A Study of Balanced Incomplete Block Designs

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Abstract

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In this thesis, we study two balanced incomplete block designs (BIBD). For the (22,33,12,8,4)-BIBD, we correct some errors in a published paper and provide the proofs for some unpublished results of Malcolm Greig. For the (28,63,27,12,11)-BIBD, we construct all 246 solutions assuming that the design is quasi-symmetric with block intersection numbers 4 and 6 and with the further assumption that the design is fixed by an automorphism of order 7 with no fixed point and no fixed block.

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Contents

Chapter 1

Introduction

Experimental design is a very active branch of statistics. W. T. Federer and L. N. Balam's *Bibliography on Experiment and Treatment Design, Pre-1969* (Hafner, New York, 1973) lists 8378 works under 44 subheadings.

Basically, there are three kinds of designs:

1. Block designs (randomized block designs, Latin squares, Greaceo -Latin squares, balanced incomplete block designs, partial balanced incomplete block designs, Youdan squares and lattices etc.)

2. Factorial designs (2^k, 3^k, orthogonal array, confounding, partial confounding, fractional factorial, nested or hierarchical nested design etc.)

3. Designs for response surface (center compositing, A-, D-, U- and various optimal designs, mixture design etc.)

In a balanced incomplete block design (BIBD), v varieties or treatments are compared in such a manner that each treatment is assigned to r experimental units. The units themselves are arranged into b blocks, each containing k experimental units. Any two treatments are required to appear together in the same block λ times, while the treatments appearing in a given block are all different. In other word, a balanced incomplete block design (BIBD) is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of b k-subsets of V, called blocks, such that each element (or called point) of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. Hence the design depends on the five parameters, v, b, r, k, and λ.

Trivial necessary conditions for the existence of a (v, b, r, k, λ)-BIBD are

$$vr = bk, \text{ and} \tag{1.1}$$

$$r(k-1) = \lambda(v-1). \tag{1.2}$$

Parameter sets that satisfy (1.1) and (1.2) are called *admissible*.

Although particular series of BIBD's were known as early as 1847, a systematic treatment was not developed until 1936 when Yates [43] introduced them as statistical designs. Since then rich contributions were made by Fisher [20, 21], Fisher and Yates [22], and Bose [8, 9, 10, 11, 12, 13, 14]. It has now become a standard part of the theory of the experimental design. Because there is no formal procedure that can construct all designs, the existence question of some BIBD's presents interesting problems [8, 19, 20].

In this thesis, we are going to construct the (28, 63, 27, 12, 11)-BIBD using orbit matrices and explore the existence of the (22, 33, 12, 8, 4)-BIBD. Let us define some concepts that will be frequently used:

We use the notation 1_p to denote a column vector with p ones, I_p to denote a $p \times p$ identity matrix, J_p to denote a $p \times p$ matrix of all ones, and the superscript ^{T} to denote the transpose operation.

1. A BIBD is called *symmetric* if $b = v$. Such a design is usually called a (v, k, λ)-design, since in this case both $b = v$ and $r = k$.

2. Two designs $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ are said to be *isomorphic* if there exists a bijection α: $V_1 \rightarrow V_2$ such that $\alpha(\mathcal{B}_1) = \mathcal{B}_2$. In other words, if $\{x_1, ..., x_k\}$ is a block in $\mathcal{B}_1$, then $\{\alpha(x_1), ..., \alpha(x_k)\}$ is a block in $\mathcal{B}_2$.

3. A (v, b, r, k, λ)-BIBD is completely determined by its *incidence matrix*, N, whose element in the i-th row and j-th column is 1 (or 0) if element i is (or is not) in block j. Hence

$$NN^T = (r - \lambda)I_v + \lambda J_v.$$

4. The *intersection matrix* S is defined to be N^TN. The entries of S are called the *intersection numbers*.

5. An *automorphism* ϕ of a design is a permutation of the points which also permutes the blocks. In other words, $\phi(\mathcal{B}) = \mathcal{B}$. The set of all automorphism forms a permutation group called the *full automorphism group of the design*.

   Let e be the identity permutation. If $\phi^p = e$ and $\phi^q \neq e$ for any $q < p$, then p is the *order* of ϕ.

6. If a group G acts on a set X and $x \in X$, the *orbit* of x is $\{g(x) : g \in G\}$.

   It is easy to see that orbits form a partition of X. In fact, one may define an equivalence relation on X by $x_1 \sim x_2$ if $x_2 = g(x_1)$ for some $g \in G$, and the equivalent classes of this relation are the orbits. The automorphism group of a design D partitions the treatment set of D into orbits, such that two treatments x, y belong to the same orbit if and only if there is an automorphism ϕ such that $\phi(x) = y$.

7. A BIBD is *quasi-symmetric* with block intersection numbers x and y ($x < y$) if any two blocks intersect in either x or y points.

8. If x is a vector and A is a symmetric matrix, when $x^TAx \geq 0$ for any x, then x^TAx is a *semi-positive definite quadratic form* and A is a *semi-positive definite (p.s.d.) matrix*.

Theorem 1.1 *Let N and S denote respectively the incidence matrix and the intersection matrix of a (v, b, r, k, λ)-BIBD, then*

$$
\begin{aligned}
N1_b &= r1_v, && (1.3)\\
1_v^T N &= k1_b^T, && (1.4)\\
NN^T &= (r-\lambda)I_v + \lambda J_v, && (1.5)\\
S1_b &= rk1_b, \quad \textit{and} && (1.6)\\
S^2 &= (r-\lambda)S + \lambda k^2 J_b. && (1.7)
\end{aligned}
$$

Proof: Relations (1.3), (1.4) and (1.5) are trivial from the definition of a BIBD. Since $S = N^T N$, from (1.3), (1.4), $S1_b = N^T(r1_v) = rk1_b$. Finally, by (1.5), $J_v = 1_v 1_v^T$, and (1.1), we have $S^2 = N^T(NN^T)N = N^T[(r-\lambda)I_v + \lambda J_v]N = (r-\lambda)S + \lambda k^2 J_b$.

Theorem 1.2 *[25, p.130] If N is the incidence matrix of a symmetric BIBD, then N satisfies the following conditions:*

$$
\begin{aligned}
NN^T = {} & N^T N = (k-\lambda)I + \lambda J, \quad \textit{and}\\
NJ = {} & JN = kJ.
\end{aligned}
$$

In Chapter 2, we concentrate on the (22,33,12,8,4)-BIBD. Although it is an admissible parameter set, it is not known at present whether or not the design exists. For any design with a smaller v, their existence or non-existence is known. For this reason the existence or non-existence of a (22,33,12,8,4)-BIBD is a challenging problem [26]. Hamada and Kobayashi [27] have made a detailed study of patterns of intersection of a block with the remaining 32 blocks. In 1988, Hall, Roth, Rees and Vanstone derived the structure of the first 5 columns of the incidence matrix. They also ruled out many of these configurations. A simplification for finding these impossible cases was made by Malcolm Greig. These studies leave a total of 7 possible 5 by 5 submatrices S_5 of S for further exploration. In the thesis, we will synthesize the above methods and results, giving their complete proofs and noting some corrections.

In Chapter 3, we construct some quasi-symmetric (28,63,27,12,11)-BIBD with intersection numbers 4 and 6, with the assumption that they are fixed by an automorphism of order 7 with no fixed points and no fixed blocks.

Chapter 2

Restrictions on the (22,33,12,8,4)-BIBD

2.1 Introduction

Hamada and Kobayashi (1978) have studied in detail the pattern of intersections of a block with the remaining 32 blocks. They reduced the nine potential patterns to four. This will be discussed in Sections 2.2 and 2.4.

Section 2.3 will deal with the structure of the first five columns of the incidence matrix N. In particular, we will show that there are only 29 possibilities for these five columns.

Section 2.4 will introduce Greig's improvement. We provide all the detailed proofs for his statements including the reduction of the nine block intersection patterns to 4 and the reduction of the number of possibilities of the first five columns of N from 29 to 7. It remains to be seen what attack will either eliminate these cases or lead to the construction of a design.

2.2 The Hamada-Kobayashi Restrictions on Block Intersections

For a (22,33,12,8,4)-BIBD, the incidence matrix N satisfies

$$NN^T = 8I_{22} + 4J_{22}.$$

Given a block B, let b_i ($i = 0, 1, \ldots, 8$) be the number of blocks, which intersect B in i points. Then

$$b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 = 32, \quad (2.1)$$

$$b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 + 6b_6 + 7b_7 + 8b_8 = 88, \text{ and} \quad (2.2)$$

$$b_2 + 3b_3 + 6b_4 + 10b_5 + 15b_6 + 21b_7 + 28b_8 = 84. \quad (2.3)$$

The Eq. (2.1) counts the remaining 32 blocks. Eq. (2.2) counts the total number of intersection points with B, that is, the remaining 11 occurrences of the 8 points. Since any pair of the 8 points occurs together 3(or $\lambda - 1$) more times we have Eq. (2.3). It follows from 3 (2.1)−2 (2.2)+ (2.3) that

$$3b_0 + b_1 + b_4 + 3b_5 + 6b_6 + 10b_7 + 15b_8 = 4, \quad (2.4)$$

where all the b_i's are non-negative integers. Therefore, the only possible non-zero terms are those with coefficients not exceeding 4. Hence, Eq. (2.4) simplifies to

$$3b_0 + b_1 + b_4 + 3b_5 = 4. \quad (2.5)$$

This leads to only 9 possible combinations as shown in Table (2.1).

In Section 2.4 we use Greig's method instead of Hamada-Kobayashi's to show that Types 5 to 9 are impossible.

Table 2.1: Type 1 to 9

Type	b_0	b_1	b_2	b_3	b_4	b_5
1	0	0	12	16	4	0
2	0	1	9	19	3	0
3	0	2	6	22	2	0
4	1	0	6	24	1	0
5	0	3	3	25	1	0
6	0	0	11	19	1	1
7	0	1	8	22	0	1
8	0	4	0	28	0	0
9	1	1	3	27	0	0

2.3 S_5 in the (22,33,12,8,4)-BIBD

In this section, we introduce the basic terminology of coding theory first. For a reference on coding theory, see [25, 39].

Definition 1 *A word is a string $x_1x_2...x_n$ where the x_i's are chosen from an alphabet of m letters $a_1, ..., a_m$.*

Definition 2 *A code of length n on m letters is a subset of the m^n words of length n on m letters. The words in a code are sometimes called codewords.*

Definition 3 *If $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$ are two words of the same length n, the Hamming Distance between them, $d_H(x, y)$, is the number of i's for which $x_i \neq y_i$.*

From now on we consider only the special case where the alphabet is a finite field.

Definition 4 *The weight $wt(x)$ of a word x is the number of letters $x_i \neq 0$.*

We define the sum of two words as the string obtained by summing the individual components. Similarly, we can define the difference of two words. Since $x_i = y_i$ if and only if $x_i - y_i = 0$, it is easy to see that $d_H(x,y) = wt(x-y)$.

The collection of all words of length n where the alphabet is a finite field forms a vector space V. We shall consider only codes C that are linear subspaces of V. Such codes C are called linear codes. A linear code C of dimension s which is a subspace of V of length n over F_q is called a linear $[n,s]$ code over F_q. If $q = 2$, it is called a binary code. Note that for binary codes, $x - y = x + y$.

Definition 5 *Let* $(u,v) = \Sigma_{i=1}^{n} u_i v_i \pmod q$ *be the inner product of* u *and* v. *If* $(u,v) = 0$, *we say that* u *and* v *are orthogonal to each other.*

Definition 6 *If* C *is a code, we let*

$$C^{\perp} = \{u \mid (u,v) = 0, \quad \textit{for all } v \in C\}. \tag{2.6}$$

It is known that if C *is an s-dimensional subspace, then* $C^{\perp}$ *is* $(n-s)$-*dimensional.* $C^{\perp}$ *is called the dual or orthogonal code of* C.
If $C \subseteq C^{\perp}$, *then* C *is self-orthogonal, and if* $C = C^{\perp}$, *then* C *is self-dual.*

Definition 7 *The weight distribution of a code is the number of vectors of any weight in the code. This is often described by the list of numbers* A_i *where* A_i *is the number of vectors of weight* i *in the code.*

The code C of a (22,33,12,8,4)-BIBD over $F_2 = GF(2)$ is the subspace of F_2^{33} spanned by the rows of the incidence matrix N. In N, every column has eight 1's and every row has twelve 1's. As $\lambda = 4$, any two rows have common 1's in exactly 4 columns. It follows that every codeword has weight a multiple of 4. Also there is no word of weight 32 in C for the following reasons. Suppose there exists such a word, by permuting columns, we can assume that the word has a zero in column 1. Choose a row of N with a one in column 1 and consequently, 11 ones in columns 2 to 33. The sum of this row with the word of weight 32 will give a word with a zero in column 1 and 21 ones in columns 2 to 33. This word has weight 22, which is impossible because 22 is not divisible by 4. Hence the weight distribution of C is $A_0, A_4, A_8, A_{12}, A_{16}, A_{20}, A_{24}, A_{28}$.

Define the weight enumerator $W_C(x, y)$ as

$$W_C(x, y) = A_0 x^{33} + A_4 x^{29} y^4 + A_8 x^{25} y^8 + A_{12} x^{21} y^{12} + \ldots + A_{28} x^5 y^{28}. \quad (2.7)$$

Since for any $x, y \in C$, $(x, y) = \sum_{i=1}^{33} x_i y_i$ is zero modulo 2, hence for $x \in C, y \in C^{\perp}$ and for $y \in C, x \in C^{\perp}$. That is, $C \subseteq C^{\perp}$ and C is self-orthogonal. As every word in C has even weight, the all 1 vector is not in C but in $C^{\perp}$. Therefore, $C \subset C^{\perp}$. Furthermore, we know $dim(C) + dim(C^{\perp}) = 33$. Hence $dim(C) \leq 16$.

If $C_0, C_1, \ldots, C_{33}$ are the weight distribution of $C^{\perp}$, then the celebrated identity of Jessie MacWilliams gives

$$W_C^{\perp}(x, y) = W_C(x + y, x - y)/|C|. \quad (2.8)$$

Here $|C| = 2^h$, when $h = dim(C)$. Hence

$$2^h\left(\sum_{i=0}^{33} C_i x^{33-i} y^i\right) = A_0(x+y)^{33} + A_4(x+y)^{29}(x-y)^4 + A_8(x+y)^{25} (x-y)^8 + \ldots + A_{28}(x+y)^5(x-y)^{28}. \quad (2.9)$$

If we expand these $(x + y)^i$ terms on the right hand side of Eq. (2.9) and compare the coefficients of different $x^i y^j$ terms on both sides, it is not hard to see that each $2^h C_i$ is a polynomial of the A_j's. The following table gives details of the $2^h C_i'$'s:

$2^h C_i'$	A_0	A_4	A_8	A_{12}	A_{16}	A_{20}	A_{24}	A_{28}
C_0'	1	1	1	1	1	1	1	1
C_1'	33	25	17	9	1	-7	-15	-23
C_2'	528	296	128	24	-16	8	96	248
C_3'	5,456	2,200	544	-24	-16	56	-320	-1,656
C_4'	40,920	11,456	1,320	-240	120	-160	456	7,600
C_5'	237,336	44,080	1,224	-288	120	-112	552	-25,021
C_6'	1,107,568	128.296	-2.912	728	-560	904	-3,584	59,192
C_7'	4,272,018	281.880	-11.968	2,088	-560	-456	5,472	-94.392
C_8'	13,884,156	417.876	-15.601	-108	1,820	-2,652	1,836	71,601
C_9'	38,567,100	429,780	5.100	-6,140	1,820	3,380	-18,868	89,700
C_{10}'	92,561,040	-45.240	47,680	-5,256	-4,368	4,264	23,712	-385.320
C_{11}'	193,536,720	-1,040.520	62.560	9.096	-4,368	-10,088	8,832	609,960
C_{12}'	354,817,320	-2.081,040	-2,760	16,896	8,008	-2,288	-56,488	-430,560
C_{13}'	573,166,400	-2,241,120	-109,480	-3,969	8,008	18,304	50,232	-270,480
C_{14}'	818,809,200	-880,440	-128,800	-27.720	-11,440	-5,720	30,912	1,090,200
C_{15}'	1,037,158,320	1,520.760	0	-11.704	-11,440	-21,736	-97,888	-1,311.000
C_{16}'	1,166,803,110	3.421.710	152.950	26,334	12.870	16,302	55,062	589,950

Since $C_i' = C_{33-i}'$ for $i = 0, 1, \ldots, 16$ and $C_0' = C_{33}' = 1$, we only list $i = 0, \ldots, 16$ in the table. It is trivial that $C_1' = 0$ because no block is empty and $C_2' = 0$ as no two blocks are identical. These yield

$$
\begin{aligned}
2^h = 2^h C_0' &= 1 + A_4 + A_8 + A_{12} + A_{16} + A_{20} + A_{24} + A_{28}, \\
0 = 2^h C_1' &= A_0\binom{33}{1} + A_4\left[-\binom{4}{1} + \binom{29}{1}\right] + A_8\left[-\binom{8}{1} + \binom{25}{1}\right] \\
&\quad + A_{12}\left[-\binom{12}{1} + \binom{21}{1}\right] + \ldots + A_{28}\left[-\binom{28}{1} + \binom{5}{1}\right], \qquad (2.10) \\
0 = 2^h C_2' &= A_0\binom{33}{2} + A_4\left\{\binom{4}{2} + \binom{29}{1}\left[-\binom{4}{1}\right] + \binom{29}{2}\right\} + A_8\left\{\binom{8}{2} + \binom{25}{1}\left[-\binom{8}{1}\right] + \binom{25}{2}\right\} \\
&\quad + \ldots + A_{28}\left\{\binom{28}{2} + \binom{5}{1}\left[-\binom{28}{1}\right] + \binom{5}{2}\right\},
\end{aligned}
$$

which give the first 3 rows of the above table.

Solving (2.10) for A_{12}, A_{16}, A_{20} we get

$$\begin{aligned} A_{12} &= 13(2^{h-6}) - 10 - 6A_4 - 3A_8 - A_{24} - 3A_{28}, \\ A_{16} &= 30(2^{h-6}) + 15 + 8A_4 + 3A_8 + 3A_{24} + 8A_{28}, \quad \text{and} \qquad (2.11) \\ A_{20} &= 21(2^{h-6}) - 6 - 3A_4 - A_8 - 3A_{24} - 6A_{28}. \end{aligned}$$

Using (2.11) to substitute A_{12}, A_{16}, A_{20} into the fourth and fifth rows of the table we have the following expressions for C_4 and C_5

$$\begin{aligned} 2^{h-9}C_4 &= (-45)2^{h-9} + 90 + 28A_4 + 5A_8 + 3A_{24} + 20A_{28}, \quad \text{and} \\ 2^{h-9}C_5 &= (-39)2^{h-9} + 474 + 92A_4 + 5A_8 + 3A_{24} - 44A_{28}. \qquad (2.12) \end{aligned}$$

Now it follows that

$$2^{h-9}C_5 = 2^{h-9}C_4 + 6(2^{h-9}) + 384 + 64A_4 - 64A_{28}. \qquad (2.13)$$

If $A_{28} \geq 6$, then $C_5 = C_{33-5} \geq A_{28} \geq 6$; if $A_{28} < 6$, then from (2.13), $2^{h-9}C_5 > 6(2^{h-9})$ and $C_5 > 6$. In either event, $C_5 \geq 6 > 0$, which means that there exist at least one codeword in $C^{\perp}$, with weight 5. Since the codeword is orthogonal to every row of the incidence matrix N, we have a submatrix of N, say N_5, formed by such 5 columns, which corresponds to the components of the five 1's in the codeword of $C^{\perp}$. The submatrix N_5 is 22 by 5 where every row has an even number of 1's, that is 0, 2, or 4 ones.

We define the projection matrix P as $P = 96I + 32J - 12S$. P satisfies

$$P^2 = 96P.$$

In [26], Hall et al showed that P is semi-positive definite. Hence $P_5 = 96I_5 + 32J_5 - 12S_5$ is also semi-positive definite. We use a computer program to find all possible N_5 subject to the condition that the row sums are even and the corresponding P_5 is semi-positive definite. Up to isomorphism we found 29 N_5's.

In [26], 31 cases of the $S_5 = N_5^T N_5$ were listed. We compared our 29 cases of the S_5, coded $L1$ to $L29$, with their 31 cases, identified with a prefix "H". We note the following errors in the latter.

$$\underset{\text{L2(H99)}}{\begin{pmatrix} 8 & 4 & 4 & 4 & 2 \\ 4 & 8 & 4 & 2 & 4 \\ 4 & 4 & 8 & 2 & 2 \\ 4 & 2 & 2 & 8 & 4 \\ 2 & 4 & 2 & 4 & 8 \end{pmatrix}} \quad \underset{\text{L1(H100)}}{\begin{pmatrix} 8 & 4 & 4 & 3 & 3 \\ 4 & 8 & 4 & 4 & 2 \\ 4 & 4 & 8 & 1 & 3 \\ 3 & 4 & 1 & 8 & 4 \\ 3 & 2 & 3 & 4 & 8 \end{pmatrix}} \quad \underset{\text{L5}}{\begin{pmatrix} 8 & 4 & 4 & 3 & 3 \\ 4 & 8 & 3 & 4 & 3 \\ 4 & 3 & 8 & 2 & 3 \\ 3 & 4 & 2 & 8 & 3 \\ 3 & 3 & 3 & 3 & 8 \end{pmatrix}} \quad \underset{\text{L28}}{\begin{pmatrix} 8 & 3 & 2 & 2 & 1 \\ 3 & 8 & 2 & 1 & 2 \\ 2 & 2 & 8 & 2 & 2 \\ 2 & 1 & 2 & 8 & 3 \\ 1 & 2 & 2 & 3 & 8 \end{pmatrix}}$$

$$\underset{\text{H105}}{\begin{pmatrix} 8 & 4 & 4 & 4 & 2 \\ 4 & 8 & 2 & 4 & 4 \\ 4 & 2 & 8 & 2 & 4 \\ 4 & 4 & 2 & 8 & 2 \\ 2 & 4 & 4 & 2 & 8 \end{pmatrix}} \quad \underset{\text{H102}}{\begin{pmatrix} 8 & 4 & 4 & 4 & 2 \\ 4 & 8 & 3 & 4 & 3 \\ 4 & 3 & 8 & 1 & 4 \\ 4 & 4 & 1 & 8 & 3 \\ 2 & 3 & 4 & 3 & 8 \end{pmatrix}} \quad \underset{\text{H104}}{\begin{pmatrix} 8 & 4 & 4 & 3 & 2 \\ 4 & 8 & 3 & 4 & 3 \\ 4 & 3 & 8 & 2 & 3 \\ 3 & 4 & 2 & 8 & 3 \\ 2 & 3 & 3 & 3 & 8 \end{pmatrix}} \quad \underset{\text{H10}}{\begin{pmatrix} 8 & 3 & 2 & 2 & 1 \\ 3 & 8 & 1 & 1 & 3 \\ 2 & 1 & 8 & 3 & 1 \\ 2 & 1 & 3 & 8 & 3 \\ 1 & 3 & 1 & 3 & 8 \end{pmatrix}}$$

The case $H105$ is isomorphic to $H99$ by interchanging columns 3 and 4, then rows 3 and 4 of $H105$. Also, $H102$ is isomorphic to $H100$ by interchanging columns 1 and 2, then rows 1 and 2, followed by columns 3 and 4 and finally rows 3 and 4 of $H102$. Thus, we reduce the 31 cases of [26] to 29. Furthermore, we note that in $H104$, the sum of the off-diagonal entries in row 1 is 13 and in $H10$, the sum of the off-diagonal entries in row 3 is 7; both of which are impossible. We suspect that there is a typographical error in $H104$ and that its (1,5)-entry should be a 3. With this change, their $H104$ is exactly our $L5$. $H10$ has to match up with $L28$, but it requires several changes. We have communicated the changes to one of the author of [26] and he agrees with our corrections.

In Appendix A, we list the 29 cases of S_5, labelled $L1$ to $L29$. For each case Li, we also list the corresponding case Hj from [26] using their labelling.

If a (22,33,12,8,4)-BIBD exists, its intersection matrix must contain a 5 by 5 submatrix isomorphic to one of the 29 cases listed in Appendix A. In the next section, we follow Greig's method to show that 22 of the 29 cases are impossible.

2.4 Greig's Improvement

In this section, we follow Greig's method to show that Types 5 to 9 are impossible in Table 2.1. We also reduce the 29 remaining cases of S_5 to 7.

Greig's fundamental observation is elementary:

> The inequality $(y - 1/2)^2 \geq 0$ implies $y(y-1) \geq -1/4$. This can be improved to $y(y-1) \geq 0$, if y is an integer. Moreover, if y is an integer-valued vector of length p, then $y^T(y - 1_p) \geq 0$ holds.

In [23], Greig give the following statements without proof. We are now providing the proofs.

Theorem 2.1 *Let y be a $p \times 1$ vector with at least t integral values, and let $F = y^T(y - 1_p)$, then*

$$F \geq (t-p)/4.$$

Proof: Let $y = (y_1, ..., y_p)^T$, then

$$F = \Sigma_{i=1}^{p}(y_i^2 - y_i).$$

If y_i is an integer, then $y_i^2 - y_i \geq 0$; otherwise $y_i^2 - y_i \geq -1/4$. Therefore

$$\begin{aligned} F &\geq -(p-t)/4, \quad \text{or} \\ F &\geq (t-p)/4. \end{aligned}$$

A vector $y = (y_1, ..., y_p)^T$ is said to be *integral* if every y_i is an integer.

In Corollary 2.2 to Corollary 2.5, we use d to denote the difference between a given real value and its closest integer. We note that $|d| \leq 1/2$.

Corollary 2.2 *Let B be a $p \times q$ matrix, and x a $q \times 1$ vector, and let $A = 1_p^T Bx$. If Bx is integral, and d is chosen such that $A/p - 1/2 + d$ is an integer, then*

$$x^T B^T Bx - A^2/p - p(1/4 - d^2) \geq 0.$$

Proof: Let $y = Bx - (A/p - 1/2 + d)1_p$. Then y is integral and from Theorem 2.1, $y^T(y - 1_p) \geq 0$. After simplification,

$$y^T(y - 1_p) = x^T B^T Bx - A^2/p - p(1/4 - d^2).$$

Therefore,

$$x^T B^T Bx - A^2/p - p(1/4 - d^2) \geq 0.$$

Corollary 2.3 *Let S be the intersection matrix of a (v, b, r, k, λ)-BIBD, and let x be any $b \times 1$ real-valued vector. Let $a = x^T 1_b$ be the sum of the entries of x and choose d such that $-\lambda va/b - 1/2 + d$ is an integer. If $[(r-\lambda)I_b - S]x$ is integral, then*

$$x^T[S - (r-\lambda)I_b]x \leq a^2\lambda v/b - b(1/4 - d^2)/(r - \lambda).$$

Proof: Let $B = (r - \lambda)I_b - S$ in Corollary 2.2. Then

$$\begin{aligned} A &= 1_b^T Bx \\ &= 1_b^T[(r-\lambda)I_b - S]x \\ &= (r-\lambda)1_b^T x - 1_b^T Sx \\ &= a(r-\lambda) - rk1_b^T x \\ &= a(r - \lambda - rk) \\ &= -a[r(k-1) + \lambda] \\ &= -a[\lambda(v-1) + \lambda] \\ &= -\lambda va. \end{aligned}$$

Since $B^T = B, S^T = S$ and $S^2 = (r-\lambda)S + \lambda k^2 J_b$, therefore

$$\begin{aligned} x^T B^T Bx &= x^T[(r-\lambda)I_b - S]^2 x \\ &= x^T[(r-\lambda)^2 I_b - 2(r-\lambda)S + (r-\lambda)S + \lambda k^2 J_b]x \\ &= x^T[(r-\lambda)^2 I_b - (r-\lambda)S + \lambda k^2 J_b]x \\ &= (r-\lambda)x^T[(r-\lambda)I_b - S]x + \lambda k^2 a^2. \end{aligned}$$

If $Bx = [(r-\lambda)I_b - S]x$ is integral and d is chosen such that $A/b - 1/2 + d = -\lambda va/b - 1/2 + d$ is an integer, then we have

$$\begin{aligned} & x^T B^T Bx - A^2/b - b(1/4 - d^2) \\ = \ & (r-\lambda)x^T[(r-\lambda)I_b - S]x + \lambda k^2a^2 - \lambda^2 v^2 a^2/b - b(1/4 - d^2) \\ = \ & (r-\lambda)x^T[(r-\lambda)I_b - S]x + a^2\lambda(\tfrac{bk^2 - \lambda v^2}{b}) - b(1/4 - d^2) \\ = \ & (r-\lambda)x^T[(r-\lambda)I_b - S]x + a^2\lambda(\tfrac{vrk - \lambda v^2}{b}) - b(1/4 - d^2) \\ = \ & (r-\lambda)x^T[(r-\lambda)I_b - S]x + a^2\lambda v(\tfrac{r-\lambda}{b}) - b(1/4 - d^2) \geq 0, \end{aligned}$$

therefore,

$$x^T[S - (r-\lambda)I_b]x \leq a^2\lambda v/b - b(1/4 - d^2)/(r-\lambda).$$

From now on, we use U_6 refer to the right hand side of the inequality in Corollary 2.3.

Corollary 2.4 *Let S be the intersection matrix of a (v,b,r,k,λ)-BIBD, x be any real valued vector and let $a = x^T 1_b$. If Sx is integral, and d is chosen such that $k^2a/v - 1/2 + d$ is an integer, then*

$$x^T Sx \geq a^2k^2/v + b(1/4 - d^2)/(r-\lambda). \tag{2.14}$$

Proof: Let $B = S$ in Corollary 2.2. Then $A = rka$ and $A/b - 1/2 + d = k^2a/v - 1/2 + d$. If Sx is integral and d is chosen such that $k^2a/v - 1/2 + d$ is an integer, then

$$x^T S^T Sx \geq (rka)^2/b + b(1/4 - d^2). \tag{2.15}$$

Since $S^TS = S^2 = (r-\lambda)S + \lambda k^2 J_b$, the left hand side of (2.15) evaluates to $(r-\lambda)x^TSx + \lambda k^2a^2$. Therefore we have

$$\begin{aligned} (r-\lambda)x^TSx \geq \ & k^2a^2(r^2/b - \lambda) + b(1/4 - d^2) \\ = \ & k^2a^2(r-\lambda)/v + b(1/4 - d^2). \end{aligned}$$

Hence

$$x^T Sx \geq k^2a^2/v + b(1/4 - d^2)/(r - \lambda).$$

We use L_7 to denote the right hand side of the inequality in Corollary 2.4.

Corollary 2.5 *Let N be the incidence matrix of a (v, b, r, k, λ)-BIBD, x be any real-valued vector and $a = x^T 1_b$. If Nx is integral, and d is chosen such that $ka/v - 1/2 + d$ is an integer, then*

$$x^T Sx \geq a^2k^2/v + v(1/4 - d^2).$$

Proof: Let $B = N$ in Corollary 2.2, then $p = v, A = 1_v^T Nx = k1_b^T x = ka$. If Nx is integral and d is chosen such that $A/p - 1/2 + d = ka/v - 1/2 + d$ is an integer, then

$$x^T N^T Nx - a^2k^2/v - v(1/4 - d^2) \geq 0.$$

Since $S = N^T N$, we have

$$x^T Sx \geq a^2k^2/v + v(1/4 - d^2).$$

From now on, we let L_8 denote the formula $a^2k^2/v + v(1/4 - d^2)$.

In the case of the (22,33,12,8,4)-BIBD, Greig computed the following table of values for U_6, L_7 and L_8, according to a and an appropriately chosen d.

Note that we have corrected the signs for some of the entries under column "$6d$" in Table 2.2. All the entries with "-1" were listed as "1" in [23] and those with "1" were listed as "-1" in [23].

Corollary 2.3 and Corollary 2.5 are used to derive the remaining results in this chapter. We use them by first choosing a specific x and computing the

Table 2.2: Values of U_6, L_7 and L_8

a	$a^2\lambda v/b$	$6d$	U_6	a^2k^2/v	$22d$	L_7	$22d$	L_8
0.0	0.00	3	0.00	0.00	11	0.00	11	0.00
0.5	0.67	-1	-0.25	0.73	1	1.75	7	4.00
1.0	2.67	1	1.75	2.91	-9	3.25	3	8.00
1.5	6.00	3	6.00	6.55	3	7.50	-1	12.00
2.0	10.67	-1	9.75	11.61	-7	12.25	-5	16.00
2.5	16.67	1	15.75	18.18	5	19.00	-9	20.00
3.0	24.00	3	24.00	26.18	-5	27.00	9	28.00
3.5	32.67	-1	31.75	35.61	7	36.25	5	40.00
4.0	42.67	1	41.75	46.55	-3	47.50	1	52.00
4.5	51.00	3	51.00	58.91	9	59.25	-3	61.00
5.0	66.67	-1	65 75	72.73	-1	73.75	-7	76.00
5.5	80.67	1	79.75	88.00	11	88.00	11	88.00

left hand side of the corresponding inequalities. Then we compare the result with the bounds guaranteed by the corollaries. To simplify the explanation, we use the notation t^{*m} to denote m copies of the value t. For example,

$$((1/2)^{*3}, (-1/2)^{*2}, 0^{*28}) = (1/2, 1/2, 1/2, -1/2, -1/2, 0, ..., 0).$$

From Table 2.2 we can see that Corollary 2.5 gives a better bound than Corollary 2.4.

Theorem 2.6 *Let*

$$\begin{pmatrix} 8 & u \\ u & 8 \end{pmatrix}$$

be a submatrix of S. *Then* $u \leq 4$.

Proof: Let $x^T = (1, 1, 0^{*31})$ in Corollary 2.3, then $a = 2$ and $U_6 = 9.75$. Now

$$x^T[S - (r - \lambda)I_b]x = (1, 1)\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}(1, 1)^T = 2u \leq 9.75,$$

which leads to $u \leq 4$ as u is an integer. Since u is the intersection number of any two blocks in a (22,33,12,8,4)-BIBD, the result shows that types 6 and 7 in Table 2.1 with $b_5 = 1$ could be eliminated.

Theorem 2.7 *Let*

$$S_1 = \begin{pmatrix} 8 & c & d & e \\ c & 8 & u & v \\ d & u & 8 & w \\ e & v & w & 8 \end{pmatrix}$$

be a submatrix of S. *Then* $c + d + e + u + v + w \geq 10$.

Proof: Let $x^T = (1^{*4}, 0^{*29})$ in Corollary 2.5, then $L_8 = 52$ as $a = 4$. Now

$$x^T S x = 1_1^T S_1 1_1 = 32 + 2(c + d + e + u + v + w) \geq 52.$$

Therefore

$$c + d + e + u + v + w \geq 10.$$

Corollary 2.8 *If* $c = d = e = 1$ *in Theorem 2.7, then* $max(u, v, w) \geq 3$.

Proof: By Theorem 2.7, $u + v + w \geq 7$ if $c = d = e = 1$. So that $\max(u, v, w) \geq 3$.

Theorem 2.9 *Let*

$$\begin{pmatrix} 8 & u & v \\ u & 8 & w \\ v & w & 8 \end{pmatrix}$$

be a submatrix of S. *Then* $u + v \geq w$.

Proof: Let $x^T = (-1, 1, 1, 0^{*30})$ in Corollary 2.3, then $U_6 = 1.75$ as $a = 1$, and

$$x^T[S - (r - \lambda)I_b]x = -2(u + v - w) \leq 1.75,$$

that is, $u + v \geq w - 0.875$. Since u, v, w are integers, Theorem 2.9 holds.

In Table 2.1, both Types 5 and 8 have $b_1 \geq 3$. Consider the S_1 formed by the four blocks B_1, B_2, B_3, B_4 where $|B_1 \cap B_i| = 1$ for $i = 2, 3$ and 4. By letting the remaining entries of S_1 be u, v and w, we get

$$\begin{pmatrix} 8 & 1 & 1 & 1 \\ 1 & 8 & u & v \\ 1 & u & 8 & w \\ 1 & v & w & 8 \end{pmatrix}.$$

Corollary 2.8 implies one of the u, v or w is at least 3. Consider the 3×3 submatrix of S_1 using this maximal entry x. We have

$$\begin{pmatrix} 8 & 1 & 1 \\ 1 & 8 & x \\ 1 & x & 8 \end{pmatrix}.$$

since $x \geq 3$, we have a contradiction to Theorem 2.9. Hence types 5 and 8 are eliminated.

Corollary 2.10 *If $u = 0$ in Theorem 2.9, then $v = w$.*

Proof: Theorem 2.9 implies $0 + v \geq w$. To show $w \geq v$, interchange blocks 2 and 3.

Corollary 2.11 *If $c = 0, d = 1, e = 2$ in Theorem 2.7, then $u = 1, v = 2$, and $w \geq 4$.*

Proof: When $c = 0, d = 1$ and $e = 2$ in the matrix S_4, then $u + v + w \geq 7$ by Theorem 2.7.

Since $c = 0$ in the first principal submatrix of S_4, we have $u = d = 1$ by Corollary 2.10. Similarly, for submatrix

$$\begin{pmatrix} 8 & c & e \\ c & 8 & v \\ e & v & 8 \end{pmatrix}$$

of S_4, $c = 0$ implies $v = \epsilon = 2$. Thus, $w \geq 4$ holds.

From Corollary 2.11, we have $u + v = 3$ and $w \geq 4$. This contradicts Theorem 2.9, and we can eliminate type 9 of Table 2.1, because $b_0 = b_1 = 1$ and $b_2 > 1$ imply we can choose 4 blocks such that the first row of the corresponding S_4 is $[8, 0, 1, 2]$.

Some of the 29 cases of S_5 can be eliminated. An S_5 contains the intersection number of five blocks. Let N_5 be the 5 columns of the incidences matrix corresponding to these five blocks. In N_5, every point has an even number of incidences and every column has eight 1's. Conventionally, we can place this N_5 in the first five columns of N.

We note the following facts:
(a) If a N_5 has f rows with 4 incidences, then it has $8 \times 5 - 4f = 2(20 - 2f)$ 1's in remaining $(22 - f)$ rows, that is, it has $(20 - 2f)$ rows with 2 ones, and $(2 + f)$ rows with no ones. To simplify the explanation, we use the "t-rows" instead of "the rows with t ones" then. The sum of the off-diagonal elements in S_5 will be $\Sigma = fP(4,2) + (20 - 2f)P(2,2) = 40 + 8f$. $P(n,r)$ is the number of permutations of n things taken r at a time.
(b) If the ith column of N_5 has t incidences in the 4-rows, then it will have $(8 - t)$ 1's in the 2-rows, and none in the empty rows. The off-diagonal sum in the first 5 columns for the ith row of $S = (s_{ij})$ will be $r_i = (4 - 1) \times t + (2 - 1) \times (8 - t) = 8 + 2t$, where $r_i = \sum_{j=1}^{5} s_{ij} - s_{ii}$, $i = 1,2,3,4,5$.
(c) If the jth column of N (with $j > 5$) has t incidences in the 4-rows, and c incidences in the 0-rows, then it will have $(8 - t - c)$ incidences in the 2-rows. The sum of the first 5 elements in row j of S, $R_j = \sum_{i=1}^{5} s_{ji}$, is $4t + 0c + 2(8 - t - c) = 16 + 2t - 2c$.

Lemma 2.12 *The values of $\Sigma/4, r_i, r_i + k$, and R_j are even.*

Proof: Since $\Sigma = 40 + 8f$, and $k = 8$, we have

$$\begin{aligned} \Sigma/4 &= 2(5+f), \\ r_i &= 2(4+t), \\ r_i + k &= 2(8+t), \quad \text{and} \\ R_j &= 2(8+t-c). \end{aligned}$$

The above fact allow us to use $x^T = (\pm 1/2, ..., \pm 1/2, 0^{*28})$ in Corollary 2.3 and 2.5, because $[(r-\lambda)I_b - S]x$ and Nx are both integral.

Lemma 2.13 $\Sigma \in \{40, 48, 56\}$.

Proof: Take $x^T = ((1/2)^{*5}, 0^{*28})$ in Corollary 2.3 and 2.5, then $a = x^T 1_{33} = 5/2$. From Table 2.2, $U_6 = 15.75, L_8 = 20$. Therefore

$$x^T[S - (r-\lambda)I_b]x = (1/2)^2 1_5^T [S_5 - (r-\lambda)I_5]1_5 = \Sigma/4 \leq U_6, \text{and}$$

$$x^T S x = (1/2)^2(\Sigma + 5 \times 8) \geq L_8.$$

That is

$$10 \leq \Sigma/4 \leq 15.75,$$

or

$$40 \leq \Sigma \leq 63.$$

Since $\Sigma = 40 + 8f = 8(5+f)$, $\Sigma \in \{40, 48, 56\}$.

Lemma 2.14 $r_i \leq \Sigma/4 - 2$, $i = 1, 2, ..., 5$.

Proof: Take $x^T = ((1/2)^{*4}, -1/2, 0^{*28})$ in Corollary 2.5, this establishes that $r_5 \leq \Sigma/4 - 2$. Here the r_5 can be substituted by any r_i for $i = 1, 2, 3, 4, 5$, by interchanging components $1/2$ and $-1/2$ in x.

Theorem 2.15 *The followings statements are true:*

(a) If $\Sigma = 40$, *then* $\{r_1, r_2, r_3, r_4, r_5\} = \{8^{*5}\}$.

(b) If $\Sigma = 48$, *then* $\{r_1, r_2, r_3, r_4, r_5\} = \{10^{*4}, 8\}$.

(c) If $\Sigma = 56$, *then* $\{r_1, r_2, r_3, r_4, r_5\} = \{12^{*4}, 8\}$ *or* $\{12^{*3}, 10^{*2}\}$.

Proof: We have $r_1 + r_2 + ... + r_5 = \Sigma$. Also, for each i, $r_i \geq 8$, r_i is even and $r_i \leq \Sigma/4 - 2$. Hence, the solution are as listed.

Lemma 2.16 *The only possible values for* s_{45} *are given by*

classification	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
Σ	*40*	*48*	*48*	*56*	*56*	*56*	*56*
r_4	*8*	*8*	*10*	*8*	*10*	*10*	*12*
r_5	*8*	*10*	*10*	*12*	*10*	*12*	*12*
s_{45}	*0,1,2*	*0,1,2*	*1,2,3*	*0,1,2*	*0,1,2*	*1,2,3*	*2,3,4*

Proof: Take $x^T = ((1/2)^{*3}, (-1/2)^{*2}, 0^{*28})$ in Corollaries 2.3 and 2.5. We have $a = x^T 1 = 1/2$ and from Table 2.2, $U_6 = -0.25$ and $L_8 = 4$. Hence,

$$x^T S x = \Sigma/4 + 10 - (r_4 + r_5 - 2s_{45}) \geq 4, \text{and}$$

$$x^T[S - (r - \lambda)I_b]x = x^T S x - 8(5/4) = x^T S x - 10 \leq -0.25.$$

That is,

$$4 \leq \Sigma/4 + 10 - (r_4 + r_5 - 2s_{45}) \leq 10 - 0.25,$$

or

$$(r_4 + r_5)/2 - \Sigma/8 - 3 \leq s_{45} \leq (r_4 + r_5)/2 - \Sigma/8 - 0.125.$$

For $\Sigma = 40$, $r_4 + r_5 = 8 + 8$; $\Sigma = 48$, $r_4 + r_5 = 8 + 10$ or $10 + 10$; $\Sigma = 56$, $r_4 + r_5 = 8 + 12, 10 + 10, 10 + 12$ or $12 + 12$. The upper and lower bounds on s_{45} then follow.

Note that the s_{45} can be substituted by any s_{ij} for $i \neq j = 1,2,3,4,5$, since the order of the elements in the vector x can be changed.

Theorem 2.17 *The only possible S_5 are those numbered L12, L15, L16, L18, L23, L26 and L29.*

Proof: The cases L1, L2, L3, L4, L5, L6 and L9 have $\Sigma = 64$. These 7 configuration are impossible by Lemma 2.13.

Classify the remaining 22 cases of S_5 by Σ, r_4 and r_5 according to Lemma-2.16. We have the following possible cases:

Classification (1):

It contains the following cases: L27, L28 and L29. From Lemma 2.16, $s_{ij} \leq 2$. In L27, $s_{35} = 3$. In L28, $s_{45} = 3$. So both L27 and L28 are impossible. Only L29 is left.

Classification (2):

It contains 3 cases, which are L24, L25 and L26. By Lemma 2.16, $0 \leq s_{45} \leq 2$. For $r_5 = 8, r_4 = 10$, since $s_{54} = 3$ in L24 and L25, both are impossible. Only L26 is possible.

Classification (3):

It contains cases L21 and L22. From Lemma 2.16, $1 \leq s_{12} \leq 3$ for $r_1 = r_2 = 10$. In both of L21 and L22, $s_{12} = 4$ for $r_1 = r_2 = 10$, so, they are impossible.

Classification (4):

It contains the cases L12 and L16. Both are still possible.

Classification (5):

It contains 7 cases. By Lemma 2.16, $0 \leq s_{ij} \leq 2$ if $r_i = r_j = 10$. Since $s_{45} = 3$ in L7, L8, L10 and L11 and $r_4 = r_5 = 10$, they are impossible. But cases L15, L18 and L23 are left.

Classification (6):

It contains five cases L13, L14, L17, L19 and L20 with $s_{53} = 4$, where $r_5 = 10$, and $r_3 = 12$. But according to Lemma 2.16, $1 \leq s_{ij} \leq 3$ if $r_i = 10, r_j = 12$, so they are all impossible.

Classification (7):

It is empty.

The 7 remaining configurations are listed as follows:

$$\underset{\text{L12(H64)}}{\begin{pmatrix} 8 & 4 & 4 & 2 & 2 \\ 4 & 8 & 2 & 4 & 2 \\ 4 & 2 & 8 & 4 & 2 \\ 2 & 4 & 4 & 8 & 2 \\ 2 & 2 & 2 & 2 & 8 \end{pmatrix}} \quad \underset{\text{L15(H82)}}{\begin{pmatrix} 8 & 4 & 4 & 2 & 2 \\ 4 & 8 & 2 & 3 & 3 \\ 4 & 2 & 8 & 3 & 3 \\ 2 & 3 & 3 & 8 & 2 \\ 2 & 3 & 3 & 2 & 8 \end{pmatrix}} \quad \underset{\text{L16(H65)}}{\begin{pmatrix} 8 & 4 & 3 & 3 & 2 \\ 4 & 8 & 3 & 3 & 2 \\ 3 & 3 & 8 & 4 & 2 \\ 3 & 3 & 4 & 8 & 2 \\ 2 & 2 & 2 & 2 & 8 \end{pmatrix}} \quad \underset{\text{L18(H81)}}{\begin{pmatrix} 8 & 4 & 3 & 3 & 2 \\ 4 & 8 & 3 & 2 & 3 \\ 3 & 3 & 8 & 3 & 3 \\ 3 & 2 & 3 & 8 & 2 \\ 2 & 3 & 3 & 2 & 8 \end{pmatrix}}$$

$$\underset{\text{L23(H74)}}{\begin{pmatrix} 8 & 3 & 3 & 3 & 3 \\ 3 & 8 & 3 & 3 & 3 \\ 3 & 3 & 8 & 3 & 3 \\ 3 & 3 & 3 & 8 & 1 \\ 3 & 3 & 3 & 1 & 8 \end{pmatrix}} \quad \underset{\text{L26(H38)}}{\begin{pmatrix} 8 & 3 & 3 & 2 & 2 \\ 3 & 8 & 2 & 3 & 2 \\ 3 & 2 & 8 & 3 & 2 \\ 2 & 3 & 3 & 8 & 2 \\ 2 & 2 & 2 & 2 & 8 \end{pmatrix}} \quad \underset{\text{L29(H1)}}{\begin{pmatrix} 8 & 2 & 2 & 2 & 2 \\ 2 & 8 & 2 & 2 & 2 \\ 2 & 2 & 8 & 2 & 2 \\ 2 & 2 & 2 & 8 & 2 \\ 2 & 2 & 2 & 2 & 8 \end{pmatrix}}$$

It remains to be seen what attack will either eliminate these cases or lead to a construction of the design.

Chapter 3

Construction of Some (28,63,27,12,11)-BIBD's

In a symmetric BIBD, the intersection matrix $S = (k-\lambda)I + \lambda J$. So, there is only one intersection number, namely λ. In a quasi-symmetric BIBD, there are two intersection numbers. In this chapter, we construct some new quasi-symmetric (28,63,27,12,11)-BIBD with block intersection numbers $y = 6$ and $x = 4$ by assuming an automorphism g of order 7 with no fixed point and no fixed block.

3.1 The Orbit Matrix

We assume that the (28,63,27,12,11)-BIBD's are invariant under an automorphism g of order 7. Suppose that the blocks are labelled B_1, B_2, ... , B_{63} and the points are labelled $x_1, x_2, ..., x_{28}$. By relabelling the blocks, if necessary, we can assume that the action of g on the blocks is given by

$$g = (1, 2, ..., 7)(8, 9, ..., 14)...(57, 58, ..., 63).$$

Similarly, we relabel the points such that g's action on the points is given by

$$g = (1, 2, ..., 7)(8, 9, ..., 14)...(22, 23, ..., 28).$$

In general, let $X_1, X_2, ..., X_m$ be the point-orbits and $O_1, O_2, ..., O_n$ be the block-orbits of a BIBD fixed by a group G. We define the orbit matrices with respect to G to be the $m \times n$ matrices $C_G = (C_{ij})$ and $R_G = (R_{ij})$ where $C_{ij} = |X_i \cap B|$ for a $B \in O_j$ and $R_{ij} = |\{B | x \in B, B \in O_j\}|$ for an $x \in X_i$. Because O_j and X_i are orbits, the values of C_{ij} and R_{ij} do not depend on the choice of $B \in O_j$ and $x \in X_i$. When g is an automorphism, we let C_g and R_g be the matrices with respect to the cyclic group generated by g. If the order of g is a prime p, then the size of the orbits are either 1 or p. In other words, a column (row) is either fixed or its orbit is of size p.

Another way of viewing the orbit matrices is that the point orbits and column orbits induces a partitioning of the incidence matrix N into an $m \times n$ block matrix. The entry C_{ij} corresponds to the sum of a column in the (i, j)-block. The entry R_{ij} corresponds to the sum of a row in the (i, j)-block. When the size of X_i equals the size of O_j, then $C_{ij} = R_{ij}$.

The following result is well known:

Lemma 3.1 *If σ_i is the size of point-orbit X_i and τ_j is the size of block-orbit O_j, then*

$$\Sigma_{k=1}^{n} C_{ik} C_{jk} \tau_k = \begin{cases} \lambda \sigma_i \sigma_j, & \text{if } i \neq j, 1 \leq i, j \leq m, \\ \lambda \sigma_i^2 + (r - \lambda) \sigma_i, & \text{if } i = j = 1, ..., m. \end{cases} \quad (3.1)$$

$$\Sigma_{k=1}^{n} C_{ik} \tau_k = r \sigma_i, \quad i = 1, ..., m. \quad (3.2)$$

Proof: We note that

$$NN^T = (r - \lambda) I_v + \lambda J_v$$

$$= \begin{pmatrix} (r-\lambda) I_{\sigma_1} + \lambda J_{\sigma_1} & \lambda J_{\sigma_1 \sigma_2} & \cdots & \lambda J_{\sigma_1 \sigma_m} \\ \lambda J_{\sigma_2 \sigma_1} & (r-\lambda) I_{\sigma_2} + \lambda J_{\sigma_2} & \cdots & \lambda J_{\sigma_2 \sigma_m} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda J_{\sigma_m \sigma_1} & \lambda J_{\sigma_m \sigma_2} & \cdots & (r-\lambda) I_{\sigma_m} + \lambda J_{\sigma_m} \end{pmatrix},$$

and

$$J_{\sigma_i\sigma_j} = 1_{\sigma_i}1_{\sigma_j}^T, \quad \text{where} \quad 1_{\sigma_i} = (\underbrace{1\cdots 1}_{\sigma_i})^T,$$

$$1_{\sigma_i}^T J_{\sigma_i\sigma_j} 1_{\sigma_j} = \sigma_i\sigma_j.$$

Let u_i be a v by 1 vector,

$$u_i^T = (\ 0_{\sigma_1} \ \cdots \ 0_{\sigma_{i-1}} \ 1_{\sigma_i}^T \ 0_{\sigma_{i+1}} \ \cdots \ 0_{\sigma_m} \ ),$$

then

$$u_i^T N N^T u_j = 1_{\sigma_i}^T(\lambda J_{\sigma_i\sigma_j})1_{\sigma_j} = \lambda\sigma_i\sigma_j, \qquad \text{if} \quad i \neq j, \text{and}$$

$$u_i^T N N^T u_i = 1_{\sigma_i}^T[(r-\lambda)I_{\sigma_i} + \lambda J_{\sigma_i}]1_{\sigma_i} = (r-\lambda)\sigma_i + \lambda\sigma_i^2, \quad \text{if} \quad i = 1, ..., m.$$

This is the right hand side of (3.1).

Since C_{ij} is the column sum within the submatrix formed by row orbit X_i and column orbit O_j, we have

$$\begin{aligned} u_i^T N (u_j^T N)^T &= (\underbrace{C_{i1}\cdots C_{i1}}_{\tau_1}|\cdots|\underbrace{C_{in}\cdots C_{in}}_{\tau_n})(\underbrace{C_{j1}\cdots C_{j1}}_{\tau_1}|\cdots|\underbrace{C_{jn}\cdots C_{jn}}_{\tau_n})^T \\ &= \Sigma_{k=1}^n C_{ik}C_{jk}\tau_k, \quad i,j = 1, ..., m. \end{aligned}$$

Since $u_i^T N N^T u_j = u_i^T N(u_j^T N)^T$, the left side of (3.1) equals to the right side, thus (3.1) holds.

It is easy to see that

$$\begin{aligned} \Sigma_{k=1}^n C_{ik}\tau_k &= (\underbrace{C_{i1}\cdots C_{i1}}_{\tau_1}|\cdots|\underbrace{C_{in}\cdots C_{in}}_{\tau_n})1_b \\ &= u_i^T N 1_b \\ &= u_i^T r 1_v \\ &= r\sigma_i, \quad i = 1, ..., m. \end{aligned}$$

These equations allow us to find solutions to the orbit matrices. In our case we have $\sigma_i = \tau_j = 7$. $C_{ij} = R_{ij}$ for all i and j. The orbit matrix of a (28,63,27,12,11)-BIBD is a 4 by 9 matrix. The equations thus reduce to

$$\Sigma_{k=1}^9 C_{ik}C_{jk} = \begin{cases} 11(7) = 77, & \text{if } i \neq j, 1 \leq i,j \leq 4, \\ 11(7) + (27-11) = 93, & \text{if } i = j = 1, ..., 4. \end{cases}$$

$$\Sigma_{k=1}^9 C_{ik} = 27, \quad i = 1, ..., 4.$$

Using a computer, we found a total of 284 orbit matrices satisfying these equations.

3.2 Non-Existence Condition on Orbit Matrices

In this section, we will try to reduce the number of possible orbit matrices.

Lemma 3.2 *In a 4 by 9 orbit matrix $C'_y = (C_{ij})$ of the (28,63,27,12,11)-BIBD, the possible values of $\sum_{i=1}^{4} C_{ij}^2$ $(j = 1, ..., 9)$ are 36, 40, 44, and 48.*

Proof: Let the incidence matrix $N = (N_1, N_2, ..., N_9) = (N_{ij})$, where the $N_j (j = 1, ..., 9)$ are the 28×7 submatrices of N and the $N_{ij} (i = 1, ..., 4, j = 1, ..., 9)$ are the 7×7 submatrices of the N. As usual, $S = N^T N = (N_i^T N_j)$. Let $N_1 = (n_0, n_1, ..., n_6)$, where each n_j is a 28×1 vectors,

$$n_j = (n_{0,j}^{(1)}, ..., n_{6,j}^{(1)}, ..., n_{0,j}^{(4)}, ..., n_{6,j}^{(4)})^T,$$

and the subscripts are taken modulo 7. Since N is fixed by an automorphism of order 7, the N_{ij} are circulants, then in N_1 we have

$$n_{p,q}^{(k)} = n_{p-1,q-1}^{(k)}, \quad k = 1, ..., 4, \quad p, q (mod\, 7) = 0, ..., 6.$$

Hence, the non-diagonal element $s_{0,l}, l = 1, ..., 6$, of the first row in the $N_1^T N_1$ are given by

$$\begin{aligned} s_{0,l} &= n_0^T n_l \\ &= \textstyle\sum_{k=1}^{4} \sum_{i=0}^{6} n_{i,0}^{(k)} n_{i,l}^{(k)} \\ &= \textstyle\sum_{k=1}^{4} \sum_{i=0}^{6} n_{i-l,0-l}^{(k)} n_{i-l,l-l}^{(k)} \\ &= \textstyle\sum_{k=1}^{4} \sum_{i=0}^{6} n_{i-l,7-l}^{(k)} n_{i-l,0}^{(k)} \\ &= \textstyle\sum_{k=1}^{4} \sum_{i=0}^{6} n_{i,7-l}^{(k)} n_{i,0}^{(k)}. \end{aligned}$$

$$\begin{aligned} s_{0,7-l} &= n_0^T n_{7-l} \\ &= \textstyle\sum_{k=1}^{4} \sum_{i=0}^{6} n_{i,0}^{(k)} n_{i,7-l}^{(k)}, \end{aligned}$$

Therefore, we have

$$s_{0,l} = s_{0,7-l}, \quad l = 1, ..., 6.$$

Now, let $s_{0,1} = s_{0,6} = b_1, s_{0,2} = s_{0,5} = b_2$, and $s_{0,3} = s_{0,4} = b_3$, where $b_i (i = 1, 2, 3)$ should be either 4 or 6, because the design is quasi-symmetric. Thus the first row sum of $N_1^T N_1$ is $\sum_{j=0}^{6} n_0^T n_j = 12 + 6(4) + 2(2t)$, for $t = 0, 1, 2, 3$, where $2t$ is the number of $s_{0,j}$ that is equal to 6.

On the other hand, this first row sum of $N_1^T N_1$ is also given by

$$\begin{aligned} 1_7^T N_1^T N_1 e_1 &= (1_7^T N_{11}^T, ..., 1_7^T N_{41}^T) N_1 e_1 \\ &= [\textstyle\sum_{i=1}^{4} C_{i1} 1_7^T N_{i1}] e_1 \\ &= \textstyle\sum_{i=1}^{4} C_{i1}^2, \end{aligned}$$

where $e_1 = (1, 0, ...0)^T$ is a 7 by 1 vector. Therefore,

$$\sum_{i=1}^{4} C_{i1}^2 = 12 + 6(4) + 2(2t), t = 0, 1, 2, 3.$$

That is

$$\sum_{i=1}^{4} C_{i1}^2 = 36, 40, 44, \text{or } 48.$$

Replacing N_1 by N_j, we get a more general equation

$$\sum_{i=1}^{4} C_{ij}^2 = 12 + 6(4) + 2(2t), j = 1, ..., 9, t = 0, 1, 2, 3,$$

which shows that the possible values of the sum of square for any column of the orbit matrix C_{ij} are 36, 40, 44, and 48.

Example. The orbit matrix of case 3 is

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\ 4 & 3 & 3 & 3 & 3 & 3 & 0 & 4 & 4 \\ 4 & 2 & 2 & 3 & 3 & 5 & 4 & 1 & 3 \\ 4 & 4 & 4 & 3 & 3 & 1 & 4 & 3 & 1 \end{pmatrix}.$$

Since the sum of square for its second column is $3^2+3^2+2^2+4^2 = 38$, which does not equal 36, 40, 44, or 48, so it can not be an orbit matrix. We can eliminate it from the 284 cases.

According to this Lemma, we reduce the 284 possible cases to 9. They are cases 1, 2, 26, 164, 171, 235, 237, 262 and 283. These cases are listed in Appendix B.

3.3 The Results

Given an automorphism g, if point j is in block i, then point j^g is in block i^g. Therefore, each block can be assumed to be a circulant. Within each block, the row sum equals to the column sum, which is a constant. So, given a particular orbit matrix, we can try to extend it to a complete BIBD by using a computer program BDX to try out all possible circulant matrices with the correct row sum. We found a total 246 new quasi-symmetric (28,63,27,12,11)-BIBD.

In Table 3.1, we give the distribution of solutions according to the size of their automorphism groups. In Table 3.2, we give the distribution of solutions according to their orbit matrices.

It would be interesting to try and construct more quasi-symmetric designs with these parameter set by assuming a different automorphism.

Table 3.1: Distribution of solutions according to size of automorphism group

Group order	Frequency
7	47
14	50
21	95
28	12
42	3
84	15
168	2
224	8
672	8
1344	4
10752	1
1451520	1

Table 3.2: Distribution of solutions according to orbit matrices

Case	Number of new solutions
1	21
2	12
26	69
161	36
171	55
235	20
237	10
262	12
283	11

Appendix A

The 29 submatrices S_5 of the incidence matrix S

L1 (H100)	L2 (H99)	L3 (H101)	L4 (H103)
8 4 4 4 2	8 4 4 4 2	8 4 4 3 3	8 4 4 4 2
4 8 4 3 3	4 8 4 2 4	4 8 4 3 3	4 8 3 3 4
4 4 8 1 3	4 4 8 2 2	4 4 8 2 2	4 3 8 2 3
4 3 1 8 4	4 2 2 8 4	3 3 2 8 4	4 3 2 8 3
2 3 3 4 8	2 4 2 4 8	3 3 2 4 8	2 4 3 3 8

L5	L6 (H107)	L7 (H90)	L8 (H89)
8 4 4 3 3	8 4 4 3 1	8 4 4 3 1	8 4 4 3 1
4 8 3 4 3	4 8 3 3 4	4 8 3 3 2	4 8 3 2 3
4 3 8 2 3	4 3 8 3 4	4 3 8 1 4	4 3 8 2 3
3 4 2 8 3	3 3 3 8 3	3 3 1 8 3	3 2 2 8 3
3 3 3 3 8	1 4 4 3 8	1 2 4 3 8	1 3 3 3 8

L9 (H108)	L10 (H94)	L11 (H91)	L12 (H64)
8 4 4 2 2	8 4 4 2 2	8 4 4 2 2	8 4 4 2 2

4 8 3 4 3	4 8 3 4 1	4 8 3 3 2	4 8 2 4 2
4 3 8 3 4	4 3 8 1 4	4 3 8 2 3	4 2 8 4 2
2 4 3 8 3	2 4 1 8 3	2 3 2 8 3	2 4 4 8 2
2 3 4 3 8	2 1 4 3 8	2 2 3 3 8	2 2 2 2 8
L13 (H80)	L14 (H87)	L15 (H82)	L16 (H65)
8 4 4 3 1	8 4 4 2 2	8 4 4 2 2	8 4 3 3 2
4 8 2 3 3	4 8 2 4 2	4 8 2 3 3	4 8 3 3 2
4 2 8 2 4	4 2 8 2 4	4 2 8 3 3	3 3 8 4 2
3 3 2 8 2	2 4 2 8 2	2 3 3 8 2	3 3 4 8 2
1 3 4 2 8	2 2 4 2 8	2 3 3 2 8	2 2 2 2 8
L17 (H88)	L18 (H81)	L19 (H78)	L20 (H73)
8 4 3 3 2	8 4 3 3 2	8 4 3 3 2	8 4 2 3 3
4 8 3 3 2	4 8 3 2 3	4 8 2 3 3	4 8 2 3 3
3 3 8 2 4	3 3 8 3 3	3 2 8 3 4	2 2 8 4 4
3 3 2 8 2	3 2 3 8 2	3 3 3 8 1	3 3 4 8 0
2 2 4 2 8	2 3 3 2 8	2 3 4 1 8	3 3 4 0 8
L21 (H35)	L22 (H36)	L23 (H74)	L24 (H52)
8 4 3 1 2	8 4 2 2 2	8 3 3 3 3	8 3 3 3 1
4 8 1 3 2	4 8 2 2 2	3 8 3 3 3	3 8 3 3 1
3 1 8 4 2	2 2 8 4 2	3 3 8 3 3	3 3 8 1 3
1 3 4 8 2	2 2 4 8 2	3 3 3 8 1	3 3 1 8 3
2 2 2 2 8	2 2 2 2 8	3 3 3 1 8	1 1 3 3 8
L25 (H55)	L26 (H38)	L27 (H2)	L28
8 3 3 3 1	8 3 3 2 2	8 3 3 2 0	8 3 2 2 1

3 8 3 2 2	3 8 2 3 2	3 8 0 2 3	3 8 2 1 2
3 3 8 2 2	3 2 8 3 2	3 0 8 2 3	2 2 8 2 2
3 2 2 8 3	2 3 3 8 2	2 2 2 8 2	2 1 2 8 3
1 2 2 3 8	2 2 2 2 8	0 3 3 2 8	1 2 2 3 8

L29 (H1)	(H104)	(H10)
8 2 2 2 2	8 4 4 3 2	8 3 2 2 1
2 8 2 2 2	4 8 3 4 3	3 8 1 1 3
2 2 8 2 2	4 3 8 2 3	2 1 8 3 1
2 2 2 8 2	3 4 2 8 3	2 1 3 8 3
2 2 2 2 8	2 3 3 3 8	1 3 1 3 8

Appendix B

The 9 orbit matrices

```
solution     1, autogp size = 2880
row_type_id =   1  1  1  1

  0  3  3  3  3  3  4  4  4
  4  3  3  3  3  3  0  4  4
  4  3  3  3  3  3  4  0  4
  4  3  3  3  3  3  4  4  0

solution     2, autogp size = 48
row_type_id =   1  1  2  2

  0  3  3  3  3  3  4  4  4
  4  3  3  3  3  3  0  4  4
  4  1  3  3  3  5  4  2  2
  4  5  3  3  3  1  4  2  2

solution    26, autogp size = 6
row_type_id =   1  2  2  2
```

```
  0  3  3  3  3  3  4  4  4
  4  1  3  3  3  5  2  2  4
  4  3  3  3  5  1  2  4  2
  4  5  3  3  1  3  4  2  2

solution   164, autogp size = 4
row_type_id =   2  2  2  2

  1  2  2  3  3  3  4  4  5
  3  2  4  3  3  5  2  4  1
  3  4  4  3  5  1  2  2  3
  5  4  2  3  1  3  4  2  3

solution   171, autogp size = 6
row_type_id =   2  2  2  5

  1  2  2  3  3  3  4  4  5
  3  2  4  3  3  5  2  4  1
  5  2  4  3  3  1  4  2  3
  3  6  2  3  3  3  2  2  3

solution   235, autogp size = 32
row_type_id =   2  2  2  2

  1  2  2  3  3  3  4  4  5
  3  4  4  1  3  5  2  2  3
  3  4  4  5  3  1  2  2  3
```

```
  5  2  2  3  3  3  4  4  1

solution    237, autogp size = 8
row_type_id =    2  2  4  6

  1  2  2  3  3  3  4  4  5
  3  4  4  1  3  5  2  2  3
  3  4  4  3  3  1  4  4  1
  5  2  2  5  3  3  2  2  3

solution    262, autogp size = 48
row_type_id =    2  2  5  5

  1  2  2  3  3  3  4  4  5
  5  2  2  3  3  3  4  4  1
  3  2  6  3  3  3  2  2  3
  3  6  2  3  3  3  2  2  3

solution    283, autogp size = 2880
row_type_id =    5  5  5  5

  2  2  2  3  3  3  3  3  6
  2  2  6  3  3  3  3  3  2
  2  6  2  3  3  3  3  3  2
  6  2  2  3  3  3  3  3  2
```

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