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A Family of Real Cubic Fields

Fares Fares

A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements for the Degree of Master of Science at Concordia University Montréal, Québec, Canada

August 1992

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ABSTRACT

A Family of Real Cubic Fields

Fares Fares

This thesis deals with a family of real cubic extensions $K_t/Q$ where $t$ is a rational integer. This family is parametrized by the cubic polynomials

$$f_t = x^3 - 3(t^2 + t + 1)x - (t^2 + t + 1)(2t + 1).$$

The ring of integers $O_{K_t}$ of $K_t$ is computed for all $t$ and the unit group $U_t$ is obtained under certain conditions (when $O_{K_t} = \mathbb{Z}[\alpha_t]$). In the general case, a bound on the index of a subgroup of $U_t$ is given. Also we investigate the arithmetic invariants of the family $K_t$, and get bounds for the regulator and the class number.
ACKNOWLEDGEMENT

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TABLE OF CONTENTS

1. Introduction ...................................................... 1
2. Ring of Integers of $K_t$ ........................................... 3
3. Group of Units of $K_t$ ............................................. 10
4. Class Number of $K_t$ ............................................... 17
5. References .......................................................... 23
1. Introduction

Consider the polynomial

\[ f = x^3 - 3\phi x - \phi' \in \mathbb{Z}[\tau][x] \]

where \( \phi = \tau^2 + \tau + 1, \phi' = 2\tau + 1 \in \mathbb{Z}[\tau] \) and \( \tau, x \) are indeterminates. Denote by \( \alpha \) one root of \( f \) in \( \overline{Q(\tau)} \) the algebraic closure of \( Q(\tau) \) and let \( K = Q(\tau)(\alpha) \).

For \( t \in \mathbb{Z}, \phi_t, \phi'_t, f_t \) and \( \alpha_t \) will refer to specializations of \( \phi, \phi', f \) and \( \alpha \) at \( \tau = t \) (hence \( f_t(\alpha_t) = 0 \)) and let \( K_t = Q(\alpha_t) \).

We list certain properties of \( f \):

**Proposition 1.1.** \( f \) is irreducible over \( Q(\tau) \), and \( [K : Q(\tau)] = 3 \).

**Proof:** For all rational integers \( t, \phi_t \equiv 1 \pmod{2}, \) and \( \phi'_t \equiv 1 \pmod{2}, \) therefore \( f_t \equiv x^3 - x - 1 \pmod{2} \). The latter polynomial has no linear factors in \( \mathbb{Z}/2\mathbb{Z}, \) hence \( f_t \) is irreducible over \( \mathbb{Z} \) and hence also over \( Q \) for all \( t \).

**Proposition 1.2.** \( K_t/Q \) is an abelian (cyclic) extension for all \( t \). In particular, \( K_t \) is a real field.

**Proof:** The discriminant of \( f_t \) is \((9\phi_t)^2\), which is a perfect square. It follows that the cubic extension \((K_t/Q)\) is a Galois extension for all rational integers \( t \). It is also abelian since its Galois group has order 3.

**Proposition 1.3.** (i) \( \phi_t \) is not divisible by 9 for all \( t \in \mathbb{Z}, \) and \( \phi_t \) is divisible by 3 if and only if \( t \equiv 1 \pmod{3} \).

(ii) \( \phi'_t \) is divisible by 9 if and only if \( t \equiv 4 \pmod{9} \)

(iii) \( \gcd(\phi_t, \phi'_t) \) equals 3 if \( t \equiv 1 \pmod{3}, \) and 1 otherwise.
Proof: The congruence $\phi_t \equiv 0 \pmod{3}$ is satisfied iff $t \equiv 1 \pmod{3}$, but $\phi_t \not\equiv 0 \pmod{9}$ for $t \equiv 1, 4, 7 \pmod{9}$. Thus 9 does not divide $\phi_t$ for all $t$ and (i) follows. To prove (ii) consider the reduction of $\phi'_t$ modulo 9. Finally, (iii) follows from (i) and the observation that $4\phi_t - \phi'_t^2 = 3$. \[\square\]
2. Ring of integers of $K_t$

Denote by $O_{K_t}$ the ring of integers of the number field $K_t = Q(\alpha_t)$.

**Proposition 2.1.** If $\phi_t = 3^{n_0}p_1^{n_1}\ldots p_s^{n_s}$, where $3, p_1, \ldots, p_s$ are distinct primes, $n_0, n_1, \ldots, n_s \in \mathbb{Z}$, and $\delta_t$ is disc $(K_t)$, the discriminant of $K_t$, then for $1 \leq i \leq s$

$$n_i \not\equiv 0 \pmod{3} \text{ implies that } p_i | \delta_t$$

and

$$t \equiv 1 \pmod{3} \text{ and } t \not\equiv 4 \pmod{9} \text{ imply that } 3 | \delta_t.$$

**Proof:** Since $f_t(\alpha_t) = 0$ we see that $\alpha_t \in O_{K_t}$ and that $\alpha_t^3 = 3\phi_t\alpha_t + \phi_t\phi_t'$. Taking ideals in $O_{K_t}$,

$$(\alpha_t)^3 = (\phi_t)(3\alpha_t + \phi_t'). \quad (1)$$

Let $n_i \not\equiv 0 \pmod{3}$. Take $q = p_i$, where $1 \leq i \leq s$, and let $Q$ be a prime ideal in $O_{K_t}$ above $q$ ($Q \cap \mathbb{Z} = q\mathbb{Z}$), with ramification index $e$. $Q$ divides $(\phi_t)$ implies that $Q$ divides $(\alpha_t)$. But $q$ does not divide $\phi_t'$ (Proposition 1.3), thus $Q$ does not divide $(\phi_t')$ (otherwise, $Q \cap \mathbb{Z} = q\mathbb{Z}$ would include $(\phi_t') \cap \mathbb{Z} = \phi_t'\mathbb{Z}$) and hence $Q$ does not divide $(3\alpha_t + \phi_t')$ ($3\alpha_t \in Q$ and $\phi_t' \not\in Q$). In terms of $Q$-valuation,

$$3\nu_Q(\alpha_t) = \nu_Q(\phi_t) \quad (2)$$

where $\nu_Q(\alpha_t), \nu_Q(\phi_t)$ are the exponents of $Q$ in the decomposition of $(\alpha_t)$ and $(\phi_t)$ into primes in $O_{K_t}$. But $\nu_Q(\phi_t) = en_i$. Then by equation (2), 3 does not divide $n_i$ implies that 3 divides $e$, and therefore that $q$ ramifies in $O_{K_t}$. This proves that in this case, $q$ divides $\delta_t$ the discriminant of $K_t$. 

3
Next, let \( t \equiv 1 \pmod{3}, t \not\equiv 4 \pmod{9} \), then both \( \phi_t \) and \( \phi'_t \) are divisible by 3 but not by 9, and so \( n_0 = 1 \). Let \( U \) be a prime ideal in \( O_{K_t} \) above 3, with ramification index \( e_3 \). Equation (1) becomes

\[
(\alpha_t)^3 = (\phi_t)(3)(\alpha_t + \frac{\phi'_t}{3}).
\]

Since \( U \) divides \( \alpha_t \) and does not divide \( \frac{\phi'_t}{3} \), \( U \) does not divide \( (\alpha_t + \frac{\phi'_t}{3}) \). Hence \( 3\nu_U(\alpha_t) = e_3 + e_3 = 2e_3 \), thus 3 divides \( e_3 \) and therefore 3 ramifies in \( O_{K_t} \). This proves that 3 divides \( \delta_t \).

The next proposition produces a \( \mathbb{Z} \)-submodule of \( O_{K_t} \), whose index is determined by Proposition 2.3. Let \( \phi_t = 3^{n_0}p_1^{n_1} \cdots p_s^{n_s} \), where \( 3, p_1, \ldots, p_s \) are distinct primes, and \( n_0, \ldots, n_s \) nonnegative integers. Write \( n_i = 3k_i + r_i \), where \( r_i, k_i \) are natural numbers, \( 0 \leq r_i < 3 \) and \( 1 \leq i \leq s \). Note that \( n_0 \) is either 1 (if \( t \equiv 1 \pmod{3} \)) or 0 (otherwise). For \( 1 \leq i \leq s \), let

\[
j_i = \begin{cases} 
0 & \text{if } r_i = 0 \\
\frac{r_i - 1}{2} & \text{otherwise }
\end{cases} \quad \text{and} \quad j_0 = \begin{cases} 
1 & \text{if } t \equiv 4 \pmod{9} \\
0 & \text{otherwise.}
\end{cases}
\]

Thus

\[
j_i = \begin{cases} 
0 & \text{if } n_i \equiv 0, 1 \pmod{3} \\
1 & \text{if } n_i \equiv 2 \pmod{3}.
\end{cases}
\]

Proposition 2.2. (i) \( \xi_t = (\alpha_t/3^{j_0}p_1^{j_1} \cdots p_s^{j_s}) \) and \( \xi'_t = (\alpha_t^2/3^{n_0+j_0}p_1^{2k_1+j_1} \cdots p_s^{2k_s+j_s}) \)
are algebraic integers.

(ii) \( \frac{1}{n} \xi_t \in O_{K_t} \ (n \in \mathbb{Z}) \) implies that \( n = \pm 1 \)

\( \frac{1}{n} \xi'_t \in O_{K_t} \ (n \in \mathbb{Z}) \) implies that \( n = \pm 1 \)

**Proof:** If \( \omega \in O_{K_t} \) has a minimal polynomial \( h(x) = x^3 + m_2x^2 + m_1x + m_0 \), where \( m_0, m_1 \) and \( m_2 \) are rational integers, and if \( a \in \mathbb{Z} \), then \( (\omega/a) \in O_{K_t} \) if
and only if \( a \) divides \( m_2 \), \( a^2 \) divides \( m_1 \) and \( a^3 \) divides \( m_0 \). Now since \( \alpha_t \) has the minimal polynomial \( x^3 - 3\phi_t x - \phi_t \phi'_t \), it follows that \( \frac{\alpha_t}{3} \in O_{K_t} \) if and only if \( 27|\phi_t \phi'_t \) and \( 9|3\phi_t \). This is equivalent to \( 3|\phi_t \) and \( 9|\phi'_t \), which is equivalent to \( t \equiv 4 \pmod{9} \).

Also, \( \frac{\alpha_t}{9} \) does not belong to \( O_{K_t} \) for any \( t \) since \( 9 \) does not divide \( \phi_t \). Hence,

\[
\begin{align*}
\frac{\alpha_t}{3} & \in O_{K_t} \quad \text{if and only if } t \equiv 4 \pmod{9} \\
\frac{\alpha_t}{9} & \notin O_{K_t} \quad \text{for all } t \in \mathbb{Z}.
\end{align*}
\tag{3}
\]

Taking \( q \) to be a prime different from 3, then for \( m \in \mathbb{Z} \), \( \frac{\alpha_t}{q^m} \in O_{K_t} \) if and only if \( q^3 \mid \phi_t \phi'_t \) and \( q^2 \mid 3\phi_t \). Thus

\[
\frac{\alpha_t}{q^m} \in O_{K_t} \quad \text{if and only if } q^3 \mid \phi_t.
\tag{4}
\]

Similarly, \( \alpha_t^2 \) satisfies the minimal polynomial : \( x^3 - 6\phi_t x^2 + 9\phi_t^2 x - \phi_t^2 \phi'_t \). Consequently, \( \frac{\alpha_t^2}{3} \in O_{K_t} \) if and only if \( 3|\phi_t \) (since \( 3|\phi_t \) if and only if \( 3|\phi'_t \)). Also \( \frac{\alpha_t^2}{3^m} \in O_{K_t}, \ m \in \mathbb{Z}, \ m > 1 \) if and only if \( m = 2, 3|\phi_t, 9|\phi'_t \). Hence

\[
\begin{align*}
\frac{\alpha_t^2}{3} & \in O_{K_t} \quad \text{if and only if } t \equiv 1 \pmod{3} \\
\frac{\alpha_t^2}{3^m} & \in O_{K_t}, \ m \in \mathbb{Z}, \ m > 1 \quad \text{if and only if } m = 2 \text{ and } t \equiv 4 \pmod{9}.
\end{align*}
\tag{5}
\]

For \( q \) a rational prime different from 3, \( m \in \mathbb{Z} \)

\[
\frac{\alpha_t^2}{q^m} \in O_{K_t} \quad \text{if and only if } q^3 \mid \phi_t^2.
\tag{6}
\]

In the above computations, we've used repeatedly the fact that \( \gcd(\phi_t, \phi'_t) = 1 \) or 3.

By equation (3) \( \alpha_t/3^{j_0} \in O_{K_t} \) and \( \alpha_t/9 \notin O_{K_t} \) for any \( t \in \mathbb{Z} \). Also equation (5) implies that \( \frac{\alpha_t^2}{3^{n_0 + j_0}} \) is an integer (since \( t \equiv 1 \pmod{3} \) if and only if \( n_0 = 1 \), and \( t \equiv 4 \pmod{9} \) is equivalent to \( n_0 + j_0 = 2 \)). By equation (4) we have that for \( m \in \mathbb{Z} \) and \( p_i \) one of the primes of \( \phi_t \) with exponent \( n_i, 1 \leq i \leq s \), \( \alpha_t/p_i^m \in O_{K_t} \) if
$p_i^{3m}$ divides $\phi_i$ that is $3m \leq n_i = 3k_i + r_i$ or $m \leq k_i + \frac{r_i}{3}$. Therefore, $\alpha_i/p_i^{k_i}$ is an integer. Also by equation (6), we have that $\frac{\alpha_i^2}{p_i^{k_i}} \in O_{K_i}$ is equivalent to $3m \leq 2n_i$.

This is true if and only if $m \leq 2k_i + \frac{2r_i}{3}$. Hence,

$$\begin{cases} \frac{\alpha_i^2}{p_i^{k_i}} \in O_{K_i} & \text{if } r_i = 0, 1 \\ \frac{\alpha_i^2}{p_i^{k_i+r_i}} \in O_{K_i} & \text{if } r_i = 2 \end{cases}$$

Therefore,

$$\frac{\alpha_i^2}{p_i^{2k_i+j_i}} \in O_{K_i} \quad \text{where } j_i = \begin{cases} 0 & \text{if } n_i \equiv 0, 1 \pmod{3} \\ 1 & \text{if } n_i \equiv 2 \pmod{3} \end{cases}$$

To complete the proof of (i), let $\theta \in O_{K_i}$, $c_1$, $c_2$, relatively prime rational integers be such that $\frac{\theta}{c_1}$ and $\frac{\theta}{c_2}$ belong to $O_{K_i}$. Then there exist $l_1, l_2 \in \mathbb{Z}$ such that $l_1 c_1 + l_2 c_2 = 1$. Therefore $\frac{\theta}{c_1 c_2} = l_2 \frac{\theta}{c_1} + l_1 \frac{\theta}{c_2} \in O_{K_i}$.

To prove (ii) note that the conditions we obtained on the primes of the denominators of an integer of the form $\frac{\alpha_i}{n}$ or $\frac{\alpha_i^2}{n}$ (equations (3), (4), (5) and (6)) are necessary conditions.

We describe the ring of integers of $K_i$.

**Proposition 2.3.** With the same notation as in proposition 2.2, an integral basis for $O_{K_i}$ is given by $\{1, (\alpha_i/3^0 p_1^{k_1} \ldots p_s^{k_s}), (\alpha_i^2/3^{n_0+j_0} p_1^{2k_1+j_1} \ldots p_s^{2k_s+j_s})\}$

**Proof:** Put $\xi_i = (\alpha_i/3^0 p_1^{k_1} \ldots p_s^{k_s})$, $\xi_i' = (\alpha_i^2/3^{n_0+j_0} p_1^{2k_1+j_1} \ldots p_s^{2k_s+j_s})$, $A = \mathbb{Z}[\alpha_i]$ and $B = \mathbb{Z}[1, \xi_i, \xi_i']$. From Proposition 2.2(i), it follows that the index of the $\mathbb{Z}$-module $A$ in $B$ is $[B : A] = 3^{n_0} p_1^{n_1} \ldots p_s^{n_s}$, where

$$n_0' = \begin{cases} 3 & \text{if } n_0 = 1 \text{ and } t \equiv 4 \pmod{9} \\ 1 & \text{if } n_0 = 1 \text{ and } t \not\equiv 4 \pmod{9} \\ 0 & \text{if } n_0 = 0 \end{cases}$$

6
and for $1 \leq i \leq s$,

\[
\begin{cases} 
  n'_i = n_i & \text{if } 3 | n_i \\
  n'_i = n_i - 1 & \text{otherwise.}
\end{cases}
\]

By proposition 2.1, the number $(3^{n''_0} p_1^{n''_1} \ldots p_s^{n''_s})^2$ divides $\text{disc}(K_t)$ (the power 2 appears because the discriminant of $K_t$ is a perfect square) where

\[
n''_0 = \begin{cases} 
  1 & \text{if } n_0 = 1 \text{ and } t \not\equiv 4 \pmod{9} \\
  0 & \text{otherwise}
\end{cases}
\]

and for $1 \leq i \leq s$,

\[
\begin{cases} 
  n''_i = 0 & \text{if } 3 | n_i \\
  n''_i = 1 & \text{otherwise.}
\end{cases}
\]

Now $\text{disc}(f_t) = [O_{K_t} : B]^2 [B : A]^2 \text{disc}(K)$ where $\text{disc}(f_t) = (9\phi_t)^2$. Hence

\[
[O_{K_t} : B] = \begin{cases} 
  1 & \text{if } t \equiv 4 \pmod{9} \\
  1, 3, \text{ or } 9 & \text{otherwise.}
\end{cases}
\]

This means that $O_{K_t} = B$ if $t \equiv 4 \pmod{9}$. We prove that this is also true when $t \not\equiv 4 \pmod{9}$.

We remark that $[O_{K_t} : B] = 3$ or 9 implies that there exists $\epsilon \in B$ such that $(\epsilon/3) \in O_{K_t} - B$. Let $\epsilon = \lambda_0 + \lambda_1 \xi_t + \lambda_2 \xi_t^2$ where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}$ be such that $\epsilon/3 \in O_{K_t} - B$, suppose that $t \not\equiv 4 \pmod{9}$. We will show that this leads to a contradiction, which in turn shows that the index of $B$ in $O_{K_t}$ cannot be 3 or 9. This shows that the index is 1 according to our proposition.

Observe that 3 cannot divide $\lambda_0, \lambda_1, \lambda_2$ simultaneously (since $\epsilon/3 \not\in B$) nor can it divide exactly 2 of those integers because $1/3, \xi_t/3$ and $\xi_t^2/3$ are not integers (Proposition 2.2(ii)).

In the remaining cases 3 divides none of the integers $\lambda_0, \lambda_1, \lambda_2$, or only one of them. Write $D_1 = p_1^{k_1} \ldots p_s^{k_s}$ and $D_2 = p_1^{2k_1+j_1} \ldots p_s^{2k_s+j_s}$. Since $j_0 = 0$ ($t \not\equiv 4 \pmod{9}$)
(mod 9)) it follows that \( \xi_t = (\alpha_t / D_1) \) and \( \xi'_t = (\alpha^2 / 3^{n_0} D_2) \). Note that \( D_1 \) divides \( D_2 \) and 3 does not divide \( D_2 \). Now

\[
\frac{\epsilon}{3} D_2 = \frac{\lambda_0 D_2}{3} + \frac{\lambda_1 D_2}{3 D_1} \alpha_t + \frac{\lambda_2}{3^{n_0 + 1}} \alpha^2_t.
\]

(7)

First consider the case where \( n_0 = 0 \) \((t \not\equiv 1 \pmod{3})\). Reading \( \frac{\epsilon}{3} D_2 \) modulo \( \mathbb{Z} \[\alpha_t\] \) we conclude that one of the following elements: \( \pm \frac{1}{3} \pm \frac{\alpha_t}{3}, \pm \frac{1}{3} \pm \frac{\alpha^2_t}{3}, \pm \frac{\alpha_t}{3}, \pm \frac{1}{3} \pm \frac{\alpha_t}{3} \pm \frac{\alpha^2_t}{3} \) belongs to \( O_{K_t} \). This list of elements could be restricted further to \((\frac{1}{3} \pm \frac{\alpha_t}{3}), (\frac{1}{3} \pm \frac{\alpha^2_t}{3}), (\frac{\alpha_t}{3} \pm \frac{\alpha^2_t}{3}), (\frac{1}{3} \pm \frac{\alpha^2_t}{3} \pm \frac{\alpha^2_t}{3}) \). We use the norms of these elements to prove that none of the above is an integer.

The following table was produced using Maple:

<table>
<thead>
<tr>
<th>((\text{integer} \times 3))</th>
<th>norm</th>
<th>(\text{norm} \equiv 0 \pmod{27})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1 + \alpha_t))</td>
<td>(2t^3 - 1)</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 - \alpha_t))</td>
<td>(-2t^3 - 6t^2 - 6t - 3)</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 - \alpha^2_t))</td>
<td>((1 - 2t^3)(2t^3 + 6t^2 + 6t + 3))</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 + \alpha^2_t))</td>
<td>(17 + 30t + 48t^2 + 40t^3 + 30t^4 + 12t^5 + 4t^6)</td>
<td>no solution</td>
</tr>
<tr>
<td>((\alpha_t + \alpha^2_t))</td>
<td>((2t^3 + 3t^2 + 3t + 1)(2t^3 - 1))</td>
<td>(t \equiv 4, 13, 22 \pmod{27})</td>
</tr>
<tr>
<td>((\alpha_t - \alpha^2_t))</td>
<td>(-2t^3 + 3t^2 + 3t + 1)(2t^3 + 6t^2 + 6t + 3))</td>
<td>(t \equiv 4, 13, 22 \pmod{27})</td>
</tr>
<tr>
<td>((1 + \alpha_t + \alpha^2_t))</td>
<td>(15t^4 + 12t^3 + 18t^2 + 9t + 9 + 4t^6 + 6t^7)</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 + \alpha_t - \alpha^2_t))</td>
<td>(-27t^4 - 20t^3 - 6t^2 + 3t + 1 - 18t^5 - 4t^6)</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 - \alpha_t - \alpha^2_t))</td>
<td>(3t^4 + 12t^3 + 12t^2 + 3t - 1 - 4t^5 - 6t^6)</td>
<td>no solution</td>
</tr>
<tr>
<td>((1 - \alpha_t + \alpha^2_t))</td>
<td>(45t^4 + 68t^3 + 72t^2 + 45t + 19 + 18t^5 + 4t^6)</td>
<td>no solution</td>
</tr>
</tbody>
</table>

The only possible solution namely \( t \equiv 4, 13, 22 \pmod{27} \) leads to \( t \equiv 4 \pmod{9} \), a case that we excluded at the start. Note that the above proves that this list of
numbers are not integers when \( n_0 = 1 \) also.

Next consider the case when \( n_0 = 1 \). If equation (7) is reduced modulo \( \mathbb{Z}[\alpha_4] \) it follows that one of the elements: \( \pm \frac{\alpha}{3} + k \frac{a^2}{9}, \pm \frac{1}{3} + k \frac{a^2}{9}, \pm \frac{1}{3} \pm \frac{\alpha}{3}, \pm \frac{1}{3} \pm \frac{\alpha}{3} + k \frac{a^2}{9} \), where \( 1 \leq k \leq 8 \), should be an integer. Since \( \alpha^2/3 \) is now an integer, \( k \) can be restricted to \( 1, 2 \) (read the above list mod \( \mathbb{Z}[\alpha_4, \alpha^2/3] \)). Finally use the norms of these numbers to prove that they are not integers. \( \square \)
3. Group Of Units Of $O_{K_i}$

We compute the roots of the polynomial $f_t$.

**Proposition 3.1.** If $\alpha_t$ is one root of $f_t$ and

\[
\begin{align*}
\beta_t &= -\alpha_t^2 + t\alpha_t + 2\phi_t \\
\gamma_t &= \alpha_t^2 - (\phi_t^2 + 1)\alpha_t - 2\phi_t
\end{align*}
\]

then $\beta_t$ and $\gamma_t$ are the other two roots of $f_t$.

**Proof:** From $\beta_t + \gamma_t = -\alpha_t$ and $\beta_t\gamma_t = (\phi_t\phi_t'/\alpha_t) = \alpha_t^2 - 3\phi_t$, we see that

\[
(\beta_t - \gamma_t)^2 = -3\alpha_t^2 + 12\phi_t. \tag{1}
\]

On the other hand, $\text{disc}(f_t) = (9\phi_t)^2$, thus

\[
(\alpha_t - \beta_t)(\alpha_t - \gamma_t)(\beta_t - \gamma_t) = \pm 9\phi_t. \tag{2}
\]

But $(\alpha_t - \beta_t)(\alpha_t - \gamma_t) = \alpha_t^2 - (\beta_t + \gamma_t)\alpha_t + \gamma_t\beta_t$, and so

\[
(\alpha_t - \beta_t)(\alpha_t - \gamma_t) = 3\alpha_t^2 - 3\phi_t. \tag{3}
\]

Equations (1), (2) and (3) imply that $(3\alpha_t^2 - 3\phi_t)(-3\alpha_t^2 + 12\phi_t) = \pm 9\phi_t(\beta_t - \gamma_t)$. Using the fact that $\alpha_t^3 = 3\phi_t\alpha_t + \phi_t\phi_t'$, the last equation gives $9\phi_t(\beta_t - \gamma_t) = \pm(18\phi_t\alpha_t^2 - 9\phi_t^2\phi_t\alpha - 36\phi_t^2)$. The following system of equations

\[
\begin{align*}
\beta_t - \gamma_t &= \pm(2\alpha_t^2 - \phi_t\alpha_t - 4\phi_t) \\
\beta_t + \gamma_t &= -\alpha_t
\end{align*}
\]

results in the expressions for $\beta_t$ and $\gamma_t$. (Choosing either the plus or minus sign on the right side of the first equation interchanges the solutions for $\beta_t$ and $\gamma_t$.) In what follows $\alpha_t$ refers to the root of largest absolute value (no two roots have the
same absolute value as $K_t$ is real) and $\beta_t$ and $\gamma_t$ are fixed by the statement of the proposition. 

Now we turn to the problem of finding the group $U_t = O_{K_t}^*$ of units. We know that $K_t$ is a real field. Therefore all of its conjugates are real, and the group of roots of unity of $K$ is $W(K) = \{-1, 1\}$. Hence the structure of $U_t$ is as such:

$$U_t = \{-1, 1\} \times \mathbb{Z}^2.$$

**Proposition 3.2.** $u_t = \alpha + t$ and $v_t = \alpha + t + 1$ are two independent units of $K_t$ for all $t \in \mathbb{Z}$.

**Proof:** Since $f_t(-t) = -1$, it follows that norm $(\alpha_t + t) = (\alpha_t + t)(\beta_t + t)(\gamma_t + t) = -(t - \alpha_t)(-t - \beta_t)(-t - \gamma_t) = -f(-t) = 1$. This proves that $u_t = \alpha_t + t$ is a unit. Also, note that

$$v_t^{-1} = \alpha_t^2 - (1/2)(2t + 2)\alpha_t - 2t - t - 2 = \gamma_t + t = u_t^{(1)}$$

where $u_t^{(1)}$ is a conjugate of $u_t$. Hence, $v_t$ is a unit.

It is sufficient to prove that $u_t$ and $u_t^{(1)}$ are independent. Consider $V_t = \langle u_t, u_t^{(1)} \rangle$ the subgroup of $U_t$ generated by $u_t$ and $u_t^{(1)}$. Let $\{1, \sigma, \sigma^2\}$ be the Galois group of $K_t/\mathbb{Q}$ where $\sigma(\alpha_t) = \gamma_t$. Let $\sigma(u_t) = u_t^{(1)}$ and $\sigma^2(u_t) = w_t$. From $w_t = 1/u_t u_t^{(1)}$, it follows that the Galois group acts on $V_t$. Suppose that $u_t$ and $u_t^{(1)}$ are dependent. Then $V_t$ has rank 1 (as a free abelian group). This makes it isomorphic to $\mathbb{Z}$, which admits only 2 isomorphisms (as an additive group), one is the identity and the other of order 2. But $\sigma$ and $\sigma^2$ act nontrivially on $u_t$ ($u_t \notin \mathbb{Q}$).

This contradiction proves that $u_t$ and $v_t$ are independent. 

11
To explicitly determine the units of \( K \) we will consider the following quantity

\[
S(\epsilon) = \frac{1}{2} \sum_{i,j=1,\ldots,3} (\epsilon^{(i)} - \epsilon^{(j)})^2
\]

where \( \epsilon^{(i)}, i = 1 \ldots 3 \), are the conjugates of \( \epsilon \), \( \epsilon \in O_K \) (\( \epsilon^{(1)} = \epsilon, \epsilon^{(2)} = \sigma(\epsilon) \) and \( \epsilon^{(3)} = \sigma^2(\epsilon) \)). If an integer \( \theta \) is given by \( \theta = m\alpha_t^2 + r\alpha_t + s \) (\( m,r,s \) being rational numbers) then one calculates using proposition 3.1 that

\[
S(\theta) = 9\phi_t(r^2 + \phi_t^2 m r + \phi_t m^2).
\]

(Compute, on Maple, recursively the powers of \( \alpha_t \) using the equality \( \alpha_t^3 = 3\phi_t + \phi_t \phi_t \).

Proposition 3.3. Let \( t \) be such that \( \phi_t \) is square free and not divisible by 3, and let \( |t| > 4 \) then

\[
U_t = \langle -1, u_t, v_t \rangle.
\]

Proof: In this case \( O_K = \mathbb{Z} [\alpha_t] \). Hence \( m, r, s \in \mathbb{Z} \) and so \( m = 0 \) and \( r = 1 \) would give the non-zero integers of lowest possible values for \( S((S(\theta)/9\phi_t) \) is here a positive rational integer since its discriminant (as a polynomial in \( r \)) is \( -3m^2 \leq 0 \).

Explicitly

\[
S(u_t = \alpha_t + t) = 9\phi_t \quad \text{and} \quad S(v_t = \alpha_t + t + 1) = 9\phi_t.
\]

Also, \( v_t \) is not a power of \( u_t \) by the previous proposition.

[Godwin's theorem [1] is stated here for reference (with his own notations).]

Any integer \( \lambda \) of a (totally real) cubic field \( (K) \) is of the form \( p + q\theta + r Q(\theta) \), where
\( \theta \) is a defining integer (i.e. 1, \( \theta \), \( Q(\theta) \) is an integral basis) \( Q(\theta) \) is a quadratic in \( \theta \) with rational coefficients and \( p, q, r \) are rational integers. \( S(\lambda) = S(q\theta + rQ(\theta)) \) is a positive definite form in \( q, r \) and we can find the values of \( q, r \), which make \( S(\lambda) \) least and then see if for any values of \( p, p + q\theta + rQ(\theta) \) is a unit. If not, we repeat this with the next lowest possible value of \( S(\lambda) \), and so on until a unit \( \epsilon_1 \) is reached. We then continue until another unit \( \epsilon_2 \), not a power of \( \epsilon_1 \) is reached.

**Theorem.** If every unit of the field has \( S(\epsilon) > 34 \) and if \( S(\epsilon_2) \geq 122 \) then either \( \epsilon_1, \epsilon_2 \) are a pair of fundamental units or there exists

(i) a unit \( \eta = \epsilon_1^{\frac{1}{2}} \epsilon_2^{\frac{1}{2}} \) such that \( S(\eta) < (81S(\epsilon_1)S(\epsilon_2)/2)^{\frac{1}{3}} \),

or

(ii) a unit \( \eta = \epsilon_1^{\frac{3}{2}} \epsilon_2^{\frac{1}{2}} \) such that \( S(\eta) < (243S(\epsilon_1^2)S(\epsilon_2)/2)^{\frac{1}{3}} \).

*End of quote*.

According to the above theorem, (and restricting \( |t| \) to values greater than 4 so that \( S(\theta) > 122 \) for all units), the above information leads to one of the following possibilities

1. \( u_t \) and \( v_t \) form a fundamental system of units,

2. \( \eta_1 = u_t^{\frac{1}{2}} v_t^{\frac{1}{2}} \) is a unit and \( S(\eta_1) \leq (6)(9\phi_t) \),

3. \( \eta_2 = u_t^{\frac{3}{2}} v_t^{\frac{1}{2}} \) is a unit such that \( S(\eta_2) \leq (2)(9\phi_t) \).

The following table gives a list of all possible candidates for \( \eta_1 \) or \( \eta_2 \) (under the restriction that \( S(\eta_1) \leq (6)(9\phi_t) \), and \( S(\eta_2) \leq (2)(9\phi_t) \)).
\[ \eta_1 \]
\[ \eta_2 \]
\[ S(\theta)/\theta \phi_t \]

\[ \theta_1 \equiv \pm[\alpha_t + c_1] \quad 1 \]
\[ \theta_2 \equiv \pm[\alpha_t^2 - t\alpha_t + c_2] \quad 1 \]
\[ \theta_3 \equiv \pm[\alpha_t^2 + (-1 - t)\alpha_t + c_3] \quad 1 \]
\[ \theta_4 \equiv \pm[\alpha_t^2 + (-t - 2)\alpha_t + c_4] \quad 3 \]
\[ \theta_5 \equiv \pm[\alpha_t^2 + (-t + 1)\alpha_t + c_5] \quad 3 \]
\[ \theta_6 \equiv \pm[2\alpha_t^2 + (-2t - 1)\alpha_t + c_6] \quad 3 \]
\[ \theta_7 \equiv \pm[2\alpha_t + c_7] \quad 4 \]
\[ \theta_8 \equiv \pm[2\alpha_t^2 - 2t\alpha_t + c_8] \quad 4 \]
\[ \theta_9 \equiv \pm[2\alpha_t^2 + (-2t - 2)\alpha_t + c_9] \quad 4 \]

where \( c_1 \ldots c_9 \in \mathbb{Z} \)

The squares of the above integers are

\[ \eta_1^2 \]
\[ \eta_2^2 \]

\[ \theta_1 \equiv \alpha^2 + 2c_1\alpha + c_1^2 \]
\[ \theta_2 \equiv (4t^2 + 2c_3 + 3t + 3)\alpha^2 - (2c_3 t + 3t + 4t^3 + 3t^2 - 1)\alpha + c_3^2 - 6t^2 - 2t - 4t^4 - 6t^3 \]
\[ \theta_3 \equiv (4 + 5t + 4t^2 + 2c_4)\alpha^2 - (5 + 4t^3 + 9t^2 + 9t + 2c_4 + 2c_4t)\alpha + c_4^2 - 8t - 2 - 4t^4 - 10t^3 - 12t^2 \]
\[ \theta_4 \equiv (2c_5 + 4t^2 + t + 4)\alpha^2 - (2c_5 t - 2c_5 + 4t^3 - 7 - 3t^2 - 3t)\alpha + c_5^2 - 4t^4 - 2t^3 + 4t + 2 \]
\[ \theta_5 \equiv (4t^2 + 7t + 7 + 2c_6)\alpha^2 - (11 + 4t^3 + 15t^2 + 15 + 2c_6 + 4c_6)\alpha + c_6^2 - 14t - 4 - 4t^4 - 14t^3 - 18t^2 \]
\[ \theta_6 \equiv (16t^2 + 16t + 13 + 4c_7)\alpha^2 - 2(2t + 1)(4t^2 + 4t + 4 + c_7)\alpha + c_7^2 - 20t - 4 - 16t^4 - 32t^3 - 36t^2 \]
\[ \theta_7 \equiv 4\alpha^2 + 4c_8\alpha + c_8^2 \]
\[ \theta_8 \equiv 4(4t^2 + c_{10} + 3t + 3)\alpha^2 - 4(c_{10} t + 3t + 4t^3 + 3t^2 - 1)\alpha + c_{10}^2 - 24t^2 - 8t - 16t^4 - 24t^3 \]
\[ \theta_9 \equiv 4(4t^2 + 5t + 4 + c_{11})\alpha^2 - 4(5 + 4t^3 + 9t^2 + 9t + c_{11} t + c_{11})\alpha + c_{11}^2 - 32t - 8 - 16t^4 - 40t^3 - 48t^2 \]

The cubes of the candidates for the unit \( \eta_2 \) are

14
\[ \eta_2^5 = \eta_1^3 + \eta_2^5 - t \eta_2^2 - \eta_1^3 \eta_2^5 + t \eta_1^3 \eta_2^5 \]

Thus, the candidates for \( \eta_1 \) are \( \theta_1 \ldots \theta_9 \), and those for \( \eta_2 \) are \( \theta_1 \ldots \theta_3 \). Using \( \eta_1^3 = u_t v_t \) eliminates possibility 1. \( S(\theta_i^3) \) is quadratic in \( c_i \) for \( 1 \leq i \leq 9 \). We put \( S(\theta_i^3) = S(u_t v_t) \), and solve for \( c_i \) in terms of \( t \). Then we check whether any of those 2 values for \( c_i \) results in \( \theta_i^2 = u_t v_t \) for some \( t \). Now use \( \eta_3^3 = u_t^2 v_t \) to eliminate possibility 2. (Reduce the terms not involving \( \alpha_t \) on both sides of the equation modulo 3. The resulting equation has no solution for any \( t \). As \( c_i \) is a rational integer for \( 1 \leq i \leq 3 \), it follows that \( \theta_i^2 \neq u_t v_t \) for \( 1 \leq i \leq 3 \) and for all \( t \).

Thus \( u_t \) and \( v_t \) form a system of fundamental units whenever \( t \in \mathbb{Z} \) is such that \(|t| > 4 \) and \( O_{K_t} = \mathbb{Z}[\alpha_t] \).

The next proposition gives a bound on the index of \( \langle u_t, v_t \rangle \) in the group of units \( U_t \) of \( K_t \). It is convenient to write \( \phi_t = A_t B_t C_t \) where \( C_t \) is a perfect cube, \( B_t \) is a square and cube free and \( A_t \) square free. Also \( \gcd(A, B) = 1 \). Let \( \mu_0 \) be 0 if \( t \equiv 1 \) (mod 3), 2 if \( t \equiv 4 \) (mod 3) and 1 otherwise.

**Proposition 3.4.**

\[ [U_t : \langle u_t, v_t \rangle] \leq 3^4\mu_0 C_t^4 B_t^3 \]
Proof: Let \( \rho_i = 3^{\nu} C_s^{2/3} B_i^{1/2} \). Then by proposition 2.2 \( \rho_i \mathcal{O}_{K_i} \subset \mathbb{Z}[\alpha_i] \).

The canonical homomorphism from \( \mathcal{O}_{K_i} \) to \( \mathcal{O}_{K_i}/\rho_i \mathcal{O}_{K_i} \) sends \( U_i \) into the group of units of \( \mathcal{O}_{K_i}/\rho_i \mathcal{O}_{K_i} \). Also, the ring \( \mathcal{O}_{K_i}/\rho_i \mathcal{O}_{K_i} \) has \( \rho_i^3 \) elements. Hence for a unit \( \varepsilon \in \mathcal{O}_{K_i} \), there exists a positive rational integer \( n < \rho_i^3 \) such that \( \varepsilon^n = 1 + \rho_i z \) where \( z \in \mathcal{O}_{K_i} \). Thus \( \varepsilon^n \) belongs to \( \mathbb{Z}[\alpha] \). This proves that the index \( [U_i : < u_i, v_i >] \leq \rho_i^s \). \( \blacksquare \)
4. Class Number Of $K_t$

Let $\phi_t = p_0^{n_0}p_1^{n_1} \cdots p_s^{n_s}$, where $n_0, \ldots, n_s \in \mathbb{Z}$ and $s \geq 0$. We know from proposition 2.3 that the discriminant of $K_t$ is $(3^{k} q_1 \cdots q_k)^2$ where $k \leq s$ is a non-negative integer, and $q_i$ for $1 \leq i \leq k$ is such that $q_i = p_j$ for some $j$ and $n_j$ is not a multiple of 3. Also $\lambda = 0$ if $t \equiv 4 \pmod{9}$ and $\lambda = 2$ otherwise. Now let $m_t = q_1 \cdots q_k$ if $t \equiv 4 \pmod{9}$ and $m_t = 9q_1 \cdots q_k$ otherwise. Thus $m_t$ is the conductor of $K_t$, and is equal to the square root of the discriminant of $K_t$. By the Kronecker-Weber Theorem, $K_t$ is contained in the cyclotomic field $Q[\zeta]$ where $\zeta = e^{2\pi i}$. Moreover the primes of $\mathbb{Z}$ which are ramified in $K_t$ are exactly those which divide $m_t$.

By the class number formula we have the following (cf [3]):

$$h_t = \frac{1}{4} \sqrt{\frac{\text{disc}(K_t)}{R_t}} \prod_{p|m} \left(1 - \frac{1}{p}\right)^{1 + \frac{1}{p^s}} \prod_{\chi \in \hat{G}} L(1, \chi).$$

where $p$ is a rational prime, $f_p$ the inertial degree of any prime ideal $P$ of $K_t$ over $p$, and $r_p$ is the number of primes of $K_t$ over $p$, $\chi$ is a character mod $m$, and $\hat{G}$ is the group of characters of the galois group $G$ of $K_t/Q$, considered as a subgroup of $\hat{\mathbb{Z}}_m^*$ the group of characters of $\mathbb{Z}_m^*$ (identified with the galois group of $Q[\zeta]/Q$).

We consider bounds on the class number $h_t$ of $K_t$. We first obtain some bounds on the regulator $R_t$. To do this we need to find bounds on the roots of $f_t$.

**Proposition 4.1.** (i) $K_t = K_{-t-1}$ for all $t$.

(ii) If $t$ is nonnegative then the roots of $f_t$ are such that

$$-t - 2 < \beta_t < -t - 1 < \gamma_t < -t < 2t + 1 < \alpha_t < 2t + 2$$

17
\[ |\beta_t + t + 1| < \frac{1}{10} \text{ for } t \geq 3 \]

**Proof:** We first note that \( f_{-t-1}(x) = x^3 - 3\phi x + \phi \phi' \). Hence \( f_{-t-1}(-x) = f_t(x) \). It follows that the roots of the polynomial \( f_t \) are the opposite of the roots of \( f_{-t-1} \), and that \( K_t = K_{-t-1} \). We can therefore restrict \( t \) to positive integers.

Also,

\[
\begin{align*}
    f_t(-t-2) &= -6t - 3 < 0 \\
    f_t(-t-1.1) &= -33t + 969 < 0 \text{ if } t \geq 3 \\
    f_t(-t) &= 1 > 0 \\
    f_t(-t) &= -1 < 0 \\
    f_t(2t + 1) &= -6t - 3 < 0 \\
    f_t(2t + 2) &= 9t^2 + 9t + 1 > 0
\end{align*}
\]

Finally, since \( \alpha_t \) is the root of \( f_t \) of largest absolute value the inequalities \( 2t + 1 < \alpha_t < 2t + 2 \) follow. It remains to show that \( \gamma_t > \beta_t \). By Proposition 3.1

\[
\gamma_t - \beta_t = 2\alpha_t^2 - \phi'\alpha_t - 4\phi_t.
\]

Thus it is sufficient to prove that \( \alpha_t \) is greater than the positive root of the second degree polynomial \( 2x^2 - \phi'x - 4\phi_t \), i.e. that \( \alpha_t > z = \frac{\phi' + \sqrt{\phi'^2 + 8\phi}}{4} \). This is true because \( z > \phi' \) and \( f(z) = \frac{-3\phi' - 3\sqrt{\phi'^2 + 36\phi} + 33}{16} < 0 \).

**Proposition 4.2.** If \( \phi_t \) is not divisible by 3 and is square free, and if \( t > 3 \) then

\[
\log(3t + 3) < R_t < [(\log(6t + 6))^2]
\]
Proof: In this case $\phi_t = p_1 \ldots p_s$ and discriminant of the field is just $(9\phi_t)^2$.

The regulator $R_t$ of $K_t$ is the determinant of
\[
\begin{pmatrix}
\log |\alpha_t + t| & \log |\beta_t + t| \\
\log |\alpha_t + t + 1| & \log |\beta_t + t + 1|
\end{pmatrix},
\]

Note that with $\alpha_t$, $\beta_t$ and $\gamma_t$ referring to the specific roots described above, we have
\[
\log |\alpha_t + t + 1| > \log |\alpha_t + t| \\
\log |\beta_t + t| > \log |\beta_t + t + 1|
\]
since $\beta_t + t$ and $\beta_t + t + 1$ are both negative.

Hence,
\[
R_t = |\log |\alpha_t + t + 1| \log |\beta_t + t| - \log |\beta_t + t + 1| \log |\alpha_t + t| |
\]
\[
= (\log |\alpha_t + t + 1|)(\log |\beta_t + t|) - (\log |\beta_t + t + 1|)(\log |\alpha_t + t|)
\]
\[
> (\log |\alpha_t + t|)(\log |\beta_t + t|) - (\log |\alpha + t|)(\log |\beta_t + t + 1|)
\]
\[
= \log |\alpha_t + t|(|\log |\beta_t + t| - \log |\beta_t + t + 1|)
\]
\[
= \log |\alpha_t + t| \log |\beta_t + t + 1|
\]
\[
= \log |\alpha_t + t| \log \left| 1 - \frac{1}{\beta_t + t + 1} \right|
\]
\[
= \log |\alpha_t + t| \log (1 + \frac{1}{|\beta + t + 1|})
\]
\[
> \log |\alpha_t + t| \text{ if } t \geq 3 \text{ (since } |\beta + t + 1| < \frac{1}{10} )
\]
\[
> \log (3t + 1)
\]
\[
> \log t \quad (t \geq 3)
\]

Also,
\[
R_t = |(\log |\alpha_t + t + 1|)(\log |\beta_t + t|) - (\log |\beta_t + t + 1|)(\log |\alpha_t + t|)|
\]
\[
\leq \log (\alpha_t + t + 1)(\log |\beta_t + t| + |\log |\beta_t + t + 1| |)
\]
\[
\leq \log (3t + 3)[\log 2 + |\log |\beta_t + t + 1| |]
\]

19
But $\beta_t + t + 1$ is a unit, so

$$1 > |\beta_t + t + 1| = \frac{1}{|\alpha_t + t + 1||\gamma_t + t + 1|} \geq \frac{1}{(3t + 3)} \quad (|\gamma_t + t + 1| < 1) \quad (4.2)$$

Equations (4.1) and (4.2) give that $R_t < \log(3t + 3)\log(6t + 6)$ Therefore

$$\log(3t + 3) < R_t < [\log(6t + 6)]^2$$

Now we obtain some bounds on the class number of $K_t$. We know that ([4])

$$|L(1, \chi)| \leq 3 \log m_t.$$ 

Also, [2]

$$\prod_{\chi \in \mathcal{O} \backslash \chi \neq 1} L(1, \chi) \geq \frac{E}{\log(\text{disc}(K_t))}$$

where $E$ is a positive constant independent of $t$ ($K_t$ has no quadratic subfield and hence $\kappa_t$ the residue of its $\zeta$-function $\zeta_{K_t}(s)$ at $s = 1$ is such that $\kappa^{-1} = O(3!\log(\text{disc}(K_t)))$).

**Proposition 4.3.** If $t > 3$, and $\phi_t$ is square free and not divisible by 3 then

$$\frac{9\phi_t}{4} \frac{E}{\log(9\phi_t)^2} \frac{1}{(\log(6t + 6))^2} < h_t < \frac{81\phi_t}{4} \frac{(\log 9\phi_t)^2}{\log(3t + 3)}$$

In particular under those circumstances, $h_t$ tends to infinity when $t$ gets larger.

**Proof:** In this case, the discriminant of $K_t$ is $(9\phi_t)^2$. Looking at the class-number formula, we note that for all $p | m$, we have $f_p = 1$ and $r_p = 1$ since $p$ ramifies in the extension $K_t/Q$. Also, $G$ the Galois group of $K_t/Q$ admits two non-trivial
characters χ and χ̅, those characters being complex conjugates of each other. It follows that the corresponding L functions are complex conjugates, and that

\[ L(1, χ)L(1, χ̅) = |L(1, χ)|^2 \]

where χ is any nontrivial character of G.

Hence

\[ h_t = \frac{9φ_t}{4} \frac{|L(1, χ)|^2}{R_t} \]

and

\[ \frac{9φ_t}{4} \frac{E}{\log(\text{disc}(K_t))} \frac{1}{[\log(6t + 6)]^2} < h_t < \frac{81φ_t}{4} \frac{(\log m_t)^2}{\log(3t + 3)} \]

Putting \( m_t = 9φ_t \) gives the desired result. \( \square \)

Using Propositions 3.4 and 4.2 one gets the following Proposition.

**Proposition 4.4.** Let \( t > 3 \). Then

\[ \frac{3^μ_1 A_t B_t^{\frac{1}{3}}}{4} \frac{E}{\log(3^μ_1 A_t B_t^{\frac{1}{3}})^2} \frac{1}{[\log(6t + 6)]^2} < h_t < \frac{3^6 μ_0}{4} C_t^4 B_t^3 \frac{3^μ_1 A_t B_t^{\frac{1}{3}}}{4} \frac{(\log 3^μ_1 A_t B_t^{\frac{1}{3}})^2}{\log(3t + 3)} \]

where \( μ_1 = 0 \) if 3 does not divide \( A_t \) and 1 otherwise.

**Proof:** The regulator \( m_t = 3^μ_1 A_t B_t^{\frac{1}{3}} = \sqrt{\text{disc}(K_t)} \). Also if \( R_t' \) is the regulator computed from \( u_t \) and \( v_t \) then \( R_t = (R_t'/[U_t : < u_t, v_t >]) \) and so by Propositions 4.2 and 3.4 the inequalities

\[ \frac{\log(3t + 3)}{3^6 μ_0 C_t^4 B_t^3} < R_t < [\log(6t + 6)]^2 \]

follow. The class-number formula completes the proof. \( \square \)

If \( t \equiv 4 \) (mod 9) and \( φ_t/3 \) is a prime then the conductor of \( K_t \) is that prime, and the computation of the \( L \) function is easy (the principal character is primitive.
and the cubes modulo \( m_t \) are exactly its kernel). Also using proposition 3.4 and 4.2 we could get 'close' bounds on \( R_t \). In fact in this case, \( [U_t : < u_t, v_t >] \leq 27^2 \).

Call \( R'_t \) the regulator computed from \( u_t \) and \( v_t \). Then

\[
\frac{R_t}{27^2} \leq R'_t \leq R_t.
\]

We compute using maple the following special cases. The class number \( h_t \) is such that \( h'_t \leq h_t \leq 27^2 h'_t \).

<table>
<thead>
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<th>( t )</th>
<th>( \phi_t )</th>
<th>( h'_t )</th>
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References


