A COMPUTER-AIDED APPROACH TO THE APPROXIMATION PROBLEM OF ELLIPTIC FILTERS

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TO MY WIFE AND CHILDREN

WHOSE PATIENCE AND ENCOURAGEMENT

HELPED ACHIEVE THIS WORK

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ABSTRACT

The approximation of the transfer function for the elliptic filter usually involves the use of elliptic functions, the use of a transformed frequency variable or iterative algorithms. These methods are generally tedious and require high precision as well as a high amount of storage in the computer. This report describes an approximation process avoiding elliptic functions by using convergent series, and establishing the transfer function of the elliptic filter. Actual computer programs are described as well as their use; some examples are given as well as design charts for the filter parameters.

CHAPTER I

INTRODUCTION

1.1 General Considerations

It is difficult to imagine a modern electronic system without a filter to select signals of specific frequencies and reject others. Applications of filters are numerous covering frequencies from less than a hertz in seismology, to the gigahertz range in the microwave transmission. Filters are at the heart of frequency-division multiplex communication systems, and are present to a lesser extent in any communication system, be it a voice television, data, telemetry, or signalling system. Filters will continue to play an important role, even with the shift of emphasis from frequency-division to time-division multiplexing.

A filter is a device which can be used to separate analog signals into well defined frequency bands. However, nowadays, any device which can be used to modify the frequency response of a system is loosely called a filter, whether it is linear or non-linear, time-invariant or time-dependent, analog or digital.

Filters may contain purely passive elements (inductors, capacitors, resistors), they may be of the active RC type, they may contain mechanical resonators (quartz, crystal, and other piezzo-electric resonators), or they may be built of distributed parameter components (microwave or active distributed RC filters). Recently, digital filters have been introduced which may be built by using gates. While all these different types of realizations have a certain number of common properties, their design procedures are substantially different, resulting in different computer aids.

1.2 Image Parameter Filters

The filter designer has several sets of tools at his disposal, the simplest and oldest being the image parameter method (1). This method is useful in the design of LC, piezzo-electric crystal, and mechanical filters and, in fact, is still being used for the latter two types extensively. It is easy to use, requiring hardly more than a slide rule and a handbook type set of tables (2). The method is based on the study of properties of certain simple networks (elementary sections) in terms of their image transfer constant, and the primary and secondary image impedances. A number of such sections having the same image impedances in both directions can be connected together in a ladder network, which will have an overall image transfer constant equal to the sum of the individual image transfer constants. With little computational work and relatively short time, a network configuration can be constructed to meet approximately the required specifications. The disadvantage of this method is the inherent assumption that the filter is terminated by its image impedances, while, in practice, the filter is most often terminated by pure resistances. Consequently, a number of correction factors and additional matching networks are needed for the filter so that the desired transmission properties may be achieved with resistive terminations. Hence, the image parameter approach is economical for small production runs only, where the design cost is a substantial part of the overall cost.

In recent years, the image parameter method has been refined extensively by Rowlands (3), Colin (4) and others, to the point where the resulting filters may compete in efficiency with filters designed by the use of more modern theories; however, the computational techniques became much more involved in the process. Unfortunately, these theoretical refinements

came along in time when modern theories have become established, consequently, they have, to a large extent, been ignored.

1.3 Insertion Loss Filters

Today, the insertion loss method is usually the basis of filter design. It tackles the filter as a lumped parameter system, and deals directly with the various parameters that characterize the filter. The "Insertion Loss" technique introduced by Darlington (5) is an exact method for LC filters working between resistive terminations; it leads to the simplest and hence the least expensive design under normal conditions, but, at the price of a vastly increased complexity in the computations. The essentials of this method have been rearranged by Saal and Ulbricht (6) in easily comprehensible steps, so that a filter design may be easily obtained without thorough understanding of the extensive proofs. The design of insertion loss filters has been simplified further by the works of Saal (7) and Skwirzynski (8). These contributions provide extensive tables of normalized low pass filters which can be easily transformed into other types of filters.

Many computer programs were written in the past on the LC ladder filters, but they lacked the numerical precision especially for high degree filters. To overcome this difficulty, two different approaches were tried. The first one is the use of a transformed independent variable z introduced simultaneously by Szentirmai (9) and Orchard and Temes (10), where the pass band is transformed from the s plane into the whole imaginary axis of the z plane; thus the critical frequencies which cluster near the two cut-offs, in terms of the conventional frequency variable, will be widely separated from each other, in terms of the new

variable. The other solution to the numerical accuracy problem, equally successful, is the product method initiated by Norek (11), and elegantly elaborated by Skwirzynski (12); in this method the polynomials are kept in factored form and multiplied out, since during the synthesis, it is the roots of the polynomials that matter and not the coefficients.

Besides LC ladder filters, a great variety of structures is available and in use; namely crystal filters, active RC filters, and digital filters.

1.4 Crystal Filters

This is the area where one of the earlier attempts (13) was made to use iterative optimization and where the transformed variable method was intended to be used (2). On the other hand, the majority of crystal filter designers still use the image parameter methods.

1.5 Active RC Filters

These filters have come a long way since their introduction. From bulky size and high power consumption of vacuum tubes, their size, cost and power consumption was reduced considerably with the advent of transistors, integrated passive and active components together with an increase in system reliability in comparison to the discrete version. They are being introduced in many areas such as telephone and data communication systems; precision instruments, etc.. Their primary application is still at low frequencies, d-c to roughly 20 KHZ, where inductors are not suitable because of their bulkiness and low quality, whereas passive filters are more attractive at higher frequencies. Active RC filters do not need elaborate computer aids in their presently popular cascade realization beyond those needed to find a factored transfer

function. A special problem, though, is pairing pole and zero pairs into biquadratic blocks and sequencing these in order to obtain optimum performance as far as dynamic range and/or sensitivity is concerned. To improve the sensitivity, two approaches are presently available to active RC filter designers: one is to employ gyrators, essentially requiring an LC ladder filter design or use multiple feedback to simulate the ladder structure somewhat differently.

1.6 Digital Filters

This is the area that has seen tremendous activity in recent years. Digital filters have been used extensively in various fields, such as spacial filtering of photographs, telemetry, communications, signal processing, biomedical electronics, just to name some. Their main advantages over analog filters are their complete freedom from drift, hum and ac pick-up, the ease of changing filter characteristics, and their capability of realizing transfer functions which cannot be realized as analog filters.

The main areas of application of the digital filter are as follows:

(a) at very low frequencies

(b) in non-real time operation

(c) wherever high precision and driftless operations or variable characteristics (equalizers) are essential

(d) where signals are discrete.

Digital filters can also be used for the processing of continuous signals by converting them into discrete signals, using analog to digital and digital to analog converters before and after the digital filter respectively, but at an increased cost. The digital filter, whose behavior

is analogous to that of the continuous time filter, is an invaluable addition to the discipline of filtering. Although it is replacing analog filters in certain applications, it is not expected to replace analog filters completely.

1.7 Comparison of Filters

Figure 1.1 shows practical operating limits for filters. The frequency and percent bandwidth for the different filters shown in the figure are representative over-all figures.

The LC filter covers the largest area of frequency spectrum and percent bandwidth. At frequencies lower than 1 KHZ, inductance values and physical dimensions of the LC filter become excessive. The active filter and the digital filter can provide filtering at frequencies even lower than a hertz. For frequencies greater than 100 MHz, discrete LC structures fail as filters because of parasitic capacitance and inductance. Microwave filters composed of distributed elements are used at the higher frequencies.

It is evident from Figure 1.1, that the operating limits for various filters overlap. Here design trade-offs must be made to obtain physically realizable and economical units. Various factors, including percent bandwidth, temperature, shape factor, and cost, influence the selection of a particular filter type. Mechanical filters are excellent devices where sharp selectivity is required. Crystal filters are well suited for narrow band applications; they are ideal for integrated circuits. Ceramic filters exhibit a reasonably good shape factor and are economical. They are not recommended for high temperature where it may be more economical to use an LC filter.

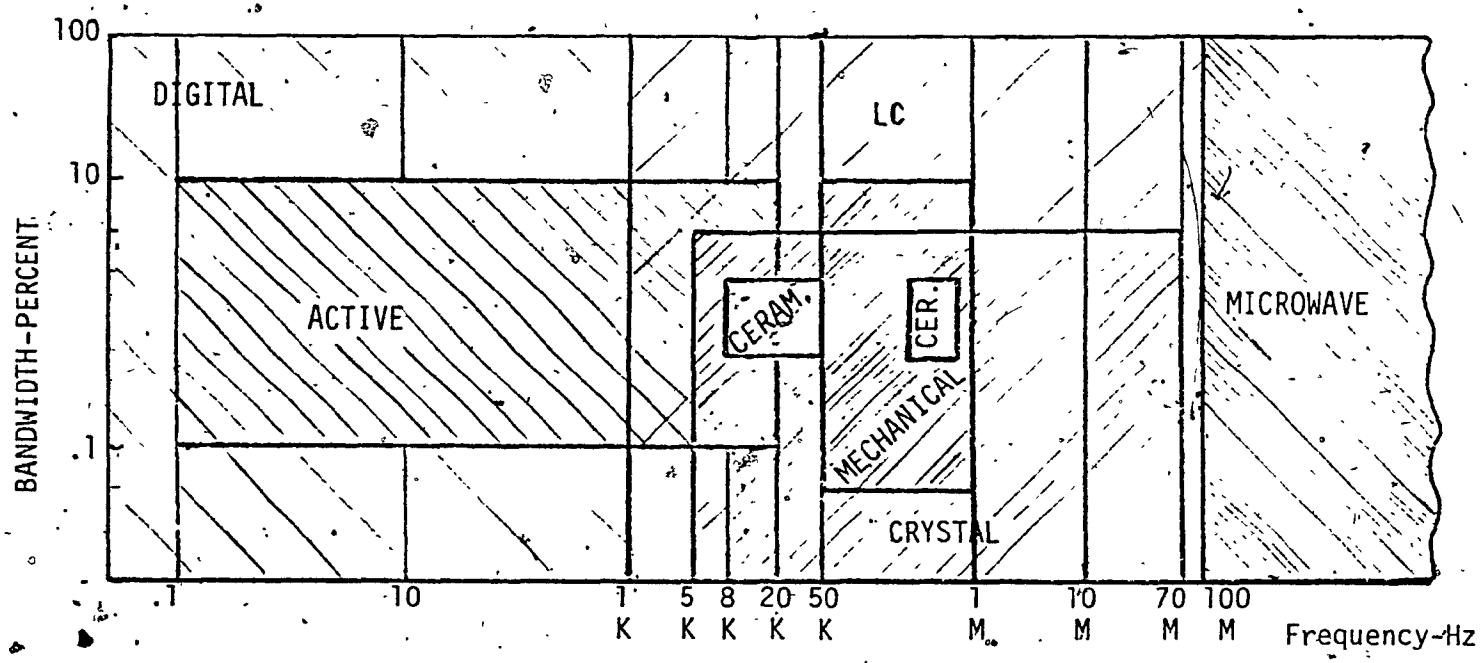


Fig. 1.1: Practical Operating Limits for Filters

1.8 The Approximation

In insertion loss filters, active filters and digital filters the first step is to obtain a suitable mathematical function for the transfer function which will satisfy the desired specifications. This is said to be the approximation problem.

The purpose of this report is to develop a computer program for "Elliptic Transfer Functions" which can be used as a computer aid for the design of Insertion loss filters, Active filters, and Digital filters.

CHAPTER II

THE APPROXIMATION

2.1 Introduction

A filter is a frequency sensitive component which is able to pass, with minimum attenuation, a select range of frequencies, while, suppressing the transmission of unwanted frequencies outside this band.

If one were able to build the ideal filter, its characteristics would be as shown in Figure 2.1. The passband has unity gain permitting signals in the range of frequencies, defined by $f_1 - f_2$, to be transmitted without attenuation. In the region outside the passband, referred to as the stopband, one hundred percent suppression of frequencies is obtained. The transition from the passband to stopband or vice-versa is over zero frequency, i.e. instantaneous. Examination of the phase characteristics shows that the phase shift changes linearly with frequency in the passband.

Actually, filters do not possess zero attenuation in the passband and suppression in the stopband is not infinite. Furthermore, the transition, or "roll off", between bands is gradual, and the phase shift is non-linear. The filter designer must approximate ideal filter characteristics as closely as possible, with combinations of passive elements like inductors and capacitors (LC filters), with resonant transducers (crystal, ceramic and mechanical filters), combination of an amplifier and passive components (RC active filters), and, at microwave frequencies, distributed, cavity and strip line filters.

A number of approximations to the ideal "brickwall" have evolved, based on the insertion loss method, to design filters using LC sections

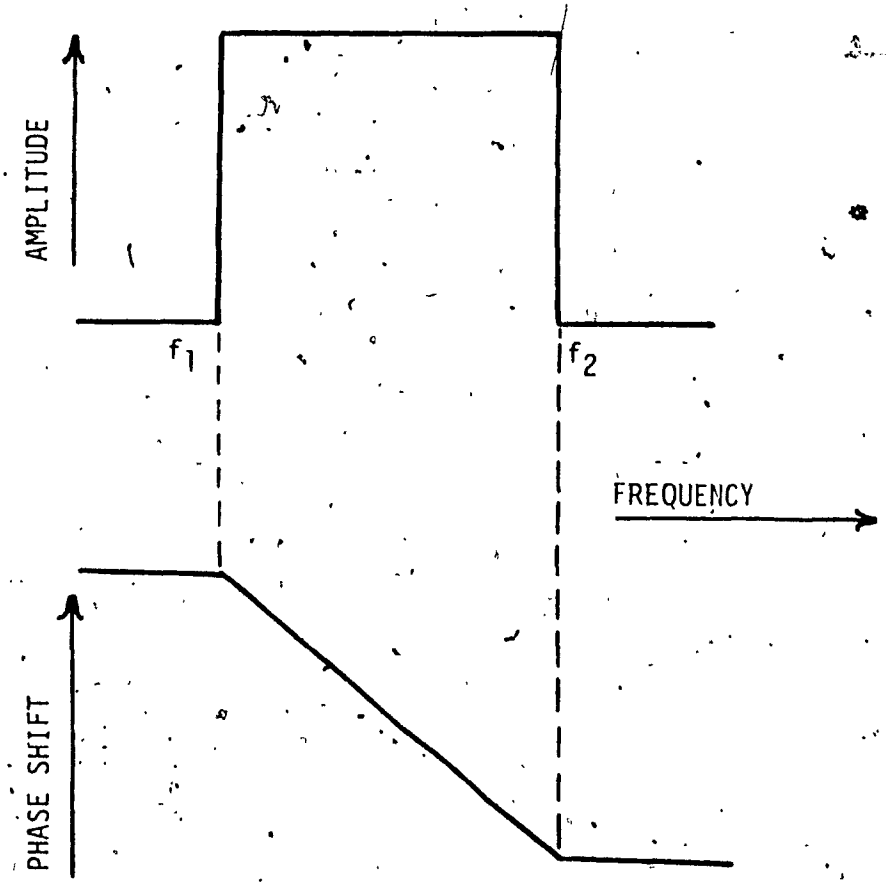


Fig. 2.1. "Ideal" Filter Characteristics.

such as the Butterworth, Chebychev and Elliptic approximations. Network synthesis methods for filter design generally start out by specifying a transfer function as a function of complex frequency ($s = jw$). From the transfer function, the imput impedance to the circuit is found as a function of s; then by various continued fraction or partial fraction expansion procedures the input impedance is expanded to give the element values of the filter. Image concepts never enter in this procedure, and the effects of the terminations are included in the initial specifications of the transfer function.

2.2 The Butterworth Approximation

The Butterwork approximation, which is also known as the maximally flat approximation, is characterized by a maximally flat amplitude response with no ripple in the passband, and monotonically increasing attenuation in the stopband as shown in Figure 2.2. The Butterworth transfer function is given by

$$T(jw) = \frac{1}{\sqrt{1 + w^{2n}}} \qquad (2.1)$$

The slope of the amplitude response in the stop-band approaches 6n db per octave, where n is the order of the approximation. The response is reduced by 3 db at the cut-off frequency for all values of η, as shown in Figure 2.3.

The poles of the normalized lowpass Butterworth transfer function are uniformly spaced along the left semicircle in the complex frequency plane and are given by:

$$S = \sigma_n \pm jw_n$$

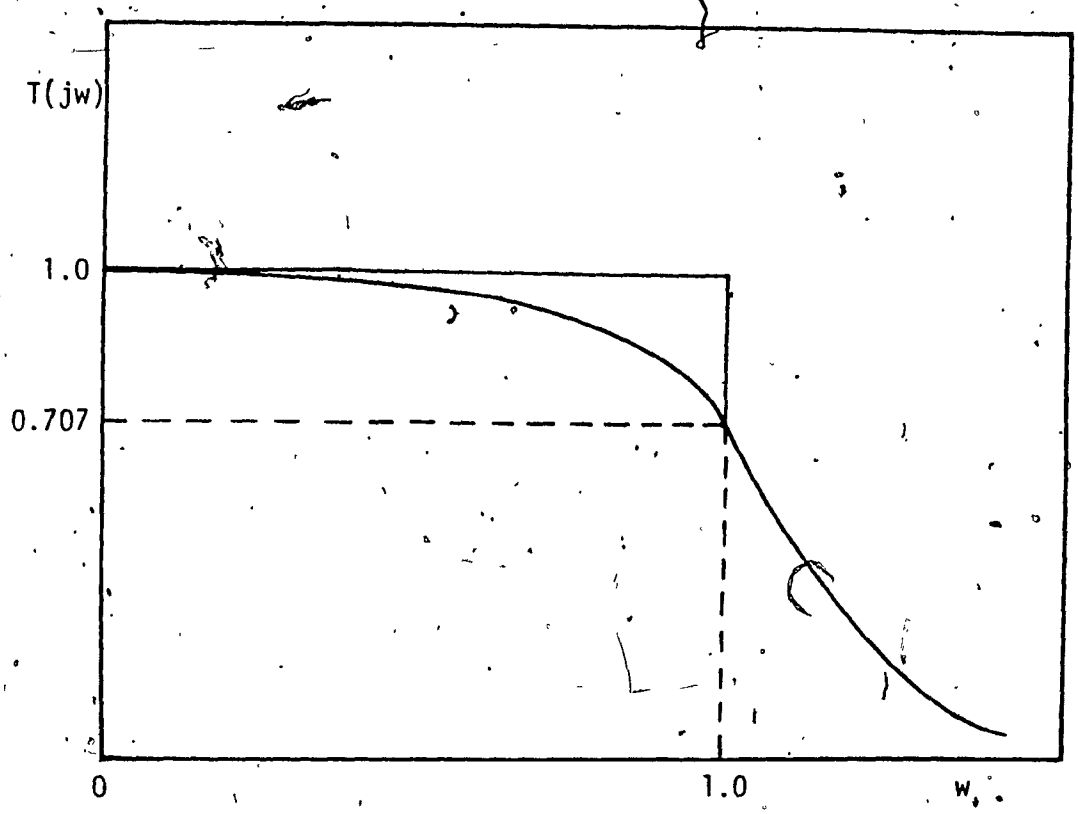


Fig. 2.2. Normalized Lowpass Magnitude Response for a Butterworth Approximation.

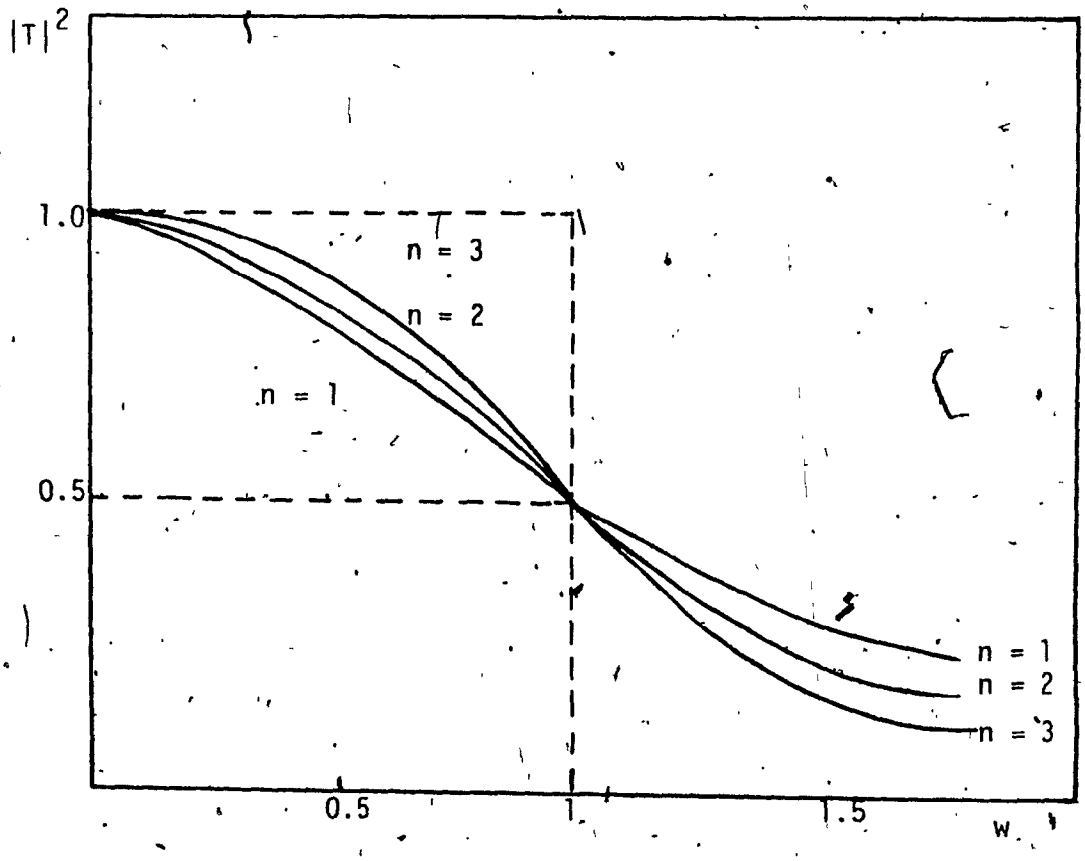


Fig. 2.3. Butterworth Approximation to the Lowpass Filter for the First Three Orders.

where

$$\sigma_n = \cos \frac{(2K - 1 + n)\pi}{2n}$$

$$w_n = \sin \frac{(2K - 1 + n)\pi}{2n}$$

$$K = 1,2,\ldots\ldots 2n$$

The above equations will give some poles in the right half s-plane which are ignored. Although the poles can be calculated easily with the aid of trigonometric tables, most often these are tabulated for different values of n (14). The Butterworth approximation is the only one that gives maximally flat magnitude about the point w = 0.

The great advantage of the Butterworth approximation is its mathematical simplicity and in this sense it is one of the best. Although it is very useful, it has limited applications. When uniform transmission of frequencies in the passband and a sharp cut-off characteristic is desired, a high order approximation must be used.

2.3 The Chebychev Approximation

The Chebychev approximation is characterized by an equal magnitude ripple in the passband, and, monotonically increasing attenuation in the stopband as shown in Figure 2.4. The number of maxima and minima ripple peaks is equal to the order of the filter. The Chebychev transfer function is given by:

$$T(jw) = \frac{1}{\sqrt{1 + \varepsilon^2 P_n^2(w)}} \qquad (2.2)$$

where ε is a real constant referred to as the ripple factor, and $P_n(w)$ is the nth order Chebychev polynomial defined as:

$$P_n(w) = \cos(n \cos^{-1} w)$$

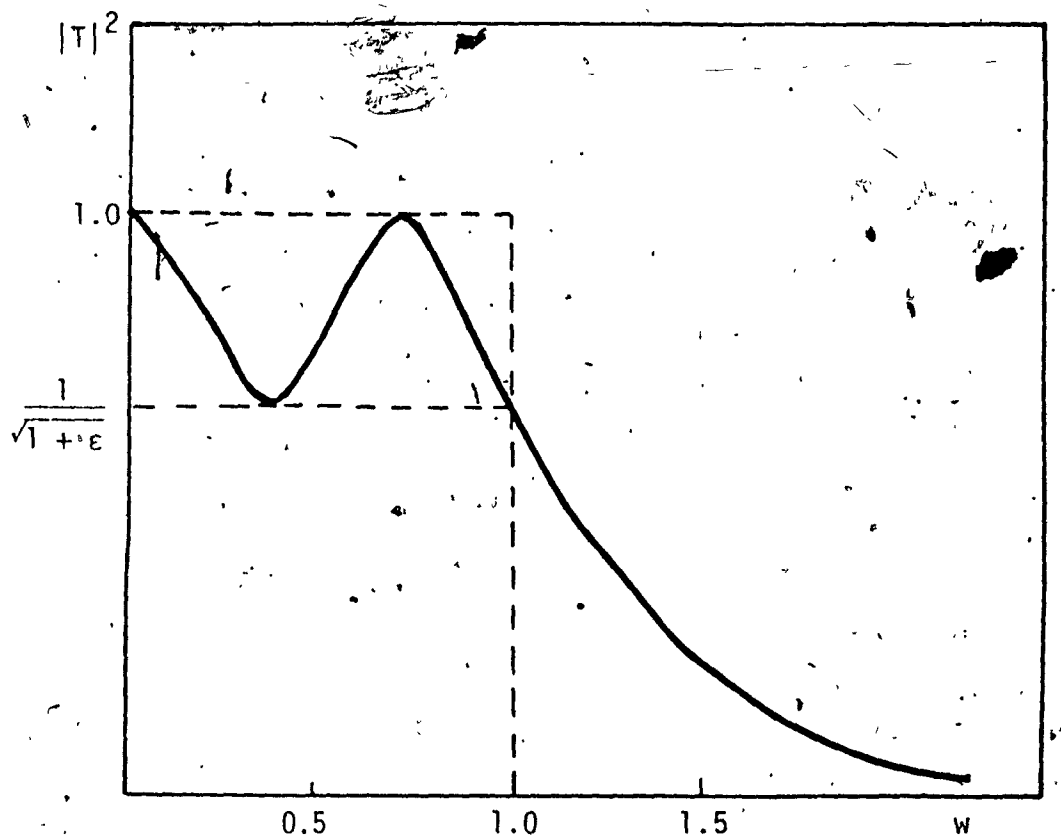


Fig. 2.4. Chebychev Approximation for the Lowpass Filter.

or preferably defined in two intervals:

$$P_n(w) = \cos(n \cos^{-1} w) \qquad 0 \le w \le 1$$

$$= \cosh(n \cosh^{-1} w) \qquad w > 1$$

As the ripple and order of the filter are increased, the rate of attenuation in the stopband increases.

The Chebychev approximation gives a sharper roll off, or squared amplitude response than the Butterworth approximation of the same order, but gives less desirable phase and group delay characteristics. It is exceedingly useful in applications where the magnitude of the transfer function is of primary concern; but wherever a constant time delay is of paramount importance, its use is precluded due to its non-linear phase characteristic and the resulting variation of its time delay. For a constant time delay filter, the Bessel approximation is used which gives a maximally flat time delay just as the Butterworth approximation gives a maximally flat magnitude response (14).

2.4 Chebychev vs Butterworth

The Butterworth and Chebychev approximations are used most frequently for telemetry filter applications because of their balanced amplitude - phase characteristics. The Chebychev approximation gives a cut-off characteristic which is superior over that given by the Butterworth approximation of the same order, that is, it gives sharper roll-off but less desirable phase and group delay characteristics. A fourth order Chebychev approximation gives an amplitude response with a slope at cut-off equal to that of a Butterworth approximation of order 16. On the other hand, the poles of the Chebychev transfer function have a

higher Q than the poles of the Butterworth transfer function, e.g. if n = 8 the Chebychev poles nearest the jw axis have a Q of 22.9, while the Butterworth poles nearest the jw axis have a Q of only 2.56. Consequently, more care is necessary in the construction of Chebychev filters. The high Q also causes the step response of Chebychev filters to ring for a longer time (15); furthermore, the passband ripple of the Chebychev filters may be objectionable in some applications.

2.5 Transitional Butterworth-Chebychev Approximation

An approximation with properties lying between those of Butterworth and Chebychev approximations would be clearly desirable. Such an approximation is described by Budake and Aronhime (16). The transfer function for this approximation is given by

$$T_{K,n}(jw) = \frac{1}{\sqrt{1 + w^{2K} C^2_{n-K}(w)}}$$

$$T_{K,n}(w) = \frac{1}{\sqrt{1 + w^{2K} C^2_{n-K}(w)}} \quad ; \quad 0 \le K \le n \qquad (2.3)$$

where the factor $w^{2K}C^2_{n-K}(w)$ is a mixture of Butterworth and Chebychev parameters. As K approaches zero or n the properties of $T_{K,n}(jw)$ approach those of the Chebychev or the Butterworth transfer function, respectively. Since $C_1(w) = w$, the new function reduces to the Butterworth function for both K = n - 1 and K = n. If (n-K) is even, then from the Maclaurin expansion of equation (3), (2K - 1) derivatives of $T_{K,n}(w)^2$ are zero at w = 0. If (n-K) is odd, then (2K + 1) derivatives are zero. The magnitude characteristic of the transitional function becomes more flat at the origin as K is made larger. The

magnitude of the slope at cut-off becomes larger as K is made smaller; the slope of $|T_{K,n}(w)|$ at $w = 1$, is given by:

$$d|T_{K,n}(1)|/dw = m_T = -2^{-3/2}\,[(n-K)^2 + K]$$

For $0 < K < n-1$, the magnitude characteristic given by the transitional approximation exhibits Butterworth-like behavior near $w = 0$ and Chebychev-like behavior near $w = 1$. The amplitude responses given by the transitional approximation for $n = 8$ and $K = 0, 2, 4, 6$, and 8, are shown in Figure 2.5. The wave for $k = 8$, corresponds to the Butterworth approximation whereas the curve for $K = 0$ corresponds to the Chebychev approximation. Poles of the transfer function $T_{K,n}(s)$ are obtained by replacing w by -js in the denominator of equation (2.3) and then solving for the left half plane roots.

The transfer function poles for $n = 8$, and with $K = 0;2,4,6$ and 8 are plotted in Figure 2.6 as K increases, the poles move away from the Chebychev pole location (ellipse) and migrate toward the Butterworth pole location; consequently the Q of the poles decrease.

2.6 Elliptic Approximation

Although the Chebychev and Butterworth approximations are the most popular, there are many others which are in use. By permitting ripples of attenuation in the passband, as in the Chebychev approximation and ripples in the stopband, it is possible to obtain an even faster transition from the passband to the stopband than can be obtained with the Chebychev approximation. Filters resulting from this approximation are usually called Elliptic Filters, mainly because elliptic functions are used in their design. The elliptic filter is of considerable importance becuase it provides simultaneously small

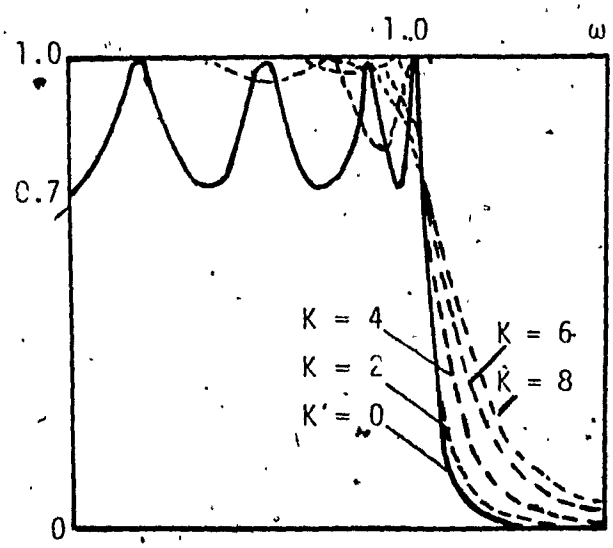


Fig. 2.5. Magnitudes of Eighth Order Transitional Filters for K = 0,2,4,6,8.

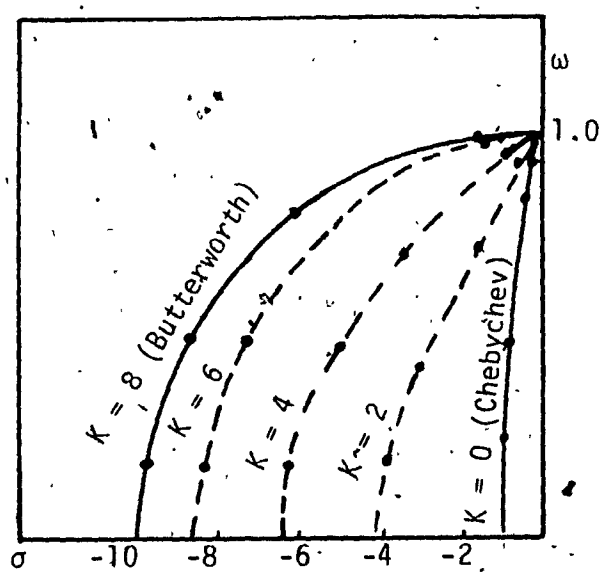


Fig. 2.6. Pole Locations of Eighth Order Transitional Filters for K = 0,2,4,6,8.

passband ripple, large stopband attenuation, and, very sharp cut-off with equal ripple behavior in both the passband and the stopband, as in Figure 2.7.

To obtain the desired approximation function, suitable values of passband attenuation A_p, stopband attenuation A_s, the degree of the filter n, and the parameter K that defines the steepness of the transition band or the selectivity of the filter must be chosen. All these parameters, are used to obtain the poles and zeros of the transfer function, by means of elliptic functions and elliptic integrals. Because of the complicated way in which these parameters are related, considerable time can be spent in juggling those parameters in order to obtain compatible values. Also considerable work is required to obtain a suitable approximation function because of certain computational difficulties such as inadequate tables of elliptic functions and the necessity of interpolation. Complete tables of elliptic filters of fourth through ninth order were compiled by R. Saal (7), giving all component values to four places, and cut-off frequencies to seven places. A much abbreviated but quite useful set of tables of elliptic filters of third to eleventh order are given by P.R. Geffe (17).

The Nomographs for Elliptic Filters

Nomographs, devised by Henderson (18), relating the design parameters of the filter reduce considerably the amount of work required and are a valuable tool for the design of elliptic filters. The nomographs enable one to choose a tentative set of parameter values. These are then established more accurately by means of tables and/or series, before evaluating the formulae leading to the approximation function. Thus, extensive trial and error

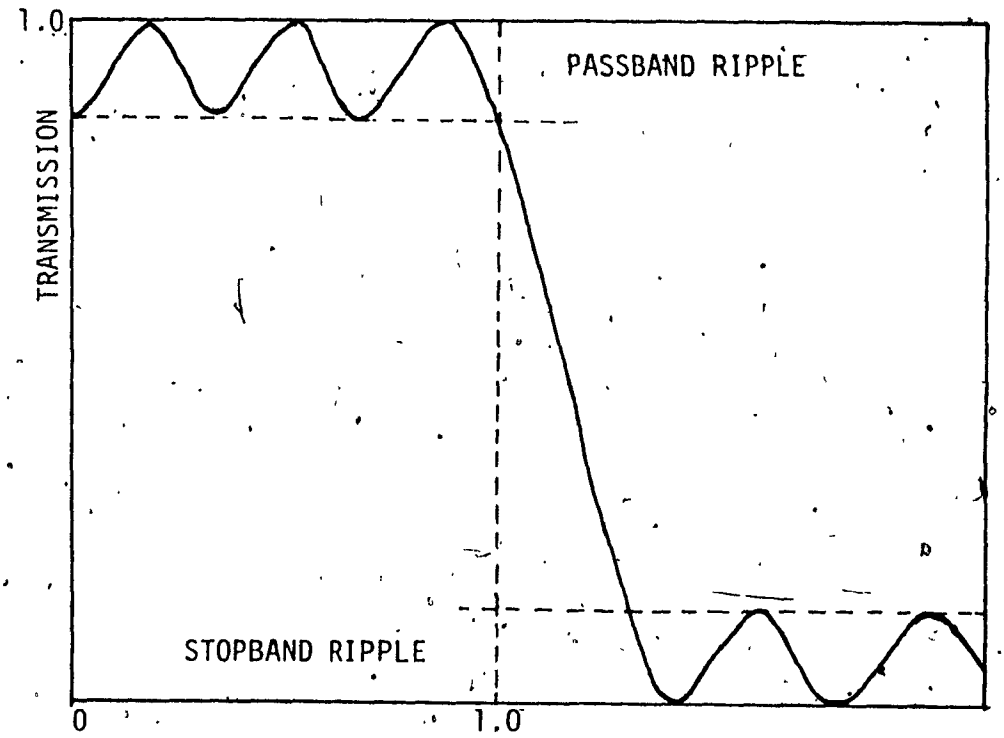


Fig. 2.7. Elliptic Filter with Equiripple Passband and Equiripple Stopband.

procedures, previously necessary in obtaining a compatible set of parameters, are eliminated. Once this is done, the approximation function is obtained in a straightforward manner. Even and odd degrees must be distinguished. In both cases, the zeros are on the jw axis of the s plane, ($s = \sigma + jw$) and the poles occur in conjugate pairs in the left half plane, with an additional pole on the negative real axis (for odd n), hence a zero at infinity, in the case of odd degree. Tables of the elliptic functions, sn, cn, and dn are readily available, however, interpolation is necessary most of the time, which makes the task tedious. It is easier then to calculate the elliptic functions directly as described by H.J. Orchard in (19) or by using converging series as described by A. Grossman (20).

CHAPTER III

ELLIPTIC TRANSFER FUNCTION

3.1 Introduction

Of the many published papers that tried to bridge the gap between Darlington's theory and its applicability in practice was one due to A. Grossman (20). This describes very simply Darlington's theory of electrically symmetrical reactive networks with particular attention to elliptic type filters which exhibit Chebychev type performance in both the passband and stopband. This paper also outlines a step by step procedure to be followed in the design of symmetrical filters using rapidly converging series in the computations in place of elliptic function tables.

This chapter uses Grossman's work for the derivation of elliptic lowpass transfer functions. A procedure is developed that allows direct calculations of the zeros and poles of the transfer function, when the parameters of the filter (number of filter sections, the ripple constant and the selectivity factor) are given. Most of the computations involved are straightforward. No transformation of the frequency variable is necessary. Furthermore, the process is very fast and yields satisfactory results without the direct use of elliptic functions. Also no interpolation or iteration is necessary.

3.2 The Insertion Loss Characteristics of the Elliptic Filter

The insertion loss characteristic for a two-section lowpass elliptic filter will now be considered.

From Figure (3-1), the passband extends from zero to the frequency f_p, the loss ripples extend between zero and a prescribed maximum α_p.

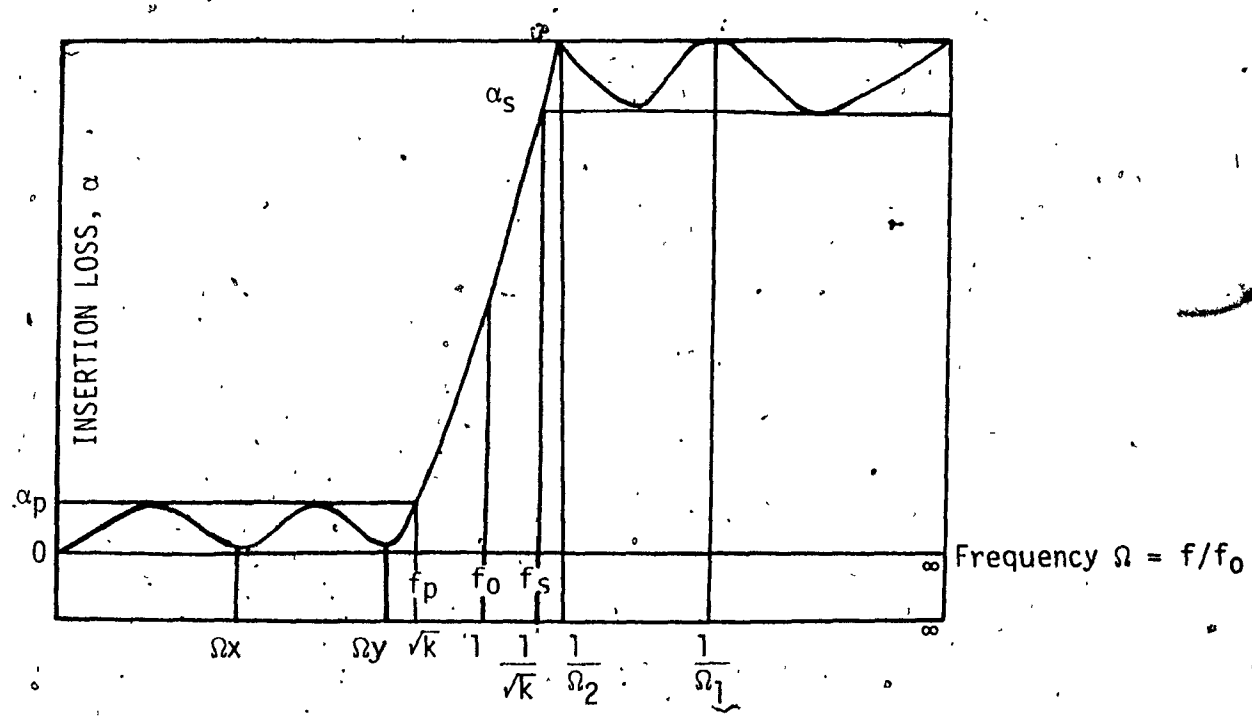


Fig. 3.1. Insertion Loss Characteristics for a Two-Section Lowpass Elliptic Filter.

As for the attenuation band which extends from the frequency f_a to infinity, the loss oscillates between an infinite value and a prescribed minimum α_s. These two bands are joined by the transition band extending from f_p to f_s. The reference frequency to the normalized frequency variable Ω and the selectivity factor k are defined as

$$f_o = \sqrt{f_p f_s}$$

$$\Omega = \omega/\omega_o = f/f_o$$

$$k = f_p/f_s$$

Hence the passband terminates at $\Omega = \sqrt{k}$ and the attenuation band begins at $\Omega = 1/\sqrt{k}$.

The insertion loss characteristic shown in Figure (3-1) may be represented by the insertion power ratio

$$e^{2\alpha} = 1 + (e^{2\alpha_p} - 1)\,[F(\Omega)]^2 \qquad (3.1)$$

where the function $F(\Omega)$ has the following characteristics:

1. $F(\Omega) = 0$ at $\Omega = 0, \Omega_x, \Omega_y$
2. Its magnitude is unity at $\Omega = \sqrt{k}$ and at two other frequencies which are separated by the zero-loss points.
3. $F(\Omega)$ is infinite at $\Omega = \frac{1}{\Omega_1}, \frac{1}{\Omega_2}, \ldots \infty$
4. Its magnitude is equal to $\sqrt{(e^{2\alpha_s} - 1)\,(e^{2\alpha_p} - 1)}$ at $\Omega = 1/\sqrt{k}$ and at two frequencies which are separated by the infinite-loss points.
5. This function is to be realized by an electrically symmetrical network.

All these requirements can be satisfied if:

$$F(\Omega) = F_0 \frac{\Omega(\Omega^2 - \Omega_x^2)(\Omega^2 - \Omega_y^2)}{(1 - \Omega_1^2\Omega^2)(1 - \Omega_2^2\Omega^2)} \tag{3.2}$$

provided that F_0 and the frequencies of the zero and infinite loss are chosen properly.

3.3 Specifying the Zero Loss Points, and the Infinite Loss Points

By studying equation (3.2), we find that:

1. The function $(1 - F^2)$ has a single root at $\Omega = \sqrt{k}$, and double roots in the passband where the magnitude of F is unity.

2. Another derived function $(1 - k_1^2F^2)$, where

$$k_1 = \sqrt{\frac{e^{2\alpha_p} - 1}{e^{2\alpha_s} - 1}} \tag{3.3}$$

has double roots at the minima of F^2 between the frequencies of infinite loss and a single root at $\Omega = 1/\sqrt{k}$.

3. The derivative of F, $dF/d\Omega$, has single roots at the maxima of F^2 in the passband and at the minima of F^2 in the attenuation band.

If we divide the derivative by the square root of the product of the other two functions, we obtain

$$\frac{dF/d\Omega}{\sqrt{(1 - F^2)\ (1 - k_1^2F^2)}} = \frac{\pm M_0}{\sqrt{(1 - \Omega^2/k)\ (1 - k\Omega^2)}}$$

because the single roots of $dF/d\Omega$ cancel the square root of the double roots of the other functions, leaving only the single roots at $\sqrt{k}$ and $1/\sqrt{k}$ and a constant multiplier M_0. This equation may be expressed in terms of definite integrals as follows:

$$\int_0^F \frac{dx}{\sqrt{(1-x^2)(1-k_1^2x^2)}} = \pm M_0 \sqrt{k} \int_0^{\Omega/\sqrt{k}} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} + C \tag{3.4}$$

where x and y are variables of integration and C is a constant of integration.

These integrals are known as the elliptic integrals of the first kind. By making the transformations:

$$x = \sin \theta_1$$

$$F = \sin \phi_1$$

$$y = \sin \theta_1$$

$$\Omega/\sqrt{k} = \sin \phi$$

Equation (4) can be put in the more desirable form:

$$\int_0^{\phi_1} \frac{d\theta_1}{\sqrt{1 - k_1^2 \sin^2 \theta_1}} = \pm M_0 \sqrt{k} \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} + C_1 \tag{3.5}$$

The solution of this equation, giving the relation between F and Ω may be written in the form of a pair of simultaneous equations as:

$$\Omega = \sqrt{k}\ \mathrm{sn}\ (u,k)$$

$$F(u) = \mathrm{sn}\ (\pm Mu + C_1,\ k_1) \tag{3.6}$$

where

$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$\pm Mu + C_1 = \int_0^{\phi_1} \frac{d\theta_1}{\sqrt{1 - k_1^2 \sin^2 \theta_1}}$$

The real period of the elliptic sine, sn (u, k) is 4 K where K is the complete elliptic integral of modulus k given by

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \phi(\pi/2,\ k)$$

The passband at positive frequencies is traversed once by values of u lying between 0 and K, the quarter period. On the other hand, the real period of sn $(\pm Mu + C_1, k_1)$ is $4K_1$ where K_1 is the complete elliptic integral of modulus k_1.

If F(u) is to correspond to $F(\Omega)$ of Equation (3.2), it must be zero for u = 0 and increase to +1 as its argument increases to K_1 and then decrease to zero at Ω_x where its argument is $2K_1$, it must trace out a similar negative cycle between Ω_x and Ω_y and then increase to +1 at $\Omega = \sqrt{k}$, or u = K. To obtain this behavior, we must set $C_1 = 0$, and M to $5K_1/K$ omitting the negative sign. Therefore, equation (3.6) becomes:

$$\Omega = \sqrt{k}\ \mathrm{sn}\ (u, k)$$

$$F(u) = \mathrm{sn}\ (5\ u\ K_1/K, k_1) \qquad (3.7)$$

By observing the plot of these functions in Figure (3-2), we find that the zero loss points are located at u = 0, 2K/5, 4K/5 and the unit maxima are located at u = k/5, 3K/5, K.

The frequencies of zero loss are

$$\Omega_i = \sqrt{k}\ \mathrm{sn}\ (2\ iK/5, k) \quad (i = 0,1,2) \qquad (3.8)$$

Hence, the zero loss points of $F(\Omega)$ in Equation (3.2) have been determined and the function will exhibit the type of performance desired in the passband.

The infinite loss points, and the choice of selectivity parameter k, and the discrimination parameter k_1, can be determined by studying the behavior of F(u) in the attenuation band, i.e. for the frequency interval

$$1/\sqrt{k} \leq \Omega \leq \infty$$

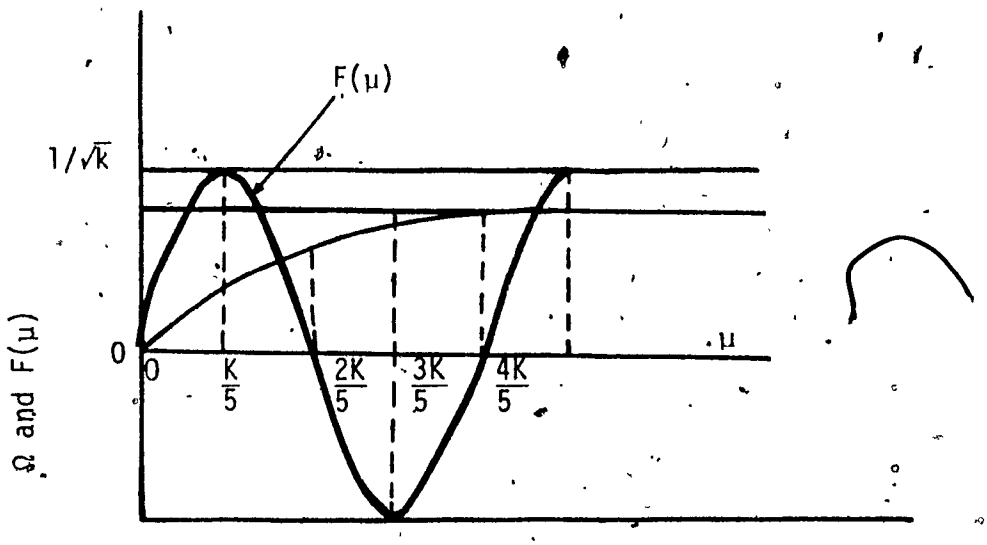


Fig. 3.2. Comparison Between the Frequency Variable and the Insertion Loss Function.

If u is replaced by $(u + j K')$ where u is real, $0 \le u \le K$, and K' is the complementary complete integral or the complete integral of modulus k' i.e.

$$K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (k')^2 \sin^2 \theta}}$$

Equation (7) becomes

$$\Omega = \frac{1}{\sqrt{k}\ sn\ (u,\ k)}$$

$$F(u) = sn\ [5\ (u + j\ K')\ K_1/K,\ k_1] \tag{3.9}$$

It is required that this function be infinite at infinite frequency and at two other undetermined frequencies to describe the loss characteristic shown in Figure (3-1).

When $u = 0$, $sn\ (u,k) = 0$ and therefore

$$\Omega = \frac{1}{\sqrt{k}\ sn\ (0;\ k)}$$

is infinity, hence $F(0) = sn\ (5j\ K'\ K_1/K,\ k_1)$ should be infinite which will be the case for

$$\frac{5\ K'}{K} = \frac{K_1'}{K_1} \tag{3.10}$$

where K_1' is the complete elliptic integral of modulus $K_1' = \sqrt{1 - k_1^2}$

The parameters K, K' and K_1, K_1' are functions of k and k_1, respectively.

Therefore, the two finite frequencies of infinite loss may be determined by introducing condition (3.10) in Equation (3.9) to get

$$F(u) = sn\ (5\ u\ \frac{K_1}{K} + j\ K_1',\ k_1)$$

The sn function for complex values of the argument shows that poles of F(u) are located at the values of the argument equal to $2iK_1 + j\ K_1'$

(i is an integer). Hence, to calculate u, we set $5\,u\,K_1/K = 2\,i\,K_1$ and obtain $u = 2\,i\,K/5$. Therefore, the frequencies of infinite loss are obtained as

$$\Omega_i = \frac{1}{\sqrt{k}\,\mathrm{sn}\,(2iK/5,\ k)} \qquad i = 0,1,2 \tag{3.11}$$

The frequencies of infinite loss in the attenuation band are thus the reciprocals of the frequencies of zero loss in the passband as seen from equations (3.8) and (3.11).

Therefore, equation (2) can be written as

$$F = F_0 \frac{\Omega(\Omega^2 - \Omega_1^2)\,(\Omega^2 - \Omega_2^2)}{(1 - \Omega_1^2\,\Omega^2)(1 - \Omega_2^2\,\Omega^2)} \tag{3.12}$$

This function has equal ripples in the passband and equal minima in the attenuation band provided that the Ω_i are selected in accordance with Equation (3.8).

The general case of an m - section filter is

$$\Omega_i = \sqrt{k}\,\mathrm{sn}\left[\frac{2\,i\,K}{2m+1},\ k\right] \qquad i = 0,1,\ldots m \tag{3.13}$$

where Ω_i are the frequencies at which the loss is zero and the reciprocals of the frequencies at which the loss is inifinite.

$C_i = \frac{1}{\Omega_i}$ will be the zero of the transfer function

$$Ts = \frac{1}{s + a_0} \prod_{i=1}^{m} \frac{s^2 + C_i}{s^2 + A_i\,s + B}$$

3.4 Determination of the Poles of the Transfer Function

Consider the circuit of Figure (3.3) in which a lossless electrically symmetrical network is inserted between equal resistance termination.

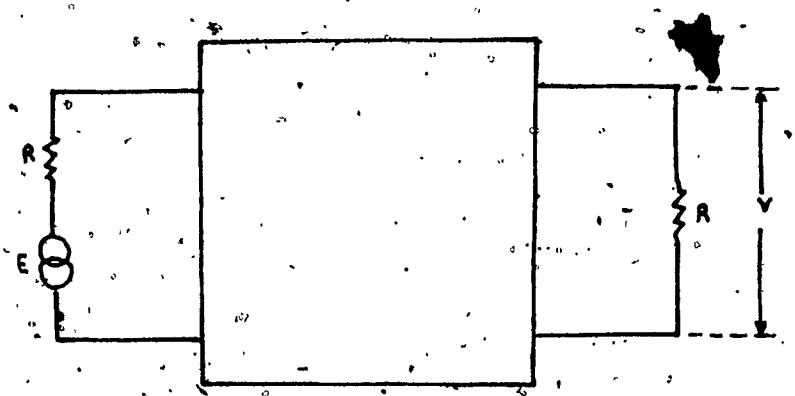


Fig. 3.3. Symmetric Terminated Network.

1. The complex insertion voltage ratio is defined as

$$e^{\Theta} = \frac{V_0}{V}$$

where $\Theta = \alpha + j\,\beta$, α being the insertion loss and

β the insertion phase shift.

V_0 = the voltage across the load resistance when the generator and load are connected directly.

V = the voltage across the load after network is inserted between the generator and the load.

2. The complex insertion voltage ratio may be written in the form:

$$\frac{V_0}{V} = \frac{A + s\,B}{P} \qquad (3.14)$$

where A, B, P are even polynomials of the frequency variable and have real coefficients. The roots of the numerator in terms of the frequency parameter $s (s = jw)$, are the natural modes of the terminated network.

3. The insertion loss α, is found from the insertion power ratio which is defined as

$$e^{2\alpha} = \left|\frac{V_0}{V}\right|^2$$

Consequently, the zeros of the insertion power ratio, are the zeros of the complex insertion voltage ratio. The latter ones are the natural modes of the network which will also be the poles of the transfer function.

Referring to Equation (3.1) after substituting Equation (3.7) for the general case we get

$$e^{2\alpha} = 1 + (e^{2\alpha_p} - 1)\, sn^2\, [(2m + 1)\, u\, K_1/K,\, k] \qquad (3.15)$$

which may be regarded as the product of two factors, one of them being

$$1 + j\sqrt{(e^{2\alpha_p} - 1)}\, sn\, [(2m + 1)\, u\, K_1/K,\, k_1]$$

The roots of this factor are the negatives of the conjugate factor. This means that the values of u which satisfy the equation

$$\mathrm{sn}\ [(2m + 1)\ u\ K_1/K,\ k_1] = \frac{j}{\sqrt{e^{2\alpha p} - 1}} \tag{3.16}$$

and their negatives are the zeros of the insertion power ratio. Hence, the solutions of Equation (3.16) will give the zeros of the insertion power ratio. Substituting $u = jv_0$ in Equation (16) will give

$$\frac{\mathrm{sn}[(2m + 1)\ v_0\ K_1/K,\ k_1']}{\mathrm{cn}[(2m + 1)\ v_0\ K_1/K,\ k_1']} = \frac{1}{\sqrt{(e^{2\alpha p} - 1)}} \tag{3.17}$$

where $k_1' = \sqrt{(1 - k_1^2)}$ is the complementary modulus. From Equation (3.17) it is possible to evaluate v_0. One value of the argument satisfying (3.16) will be $(2m + 1)\ j\ v_0\ K_1/K$ with a real period of $4\ K_1$ for the elliptic sine; hence the solutions are:

$$(2m + 1)jv_0K_1/K + 4iK_1 = (2m + 1)\ \frac{K_1}{K}\ \left[jv_0 + \frac{4iK}{2m + 1}\right]$$

where $i = 0,1,2\ldots 2m$

Therefore, the zeros of the insertion power ratio in terms of u are

$$u = jv_0 + \frac{4ik}{2m + 1} \qquad i = 0,1,2,\ldots,2m \tag{3.18}$$

In terms of the frequency parameter $s = j\Omega = j\sqrt{k}\ \mathrm{sn}\ (u,k)$, the real root is given by

$$a_0 = j\sqrt{k}\ \mathrm{sn}\ (jv_0,k) = -\sqrt{k}\ \frac{\mathrm{sn}(v_0,k')}{\mathrm{cn}(v_0,k')} \tag{3.19}$$

and the remaining roots are given by

$$a_i + jb_i = (-1)^i\ j\sqrt{k}\ \mathrm{sn}\ \left[jv_0 \pm \frac{2iK}{2m + 1},\ k\right]\ (i = 1,\ldots,m) \tag{3.20}$$

Noting that the roots come in conjugate pairs, and knowing that

$$sn(x+y) = \frac{sn\ x.cny.dn\ y + cn\ x.sn\ y.dn\ x}{1 - k^2 sn^2 x.sn^2 y}$$

and with the aid of the standard relations between elliptic functions, equation (3.20) may be written as

$$a_i + jb_i = \frac{(-1)^i a_0 V_i \pm j\Omega_i W}{1 + a_0^2\ \Omega_i^2} \qquad i = 1,2...m \tag{3.21}$$

where

$$W = \sqrt{(1 + ka_0^2)(1 + a_0^2/k)}$$

$$V_i = \sqrt{(1 - k\Omega_i^2)(1 - \Omega_i^2/k)}$$

$$\Omega_i = \sqrt{k} \cdot sn\ (\frac{2ik}{2m+1}, k)$$

For even values of i the sign of the roots should be reversed. For the two section filter under study, the polynomials will be

$$(s + a_1 + jb_1)(s + a_1 - jb_1)$$

and $(s + a_0)(s + a_2 + jb_2)(s + a_2 - jb_2)$

The transfer function can, in general, be written as

$$T(s) = \frac{1}{s + a_0} \prod_{i=1}^{m} \frac{s^2 + 1/w_i^2}{(s + a_i + jb_i)(s + a_i - jb_i)}$$

The parameters of T(s) can be computed by using elliptic function tables. Since the value of an elliptic function depends not only on the argument but also on the modulus, these tables must be double entry tables. Consequently, interpolation between tabulated values is laborious and subject to error. The need for elliptic function tables can be removed completely by

using auxiliaryfunctions called Theta Functions, which are represented by rapidly converging series as defined below:

$$\Theta_0(u/2K,q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n \frac{\pi u}{2K}$$

$$\Theta_1(u/2K,q) = 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)} \sin(2n+1) \frac{\pi u}{2K}$$

$$\Theta_2(u/2K,q) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1) \frac{\pi u}{2K}$$

$$\Theta_3(u/2K,q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2n \frac{\pi u}{2K}$$

where $q = e^{-\pi K'/K}$ = modular constant

K = complete elliptic integral of modulus k

K' = complete elliptic integral of modulus $\sqrt{1 - k^2}$

The elliptic functions are expressed in terms of theta functions as follows:

$$sn(u,k) = \frac{1}{\sqrt{k}} \frac{\Theta_1(u/2K,q)}{\Theta_0(u/2K,q)}$$

$$cn(u,k) = \sqrt{k'/k} \cdot \frac{\Theta_2(u/2K,q)}{\Theta_0(u/2K,q)}$$

$$dn(u,k) = \sqrt{k'} \frac{\Theta_2(u/2K,q)}{\Theta_0(u/2K,q)} \tag{3.22}$$

To determine the modulus q in terms of k, the approximation

$$\varepsilon = \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}$$

is formed based on the observation that

$$dn(0,k) = \sqrt{k'} \frac{\Theta_3(0,q)}{\Theta_0(0,q)} = 1$$

Therefore,

$$q = \varepsilon + 2\varepsilon^5 + 15\varepsilon^9 + 150\varepsilon^{13} + \ldots\ldots \tag{3.23}$$

Also, by noting that:

$$sn(K,k) = \frac{1}{\sqrt{k}} \frac{\Theta_1(1/2,q)}{\Theta_0(1/2,q)} = 1$$

we obtain

$$k = 4\sqrt{q} \; \frac{1 + q^2 + q^6 + \ldots}{1 + 2q + 2q^4 + \ldots}$$

These approximations are equally valid for q_1 and k_1.

Usually the filter design problem consists of the specification of the number of filter sections that is required to attain prescribed minimum stopband loss, a tolerable passband ripple and a prescribed transition band.

In terms of the Theta function series, the frequencies [illegible]ch the passband loss is zero, and the reciprocals of the freq[illegible] at which the attenuation band-loss is infinite, are:

$$\Omega_i = 2q^{1/4}\left[\frac{\sin\frac{i\pi}{2m+1} - q^2\sin\frac{3i\pi}{2m+1} + q^6\sin\frac{5i\pi}{2m+1}}{1 - 2q\cos\frac{2i\pi}{2m+1} + 2q^4\cos\frac{4i\pi}{2m+1}}\right] \qquad (3.24)$$

where $i = 0,1,\ldots m$

and $z_i = 1/\Omega_i$ will be the zero of the transfer function. Similarly, the frequencies at which the passband loss is equal to the maximum α_p, and the reciprocals of the frequencies at which the stopband loss is equal to the minimum α_s, are:

$$\Omega_t = 2q^{1/4}\left[\frac{\sin\frac{(2t+1)}{2(2m+1)}\pi - q^2\sin\frac{3(2t+1)}{2(2m+1)}\pi + q^6\sin\frac{5(2t+1)}{2(2m+1)}\pi}{1 - 2q\cos\frac{(2t+1)}{(2m+1)}\pi + 2q^4\cos\frac{2(2t+1)}{(2m+1)}\pi}\right] \qquad (3.25)$$

where $t = 0,1,\ldots m$

As for the real root a_o it may be written as

$$a_o = j\,\frac{\Theta_1(jv_o/2K,q)}{\Theta_o(jv_o/2K,q)}$$

$$= 2q^{1/4}\,\frac{\sinh\Lambda - q^2\sin 3\Lambda + q^6\sinh 5\Lambda + \ldots\ldots}{1 - 2q\cosh 2\Lambda + 2q^4\cosh 4\Lambda + \ldots\ldots\ldots} \tag{3.26}$$

where $$\Lambda = \frac{v_o\pi}{2K} = \frac{1}{2(2m+1)}\left(\log_e\frac{2}{a_p} + \frac{a_p^2}{12}\right) \tag{3.27}$$

Using this value of a_o in terms of the Theta function, the complex conjugate roots will be:

$$a_i \pm jb_i = \frac{a_o\sqrt{(1-k\Omega_i^2)(1-\Omega_i^2/k)} \pm \Omega_i\sqrt{(1+ka_o^2)(1+a_o^2/k)}}{1 + a_o^2\Omega_i^2} \tag{3.28}$$

which with as a_o constitute the poles of the transfer function. The transfer function now can be written in the following form:

$$T(s) = \frac{1}{s + a_o}\prod_{i=1}^{m}\frac{s^2 + z_i^2}{s^2 + 2a_i s + (a_i^2 + b_i^2)} \tag{3.29}$$

CHAPTER IV

THE COMPUTER AID

4.1 Introduction

The theoretical approach of Chapter III was used to develop two computer programs, one for the evaluation of elliptic transfer functions and the other, for frequency domain analysis. The first program uses the desired filter specifications as input data and prints the parameters of the transfer function. The second program evaluates the amplitude response of the desired filter. Design charts can also be obtained using the first program. All calculations are carried out in 6 digit precision. This method requires little storage space and is extremely fast. Various computer runs on the CDC 6600 at Concordia University yielded very good results.

4.2 The Programs

The Transfer Function Program

PROGRAM CALTRAN in Appendix A calculates the zeros and poles of the transfer function for the normalized lowpass elliptic symmetric filter. It also provides the coefficients of the transfer function. The program is described by the flow chart of Figure 4.1. From equation 3.26 the pole on the $j\omega$ axis is calculated (symmetric filter). The reciprocal of equation 3.24 gives the zeros of the transfer function, while equation 3.28 gives the poles. The coefficients of the transfer function are calculated from the poles and zeros as shown in the flow chart.

The input data for the program includes the order of the filter n, the pass-band ripple α_p in db and the selectivity factor k. The program will print out n, α_p, k, the stop-band attenuation in db, the zeros and poles of the filter and the transfer function in the product form. A typical output is shown in Figure 4.2.

START

$L = 1$

(1)

Read n, k, α_p

$m = (n-1)/2$

$k' = \sqrt{1 - k^2}$

$$\varepsilon = \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}$$

$$q = \varepsilon + 2\varepsilon^5 + 15\varepsilon^9$$

$$\alpha_s = 10 \log_{10} 10^{\alpha p/10} - 10\, n \log_{10} q - 12$$

Print Title, n, α_p, k, α_s

$$\Lambda = \frac{1}{n} [1.427424 - 1.151292 \log \alpha_p + .00052\, \alpha_p^2]$$

$$A_o = 2q^{1/4} \frac{\sinh\Lambda - q^2 \sinh 3\Lambda + q^6 \sinh 5\Lambda}{1 - 2q\cosh 2\Lambda + 2q^4 \cosh 4\Lambda}$$

(A)

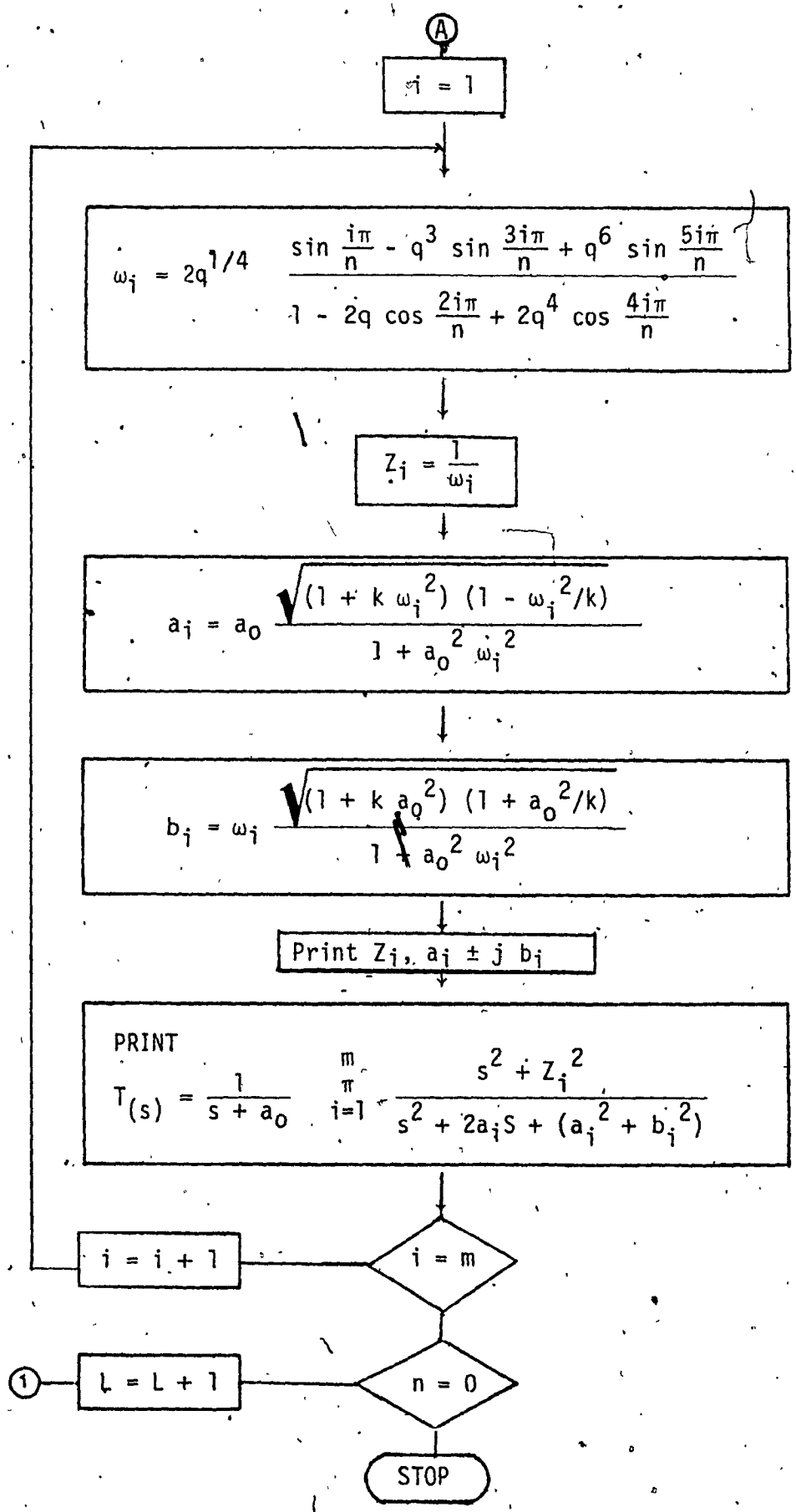


Fig. 4.1: Flow-Chart for Program CALTRAN.

```
                    THE TRANSFER FUNCTION
                          OF
                THE ELLIPTIC LOWPASS FILTER

N = 5     RIPPLE = .10  DB     SELECTIVITY FACTOR=  .70
STOPBAND ATTENUATION =-14.558709   DB

  ZEROES                POLES

                          -  .558428
-  1.835682      -   .345301 +J  .661678
-  1.238973      -   .090161 +J  .890229

                 1
T( S) = ------------
        S+   .558428

                 2
              S +3.369729
      * ----------------------
                 2
        S +  .690602*S + .557051

                 2
              S +1.535053
      * ----------------------
                 2
        S +  .180322*S + .800638
```

Fig. 4.2: Typical Output of Program CALTRAN.

SIR GEORGE WIL

The Frequency Analysis Program

PROGRAM FREQAN in Appendix B calculates the amplitude response for the normalized lowpass elliptic symmetric filter. The flow chart in Figure 4.3 describes the necessary equations and steps used in this program. PROGRAM FREQAN uses PROGRAM CALTRAN as a subroutine to evaluate the parameters of the transfer function. These parameters are used to compute the amplitude response for the normalized frequency range. From Equation (3.29), the constant multiplier H is computed as follows:

$$H = \frac{a_o\,(a_i^2 + b_i^2)}{Z_i^2}$$

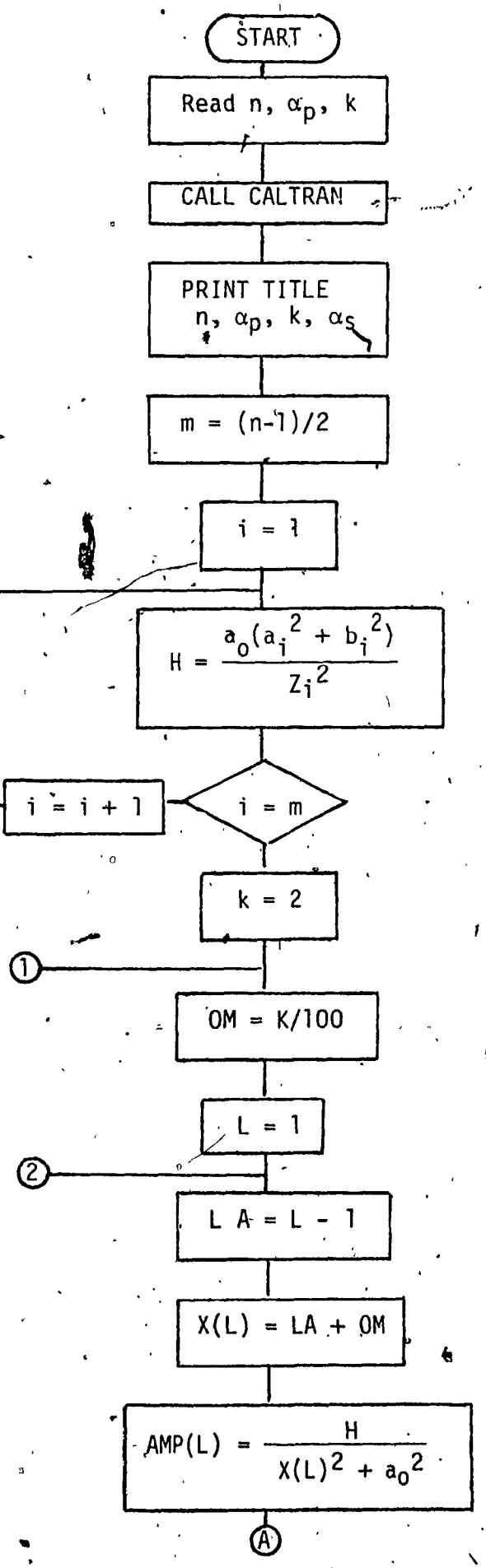
Replacing s by jw in Equation (3.30), the amplitude AMP is obtained as follows:

$$Amp = \frac{H\,.(Z_i^2 - \omega^2)}{\omega^2 + a_o^2\,.\,\sqrt{(a_i^2 + b_i^2 - \omega^2)^2 + (2a_i\omega)^2}}$$

from which the amplitude in dbs calculated as

$$\text{Amplitude in db} = 20\,\log_{10}\,|Amp| \quad \text{dbs}$$

The input data required by this program includes the order of the filter n, the passband ripple α_p in db, the selectivity factor k, as well as the normalized frequency range and frequency increments. A typical output is shown in Figure 4.4.


START
Read n, α_p, k
CALL CALTRAN
PRINT TITLE
n, α_p, k, α_s
m = (n-1)/2
i = 1
$H = \frac{a_0(a_i^2 + b_i^2)}{Z_i^2}$
i = i + 1
i = m
k = 2
1
OM = K/100
L = 1
2
L A = L - 1
X(L) = LA + OM
$AMP(L) = \frac{H}{X(L)^2 + a_0^2}$
A

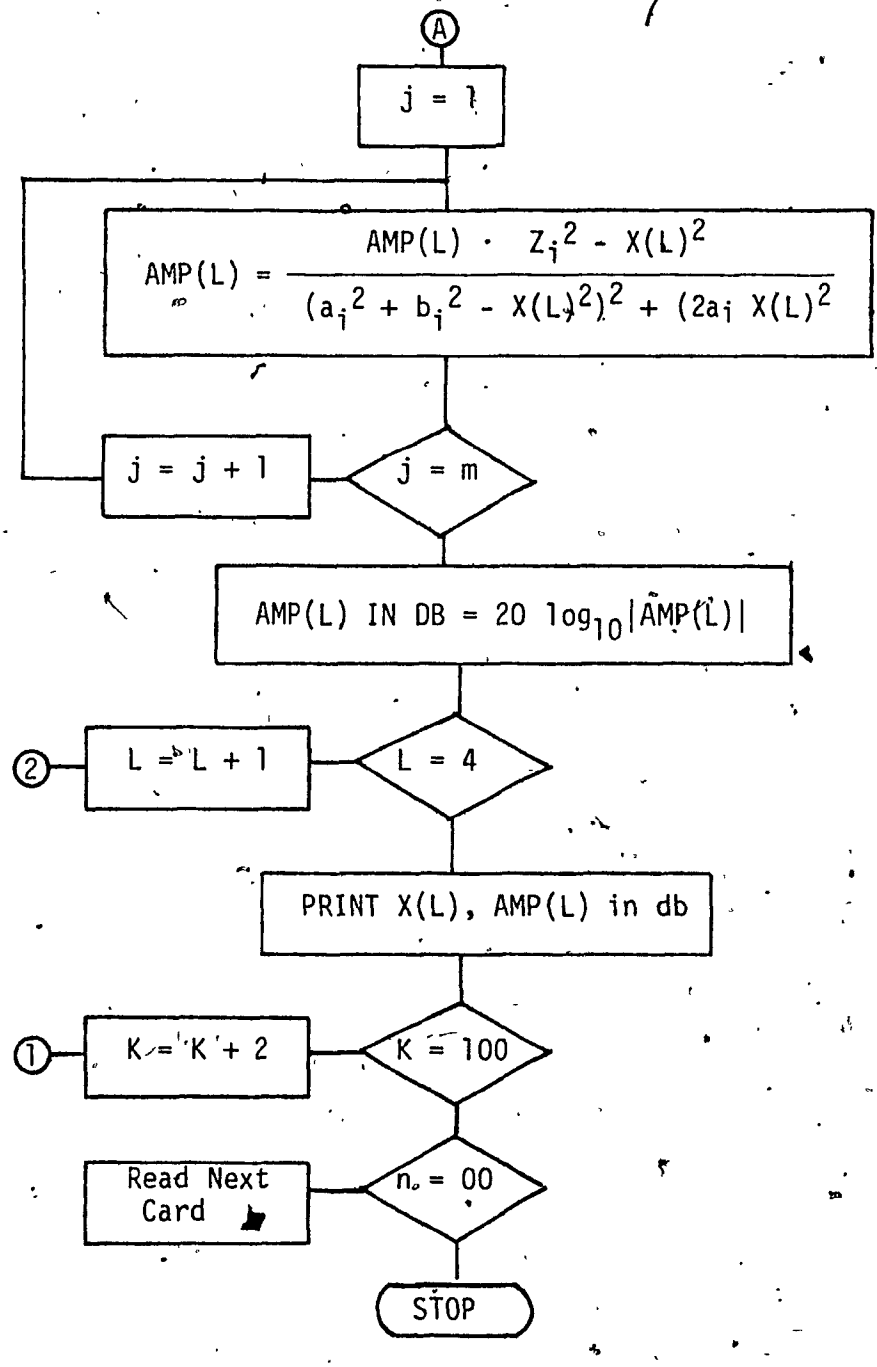


Fig. 4.3: Flow-Chart for Program FREQAN.

THE FREQUENCY ANALYSIS
OF
THE ELLIPTIC LOWPASS FILTER

N = 7 RIPPLE = .50 DB SELECTIVITY FACTOR= .700000
STOPBAND ATTENUATION = 75.247521 DB

OMEGA	AMP DB	OMEGA	AMP DB	OMEGA	AMP DB	OMEGA	AMP DB
.02	-.01	1.02	-36.46	2.02	-78.49	3.02	-79.19
.04	-.04	1.04	-39.86	2.04	-78.95	3.04	-79.00
.06	-.09	1.06	-43.27	2.06	-79.43	3.06	-78.83
.08	-.15	1.08	-46.75	2.08	-79.95	3.08	-78.66
.10	-.22	1.10	-50.36	2.10	-80.51	3.10	-78.50
.12	-.29	1.12	-54.18	2.12	-81.11	3.12	-78.35
.14	-.36	1.14	-58.35	2.14	-81.75	3.14	-78.21
.16	-.42	1.16	-63.10	2.16	-82.43	3.16	-78.07
.18	-.47	1.18	-68.98	2.18	-83.18	3.18	-77.94
.20	-.49	1.20	-77.87	2.20	-83.98	3.20	-77.82
.22	-.50	1.22	-95.01	2.22	-84.87	3.22	-77.70
.24	-.49	1.24	-79.54	2.24	-85.84	3.24	-77.58
.26	-.45	1.26	-76.33	2.26	-86.91	3.26	-77.47
.28	-.40	1.28	-75.30	2.28	-88.13	3.28	-77.37
.30	-.33	1.30	-75.30	2.30	-89.52	3.30	-77.27
.32	-.25	1.32	-75.97	2.32	-91.15	3.32	-77.17
.34	-.17	1.34	-77.19	2.34	-93.13	3.34	-77.08
.36	-.10	1.36	-78.96	2.36	-95.63	3.36	-76.99
.38	-.04	1.38	-81.44	2.38	-99.09	3.38	-76.91
.40	-.01	1.40	-85.02	2.40	-104.78	3.40	-76.83
.42	-.00	1.42	-91.10	2.42	-125.32	3.42	-76.75
.44	-.03	1.44	-121.98	2.44	-106.78	3.44	-76.67
.46	-.09	1.46	-92.19	2.46	-100.36	3.46	-76.60
.48	-.17	1.48	-86.19	2.48	-96.77	3.48	-76.53
.50	-.27	1.50	-83.20	2.50	-94.28	3.50	-76.47
.52	-.36	1.52	-81.10	2.52	-92.39	3.52	-76.41
.54	-.44	1.54	-79.58	2.54	-90.87	3.54	-76.34
.56	-.49	1.56	-78.45	2.56	-89.61	3.56	-76.29
.58	-.50	1.58	-77.58	2.58	-88.53	3.58	-76.23
.60	-.46	1.60	-76.91	2.60	-87.59	3.60	-76.18
.62	-.37	1.62	-76.39	2.62	-86.76	3.62	-76.13
.64	-.25	1.64	-75.99	2.64	-86.02	3.64	-76.08
.66	-.12	1.66	-75.69	2.66	-85.36	3.66	-76.03
.68	-.02	1.68	-75.47	2.68	-84.76	3.68	-75.99
.70	-.00	1.70	-75.32	2.70	-84.21	3.70	-75.94
.72	-.09	1.72	-75.24	2.72	-83.71	3.72	-75.90
.74	-.26	1.74	-75.21	2.74	-83.24	3.74	-75.86
.76	-.44	1.76	-75.22	2.76	-82.81	3.76	-75.82
.78	-.49	1.78	-75.28	2.78	-82.42	3.78	-75.79
.80	-.30	1.80	-75.38	2.80	-82.05	3.80	-75.75
.82	-.00	1.82	-75.51	2.82	-81.70	3.82	-75.72
.84	-.86	1.84	-75.68	2.84	-81.38	3.84	-75.69
.86	-4.27	1.86	-75.88	2.86	-81.08	3.86	-75.66
.88	-9.09	1.88	-76.11	2.88	-80.79	3.88	-75.63
.90	-13.80	1.90	-76.37	2.90	-80.52	3.90	-75.60
.92	-18.15	1.92	-76.65	2.92	-80.26	3.92	-75.58
.94	-22.17	1.94	-76.96	2.94	-80.03	3.94	-75.55
.96	-25.94	1.96	-77.31	2.96	-79.80	3.96	-75.53
.98	-29.55	1.98	-77.67	2.98	-79.58	3.98	-75.50
1.00	-33.04	2.00	-78.07	3.00	-79.38	[illegible]	[illegible]

Fig. 4.4: Typical Output of Program FREQAN

The Charts Program

The program given in Appendix C is in fact a part of PROGRAM CALTRAN. Its purpose is to relate the different filter parameters namely: filter degree n, selectivity factor k, passband ripple α_p and stopband attenuation α_s. The input data needed to run the program comprises n, α_p and k. Two of these parameters are kept constant and the third one is varried as follows:

a- Keeping α_p and k constant, varying n and computing the corresponding stop-band attenuation α_s.

b- Keeping n and α_p constant, varying k and computing the corresponding α_s.

c- Keeping n and k constant, varying α_p and computing α_s.

Typical outputs for the 3 cases are shown in Figure 4.5.

4.3 Results Obtained

The Frequency Analysis:

The frequency analysis for various filters has been plotted in Figure 4.6 through Figure 4.8.

Figure 4.6 shows the amplitude response for filters of degree 3,5,7 all having the same passband ripple 0.5 dbs, and a selectivity factor 0.7. The scale in the passband is exaggerated to show the ripples. The higher the degree of the filter the greater is the minimum stopband attenuation.

Figure 4.7 is for a filter of degree 7, pass-band ripple 0.5 db and different selectivity factors 0.6, 0.7, 0.8 to compare the minimum stop-band attenuation for the corresponding selectivity factor. The less the selectivity factor the higher the stopband attenuation.

PROGRAM CHARTS

FILTER DEGREE	RIPPLE DB	SELECTIVITY FACTOR	STOPBAND AT DB
3	.10	.70	12.98
5	.10	.70	40.52
7	.10	.70	68.06
9	.10	.70	95.59
11	.10	.70	123.13
5	.30	.50	63.81
5	.30	.60	54.29
5	.30	.70	45.39
5	.30	.80	36.40
5	.30	.90	26.04
5	.30	.92	23.50
5	.30	.94	20.62
5	.30	.96	17.15
5	.30	.98	12.37
5	.10	.70	40.52
5	.20	.70	43.58
5	.30	.70	45.39
5	.40	.70	46.69
5	.50	.70	47.71
5	.60	.70	48.55
5	.70	.70	49.27
5	.80	.70	49.90
5	.90	.70	50.47

Fig. 4.5: Typical Output of Program Charts.

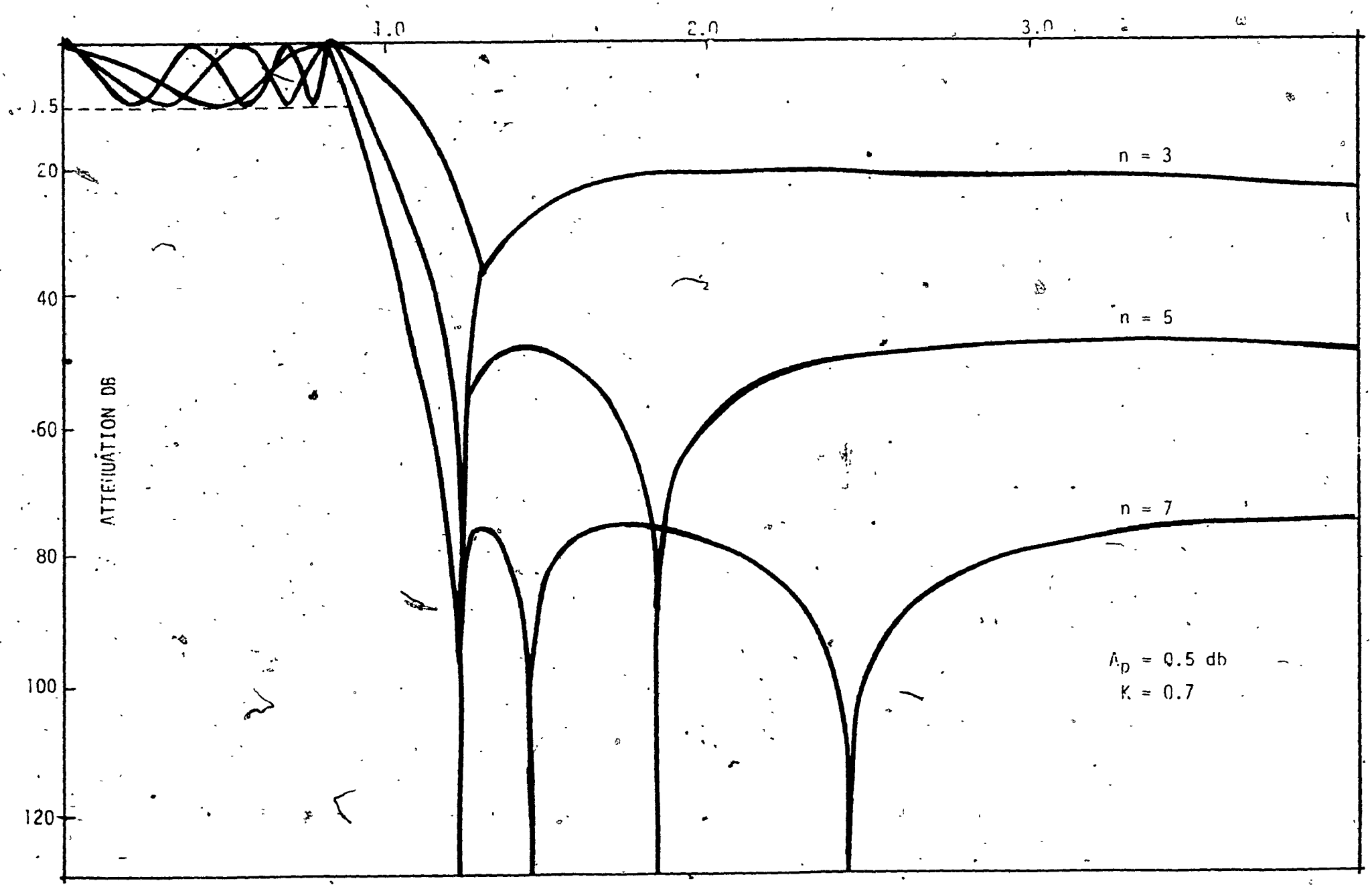


Fig. 4.6: Amplitude Response for Various Filter Degrees with 0.5 db Passband Ripple and 0.7 Selectivity Factor.

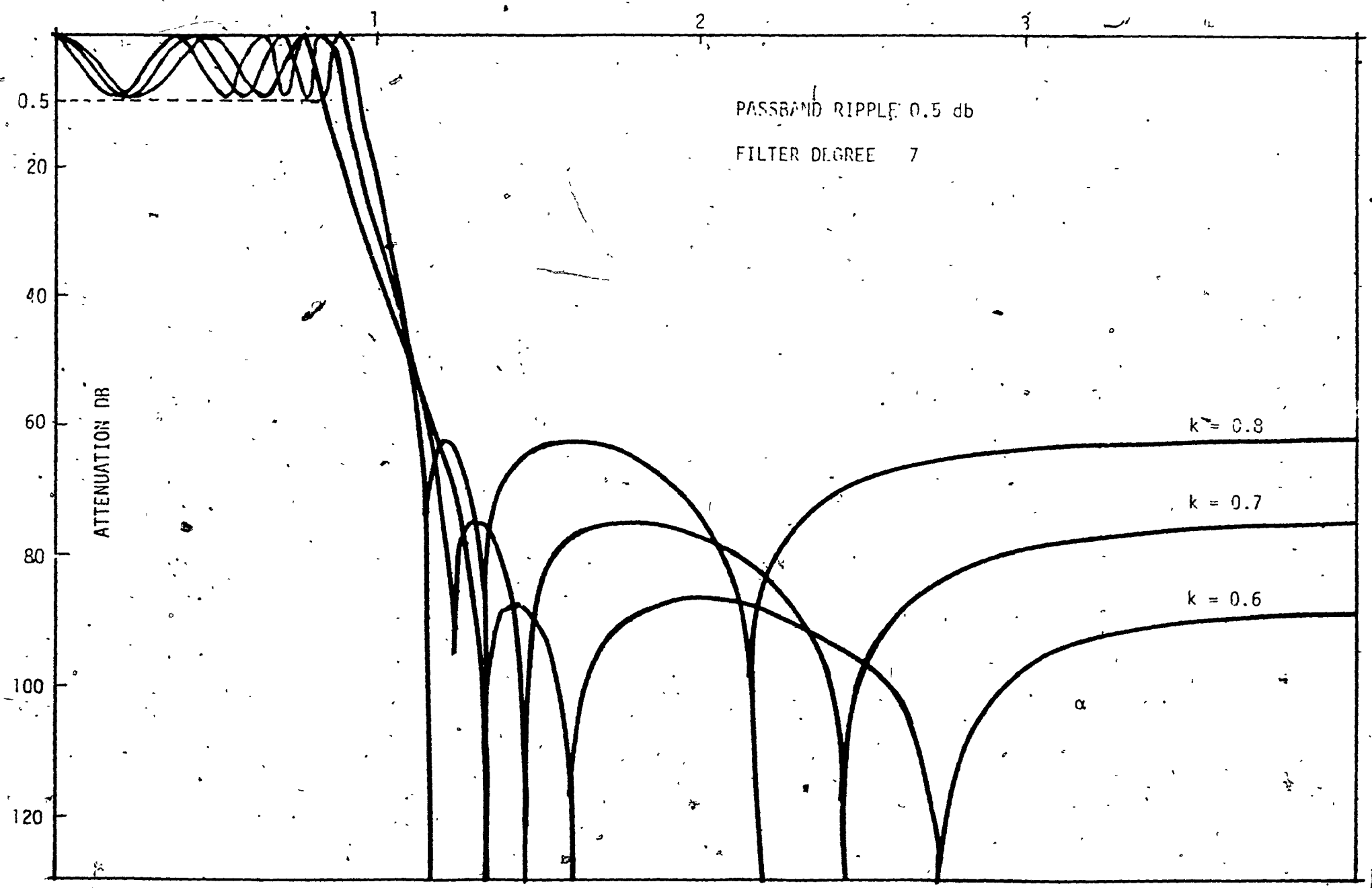


Fig. 4.7: Amplitude Response for the 7th Order Lowpass Filter with 0.5 db Passband Ripple and 0.7 Selectivity Factor.

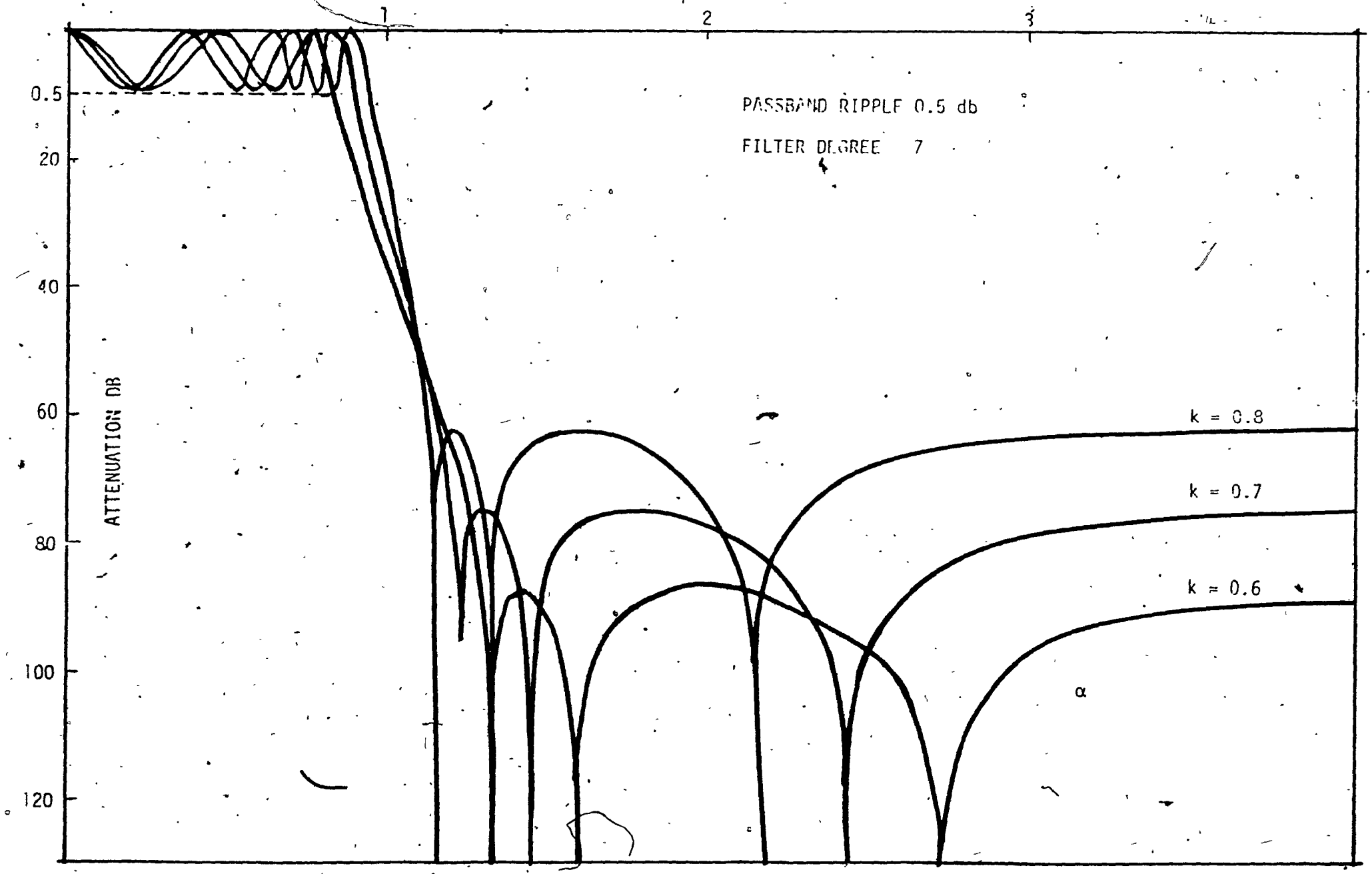


Fig. 4.7: Amplitude Response for the 7th Order Lowpass Filter with 0.5 db Passband Ripple and 0.7 Selectivity Factor.

Figure 4.8 shows the response at various passband ripples 0.1, 0.3 and 0.5 dbs for the same filter degree 7 and k = 7.

Design Charts

The design charts in Figure 4.9 through Figure 4.12 are very helpful in the selection of the optimum design parameters which comply to the specifications. The chart in Figure 4.9 relates filter degree to the attenuation band for constant passband ripple and selectivity factor k.

From the chart in Figure 4.10, one can select the proper filter degree for a specified stopband attenuation passband ripple and selectivity factor. As for the Charts in Figure 4.11 and 4.12, each relate the selectivity factor to the attenuation band for a given filter degree and a fixed value of the passband ripple. The charts have been drawn for a passband ripple of 0.1, 0.2, 0.3 and 0.5 dbs.

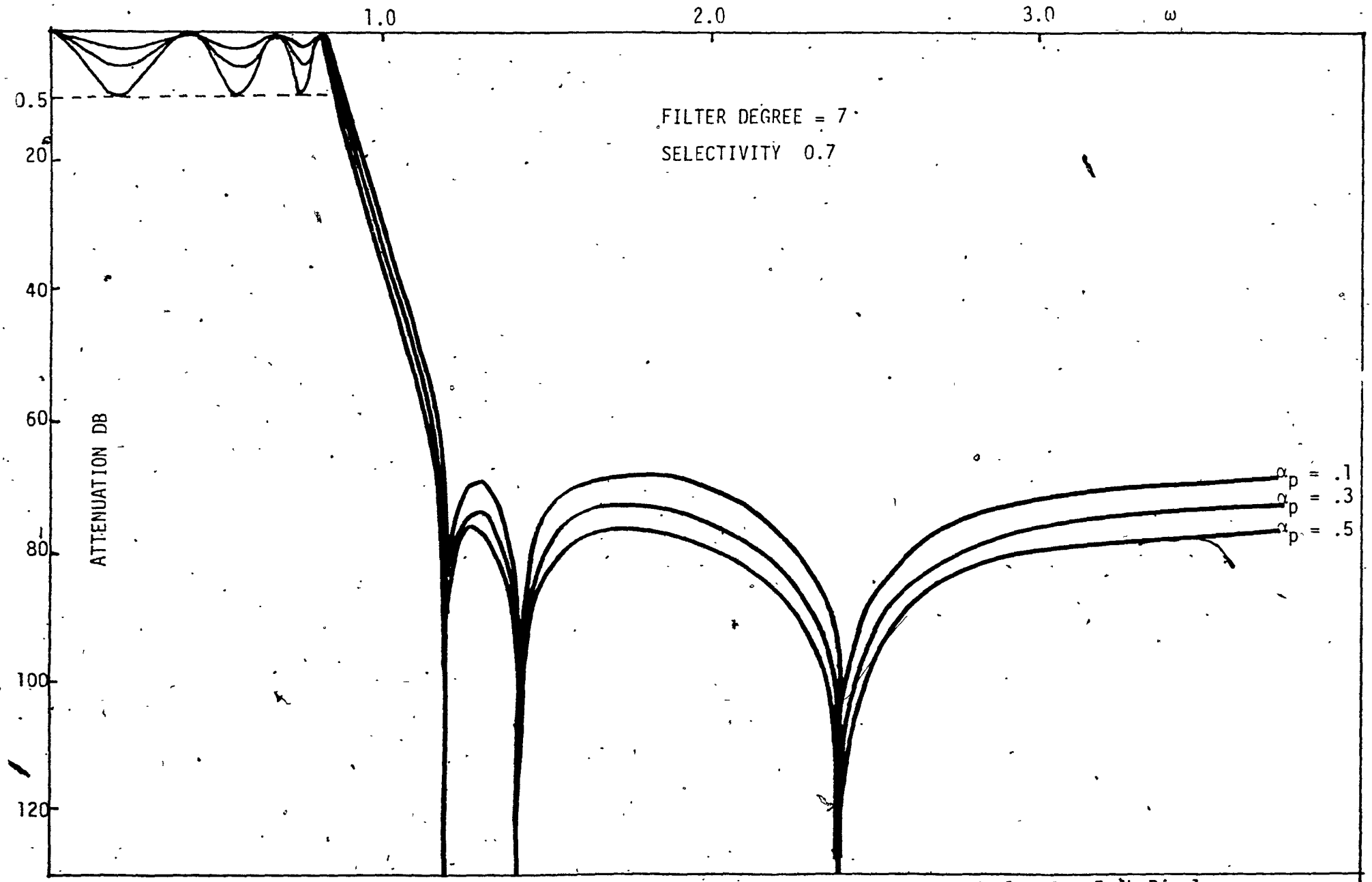


Fig.4.8: Amplitude Response for the 7th Order Filter at 0.7 Selectivity Factor and .1, .3, .5 db Ripple.

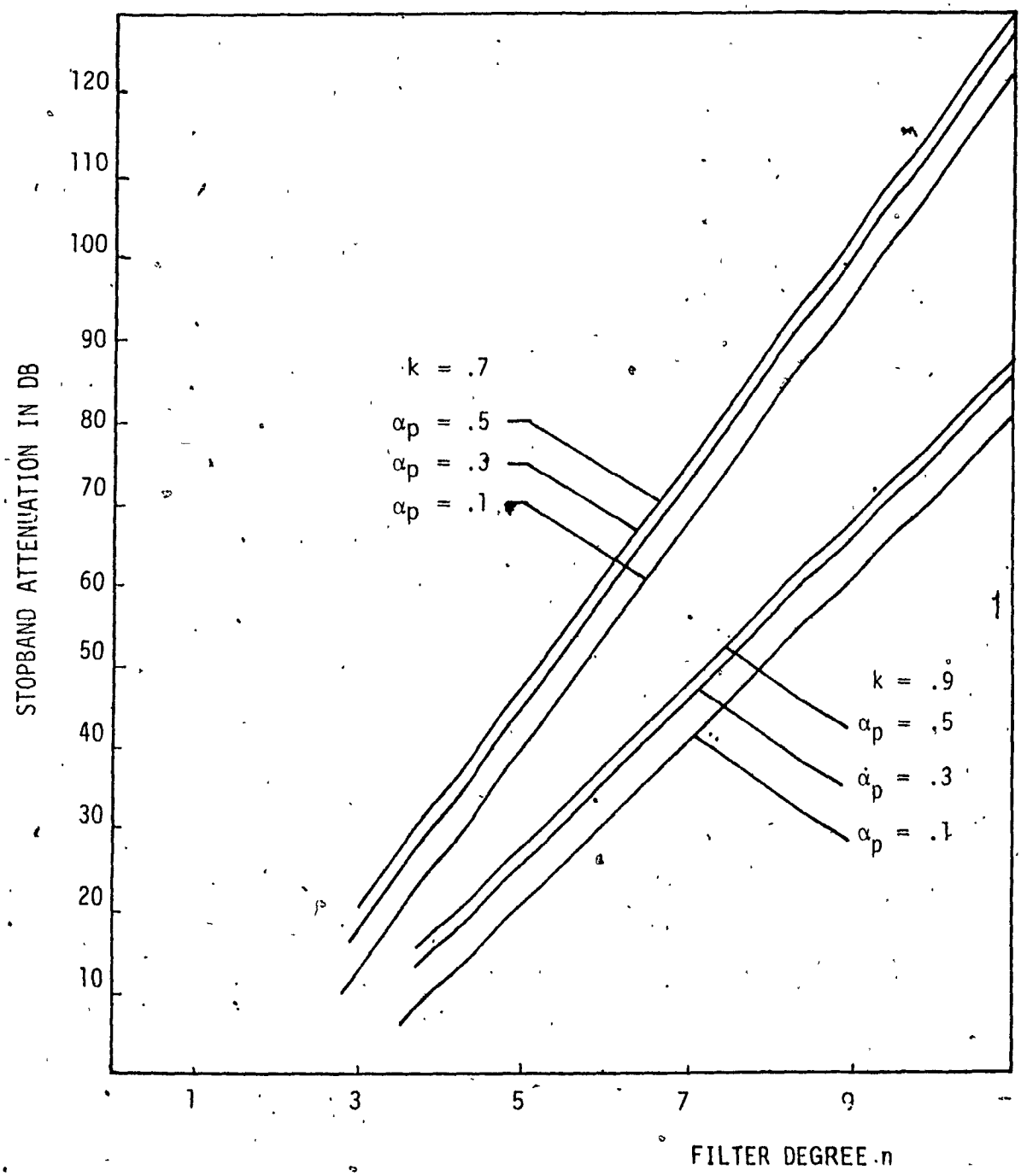


Fig. 4.9. Design Chart For Selecting Filter Parameters When k = .7 or .9 at Passband Ripples of .1, .3, .5 db.

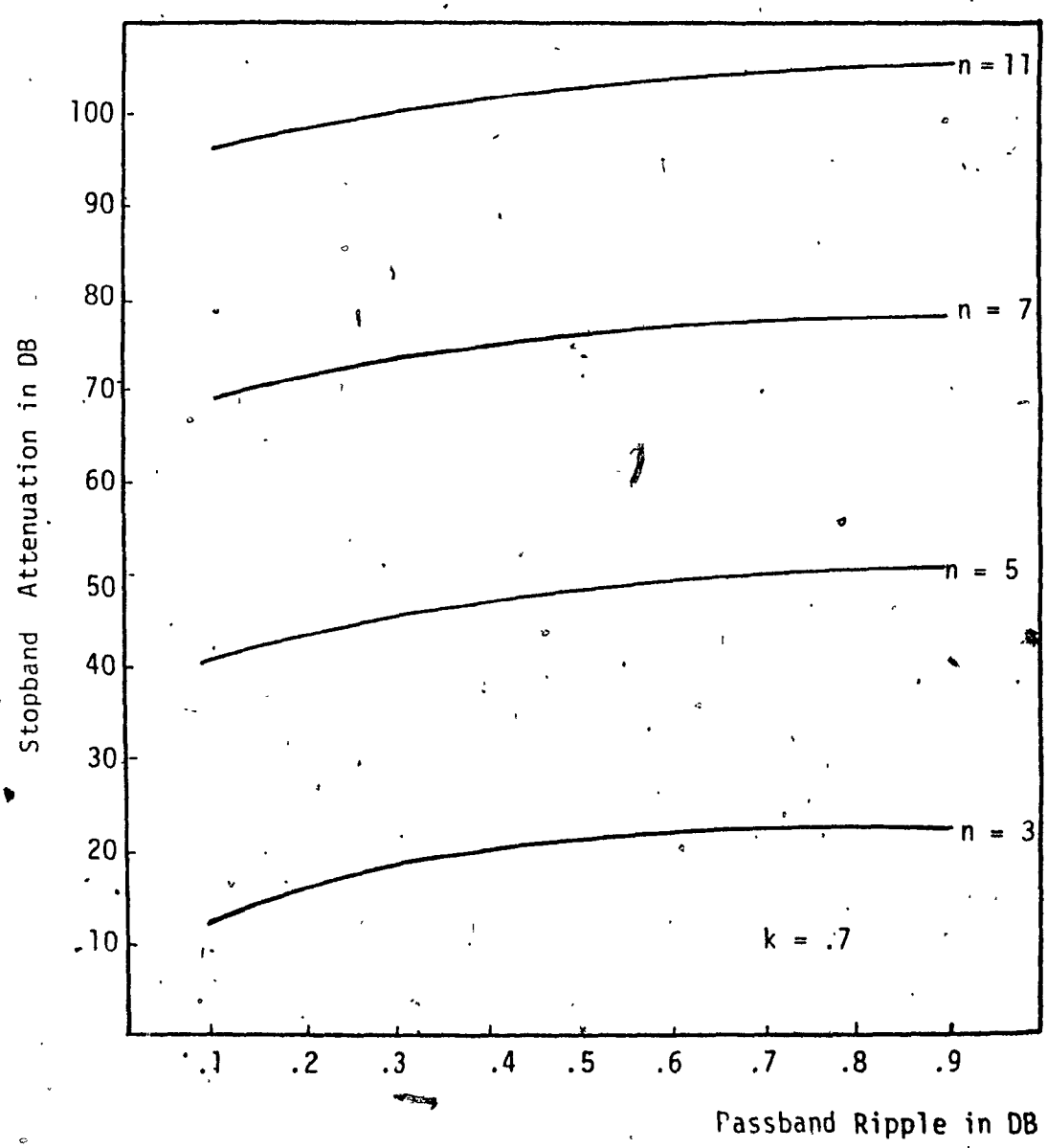


Fig. 4.10. Design Chart For Elliptic Filters of Degree 3, 5, 7 and 11 when k = .7

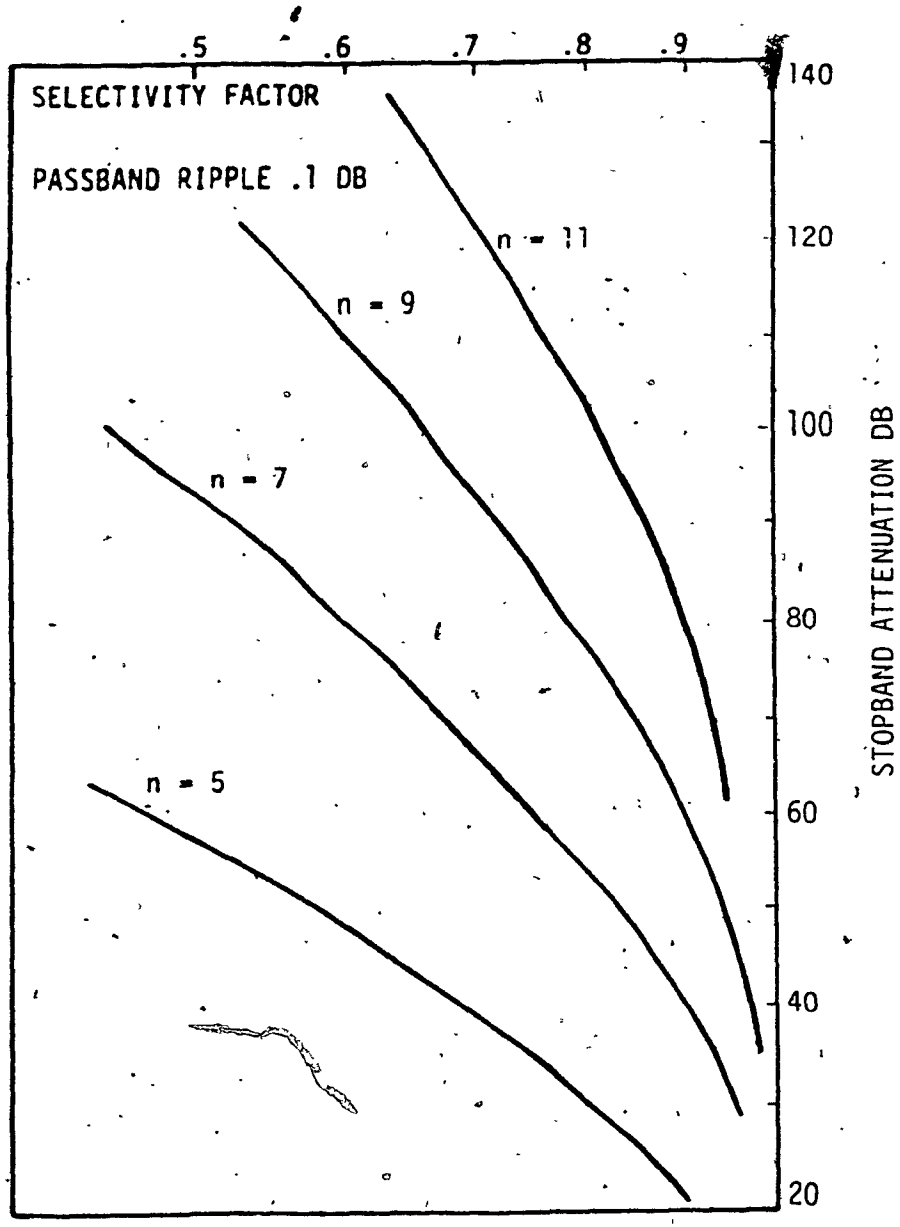


Fig. 4.11(a): Design Chart for Elliptic Filters with 0.1 db Passband Ripple.

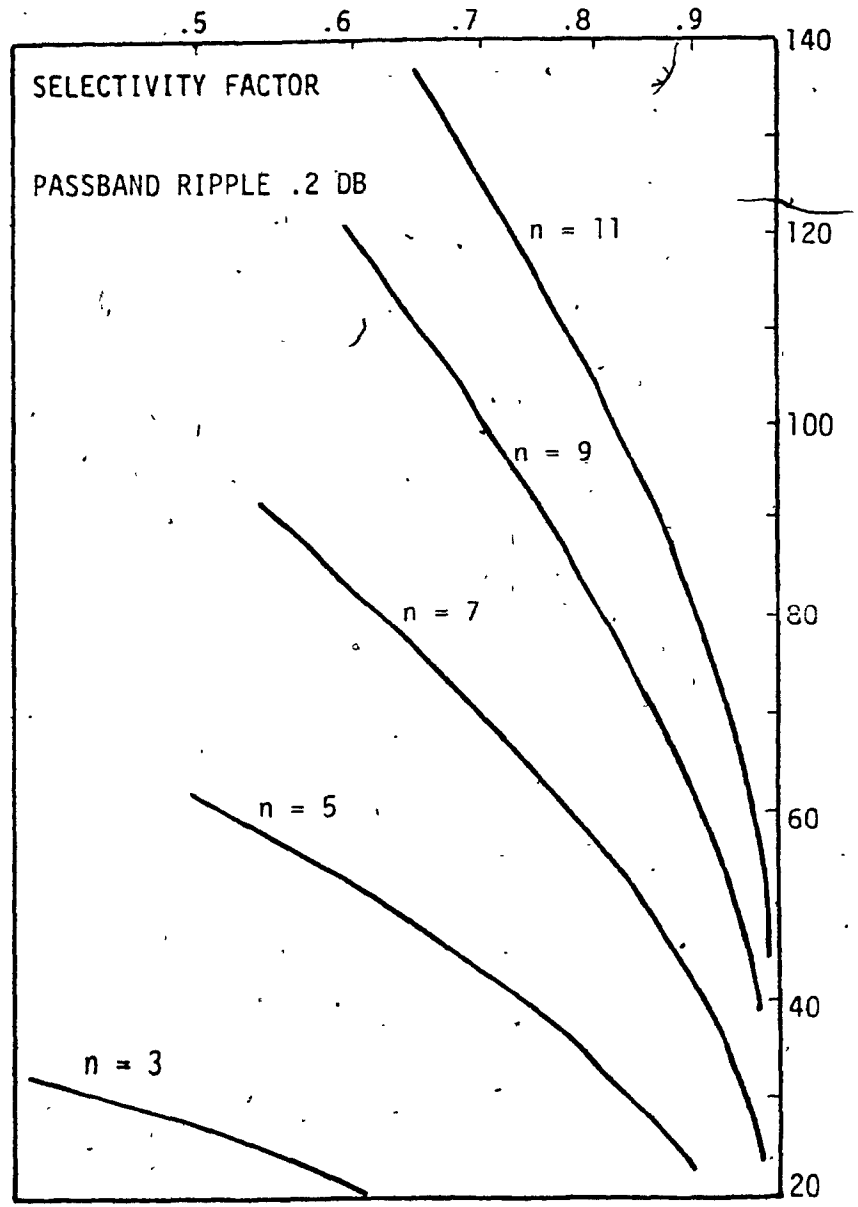


Fig. 4.11(b): Design Chart for Elliptic Filters with 0.2 db Passband Ripple.

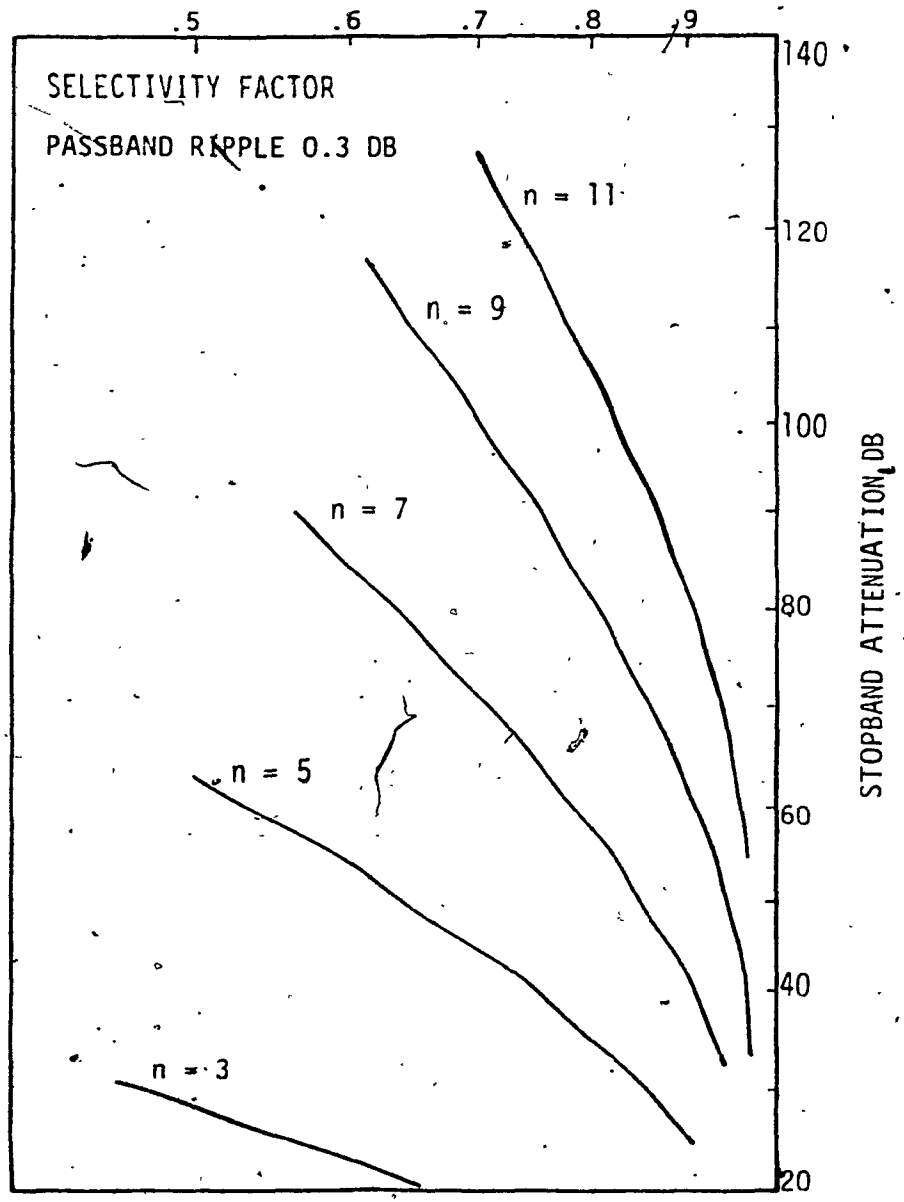


Fig.4.12(a): Design Chart for Elliptic Filters with 0.3 db Passband Ripple.

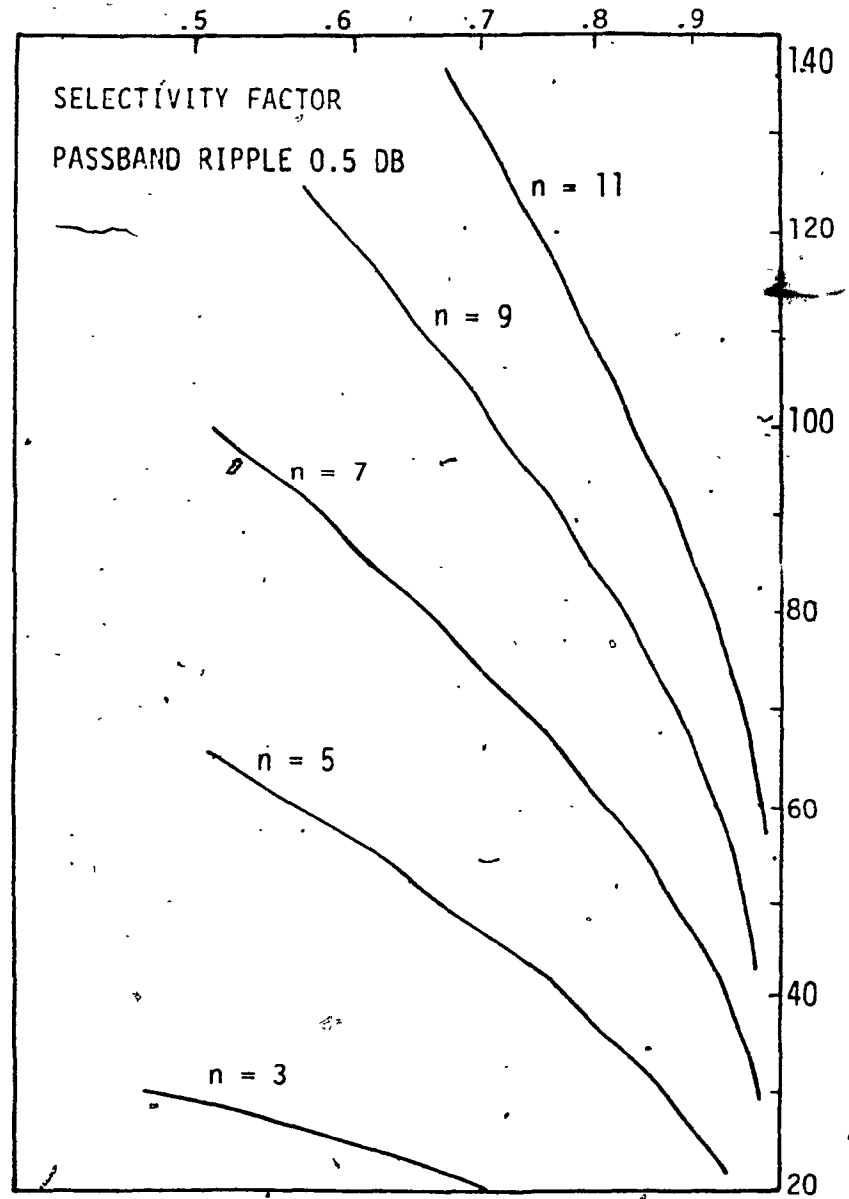


Fig.4.12(b): Design Chart for Elliptic Filters with 0.5 db Passband Ripple.

CHAPTER V

CONCLUSION

An approximation method has been described which leads directly to the transfer function of the elliptic lowpass filter. The method was used to develop computer-aided design programs. The structure and use of these programs is described. Practical examples are included to illustrate the speed and accuracy of the method. Design charts are also derived from which the design parameters can be selected.

The method involves little storage space in the computer and is extremely fast by comparison to the iteration algorithm methods or the method using the transformed frequency variable. The programs are very simple and very easy to use.

REFERENCES

1. J.H. Mole, "Filter Design Data for Communication Engineers", New York, Wiley, 1952.
2. A. Grossman, "Electric Wave Filters", in H. Pender and K. McIlwain, Ed., Electrical Engineers Handbook; Communication Electronics, New York: Wiley, 4th edi., 1950.
3. R.O. Rowlands, "Double-derived Terminatios", Wireless Engineer, Vol. 23, Nov. 1946, pp. 292-295.
4. J.E. Colin, "Généralisation des Filtres en échelle du Genré Zobel", Cables et Transmission, Vol. 12, July 1958, pp. 185-205.
5. S. Darlington, "Synthesis of Reactance Four-Poles", Journal of Mathematics and Physics, Vol. 18, Sept. 1939, pp. 257-353.
6. R. Saal and E. Ulbricht, "On the Design of Filters by Synthesis", IRE Trans. C.T., Dec. 1968.
7. R. Saal, "The Design of Filters Using the Catalogue if Normalized Low Pass Filters", Telefunken.
8. J.K. Skwirzynski, "Design Theory and Data for Electrical Filters", Van Nostrand, 1965.
9. G. Szentirmai, "Computer Aids in Filter Design", IEEE Trans. on Circuit Theory, Vol. CT-18, No.1, Jan. 1971, pp. 35-40.
10. H.J. Orchard, G.C. Temes, "Filter Design Using Transformed Variables", IEEE Trans, on Circuit Theory, Vol. CT-15, No. 4, Dec. 1968, pp.385-408.
11. C. Norek, "Product Method for the Calculation of the Effective Loss LC Filters", in Proc. Int. Symp. Network Theory, Belgrad, 1968, pp. 353-365.
12. J.K. Skwirzynski, "On Synthesis of Filters", IEEE Trans. on Circuit Theory, January 1971.

13. H.J. Orchard and D.F. Sheahann, "Inductorless Band-pass Filters", IEE J., Solid-State Circuits, Vol. SC-5, June 1970, pp. 108-118.

14. L. Weinberg, "Network Analysis and Synthesis", McGraw-Hill.

15. K.W. Henderson and W.H. Kautz, "Transient Response of Conventional Filters", IRE, Trans. Circuit Theory, Vol. CT-5, Dec. 1958, pp. 333-347.

16. A. Budake and P. Aronhime, "Transitional Butterworth-Chebychev Filters", IEEE, C.T. May 1971, pp. 413-415.

17. R.R. Geffe, "Simplified Modern Filter Design", Rider, 1963.

18. K.W. Henderson, "Nomographs for Designing Elliptic Function Filters", IRE Proc. Circuit Theory, Nov. 1958, pp. 1860-1864.

19. H.J. Orchard, "Computation of Elliptic Function of Rational Fractions of a Quarter Period", IRE Tr. C.T., Dec. 1958, pp. 352-355.

20. A. Grossman, "Synthesis of Tchebycheff Parameter Symmetricl Filters", Proc. of the IRE, April 1957, pp. 454-473.

APPENDIX A

```
PROGRAM     CALTRAN                                    CDC 6600 FTN                                    PA
            PROGRAM CALTRAN(INPUT,OUTPUT)
            DIMENSION W(20),A(20),B(20),C(24),P(20),P1(20),Z(20)
            DO 40 L=1,200
            READ10,N,TP,AK
   10       FORMAT(I2,2F10.6)
            AKP=SQRT(1.-AK**2)
            SAK=SQRT(AKP)
            E=.5*(1.-SAK)/(1.+SAK)
            Q=E+2.*E**5+15.*E**9
            TP=TP/10.
            TA=10.*ALOG10(10.**TP-1.)-10.*(2.*M+1.)*ALOG10(Q)-12.
            M=(N-1)/2
            PRINT 1
    1       FORMAT(4H1//,40X,*THE TRANSFER FUNCTION*,/,49X,*OF*,/,37X,*THE ELL
           1IPTIC LOWPASS FILTER*//)
            PRINT11,N,TP,AK,TA
   11       FORMAT(//20X,*N =*I2,5X,*RIPPLE =*,F4.2,2X,*DB*,5X,*SELECTIVITY FAC
           1TOR= *,F4.2//20X,*STOPBAND ATTENUATION =*,F10.6,5H   DB)
            PI=3.141592653589793/(2.*M+1.)
            TA=TA/(2*M+1)
            V=2.854548/TA-2.302584/TA*ALOG10(TP)+.00104/TA*TP**2
            A0=2**.25*(EXP(V)-EXP(-V)-Q**2.*(EXP(3.*V)-EXP(-3.*V))+Q**6*(EXP(5
           1.*V)-EXP(-5.*V)))/(1.-Q*(EXP(2.*V)+EXP(-2.*V))+Q**4*(EXP(4.*V)+EXP
           2(-4.*V)))
            PRINT 12,A0
   12       FORMAT(//22X,*ZEROES*,15X,*POLES*//4HX,*-*,F9.6)
            DO 20I=1,M
            W(I)=2.*Q**.25*(SIN(I*PI)-Q**2.*SIN(3.*I*PI)+Q**6*SIN(5.*I*PI))
           12.*Q*COS(2.*I*PI)+2.*Q**4*COS(4.*I*PI))
            P(I)=1/W(I)
            Z(I)=1.-(W(I)**2.)
            A(I)=A0*SQRT((1.-AK*W(I)**2)*(1.-W(I)**2/AK))/(1.+A0**2*W(I)**2)
            B(I)=W(I)*SQRT((1.+AK*A0**2)*(1.+A0**2/AK))/(1.+A0**2*W(I)**2)
            P1(I)=A(I)**2+B(I)**2
            C(I)=2.*A(I)
            PRINT 3,P(I),A(I),B(I)
   20       CONTINUE
    3       FORMAT(22X,*-*,F9.6,5X,*-*,F9.6,* +J*,F9.6)
            PRINT100,N,A0
  100       FORMAT(//32X,*I*/2X,*T(*,I2,*) = *,12(1H-)/30X,*S+ *,F10.6)
            DO111I=1,M
  111       PRINT110,(Z(I),C(I),P(I))
  110       FORMAT(//36X,*Z*/35X,*S +*,F8.6/25X,** *,27(1H-)/30X,*Z*/29X,*S +
           1*,F8.6,**S* +*,F8.6)
   40       CONTINUE
   41       STOP
            END
```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APPENDIX B

```
PROGRAM     FREQAN                                          CDC 6600 FTN                                    PA

            PROGRAM FREQAN(INPUT,OUTPUT)
            DIMENSION A(20),B(20),P(20),Z(20),C(20),X(20),AMP(20),ADB(20)
            DO40LL=1,200
            READ10,N,AP,SK
            IF(N.EQ.0)GOTO41
      10    FORMAT(I2,2F10.6)
      C     N=FILTER DEGREE, AP IS PASSBAND RIPPLE IN DBS, SK IS THE SELECTIVITY
      C     FACTOR
      C THE FREQUENCY ANALYSIS (ATTENUATION RESPOSE) FOR THE NORMALISED LOWPASS
      C ELLIPTIC FILTER
            CALL CALTRAN(N,AP,SK,AO,P,Z,A,B,C,AS)
            PRINT1
      1     FORMAT(1H1//,40X,*THE FREQUENCY ANALYSIS*,/,49X,*OF*,//,37X,*THE E
            1LLIPTIC LOWPASS FILTER*,//)
            M=(N+1)/2
            PRINT 11,N,AP,SK,AS
      11    FORMAT(20X,*N =*I2,5X,*RIPPLE =*,F4.2,2X,*DB*,5X,*SELECTIVITY FACT
            1OR= *,F9.6,/20X,*STOPBAND ATTENUATION =*F10.6,*   DB*)
      C M IS THE NUMBER OF FILTER SECTIONS
      C AS IS THE MINIMUM STOPBAND ATTENUATION IN DBS
            H=AO
            DO19I=1,M
      19    H=H*P(I)/Z(I)
            PRINT36
      36    FORMAT(//,10X,4(6X,*OMEGA*,3X,*AMP DB*))
            DO 30 K=2,100,2
            OM=K/100.
            DO 400 L=1,4
            LA=L-1
            X(L)=LA*OM
            AMP(L)=H/SQRT(X(L)**2.+A0**2)
            DO 32 J=1,M
            UM=Z(J)-X(L)**2
            DEN=SQRT((P(J)-X(L)**2)**2+(C(J)*X(L))**2)
      32    AMP(L)=AMP(L)*UM/DEN
      400   ADB(L)=20.*ALOG10(ABS(AMP(L)))
      30    PRINT34,(X(L),ADB(L),L=1,4)
      34    FORMAT(10X,4(6X,F4.2,3X,F7.2))
      40    CONTINUE
      41    STOP
            END
```

```
SUBROUTINE  CALTRAN                                         CDC 6600 FTN V3.0-P296 OPT=1  75/03/24. 16.48.16.   PA

            SUBROUTINE CALTRAN(N,AP,SK,AO,P,Z,A,B,C,AS)
            DIMENSION W(20),A(20),B(20),P(20),Z(20),C(20)
            M=(N-1)/2
            SSK=SQRT(1.-SK**2)
            SAK=SQRT(SSK)
            E=.5*(1.-SAK)/(1.+SAK)
            Q=E+2.*E**5+15.*E**9
            TB=AP/10.
            AS=10.*ALOG10(10.**TB-1.)-10.*N*ALOG10(Q)-12.
            D=3.14159265/N
            IA=2*N
            V=2.85484R/IA-2.302584/IA*ALOG10(AP)+.001104/IA*AP**2
            AO=Q**.25*(EXP(V)-EXP(-V)-Q**2.*(EXP(3.*V)-EXP(-3.*V))+Q**6*(EXP(5
           1.*V)-EXP(-5.*V)))/(1.-Q*(EXP(2.*V)+EXP(-2.*V))+Q**4*(EXP(4.*V)+EXP
           2(-4.*V)))
            DO 201=1,M
            W(I)=2.*Q**.25*(SIN(I*D)-Q**2.*SIN(3.*I*D)+Q**6*SIN(5.*I*D))/(1.-
           * 2.*Q*COS(2.*I*D)+2.*Q**4*COS(4.*I*D))
            Z(I)=1./(W(I)**2.)
            A(I)=AO*SQRT((1.-SK*W(I)**2)*(1.-W(I)**2/SK))/(1.+AO**2*W(I)**2)
            B(I)=W(I)*SQRT((1.+SK*AO**2)*(1.+AO**2/SK))/(1.+AO**2*W(I)**2)
            P(I)=A(I)**2+B(I)**2
            C(I)=2.*A(I)
      20    CONTINUE
            RETURN
            END
```


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APPENDIX C

```
PROGRAM    CHARTS                                        CDC 6600 FTN

           PROGRAM CHARTS(INPUT,OUTPUT)
           PRINT 1
      1    FORMAT(1H1//31X,*PROGRAM CHARTS*,//15X,*FILTER*,5X,*RIPPLE*,5X,*SE
          1LECTIVITY*,5X,*STOPBAND*/15X,*DEGREE*,7X,*DB*,9X,*FACTOR*,10X,*AT
          2DB*//)
           DO 40 L=1,200
           READ10,N,AP,SK
           IF(N.EQ.0)GOTO41
      10   FORMAT(I2,2F10.6)
           SSK=SQRT(1.-SK**2)
           SAK=SQRT(SSK)
           E=.5*(1.-SAK)/(1.+SAK)
           N=E+2.*E**5+15.*E**9
           TH=AP/10.
           AS=10.*ALOG10(10.**TH-1.)-10.*N*ALOG10(Q)-12.
C AS IS THE MINIMUM STOPBAND ATTENUATION IN DBS
      40   PRINT 11,N,AP,SK,AS
      11   FORMAT(16X,I2,9X,F4.2,10X,F4.2,11X,F6.2)
      41   STOP
           END
```