, in general,

$$
\beta(RS_n,\rho) = (1+3\rho)\beta(RS_{n-1},\rho) + \sum_{i=1}^{n-2}(-1)^i\rho^i(1+4\rho)\beta(RS_{n-1-i},\rho), \qquad \text{(E.17)}
$$

with $n \geq 3$, and

$$
\beta(RS_n,\rho) = (1+2\rho)\beta(RS_{n-1},\rho) - \rho^2\beta(RS_{n-2},\rho), \qquad \text{(E.18)}
$$

with $n \geq 4$.

If we define R_n as the network generated from RS_n by removing one edge from it, then

$$\begin{aligned}\beta(RS_n,\rho) &= \beta(R_n,\rho)+\rho\beta(R_{n-1},\rho);\\ \beta(R_n,\rho) &= \beta(RS_n,\rho)-\rho\beta(R_{n-1},\rho);\\ \beta(R_{n-1},\rho) &= \beta(RS_{n-1},\rho)-\rho\beta(R_{n-2},\rho);\\ \beta(R_n,\rho) &= \beta(RS_n,\rho)-\rho\beta(RS_{n-1},\rho)+\rho^2\beta(R_{n-2},\rho)\\ &= \sum_{i=0}^{n-1}(-1)^i\rho^i\beta(RS_{n-i},\rho).\end{aligned}$$

Since

$$\beta(RS_n,\rho)=(1+3\rho)\beta(RS_{n-1},\rho)+(1+4\rho)\sum_{i=1}^{n-2}(-1)^i\rho^i\beta(RS_{n-1-i},\rho),$$

if we let

$$J=\sum_{i=1}^{n-2}(-1)^i\rho^i\beta(RS_{n-1-i},\rho),$$

and $k=i+1$, we have

$$\begin{aligned}J &= \sum_{k=2}^{n-1}(-1)^{k-1}\rho^{k-1}\beta(RS_{n-k},\rho)\\ &= -\frac{1}{\rho}\sum_{k=2}^{n-1}(-1)^k\rho^k\beta(RS_{n-k},\rho).\end{aligned}$$

Thus,

$$\begin{aligned}\beta(R_n,\rho) &= \beta(RS_n,\rho)-\rho\beta(RS_{n-1},\rho)+\sum_{i=2}^{n-1}(-1)^i\rho^i\beta(RS_{n-i},\rho)\\ &= \beta(RS_n,\rho)-\rho\beta(RS_{n-1},\rho)-\rho J.\end{aligned}$$

Further, from Equation (E.17),

$$J=\frac{1}{1+4\rho}[\beta(RS_n,\rho)-(1+3\rho)\beta(RS_{n-1},\rho)].$$

Therefore, we have

$$\beta(R_n,\rho)=\frac{1+3\rho}{1+4\rho}\beta(RS_n,\rho)-\frac{\rho^2}{1+4\rho}\beta(RS_{n-1},\rho), \tag{E.19}$$

with $n \geq 2$, where

$$\begin{aligned} \beta(R_1,\rho) &= 1+\rho; & \beta(R_2,\rho) &= 1+3\rho+\rho^2; \\ \beta(R_3,\rho) &= 1+5\rho+6\rho^2+\rho^3. \end{aligned}$$

Substituting Equation (E.18) to Equation (E.19) leads to

$$\beta(R_n,\rho) = \frac{1}{1+4\rho}\beta(RS_{n+1},\rho) + \frac{\rho}{1+4\rho}\beta(RS_n,\rho) \tag{E.20}$$

If we apply Equation (3.18) to the R_n networks of different sizes, we also generate

$$\beta(R_n,\rho) = (1+2\rho)\beta(R_{n-1},\rho) - \rho^2\beta(R_{n-2},\rho). \tag{E.21}$$