The Construction of Meaning for Algebraic Expressions

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ABSTRACT

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A review of current textbooks used in secondary schools indicates how difficult it is to introduce algebraic expressions, the starting point for most introductory algebra courses, in a way which might be meaningful to beginning students. This study aims at designing and experimenting a new approach for the introduction of these expressions.

A teaching outline, attempting to guide students to construct meaning for algebraic expressions is formulated taking into account the cognitive obstacles uncovered in prior research. The students are led to identify algebraic expressions as "answers to problems". The problems to be used are highly pictorial in nature, easy to visualize, and would not in themselves create cognitive obstacles. The three problem types selected involve the quantification of a rectangular array of dots, the length of a line divided into segments and the area of a rectangle.

In the experimental design, an exploratory phase involving an Initial Pilot Study with three students and one Case Study is conducted. Both these studies are essential in the actual construction of the Teaching Outline. A final version of the Teaching Outline, consisting of a Pretest, three lessons and a Post Test, is developed and experimented.

The methodology that is used in this study is a version of the

Soviet "teaching experiment". This method allows for the teaching and learning processes to be studied simultaneously. Six students of various abilities from Grades 6 and 7 are interviewed individually. Each interview is audio-taped, fully transcribed and then is correlated with the written work. Thus the analysis is based both on the written work and the student's comments about it.

The analysis shows that all the subjects are able to construct meaning for algebraic expressions, as intended in the Teaching Outline. Interesting results concerning the novice's perception of algebraic symbolism are revealed. New insights are gathered regarding cognitive obstacles uncovered in prior research, such as, the name-process dilemma, the acceptance of lack of closure, specific unknown versus generalized number, and the conflict between the learner's algebraic and arithmetic frames of reference.

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INTRODUCTION

"I was always good in mathematics until I got to High School".

This familar statement, spoken by many people, implies that their mathematical failure began with the introduction of algebra, since it is at the high school level that algebra is first presented. As a high school mathematics teacher, I evidenced the frustration of many students with their initial acquisition of this subject. During interviews with parents, I found myself continually reassuring many parents that their child still had capabilities in mathematics, it was algebra that appeared to be his stumbling block. Although I was aware of the negative impact that algebra had on these students, I could not provide any possible solution for their problems. Consequently upon entering the program at Concordia University, Masters in the Teaching of Mathematics, my immediate intent was that the subject for my thesis would involve some aspect of the teaching and learning of algebra.

The topic chosen for this thesis "The Construction of Meaning for Algebraic Expressions" and the Teaching Outline developed was the result of many hours of thinking and discussion with Prof. N. Herscovics. It was finally concluded that such a construction was basic in the learning of algebra and should be a first topic in its introduction.

The knowledge I have acquired by doing this research has made me significantly more aware of the cognitive problems that algebra students encounter with their first introduction to this subject. By sharing this work with others, I hope that the Teaching Outline presented in this thesis will be only a beginning in an attempt to make algebra easier and more meaningful for the beginning student.

The following is a brief summary of the contents of each chapter.

Chapter I points out the pedagogical problems represented by the learning of algebraic expressions and points out how widespread this problem is. An evaluation of three current or recently revised textbooks' treatment of this topic is presented, justifying the need to design a new approach for the construction of meaning for algebraic expressions.

Chapter II reviews the recent research literature concerning the difficulties pupils encounter in assigning meaning for algebraic expressions and operations. R. Davis' and M. Matz's work on the incongruencies between arithmetic and algebra are cited, along with recent research done by K. Collis and D. Kuchemann. Based on this review the possibility of using a geometric approach for the introduction of algebraic expressions is proposed.

Chapter III outlines the pedagogical considerations in the preparation of the Teaching Outline. The theoretical basis of the Soviet "teaching experiment" is presented, justifying a version of it as the methodology most appropriate for our investigation. This chapter also describes the selection of the subjects and their background, as well as the planned analysis of the data collected in the interviews.

Chapter IV details the Construction of the Teaching Outline. Included in this chapter is the analysis of an Initial Pilot Study and an Exploratory Case Study, showing how these two studies were instrumental in the development of the final Teaching Outline used in this experiment.

Chapter V first describes the Pretest administered to each student and the rationale for each section of the test. This is followed by the responses given by the six subjects to these questions, which are then analyzed, bringing out the prealgebraic interpretations assigned by our subjects to algebraic symbolism and notation.

Chapters VI, VII and VIII analyze the protocols of the individual interviews with the six subjects, for Lessons 1, 2, and 3, respectively. The six students' responses are examined in detail, showing their thinking about the new material during the actual teaching and learning situation. A comparison of the written and verbal responses of these students is presented, including a description of the common thinking patterns which emerged. Any cognitive obstacle inherent to the Teaching Outline is also pointed out and some suggestions made.

Chapter IX analyzes the subjects' responses in the Post Test in the form of case studies. An attempt is made to determine how the Teaching Outline had affected each individual subject's knowledge. That is, each student's responses are compared with his answers from the Pretest, and verified to see if the difficulties he experienced at the end were similar to those he had experienced during the three lessons.

Chapter X first presents a summary of the discussions and results from the preceeding nine chapters. The second part examines more generally questions such as the quality of the Teaching Outline, the appropriateness of the methodology used in this study and the implications of our results for further research as well as for teaching.

As a final comment I would like to point out that this thesis came out of a project funded by the Quebec Ministry of Education (FCAC grant EQ-1741). N. Herscovics directed the part of the project dealing with algebra.

CHAPTER I

STATEMENT OF THE PROBLEM

While one may believe that the difficulties students experience in Secondary School Mathematics are mainly due to the complexity of the more advanced algebraic concepts, two recent studies, NAEP (Carpenter, et al., 1981) and CSMS (Kuchemann, 1977) indicate that a great many of them have not been able to assimilate even the most fundamental ideas introduced at the very beginning of a first formal course in algebra, those involving algebraic expressions and their manipulations. These two assessment studies have provided data showing that a significant number of students who have had one or two courses in algebra still have difficulties with concatenation as used in algebra, numerical substitutions in algebraic expressions, and simple tasks such as combining like terms and multiplying algebraic expressions. That so many students continue to experience difficulties with these elementary notions brings into question whether or not students have been able to construct meaning for algebraic expressions and the simple operations they involve. Of course, the kind of meaning they are likely to construct depends on the type of instruction they receive. To some extent, one can gather information about the teaching of this topic by examining different approaches used in current textbooks.

To substantiate the problem represented by the difficulties that students experience in learning algebraic expressions, this chapter will be divided into two parts. The first part will deal with the learning aspect of the problem by reporting and examining the results of the two assessment studies. The second part will deal with the instructional aspect through a careful analysis of three current or recently revised textbooks: Mathematics 9 (Kelly et al., 1981); Algebra in Easy Steps, (Stein, 1982); Algebra 1 (Dolciani et al., 1980). These books will be compared for their treatment of the definition of algebraic expressions, concatenation, and the evaluation of algebraic expressions through numerical substitutions.

A. What have the Students Learned?

An answer to the question, "What have the students learned?" can be gathered from two major assessment studies which deal primarily with the extent of the problems and not particularly with their causes. Research aimed at uncovering these causes will be the subject of Chapter 2.

The first study in question is the National Assessment of Educational Progress (NAEP) which compared the mathematical performance of 70,000 American students aged 9, 13, and 17 between 1973 and 1978. These students were examined on five content areas, one of which was algebra. Part of the data collected was used in the evaluation of the performance of 17-year-olds based on their mathematical background: 67% of the students surveyed reported completing one year of algebra and 35% reported taking at least a half year of second year algebra. (Carpenter et al., 1981)

The second study, Concepts in Secondary Mathematics and Science (CSMS) originates from Great Britain and tested the mathematical understanding of 10,000 children throughout England (Hart, 1981). The algebra component involved 3000 students aged 13, 14, and 15. (Kuchemann, 1977)

1. Overview of the Problems

The two studies mentioned above can be used to determine how widespread the problems are after an introductory course in algebra. Since most British students start their secondary school between the ages of 11 and 12, by examining the Kuchemann data for 14-year-olds we are assured that most of them have been exposed to one or two years of algebra (Kerslake, 1977). The same applies to NAEP data describing 17-year-olds with one or two courses in algebra. However, as Carpenter et.al., (1981) point out, the assessment was for many 17-year-olds an inventory of algebraic skills and understanding retained one or two years after studying elementary algebra. Furthermore, they note that although the results of 17-year-olds with two years of algebra were consistently 5 to 10% above the average of those with only one course, one cannot conclude that the students' performance is related merely to the number of courses taken for only the better students take the additional course.

Concatenation

That concatenation does indeed present cognitive obstacles to the beginning student is illustrated by the NAEP study in the context of the solution of simple equations in one unknown; 91% of 13-year-olds were able to solve '$4 \times \square = 24$', but only 65% could find the correct answer to '$6m = 36$'. Although Kuchemann has not examined directly the difficulties students might have with concatenation, one of his results bears some relation to this question. To the problem:

> Cabbages cost 8 pence each and turnips cost 6 pence each. If c stands for the <u>number</u> of cabbages bought and t stands for the <u>number</u> of turnips bought, what does 8c + 6t stand for?

only 1% of the children gave a clear correct answer. Most children thought '8c + 6t' meant "8 cabbages and 6 turnips" (52%). Thus, if the letters used are perceived as shorthand notation for various vegetables, the problem of concatenation does not even arise for <u>8c</u> does not denote a multiplication at all.

This also illustrates one of the major obstacles pupils encounter with word problems. Even where the letters are clearly specified as representing numbers this is not necessarily the interpretation given to them by the learner. As shown by Clement (1979) this persists even at the university level where a large number of the undergraduates tested (37%) translated "There are six times as many students as professors at this University" by "6s = p". Thus it is not surprising that many 17-year-olds cannot translate simpler problems not even involving concatenation as shown by the NAEP question:

> Carol earned D dollars during the week. She spent C dollars for clothes and F dollars for food. Write an expression using D,C, and F that shows the number of dollars she had left.

Only 57% of the students with one year of algebra and 67% of those with two years responded correctly. This illustrates a major dilemma with word problems. On one hand they are essential to create relevance for the study of algebra. But on the negative side the translation problem involves major cognitive obstacles. If letters are not perceived by the beginning student as numerical symbols he cannot possibly be expected to translate the numerical relationships implied in a word problem in terms of algebraic expressions.

Substitution

Regarding problems encountered with the evaluation of algebraic expressions only one result from the NAEP deals directly with it. To the question "What is the value of 'a + 7' when a = 5?", 70% of the 13 year-olds and an average of 97% of the 17 year-olds with one or two courses in algebra provided correct responses. This trivial problem contained only one unknown and did not involve any concatenation. Thus the possible difficulties students may experience with more complex expressions such as '3a + 2b + 5' if a = 4 and b = 5, can only be inferred indirectly from problems of evaluation within the context of functions. Carpenter et al. state that between half and two-thirds of the students (13 to 17) could find w when given a value for a in 'w = 17 + 5a'. This is in line with equivalent questions in Kuchemann's test: 62% of 14 year-olds could evaluate m where 'm = 3n + 1', given that n = 4. However, the presence of a dependent variable might have had an effect on the evaluation, and thus differences might be expected in the evaluation of an algebraic expression, since no dependent variable is present explicitly.

Combining like Terms

Kuchemann reports that the simplifications involved in expressions containing like terms causes few problems. For instance, 86% of the 14-year-olds tested could simplify '2a + 5a'. Perhaps this rate of success can be attributed in part to the interpretation of the literal symbol as an "object" (2 apples + 5 apples). As soon as unlike terms are present and have to be regrouped, there is a sharp decrease in correct responses:

PROBLEM	% CORRECT RESPONSE
2a + 5b + a =	60%
(a + b) + a =	53%
3a - b + a =	47%
(a - b) + b =	23%

A surprisingly large number of students (20%) responded with "8ab" to the first question in the table. This certainly reflects the difficulties students have with interpreting such algebraic expressions and perhaps it stems from their inability to accept '3a + 5b' as an answer. Of course, as the results indicate, the cognitive problems are increased with the introduction of subtraction and brackets.

EXPANSIONS

Kuchemann's study illustrates some difficulty students have in expanding algebraic expressions. Only 14% of the 14-year-olds were able to respond correctly to the problem 'Multiply n + 5 by 4'. Common wrong answers were 'n + 20' (31%), '4n + 5' (19%) and '20' (15%). Although this topic is often introduced with reference to the distributive axiom in a beginning algebra course, the results clearly indicate that many students are unaware that both elements of the expression must be multiplied by 4.

Another area of difficulty in expanding is evident in the NAEP study's example requiring the student to simplify '9(1 + 5x) + 3'. Once again many students did not multiply each element in the expression '1 + 5x' by 9, but rather first interpreted '1 + 5x' as <u>6x</u> and then performed the multiplication. In this example the pupil's inability to distinguish unlike terms prevents any perception of the problem as one involving expansion.

<u>Summary</u>

The results of these two studies indicate clearly that the problem is widespread, that is a large number of students in the United States and Great Britain experience difficulties in learning algebraic expressions. Concatenation and substitution appear to involve unsuspected cognitive obstacles which bring into question the kind of meaning they have constructed for these expressions.

B. <u>Textbook Analysis</u>

1. <u>Defining Algebraic Expressions</u>

Most high school textbooks use algebraic expressions as the first topic in their introduction to algebra. Inherent to these expressions is the notational problem of concatenation (the replacement of the multiplication symbol by mere juxtaposition, that is, '2 · a = 2a', '2 × a = 2a'). We now examine the treatment of this topic in <u>Mathematics</u>, (Kelly et al., 1981); <u>Algebra in Easy Steps</u> (Stein 1982); <u>Algebra</u>, (Dolciani, et al., 1980).

Kelly et al., start by introducing the formula for area of a rectangle, 'A = 1 × w' and then state that "letters such as <u>A</u>, <u>1</u> and <u>w</u>, which represent numbers, are called <u>variables</u>". The authors continue by defining formulas as "the result of combining variables and numbers using the basic operations of arithmetic", which they illustrate by examples such as $x + 7$, $n - 5$, $7n$, $\frac{6x}{5}$. The latter two are followed by the comment, "The signs $\times$ and $\div$ are usually omitted with variables". (p. 6)

From formulas, Kelley et al. proceed to introduce algebraic expressions as the right hand side of a formula. An example is used to illustrate the terminology:

> 1-3 Introduction to Algebra
>
> One use of the language of algebra is to express and solve problems of arithmetic. In order to use it for this purpose, we need to know the words of the language.
>
> Consider the formula $A = 1 + x - 2z + 7xy$
>
> These are terms. $\underline{1} + \underline{x} - \underline{2z} + \underline{7xy}$
>
> These are coefficients. $1 + \underline{1}x - \underline{2}z + \underline{7}xy$
>
> These are variables. $1 + \underline{x} - 2\underline{z} + 7\underline{xy}$
>
> This is an expression. $1 + x - 2z + 7xy$
>
> (Kelly et. al., p. 10)

Although, as can be seen above, the authors use an exemplar in their introduction, they nevertheless provide the student with a formal definition of the glossary. (p. 422): "Expression: any combination of numbers, variables, and symbols" with examples $3x + 5$, $\sqrt{x^2 + y^2}$.

On the other hand Stein (1982) first presents "numerical expressions" (p. 15) such as '$4 + 7$' and '6×9' . He then proceeds to define "variable as a placeholder" ("a variable holds a place open for a number" (p. 23) and finally defines an algebraic expression as a translation of a mathematical verbal phrase. The following problem is used as an example:

> "Write as an algebraic expression the sum of <u>n</u> and six" which he answers by 'n + 6'. (p. 23)

Within the procedure outlined for the "translation of a mathematical verbal phrase into an algebraic expression", Stein points out that "No multiplication symbol is necessary when the factors are two variables or are a number and a variable named by a letter. In the latter case the numeral always preceeds the variable". (p. 23) As an illustration the following example is used. "Write <u>x</u> multiplied by nine as an algebraic expression" which is answered as <u>9x</u> (p. 24).

Dolciani, et al.,(1980) also start with "numerical expressions" but go further than mere examples by defining them as "a name for a number". For instance "4 × 9 is a numerical expression for the number 36" (p. 1). The authors then continue by defining <u>variable</u> as a "letter such as <u>n</u>, <u>a</u> or <u>x</u> or a symbol such as <u>?</u>, used to stand for a number or numbers" and <u>variable expression</u> as "an expression containing one variable but may also contain other symbols, including numerals" (p. 2).

The authors then deal with concatenation by merely stating that "in a variable expression like '3 × n', the multiplication sign is often omitted". An example follows to illustrate this idea:

> evaluate 5y; y = 8. The solution is as follows: 5y = 5 × y = 5 × 8 = 40.

These examples illustrate how difficult it is to define algebraic expressions in terms that are meaningful to the students. Even the introduction of a numerical expression as a "name for a number" meets with cognitive problems. Davis (1975, p. 18) points out that to a pupil '3 + 5' is a problem or question and 8 is the answer. Thus constructing meaning for algebraic expressions on the basis of numerical ones seems questionable.

On the other hand, Kelly et al., introduce algebraic expressions by detaching them from meaningful formulas. Implicit in their presentation is the idea of an algebraic expression being the answer to a problem. For indeed, if <u>a</u> stands for the area of a rectangle, then '$1 \cdot w$' is <u>the answer</u> to the problem of finding the area. However, there is no attempt here to construct such meaning explicitly.

Following this textbook analysis, one could raise the question, "Do teachers have any other means to deal with the problem of definition?" Judging from the lack of articles in mathematics education journals, this appears to be doubtful. The complexity of the problem is alluded to in the Teacher's Manual of an earlier edition of the Dolciani et al., text. Here the authors discuss the analogous problem of defining polynomials:

> This chapter presents the formal techniques of calculating with polynomials. In advanced courses, polynomials may be regarded simply as <u>forms</u> for which suitable operations of addition and multiplication are defined. From this point of view, the letter x in $5 + 4x + 3x^2$ does not denote an element of some set of numbers, and the + signs do not indicate addition. Rather, the + signs as well as x and x^2, function as "bookkeeping devices" to keep track of the coefficients 5,4, and 3. This is your meaning when you say that a polynomial is determined by listing its coefficients in order.

> On the other hand, if you think of x as denoting an element of a set of numbers, then the polynomial $5 + 4x + 3x^2$ represents some number for each value of x. For example, if you replace x by 2, then $5 + 4x + 3x^2$ represents the sum of 5, 8, and 12 or 25. In this case, you regard polynomials as functions. With this interpretation, the commutative, associative, and distributive properties of operations with numbers determine operations with polynomials.
>
> Both interpretations appear in elementary algebra, although the concept of a polynomial as a form, while implicit in manipulations like factoring and simplifying, is usually not discussed explicitly with immature pupils. The distinction should be clear to you, but pupils at this level are easily confused by the idea of x as an indeterminate. Furthermore, for polynomials over the set of real numbers (or over any infinite set) the two concepts are abstractly identical in the sense that the set of polynomial forms and the set of polynomial functions are isomorphic.
>
> (Dolciani, et al., 1965, p. 22)

Similarly to polynomials, algebraic expressions can be interpreted mathematically as forms and functions but such interpretations are beyond the reach of beginning students for obvious reasons. Whereas advanced students may have the mathematical maturity to handle mathematical forms by relating them to their study of formal systems such as polynomial rings (Dean, 1966, Chapter 6), for novices this interpretation reduces algebra to the meaningless manipulation of meaningless symbols. Nor can they perceive such forms as functions, a topic which they have not yet seen. One could always argue that this may be an appropriate way to introduce this concept. However, algebraic expressions such as '2x + 3' convey the concept of function only implicitly since they lack a dependent variable (as in 'y = 2x + 3') which conveys explicitly the notion of relationship. In this sense, algebraic expressions are but "incomplete" mathematical statements.

2. Numerical Substitutions for Literal Symbols

After their initial presentation of algebraic expressions, textbook authors then proceed to have the students evaluate these expressions by substituting numerical values for literal symbols. This step is an important one since it assists the student in bridging the gap from arithmetic to algebra. It may be thought that this is a trivially simple process; however, NAEP results indicate otherwise. For example, for a formula like $W = 17 + 5A$, only about half the 13-year-olds and two thirds of the 17 year-olds were able to find W when given the value for A (NAEP, 1981, p. 69).

As pointed out earlier, Kelly et al., start with algebraic formulas rather than algebraic expressions. They then proceed to have the students evaluate formulas by substituting given values for all but one of the variables. Three of the four examples are of a geometric nature such as the circumference of a circle, '$c = \pi d$'. This is followed by 11 problems containing formulas which the students are asked to evaluate. Only then are algebraic expressions introduced as the right hand side of a formula.

Students are then guided into seeing a need to evaluate expressions. For instance, an algebraic expression is first obtained from a word problem:

> Victoria Falls is twice as high as Niagrara Falls. Write an expression for the height of Victoria Falls if the height of Niagara Falls is x metres.
>
> Answer: 2x metres high.

(Kelly et al., 1981, p. 11)

However, it is pointed out that to find the actual height of the falls, a value has to be substituted for <u>x</u>. Thus, within the context of a word problem the substitution process is not reduced to a mechanical exercise, but instead gives the answer to the problem. Once the authors have established the relevance of evaluating expressions, most of the eleven problems which follow have students evaluate expressions without referring to word problems.

In contrast to Kelly et al., Stein does not attempt to justify any need for substitution, putting as he does all the emphasis on the relevant procedures:

> I. AIM: To find the value of simple algebraic expressions when numbers of arithmetic are assigned to the variables.
>
> II. PROCEDURE:
>
> 1. Rewrite the given expression.
> 2. Substitute in order the given numerical value for each variable.
> 3. Perform the necessary operations as indicated to get the answer. See sample solutions 1, 2, 3 and 4.
> 4. When there is a numerical coefficient, it is rewritten and used as a factor to find the numerical value of the expression. See sample solution 5.
> 5. When there is more than one term (part of expression separated) by the + or - sign, first find the value of each part. Then combine as indicated. See sample solutions 6 and 7.
> 6. If the expression is a fraction, simplify both the numerator and denominator separately, and then express the fraction in simplest terms. See sample solution 8.
>
> (Stein, 1982, p. 96)

Eight sample solutions are given in the text and these are followed by eighty exercises.

Of the three textbooks, the Dolciani et al. one gives substitution the most cursory treatment. Within the same paragraph introducing variables and algebraic expressions, the authors define "replacement set" or "domain" as the set of values of the variable and illustrate it with the following example:

> If n can stand for 2 or 5 then you write $n = 2$ or $n = 5$, $3 \times n$ is a variable expression.
>
> If $n = 2$, then $3 \times n = 3 \times 2 = 6$
> If $n = 5$, then $3 \times n = 3 \times 5 = 15$
>
> This is called "evaluating the expression".

(Dolciani, et al., 1980, p. 2)

This example is followed by 30 exercises involving numerical substitutions without any further reference to replacement sets.

For numerical substitution, we see that Kelly et al. address themselves to the motivational aspects by presenting substitutions within a context which is relevant to the student, but ignore the procedural difficulties inherent to the symbolic notation. This latter aspect is the only one emphasized by Stein whereas the Dolciani treatment does not indicate any awareness of the cognitive problems associated with substitution.

<u>Summary</u>

This section has been devoted to a careful analysis of textbooks used in a first course on high school algebra. It has dealt with the instructional aspect of defining algebraic expressions. The authors of the three books have used very distinct approaches. The Kelly et al.

text has consistently dealt with problems of justification and motivation showing a great awareness for the learner's problems. These problems have been ignored in Stein's procedural approach which, nonetheless, has brought to light some possible difficulties due to notation and symbolism. Dolciani et al's formalism has shown little concern for the beginning student's need to construct meaning for algebra.

CONCLUSION

The objective of this first chapter was to establish not only the existence of the pedagogical problem represented by the learning of algebraic expressions but also to determine how widespread this problem was. The two assessment studies clearly show that many students do not manage to deal with the very first concept introduced in their initial formal course in algebra, that of algebraic expressions. These studies also indicate that concatenation and substitution represent cognitive problems far more serious than suspected. Thus one must question whether or not these students have been able to attach any significance to these expressions.

The difficulties in constructing meaning for algebraic expressions become quite evident in the textbook analysis. Kelly et al's presentation seems to be the more accessible one for the novice since they define expressions as a right-hand side of a formula. It is hard to see how Stein's definition of an algebraic expression as a translation of a mathematical verbal phrase ("'n + 6' is an algebraic expression for the sum n and six"), can be relevant to a beginning student. And Dolciani et al's definition of a 'variable' expression as "an expression containing one variable but may also contain other symbols, including numerals" is purely formal and as such cannot be related to the students' experience. Quite clearly the last two definitions are beyond a beginner's ability to construct meaning. While Kelly et al.'s definition is meaningful, they do not develop it sufficiently before moving in to word problems and subsequent evaluations.

Considering the scope of the problem as well as the lack of appropriate presentations, the need for new ways of introducing this topic is warranted. This will be the object of the research presented in this thesis, that of designing and experimenting a new approach to the construction of meaning for algebraic expressions. Of course, before suggesting a new approach one has to study carefully the research which has uncovered the possible causes of the students' learning difficulties. This research is discussed in Chapter II.

CHAPTER II

COGNITIVE OBSTACLES

Whereas the first part of Chapter I was devoted to the students' performance and the second part to an analysis of textbook presentations, this chapter will deal with the cognitive obstacles uncovered in the existing research literature. This review will focus on the difficulties pupils encounter in assigning some meaning for algebraic expressions.

One can distinguish two stages in the learning of algebraic expressions, that of an initial assignment of meaning for the algebraic form and a second phase involving transformations such as substitutions and manipulations. As has been shown in the textbook analysis, little emphasis is given to the first part. Such a presentation assumes that the transformations will ultimately provide the necessary significance. However, the two assessment studies reported in Chapter I indicate that for a large number of students, this approach leads to the manipulation of meaningless symbols.

On the other hand, the student might have accepted the meaning "an algebraic expression is part of a formula" as suggested by Kelly et al. (1981) but this isolated piece of knowledge is far from sufficient by itself to handle the solution of algebraic problems. In fact, such problems involve substitutions and manipulations and thus, the "constructions of meaning for algebraic expressions" has to be interpreted in a larger sense, in the sense of an operational schema which includes the necessary transformations.

A. Algebraic Expressions

Textbook authors use algebraic expressions as a starting point in a first year algebra course. However, in light of the importance of this topic in the further learning of algebra, it is necessary to consider some of the cognitive obstacles students meet when first attempting to assign some meaning to these expressions.

Kieran (1981) asked ten children aged 12½ - 13½, the meaning of '3a', 'a + 3', '3a + 5'. For 'a + 3', seven out of ten subjects could not assign any meaning to the expression, because they could not find the value of 'a'. The children did not accept the expression on its own and were searching for some equality in order to arrive at what they perceived as an answer. For 3a one student said, "If we had the answer, like '3a = 30', we could do it." Thus for these students, the expressions on their own were meaningless. The inability of these children to attach any meaning to these expressions indicates some of the difficulties students have with their first introduction to algebra.

Davis (1975, 1978) points out the incongruencies between arithmetic and algebra, and the subsequent inability of children to regard algebraic expressions as legitimate 'answers'. In their article "Cognitive Processes in Learning Algebra" (Davis, et al., 1978), the authors illustrate the distinction between "algebraic addition" and 'arithmetic addition'. In algebraic addition, 'x + 7' is "the name for the answer we get if we do it" (p. 17) and in arithmetic addition the process requires to "state a standard numeral for the sum", as in '5 + 2 = 7'.

Davis (1975) in his first interview with a twelve-year-old student, Henry, states that in standard elementary school '3 + 5' is the problem or question and '8' is the answer. However, in standard high school algebra '3 + 5' is both the process and the name of the answer. For example, Henry was unable to accept <u>6x</u> as a name for the answer of what you get when you multiply 6 by <u>x</u> . To Henry, <u>6x</u> , in his elementary language, still was a statement of the process of 6 multiplied by <u>x</u> .

Further on in the interview, Henry was unable to multiply by '3x + 1', because to him '3x + 1' was not a number and he could not perceive of multiplication in any other framework.

To Davis, the child's inability to accept algebraic expressions as 'answers' presents a formidable cognitive adjustment for many beginning algebra students — one requiring accommodation rather than assimilation. He feels the student must accept that the notation in an expression "indicates both <u>a process</u> and also <u>the result that will be obtained</u> when one carries out the process". (Davis 1975, p. 28)

Marilyn Matz (1979) confirms these findings: "Commonly, naive students respond to a request like "multiply by <u>x</u>" with a bewildered "you can't do it, you don't know what <u>x</u> is!" (Matz, 1979, p. 4) Similarly, she states that accepting expressions such as '2x' as a 'name-process' (reference to Davis), requires that the student changes his expectation about well-formed answers, that is, an 'answer' is not just a number. Thus algebra is not a simple generalization of arithmetic.

The arithmetic-algebra transition can be highlighted by an examination of the different interpretations which must be given to concatenation. "In arithmetic, concatenation denotes implicit addition as in both the place-value number system and mixed fraction notation". (Matz, 1979, p. 8) For example, '43' implies '40 + 3' as does $4\frac{3}{4}$ which involves addition of units to a fractional part. However, in algebra concatenation denotes multiplication, as in 'xy' or '3x'. Data has shown that students using their arithmetic interpretation will commonly make the following typical errors. For instance, when asked to substitute '6' in '4x', they may give '46' as an answer. Similarly when the values $x = -3$ and $y = -5$, they may conclude that $xy = -8$. In the interpretation of concatenation, we see once again that algebra is not a straightforward generalization of arithmetic. Thus both Davis and Matz attribute some of the cognitive obstacles with algebraic expressions to a student's strong attachment to an arithmetic frame of reference. (e.g. the name-process distinction and concatenation).

Collis (1974, 1975) proposes another explanation for the difficulty students have in accepting algebraic expressions as 'legitimate answers'. To Collis it is the inability of some students to hold unevaluated operations in suspension. The expressions '$x + 7$' and '3x' are incomplete sentences - that is, not closed. He notes that this changes gradually and that eventually (by age 15) pupils can indeed hold unevaluated operations in suspension to which he refers as the "Acceptance of Lack of Closure".

On the basis of his observations of children's mathematical behaviour, Collis distinguishes four levels of cognitive sophistication with respect to children's 'Acceptance of Lack of Closure'. (Collis, 1974). The level of <u>closure</u> at which the child is able to work with operations depends on his ability to regard the <u>outcome</u> of an operation (or a series of operations) as unique and "real".

At the lowest level (age 7+), he requires that two elements connected by an operation be <u>actually replaced</u> by a third element.

At the middle level (age 10+) he regards the outcome as unique but does not need to make the actual replacement to guarantee this. He can go beyond his empirically verifiable range (273 + 472) and can use two operations (6 + 4 + 5).

Later (age 12+) he can refrain from actual closure and is capable of working with formulas (v = 1 × b × m) "provided each letter stands for a unique number and each binary operation may be closed at any stage".

At the final stage of development, (age 15+), the adolescent is able to "consider closure in the formal sense because he is able to work on the operations themselves and does not need to relate either the elements or the operations to a physical reality". (Collis, 1974, p. 6)

Collis concludes that the development of higher levels of reasoning is closely related to the child's tolerance for unclosed operations. That is, the closer to early concrete reasoning (referring to the Piagetian sub-stages), the more the child depends on an immediate closure of the operation in order to make the situation more meaningful to him. On the other hand at the top level of adolescent reasoning (formal) the subject is able to withhold closing while he considers the effect of the variables in the problem.

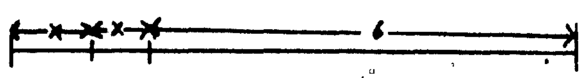
Considering the developmental focus which Collis places on the learning of mathematics, one might be tempted to conclude that algebraic expressions can become a meaningful form only when the student has reached the cognitive level of formal reasoning which is necessary for acceptance of lack of closure. However, a closer look at Collis' tests reveals that the questions asked favoured the formal thinker since the problems involved the manipulation of formal algebraic statements (such as "$\overleftrightarrow{x \circ y}$, means x and y must be in the expression but no other letter"). This type of questioning might have led Collis to conclude that formal reasoning was required for the acceptance of lack of closure. However, since no concrete support was used, such conclusions may well be unwarranted. Nevertheless, Collis' Acceptance of Lack of Closure is an interesting theoretical construct which can explain why young children cannot accept '2 + 3' as "another name" for 5 , or why some children are unable to accept or attach any meaning to algebraic expressions.

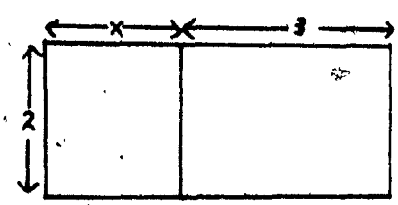
Discussion

The many ideas which emerge thus far from the review of the research literature need to be related to each other and their significance analyzed in the search for new directions of investigation.

Kieran's work clearly demonstrates the student's need for some referent in his construction of meaning for algebraic expressions. Davis and Matz have shown some of the cognitive obstacles created by the usual arithmetic frame of reference in the learning of algebra. The name-process dilemma they have identified obviously relates to Collis' Acceptance of Lack of Closure. If '2 + x' is viewed as an operation to be performed, how can the pupil be expected to hold it in suspension if he is looking for an answer?

To a learner of algebra whose main exposure to mathematics has been arithmetic '2 + 3' is interpreted as a problem and '5' as the answer. Thus he cannot view an algebraic expression such as '2x + 6' as both the problem and the answer to the problem. Can he view the algebraic expression as a problem? Only in the very limited case involving substitutions. Can an algebraic expression be interpreted as an "answer"? Of course, all kinds of problems can be set yielding algebraic expressions as the answer. For instance the expression '2x + 6' will be the answer to the following problems:





Length = 2x + 6 Area = 2x + 6

In these examples the algebraic expression '2x + 6' becomes the answer to problems which can be easily represented pictorially. To a student acquainted with length and area, such problems are readily visualized and allow him to focus his attention on the answer instead of having to cope with the difficulties of grasping the question.

Of course this is not the first time that geometric problems have been suggested in order to teach algebra. Horak and Horak (1981) illustrate the use of geometric proofs for some algebraic identities. However, such use is at most sporadic, no continuous geometric treatment having been attempted. Furthermore, in the isolated instances where geometry is used, only one type of problem (cf. area) is selected conveying to the student a single limited use of algebra. It would seem rather obvious that the very general nature of algebra can only be conveyed by selecting many different types of problems (discrete uses, lengths, areas, etc.). Of course, if these different types of problems yield the same algebraic expression as an answer, their general nature is inescapable.

B. Algebraic Operations

In the previous section we dealt with the cognitive problems students meet in attempting to assign some meaning to algebraic expressions. This part will focus on the cognitive problems associated with the algebraic manipulations of these expressions.

As stated in the introduction to this chapter, it is not realistic to view the cognitive problems associated with algebraic manipulations as a separate topic from those associated with algebraic expressions. This artificial separation is only being used as an analytic tool. The necessity for the student to construct meaning for algebraic expressions should once again be emphasized because without this initial understanding, algebraic operations would become for many students merely mechanical procedures, to be learnt by rote and thus easily forgotten and confused.

Marilyn Matz (1979) in her theory of errors in high school algebra views the difficulties students encounter in terms of the arithmetic-algebra transition, and states that cognitive changes to concepts and reasoning styles are required of the student as he moves from arithmetic to algebra. She claims that many of the errors students make and the difficulties they encounter correspond precisely to these changes that are not made. (Matz, 1979, p. 1)

K. Collis adheres to his premise that the child's level of success with algebra will be a reflection of his developmental level. According to Collis, the student's ability to accept lack of closure and the interpretation he assigns to the literal symbol will be determining factors in his ability to deal with algebraic operations.

Not only did Collis identify four levels of development with respect to the child's "Acceptance of Lack of Closure" (Chapter II, Part A), he also identified three different ways in which pronumerals can be viewed by children (Collis, 1975):

- Initially a child will attach a specific number to the letter symbol. If this one trial does not give a satisfactory result, the child gives up working on the problem. For instance, given the problem '6 + a = 9' , he will attempt one value for 'a' to see if it is a solution, if not he simply gives up on the problem.

- At the second level, the child will map a group of numbers onto the literal symbol, using a trial and error technique.

- At the top level, the child has what Collis refers to as the generalized number concept. That is the symbol b (say) could be regarded as an entity in its own right but having the same properties as any number with which they had previous experience. Collis believes a child should acquire this concept by the age of 14-15. (Collis, 1975, p. 4)

There seems to be fairly little difference between Collis' first two levels of interpretation of pronumerals. In fact, as shown by the example '6 + a = 9', the child is essentially treating the symbol as an <u>unknown</u> which he may or may not determine by mere substitution. The difference between these levels seems to be only one of perseverance.

The third level is indeed markedly different, for Collis' statement "<u>b</u> could be regarded as an entity in its own right" brings out one of the criteria implied in Matz' observation about the need for cognitive changes that are to be made in the transition from arithmetic to algebra.

Moreover, by defining "generalized number" as a pronumeral endowed with "all the properties of numbers" Collis achieves a characterization which seems to be very useful in describing the students' interpretation of literal symbols. This definition is far superior to a definition of variable as a "set representative" (Herscovics, 1982), a rather <u>static</u> interpretation which does not reflect the operational nature of the child's thinking. Thus, in our own work we will use Collis' definition for it justifies the inclusion of algebraic operations in any construction of meaning for algebraic expressions.

The one aspect of "generalized number" which must be brought into question is again the developmental nature that Collis assigns to it. Although he observes that this level is evidenced at the age of 14-15, it is difficult to argue a relation with maturation in view of the lack of control over the "instruction" variable. Indeed, the late development of "generalized number" just might be a product of the arithmetic-algebra linkage. However, by using consistently a geometric frame of reference to construct algebraic expressions (cf. the examples page 26).

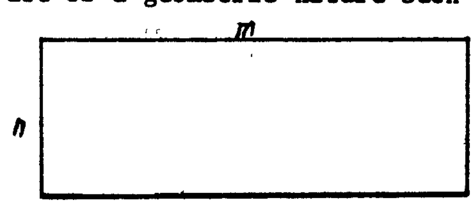
the child may prove to be able to reach this level of interpretation at an earlier age.

Expanding on Collis' work, Kuchemann (1978, 1981) devised an algebra test in which he describes six different ways the letters can be used. A detailed discussion of his taxonomy is essential for it bears directly on this thesis and furthermore only a detailed analysis brings out the fact that his terminology conflicts with current usage.

Kuchemann's first category is "letter EVALUATED" which he illustrates with the problems "if $a + 5 = 8$, $a = \ldots$" or "if $u = v + 3$ and $v = 1$, $u = \ldots$". It should be noted that in the current literature "a" is referred to as a "placeholder" or "unknown" and either u or v as functional variables.

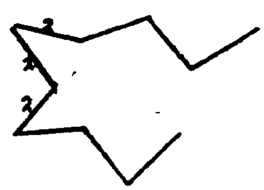
His second class, "letter IGNORED" merely reflects the fact that in some problems the child needs only to focus on the arithmetic part of the operation as in "if $a + b = 43$, $a + b + 2 = \ldots$".

His third usage is classified as "letter as OBJECT" which he describes by "the letter can be operated upon without first having to be evaluated, but the letter is regarded not as an unknown number but as an object or a name or shorthand for an object." The examples used to illustrate this are of a geometric nature such as,

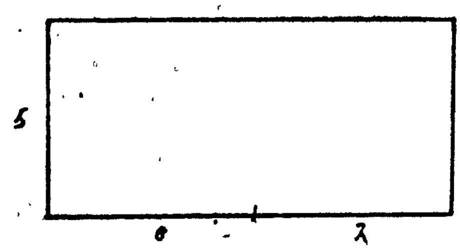


$A = \ldots$, Kuchemann states that the letters are names or labels for the sides rather than unknown lengths. This comment brings to light one of the major drawbacks of Kuchemann's taxonomy. His classification is not at all based on "meaningful uses" but on purely mechanical and formal usage of symbols. For indeed, what is the meaning of area if n and m represent merely sides and not their length? In fact, as will be seen in the example illustrating the next category, a slight change in the area problem forces Kuchemann to view the symbol as a "specific unknown".

Kuchemann's fourth category is "letter as SPECIFIC UNKNOWN" where the letter is thought of as a specific but unknown number "which can be operated upon without having to be evaluated". For example in the problem



"Part of this figure is not drawn. There are n sides altogether, all of length 2. p = ..." A second example, announced in the last paragraph, deals with area:



A = ...

This last example highlights one of the dilemmas in a taxonomy based purely on a rote use of symbols. Any such classification has to break down since any non-trivial level of algebra necessitates a meaningful approach. For instance to classify as an object a letter representing the side of a rectangle and then calling "specific unknown" a letter representing part of a side seems somewhat contradictory. It also illustrates the very limited value of the "object" category and how inevitably meaning has to be brought in. Nevertheless, the fact that a child could interpret the symbol as a label and not a length must be kept in mind and verified in any teaching experiment using geometry as a basis for algebra.

Kuchemann's "specific unknown" seems very close to Collis' "generalized number" for in order to operate on the literal symbols the student must view them as entities in their own right and as being endowed with the properties of numbers. Surprisingly, Kuchemann's definition of generalized number ("the letter is seen as being able to take, or as representing, a series of values rather than one value only" differs from Collis', for it emphasizes set representation. In the example "$c + d = 10$; $c < d$, $c = ?$" the letter c does indeed represent a set of values but the context of an algebraic relation seems to be ignored.

Finally, in his last category, "letter as VARIABLE", Kuchemann very clearly implies a functional context as illustrated by "which is larger, $2n$ or $n + 2$?". Thus it is surprising that in the case of a function the letter is taken as a variable but in the case of a relation, it remains a generalized number.

Using a population of 3000 subjects, Kuchemann determined the students' rate of success for each type of question and these rates were then used to formulate a hierarchy of levels of understanding which were then related to Piagetian stages of development. This work remains open to serious criticism. For instance, it seems quite questionable whether the success rate on a problem does indeed reflect the level of understanding. In fact, it is difficult to see how the answer on a written test can convey the student's thinking and furthermore how this thinking could be evaluated without a prior epistemological analysis of the concepts involved.

This critique of Kuchemann's work should not take away his valuable contribution. In fact, the distinction of letter as object (label) and letter as a specific unknown will prove to be useful in the present work both at the level of experimental planning and experimental verification. That such distinctions are important seems also to be evidenced by the recent investigation of James Kaput (1982).

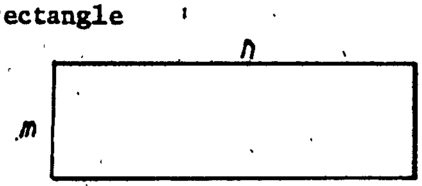
Kaput attempted to measure how mind sets created by first doing arithmetic or geometry differ in their impact on error patterns in word-to-equation translations. Two different groups of students, one proving a theorem in geometry and the second one doing non-trivial decimal calculations, were asked to deal with Clement's Students-Professors Problem (Clement, 1982). In this problem the subjects are given to translate into an equation:

> At a certain university there are six times as many students as there are professors. If S stands for the number of students and P stands for the number of professors, write an equation that describes the relationship between the number of students and the number of professors.

In a number of experiments Clement has shown that a large percentage of college freshmen (about 38%) mistakenly reverse the equation, obtaining '6S = P' instead of 'S = 6P'. To a large extent, the explanation of this reversal is that the students use the letters as labels for Students and Professors and not as letters representing their number. Kaput's hypothesis was that a prior geometric activity would encourage the label interpretation and thus cause an increase in reversals. His results have confirmed his hypothesis and this raises the question of the value of a geometric framework for the teaching of algebra.

Kaput justifies his results by pointing out that in Euclidean geometry, connections between the symbols and the referent tend to be direct, concrete and figural, "most often mediated by visual perceptual or imagistic processes". He notes that "the role of quantity is secondary even when measures are involved". (Kaput, 1982, p. 3) And in fact, in the task given to the geometry group, which had to prove the equality of angles in an isosceles triangle, the equal <u>length</u> of the sides is implied by definition but in fact plays hardly any role in the proof.

Without doubt, in formulating a proof, as in the above example, a student need not be concerned with the quantitative measurements of the figure. Consequently, a mind set based within this framework could lead to Kaput's results and conclusions. However, these have but a limited value with respect to the geometric construction of algebra suggested on page 26. For in fact, problems of length and area cannot be perceived without an explicit reference to quantification in the sense of measure. Given the rectangle



it is meaningless to say that the area is 'm × n' without first realizing that the letters represent the length of the sides. Surely one cannot multiply sides, only numbers.

This analysis of the literature has been essential for any coherent discussion of algebraic operations. Collis' work has shown the cognitive change which has to occur in the student. In order that the operations become "meaningful" the literal symbol must be perceived as a "generalized number" in the sense of having the same properties as numbers. Kuchemann's taxonomy has provided the important distinction between the letter as a label and the letter as a specific unknown. And, although Kaput's study dealt with geometric proof, it does signal that caution must be used in developing a geometric framework for the learning of algebra.

C. Evaluating Expressions

As pointed out in Chapter I, practically all textbooks follow up the introduction of algebraic expressions by exercises requiring the student to evaluate them by substituting numerical value(s) for the literal symbol. This should not be confused with the activities used by Kuchemann and Collis where it is not the evaluation of the algebraic expression which is involved but rather the solution of an equation (if $a + 5 = 8$, $a = ?$) or the determination of a dependent variable (if $u + v = 3$ and $v = 1$, $u = ?$) .

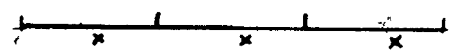
Although evaluating expressions by substituting numerical value(s) for the literal symbol(s) is viewed by Matz as a means of bridging the gap temporarily between arithmetic and algebra, she feels this type of exercise merely "slap a veneer of names on an arithmetic base" (Matz,1979,

p. 4). She claims that the difficulties associated with algebraic symbolism are not overcome by this method of substitution. The work still remains in arithmetic and the student finds it difficult to operate on or with a letter symbol without first being given its numerical value. Consequently the student may respond to "multiply by $\underline{x}$" with a bewildered "you can't do it, you don't know what $\underline{x}$ is!" (Matz, 1979, p. 4)

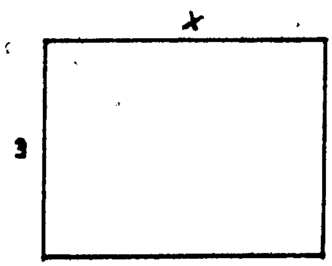
As discussed in the previous section, cognitive changes are required of the student in the transition from arithmetic to algebra. One important change is the student's development of the 'generalized number' concept. However, it is questionable whether the continuous substitution of numerical values for the literal symbol is sufficient in helping the student develop this much needed concept, since the student must be able to treat the literal symbol as an 'entity on its own'.

Matz' comments put into perspective the limiting aspect of these evaluation problems, since ultimately the algebra student must perform algebraic operations which require the manipulation of the literal symbol without any prior evaluation. Thus although these evaluation problems may be necessary, (to bridge the gap from arithmetic to algebra and for a first view of letters as representing numbers), this type of activity does not appear sufficient in assisting the student in his development of the 'generalized number' concept.

The question then arises as to how can the student develop the 'generalized number' concept. Or more specifically, how can he learn to regard it as "an entity in its own right endowed with all the properties of numbers?" One could venture here the hypothesis that this might be achieved through situations in which the literal symbol is clearly perceived both as having a numerical referent and also requiring its manipulation without the possibility of prior evaluation. Problems involving length and area, such as



and



reflect these requirements. For indeed, the literal symbol clearly indicates a specific but undetermined length. And presuming a minimal understanding of the concepts of length and area, the student should be able to make the transition from purely numerical problems to algebraic expressions.

CONCLUSION

This chapter has dealt with the cognitive obstacles encountered by students in assigning meaning to algebraic expressions and in performing algebraic operations. The importance of constructing meaning for algebraic expressions was emphasized, particularly its significance with respect to the pupil's performance of algebraic manipulations.

The review of the research literature has demonstrated some of the difficulties inherent in the transition from arithmetic to algebra, also illustrating some of the limitations of an arithmetic referent. (Davis, Matz). Highlighted in this review was the importance of the student's attainment of two concepts for his learning in algebra, the 'generalized number concept' and the 'acceptance of lack of closure'. Although Collis maintains that a child's level of development will determine the extent to which he can acquire these concepts, we have suggested that perhaps a new direction in the presentation of algebraic expressions and operations, a geometric frame of reference, would assist the student in forming these concepts at an earlier age and also assist him to overcome some of the cognitive difficulties associated with algebraic operations.

CHAPTER III

METHODOLOGY

An analysis of current textbooks (Chapter I) has shown how difficult it is to introduce algebraic expressions in a way which might be meaningful to beginning students. A major cognitive obstacle encountered by them is the "name-process" dilemma uncovered by Davis and Matz (Chapter II), who showed that these students have difficulty perceiving algebraic expressions as "answers". Thus the need to find alternate approaches was more than warranted and justified the research problems investigated in this thesis. The research problems were twofold: first, to design a new approach in the construction of meaning for algebraic expressions and secondly, to experiment it with students.

In designing a new approach we wanted to overcome the meaninglessness of algebraic expressions resulting from their introduction as indeterminate forms. We aimed at finding situations in which these expressions could be viewed as answers to problems. And since the focus was to be on the algebraic expression, the problems selected had to be readily accessible to the student without being trivial. To meet this criteria, we selected problems which were visual by their nature such as the area of a rectangle, the length of a line segment and the number of dots in a rectangular array.

These ideas were developed in a sequence of lessons, our Teaching Outline, the design of which was tested step by step as it was being formulated. Thus, each pedagogical intervention was tested as soon as it was conceived in a pilot study involving three students. This constant feedback enabled us to prepare a first draft of our Teaching Outline consisting of three lessons. These lessons were then tested on a single subject as an Exploratory Case Study which again provided feedback justifying further modification of the Teaching Outline. A full account of the construction of this Teaching Outline is reported in Chapter IV.

While the first problem researched in this thesis was the design of a teaching outline incorporating a new approach for the construction of meaning for algebraic expressions, the second research problem consisted in experimenting it with students. The objectives in this experimentation were to assess the accessibility of our new approach, to determine if it enabled the students to overcome some of the cognitive obstacles uncovered in prior research, and finally, to uncover if new cognitive obstacles might be inherent to the teaching outline.

Since the object of the experiment was the "construction" of meaning for algebraic expressions, this needs to be distinguished from the mere "assignment" of meaning. The latter suggests that the role of the student is that of a "passive recipient", the emphasis being placed on instruction. In the "construction" of meaning, although the teacher still plays a prominent role, the subject is more involved in the activity in the sense of actively participating in establishing the meaning. Thus to investigate this construction one needs to find a methodology which will enable the experimenter to observe the way the learner understands and thinks about specific content during the actual teaching process. The only method available meeting this criterion is a version of the Soviet Teaching Experiment applied to single case studies.

This chapter will further detail the pedagogical considerations involved in the preparation of the teaching outline, the theoretical basis of the Soviet Teaching Experiment, the selection of the subjects and their background, as well as the planned analysis of the data collected in these interviews.

A. The Teaching Outline

The Teaching Outline used followed Herscovics' principle of "Didactic Reversal" (Herscovics, 1979). Algebraic expressions were shown as representations of visualizable ideas (geometric and pictorial). Thus we started from the students' cognition and attempted to achieve accommodation of the new concepts through assimilation. This method required that the student be guided in small manageable steps in order to construct meaning for algebraic expressions using his knowledge of geometry and arithmetic. It was expected that the new algebraic forms should then become more meaningful and manageable to the student since they would be an extension of his existing knowledge. Once the student acquired some meaning for algebraic expressions, a reversal was encouraged. That is, from the now 'meaningful' algebraic expression, the student was asked to find a geometric and pictorial representation. Not only does this method assist the students in assimilating the new algebraic form, but also, the experimenter can evaluate the students' acquisition of the new mathematical idea, on the basis of these correct reversals.

B. The Soviet Teaching Experiment

This study intended not only to investigate the cognitive processes involved in the students' acquisition of a new algebraic concept, but also intended to evaluate a new teaching method. As Menchenskaya points out, there is an "inseparable link between two processes: the development of concepts in the students' minds, and the process of forming concepts as one aspect of the teacher's activity." (Menchenskaya, 1969, p. 76) Thus a methodology was required which enabled the experimenter to observe the way the learner understands and thinks about specific content during the actual teaching process, since learning and instruction must be considered simultaneously.

Piaget's clinical method was developed to provide a situation which would enable the exploration of the thought processes of children. The clinical method takes the form of a dialogue or conversation held in an individual session between the interviewer and the subject. The essential character of the method is that it constitutes a hypothesis-testing situation, permitting the interviewer to infer rapidly a child's competence in a particular aspect of reasoning by means of observation of his performance at certain tasks. (Opper, 1977, p. 92)

The clinical method has undergone a number of alternations and revisions. One outcome is the partially standardized clinical method. According to Opper, in this method the subjects are presented with a standard problem and certain identical manipulations are applied to this material. The subjects are then asked a number of identical questions relating to both the material and the manipulations. Having presented these identical situations and questions, the interviewer may then conduct the experiment as he deems appropriate, thus retaining some of the freedom of the clinical interview. (Opper, 1977, p. 94) This version of the clinical interview allows for comparability of results between case studies.

In our study we used the semi-standardized clinical interview since it allowed the interviewer to discover the learner's cognitive processes, and at the same time allowed for comparability of results. However, since we wished not only to consider the learning process, but also to evaluate this process within a specific teaching situation, the teaching component had to be taken into consideration.

The Soviet "Teaching Experiment" is an example of an experiment in which both the learning and teaching process can be evaluated. This experimental method is "primarily directed at disclosing and elucidating the very process of learning, as it takes place under the influence of pedagogy." (Menchenskaya, 1969, p. 89) By means of clinical interviews, the researcher attempts to develop essential knowledge, abilities, skills and modes of activity in the child during the experiment. This is done in order to reveal the child's psychological traits in the making, that is, it explores the dynamics involved in the learning process. The teaching experiment requires that the abilities and skills needed to master the new material be arranged in a hierarchy. (Menchenskaya, 1969, p. 6)

This experimental method is divided into two categories 1) experiments of assessment and 2) experiments of instruction. The former investigates a "produce" of instruction which has already been formed (mastered) and the ability of the student to use it to solve problems. In the second category, instruction is direct, and it thus is possible to investigate knowledge during the process of transmission. The latter one enables the investigator to observe development and changes of particular mental processes and, because the same pupils are investigated, the direct influence of teaching on the process is easily displayed.

The "experiment of instruction" has different forms; two basic ones are: the 'experiencing' method and the 'testing' method. In the first cases the experimenter uses only one precise method of instruction, and evaluates the influence of this one method. However, in the second case, different methods are used and the investigator attempts to discover which one promotes the most effective mastering of information.

For our research the version of the Soviet Teaching Experiment used was the "Experiment of Instruction", an example of the 'experiencing mode', since we wished to investigate the influence of our teaching method on the students' understanding of algebraic expressions. And as mentioned before, this "experiment of instruction" was conducted within the confines of 'semi-standardized clinical interviews'.

This methodology was chosen because it enables the interviewer, by means of individual interviews, to discover changes within the learner's thinking, and the effects of the planned instruction. Another reason for the choice of this method, was that although the instruction is planned in advance, there is a built in flexibility, since the interviewer can alter his plan or pursue any unforeseen or interesting responses which may come out during the interview. This flexibility allowed for more depth in our examination of the levels of understanding of the new concepts presented to our subjects and the effects of our teaching outline.

C. Selection of the Subjects

The number of students involved in this study was six: three of them were from Grade 6 and three of them were from Grade 7. The students were selected from these grade levels for we wished our subjects to have a minimum exposure to algebra. The study was limited to six students since we were not trying to generalize to an entire population, but rather, the object was to evaluate our teaching outline in the construction of meaning for algebraic expressions, and to uncover any cognitive obstacles inherent in it. The students were selected to represent different levels of mathematical ability (Weak, Average, Strong) as identified by their teachers on the basis of their previous school performance.

The following table describes the relevant information about the six subjects selected for this experiment.

Name	Frankie	Wendy	Antoinetta	Yvette	Filippo	Gail
Age at First Interview	12.6	11.6	11.3	12.9	12.7	12.6
Grade	6	6	6	7	7	7
Teacher Rating	Weak	Average	Strong	Weak	Average	Strong

D. <u>The Subjects' Background</u>

Our Grade 6 subjects attended St. Ignatius Elementary School and our Grade 7 subjects attended Marymount High School, both schools being part of the Montreal Catholic School Commission. An examination of their program indicates that a formal course in algebra is introduced in Grade 8. Furthermore, discussions with the Grade 6 class teacher and the Grade 7 mathematics teachers gave us more information regarding the students' backgrounds.

An analysis of the textbook used by these Grade 6 subjects (<u>Investigating School Mathematics</u>; R. Eicholz, et al.,1974) indicated that all the subjects had been exposed to letters within various contexts. The literal symbols used were not restricted only to the letter <u>n</u>, but rather a wide variety of letters were presented to the students.

The contexts in which the literal symbols were presented varied as well. The letters served as 'specific unknowns' within the context of equations such as $6 + 7 = a$, $s - 7 = 3$, $5 + x = 10$; in problems involving axioms, $68 + 27 = n + 68$, powers, $10^2 \times 10^2 = 10^b$, and bases $23_{(6)} + 4_{(6)} = s$.

The literal symbol was not merely presented as a specific unknown, but was also introduced within the context of 'functional variable', as illustrated by the 'Function Rule'. For example students were presented with the problems of the following type:

$n + 6$

n	f(n)
7	≡
9	≡
5	≡

In conclusion it can be said that the Grade 6 subjects in this experiment had some exposure to letters in many different contexts, with more than one interpretation. Their teacher indicated that this work was to be covered between September and January, for the following session was to be devoted to arithmetic work with fractions and some geometry.

Since our Grade 7 students were in the same school board they had all used the same text in their prior grade. Discussions with their mathematics teachers confirmed that the school year was devoted essentially to the introduction of integers and to a review and consolidation of the notions introduced at the elementary school level.

E. The Experiment

We conducted five individual interviews with each subject, each approximately 30-40 minutes long: Pretest, three Lessons, and Post Test. Each interview was carried out in the schools each subject being released from his class for the allowed time. In order to have continuity and to avoid memory related problems, it was decided to conduct two interviews per week with each child. The exception, however, was the timing of the Post Test, which took place approximately one month after the third lesson. Each interview was audio-taped on a cassette recorder, following which the tape was fully transcribed.

F. The Analysis

The analysis of the protocols for five students (all students except Wendy) took place after the completion of the five interviews. In the case of Wendy, after her last interview, the Post Test, an immediate analysis was deemed necessary, since after her interview we felt that some changes were required in the questionning of the Post Test. In fact some significant changes were made based on that interview. These changes and the rationale for them is described in detail in Chapter IX.

The analysis of the protocols was made in two different manners.

for the Pretest and the three Lessons, we did not follow a case study approach to the analysis, but rather we reported the responses of all the six subjects' for each question. Thus we were able to compare these responses, and to determine their consistency. Also, based on the number of students who were experiencing the same difficulty, we were able to determine if a cognitive obstacle was particular to an individual, or of a common nature, or due to the poor wording of our question.

However, we changed the method of analysis for the Post Test, because at that point we were no longer interested in the cognitive obstacles inherent in our teaching outline, but rather, how the teaching outline had affected each individual subject's knowledge. Thus a case study approach was used in the analysis of the Post Test. That is, we analyzed each student's responses in the Post Test, compared them with his answers from the Pretest, and verified if the difficulties he experienced at the end were similar to those he had experienced during the three lessons.

CHAPTER IV

CONSTRUCTION OF THE TEACHING OUTLINE

INTRODUCTION

The preparation of the teaching outline occurred in stages. The first step was to study in detail the current approaches to the introduction of algebra. This was done by analyzing three new or recently revised textbooks (See Chapter I). While these textbooks were quite different in their approach, all three introduced algebraic expressions before the topic of algebraic equations. Although the possibility of introducing equations before expressions was considered, this alternative was eliminated in the light of Kieran's results indicating that within the context of equations the problems involved in algebraic manipulations were still present (Kieran, 1981). Thus, there was no evidence suggesting that a change in the conventional sequence of presentation would prove to be of any pedagogical value. Hence, the teaching outline developed for this experiment used algebraic expressions as a starting point.

The second phase in the preparation of the teaching outline concentrated on identifying the cognitive obstacles involved in the learning of algebraic expressions as uncovered by prior research (see Chapter II). Among these major obstacles which had to be taken into account were: the fact that algebraic expressions had no intrinsic meaning to the novice (Kieran, 1981); Davis' (1975) and Matz's (1979) name-process dilemma, (the inability to perceive an algebraic expression as an answer as well as an operation); Collis' (1974) acceptance of lack of closure, (the ability to hold unevaluated operations in suspension); problems associated with concatenation, (Matz, 1979).

As a possible solution to some of these cognitive obstacles we intended to design a teaching outline which would enable the student to interpret an algebraic expression as an answer to a problem or to different types of problems. Of course, in order to have the student focus on the "answer to a problem" interpretation, one needed types of problems which would not in themselves create cognitive obstacles. Thus the problems selected were of a highly pictorial nature, easy to visualize. They involved the quantification of an array of dots, the length of a line divided into segments, as well as area problems.

In order to ascertain that these problems were readily accessible to the student, a first pilot study was undertaken. This pilot study involved both a selection interview and an initial draft of Lesson 1. The selection interview aimed at determining whether or not the student perceived the multiplicative nature of the above problems within an arithmetical context. The initial draft of Lesson 1 consisted of the same types of problems but yielding algebraic expressions. Three Grade 6 students were selected since, at this level, none had received any formal instruction in algebra, and thus could provide us with spontaneous responses unaffected by prior exposure to the subject.

This initial pilot study provided us with very strong evidence about the difficulties students encountered in using a letter as an unknown number. This led us to modify the initial lesson plan by introducing the "box" as a placeholder prior to the introduction of letters, both within the context of "a hidden number". The use of a letter as an un-

unknown number was thus delayed until Lesson 2 which also introduced concatenation. Finally, a third lesson was designed to motivate the use of different operations (multiplication and addition) as well as the use of two different letters. These three lessons were tested in an exploratory case study involving another Grade 6 student. The analysis of this case study allowed us to improve these lessons by expanding on some questions where more clarification was deemed necessary as well as some minor changes in the sequencing of questions.

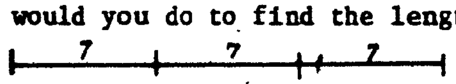
A. THE INITIAL PILOT STUDY

The objectives of the initial pilot study were to test the selection interview as well as the first draft of Lesson One of the proposed teaching outline. The initial pilot study consisted of one individual interview with each of three grade 6 students deemed average by their mathematics' teacher. (In this particular private elementary school, mathematics instruction is given in French, by the French teacher.) Each interview was approximately forty minutes long and was tape recorded; the relevant parts were transcribed.

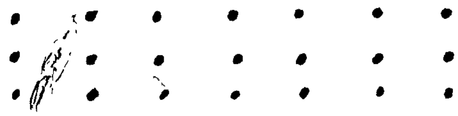
1. The Selection Interview

In view of the problems selected for the teaching outline, it was essential to determine if the subjects to be chosen for the case studies, perceived the multiplicative nature of the various problems. To this effect, the following three problems were presented; within a purely arithmetic context.

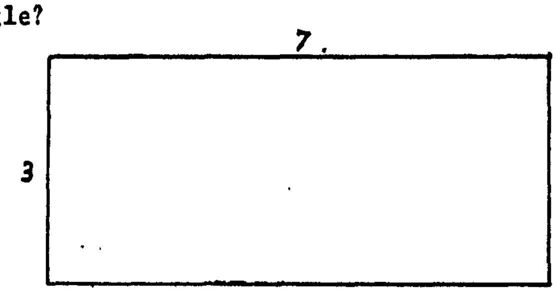
Problem 1: What would you do to find the length of this line?



Problem 2: How do you find the number of dots in this picture?



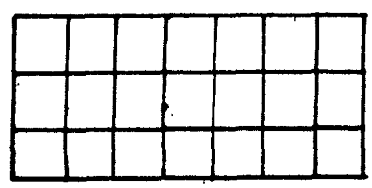
Problem 3: Can you tell me what you would to to find the area of this rectangle?

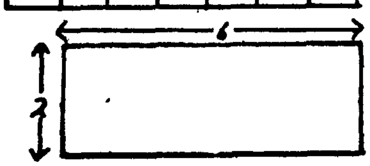


The area question was to be presented last since the first two problems could be answered by using repeated addition, or, especially with the array of dots, by straight counting.

In the case where the student would be unable to solve the area problem the rectangle was to be divided into square units:

I: Can you tell me what is the area?
If No: Are all squares equal? How many squares do you need to cover the rectangle? (Student may use repeated addition or straight counting.)
(Reinforce: So area means the number of squares needed to cover. — then give another problem.
I: Can you tell me the area of this rectangle?





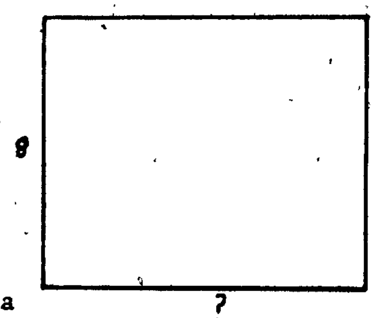
Two further area problems (rectangles, 9 by 8, and 13 by 7) were given to the student, the dimensions being large enough to discourage counting.

2. First Draft of Lesson 1

Lesson 1 attempted to construct meaning for algebraic expressions such as $3x$, by presenting these expressions as answers to three different types of problems, - area, length, and number of dots. In each problem a letter was used to represent some unknown number (the base of a rectangle; the length of a part of a line; the number of dots in a row.)

Prior to the introduction of these problem types, we felt it was essential to create a situation which provided some rational explanation for the use of the literal symbol. Thus the following problem was presented to the students.

> Suppose I want to carpet a room, and I need to know the floor area in order to find out how much carpet to buy. This is the drawing of the floor. (Interviewer sketches the rectangle showing only the height as 8 units). I know that one side is 8 units, however, I realize I forgot to measure the base. Until I can measure it I will assign a letter for the length of the base. Choose a letter. (The student chooses a letter and the rectangle is subsequently labelled accordingly).



This problem illustrates the use of a literal symbol as an "unknown number". The explanation "I forgot to measure the base" provides a need to represent this unknown quantity until it has been measured. This introduction of letter as a specific but unknown number rather than its use as "generalized number" (multiple valued) was considered preferable since as both Collis (1975) and Kuchemann (1978) have shown, the latter interpretation is much more sophisticated and usually occurs at a later stage. The request for the student to choose any letter was motivated by the need to convey the arbitrary nature of the symbol as has been shown by Wagner (1981).

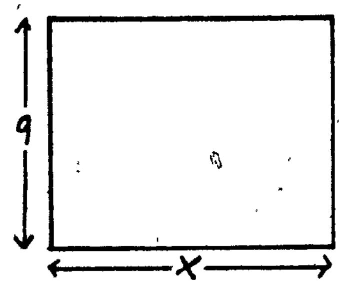
The interview proceeds with the following line of questioning and the projected answers.

I: Can you write the area of this room recalling the formula for area?
S: (8 times x, or 8 × x)
I: What does x represent?
S: ?
I: What does 8 × x represent?
S: ?
I: So 8 × x is the area of the room. If I were to measure the base, I would know exactly how much carpeting to buy. Let's say that the base is 12, that is x = 12. How much carpeting would I need? Please show your work.
S: (8 × 12 = 96)

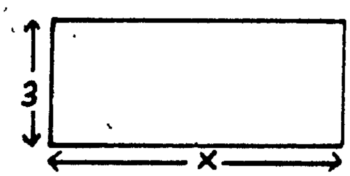
This last question was to stress the fact that the letter was of a temporary use in the initial problem. The process of numerical substitution was intended to underline the numerical referent.

Following this justification of the literal symbol, the three different problem types were introduced:

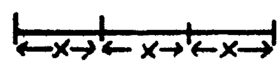
I: Suppose I have this rectangle. I know the height is 9 and the length of the base is some unknown number x . Does it matter which letter we use?
S: ?
I: Write the area of this rectangle.
S: (9 × x)



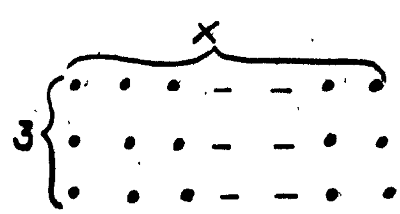
I: Can you write the area of this rectangle?
S: ?



I: What would you say is the length of this line?
S: ? (3 × x)



I: Now I have a row of dots, but I do not know exactly how many dots are in that row. (Show ...———....)
We will say that there are x dots in the row:



I: What does the x represent?
S: ?
I: Now suppose I had 3 rows with the same number of dots. I can draw it like this. How many dots are there?
S: ?
I: Look at these last three problems. What is the answer?
S: $(3 \times x)$
I: Can you see that '$3 \times x$' can be the answer to different types of problems?
S: ?

It should be noted that in the dot problem it was necessary to illustrate a convention used to indicate an unknown number of dots in a row, that is, some dots followed by dashes and then more dots. The dashes were used to represent the unknown part. The total number of dots in the row was labelled with an <u>x</u> .

After the presentation of the three different problem types, the planned lesson continued as follows:

I: Can you make up an area problem where the answer would be '$5 \times x$'?
S: ?
I: A line problem?
S: ?
I: A dot problem?
S: ?

This generation of problems by the students was deemed important in terms of 'didactic-reversal' (Herscovics, 1979). For having constructed meaning for these algebraic expressions based on the problem types, the subsequent generation of the problems by the student might indicate a perception on his part of the equivalence of the two representations (the algebraic and geometric form). This 'reversal' can also be used to determine the extent to which the student has understood the new concept, for his ability to generate such problems implies a greater grasp than mere recognition of the problems.

Since the three problem types directed the students to write algebraic expressions such as '3 × x', it seemed appropriate at this point to introduce the notion of concatenation; in this case, the juxtaposition of a letter and a number to indicate multiplication. Thus Part II of Lesson 1 begins as follows:

I: How many ways in arithmetic can you write six times seven?
S: (6 × 7 ; 6 · 7) (If just one, no need to go further.)
I: In algebra we can use all the letters of the alphabet - capitals and small letters. We often use the letter x . This can get a little confusing especially if you have 2 x's as in

8 × x (say eight times x)

In order not to have any confusion, in algebra, we can just leave out the multiplication sign and write

8x for 8 × x

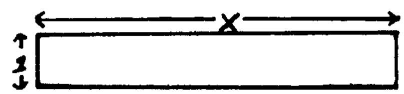
Whenever you see a letter attached to a number there is a hidden multiplication sign there.
What does 5a mean? 4b?
S: ?
I: Can we do this in arithmetic? That is, can I write '6 × 7' without the multiplication sign? Why not?
S: (67)

The last question was included in the lesson to ensure that the students would not generalize this concept to arithmetic. We expected the students to perceive the result of 'sixty-seven' and thus conclude that concatenation was only possible in algebra.

The opportunity now existed to reintroduce two of our original problems (area and length) however, with the inclusion of concatenation.

I: Can you make up an area problem where 5a is the answer?
S: ?
I: Can you make up a line problem where 4b is the answer?
S: ?

The final topic in the initial lesson was the illustration of the convention in algebra of writing only the literal symbol when one times a letter is the answer (the expression), that is, '1 × x , 1x' written as x . This convention was illustrated within the context of an area problem.



I: Here is a rectangle, can you tell me what the area is?
S: (1 × x or 1x)
I: What do we do in arithmetic with a product 1 × 3?
S: ? (We just write 3)
I: The same thing in algebra, we omit the 1 and just write x.

3. Analysis of the Initial Pilot Study

In accordance with the objectives of this initial pilot study, the analysis of the interviews will be presented in sections dealing with specific parts such as the testing of the selection interview, the introduction of a letter as an unknown, the testing of the three problem types, and concatenation. The cognitive obstacles uncovered in these sessions will be substantiated by quotes from the three students who will be labelled student A , B , and C , respectively.

a) Testing the Selection Interview

The testing of the selection interview on the three Grade 6 students revealed that only the area problem created some difficulty. Student A did not remember how to find the area of a rectangle ("I just learned it yesterday and I forgot"). The interviewer then proceeded to cover the rectangle by a grid. Student A counted the square units in this rectangle but then multiplied the height by the base in all subsequent area problems. Student B confused area with perimeter and required numerous examples using a grid. Once their difficulties were overcome, the multiplicative nature of the area problem became as evident as with the line and the dot problems. Thus, on the basis of these interviews, the type of questions, as well as their sequence, in the selection interview seemed appropriate.

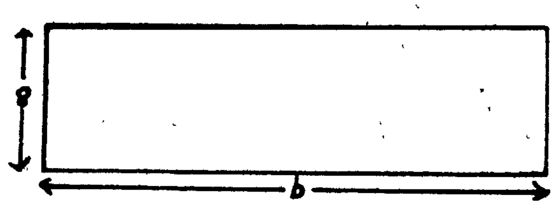
b) <u>Introduction of Letter as an Unknown</u>

Despite the attempt to create a rational situation (the carpet problem) for the use of a letter as an unknown number, the interviews reveal that all three students experienced major cognitive problems. In fact, the evidence gathered here is even more striking than that shown by Davis' interview with Henry ("How can we multiply <u>x</u> when we don't know what <u>x</u> is?"). (Davis, 1975, p. 17).

<u>Student A</u>

When presented with the carpet problem involving a missing dimension, the student was asked:

I: ... Now until I can measure it, I will assign a letter for the length of the base. You choose a letter.
SA: Of the alphabet?
I: Yes.
SA: <u>b</u>
I: We will call the length of this base '<u>b</u>' (completes the drawing)



I: Can you write the area of this rectangle? Keep in mind the formula for area of a rectangle.
SA: (pause) (Writes '8 × 8 = 64')

Even if on the surface Student A seemed to accept the use of a letter as an unknown number, when it came to using it in expressing the area of the rectangle, she estimated the length of the base by comparing it with the height and provided a numerical answer, thus avoiding the use of the letter. This avoidance became even more evident as the interview progressed:

I: What was the formula for the area of a rectangle?
SA: Length of the height times length of the base.
I: What is the length of the height in this rectangle?
SA: 8
I: What is the length of the base?
SA: b
I: So you wrote 8 times 8. Is the base 8?
SA: No. (changes her answer to '8 × 0').
I: There is no base? What is the length of the base?
SA: b. But we can't multiply 8 with b.
I: Yes, you can.
SA: 8 times b.

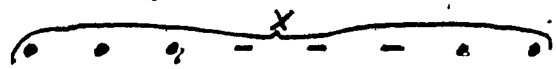
This comment, "But we can't multiply 8 with b", shows very clearly Student A's inability to deal with a letter as an unknown number. In fact, this probably explains the change in her answer to '8 × 0', for then, zero is used to symbolize the lack of any number to represent the length of the base.

Student A did not experience any difficulty with a numerical substitution in this carpet problem (b = 12). The next two area problems ('9 × x' and '3 × x') as well as the length of a line ('3 × x') were handled easily. However, the difficulty in using the letter as an unknown resurfaced in the array of dots where the letter represented the number of dots in a row:

I: I have a row of dots but I don't know the exact number of dots.

(Shows) • • • — — — • •

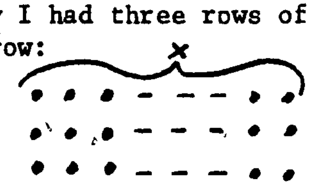
The dashes mean that I don't know how many dots are in the middle. I will say that there are x dots in the row.



What does x represent?
SA: It is a letter
I: I have a row of dots but I don't know how many dots are in that row. I will say that there are x dots in that row. What does x represent?
SA: (no response)
I: How many dots are in the row?
SA: We don't know.
I: So x stands for...?
SA: We don't know how many there is.

Student A's response to the question "What does x represent", "It is a letter", indicates that she had not identified it with the unknown number of dots. In fact her interpretation of the letter becomes even more evident in her last comment, that x stands for "We don't know how many there is". To her the letter does not imply how many there is, but rather "we don't know" how many there is. Thus for Student A, a letter is used to indicate a situation in which one quantity is unknown, and not the unknown quantity. Her avoidance of the use of a letter is further evidenced in the following problem:

I: Let's say I had three rows of dots with x dots in each row:



How many dots would I have altogether?
SA: (Pause) Can I say the whole answer or how many dots?
(Student writes $3 \times 8 = 24$)
I: How do you know there are 8?
SA: Because you said 3 rows and the first row has 3, and the other has 3... (referring to the dashes)
I: How many dots are in the row? Do we know the number of dots?

SA: No

I: How many dots did we say there were?

SA: We said 'x'.

I: How do we find the total number of dots?

SA: 3 × x

This part of the interview indicates that the student had difficulties with the convention used to represent the unknown number of dots (the use of dashes). Her question "can I say the whole answer, or how many dots?" shows that she did not interpret the dashes as representing a missing part of the array. In giving "the whole answer" she counted the number of dots and the number of dashes in each row. Although this clearly shows her different interpretation of the intended convention, nevertheless, it also provides further evidence of her avoidance in the use of the literal symbol.

The stongest indication of Student A's struggle with the use of a literal symbol representing an unknown number appears at the end of the interview, in the context of concatenation. Although she seemed to accept the concatenational notation:

I: We said <u>8x</u> meant?

SA: 8 times x

I: So 5a?

SA: 5 times a

I: And what about 4b?

SA: 4 times b

I: Can we do this in arithmetic? That is, can I write '6 × 7' without the multiplication sign?

SA: No

I: Why?

SA: (no response)

I: In algebra we have a letter with a number, so we can leave out the multiplication sign. But in arithmetic we have this problem. (Writes 67)

SA: How can you multiply a number with a letter?

It is probably impossible to gather clearer evidence of the novice's difficulty in accepting the use of a letter as an unknown number. Even at the end of a forty minute interview during which Student A appeared to accept it and use it, her final comment "How can you multiply a number with a letter" reveals the extent of this cognitive obstacle.

Student B

Due to the difficulties Student B experienced with the area of a rectangle, there was insufficient time to cover the entire first lesson. However, the carpet problem was presented to her and it provided more evidence concerning the difficulties students first meet in using a letter as an unknown number.

Student B was asked to choose a letter to represent the base:

I: ... choose a letter.
SB: d
I: and d is going to stand for what?
SB: stand for?
I: uh huh
SB: I don't understand the question.
I: What is the 8 here?
SB: The height.
I: The length of the height. Do we know the length of the base?
SB: No.
I: So what does d stand for?
SB: The length of the base.

Her initial difficulty to explain what the letter d stood for appeared to be overcome in her last statement: "... the length of the base". However, her actual interpretation of the literal symbol becomes apparent when she is asked to write the area of the rectangle.

I: Remembering the formula for area, can you write for me what the area of the rectangle is?
SB: The area of the rectangle?
I: uh huh
SB: The formula?
I: No, we have the formula here (shows Area = length of height × length of base). Using this formula and looking at this picture (referring to the rectangle), can you tell me what the area of the rectangle is?
SB: (No response)
I: How do you find the area?

SB: Multiply the height and the base. (writes 32)
I: How did you get that?
SB: I am not sure if it is right, but <u>d</u> is number 4 in the alphabet, so I multiplied 8 times 4.

Although Student B knew the formula for area and stated previously that <u>d</u> stood for the length of the base, it is not until after some hesitation that she finally provided the response '32' , ignoring the letter. Similarly to Student A, she seemed compelled to give a numerical answer. However, her explanation is quite different from Student A, who used comparison to evaluate the literal symbol. She determined the value of <u>d</u> by relating it to its position in the alphabet. Thus once again we have evidence of the tendency for the novice to avoid the use of the literal symbol, by providing for it a numerical value.

Towards the end of the interview the student is given the correct response '8 × d' by the interviewer.

I: So you write '8 × d'...
What bothers you about this?
SB: Oh, it is not important.
I: What did the <u>d</u> stand for?
SB: The length...the base.
I: Is it a number?
SB: No.

The last statement that <u>d</u> is not a number clearly shows that Student B had not accepted the letter as an unknown number, even though she stated, it stood for the length of the base. To her <u>d</u> is a letter, which probably explains her inability to give the answer '8 × d', since to her <u>d</u> is not a number.

Student C

As in the previous two interviews the student was asked to choose a letter to represent the unknown length of a side of the room in the 'carpet problem'.

I: Choose a letter.
SC: r
I: What is the r ?
SC: a letter
I: And what does it stand for?
SC: The length
I: of what?
SC: The base

The student's immediate response that r is "a letter", indicates that there is not a spontaneous connection made between the letter and some unknown number - r is a letter. She said it is the length of the base, but it is questionable whether she thinks of it as a number. The difficulties she subsequently met in writing the area of the rectangle demonstrates her inability to see and use the letter as an unknown quantity.

I: ... Is there some way to write the area of this rectangle?
SC: 7?
I: How did you get 7?
SC: Because I turned it into squares again, I would get 7.
I: Show me.
SC: (Makes a 8 by 7 grid)
I: How did you get 7? What is the length of the base?
SC: 7
I: How?
SC: Normally it is one less than the height.
I: Is that the type of problems we have been doing all along? Shows student previous problems; student sees that in the other problems the bases are not one less than the height)
SC: That is the number I think it is.

As in the cases of Student A and Student B, Student C rejected the use of the letter in writing the area of the rectangle. She said the base is '7', explaining that "normally, it is one number less than the height". After being shown evidence to the contrary, she still maintained her answer by saying, "that's the number I think it is".

Her avoidance of the use of the letter becomes more evident further in the interview:

I: ...What is the length of the height?
SC: 8
I: And what is the length of the base?
SC: We don't know yet.
I: What does the r stand for?
SC: The length of the base.
I: So what did we say - the area of a rectangle equals...
SC: 6
I: the length...
SC: The length of the height times the length of the base.
I: Can you write the area?
SC: (no response)
I: What is troubling you?
SC: So it is 8 and 8.

Thus Student C continued to provide numerical values for the letter r, 6 and 8, which indicated a very strong rejection of the use of the literal symbol. At no point did she seem able to provide an expression for the area involving its use, but preferred to attempt to evaluate the letter. The answer '8 × r' was finally given to her but the difficulty she had in accepting this as an answer becomes apparant when she was asked the meaning of '8 × r':

I: What does 8 stand for?
SC: The length of the height.
I: 'r'?
SC: The length of the base?
I: and '8 × r'?
SC: equals 8...we can't...we don't know the length of the base yet.

Her response to the question "what does '8 × r' stand for", "equals 8", once again reveals her search for a numerical answer. Her final rejection of the algebraic expression, '8 × r', appears when she said "we can't...we don't know the length of the base."

The interpretation she assigned to the literal symbol becomes very apparant in the subsequent part of the interview:

I: What does the '8 × r' stand for?
SC: The rectangle?
I: What of the rectangle?
SC: (no response)
I: What were we trying to find?
SC: The length of the base.
I: What were we trying to find out?
SC: The area
I: So '8 × r' stands for what?
SC: The area

It appears that the introduction of the letter may have signalled to her that being an unknown, its numerical value had to be found. The problem was "find the length of the base" - not "the area of the rectangle", since writing the area of the rectangle is an impossible task for her without knowing the length of the base, "we can't...we don't know the length of the base...".

Further evidence of Student C's avoidance of the letter can be seen when she was presented with the dot problem.

I: Let's say I have a row of dots, but I don't know how many dots are in a row. (draws • • • — — — • • •) I am going to say there are <u>x</u> dots in the row.

I have 3 rows of these dots.

How many dots do I have altogether?
SC: Are these suppose to be dots, too? (referring to the dashes)
I: No, they show I don't know how many dots there are... How many dots are in that row? (referring to one of the rows of dots)
SC: 9
I: Do we know how many dots are in that row?
SC: No
I: How did you know there were nine dots in that row?

SC: I counted it.

Student C was experiencing some difficulty with the convention shown for an unknown number of dots in a row. However, even though she stated that she did not know the number of dots in the row, she still provided a numerical value for <u>x</u> by counting all the dots and dashes.

In an attempt to try to overcome some of the difficulties associated with this convention, the student was asked to develop her own method to illustrate an unknown quantity of dots in a row.

I: Can you draw me something that would show a row of dots, where I don't know how many dots there are?
SC: (draws • • • • • • • • _ _)
I: We are going to say there are <u>x</u> dots in that row. What does the <u>x</u> stand for?
SC: How many dots there is.
I: Make 3 rows of these dots.
SC: (Draws
• • • • • • • • _ _
• • • • • • • • _ _
• • • • • • • • _ _

I: How many dots do we have altogether?
SC: (writes '3 × 8')
I: Where did you get the 8?
SC: There are 8 dots so far that I know.
I: Is that how many dots there are altogether?
SC: No.
I: (gives answer '3 × x')

This part of the interview demonstrates clearly Student C's avoidance of the use of the literal symbol. Even within her own convention for an unknown number of dots, she still determined a numerical value for <u>x</u>, rather than use it to represent some unknown quantity.

Summary

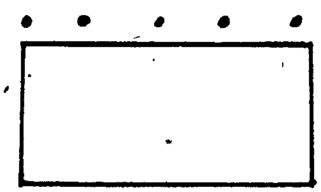
In summary despite the rationale for the use of the unknown, we witnessed among the three students, not just a mild avoidance but a very strong avoidance of the use of the letter, and within many different contexts (rectangle, dots, concatenation). All three students felt the need to provide numerical values rather than use a letter as an unknown. Student B's avoidance brought her to rationalize 'd' as the fourth letter of the alphabet. However no stronger evidence can be found that Student A's response, after a forty minute interview, "How can you multiply a letter with a number?"

This evidence indicated that the introduction of the literal symbol as a specific unknown presented far greater cognitive problems than was expected. The need for an alternative approach seemed essential. Thus a preliminary phase was added to the teaching outline using the empty box as a placeholder for the unknown, for it was felt that the use of this familiar convention might bridge the gap between the notion of an unknown quantity and its symbolization by a letter. Furthermore, as was the case in Herscovics' and Kieran's (1980) "Construction of Meaning for Equation", the idea of at first using the box in conjunction with the "hiding" of one of the dimensions of the problem was introduced as a means of providing an enactive representation (Bruner, 1966) to the concept of unknown. This change will be dealt with in the section covering the Exploratory Case Study (Philip).

c) Testing the Presentation of the Three Problem Types

The results of the testing of the presentation of the three problem types indicated that only the dot problem created difficulties for the students. The convention used to denote an unknown number of dots in a row, that is, dashes representing the missing part, was not understood by the subjects. Both Student A and Student B in giving their answer for the total number of dots in the array, and the number of dots in a row, respectively, counted the number of dashes and included them as a part of the number of dots.

Thus the need for a modification in the dot problem became necessary. Instead of the number of dots in a row being the unknown quantity, the problem type was altered so that the number of rows became the unknown part of the problem. This was done by showing to the student only the number of dots in the row, and using a square to denote rows that were covered.



Prior to the use of a square, in the previous lesson a piece of cardboard was used to cover the rows in the array. Thus this new convention for the dot problem would be merely an extension of these problems, that is, the square would replace the piece of cardboard.

d) Concatenation

Although we only managed to test our presentation on concatenation with one student, the Initial Pilot Study indicated that no change at this point was required. Therefore, the sequence and structure of the questions was maintained for the Exploratory Case Study.

B. AN EXPLORATORY CASE STUDY

Based on the results of the Initial Pilot Study, the first three lessons of the teaching outline were prepared and experimented in an exploratory case study. The student selected for this study was a sixth grade student, (Philip, aged 12.1 years) judged average by his teacher. In order to avoid repetition, the rational of the lessons and questions asked will be presented in Part C, the final version of the teaching outline. We will limit ourselves in this part to bringing out those problems experienced by Philip, which warranted a revision of the teaching outline.

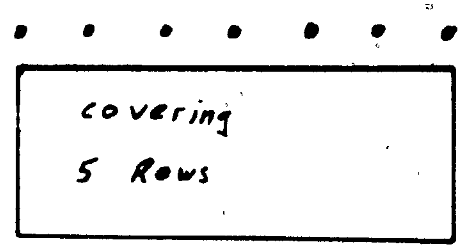
1. Selection Interview

The selection interview revealed that Philip perceived the multiplicative nature of the three problem types. Although initially when writing the area of the rectangle, he had a mild confusion between area and perimeter, he, himself, corrected his error. Thus Philip was considered a suitable subject for the experiment.

2. Lesson 1 - Transition From Placeholder to Literal Symbol

The rational for Lesson 1 was based on the results of the "Initial Pilot Study". An empty box as a placeholder was introduced prior to that of the literal symbol. Furthermore the box was used in conjunction with the "hiding" of one of the dimensions of the problem. A dot problem was presented to the student first, followed by a line problem and then an area problem. This change in sequence, that is, placing the dot problem first as opposed to last, as done previously, in the Initial Pilot Study, was due to the new orientation of the problem types - the hiding of one of the parts of the problem. It was believed that the array of dots provided an excellent situation to introduce the notion of hiding, and could be initially more easily visualized than within the context of the line or area problem. The presentation of the three problem types was as follows:

<u>Dot Problem</u>

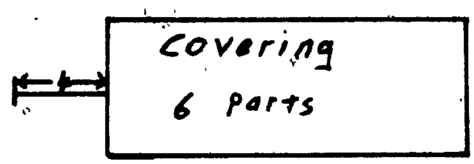


Here is a card. You can see a row of 7 dots. And I have hidden more rows each with 7 dots. Here is the problem. How would I write the total number of dots if I don't know the exact number of rows. So let me show you how I do it.

Number of dots = $7 \times \square$

Since I don't know the exact number of rows I am meanwhile using a box. Now I am going to let you have a peak at what I have covered and ask you to fill the right number in the box. "How many rows are there altogether?"

<u>Line Problem</u>



Now look at this line problem. Each part is 4 units long. But you don't know how many parts there are. Can you complete this equation?

Length =

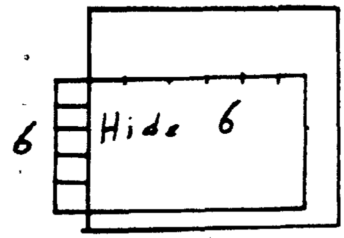
(Uncover)
So what number has to go into the box?

Area Problem

Now I will draw an area problem, in which I will only show you the height and I will cover the rest of the rectangle. (Draws rectangle and covers the appropriate part)
Can you complete this equation?

Area =

(uncover and have student fill in box)

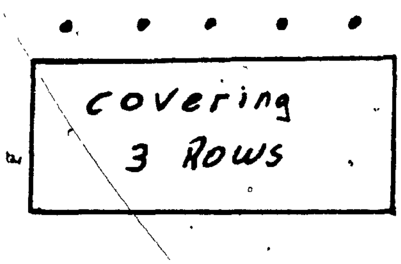


Each problem required the evaluation of the placeholder (the empty box), that is, the 'uncovering' and the 'filling in' with the correct number. This evaluation was deemed necessary in order to guarantee a numerical referent for the placeholder. After the presentation of each problem type and the subsequent evaluation, the student was required to generate his own "problems" corresponding to given "answers", as for example, "Can you make up a dot problem like we just did where the total number of dots is $5 \times \square$?".

All three problem types presented no difficulties for Philip. His spontaneous and correct responses indicated that the presentation was clear and that the hiding of one of the dimensions proved to be enough justification for the use of the box. He was also able to generate all three problem types with equal ease. Thus at this point no modification in the lesson appeared necessary.

The second part of Lesson 1 involved the transition from boxes to letters for the hidden dimension. The three problem types were presented within the same context as the first part, however, in answering the problems the box was replaced by a letter - one selected by the student himself. The introduction to these problem types was as follows:

<u>Dot Problem</u>



I: Let's do another dot problem. You can see that I have a row of 5 dots. But you can't see the exact number of rows. This time, I am going to use a letter instead of a box for the number of rows. Choose a letter.

S: Chooses a letter.

I: So let me show how I write the total number of dots using your letter.

number of dots = 5 × a
(or whatever letter is chosen)

Now I will let you look at the number of rows. How many rows are there?

S: 4

I: So what number does your letter stand for?

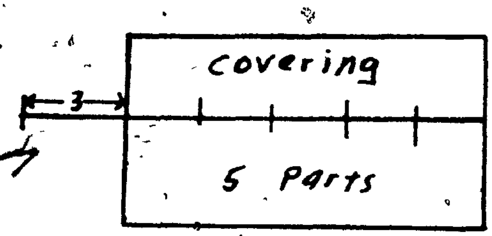
S: 4

I: Can you now complete this equation?

number of dots = 5 ×

<u>Line Problem</u>

Now let's try a line problem, where we will use a letter instead of a box for the number of parts. Each part is 3 units long but you don't see how many parts. Can you complete this equation using a letter instead of a box.

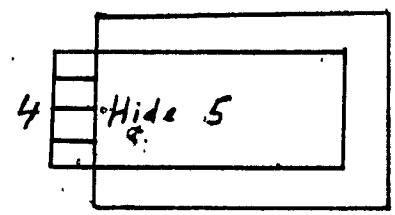


Length =

(uncover)
So what number does the letter stand for?

Area Problem

Now I will draw an area problem, where you will see only the height, because I will cover the rest of the rectangle. (Draws rectangle and covers the appropriate part) Can you complete this equation?



Area =

(Uncover)
So what number does the letter stand for?

Following the same format as the previous section, the presentation of these problems was followed by an evaluation of the literal symbol and the subsequent generation of each problem type by the student.

Philip experienced no difficulties in the transition to letters from the 'empty box', and easily wrote the required algebraic expressions using many different letters. None of the cognitive obstacles encountered in the Initial Pilot Study were experienced by Philip in his first use of the literal symbol. Furthermore, the fact that Philip used a wide variety of letters indicated that he was very aware of their arbitrary nature.

The final section of Lesson 1 dealt with a series of questions designed to determine the student's perception of the differences and similarities between the use of letters and the use of boxes, and also to uncover any hesitation on the part of the student in his use of the letter in an expression such as '3 × a', as evidenced in the initial Pilot Study. The questions were as follows:

1. Do you find it more difficult to use letters than boxes?
2. Can you tell me why?
3. Does it bother you to see a number multiplying a letter? like in '3 × a'?
4. Can you tell me why?

5. What about boxes, does it bother you the same when you see '$3 \times \square$'?
6. What difference do you see between $3 \times a$ and $3 \times \square$?
7. Can we use a letter like a box?
8. When we uncover the part that gives us the missing number, do we do the same thing when we use a box or a letter?

Philip's responses to these questions indicated that to him letters and boxes were "the same thing". He saw no difference between '$3 \times a$' and '$3 \times \square$'.

Since Lesson 1 proceeded very smoothly with Philip, no modification of this lesson was deemed necessary for the final version of the teaching outline.

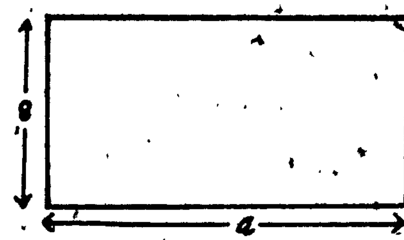
3. <u>Lesson 2 - The Literal Symbol as an Unknown Quantity</u>

The aim of Lesson 2 was to introduce the conventional usage of the literal symbol as an unknown quantity, not just a hidden one. The basis for this construction had been prepared in Lesson 1. The introduction of the box as a placeholder seemed a useful intermediate step prior to the introduction of letters. Moreover, their use as representing an actually <u>hidden quantity</u> also eliminated the rather artificial justification as done previously (the carpet problem). Thus the transition from a <u>hidden</u> quantity to an <u>unknown</u> one seemed but a small step. Of course this could only be managed in the area and line problems where the unknown dimension would remain unknown until they were measured. However, due to the presentation of the dot array problem (a drawing of a cardboard hiding some of the rows), the transition from hidden to unknown was not possible.

The presentation of the problem types was as follows:

Area Problem

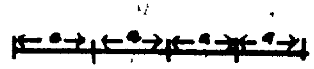
Let's look at this area problem. Do you think you could write down the area of this rectangle?



So this problem is different in that neither of us knows the length of the base. But the letter <u>a</u> can be used to stand for it. Can you now write the area?

Line Problem

Here is a line problem. How many parts do you see?

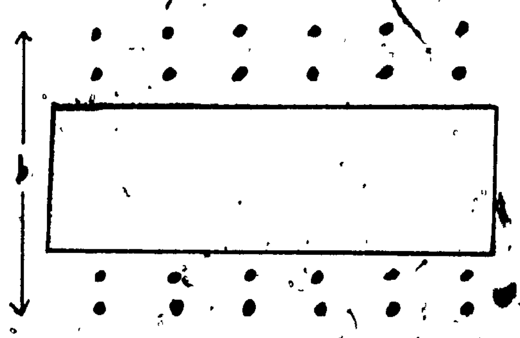


What do you think the letter <u>a</u> stands for?

Can you write the length of the line?

Dot Problem

Here is a dot problem. How many dots are in each row?



What do you think the letter <u>b</u> stands for?

Can you write the total number of dots?

After the introduction of each problem type, the student was required to generate a similar problem given an algebraic expression such as '5 × a'. The numerical referent of the literal symbol was also emphasized by having the student measure the unknown dimension in one of the area and one of the line problems. With this measure the student then evaluated the initial algebraic expression. Obviously this evaluation process could not occur in the dot problem due to the nature of the problem. As seen in the diagram above in the dot problem one cannot assign to the literal symbol a measureable value. In this class of problems the letter remains associated with a "hidden unknown".

In the area problem, Philip easily adjusted to using the letter as the unknown length of the base, and at no point demonstrated any difficulties in any of the aspects of these problems; the writing of the 'answer' (the algebraic expression), the generation of his own area problem and the evaluation of the expression.

However, the line problem created some confusion for Philip due to the change in its presentation. In Lesson 1, the letter represented the <u>unknown number of parts</u>, but in this new version of the line problem, the letter represented the <u>unknown length of each part</u>. Philip's confusion is evident in the following part of the interview:

I: Here is a line problem.
How many parts do you see?
S: 4
I: What do you think the letter <u>a</u> stands for?
S: 4
I: How many parts are there?
S: 4
I: And what do you think the letter <u>a</u> stands for?
S: How many parts...how many centimeters...
I: How many centimeters...
S: between each line (referring to the separation on the line)
I: So "<u>a</u>" stands for the length of each part.
S: Yes
I: O.K. Can you write the length of the line?
S: (begins to measure <u>a</u>)
I: Without measuring
S: 4 centimeters

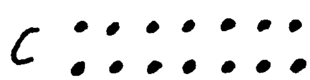
Lines 4, 8 and the final answer Philip provided for the length of the line, 4 centimeters, illustrates that Philip had not adjusted to the new orientation of the line problem. It would appear that he determined the length of the line to be four centimeters, by estimating the length of each part to be one centimeter. After a further explanation by the interviewer, he managed to correctly answer this line problem and all subsequent line problems presented to him.

This confusion of Philip's led to a minor change in the final version of the teaching outline. A preamble was added to the introduction of line problems which pointed out the difference between the line problems in Lesson 1 and those in Lesson 2.

Philip had no difficulty in writing an algebraic expression for the number of dots in the dot problem. However, in generating a dot problem he experienced some minor difficulty in his interpretation of what the letter represented.

I: Can you make up a dot problem where the total number of dots is '7 × c'?
S: (Draws a problem with 7 dots in each row and 2 rows)

I: I would say that the number of dots in your drawing is '7 × 2'.
S: Yes
I: Can you draw a dot problem where the total number of dots is '7 × c'?
S: Here is C (Writes c next to the two rows)



I: In your problem I know the number of rows so I don't need to use the letter. Can you make a problem similar to the one I showed you, where we used the letter b. Why did we use the letter b ?
S: Because we didn't know the number of rows. ...so that should be more rows.
I: Do we know the number of rows in your drawing?
S: Yes
I: So is it necessary to use a letter?
S: No, we can see it right away.
I: Can you make up a problem where we wouldn't know the number of rows?

S: Should I use a box? (referring to the square)
I: uh huh
S: (draws problem correctly)

Although Philip was eventually directed to draw a correct dot problem, this part of the interview alerted us to possible difficulties we might encounter in future interviews. However, seeing the ease with which it was resolved no modification was made in the teaching outline.

Part 2 of Lesson 2 involved the introduction of concatenation. The questions and their sequence were that of the Initial Pilot Study. However, the interview with Philip illustrated a possible area of confusion. After the introduction of the convention of writing '8 × x' as <u>8x</u>, and the subsequent examples, <u>5a</u> for '5 × a' and <u>4b</u> for '4 × b', the interview proceeded as follows:

I: Can we do this in arithmetic? What is can I write 6 times 7 without a multiplication sign?
S: No, because it could be minus or addition, you would never know.
I: If you remove the multiplication sign and put the numbers next to each other, what do you get?
S: 13
I: (says and writes) '8 × x' we said was <u>8x</u>. Can we write '6 × 7', leaving out the multiplication sign and putting the numbers next to each other?
S: No, I don't think so.
I: What do you get?
S: 13, you just get like, if you didn't have the times sign you wouldn't know which one it was.
I: In algebra we put the number and the letter next to each other. We don't leave a space in between. So if we did that in arithmetic it would...
S: 13
I: Why 13?
S: 6 plus 7
I: Where is the plus sign?
S: (no response)
I: It is just the number...
S: (no response)
I: (writes the 6 and the 7 next to each other) sixty-seven
S: Oh yes!!

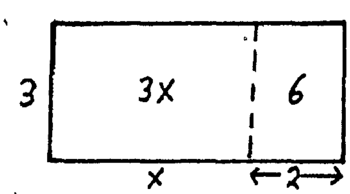
At no point in the interview did Philip write the 6 and the 7 next to each other. Perhaps the empty space between them led Philip to consider an alternate operation, in this case addition. Thus in the final version of the teaching outline, an initial greater insistance on putting the numbers together was added, since it was judged necessary for a contrast with algebraic concatenation. Furthermore, the convention of writing '$a \times 3$' as <u>3a</u> was also included. This addition was deemed necessary in light of the nature of the type of problems the student would be meeting in the subsequent lessons.

4. <u>Lesson 3 - Algebraic Expressions Involving Both Multiplication and Addition</u>

The algebraic expressions in Lesson 2 were of a purely multiplicative nature (3a). Lesson 3 introduced more complex ones, involving both multiplication and addition, presented within the context of the three problem types. This lesson was divided into two sections. The first part concentrated on algebraic expressions with only one unknown quantity, such as '$3x + 6$'. The second part introduced algebraic expressions with two unknown quantities, such as '$3x + 3y$'. Only the area and the line problem were used in the second part since the nature of the dot problem did not lend itself easily to be used within this context. In part one, the problem types were introduced as follows:

<u>Area Problem</u>

I: Here is a rectangle (outlined in blue). The length of the height is 3 units. The length of the base is in two parts, one is unknown, so I marked <u>x</u>, and the other part is 2 units. In order to find the area, of this rectangle,



I: I'm going to simplify the problem. Here, I'm drawing a line, splitting it up into two smaller rectangles.

What is the area of the rectangle on the left?

S: 3x (recall if needed, area and concatenation. Write 3x in rectangle).

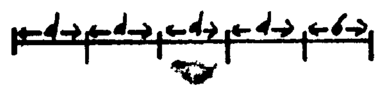
I: What is the area of the rectangle on the right?

S: 6 (Write 6 in smaller rectangle)

I: What is the area of the blue rectangle? (that is the rectangle we started with)

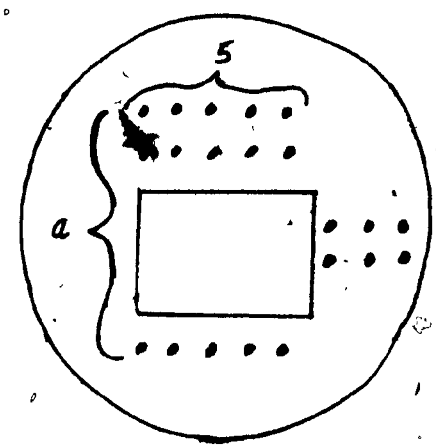
Line Problem

I: Here is a line. What is the length of this line?



Dot Problem

I: Here is a dot problem. What is the total number of dots in the circle?



The presentation of each problem type was followed by a 'reversal', which required the student to generate his own problem given an algebraic expression such as '3a + 9'.

During the interview, Philip experienced some minor and major difficulties with the area problem. When the rectangle was divided into smaller rectangles he easily responded that the area of the rectangle on the left was 3x. However, some confusion arose when he had to write the area of the rectangle on the right.

I: What is the area of the rectangle on the right?
S: Three two umm two
I: What is the length of the height?
S: 3
I: What is the length of the base?
S: 2
I: So what is the area of the rectangle?
S: Three two or six (writes 6 in rectangle)
I: Six. When you said "three two" what were you thinking?
S: Like 3 times x.
I: How would you write that - three two?
S: (writes - 32)

Philip has some confusion with the use of concatenation. He had generalized this convention to arithmetic by writing '32' (three two) meaning three times two. He maintained this misinterpretation when he was first asked to write the area of the rectangle.

I: Now can you tell me the area of the blue rectangle, that is, the rectangle we started with?
S: 6
I: Why would you say 6?
S: Three x and three 2.

Although his final response did not justify the answer '6', it is nonetheless evident that he was still holding onto the expression 'three-two'. Not only did Philip experience difficulty with concatenation but he also had further difficulties in the writing of the area of the rectangle. After he had responded that the area of the rectangle was '6', the interview proceeded as follows:

I: You said the area of the rectangle was 6, what is the area of this rectangle (showing only the one on the right)
S: 6
I: Is the area of the blue rectangle the same as that of the right?
S: No.
I: So the area of the rectangle on the left is 3x and the one on the right is 6, so if you wanted to write the area of the blue rectangle it would be?
S: three times 6
I: Why is it three times six?

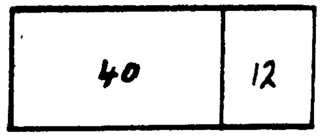
S: There is 3x and 6, so it is 3 times six.

Philip's last statement indicates that within the context of area the idea of multiplication was so strong, that it interferred with his ability to perceive the additive aspect of the problem. It should also be noted that he ignored the presence of the letter (in 3x). After a further attempt by the interviewer to emphasize the area of the two smaller rectangles, he then responded with '18', once again focusing on the 'multiplicative' aspect of area. At that point the interviewer directed Philip to the correct response as follows:

I: The area of the blue rectangle is equal to the area of this rectangle which is 3x, plus...
S: 6
I: That's it. The area of the rectangle on the right. So the area of the rectangle is 3x plus (the interviewer presents: Area = 3x +).
S: (student adds 6)
I: You seem to be bothered by the 3x + 6, what is troubling you?
S: Well, I am not used to it. I am used to 3 times 6, or something like that.
I: In 3x + 6, which part are you not used to?
S: The plus, we use 3 plus 6, or 3 times 6, or 3 times x, or three plus 6.

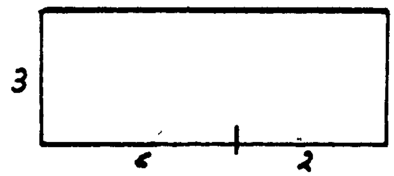
Philip was obviously troubled by the sudden use of two operations. Although he could accept the lack of closure in one operation (3x), he was upset by the presence of an expression containing two operations.

In view of Philip's difficulties (the strong adherence to multiplication in area problems, and the difficulty in his acceptance of two operations), the final version of the teaching outline was modified to include an additional step. In the event that a student would experience similar difficulties, an area problem showing the additive aspect of these problems at a purely numerical level was added. For example, this rectangle,

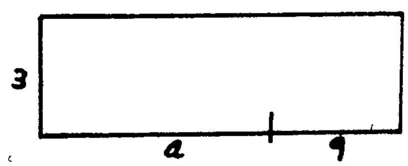


followed by the question, "What is the area of the large rectangle?" It was expected that the student would see the similarity between this purely numerical problem and the algebraic one.

Another minor change was made to this problem. This change however, was not a direct result of the interview with Philip. In order to avoid any possible confusion with the <u>x</u> as a multiplication sign and <u>x</u> as some unknown quantity, it was decided to change the letter in the area problem from an <u>x</u> to a <u>c</u>.



Further in the interview, when Philip was asked to generate his own area problem involving two terms '3a + 9', he drew the problem as follows:



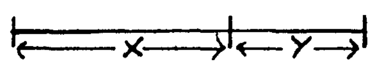
A new element is brought out here in that for these problems the student requires a fairly good knowledge of multiplication factors. Philip associated the 9 with only one dimension, that is, the length of the base of the second rectangle. This could be due to the fact that <u>3x</u> represents an obvious problem but nine has to be decomposed.

In writing the length of the line in the line problem, and the number of dots in the array (see page 81), Philip responded with D6 (meaning D times 6) and <u>5a</u>, respectively. This is further evidence of his avoidance of algebraic expressions involving two operations. Philip easily overcame his difficulties once he was directed to see the two parts of each problem which had to be added, and responded correctly to subsequent line problems. Philip also did not experience any difficulties in generating either a line problem or a dot problem.

Thus only a minor change was necessary in the final version of the teaching outline. Initially the presentation of only one line problem seemed sufficient. However, the interview with Philip pointed out the necessity of the inclusion of an additional one in such cases where the student was experiencing some difficulties. In the dot problem no change was deemed necessary at this point.

Part 2 of Lesson 3 consisted of the introduction of two unknown quantities. A line problem was first used to convey the notion of using two different letters to denote two different numbers. The problem was presented as follows:

I: Suppose I had a line as follows:
You notice that it's in two parts.
Why do you think I used different letters?



S: (two different lengths)
I: What is the length of this line?

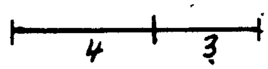
Length of line =

S: $x + y$

Philip understood immediately that the use of two different letters represented two different lengths. However, when he was asked to write the length of the line he responded as follows:

I: Do you think you can write the length of this line?
S: xy (says it without the times)
I: When there is no sign between, what sign was hidden?
S: The x (student refers to the multiplication sign)
I: Which means times or multiplication

S: yes
I: So the length of the line would be x times y ?
S: There is an invisible x (referring to the multiplication sign)
I: So what is the length of this line?
S: x times y
I: Suppose I had a line 4 units and 3 units. What is the length of the line?

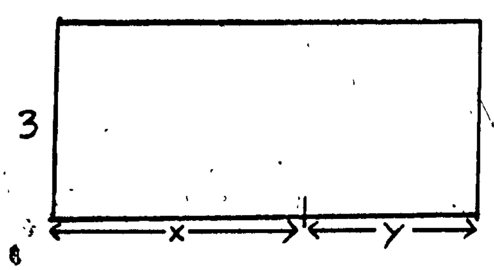


S: 4 times 3
I: So that is 12; ...if this were centimeters...
S: Oh, 4 plus 3
I: 4 plus 3
S: Oh, so x plus y (writes a '+' between the x and the y)

Philip multiplied instead of adding (xy) and he did this even at a purely numerical level (4 times 3). It is only when the idea of measurement is suggested that Philip was sparked into seeing the additive aspect of both the numerical and algebraic problems presented to him.

The introductory line problem was then followed by an area problem in which the base of the rectangle was divided into the same two unknown lengths as in the previous line problem.

I: Here is a rectangle. Can you write the area of this rectangle?



Philip initially gave the area as '3x + y', an error similar to the previous area problem he generated, that is, he used the y as representing the area of the rectangle on the right. Once he had divided the rectangle into two smaller rectangles and had written their respective areas, Philip managed easily to correct his error and wrote '3x + 3y'. The subsequent area and line problems of this type (two unknowns) presented no further difficulty for Philip. He easily wrote the required algebraic expressions, and correctly generated both an area and a line problem.

Since Philip experienced only some minor problems with this section of Lesson 3, no changes were deemed necessary for the final version of the teaching outline.

C. THE TEACHING OUTLINE

Lesson 1

The use of boxes as placeholders was motivated by the results of exploratory interviews which showed that the direct introduction of letters (standing for the number of rows in an array of dots, or the number of equal segments in a line or the length of a base of a rectangle) raised major cognitive obstacles ("How can you multiply a number by a letter?"). This brought about changes in the planning of the teaching experiment:

1. Placeholders have been introduced first because of the students' familiarity with them from the elementary school work.

2. The placeholders have been introduced in conjunction with the hiding (covering with a carton) one element of the problem, thus illustrating correctly the idea of "unknown quantity".

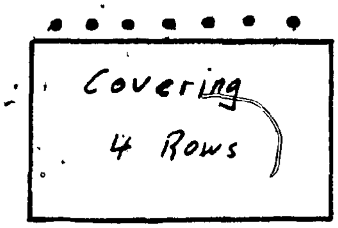
3. By eventually removing the cardboard, the unknown element of the problem is revealed, thus allowing the student to fill in or substitute, thus linking the placeholder or letter to a numerical referent.

Lesson 1

Transition From Placeholder to Literal Symbol (Letter)

Dot Problems

A. I: Here is a card. You can see a row of 7 dots. And I have hidden more rows each with seven dots. Here is the problem. How would I write the total number of dots if I don't know the exact number of rows. So let me show you how I do it.



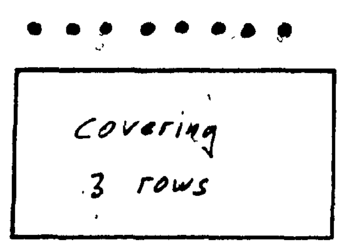
Number of dots = 7 × □

Since I don't know the exact number of rows I am meanwhile using a box. Now I am going to let you have a peak at what I have covered and ask you to fill the right number in the box.

Comment: This first problem is of a demonstrative nature. We are showing the student how to use the box as a placeholder. This function of the box is translated into action by the student when he actually fills it. The other purpose of the question is to verify if the student will use the total number of rows or only those that have been covered. In such a case we ask,

"How many rows are there altogether?"

B. I: (Presenting another similar problem) You can now see 8 dots in each row, but once again I am covering the other rows so you don't see the exact number of rows. Can you complete this equation the way I did before? (Present student with a sheet of paper on which is written:



number of dots =

S: (No response)

I: Remember what we did in the

I: other problem when we didn't know the number of rows. What did we use for the number of rows?

S: 8 × □

I: (uncover)
How many rows are there altogether? What number has to go in the box?

Comment: The second problem serves to verify if the student can himself use the box as a placeholder. The purpose of filling in the box is to keep the idea of numbers constantly present.

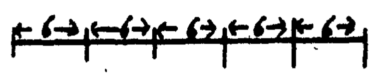
C. I: O.K. Now it's your turn. Can you make up a dot problem like we just did, where 5 × □ is the total number of dots.
(Encourage student to hide her work while she is drawing the problem)

S:

Comment: The purpose here is to have the student generate a problem given an algebraic expression. -- Didactic Reversal between expression and problem.

Line Problem

I: Now let's look at another type of problem. Do you remember how you found the length of this line?

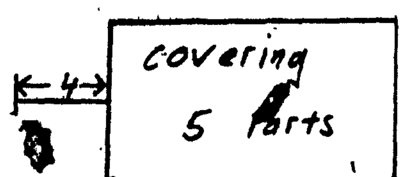


S:

I: (Writes formula:
Length of line =

length of each part × number of parts.

A. I: Now look at this line problem. Each part is 4 units long. But you don't know how many parts there are. Can you complete this equation:-



length =

S: No answer
I: length = 4 ×
S: No answer
I: What do you think I should write for the number of parts?
S: No answer
I: Do you remember what we did in the dot problem?
S: No answer
I: Because we didn't know the number of rows we used?
S: A box.
I: Could you use a box here?
S:
I: (If no, interviewer completes the box.)
(Uncover)

So what number has to go into the box?
S: (7)
I: (If incorrect)
How many parts are there altogether?

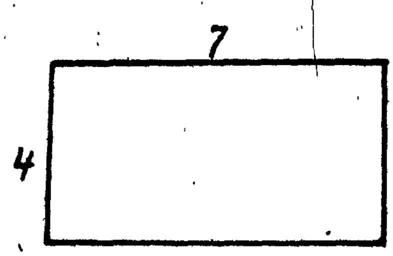
Comment: The purpose of this problem is to illustrate another situation in which an algebraic expression is needed. The initial question verifies if there is any immediate transfer from the previous type of problem. The questioning leads as slowly as possible the student to a possible transfer.

B. I: Your turn now to make up a line problem where $3 \times \square$ is the length of the line. Hide your work and show only the first part of your line.

Area Problems

I: Do you remember how to find the area of a rectangle?
S:
I: Writes

Area = length of height × length of base

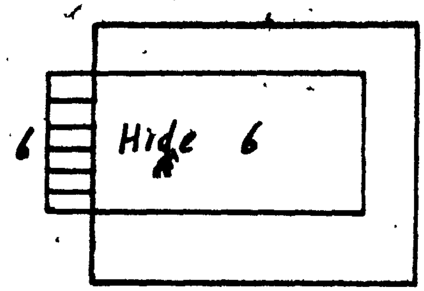


A. I: Now I will draw an area problem, in which I only show you the height and I will cover the rest of the rectangle.

(Draws rectangle and covers the appropriate part)

Can you complete this equation?

Area =



S: no response
I: Area = 6 ×
S: no response
I: What do you think I should write for the length of the base?
S: no response
I: Do you remember what we did in the dot and line problem?
S: no response
I: Because we didn't know the number of rows and the number of parts we used?
S: A box.
I: Could you use a box here?
S:
I: (If no, interviewer completes the box)

(Uncover)
(Have student fill in box)

So what number has to go into the box?
S:
I: (If incorrect)
What is the length of the rectangle?

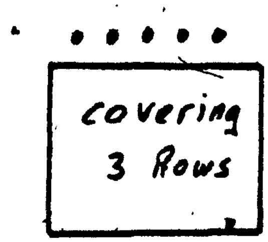
B. I: Now it is your turn. Make up an area problem, where I can only see the height of the rectangle. Don't forget to hide your work.

<u>Comment</u>: Exploratory work has indicated that students like to make their own problems. Therefore, problem B is left open to the subject.

Transition to Letters

Dot Problems

A. I: Let's do another dot problem. You can see that I have a row of 5 dots. But you can't see the exact number of rows. This time, I am going to use a letter instead of a box for the number of rows. Choose a letter.



S: Chooses a letter.

I: So let me show how I write the total number of dots using your letter.

number of dots = 5 × a

(or whatever letter is choosen)

Now I will let you look at the number of rows. How many rows are there?

S: 4

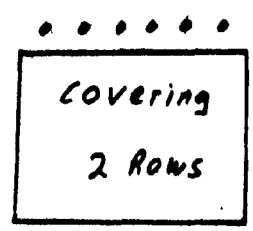
I: So what number does your letter stand for?

S: 4

I: Can you now complete this equation?

number of dots = 5 ×

B. I: (Presenting another similar problem) You can now see 6 dots in each row, but once again I am covering the other rows so you don't see the exact number of rows. Can you complete this equation using a letter instead of a box?



number of dots =

S: No response

I: Remember what we did in t[illegible]er problem when we didn't kn[illegible] number of rows. What did we use for the number of rows?

S: 6 × a

I: (Uncover)
How many rows are there altogether?

S:

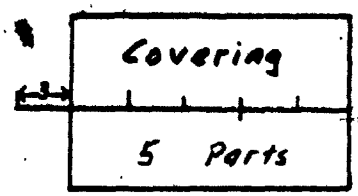
I: What number does the letter stand

I: for?
S:

C. I: Now it is your turn. Make up a dot problem where the answer will be $9 \times c$.

Line Problems

A. I: Now let's try a line problem, where we will use a letter instead of a box for the number of parts. Each part is 3 units long but you don't see how many parts. Can you complete this equation using a letter instead of a box.



Length =

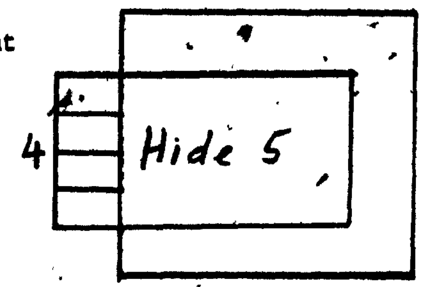
S: No answer
I: Length = $3 \times$
S: No answer
I: Do you remember what we did in the previous dot problem?
S: No answer
I: Because we didn't know the number of rows we used?
S: (Should say a letter, if student says a box, remind him that we are now using letters instead of boxes)
I: Could you use a letter here?
S:
I: (If no, interviewer completes equation with a letter.)

(Uncover)
So what number does the letter stand for?
S: (6)
I: (If incorrect)
How many parts are there altogether?

B. I: Make up a line problem where the answer is $7 \times d$.

Area Problems

A. I: Now I will draw an area problem, where you will see only the height because I will cover the rest of the rectangle.
(Draws rectangle and covers the appropriate part)



Can you complete this equation?

Area =

S: no response
I: Area = 4 ×
S: no response
I: What do you think I should write for the length of the base?
S: no response
I: Do you remember what we did in the dot and line problems?
S: No response
I: Because we didn't know the number of rows and the number of dots we used?
S: (a letter — if student responds with a box, remind him that we are now using letters instead of boxes.)
I: Could you use a letter here?
S:
I: If no, interviewer completes with a letter)

(Uncover)
So what number does the letter stand for?
S:
I: (If incorrect)
What is the length of the base of the rectangle?

B. I: Now it is your turn. Make up an area problem where I can only see the height of the rectangle.

Comment: Although the only change in these problems is the transition from box as placeholder to letter, the difficulties students have in their initial use of letters warrants a slow approach. Thus three dot problems are used in the initial introduction of letters. In the first problem the student is shown how, in the second he has to complete the equation, in the third he has to generate a problem.

Although we have not raised with the student the difference in the use of a box and that of a letter, the mere evaluation process in the case of the box simply requires filling-in whereas in the case of a literal symbol, the letter has to be substituted by a number.

Boxes vs. Letters

1. I: Do you find it more difficult to use letters than boxes?
   S:

2. I: Can you tell me why?
   S:

3. I: Does it bother you to see a number multiplying a letter like in $3 \times a$?
   S:

4. I: Can you tell me why?
   S:

5. I: What about boxes, does it bother you the same when you see $3 \times \square$?
   S:

6. I: What difference do you see between $3 \times a$ and $3 \times \square$?
   S:

7. I: Can we use a letter like a box?
   S:

8. I: When we uncover the part that gives us the missing number, do we do the same thing when we use a box or a letter?
   S:

Homework

I: For our next meeting, please make

a dot problem where the answer is $3 \times a$
a line problem where the answer is $5 \times b$
an area problem where the answer is $6 \times d$

Comment: Questions (1) to (5) attempt to uncover the student's thinking about the use of letters vs the use of boxes. Questions (6) to (8) try to bring out the differences between the two symbolizations: a letter can be used the same way as a box except for their evaluations.

The assignment of three homework problems should lead the student to continue thinking about the lesson but moreover, it provides for a review at the beginning of the next lesson.

Lesson 2

Lesson 1 introduced the literal symbol as an extension of the place-holder which was used to express a hidden dimension of a problem. This was essential, since students objected in the pilot study to arithmetic operations using both letters and numbers.

The object of Lesson 2 is:

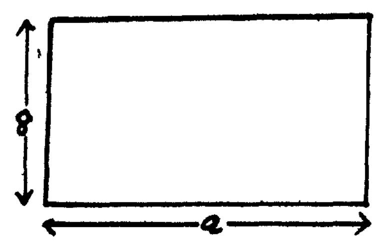
1) to introduce the conventional usage of the literal symbol as an unknown quantity, not just a hidden one. There is no reason why this convention needs to be re-invented by the student. However, its linkage to the previous work should be explicit. Furthermore, whenever possible, that is with the length and area problems, the motivation for substitution can be generated by measuring. (Of course, this is impossible with the dot problem).

2) to introduce concatenation.

Review of Previous Meeting

I: At our last meeting, I had asked you to makeup 3 problems. First a dot problem, where the total number of dots is $3 \times a$.
S:
I: What does the a stand for?
S:
I: And a line problem, where the length of the line is $5 \times b$.
S:
I: What does the b stand for?
S:
I: And an area problem, where the area of a rectangle is $6 \times d$.
S:
I: What does the d stand for?
S:
I: Do you recall the formula for area?
S:
I: (writes Area of Rectangle = length of height × length of base)

A. I: Let's look at this area problem. Do you think you could write down the area of this rectangle?



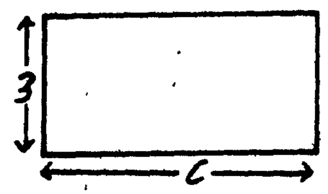
S: No response
I: What is the length of the height?
S: 8
I: What do you think is the length of the base?
S: No response. (if yes - still go through appropriate explanation)
I: What do you think the letter a stands for?
S: No response
I: This problem is a bit different from what we used to do. In our other problems we used to cover the base so that you didn't know the length of the base. Do you remember, what you did then to write the area?

Area = 8 ×

S: (completes with a letter or box, if box continue...)
I: And what did you use after the box?

S: (completes with a letter)
I: So this problem is different in that neither of us knows the length of the base. But the letter <u>a</u> can be used to stand for it. Can you now write the area?
S: Area = 8 × a

B. I: What is the area of this rectangle?
S: Area = 3 × c

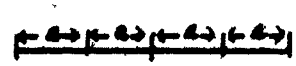


C. (Reversal)
I: Can you make up a problem where the area of a rectangle is 5 × d ?
S:

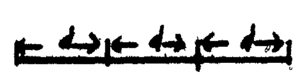
D. (Substitution)
I: Here is a ruler. Can you measure the length of the base? (1st problem)
S: (measures)
I: So what does the letter <u>a</u> stand for?
S: (number)
I: Can you replace it in the formula?
S: Area = 8 × (the number)

<u>Line Problems</u>

A. I: Here is a line problem. It is slightly different from the ones you did before. In this case we know the number of parts but we do not know the length of each part. That's why we use a letter.



How many parts do you see?
S: 4
I: What does the letter <u>a</u> stand for?
S:
I: Can you write the length of the line? (If student is unable to respond, have him recall the



I: formula for determining the length of a line)
S: Length of line = $4 \times a$

B. I: Here is another line problem. Can you complete this equation:

Length of line =

C. I: Can you make up a line problem with the

Length of line = $5 \times b$

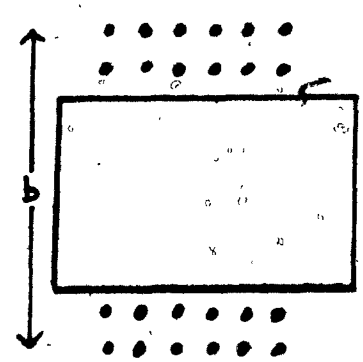
(reversal)

D. (Substitution)

I: Here is the ruler, can you measure the length of a part? Problem A.
S:
I: What does <u>a</u> stand for?
S: (number)
I: Can you replace it in the formula?
S: Length of line = $4 \times$ (the number).

<u>Dot Problem</u>

A. I: Here is a dot problem. How many dots are in each row?
S:
I: What do you think the letter <u>b</u> stands for?
S:
I: Can you write the total number of dots?
S: Total number of dots
S:



B. (Reversal)
I: Can you make up a dot problem where the total number of dots is $7 \times c$?

PART II Concatenation - Numeral and Literal Symbols

A. I: How do you write 6 times 7 in arithmetic?

S:

I: In algebra we can use all the letters in the alphabet - capitals and small letters. We often use the letter x . This can get a little confusion especially if you have 2 x's as in

$8 \times x$ (8 times x)

In order not to have any confusion, in algebra we can just leave out the multiplication sign and write:

$8x$ for $8 \times x$

Whenever you see a letter attached to a number, there is a hidden multiplication there.

What does 5a mean?

S:

I: What does 4b mean?

S:

I: Can we do this in arithmetic? That is, can we remove the multiplication sign in 6×7 and write the numbers next to each other as in 8x ?

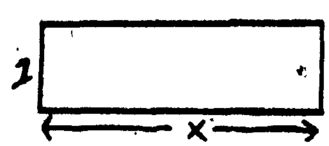
S: (67)

I: Can you make up an area problem where 4b is the answer?

S:

I: Can you make up a line problem where 5a is the answer?

B. I: Here is a rectangle, can you tell me what the area is?



S: ($1 \times x$ or $1x$)

I: What do we do in arithmetic with a product 1×3?

S: We just write 3.

I: The samething in algebra, we omit the 1 and just write x .

C. I: One other convention that I want to show is the following one:

$a \times 3$ we also write as $3a$

Does it make any sense? Do you think we should write $a3$?

S:

I: It does not make much difference since

$$a \times 3 = 3 \times a$$

Is this true for any value of <u>a</u>?

S:

I: Can you check it with some number?

S:

I: And another?

S: From now on we write

$b \times 5$ as

$c \times 6$ as

(Note: Some transition is taking place here, since in substituting different values for <u>a</u>, the idea of generalized number is starting to appear).

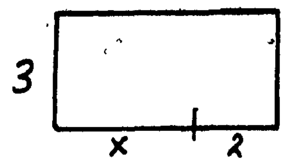
Homework

For our next meeting, please make up

- an area problem where the answer is $6a$
- a dot problem where the answer is $4d$
- a line problem where the answer is $6b$
- a line problem where the answer is $1c$ or c.

Lesson 3 RATIONALE

In Lesson 2 all problems were of a purely multiplicative nature. Lesson 3 introduces more complex ones, involving both multiplication and addition. Although the possibility exists here to present a situation to the student where a contradiction leads him to see the need for brackets (e.g.



area = $3(x + 2)$), such an introduction of this new concept at this point may cause some confusion. Thus the topic of bracketing is not introduced in this lesson.

In this lesson, the area of the rectangle will be determined by subdividing it into smaller parts, making it less confusing and easily visualized.

A. I: At our last meeting I had asked you to make up 4 problems.
First an area problem where the answer is <u>6a</u> , that is the area of the rectangle is <u>6a</u>.

S:

I: And a dot problem where the total number of dots is <u>4d</u> .

S:

I: And a line problem, where the length of the line is <u>6b</u> .

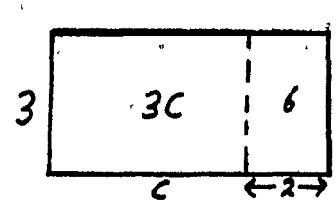
S:

I: And a line problem where the length of the line is <u>1c</u>.

<u>PART I: One Unknown</u>

I: Here is a rectangle. The length of the height is 3 units. The length of the base is in two parts, one part is unknown, so I marked <u>c</u> , and the other part is 2 units. In order to find the area of this rectangle, I'm going to simplify the problem. Here, I'm drawing a line, splitting it up into two smaller rectangles.

What is the area of the rectangle on the left?



S: <u>3c</u> (Recall if needed area and concatenation. Write <u>3c</u> in rectangle.)

I: What is the area of the rectangle on the right?

S: 6 (Write 6 in smaller rectangle)

I: What is the area of the blue rectangle? (that is the rectangle we started with)

S: (adds up) We can expect student to perceive additive nature of areas)

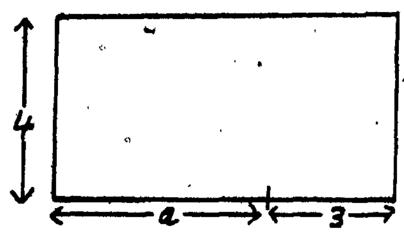
I: O.K. We can write:

Area of blue rectangle =

S: Fills in $3c + 6$

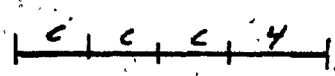
B. I: Here is another rectangle. What is the area of this rectangle?

S: (If no response guide student into separating large rectangle into two smaller ones and follow similar line of questioning as in previous problem.)



C. I: Now it is your turn, can you make up an area problem where the area of the rectangle would be $3a + 9$?

D. I: Here is a line. What is the length of this line?

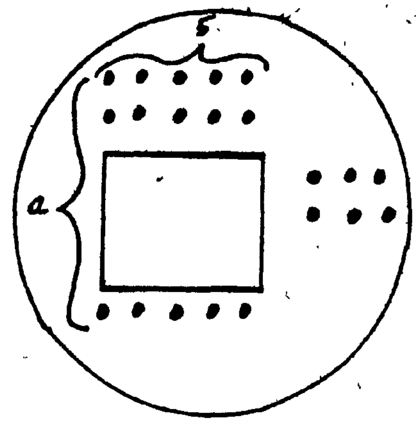


S:

I: (If student is having difficulty, present him with another line problem as follows;)

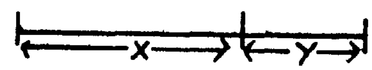
E. I: Make up a line problem where the length of the line would be $4c + 8$

F. I: Here is a dot problem. What is the total number of dots in the circle?



PART II. Two Unknowns

A. I: Suppose I had a line as follows: You notice that it's in two parts. Why do you think I used different letters?

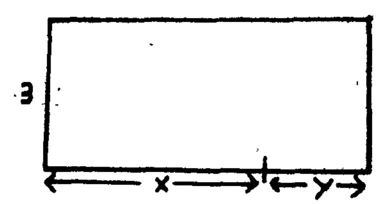


S: Two different lengths
I: What is the length of this line?

Length of line =

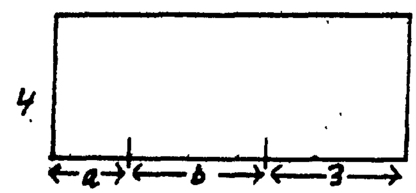
S: x + y

B. I: Here is a rectangle. Can you write the area of this rectangle?



S: no response
I: Recall how we found the area of the rectangles in the previous problems. (Draw line)
S:
I: What is the area of the rectangle on the left?
S: 3x (Fill in the drawing) (If the student does not answer correctly, have him go through the line questionning where he must tell the interviewer the length of the height and the length of the base.)
I: What is the area of the rectangle on the right?
S: 3y (Fill in the drawing)
I: Area of rectangle =
S: Completes 3x + 3y

C. I: Look at this rectangle. Recalling how we found the area of the previous rectangles, can you write the area of this rectangle?

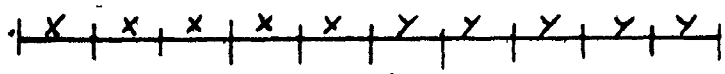


S:

D. I: Now it is your turn. Make up an area problem where the answer would be:

$5d + 5e + 10$

E. I: Can you write the length of this line?



S:

F. I: Can you make up a line problem where the answer would be $5a + 3b$?

CONCLUSION

The purpose of this chapter was to describe in detail the construction of a teaching outline. This was deemed to be useful since the essence of a teaching experiment is the experimentation of a new approach in the teaching of a particular topic. Thus, the teaching outline becomes the principle area of investigation. The aim in such an experiment is not to uncover unrelated cognitive difficulties but in fact designing an innovative presentation to overcome the known cognitive problems involved in the learning of the given topic and then to verify if it has not created new learning difficulties.

It follows that the preparation of the teaching outline requires an initial investigation of the current teaching practices as well as a search of the research literature related to the cognitive problems involved. This was presented in Chapters I and II, respectively. On the basis of this work, a new presentation based on a systematic geometric approach was judged as an alternative which might eliminate some of the known problems.

However, the construction of such a teaching outline needs to be constantly tested and revised during its development. Such an outline must be the result of an interactive process whereby the elements of the planned lessons, as well as their order, must constantly be verified with subjects. In this experiment, the initial pilot study as well as the exploratory case study provided this input in the elaboration of the outline to be experimented.

CHAPTER V

ANALYSIS OF PRETEST

A. Objective of Pretest

Although Kuchemann (1981) examined the various interpretations British students assigned to literal symbols, his study was more general than ours for it covered equations, problem-solving, functions, etc., whereas the present investigation focuses on algebraic expressions. And thus, the pretest has been designed to study how the students' perception of the letter used in algebraic expressions varies according to the context.

B. The Pretest

1. Have you ever used letters in mathematics?
   Can you show me?

2. When I show you something like this $\boxed{3a}$, can you tell me what does it mean to you?

   (Multiplication? 3 a's? , 3 apples? 3 "times" a number)

2a. Can you give me an example?
   Can you show me?

3. If I show you something like this $\boxed{3n}$, does it mean the same-thing to you?

3a. Can you give me an example?
   Can you show me?

4. If I ask you to replace here $\boxed{3a}$, the letter a by the number 2, can you tell me what you get?

Rationale for First Four Questions

In these 4 questions, no arithmetic operational symbols are used. The only possible connection is the concatenation of a number and a letter.

We wish to investigate how the student interprets the letter in this context (number, label (apple), object (a's) and whether the concatenation is spontaneously interpreted as multiplication.) The letter is changed to verify if the "label" interpretation might not induce the student to think of <u>n</u> as a number. In question 4, the substitution should determine whether the student is multiplying or not, if the problem of concatenation is always present for the beginner.

5. When I show you something like this $\boxed{4 + b}$, can you tell me what it means to you?

   a) 4 plus b
   b) I don't know
   c) 4 plus some number

5 (a or b) What do you think the letter could mean?

   a') I don't know
   b') some number

5a' Do you think it is possible to add a letter with a number?

5 (b',c) Can you give me an example?
Can you give me another example?
How many examples could you think of?

6. When I show you this $\boxed{4 + n}$ can you tell me what it means to you? (a,b,c as in 5)

6 (a or b) What do you think the letter could mean?

6c Do you think there is any difference between

$\boxed{4 + b}$ and $\boxed{4 + n}$

7. When I show you something like this $\boxed{5 + \square}$ can you tell me what does it mean to you?

(a box to fill in with a number)

7a. Can you give me an example? (puts in a number)

7b. Do you think that these two mean the same thing?

$\boxed{5 + \square}$ and $\boxed{5 + c}$

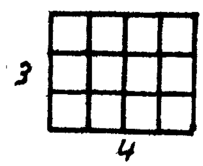
a) No b) Yes

7c. (If no) What do you think is the difference?
(If yes) Do you think the letter c and the box can be used the same way? Example?

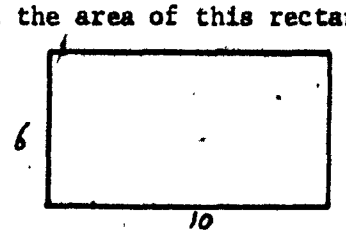
Rationale for Questions 5 - 7

In these questions all possible problems due to concatenation are avoided. However, the symbol for addition now places the letter in an arithmetic context. The addition can only make sense if the letter is interpreted as a number. This could depend on the letter used, which explains the transition from [illegible] to n. The difference between the letter and the placeholder is explored in question 7.

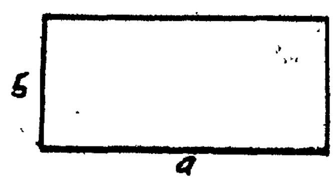
8a. Can you tell me the area of this rectangle?



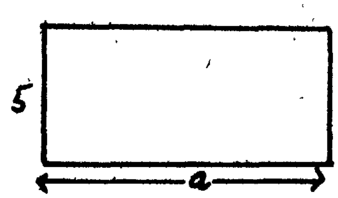
8b. Can you tell me the area of this rectangle?



9. Can you tell me the area of this rectangle?



9a. What do you think the letter a stands for? (side, the base, the length)



9b. Do you think the letter a means the same thing in these two rectangles?

Rationale for Questions 8 and 9

Questions 8 and 9 are to verify the meaning of a letter in a geometric context (label or length). Question 9 should indicate if the student can use the letter in formulating the area. Question 9b is to verify if students interpret the arrows as conventions to denote length.

10. Look at this work card:

Simplify 2a + 5a

Can you tell me what you think it means? (Put them together - or 7a)

10a. Can you do it?

10b. Can you explain how you got that answer?
(a's or apples?)

10c. (If a's) What do you mean by 2 a's and 5 a's?

11. Can you do this one?

Simplify
4n + 3n

11a. What do you think the letter n stands for? (number)

11b. Can you give an example? Another?

Rationale for Questions 10 and 11

Combining like terms does not require the perception of letter as number, the object and label interpretations are sufficient. Note that this question also verifies the answers obtained in Question 1. The switch from a to n is to verify if the label interpretation may lead to "number".

12. Here are some workcards:

a) Add 4 onto 8 Answer ____________
b) Add 4 onto n + 5 Answer ____________
c) Add 4 onto 3n Answer ____________

13. a) Multiply 8 by 4 Answer ______________
    b) Multiply $n + 5$ by 4 Answer ______________
    c) Multiply $3n$ by 4 Answer ______________

<u>Rationale for Questions 12 and 13</u>

This is question 4 from the Kuchemann test, which will give us some comparison with his results.

14a. Can you simplify

$$\boxed{3a + 4a + 5}$$

14b. Can you simplify

$$\boxed{2c + 3d + 4c}$$

15. Why do you think two different letters are used in the second problem?

15a. What do you think <u>c</u> could mean?

<u>d</u> could mean?

<u>Rationale for Questions 14 and 15</u>

In these problems we are verifying if the student not only combines like terms but also would "collapse" unlike ones. We are also investigating what meanings he would give to the presence of two different letters.

16. Can you work out

$$\boxed{6 \times (2 + 3)}$$

16a. What do you think is the meaning of brackets? (Do the operation in the brackets first)

17. Can you work out

$$\boxed{3 \times (a + 5)}$$

17a. Is it possible to do the operation in the brackets here?

17b. Even if we can't do the operation in the brackets, can we multiply it out? Show me.

If he can do it try:

$$\boxed{4 \times (3 + 2b)}$$

$$\boxed{5 \times (2a + 3b)}$$

Rationale for Questions 16 and 17

In these questions the multiplication sign has been included to avoid any confusion. Clearly the meaning of the brackets changes in the context of algebra and this might prevent the student from using distributivity.

For the purposes of the analysis the pretest has been divided into seven parts, each one restricted to a particular aspect.

1. Open Question (Question 1).
2. Concatenation (Questions 2-4)
3. Letter in the Context of Addition (Questions 5-7)
4. Geometric Context (Questions 8 & 9)
5. Kuchemann Test (Questions 12 & 13)
6. Simplification and Combining Like Terms (Questions 10 & 11; 14 and 15)
7. Distributivity (Questions 16 & 17)

C. Analysis of Pretest

1. Open Question

"Have you ever used letters in mathematics?"

"Can you show me?"

This question aimed at eliciting the students spontaneous interpretation of the use of letters. The question was open ended and did not provide any clues.

Frankie responded by stating that he could use n for a number in 'n × n' or as a subdivision label ("like when you start a question you have a1").

Wendy indicated that letter could be used as an answer to an arithmetic operation or as a specific unknown in an equation $(25 + 20 = n)$

Antoinetta referred to the use of a letter as an answer to an arithmetic operation $(9 \times 9 = n)$, and also gave as an example expressing commutativity $(30 + 40 = n + 30)$.

Yvette interpreted letter as an answer to a numerical operation $(1238 + 5678 = n)$ and as a specific unknown

$$\left(\frac{2}{4} = \frac{n}{8}\right).$$

Filippo used a letter as a specific unknown $(x + 1 = 2)$, and as an answer to an arithmetic operation $(10 \times 15 = y)$

Gail used a letter as a specific unknown in an equation $(4 \times n = 12)$

The answers show a prevalence of two interpretations: letter as a specific unknown in the context of an equation and letter as an answer to an arithmetic operation. Each one of these interpretations was given by four students, letter as an unknown in an equation was identified by the three Grade 7 students and by one Grade 6 student; letter as an answer to a numerical operation was used by two students from each grade.

The other interpretations - use of letter in an algebraic expression (Frankie), use of letter as subdivision label (Frankie, use of letter in axiom (Antoinetta) - cannot be considered here as common since each was expressed by only one student.

2. Concatenation

The second part of the pretest was aimed at verifying whether or not the problem of concatenation is always present for the beginning student.

Q. 2. When I show you something like this $\boxed{3a}$ (a card with the expression 3a written on it) can you tell me what does it mean to you?

Q. 3. Can you give me an example? Can you show me?

Based on Kuchemann's study, the expected responses were that the student might interpret the letter as an object (three a's) or as a first letter abbreviation (three apples), or as a number, or the intended meaning (3 times the number a). The interpretations of the six students were as follows:

Frankie interpreted 3a as a subdivision label: "third problem, first part".

Wendy thought of 3a as 31, a place value interpretation of concatenation, "three one because a is the first letter of the alphabet".

Antoinetta first responded with a subdivision label interpretation ("Number 3, a is the first one), and later with a first letter abbreviation (3 ants).

Yvette interpreted 3a as a subdivision label ("like 1, a, b, c...) and also in terms of place value (90) .

$$\begin{array}{r} 90 \\ -60 \\ \hline 3A \end{array}$$

Filippo assigned to 3a a subdivision label (3a, 3b)

Gail spontaneously assigned three meanings, first that of a subdivision label (3a,3b,...), second that of a first letter abbreviation (a can stand for annex), and third that of place value (30).

These responses are very different from the ones obtained by Kuchemann. Five out of six students interpreted 3a in terms of a subdivision label, an interpretation which never came to light in the British study. Another unexpected response provided by three students (Wendy, Yvette and Gail) was a place value assignment to the letter a . Two of the six students showed evidence of a first-letter abbreviation interpretation.

Q. 3. If I show you something like this $\boxed{3n}$, does it mean the same thing to you? Can you give me an example? Can you show me?

This question was raised in order to verify if the letter n, as opposed to letter a, would induce the students to think of n as a number (based on a first letter abbreviation) and thus result in a subsequent change in their interpretation of concatenation. Another reason for the change of letter stems from Wagner's (1981) discovery of some students assigning a numerical value to a letter according to their rank in the alphabet (as for instance Wendy above).

Frankie did not perceive 3n as being the same as 3a and stated that 3n is "three times a number or three times n".

Wendy thought that the expressions 3a and 3n were analogous in the sense that the letters could be replaced by their rank in the alphabet ("yes, because n can stand for ... whatever number it is in the alphabet") but the two expressions were not the same since the value assigned to the letters were different. Whereas for Wendy 3a was 31, for 3n, "Let's just say it was 13 or 14, I'm not sure, so 313 or 314".

Antoinetta could not at first assign any meaning to 3n since it did not fit into the subdivision label associated with 3a because to her, a classification "It usually doesn't go all the way to n". However, later on going back to viewing 3a as "3 ants" she continued this first letter abbreviation and explained that the n in 3n could be "a number, a missing number or something".

Yvette felt that the change of letter did not make any difference. She maintained the two interpretations she assigned to 3a : thus in the context of a subdivision label, 3n was "like 3a, b, c up till n"; in the context of place value she wrote

$$\begin{array}{r} 681 \\ -652 \\ \hline 3n \end{array} .$$

Filippo continued initially with his subdivision label meaning of 3n, "In question 3 and it goes to a,b,c,... and so on till n". However, the letter n triggered another interpretation as indicated by his statement "Sometimes I guess like 3 times something equals 12 or 13 and 4 ..." which he wrote up as "3n4 = 12". Filippo thought some letters such as n and h (but not a) could be used "like a blank" to express some unknown operation as in the example above "but x they wouldn't use, because it would mean multiplication."

Gail did not think that 3n was the same as 3a since in the subdivision label interpretation "in elementary we don't have n's, we only have 3a or 3b". When asked to assign meaning to 3n, she reverted to her first letter abbreviation interpretation and wrote "3 numerators" for which she gave as example

$$\frac{n + n + n}{12}$$

for which she later assigned different values

$$\frac{\overset{2}{n} + \overset{1}{n} + \overset{4}{n}}{12}.$$

These responses bring out the strength of the subdivision label interpretation. Of the five students who initially used this interpretation for 3a, three of them maintained it for 3n, while for the other two (Antoinetta and Gail) this initial perception created a conflict. Two of the three students who had initially indicated a place value interpretation with 3a gave further evidence of this with 3n (Yvette and Wendy).

The change in letter from a to n appears to have resulted in some unexpected changes in interpretation of the letter in the context of concatenation. Only in the cases of Frankie, who used the intended meaning for 3n (3 times a number), and of Antoinetta, who referred to n as the missing number, was there any evidence that students spontaneously think of n as opposed to a as a number based on first letter abbreviation. The n in 3n resulted in an entirely different interpretation for Filippo, that of a 'missing operation', and for Gail, thinking in terms of first letter abbreviation did not lead her to interpret 3n as 3 numbers, but rather 3 numerators.

Wendy's response reinforces Wagner's discovery. To Wendy 3a is

31 and 3n is 313 or 314. It is interesting to note that for Wendy the letter in 3a and 3n, is not interpreted just as place values, but rather as place holders for numbers in some place value system. The value of the letter is always determined by it's position in the alphabet.

Of the six students only Yvette thought that the expressions 3a and 3n had the same meaning; to Wendy, they were similar in the sense that the letters represented their position in the alphabet, but for the other four students the two expressions could not be accepted as equivalent.

Q. 4. If I ask you to replace here $\boxed{3a}$ the letter a by the number 2, can you tell me what you get?

This question was aimed at determining whether or not those students using a first-letter abbreviation (a for apples) or a letter-as-object (3 a's) interpretations would in fact use multiplication upon being asked to substitute.

Frankie at first responded "I would put 3 times or plus by 2" and wrote '3 × 2'. But substitution brought other interpretations: "or there is another thing (he wrote 3^2), the base 3" and "or also, sometimes there is a number on the bottom, it would show you (he wrote 3_2)". Identical interpretations were obtained by substituting 2 in 3n.

Wendy responded "O.K. thirty-three, because the a is one and adding on the two would be three, or if you just drop the a and put 2 it would be 32". When asked to substitute 2 in 3n she wrote "32 or 17" the latter answer being justified by "I just counted n is 14...14 plus 3...". However, she later reverted to the initial procedure she used for substituting in 3a, "because n is 14 plus the 2, is 16, that would be 316".

Antoinetta departed from her initial interpretations of concatenation and upon substituting 2 in either 3a or 3n responded with "32" thereby indicating a place value interpretation.

<u>Yvette</u> continued with the place value interpretation she had indicated before and wrote "3a = 32".

<u>Filippo</u> also diverted from his initial interpretations of concatenation and responded with 32, a place value meaning.

<u>Gail</u> upon being asked to substitute responded with "thirty-two or three times two or three plus two...or three minus two" and wrote '3 - 2, 3 × 2, 3 + 2, 32'.

The substitution question proved to be most interesting in terms of unexpected responses. All students, except Frankie, used a place value interpretation. Although Wendy, Yvette and Gail had given some prior evidence of this, Antoinetta and Filippo had not. The two students (Antoinetta and Gail) who had indicated that they could use a first letter abbreviation did not continue this when they substituted although Gail as well as Frankie did mention "3 times 2" along with other possible operations.

TABLE 1 Students' Responses to the Three Concatenation Questions

	Frankie			Wendy			Antoinetta			Yvette			Filippo			Gail		
	3a	3n	sub	3a	3n	sub	3a	3n	sub	3a	3n	sub	3a	3n	sub	3a	3n	sub
Subdivision Label	✓						✓	✓		✓	✓		✓	✓		✓	✓	
Place Value				✓	✓	✓			✓	✓	✓	✓			✓	✓		✓
1st Letter Abbre.		✓					✓	✓								✓	✓	
Multiplication		✓	✓															✓
Other			✓											✓				

(sub - substitution)

Summary

What is striking in these results is that the two most prevalent interpretations chosen by five out of the six students (subdivision label and place value) did not appear at all in Kuchemann's study. The strength of these interpretations is evidenced by their consistent use: the subdivision label was used twice by four of the students, whereas the place value interpretation was used three times by two students and twice by another.

While none of our subjects used letters as objects in this context, three students did indicate a use of literal symbol as a first letter abbreviation. Two of these three students did in fact indicate that 3a meant multiplication when a was replaced by 2. However, this substitution task brought the place value interpretation among two other students who previously had not made any reference to it.

Interestingly enough the change of letter from 3a to 3n only induced one student (Frankie) to think of multiplication while another one (Antoinetta) did mention "number" without any reference to multiplication and, a third one (Gail) looked for a word starting with the letter n and found "numerator". Thus the change of letter did not provide the expected evidence of change to a numerical referent in the students.

Almost all the students (5 out of 6) did not perceive 3a as being the same as 3n. Antoinetta and Gail explained it in terms of classification not usually extended to n parts; Frankie shifted from subdivision label to multiplication; Filippo shifted from subdivision label to an arithmetic operation; Wendy perceived them as different since to her the letter represented a number determined by its position in the alphabet.

3. Letter in the Context of Addition

In these questions all possible problems due to concatenation were avoided. However, the symbol for addition now replaced the letter in an arithmetic context. The addition could only make sense if the letter is interpreted as a number. This could depend on the letter used, which explains the transition from b to n. The difference between the letter and the placeholder was also explored.

Q. 5. When I show you something like this $\boxed{4 + b}$, can you tell me what it means to you?

The following three possible answers were expected:

a) 4 plus b b) I don't know c) 4 plus some number

In the case of answers (a) and (b), where the word "number" was not used explicitly, the questioning proceeded with:

Q. 5.a,b. What do you think the letter could mean?

The expected responses were:

a') I don't know b') some number

In the case of answer (a'), the student was further prodded with:

Q. 5a' Do you think it is possible to add a letter with a number?

As soon as the student referred to the letter as representing a number he was asked:

Q. 5b',c. Can you give me an example?
Can you give me another example?

Frankie explained "it means 4 plus a number, equals something - the answer" and gave as an example "Like 4 plus 2 equals 6".

Wendy answered "yes, 4 plus b would be 4 plus 2, because b is the second letter in the alphabet." She thought that b had to be 2 but that "b can be the abbreviation for bananas" provided that it was preceeded by a numeral like "4 bananas".

Antoinetta replied "...it means 4 plus like a number, like a number is missing" and gave us examples 4 + 44 and 4 + 11 .

Yvette thought "The b represents a number, you have to find the number b" but claimed that "it can be 4 plus any number" and wrote '4 + 6' and '4 + 5'.

Filippo replied, "It would mean 4 plus something equals a certain number", and wrote '4 + b = 12', and explained that "it could mean a blank space of missing number". He thought that the value of b could be different as in his other example '4 + b = 32'.

Gail's initial response was "just b, but you can put any number here but you have to look for an answer" and she wrote '4 + b = 7'. She later stated "You can put any number if it is not an equation" and wrote '4 + 2, 4 + 1, 4 + 7'.

Without any exception, all six students responded to the context of addition and thus perceived b as a number. None of them expressed any cognitive conflict due to the presence of a letter and a number, as might have been expected on the basis of the pilot study.

Noteworthy is the fact that three students (Frankie, Filippo and initially Gail) transformed the expression into an equation. Whether this was due to their inability to accept the lack of closure (Collis' ALC) or to their prior exposure to equation cannot be determined.

In terms of the students' perception of the literal symbol, we find here a wide array of interpretations. To Wendy, since letters are just representing their rank in the alphabet, b is not even an "unknown number", it is more a constant value. To Frankie and Filippo, the expression '4 + b' needs to be "closed" into an equation, and within such a context, the letter becomes a "specific unknown" (c.f. Kuchemann, Collis) since it has but one numeral solution in any given equation.

The three other students (Antoinetta, Yvette, and Gail) indicate that they have a broader perspective, that of <u>generalized number</u> in the sense that the letter can represent either more than one number or "any" number. Most interestingly, it is also these last three students which gave evidence that they could accept the lack of closure for '4 + b'.

Question 6 was the same as Question 5 except that the letter was changed from <u>b</u> to <u>n</u> in order to determine if more students would view the letter as a number.

Q. 6 When I show you this $\boxed{4 + n}$, can you tell me what it means to you?

(a, b, c as in 5)

6(a) or (b) What do you think the letter could mean?

6c Do you think there is any difference between $\boxed{4 + b}$ and $\boxed{4 + n}$?

<u>Frankie</u> stated, "They mean the same thing, because they both stand for a number".

<u>Wendy</u>'s interpretation shifted from her previous "alphabetical" one. She now thought "that would be 4 plus <u>n</u>, you would have to have another number after the <u>n</u>, like an equal sign" and wrote '4 + n = 6'. She did not think there was any difference between '4 + b' and '4 + n' since in both cases one had "to replace the number".

<u>Antoinetta</u> thought that '4 + n' was "the same as the other one" and that there was no difference between the two expressions.

<u>Yvette</u> responded that '4 + n' was "4 plus n, that could be any number". She felt there was no difference between '4 + b' and '4 + n' because "they both stand in for a number".

<u>Filippo</u> answered "It means 4 plus something equals something" exactly the same response as for '4 + b'.

<u>Gail</u> again first thought of an equation, as evidenced by her reaction to '4 + n' : "The same thing, like you have to find another number to fit the equation" as in "4 plus what equals, you can put any number there". When asked why in the present example she saw no difference while she did for <u>3a</u> and <u>3n</u>, she responded "because there is no sign".

Since in '4 + b' all students perceived the letter as a number, it is not surprising that for all but one pupil (Wendy) no change of interpretation resulted from the change in letter. For Wendy, this change initiated a new interpretation, that of an unknown number in an equation rather than the alphabetic rank of the letter. Gail's last comment hints at a plausible explanation for the difference perceived between <u>3a</u> and <u>3n</u> and the lack of it between '4 + b' and '4 + n' since in the former "there is no sign" indicating a possible operation.

The final set of questions in this section intended to determine whether the students' interpretations of letters were similar to that of placeholders, the box, or if not, what were the differences.

Q. 7 When I show you something like this

$\boxed{5 + \square}$

Can you tell me what does it mean to you?
(expected response was 'a box to fill in with a number')

Q. 7a Can you give me an example?
(expected response - puts in a number)

Q. 7b Do you think that these two mean the same thing,

$\boxed{5 + \square}$ and $\boxed{5 + c}$?

a) no b) yes

Q. 7c (If no) What do you think is the difference?
(If yes) Do you think the letter <u>c</u> and the box can be used the same way? Example?

<u>Frankie</u> responded that 5 + □ meant "also I think it means the same thing, like 5 plus something equals". He gave as examples '5 + c = ' and '□ + c', the latter he transformed into an equation '5 + 2 = 7'. Like um, mostly when you have an equation, you can put a <u>c</u> or a box".

<u>Wendy</u> sought a numerical value for the box "this would be one (referring to the length of each side of the box), each side would be one, so that would be 4". She maintained that the numerical value of the box was

determined by its perimeter. '5 + □' and '5 + c' were the same. "For here you just add up the sides like I did, 4, here 5 plus c, when I said before it was the third letter in the alphabet, according to this I think you could put 4 also, because there are both 5, so maybe you put 4". In this context the box and the c could be "just 3 or 4".

Antoinetta answered that "usually when you have a box it means to put another number there", and gave as example '5 + 5'. She responded that '5 + □' and '5 + c' were not the same since "that's a letter and that's a box". When prodded further as to their differences she stated, "well that one means there is a missing number (referring to the '5 + □') and that one you have to say what number c represents.

Yvette stated that 5 + □ meant "to put the number you get inside the box." and gave as examples: 5 + □ , 5 + 4 ." or it can be 3, any number." She showed a fine distinction between '5 + □' and '5 + c' "this means you have to put the number in the box; but this means to put the number there." As examples she wrote '5 + c = 5 + 4'; and she wrote '5 + □ and 5 + 4 ', side by side.

Filippo responded that '5 + □' meant "Find the number that goes in the square" and as in the previous questions he transformed the expression into an equation and wrote 5 + □ = 28 . "And it would equal 23". He stated that '5 + □ and 5 + c' "both mean you have to find out the number that is in the space there." He gave the following equations as examples:

$$5 + \underset{\textstyle 5}{\underset{\downarrow}{\square}} = 10 \qquad\qquad 5 + \underset{\textstyle c = 13}{\underset{\downarrow}{c}} = 18$$

Gail maintained the same two interpretations she had for the literal symbol, and initially stated "you can put any number here, just like the same thing, instead of a letter you put a square, you just have to find the number that fits in the equation." But she later added that if there were no equation "then you can put any number."

Except for Wendy, the change in symbol from a letter to a box did not result in any significant change in interpretation of the symbol used in this additive context. Even in the case of Wendy, it was merely a question of how she would determine the specific or 'constant' value of the different symbol - the box. While the value of a letter could be determined by its position in the alphabet, the value of a square required the measurement of its perimeter.

The three students who had perceived the prior problems within the context of equation (Frankie, Filippo and initially Gail) consequently saw no difference between the box and the letter. To Wendy also the box and the letter were the same since they both represented specific values. Antoinetta and Yvette, however, who had used a "generalized number" interpretation of the letter in '4 + b', perceived subtle differences between the two symbols. To Antoinetta the box initiated a more general response that of "missing number", a void to fill, but for c in '5 + c', "you have to say what number c represents". From this last comment we cannot infer whether Antoinetta was thinking of a specific unknown or of a generalized number. Yvette saw the difference only in terms of form, that is, in the case of the box the answer had to be written inside a box.

TABLE 2 Students' Interpretation of the Literal Symbol within the Context of Addition

	Frankie					Wendy					Antoinetta					Yvette					Filippo					Gail				
	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c	4 + b	4 + n	4 + b same as 4 + n	5 + □	5 + □ same as 5 + c
Specific Unknown	✓			✓			✓														✓	✓		✓		✓	✓		✓	
Generalized Unknown											✓					✓	✓		✓							✓			✓	
b and n same or different			✓					✓					✓					✓					✓					✓		
Alphabetical Rank						✓																								
Box as Perimeter									✓																					
Box and c same or different					✓					✓ ✓					✓					✓ ✓					✓					✓
Insufficient Data for Classification		✓										✓		✓																

Summary

In contrast with algebraic expressions involving concatenation, those expressions in which the algebraic operation is explicit elicit from the students a numerical referent. For the expression '4 + b' , all six students thought of b as a number. The openness of the expression did not seem to cause any cognitive conflict in the sense of being rejected by the pupils. We did not get any response such as, "you can't add a letter to a number". The numerical referent for the letter was used immediately. However, three of the six students (Frankie, Filippo and initially Gail) felt the need to close the expression by transforming it into an equation. Gail, along with two other students (Antoinetta and Yvette) expressed their interpretation of letter as "generalized number" by either referring to the letter b as "any number" or by substituting more than one value for b in the expression without closing it.

All six subjects signified that '4 + n' was to them the same as '4 + b'. This is not surprising since the initial referent for the letter was numerical. We did not go further than this response to gather more data which might have enabled us to ascertain in all cases if the initial interpretations for b as a specific unknown or generalized number were carried over for n .

This last comment must be qualified in the cases of Wendy and Gail. For indeed, both expressions ('4 + b' and '4 + n') were viewed as "the same" only in the sense that the letters represented numbers. However Wendy shifted her alphabetical rank interpretation for b to that of specific unknown for n . Gail expressed both a specific unknown and a generalized number interpretation for <u>b</u> but only the former for <u>n</u>.

Regarding the question of whether or not students perceive a letter as <u>equivalent</u> to a placeholder in an expression, the data seems to indicate that in general they do to some extent but also that there is some ambivalence. Five out of six students give such evidence although Wendy's and Yvette's responses can be viewed as contradictory. Although Wendy claims that '5 + □' and '5 + c' are the same, she determines the value of the box in terms of the box's perimeter, while the letter <u>c</u> is associated with the number 3, it's alphabetical rank. On the other hand, Yvette claims that the two expressions are different but the difference appears to be one of form, not of meaning, since she uses both expressions in a similar manner.

As pointed out earlier, in the case of '4 + b', three students (Frankie, Filippo and initially Gail) needed to transform the expression into an equation. Wendy followed a similar pattern with '4 + n'. This need to change an expression into an equation can be interpreted as evidence of the students inability to accept the lack of closure involved in an algebraic expression. Collis' criterion of the number of substitutions made by a pupil does not apply here since such a transformation reduces the letter to a specific unknown. As mentioned previously, although this transformation seems to be another criterion for ALC, nevertheless, one cannot determine if it is due to their prior exposure to equations.

Two of the students (Antoinetta and Yvette) did not transform the expressions into equations and thus could assign different values to the literal symbol, hence indicating a 'generalized number' interpretation.

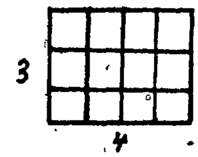
Gail, on the other hand, could move from specific unknown to generalized number by simply considering the expression either as an equation, or if there is no equation, putting in "any number". From these observations, it appears that our criteria for ALC, the need to transform an algebraic expression into an equation, is a determining factor in the student's perception of the literal symbol. If he needs to perform this transformation, he is inevitably bound by the letter's role of a specific unknown. If this transformation is not performed then the context remains open for a generalized number interpretation.

This discussion brings into question the validity of the approach followed in our teaching outline where the algebraic expressions use letters as specific unknowns. For indeed, one could argue that Antoinetta, Yvette, and Gail had the same background as the other students and yet managed to forego the closure of the expression thus achieving a perception of letter as generalized number. But we have absolutely no idea of how this was achieved and the problem of bringing the other students to this level remains complete. However, the fact that in our teaching outline the "answers" to the problems remain in the form of algebraic expressions seems to be a possible stepping stone since these expressions need not be transformed into equations.

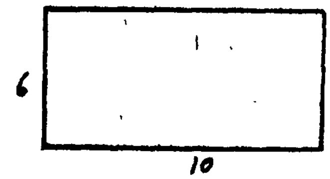
4. Geometric Context

This part of the pretest changed the context of the literal symbol and placed it within a geometric one. The questioning proceeded as follows:

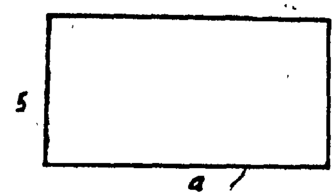
Q. 8a. Can you tell me the area of this rectangle?



Q. 8b. Can you tell me the area of this rectangle?



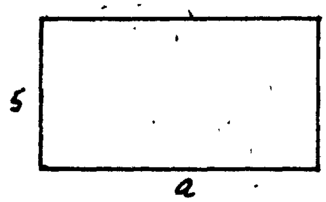
Q. 9. Can you tell me the area of this rectangle?

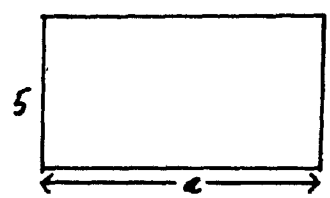


Q. 9a. What do you think the letter <u>a</u> stands for?

The following three possible answers were expected: side, the base, the length.

Q. 9b. Do you think the letter <u>a</u> means the same thing in these two rectangles?





The questions aimed at determining whether the students used a geometric label (side) or a length (numerical) interpretation for the literal symbol, and also whether the students could use the letter in formulating the area of the rectangle. The last question was included to verify if the students would interpret the arrows as a convention to denote length. The responses were as follows:

Frankie viewed the a as the length of the base. He stated "you would have to measure this side, to find the number you would times it by". He hinted at a possible distinction between the use of arrows and their omission "this shows you a the length (with arrows) and this just shows you you have to get the number (without arrows), they don't show you the arrows." However, he ultimately saw no difference.

Wendy maintained her 'alphabetical' interpretation for letters and responded that the area of the rectangle was 'five'. "In area you times, and it is five times a and a is one because it is the first letter in the alphabet, so five times one is five." The presentation of the two rectangles, with and without arrows, resulted in a first letter interpretation for a - to add, - an interpretation she did not pursue. However, the arrows brought on the notion of measurement "You would have to measure, I guess". Ultimately Wendy concluded that a could be "one or the measure of the length".

Antoinetta viewed a as the length of the base and estimated it to be "6 or something like that". She responded to the two different notations for the length of the side "yes, well I think they could mean the whole ...yes, it means the same".

Yvette focused on the definition of area as the number of squares it takes to cover a given surface and responded that "there are 5 rows across, but there isn't any number, you don't know how many are there, so you have to find that number there". The a respresented "the amount of lines going up." The arrows induced Yvette to think in terms of horizontal as opposed to vertical lines.

Filippo estimated the length of the base to be 9. And as he had done previously formed an equation and for the area wrote '$5 \times a = 45$'; '$5 \times 9 = 45$'. The letter a in both contexts, with and without the arrows, to him meant the length of the base.

Gail estimated the a, with and without the arrows, to be 6, and responded for the area of the rectangle "it could be about maybe 30".

Within this geometric context, all students used a numerical referent for the literal symbol. Five of the students thought of the a as the length of the base, only Yvette used a different numerical meaning, "The amount of lines going up".

Since four of the six students initially perceived the a in the rectangle as representing the length of the base, the additional inclusion of the convention of arrows to denote length did not result in any change of interpretation. For Wendy, however, the arrows definitely sparked the idea of measurement, that of length, and the letter then took on two values, either, one, the letter's rank in the alphabet, or the measurement of the length of the base. Yvette viewed the <u>a</u> in terms of the number of "lines going up".

It is interesting to note that none of the students wrote '$5 \times a$' as the area of the rectangle - an open algebraic expression. Three of the students substituted estimated values for <u>a</u> (Antoinetta, Filippo aand Gail). Frankie and Wendy suggested the need for measurement. However, Wendy claimed that <u>a</u> could also have a value of one, based on its rank in the alphabet. Yvette did not seem to be able to express the area of the rectangle, since the letter indicated she was missing a number. Filippo, was the only subject who did write at one point '$5 \times a$', however, as he had done previously, he closed the expression by transforming it into an equation, '$5 \times a = 45$'.

The situation here is in contrast to the previous additive problem where two of the six students (Antoinetta and Yvette) were able to accept the lack of closure of expressions such as '$4 + b$' and '$4 + n$' since they did not have to transform these expressions into equations. In the geometric context, none of the students were able to accept the lack of closure as evidenced by either their need to substitute values for <u>a</u> or their reference to the need for measurement. But this difference could be explained by the fact that the pupils viewed <u>a</u> as the length of the base (even Yvette "amount of the lines going up" is associated with length), thus implying a specific but unknown number.

Hence the students were more prone to substitute an estimated value or simply leave the area question open until the measure was determined. Another factor which cannot be ignored in comparing the two situations is that in the area problem the student was expecting to generate the algebraic expression whereas in the additive situation the algebraic expression was presented to him, notwithstanding the fact that the operations were different.

These results again bring into question the validity of our approach used in preparing the teaching outline. After all, if a geometric context induces a greater need for closure, why base a whole teaching outline on it?

If we had attempted, as we did in the pilot study, to move directly from a standard geometric approach in order to induce the algebraic expressions, this objection would have been quite valid. In fact, the pilot study showed that it created cognitive problems such as "How can you multiply a letter with a number?" The intermediate step which was added to the revised outline involves problems in which one of the dimensions is not only unknown but actually hidden. This hiding by force, prevents the student to estimate or measure the unknown quantity which in his mind nevertheless exists, even if he cannot see it. Thus the need for a convention to express the total number of dots, the length of a line, or the area of a rectangle is being established. And since initially this hidden quantity is represented by the usual open box (□), the ensuing introduction of the letter is within the context of a placeholder for a hidden quantity. Thus the letter takes on a meaning different from the one in the geometric situation involved in the pretest where the data shows students view the literal symbol as a specific unknown but visible measure.

5. Kuchemann Test

Question four of Kuchemann's test was included in the pretest in order to give us some comparison with his results. Since the students in our study were 11 and 12 years old and had little if any exposure to formal algebra, it was decided to compare their responses with Kuchemann's 'Level 2' students. (Personal communication to N. Herscovics, 1975) In his study the 'Level 2' students were 13 year-olds who had had two years experience in algebra.

The following questions were presented to the students, each question was written on a card.

Q. 12a. Add 4 onto 8 All students gave the correct response - 12.

Q. 12b. Add 4 onto n + 5

		Explanation
Frankie	4 + 5 = 9	No explanation was given.
Wendy	4 + 5 = 9	"I took the n and changed the n to 4 and 4 plus 5 equals 9"
Antoinetta	13; 4 + 4 + 5	"There is a 4 there so maybe I thought..."
Yvette	4 + n + 5; $4 + \overset{6}{n} + 5 = 15$	"The n can be anything,... let's say it is a 6—15. That can be any number. It just tells you, you have to find the number."
Filippo	4 + n + 5 = 15 4 + ▩ + 5 = 15	To Filippo, "the value of n depends on what you are going to put here (referring to the 15)..."

Gail	'nine'	Referring to the n "It can be zero or you can put that number (4)"

In Kuchemann's study 61% of the Level 2 students provided the correct response to this question, whereas 20% gave '9' as an answer, and 3% answered by giving "a specific obvious numerical answer". In our study, only one student, Yvette, was able to write the answer in the form of an unevaluated algebraic expression '4 + n + 5'. Although she ultimately assigned a numerical value for the n, (6), she maintained her previous interpretation of the literal symbol that "the n can be anything". Three of the subjects (Frankie, Wendy and Gail) ignored the letter and gave '9' as an answer corresponding to the 20% of Kuchemann's students with the same response. A "specific obvious numerical answer" was given by Antoinetta that is, 13. Filippo, as he had done previously, closed the expression '4 + n + 5' by transforming it into an equation, '4 + n + 5 = 15', and subsequently solved for n.

Q. 12 Add 4 onto 3n

		Explanation
Frankie	For 3n wrote: $4 + 3_2$ For 3N wrote: 4 + 35	Frankie interpreted small letters as subscripts, which may reflect his prior introduction to bases other than 10. Thus the question was changed to '4 + 3N'.
Wendy	4 + 3 = n	"I took the n as the answer and I made it equal n...since Grade 1 we used letters to show what plus what equals... we would usually use n..."

Antoinetta	40 4 + 38 = 40	She questioned whether n "always have to be the same number" because when she worked with n it wasn't a number, it was always the answer".
Yvette	4 + 3n 4 + 31 = 35	"That could mean thirty something. (referring to the 3n) 35, 36, or 37... It could be any number. That tells you there is a number there."
Filippo	4 + 3n = 39 4 + 35 = 39 4 + 38 = 42	He claimed you could have put any number (referring to the 39) "as long as you put the right number there".
Gail	4 + 30 = 70 4 + 32 = 36	"You can put any number there".

Kuchemann's results indicate the following: 7n or 12n was the most common response (41%) while 21% gave the correct answer and 17% ignored the letter and wrote 7 or 12. Wendy's response, '4 + 3 = n', appears to be similar to the latter, however, the equation she formed included the letter n, because as she stated "...since Grade 1 we used letters to show what plus what equals...we would usually use n ...". The five remaining students provided an interpretation for the n which Kuchemann did not observe, they used the place value interpretation for the n in 3n, as evidenced by the fact that all their numerical values remained in the 30 to 39 range.

Within this place value interpretation, one must note some differences between Gail and Yvette on one hand and Filippo on the other hand. While remaining in the place value context, Gail and Yvette still maintained that n could be "any number", but they both first substituted and then evaluated the expression. In contrast, Filippo first transformed the expression into an equation ($4 + 3n = 39$) and then proceeded to solve for <u>n</u> in <u>3n</u>.

That the students used a place value interpretation in this task is not too surprising. If, as shown by the answers to Part 2 of the pretest, an expression such as <u>3a</u> is viewed by most students as a classification label, within the numerical context provided by the request to substitute, all the students fell back on the only arithmetic perspective they knew, that of place value. Hence, the instruction "add 4 onto <u>3n</u>" refers them to an arithmetic operation and immediately brings them back into a numerical framework initiating the place value interpretation.

Q. 13a. Multiply 8 by 4 All students responded correctly - 32.

Q. 13b. Multiply $n + 5$ by 4

		Explanation
<u>Frankie</u>	$4 + 5 = 9$	"You take the 5 (pointing to the 4) and you put it here."
<u>Wendy</u>	No response	"How can you multiply 'n + 5' ...when there is a plus?...You can't add and multiply 5, you can add and multiply 5 but you would need another number".
<u>Antoinetta</u>	$3 + 5 \times 4$	"I picked 3 because the other one was 5 so..."
<u>Yvette</u>	$4 \times n + 5$ $4 \times n + 5 = n$ $4 \times 5 + 5 = n$ $4 \times 10 = 40$	"...it means you times <u>n</u>. I guess you take 4 times, this could equal 4 times n + 5, this could equal 5, or 6 or 7 (referring to the first <u>n</u>), could be any number..."

<u>Filippo</u>	$(n + 5) \times 4 = 28$ $(2 + 5) \times 4 = 28$	
<u>Gail</u>	20 $3 + 5 = 8 \times 4 = 32$ $7 + 5 = 12 \times 4 = 48$ $4 + 5 = 9 \times 4 = 36$	"<u>n</u> can be anything"

Kuchemann's study revealed a wide variety of responses to this question, with only 8% giving a correct one. The most common answer was n + 20 , n + 9 (39%), while 16% chose to ignore the letter and gave the answer 9 or 20. Another 12% gave answers such as '4n + 5 , n + 5 × 4 , 4n5 , n54 , n + 54', and 7% provided specific numerical answers.

None of the students provided the response that was most common in Kuchemann's study. Only one student, Frankie, chose to ignore the letter (in the sense of Kuchemann) and gave '9' as an answer. This is in contrast not only with Kuchemann's results, but also to the first question in this section 'Add 4 onto n + 5' where 3 students (Frankie, Wendy and Gail) had ignored the letter. The change in operation from addition to multiplication could in some way account for this difference.

For Wendy the presentation of two unclosed operations appears to have created a cognitive conflict which interferred with her ability to provide a response, as she explained: "How can you multiply 'n + 5'... when there is a plus?... You can't add and multiply 5, but you would need another number". It is quite obvious that Wendy does not view <u>n</u> as a number and hence she cannot perform two operations on two numbers. Thus Wendy also ignored the letter here, just as she did previously in the question "add 4 onto n + 5" where she responded '4 + 5 = 9'.

Yvette once again managed to initially write an open expression (even if incorrect) viewing n as "any number". However, as shown in the second line of her work $(4 \times n + 5 = n)$ she used n simultaneously to represent both "any number" and also the answer. Yvette's use of n as an "answer" reflects many such comments obtained in the first question of this pretest. Quite interestingly, Yvette substituted 5 for n in the expression yielding 40 as the answer for the other n.

Filippo was consistent in this problem. He once again transformed the algebraic expression into an equation, drawing in the necessary brackets, and solved for n.

Similarly to the 7% of Kuchemann's students, three students in this study provided specific numerical responses, Antoinetta, Yvette and Gail. Antoinetta, however, did not arrive at a final numerical answer, but left an open arithmetic expression. Gail, however, suggested that "n also could be anything", and subsequently illustrated this by writing three more different numerical substitutions. Her written work is quite interesting for it reflects the operational nature of her thinking. Although $3 + 5 \neq 3 \times 4$ she simply wrote the answer '3 + 5' and proceeded with the multiplication. This kind of evidence is similar to Kieran's findings (Kieran, 1979).

Q. 13c. Multiply 3n by 4

		Explanation
Frankie	$32 + 4$	
Wendy	$3 \times 4 = 12$	"The n became a 4...and then I multiplied."
Antoinetta	$32 \times 4 = 128$	

Yvette	$3n \times 4 = 32$ $\underline{\times\ 4}$ 126	"The <u>n</u> means a number you are suppose to put there".
Filippo	$3n \times 4 = 128$ $3\square \times 4 = 128$	"If you put the <u>n</u>, then you can't put the answer there, so you just put a box, you are using that for an empty space, and then you just put the number in".
Gail	$30 \times 4 = 120$ $31 \times 4 = 124$	"<u>n</u> can be any number".

Kuchemann's results for this question are: 39% correct responses. 17% ignored the letter, 11% gave as answers <u>7n</u> or <u>12n</u>. Our results contrast with those of Kuchemann's since all our students, except Wendy, used a place value interpretation as shown by the fact that they wrote numbers between 30 and 39. Wendy continued to ignore the letter and '$3 \times 4 = 12$'.

The following table summarizes the responses given by our six students and will facilitate a comparison of the various tasks.

TABLE 3 Responses given by the Six Students to Kuchemann's Question 4

	Frankie				Wendy				Antoinetta				Yvette				Filippo				Gail			
	12b	12c	13b	13c	12b	12c	13b	13c	12b	12c	13b	13c	12b	12c	13b	13c	12b	12c	13b	13c	12b	12c	13b	13c
Ignores letters	✓		✓		✓	✓	✓ *	✓													✓			
Uses a Place value Interpretation		✓		✓						✓		✓		✓		✓		✓		✓		✓		✓
Substitutes &		✓		✓					✓	✓	✓	✓	✓	✓	✓	✓						✓	✓	✓
Evaluates									✓	✓		✓	✓	✓	✓	✓						✓	✓	✓
Transforms to Equation and then solves for n																	✓	✓	✓	✓				
Writes an open algebraic expression													✓	✓	✓	✓								

* no answer given; interpretation based on her statements

12b Add 4 onto n + 5
12c Add 4 onto 3n

13b Multiply n + 5 by 4
13c Multiply 3n by 4

This table indicates a striking consistency in the students' responses. Four of the six students responded to requests for operations on the algebraic expressions by substituting numerical values for the literal symbol and then evaluated the resulting arithmetic expressions. The other two students were also very consistent. Wendy ignored the letter in all four problems, Filippo transformed all the expressions into equations and solved for n .

This consistency is even more obvious if we compare questions 12c (add 4 onto 3n) with 13c (multiply 3n by 4). In both problems the students' responses were absolutely identical. Except for Wendy who ignored the letter, all other students used a place value interpretation. It is interesting to note that Kuchemann's results did not indicate such a strong similarity in the responses given to these two questions. However, the explanation for this disparity is quite simple. To our students, there was but one operation involved since they did not view 3n as $3 \times n$ whereas the British students did.

Although there is consistency between 12b (add 4 onto $n + 5$) and 13b (multiply $n + 5$ by 4), it is not as strong as in the other two problems. As shown by the table, four students (Frankie, Antoinetta, Yvette, and Filippo) gave identical responses to both questions. Only Wendy and Gail showed some change. While for 12b Wendy ignored the letter (she gave 9 as an answer) for 13b we can only infer this from her statement - "...you would need another number". Gail on the other hand ignored the letter in 12b (she gave 9 as an answer) but for 13b used the letter n as indicated by the multiple values she assigned to it.

Summary

The difference in our results and those of Kuchemann's are to be expected since the students in our experiment had little, if any, formal instruction in algebra, whereas Kuchemann's students had two years of such instruction. Thus the responses we observed are more an indication of the spontaneous interpretations assigned by the beginning algebra student to algebraic notation and operations, rather than a measurement of the students' retention of learnt material as was the case for Kuchemann's test. For instance, unlike the British students, our novices gave a place value interpretation for concatenation.

What is quite striking in this part of the pretest is the consistency of our students' responses to the four questions. It indicates that our subjects are not responding erratically nor arbitrarily. They assign a logical meaning to the problem at hand and solve it in accordance with this meaning.

It would be tempting to compare here the changes in our students' acceptance of lack of closure (ALC) and in their notion of generalized number observed in Section C.3 (What does '4 + b' mean to you?). For instance in that section three students showed their inability to accept the lack of closure by transforming the algebraic expression into an equation whereas in the Kuchemann questions only one student (Filippo) maintained this pattern. However, a comparison is not recommended in view of the different instructions given to the students. In the questions of part C.3 the students were asked for "meaning" whereas in C.5 they were asked to perform operations on algebraic expressions. If future comparisons are desired, they should be of the form "What is the meaning of '4 + (n + 5)'? ;'(n + 5) × 4'? These expressions involve two operations and the brackets clearly indicate the sequence to follow.

6. <u>Simplification and Combining Like Terms</u>

This part of the pretest focuses on two aspects of simplification: 1) given two like terms, to simplify (to combine) them into one term, and 2) the combining of like terms in the presence of unlike ones. The following questions focus on the first aspect. (It should be noted that in the pretest these questions preceded those of Section C.4)

i) <u>Combining Two Like Terms</u>

Q. 10 Look at this card: $\boxed{\text{Simplify } 2a + 5a}$
Can you tell me what you think it means?
(expected responses were - Put them together or 7a)

Q. 10a Can you do it?

Q. 10b Can you explain how you got that answer?
(expected explanation a's or apples)

Q. 10c (if a's) What do you mean by 2 <u>a's</u> and 5 <u>a's</u>?

Q. 11 Can you do this one? $\boxed{\text{Simplify } 4n + 3n}$

Q. 11a What do you think the letter <u>n</u> stands for?
(expected response 'number')

Q. 11b Can you give an example? Another?

These questions were included in the pretest because a simplification problem does not require the perception of letter as a number. We expected to possibly observe different interpretations for the literal symbol such as its use as a 'first-letter abbreviation'. Also, since concatenation is involved in these expressions, these questions could also be used to verify the responses the students gave to Question 2, where they were asked the meaning of <u>3a</u>:

Once again we changed the letter from a to n in order to determine if the letter n would spark the idea of number, and that consequently the students would change their responses from a 'first-letter abbreviation' for a to a numerical one for n .

The following were the responses given by the six students:

Frankie initially responded to the simplification of '2a + 5a' by writing $2_3 + 5_4$, again using his 'base' interpretation as previously observed, and maintained that '4n + 3n' meant the same thing. However when the n was changed to a capital letter, '4N + 3N', he wrote '44 + 33 = 77'. Even within a fixed context he did not feel that the same letters had to have the same value.

Wendy returned to her 'alphabetical' interpretation of literal symbols and wrote '2a + 5a = 72' with the explanations "I used the a as one in both cases, 21 plus 51 equals 72". Similarly for '4n + 3n', she wrote: 'n = 14, 414 + 314 = 728'. "I put n equal to 14, because it was the fourteenth letter in the alphabet." She felt a had to be 'one' and n had to be 14.

Antoinetta responded to 'simplify 2a + 5a' by saying, "Maybe it is trying to tell you to find the answer..." and wrote '25 + 52'. After further questioning she concluded referring to the two a's "Probably it has to be the same...that is why they have the other letter the same." and then wrote '21 + 51'. For '4n + 3n' she wrote '48 + 38'. She stated that she didn't know what "to simplify" meant, but finally concluded that her answers were simplified, "I think this (48 + 38) is simpler than this one, (4n + 3n) because they are the same numbers."

Yvette answered by saying "I think it means 2, and the a can stand for any number." She wrote '2a + 5a; 24 + 55'. She felt that the a's could be different numbers within the same problem, and went further and said "It can be 2a and 5n or something, and it would be the same thing, it would be two different numbers, it doesn't matter what you put." For '4n + 3n' she wrote '44 + 33' and '44 + 35'.

Filippo stated that it meant "...twenty something plus fifty something" and wrote '2a + 5a = 78; $2\boxed{2} + 5\boxed{6}$ = 78'. He felt the a's could be the same or different "...as long as you put the number that would equal the last number there, 78." For '4n + 3n', "It's the same thing as the last one" and wrote '4n + 3n = 71: $4\boxed{0} + 3\boxed{1}$ = 71'. He stated that any letter can be used "It could have been an n , it could have been an a".

Gail initially wrote 23 + 51 = 74, but when questioned further as to the use of the same letter she said "yes, it could be the same number, this could be 3" and then concluded "no...it has to be the same number."

changing her answer to read '23 + 53 = 76'. Then for '4n + 3n' she felt "It is the same thing, but you have to get the same number." However, she ultimatley concluded "it could be the same number, or different numbers." depending "on how you were taught to do it. If they had directions to fill in n , the same number, say n is equal to 3, that has to be 3 and that has to be 3." An attempt by the interviewer to initiate a first-letter abbreviation interpretation (as she had used for 3a) was rejected by her.

It is interesting to note the consistency of the responses observed. All students used the place value interpretation for concatenation. For Wendy it was a placeholder for numbers in some place value system, as for example 4n which she interpreted as 414. The previously observed 'subdivision label' interpretation and the expected and previously observed 'first letter abbreviation' interpretation for concatenation (ref. Sect.C.2) was not used by any of the students. However, the expressions in this section differed from those of Section C.2 in that these explicitly indicated an operation, to add; (2a + 5a). Therefore, it is not surprising that the students fell back on the only numerical referent they had for concatenation, that of 'place value'.

The responses given show that a majority of the students (4 out of 6) did not feel that the use of the same letter within the same algebraic expression had to have the same value. Two students, Antoinetta and Wendy, thought that the repeated use of the same letter in an expression meant the same number. Antoinetta said "Probably it has to be the same... that is why they have the same letter." However, for Wendy to use the same letter was merely a natural consequence of her premise that the value of a letter is determined by its rank in the alphabet. Thus, to Wendy, a fixed context is not the determining factor, for a and n would have the same value in any context - any expression.

It is not surprising that Yvette maintained that the same letter in a fixed context can take on different values. This response reflects her perception of the literal symbol in the sense that it can represent 'any number', with no inherent restrictions as to context.

All students felt that there was no difference between the two problems, '2a + 5a' and '4n + 3n'. This can be explained by the fact that the students had all used initially a numerical referent for the <u>a</u> in '2a + 5a', and consequently the change in letter from <u>a</u> to <u>n</u> did not result in any change in interpretation. They merely maintained their same numerical referent - that of place value.

The responses given by the six subjects also indicate that the instruction to 'simplify' is not easily understood by the beginning student. Five students responded to this instruction by substituting values for the letters. Frankie, Wendy and Gail followed the substitution with an evaluation of the expression, while Antoinetta and Gail left open arithmetic expressions. The instruction to 'simplify' also did not alter Filippo's procedure of forming equations.

ii) <u>Combining Two Like Terms in the Presence of Unlike Ones.</u>

The next set of simplification questions involved the combining of like terms in the presence of unlike ones. These questions were included to determine if the student would not only combine like terms but also would 'collapse' unlike ones, that is, write '5a + 3b as 8ab'. (See Chapter 1, p. 9). We were also interested in investigating what meanings the student would give to the presence of two different letters in the same expressions.

Q. 14a Can you simplify $\boxed{3a + 4a + 5}$?

Q. 14b Can you simplify $\boxed{2c + 3d + 4c}$?

Q. 15 Why do you think two different letters are used in the second one?

Q. 15a What do you think <u>c</u> could mean?
<u>d</u> could mean?

The responses given by the students were as follows:

<u>Frankie</u> wrote '45 + 33 + 5' and explained that the value of the letters in '3a + 4a + 5' would be determined by extra problems, for example "they would write 2 plus 1 equals <u>a</u> and that is how you would get the <u>a</u>." For the 5, he wrote '2 + 3 = n'. He continued a similar procedure for the second problem, writing '24 + 33 + 45'. He felt the same letters in a fixed context did not have to be the same but that they could. "It depends on what it says in the paper."

<u>Wendy</u> maintained her 'alphabetical' interpretation and wrote '31 + 41 + 5' for '3a + 4a + 5' and '23 + 34 + 43' for '2c + 3d + 4c'. She thought different letters were used "to get children mixed up if they did the problem, they would just put any numbers instead of thinking what number <u>c</u> or <u>d</u> come in the alphabet."

<u>Antoinetta</u> wrote '35 + 45 + 5' for '3a + 4a + 5' and '23 + 34 + 43' for '2c + 3d + 4c'. She claimed she did not have 'any idea' why different letters were used, but in both instances used the same value for the same letter.

<u>Yvette</u> wrote '31 + 40 + 5' and explained "the <u>a</u> in this part is one, or could be any other number, and the same goes for this, the <u>a</u> is zero." For the second problem she initially wrote '2c + 3d + 4c = n' and then substituted numerical values for the letters, and wrote '22 + 31 + 44 = ____', and stated that "...but it (the letters could be any number." Her response to why different letters were used in the second problem was "...that could be any other letter, the <u>c</u> you could have put an <u>a</u> or a <u>d</u>."

<u>Filippo</u> once again transformed the expression into the equation '3a + 4a + 5 = 79' which he then solved by writing '3$\boxed{2}$ + 4$\boxed{2}$ + 5 = 79'. For '2c + 3d + 4c' he wrote '2c + 3d + 4c = 99', which he solved by writing '2$\boxed{2}$ + 3$\boxed{4}$ + 4$\boxed{3}$ = 99'. Although he used the same value for the <u>a's</u> in the first problem, this was merely accidental as he showed by using different values for the <u>c's</u> in the second example. Which he explained by "...you can put any letter you want, you can put an <u>a</u>, a <u>d</u>, a <u>c</u> or you can put all c's. But you can't put an <u>x</u>, that is for multiplication, you could get all mixed up."

<u>Gail</u> wrote '33 + 43 + 5 = 81' and explained "You can put it to any number, the same number here will be the same number here (referring to the 2 a's). As another example she wrote '31 + 41 + 5 = 77'. For '2c + 3d + 4c' she wrote '23 + 32 + 43 = 98'. She claimed that different letters were used "because they wanted a different

number from c " and that c and d could be 'any number' but the c's "both have to be the same".

The introduction of unlike terms in the simplification problems did not result in any significant change in the responses of five of the six subjects. Only Frankie answered differently by introducing a new aspect to the simplification problem. To him the problem was incomplete, and required additional equations in order to determine the value of the letters, such as '2 + 1 = a'.

None of the students provided the expected response, that of "collapsing" of the unlike terms. This is not surprising since all the students had maintained the place value interpretation for concatenation; they merely substituted numerical values for the letters, consequently no letters remained to "collapse".

The students who had previously claimed that the same letters had to have the same value in a fixed context (Antoinetta and Wendy) maintained such responses for these problems, even though Antoinetta claimed she did not know why different letters were used. Wendy continued to use her 'alphabetical' interpretation of literal symbols. Gail on the other hand had shown some ambivalence in the previous problems, but the presence of the unlike terms finally led her to conclude that the same letters in the expression had to have the same value and saw the use of the different letters as a means to indicate "...they wanted a different number from c".

The other students (Frankie, Filippo and Yvette) still used different values for the same letter, with Filippo and Yvette reemphasizing that 'any letter' could have been used, that is, the c's did not have to be the same. It is interesting to note that although Filippo feels you can use any letter he excludes the use of the letter <u>x</u>, one commonly used in algebra, with the very reasonable justification "...but you can't put an <u>x</u>, that is for multiplication, you could get all mixed up."

<u>Summary</u>

The questions presented to the subjects in this part of the pretest (the simplification and the combining of like terms), bring out the strength of the place value interpretation for concatenation by the beginning algebra students. All six subjects responded to the questions using this interpretation.

This part of the pretest also demonstrated that the algebraic concept that the same letter within the same expression must have the same value is not perceived as necessarily the case by some of the beginning algebra students. The following table summarizes the students' perception of the same letter within the same expression whether they represented the same value or different values.

TABLE 4 Students' Perception of the Same Letter within the Same Expression

		2a + 5a	4n + 3n	3a + 4a + 5	2c + 3d + 4c
Frankie	same				
	different	✓	✓	✓	✓
Wendy	same	✓	✓	✓	✓
	different				
Antoinetta	same	✓	✓	✓	✓
	different				
Yvette	same				
	different	✓	✓	✓	✓
Filippo	same				
	different	✓	✓	✓	✓
Gail	same	✓	✓	✓	✓
	different	✓	✓		

The consistency of the subjects responses is remarkable. Even with the change in questioning from presenting expressions containing only like terms to expressions containing like terms in the presence of unlike ones, Gail was the only subject to alter her response.

Another interesting aspect which was brought to light in this part of the pretest was the subjects' different perceptions of the instruction to 'simplify', one used very often in algebra textbooks.

7. Distributivity

The last part of the pre-test was to determine the students' spontaneous responses to problems involving distributivity and also to explore the students' interpretations of brackets within an algebraic context. In the following questions the multiplication sign was included in order to avoid any confusion:

Q. 16 Can you work out $\boxed{6 \times (2 + 3)}$?

Q. 16a What do you think is the meaning of brackets?
(Expected response: "Do the operation in the brackets first")

Q. 17 Can you work out $\boxed{3 \times (a + 5)}$?

Q. 17a Is it possible to do the operation in the brackets here?

Q. 17b Even if we can't do the operation in the brackets, can we multiply it out? Show me.

If he can do it, try: $\boxed{4 \times (3 + 2b)}$

$\boxed{5 \times (2a + 3b)}$

The students' responses were as follows:

Frankie stated "first you do the ones in the brackets," referring to the problem '6 × (2 + 3)'. When he was presented with '3 × (a + 5)', he claimed, "in the book they would tell you what would be the figure there (referring to the a)". Then he substituted 4 for a and wrote '4 + 5 ; 3 × 9 = 27'. For the problems which included concatenation, he again used a place value interpretation and wrote '3 + 22 ; 25 × 4 = 100' for '4 × (3 + 2b)'. He claimed that the problem '5 × (2a + 3b)' would be the same except that two numbers had to be found for the two letters.

Wendy claimed "...You have brackets to separate the numbers. So you would work with the brackets first." However, for the problem '3 × (a + 5)' she claimed the a could have two meanings. "I can either use a like I did all the time as one; or if there was an answer...Let's say it was 19, (and wrote '3 × (a + 5) = 19' I would have to find what number this is (referring to the a)". The other questions were not presented to her.

Antoinetta thought brackets "...make it easier, that is what you do first." She wrote '3 × (2 + 5) = 21' for '3 × (a + 5)'; '4 × (3 + 22) = 100' for '4 × (3 + 2b)'; and '5 × (23 + 32)' for '5 × (2a + 3b)'.

Yvette said that brackets meant, "you start with that first". She wrote the expression '3 × (a + 5)' and for the a she claimed "... you could put a 2 or any number, so 7 times 3 is 21", (she wrote the 2 over the a). For '4 × (3 + 2b)' she wrote '4 × (3 + 22)'; and then '4 × 25 = ', once again stating the b could be any number.

Filippo stated "you always do the brackets first". He once again transformed the expression into an equation and wrote '3 × (a + 5) = 39', stating "you just put an answer and then find the value for a ". He then wrote '8 + 5' for 'a + 5', and finally '3 × 13 = 39'. For '4 × (3 + 2b)' he wrote '4 × (3 + 2b) = 108' and solved for b . Whereas for '5 × (2a + 3b)' he first substituted values for a and b , '5 × (23 + 32)' and then evaluated.

Gail stated the brackets "come first". For '3 × (a + 5)' she answered '24' but stated, "it can be anything. It can be zero, (she wrote '3 × (0 + 5) = 3 × 5 = 15', or (and wrote '3 × (4 + 5) = 3 × 9 = 27') twenty-seven." In the problem '4 × (3 + 2b)' she wrote '23 × 4 = 93' and explained, "well b can be anything, any number, I just put it as 3 for an example. It was zero, so 20 plus 3 is 23", and gave another example '4 × (3 + 21) = 4 × 24 = 96'. For '6 × (2a + 3b)' she wrote '5 × (21 + 32) = 5 × 53 = 265'.

The distributive axiom is initially introduced to students within an arithmetic framework. Thus we expected the students to respond to the expression '3 × (a + 5)', and similar other problems, by either claiming they could not perform the operation (due to the algebraic context) or by applying their arithmetic knowledge of the distributive axiom to this algebraic situation.

However, the students responded in a totally unexpected manner. They spontaneously assigned numerical values for the letter(s), consequently the problems were no longer algebraic, but arithmetic. The need for the distributive axiom no longer existed for these students. They merely performed the operations required with the numbers they had.

The consistency of the students' responses was still evident since all six subjects used the 'place value' interpretation for concatenation where applicable. Also Yvette and Gail still maintained that the letters could be 'any number', a further indication that they had the 'generalized number' concept.

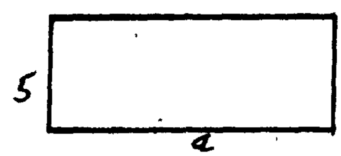
Noteworthy is the slight change in Wendy's response. The letter is not only the specific value as determined by its position in the alphabet, but she also claimed it could have other values if the expression was placed in an equation. This response is not inconsistent with her previous work. The idea of two possible values for the latter, came up when Wendy was describing the meaning of the letter <u>a</u> in the geometric area problem (it was one of the measurements of the side). She had also deviated before from her alphabetical interpretation when she was presented with the expression '4 + n' . She transformed the expression into an equation '4 + n = 6', and the letter thus became a specific unknown.

D. <u>A Summary of Each of the Subject's Responses</u>

<u>Frankie</u>

Frankie's preception of concatenation was not consistent throughout the pretest. For the expression <u>3a</u> he used a 'subdivision label' interpretation. The introduction of the letter <u>n</u> in place of the letter <u>a</u> in <u>3a</u> did result in a change in his interpretation. He stated it meant "3 times a number". He maintained this latter interpretation when he was required to replace the letter <u>a</u> by the number 2 in <u>3a</u> , he wrote 3 × 2 . However, in none of the succeeding questions did he again refer to this operational definition for concatenation, but rather he proceeded to use a 'place value' interpretation for all concatenated expressions in the remainder of the pretest.

Except for his initial 'subdivision' label interpretation for 3a, Frankie always used a numerical referent for the literal symbol. He either substituted numerical values for the letter (as in the problem multiply 3n by 4, he wrote '32 × 4') or he would make some reference that the letter represented a number (for example, he explained the meaning of the expression '4 + b' by saying it meant "4 plus a number equals something". Even within a geometric context he used a numerical referent for the letter as seen in the discussion of the area of the rectangle



he indicated that the a represented the measure of the base. This numerical referent for letters was also extended by Frankie to expressions containing like and unlike terms, for example '2a + 5a' and '2c + 3d + 4c'. However, at no time did he indicate that the same letter within the same expression represented the same number.

Frankie also tended to close open expressions such as '4 + b'. He handled these open expressions in two ways; he either transformed them into equations (4 plus b equals something) or he substituted numerical value(s) for the letter(s) and then evaluated the expression.

The section of the pretest used for a comparison with some of Kuchemann's test items, indicated a tendency of Frankie's to ignore the letter in problems such as 'add 4 onto n + 5'; he wrote '4 + 5 = 9'. It was also in this part of the test that Frankie first began to use a place value interpretation for concatenation.

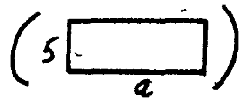
In general Frankie definitely has the perception of letters as representing numbers in mathematical situations. However, he does not appear to have a generalized number concept in the sense that he never indicated that the letter could represent 'any number'. Nor does he appear to have an acceptance of lack of closure. He interprets concatenation in two different ways; either as signifying multiplication (for simple expressions) or place value (for complex expressions).

Wendy

Wendy's responses throughout the pretest were remarkably consistent. She repeatedly used the place value interpretation for concatenation. To Wendy letters represented numbers, - not just 'any number' but each letter had a specific value. This value was determined by the letter's position in the alphabet.

Wendy deviated from this alphabetic interpretation in only a few problems. In one instance when she explained the meaning of '$4 + n$' she formed an equation. She wrote '$4 + n = 6$'. This deviation could be explained by her prior experience with the letter n. That is, the letter n was usually presented to her within the context of equations. She also formed an equation for the expression '$3 \times (a + 5)$' and said that the a could be one or another value determined by the equation '$3 \times (a + 5) = 19$'.

The Kuchemann questions also resulted in a change in her perception of the letter. In all four problems she ignored the letter. This resulted in a cognitive conflict for her in the problem 'multiply $n + 5$ by 4', since she could not "add and multiply by 5".

The geometric context for the letter  resulted in Wendy suggesting two interpretations for the letter a. Either the a was one, according to her alphabetic interpretation or it was the measure of length of the side.

In general Wendy does not have the generalized number concept since the letters do not represent 'any number', but rather their value is usually determined by their rank in the alphabet. She also does not appear to have an acceptance of lack of closure. For most expressions she usually substituted a numerical value for the letter (based on its rank in the alphabet) and then evaluated the resulting numerical expression.

Antoinetta

Antoinetta used three different interpretations for concatenation. Initially she suggested a 'subdivision label' interpretation and then that of 'first letter abbreviation'. However, when more complex expressions were introduced and when operations on these expressions were required, such as 'add 4 onto 3n', she switched to a place value interpretation and maintained this interpretation for the remainder of the pretest.

Antoinetta used a numerical referent for the literal symbol. She did not transform any expressions into equations but rather substituted a numerical value for the given letter. She claimed that the letter could be 'any number'. This suggests that she may have the 'generalized number' concept. However, within the geometric context, 

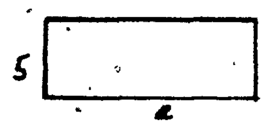
she definitely did not perceive the letter as representing 'any number', but rather it was a specific unknown measurement. Not only did she use a numerical referent for the literal symbols but also the problems requiring the simplification and combining of like terms led her to conclude that the same letters in the same expression had to have the same numerical value.

In the problems related to the Kuchemann test, Antoinetta solved them by substituting numerical values for the letters, and then either evaluated the resultant arithmetic expression or left the expression as an open one. She chose different values for the letter <u>n</u> in each of the problems and indicated that she had merely chosen these numbers randomly.

In general, Antoinetta appears to have the generalized number concept. Her primary interpretation for concatenation is that of 'place value'. Also it appears she has some acceptance of lack of closure. This is evident by the fact that she does not transform expressions into equations and often leaves open arithmetic expressions unevaluated.

<u>Yvette</u>

Yvette responded consistently throughout the pretest. She maintained a place value interpretation for concatenation in all examples. Also in all situations she used a numerical referent for the literal symbol. Furthermore, she stated that the letter could represent 'any number', hinting at a possible 'generalized number' concept. Only within the geometric context,



did she deviate from a 'generalized number' interpretation for the letter

<u>a</u> to that of 'specific unknown'. That is, the <u>a</u> to Yvette represented 'the number of lines going up vertically from the base', thus suggesting some specific unknown length.

Yvette also demonstrated an ability to accept open algebraic expressions. For example, she viewed the expression '4 + b' as '4 plus any number' and did not express the need to transform the expression into an equation. In the section on the Kuchemann test, she also indicated this acceptance. She would always initially write an open algebraic expression such as '4 + n + 5', and only then would she substitute a number for the letter and evaluate it, each time stating that the letter could be 'any number'.

She maintained this procedure of substituting numerical value(s) for the literal symbol(s) in the problems involving the simplification and combining of like terms (suggesting each time the letter could be 'any number'). However in these problems she did not evaluate the resulting arithmetic expressions, but rather left them as open expressions.

Since she maintained that a letter could represent any number, she did not assume that the same letter in the same expression had to have the same value — a letter could be 'any number'.

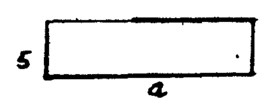
In general, Yvette appears to have the generalized number concept. There are also numerous examples which hint that she has some acceptance of lack of closure since she is able to write both open arithmetic and open algebraic expressions. Her predominant interpretation for concatenation is that of place value.

Filippo

Filippo's responses throughout the pretest were also very consistent. Although he had initially provided a 'subdivision label' interpretation for concatenation, he switched to a place value interpretation as soon as operations were indicated for these expressions and maintained this interpretation for the remainder of the test.

Filippo usually viewed the letter within a numerical context, except for his initial interpretation of concatenation. Not only did he view the letter as a number, but it was a placeholder for some specific unknown number. He indicated this by writing the numerical value he assigned to the letter in a box. For example, he wrote '$2a + 5a = 78$; $2\boxed{2} + 5\boxed{6} = 78$'. As can be seen from this example and in the other simplification problems, Filippo did not believe that the same letters in the same expression had to have the same value.

Filippo did not demonstrate any acceptance of a lack of closure. In all expressions which explicitly indicated an operation (eg. $2a + 5a$), he would transform them into equations, which he would then solve. Thus to Filippo the letter is not a generalized number, it is a specific unknown number. Even within the geometric context



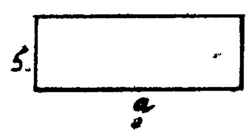
for the area of the rectangle he wrote: '$5 \times A = 45$; $5 \times 9 = 45$'.

In general, to Filippo the introduction of a literal symbol causes him to immediately think of a specific unknown number. He feels the necessity to close all open algebraic expressions by transforming them into equations. Also his primary interpretation for concatenation is that of 'place value'.

Gail

Gail assigned three interpretations for concatenation: first letter abbreviation, subdivision label and place value. However, as some of the previous subjects had done, she used only the place value interpretation for concatenated terms, when these terms were part of more complex expressions (2a + 5a).

Throughout the pretest, Gail repeatedly stated that the letter could represent 'any number' and for each problem gave more than one example illustrating this point. Thus it can be concluded that Gail has the generalized number concept. It was only within the geometric problem the area of



that Gail suggested that the letter represented some specific number, the length of the base.

Although Gail maintained that the letter could represent any number, she ultimately concluded that in expressions where the same letter appeared more than once, this same letter had to have the same numerical value.

In the questions related to the Kuchemann test, Gail's responses were basically consistent with her previous responses. In the problems involving concatenation she used a place value interpretation, emphasizing that the letter could be any number. However, in two problems, 'add 4 onto n + 5', and 'multiply n + 5 by 4', she ignored the letter and answered 'nine' and 'twenty', respectively. For the latter she also suggested alternate responses such as '3 + 5 = 8 × 4 = 32'.

Gail gave some indication that she could accept the lack of closure of some open expressions by leaving open arithmetic ones such as

'4 + 2', and '4 + 1' when writing the meaning of the expression '4 + b'. However, at times she would close some expressions by either transforming them into equations (4 + b = 7) or by solving the open arithmetic expression (for '2a + 5a' she wrote '23 + 51 = 74').

In general Gail appears to have the generalized number concept and demonstrates some acceptance of lack of closure of open expressions. As with the previous subjects, her primary interpretation for concatenation is that of place value.

CONCLUSION

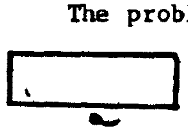
The result of the Pretest clearly demonstrates our subjects' perceptions of algebraic symbolism. None of them had any formal instruction in algebra, thus it can be concluded that these prealgebraic interpretations were spontaneous ones, not reflections of learnt material.

One of the most interesting results was the interpretation the subjects assigned to concatenation. The two most prevalent ones were of 'subdivision label' and 'place value'. Although the 'place value' interpretation for concatenation had been suggested by M. Matz, in our experiments, this interpretation appeared not as a mere possibility but was the strongest prealgebraic interpretation for concatenation. The strength of this interpretation became even more evident within a numerical context (replace <u>a</u> by 2 in <u>3a</u>) or when the simplification of two or more terms was required (simplify 3a + 5a), where <u>all</u> the subjects used only this interpretation.

The presentation of the expressions '4 + b' and '4 + n' brought about some very interesting responses with respect to the students' interpretation of the literal symbol. Three students (Frankie, Wendy, Filippo) closed these expressions by transforming them into equations, while Antoinetta and Yvette left the expressions as 'open' expressions and merely substituted numerical value(s) for the letter. Gail suggested both possibilities. The former approach indicated that the subject perceived the letter as a *specific unknown*. The second method seemed to reflect a broader interpretation of the letter, that of *generalized number*. Not only do these two approaches reflect different interpretations of the literal symbol, but they could also be a basis for suggesting whether or not the students had an acceptance of lack of closure. The need to transform the expressions into an equation appeared to suggest that the student could not accept its openness and consequently did not have an acceptance of lack of closure. Conversely, those students who left the expression as an open one appeared to have acceptance of lack of closure. Thus we conclude that a possible criteria for ALC is the absence of the need to transform algebraic expressions into equations.

The fact that three students gave evidence of a 'generalized number' interpretation for the literal symbol, brought into question our teaching outline in which the letter is presented as a specific unknown within the algebraic expression. However, since not all the students had achieved this level of interpretation, that is 'generalized number', and since in the teaching outline the 'answers' to the problems remain in the form of algebraic expressions, this seems to be a possible stepping stone, since these expressions do not need to be transformed into equations. Thus we felt the validity of our teaching outline still remained.

The problems involving a geometric context (what is the area of

5 )

also brought into question our teaching outline. The responses indicated that within this context none of the students gave any evidence of an acceptance of lack of closure as evidenced by either their need to substitute a numerical value for the letter and then give a single numerical answer for the area or by their reference to the need for measurement. Thus it appeared that the geometric context induced a greater need for closure. It could be questioned why we based a whole teaching outline on it. It should be recalled that in our Initial Pilot Study the direct geometric approach to introducing algebraic expressions uncovered some cognitive difficulties: 'How can you multiply a number by a letter?' Thus we changed the initial introduction by adding an intermediate step in which one of the dimensions of the problem was not only unknown, but actually hidden. This hiding eliminates the possibility that the student will seek to estimate or measure the unknown quantity which in his mind nevertheless exists, even if he cannot see it. The need for a convention to represent this unknown quantity is established - at first the conventional box is used, and then the box is replaced by a letter within the context of placeholder for a hidden quantity. Thus the meaning of the letter in our teaching outline is different from the one in the geometric situation involved in the pretest where to the student the literal symbol was a specific but a visible measure.

The responses given by our subjects to the questions from the Kuchemann test were markedly different from those obtained in the original test administered in Great Britain. However, it must be kept in mind that the subjects in our experiment had little or no formal instruction in algebra, whereas the British students had two years of such instruction.

Therefore the responses given by our subjects were spontaneous interpretations of algebraic symbolism and notation rather than a reflection of learnt material. Four of the six students responded to requests for operations on the algebraic expressions by substituting numerical values for the literal symbols and then evaluating the resulting arithmetic expression. The other two students were also very consistent. Wendy ignored the letter in all four problems. Filippo transformed all the expressions into equations and solved for n. For the questions which involved concatenated expressions (add '4 onto 3n' and multiply '3n by 4') five of the students used a place value interpretation in their response.

The simplification problems did not bring out any new interpretations. The students again were consistent and maintained the interpretations they had assigned throughout the Pretest. Of interest to note, is that three subjects felt that the same letter within the same expression did not have to have the same numerical value. Antoinetta felt they had to be the same; Gail was ambivalent; and Wendy claimed they had to be the same based on her alphabetic interpretations of letters. The different perceptions the students had of the instruction to 'simplify' was also highlighted in this part of the pretest.

What proved to be most remarkable in the pretest was the consistency of all our subjects. Each one provided some interpretation for algebraic expressions and tried to answer all the questions within the framework of his interpretation. For instance, the place value interpretation of concatenation was used in each part of the pretest indicating that the students' thinking was not erratic nor arbitrary but rather that they assigned logical meanings to the problems at hand.

CHAPTER VI

ANALYSIS OF LESSON 1

The Transition from Placeholder to Literal Symbol (Letter)

INTRODUCTION

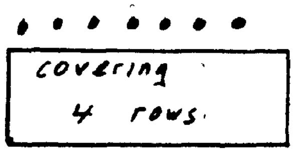
The initial pilot study revealed that the direct introduction of letters standing for some unknown quantity (such as, number of rows in an array of dots, or the number of line segments in a line, or the length of the base of a rectangle) raised major cognitive obstacles. ("How can I multiply a number by a letter?") Consequently, an intermediate step was added. The use of a placeholder, the familiar box, was first intoduced to represent the unknown quantity. This introduction was done in conjunction with the hiding (covering with a piece of cardboard) of one element of the problem, thus illustrating concretely the idea of "unknown quantity". The cardboard was then removed and the "unknown quantity" revealed allowing the student to fill in or substitute a numerical value for the placeholder, thus linking the placeholder to a numerical referent.

Once the three problem types (number of dots, length of line, and area of rectangle) had been introduced using the placeholder, they were reintroduced using a letter in place of the box to represent the unknown quantity. The presentation followed the same pattern of covering and uncovering that was done when illustrating the use of the placeholder. Thus the literal symbol became an extension of the placeholder, both representing unknown quantities, both linked to a numerical referent.

A. The Introduction of the Three Problem Types Using a Placeholder

1. Dot Problems

The dot problem was presented as follows:



Here is a card. You can see a row of 7 dots. And I have hidden more rows each with 7 dots. Here is the problem. How would I write the total number of dots if I don't know the exact number of rows? So let me show you how I do it.

Number of dots = $7 \times \square$

Since I don't know the exact number of rows, I am meanwhile using a box. Now I am going to let you have a peak at what I have covered and ask you to fill the right number in the box.

How many rows are there altogether?

The students' responses were as follows:

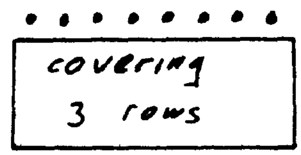
Wendy upon seening the expression '$7 \times \square$' , stated "If you wrote 7 times box I would have to say what the box stands for." Upon uncovering, she filled the box with the number 5.

Frankie followed the presentation and filled in the box with the number 5.

The remaining four students responded similarly to Frankie.

The presentation of the box representing the unknown quantity did not appear troublesome to any of the students, except initially for Wendy, who felt the expression was incomplete and had to provide a numerical value for the box immediately upon seeing it, as indicated by her statement, "If you wrote '$7 \times \square$' I would have to say what the box stands for." However, all six subjects correctly wrote the number 5 inside the box, thus indicating that they correctly perceived the box as representing the total number of rows, not just those that were covered.

The succeeding problem was aimed at verifying whether the student himself could use the box as a placeholder.



> You can now see 8 dots in each row, but once again I am covering the other rows. Can you complete this equation the way I did before?
>
> Number of dots =
>
> (Uncovering) How many rows are there altogether? What number has to go into the box?

A sample response was Antoinetta's:

Antoinetta completed the equation by writing '8 × ☐', and after uncovering, filled the number 4 in the box.

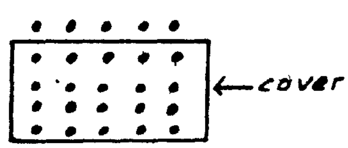
The other five students responded in a similar manner.

None of the subjects experienced any difficulty with this problem. They demonstrated that they could correctly use the box as a placeholder by writing '8 × ☐'. They also, once again, illustrated that they understood that the box represented the total number of rows, by stating there were four rows and that the box stood for the number four.

Finally the students were given an expression, and were then required to generate a similar dot problem.

> Now it is your turn. Can you make up a dot problem like we just did, where 5 × ☐ is the total number of dots?

A typical response was that of Gail:



Gail hid her work from the interviewer and when she turned around, she showed a row of 5 dots and covered the remaining rows.

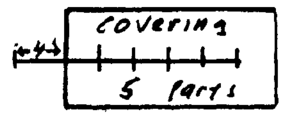
The other five students behaved in a similar manner.

The reversal was done easily by all six subjects. However, their behaviour showed that they had grasped the meaning of the box as representing a hidden number, and therefore had to hide the number of rows from the interviewer.

2. Line Problem

The line problem presented another situation in which an algebraic expression could be used as an answer to a problem. The problem type was introduced as follows:

Now look at this line problem. Each part is 4 units long. But you don't know how many parts there are. Can you complete this equation?



Length = _

(Uncover) So what number has to go into the box?

If the student was unable to respond, questions were prepared to guide the student into using the box.

What do you think I should write for the number of parts?
Do you remember what we did in the dot problem?
Because we didn't know the number of rows we used...?
Could you use a box here?

The students' responses were as follows:

Wendy did not spontaneously think of using a box. When asked what she should write for the number of parts, she responded 4, but then corrected herself by saying, "No, there are not 4 parts, the length is 4 in each part, so that is your 4." The succeeding questions did not lead her to correctly answer the problem, thus she had to be reminded that a box was used to represent the unknown number of rows in the previous problem type. She then stated that a box could be used in this problem, and correctly wrote '4 × □', and later wrote 6 in the box.

Antoinetta after first questioning "Do you still have to put a box like the other one?" wrote '4 × □', and substituted correctly the number 6. When questioned why we were using a box, she responded, "It's for how many parts."

Gail, Filippo, Frankie and Yvette experienced no difficulties and immediately wrote the correct response, '4 × □', and subsequently filled 6 in the box.

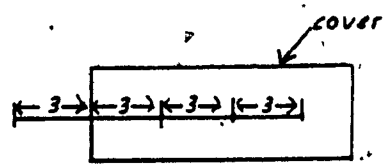
The introduction of the line problem demonstrated that 5 of the 6 students were able to spontaneously transfer the knowledge they had acquired in the previous dot problem (that is, the use of a box to represent an unknown quantity). Only Wendy was unable to see this link between the two problem types. However, after being reminded about using the box in the previous problem, she had no difficulties in writing the expression and also provided the correct substitution.

The second problem in this section required a reversal. The problem was as follows:

> Now it is your turn to make up a line problem where '3 × □' is the length of the line.

The responses were as follows:

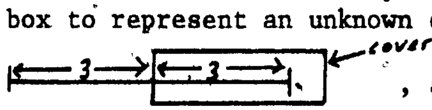
<u>Gail</u> turned around to hide her work and then presented the following drawing to represent the expression '3 × □'.



<u>Wendy, Frankie, Antoinetta and Yvette</u> behaved similarly and drew correct variations of the problem.

<u>Filippo</u> at first drew 

claiming "I thought you meant there were three spaces and I had to find what was in the middle of the space". He then responded to the question "What did we use the box for?" by stating "we used the box to find the missing number." After a review of the previous problems, illustrating the use of the box to represent an unknown quantity, Filippo then drew



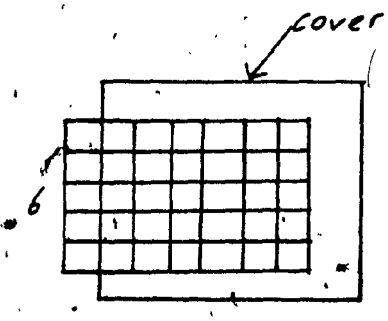
, and concluded that the box represented "the number of parts".

Five of the six subjects easily generated a line problem where the length of the line was '3 × □'. Only Filippo experienced some difficulty with the reversal, by initially drawing a line divided into 3 segments each one unit long. His statement "I thought you meant there were 3 spaces and I had to find out what was in the middle of the space", indicated that he did not appear to comprehend the meaning of the box in this problem, as representing the unknown number of parts. He seemed to have interpreted the box as standing for the length of each line segment. However, the final problem he generated indicates he had accepted the convention.

3. Area Problem

The area problems were also introduced to provide another situation in which algebraic expressions could be presented as answers to a problem. The presentation went as follows:

> Now I will draw an area problem, in which I will only show you the height, and I will cover the rest of the rectangle.
> Can you complete this equation?
> (Uncover) So what number has to go into the box?



Filippo's response is representative of all the students.

Filippo completed the equation by writing '6 × □', and then after the uncovering filled in a 7 in the box.

The remaining 5 students responded similarly to Filippo.

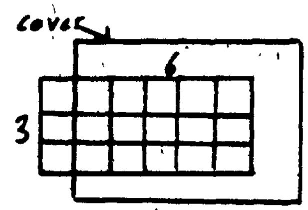
Thus all six subjects easily transferred the notion of using a box to represent an unknown quantity by using it to represent the length of the base of the rectangle in writing its area.

The reversal problem in this section was altered slightly in that the student was encouraged to make up his own problem entirely, by not being given any numerical suggestions. The problem was presented in this open format since exploratory work had indicated that students like to make up their own problems. The problem was stated as follows:

> Now it is your turn. Make up an area problem where I can only see the height of the rectangle.

The responses were as follows:

<u>Filippo</u> after first hiding his work, presented the following problem.



The remaining five students behaved similarly, using a variety of numbers.

The responses indicated that none of the subjects experienced any difficulties in generating their own area problems.

Summary

Except for a few minor instances in the line segment problems, the use of a placeholder, the empty box, to represent some hidden and therefore unknown quantity in three different problem types did not present any cognitive difficulties for the subjects. Ultimately all six subjects were able to write the correct expressions to represent the 'answers' to the given problem types. The generation of the three problem types was also accomplished by all the subjects. Thus, at this point, we can conclude that the teaching outline was successful in introducing placeholders to represent hidden quantities, and could be used by the students in completing equations, such as 'number of dots = $8 \times \square$'.

B. The Transition to Letters

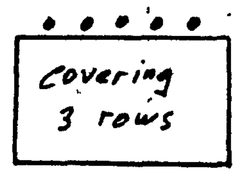
The problems in this section of the teaching outline, were presented exactly as in the previous section, except that in these problems a letter was used instead of an 'empty box' to represent the hidden quantity.

1. Dot Problems

The difficulties the students experienced in their first use of letters, as evidenced in our initial pilot study, warranted a slow approach to the introduction of letters. Thus three dot problems were used. In the first problem, the student was shown how to use the letter in the expression. The second problem required the student to complete an equation by writing the correct algebraic expression representing the problem. And finally the student was asked to generate his own problem given the algebraic expression.

The first problem was presented as follows:

Let's do another dot problem. You can see that I have a row of 5 dots, but you can't see the exact number of rows. This time I am going to use a letter instead of a box for the number of rows. Choose a letter. So let me show you how I write the total number of dots using your letter.



Number of dots = 5 × a (or whatever letter is chosen)

Now I will let you look at the number of rows. How many rows are there? What does your letter stand for? Can you complete this equation?

Number of dots = 5 ×

The responses were as follows:

Wendy chose the letter a. Then, after uncovering, she stated that the a stood for 4, and wrote 4 next to the a (5 × a). She completed the equation 'Number of dots = 5 × ' by writing the number 4.

Frankie chose the letter n, and then after the uncovering answered that the letter n stood for the number 4 and then completed the equation, 'Number of dots = 5 × 4'.

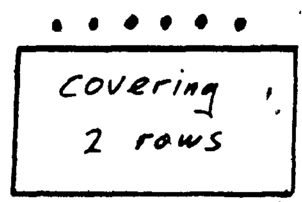
The remaining 4 students responded like Frankie: Antoinetta chose the letter c, Gail, the letter a, Filippo chose x, and Yvette, n.

A variety of letters was chosen by the subjects. All students appeared to accept the use of the letter they chose as part of the algebraic expression, 5 × (the letter they chose), to represent the total number of dots. They exhibited no difficulty in understanding that the letter stood for the total number of rows, since all students completed the final equation by writing a 4, (number of dots = 5 × 4).

Of interest to note is Wendy's quick response that the letter a (the letter she chose) stood for the number 4. Throughout the pretest she had maintained that the value of a letter is determined by its rank in the alphabet. Yet at no point did she suggest this alternate value, one, for the letter a.

The next problem aimed at determining whether the student, himself, could use a letter to represent the unknown quantity in writing the answer to a dot problem.

> You can now see 6 dots in each row, but once again I am covering the other rows so you don't see the exact number of rows. Can you complete this equation using a letter instead of a box?



> Number of dots =
>
> The rows are uncovered, followed by the questions:
> How many rows are there altogether?
> What number does the letter stand for?

The responses were as follows:

Wendy completed the equation by writing '6 × a' and then after uncovering wrote '6 × 3', under the '6 × a'.

Frankie (using the letter n), Gail (using the letter a) and Yvette (using the letter n) responded the same way as Wendy. Without being asked, they wrote the numerical expression '6 × 3' under the algebraic one.

<u>Antoinetta</u> wrote '6 × d'. After uncovering, when asked what the <u>d</u> stood for, she stated 3. She did not rewrite the expression substituting 3 for the letter <u>d</u> as had been done by the two previous students.

<u>Filippo</u> wrote '6 × y' to complete the equation. When he was asked what the <u>y</u> stood for, he wrote 'y = 3'.

The six subjects easily wrote a correct algebraic expression to represent the total number of dots. They used a letter without any hesitation. None of the cognitive dilemmas evidenced in the Initial Pilot Study surfaced, such as, "How can you multiply a number with a letter?" Four of the students (Wendy, Frankie, Gail and Yvette) used the same letter as the one they had chosen in the previous problem. The other two students, Antoinetta and Filippo, chose different letters, indicating explicitly that they understood the randomness of the selection of the letter.

All subjects perceived the correct numerical referent for the letter (3). Four of the students (Wendy, Frankie, Gail and Yvette) spontaneously rewrote the equation substituting the number 3 for their letter. Filippo wrote y = 3, and Antoinetta merely stated that the letter stood for the number three. However, in all cases it was very clear that the subjects maintained a numerical referent for the letter they chose.

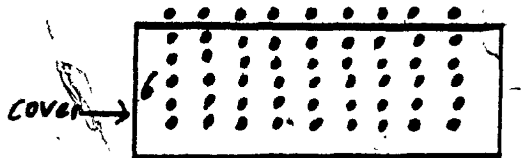
Once again it is interesting to note that Wendy was able to provide a different numerical value for the letter <u>a</u>, that is 3, without any evidence of a cognitive dilemma.

In the next problem the students were given the algebraic expression and they were asked to draw a representative problem.

> Now it is your turn. Make up a dot problem where the answer will be '9 × c'.

The responses were as follows:

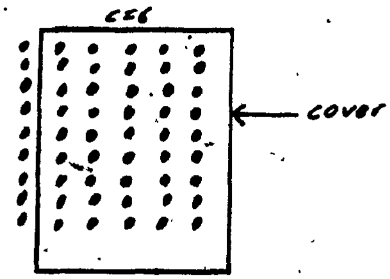
Frankie after first hiding his work, presented the following problem:



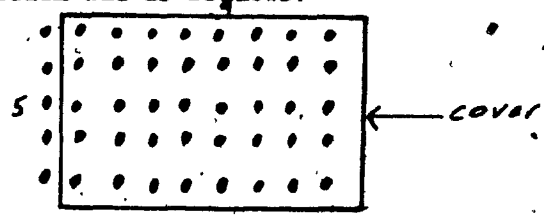
and then after uncovering wrote '9 × 6'.

Antoinetta, Gail and Yvette behaved similarly.

Wendy presented the following problem.



Filippo's problem was as follows:



Four students (Frankie, Anotinetta, Gail and Yvette) drew a correct dot problem similar to the previous examples. Wendy demonstrated some originality in her problem. She drew a correct representation for the expression '9 × c' , however, unlike the previous examples, the letter c , in her problem, stood for the unknown number of dots in each row not the number of rows. Later upon uncovering she wrote that 'c = 6', since in each row there were six dots. Thus Wendy preceived that the letter represented some hidden quantity but that it did not necessarily have to be the number rows.

Filippo, on the other hand, drew a 9 by 5 array of dots, but covered the number of dots in each row, showing 5 rows. Unlike Wendy's representation, his drawing represented the expression '5 × c' , rather than '9 × c'. However, once the interviewer remarked that the answer to his problem would be '5 × c', he responded "Oh, I showed you the answer" and stated that the c stood for 5, the number of rows. Thus Filippo's error seems to have been accidental since stating, "Oh, I showed you the answer ", implies he understood the problem and realized his error.

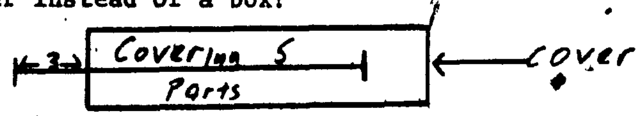
2. Line Problems

The line problems were presented as in the previous section, but once again the student was asked to use a letter instead of a box to represent the hidden quantity.

The presentation was as follows:

Now let's try a line problem where we will use a letter instead of a box for the number of parts.
Can you complete this equation using a letter instead of a box?

Length =



(Uncover) So what does the letter stand for?
(If incorrect) How many parts are there altogether?

The responses were as follows:

Antoinetta completed the equation by writing '3 × b'. After uncovering, she stated the letter b stood for the number 6.

Gail wrote '3 × m' and after uncovering substituted the number 6 for the m by spontaneously writing '3 × 6' under the '3 × m' when questionned what the letter m stood for.

Wendy, Frankie and Yvette responded similarly to Gail.

Filippo wrote '3 × y', then after the uncovering answered the questions by writing: 'y = 6 , 3 × 6 = 18".

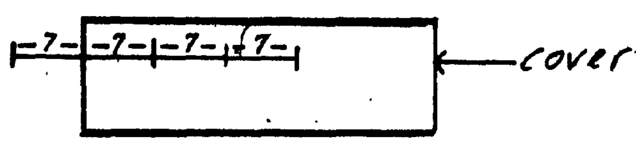
All six subjects transferred the knowledge they had acquired in the previous dot problems, and experienced no difficulties in writing algebraic expressions as answers to the given line problem. A wide variety of letters was selected by the subjects to represent the unknown number of parts. When they were asked what the letter stood for, all students provided a numerical referent for the letter. The same four students (Wendy, Frankie, Gail and Yvette) wrote '3 × 6', thus substituting the number 6 for the letter in their expression. Filippo wrote 'y = 6', and then evaluated the expression, while Antoinetta merely stated that her letter stood for the number 6.

This initial line problem was succeeded by a problem requiring a reversal:

Make up a line problem where the answer would be '7 × d'.

The sample response was as follows:

Yvette after first hiding her work presented the following problem



and then wrote '7 × 4' after uncovering.

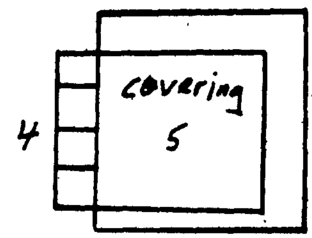
All other subjects responded in a similar manner.

The responses indicate that no student experienced any difficulty in drawing a correct line problem to represent the algebraic expression '7 × d'.

3. Area Problems

The area problem was also reintroduced but without any suggestion that the use of a letter was required. The problem was presented as follows:

Now I will draw an area problem where you will see only the height, because I will cover the rest of the rectangle. Can you complete this equation?



Area =

(If the student used a box, he was to be reminded that we were now using letters)

(Uncovering) So what does the letter stand for?
(If incorrect) What is the length of the base of the rectangle?

The responses were as follows:

Wendy first questionned "any letter or a box?" then wrote '4 × c'. Later after uncovering substituted 6 for the c and wrote '4 × 6' when asked what the letter c stood for.

Frankie wrote '4 × c', then after uncovering wrote '4 × 6', stating that the c stood for 6.

Antoinetta after first questionning "I put a letter?" wrote '4 × a', and then stated that the a stood for the number 6.

Filippo completed the equation by writing '4 × d', and as he had done previously, he wrote 'd = 6'. When he was asked to write the area, he wrote '4 × 6 = 24'.

Yvette completed the equation by writing '4 × b', and then after uncovering substituted 6 for the b and wrote '4 × 6'.

Gail responded similarly to Yvette but used the letter m.

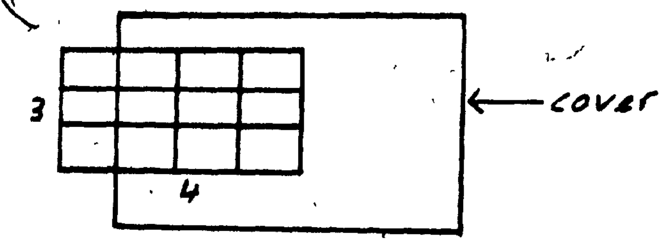
Only two students (Wendy and Antoinetta) questionned whether they should use a letter. Ultimately all six students wrote a correct algebraic expression to represent the area of the given rectangle. When asked what the letter stood for, they responded by giving the numerical value, 6. All students, except Antoinetta, rewrote the expression substituting the number 6 for their letter. Filippo again went further by providing a final answer, 24.

The final problem in this lesson required the student to make up his own area problem.

> Now it is your turn. Make up an area problem
> where I can only see the height of the rectangle.

A sample response is the one of Gail.

<u>Gail</u> after first hiding her work, presented the following area problem:



and wrote '3 × n' .

The other students presented similar correct problems but used different numbers and letters.

All six students drew correct area problems accompanied by an appropriate expression, demonstrating no difficulties in understanding the use of letters in this problem type.

<u>Summary</u>

The subjects did not experience any cognitive dilemmas in their use of a letter to represent a hidden quantity and in the writing of algebraic expressions. The problem evidenced in the Initial Pilot Study (How can you multiply a number by a letter?) did not surface at any point in this Lesson. The presentation used in Lesson 1 provided a situation where the letter became a natural extension of the student's use of placeholder, the box, and both were explicitly linked to a numerical referent (the number of rows; the number of parts; the length of the base). Consequently, the dilemma in writing a number multiplying a letter did not occur, since the letter was always tied to a numerical referent. The ease with which the subjects were able to use literal symbols cannot be solely attributed to our presentation, ~~since~~ the pre-test had indicated that the students, prior to this lesson, already had a numerical referent for letters used in mathematics.

It is interesting to note Wendy's ability to use different letters and to abandon her theory that the numerical value of a letter is determined by its rank in alphabet.

C. Letters vs. Boxes

The final section of Lesson I dealt with a series of questions aimed at determining the students' perceptions of the difference and similarities between the use of letters and the use of boxes.

The first two questions were as follows:

1) Do you find it more difficult to use letters than boxes?

2) Can you tell me why?

Wendy, Frankie and Antoinetta felt that letters and boxes were the 'same thing', thus one was not more difficult to use than the other.

Gail claimed she found letters more difficult to use than boxes "because they are different letters and you can get mixed up...sometimes I think that every letter is a different number". When questionned whether she thought that every letter had to be a different number, she replied "no".

Filippo stated he found it more difficult to use letters, using as an example the expression, '5 × x'. "You are not going to be sure what does the x mean - multiplication or something". He also cited another possible confusion, "You might think it meant a letter or something...Like when I saw it (an expression with a letter and a number) I thought it meant a number times a letter. He later stated "And I would ask what is a letter, does it represent a number, a symbol or something." (He was referring to his earlier experiences with letters.) But now he claimed he knew "They represent the missing number." He felt the box was clearer, "You know that something belongs in there... But if it is a letter...~~you don~~'t know what it means, it makes it more difficult."

Yvette preferred letters. She explained by writing, '6 × n' and '6 × □'. "I prefer the n because you know the n stands for a missing number." When further questionned if the letter a was used and not n, she claimed she still preferred letters.

The responses indicate that only two of the subjects, Gail and Filippo, had more difficulty in using letters than in using boxes, each one highlighting possible confusions which may arise from the use of letters. Gail pointed out that the use of different letters could imply perhaps different numbers resulting in a possible 'mix-up'. Filippo astutely brought attention to the dilemma that could be experienced with the letter <u>x</u>, whether it stood for multiplication or represented a number. He also hinted at the cognitive difficulty "How can you multiply a number by a letter" exposed in the Initial Pilot Study. It is interesting to note that one student, Yvette, actually found the use of letters to be simpler than boxes, in terms of what she believed was their direct reference to numbers. That she did not limit this preference to the letter <u>n</u>, which could be linked to the word '<u>n</u>umber', was evidenced by the fact that she still preferred the use of letters when questionned about the use of the letter <u>a</u>.

The next questions aimed at verifying <u>explicitly</u> whether or not the cognitive dilemma, associated with the use of letters and numbers, evidenced in our Initial Pilot Study (How can you multiply a number with a letter?) was present for any of the subjects after the presentation of Lesson 1. It must be recalled that Lesson 1 was included in the teaching outline in order to avoid the cognitive problems associated with a <u>direct</u> approach to the use of letters.

The questions were as follows:

3) Does it bother you to see a number multiplying a letter, like in '3 × a?'

4) Can you tell me why?

5) What about boxes, does it bother you the same way when you see

'3 × □'?

All six subjects stated that it did not bother them to see a number multiplying a letter, thus confirming that no cognitive dilemma existed for them and suggesting that the primary aim of Lesson 1 had been achieved. This positive result could be attributed to the presentation of the lesson, where the letter was continually linked to a numerical referent, coupled with the subjects' prior apparent understanding that letters represented numbers, as evidenced in the pretest.

Naturally since numbers multiplying letters were not presenting any cognitive difficulties, and since Lesson 1 presented letters as an extension of the placeholder, the box, the expression '3 × □' did not present any difficulty to the subjects.

Since Lesson 1 presented the literal symbol as the extension of the placeholder, the next two questions were included to determine to what extent the student did view the letter as playing the same role as the placeholder, and whether or not the subject saw any significant differences between the two symbolisms.

The questioning proceeded as follows:

6) What difference do you see between '3 × a' and '3 × □'?

7) Can we use a letter like a box?

The subjects responded as follows:

<u>Wendy</u> felt there was no difference between the two expressions "... because in both cases you just have to fill in the number, the <u>a</u> and the box". She also responded that a letter could be used like the box.

<u>Frankie</u> saw no difference between the two expressions, and stated that a letter could be used like a box. "The box can be either an <u>a</u> or a box." and he wrote as an example '3 × $\overset{A}{□}$ × 4 = '

"You can put a <u>A</u> or a box , there's no difference."

Antoinetta claimed the only difference in the two expressions was that one was a letter and one was a box, however both represented a 'missing number'. To the question, can we use a letter like a box, she responded hesitantly, "I guess so."

Gail saw the difference in the two expressions in terms of 'the shape' and also that the letter and the box could (but not necessarily had to) represent different numbers. She stated that a letter could be used like a box.

Filippo viewed the difference between the two expressions in terms of the evaluation of the symbol. "I don't think there really is a difference, only where the number goes. That's all this would mean, a = 7 or something (referring to the '3 × a') and this would mean 3 times something (wrote '3 × $\boxed{8}$ ')

Yvette again repeated her preference for letters when questioned about the difference between '3 × a' and '3 × □'. "I like it better with a letter, the letter let's you know there has to be a number there and you don't have to stick it (the number) in a box."

Two students (Wendy and Frankie) explicitly claimed there was no difference between the two expressions '3 × a' and '3 × □'. The difference cited by two other students, Antoinetta and Gail, reflect a difference of form rather than one of meaning, "one is a letter, one is a box", and "the shape". Filippo's response went further than merely a question of form, touching on a difference in the evaluation procedure - the box required a 'filling in', whereas the letter required an equality statement, 'a = 7'. (The different evaluation procedures will be dealt with in the next question.) Yvette continued to maintain that the only difference was that she "liked it better with a letter". Her final statement "you don't have to stick it in a box", like Filippo's response, also touched on the evaluation procedure. Although some differences were cited by four of the students, the meaning of both expressions to the students wes essentially the same; both meaning 3 times some number. Thus the subjects grasped the idea that the letter was an extension of the placeholder.

In order to determine further whether the students were able to perceive that the only difference between the two symbolizations was in their evaluation, the following question was added.

8) When we uncover the part that gives us the missing number, do we do the same thing when we use a box or a letter?

The responses were as follows:

Wendy claimed that the same thing was done when using the letter or the box.

Frankie stated for "the a we wrote '3 × 7' and for the box, we put 3 times 7 (put 7 in the box)"

Antoinetta stated that we do the same thing for the letter as the box. After being further questionned by the interviewer, she said that for the box we "filled it in" and for the letter "filled it in, or we said what the letter stood for."

Gail also claimed we did the same thing for the letter as we did for the box, she said "We were trying to find the number that fit...they're both the same thing."

Filippo as a response to the previous question had discussed the evaluation of the two different symbols. For the letter, he claimed, an equality had to be written, (a = 7) and for the box, no equality was required, that is a number merely had to be filled in the box.

Yvette following her belief that "letters are better", followed her previous explanation by pointing out that for letters you merely have to write "3 × 6 below 3 × a" (she wrote '$\begin{matrix} 3 \times 6 \\ 3 \times a \end{matrix}$') but for the box you have to write '3 × 6' and make a box". She illustrated her statement by writing

$$\begin{matrix} 3 \times \square \\ 3 \times \boxed{6} \end{matrix}$$

The subtle difference in the evaluation procedure of the two symbolizations, that is, letters required the rewriting of the expression whereas the box merely required a 'filling in', was perceived by three of the subjects (Frankie, Antoinetta and Filippo). Wendy's and Gail's responses reflect that both were either not aware of this difference or did not assign much significance to it, with Gail concentrating on the fact that both represented a number. Yvette viewed the evaluation procedure quite differently than all other subjects. She felt that both the letter and the box required a rewriting of the equation, in each case substituting a numerical value for the symbol. However, the evaluation for the box was more tedious, since this symbol required that a box be drawn around the substituted number.

Summary

The fundamental aim of Lesson 1 was to introduce the literal symbol as an extension of the placeholder, 'the empty box' - both representing hidden quantities, both tied to numerical referents. This was done in order to eliminate the cognitive dilemma evidenced in the Initial Pilot Study (How can you multiply a number by a letter?)

The responses to the eight questions in this section confirm that the aim of Lesson 1 was achieved. The subjects ultimately perceived the two symbolisms, 'the empty box' and the letter, as both having the same meaning since they both were tied to a numerical referent - a hidden or 'missing' number. The differences that were perceived were that of form (one is a letter and one is a box), the possibility of some confusion that may arise from their use ($\underline{\times}$ for multiplication; is it an operation or a number?) and the different evaluation procedures.

The ultimate test question used to evaluate the success of Lesson 1 was "Does it bother you to see a number multiplying a letter?", to which <u>all</u> subjects responded "No". This suggests that the presentation in the Lesson successfully maintained a close link between the literal symbol and a numerical referent. This consequently allowed the subjects to accept a number multiplying a letter since to them the letter always represented a number. Thus the cognitive difficulties evidenced in our Initial Pilot Study, the basis of Lesson I, had been successfully dealt with in this Lesson.

D. Analysis of Homework

At the end of the Lesson problems were assigned to the students, and reviewed with them at the beginning of Lesson 2 about 2 days later.

Please make up a

- dot problem where the answer is '3 × a'
- line problem where the answer is '5 × b'
- an area problem where the answer is '6 × d'

The assignment of homework problems had two aims. One, it allowed the student, himself, to review the topic covered in the lesson, and two, it set the stage for a review in the succeeding lesson of the previous one, which then provided for a stepping stone into the next topic.

The questions prepared for this review were as follows:

At our last meeting, I had asked you to make up 3
problems. First a dot problem, where the total
number of dots is '3 × a' .
What does the <u>a</u> stand for?

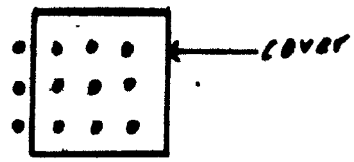
And a line problem, where the length of the
line is '5 × b' .
What does the <u>b</u> stand for?

And an area problem, where the area of a rec-
tangle is '6 × d'.
What does the <u>d</u> stand for?

The following is a summary of the review as it proceeded with each subject.

<u>Wendy</u> drew three correct problems.

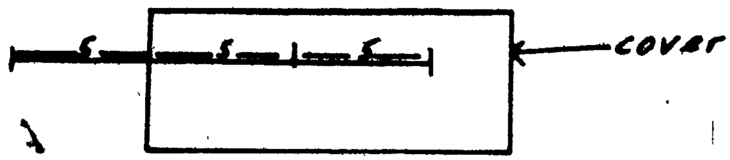
Dot Problem:



and wrote '3 × a'.

Then she uncovered and said the <u>a</u> stood for 4, and wrote '3 × 4'.

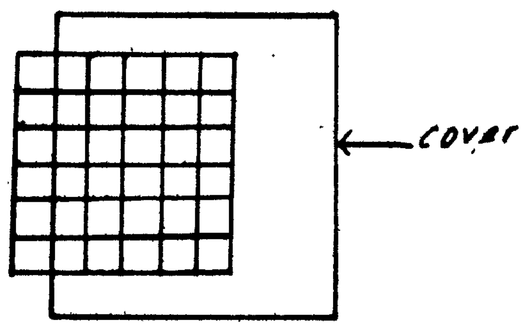
Line Problem:



and wrote '5 × b'.

Then she uncovered and said the <u>b</u> stood for 3 and wrote '5 × 3'.

Area Problem:

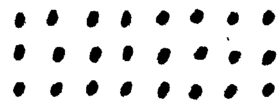


and wrote '6 × d'.

Then she uncovered and said that the "<u>d</u> stands for 6", and wrote '6 × 6'.

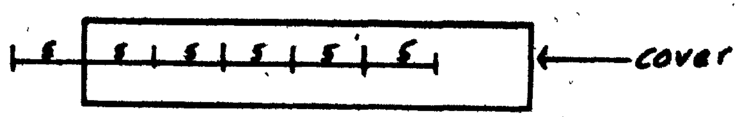
Antoinetta

Dot Problem: She showed three rows, eight dots in each row, '3 × 8 = 24'.



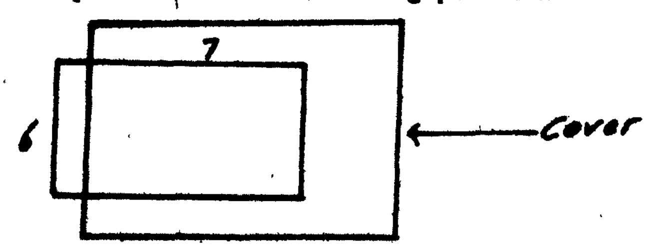
The interviewer then gave her a piece of cardboard to cover the appropriate part, however, she did not use the cardboard. She had written '3 × 8 = 24', so she covered the 8 with her finger. The interviewer reminded her that in the previous lesson we had covered a part of the problem, and proceeded to cover the bottom two rows and asked "How would you write the total number of dots?" Antoinetta responded by writing -'total number of dots = '. However, she was unable to continue. She was then prompted into completing the equation with '3 × a', by the suggestion that she was supposed to draw a problem where the total number of dots was '3 × a'. When she was asked what the <u>a</u> stood for, she replied 8. From here on Antoinetta was able to present the problems she had prepared for her assignment in a correct manner, that is, by hiding one of the dimensions.

Line Problem: She presented the following:



She covered the parts showing only the length of one of the parts, and wrote 'length = 5 × b'. After uncovering, she wrote '5 × 6 = 30'.

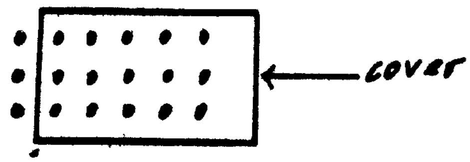
Area Problem: She presented the following problem:



After uncovering, she wrote, '6 × 7 = 42'.

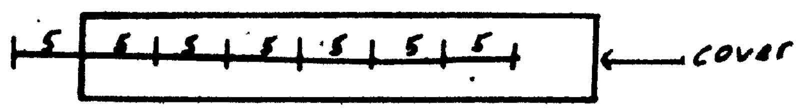
Frankie

Dot Problem: When he was asked to show his dot problem, he questionned "Should I cover it up". After a positive response, he presented the following problem.



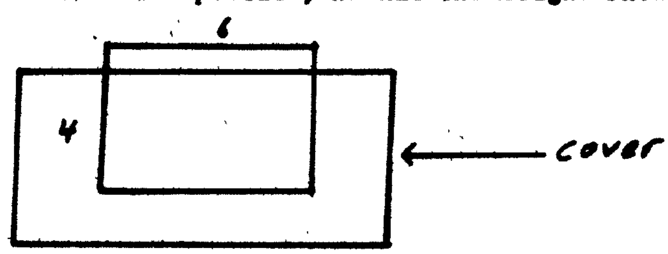
After uncovering, he stated that the a stood for the number 6. At no point did he write the expression '3 × a', nor substitute a 6 for the a.

Line Problem: He also presented a correct line, covering the appropriate parts.



Again at no point did he write the algebraic or arithmetic expression, he merely stated that the b stood for "the number of parts, seven".

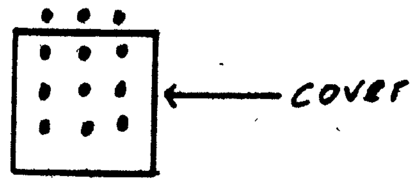
Area Problem: For the area problem, he hid the height showing the base to be 6.



This was a correct representation of '6 × d', however, in his problem the letter stood for the height not the base (as shown in Lesson 1). After uncovering, he said "the number would be 4". (referring to the letter d.)

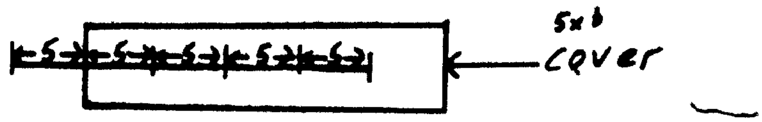
Gail

Dot Problem: She showed 3 dots in a row, hid the other rows and wrote '3 × a'.



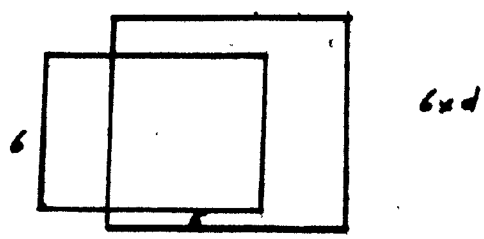
When she was asked what the letter a stood for, she questionned whether she should lift the paper. She was encouraged to respond without lifting. She then said "You have to find another number that fits the operation". And then being asked "It stands for a number and it is the number of .....?" She answered "rows".

Line Problem: Her line problem showed one part to be 5 units long, she hid the remaining parts and wrote '5 × b'.



When asked what the b stood for, she replied "the number of pieces".

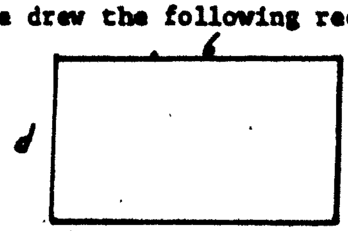
Area Problem: She presented a rectangle showing only its height to be 6, and covered the base, and wrote '6 × d' next to the rectangle.



When asked what the d stood for she said "the length of the base".

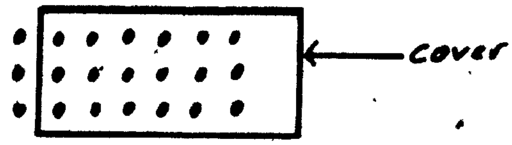
Filippo did not prepare the homework problems. He was then asked to prepare the problems while the interviewer waited. The problems were not presented in the same order as the other subjects, since the interviewer did not ask them in that order.

Area Problem: He drew the following rectangle



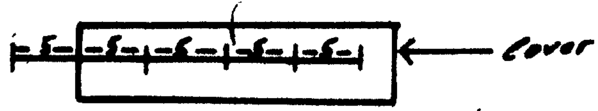
and wrote 'AREA = 24'.

Dot Problem: He showed the number of rows, but hid the number of dots in each row,



and wrote 'Total = 21'.
When asked what the a stood for, he uncovered and said 7.

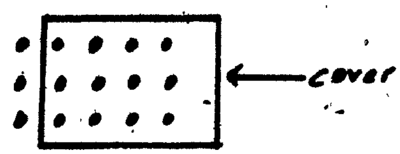
Line Problem: He showed one part 5 units long, and covered the other parts,



and wrote 'Total = 25'.
When he was asked what the b stood for, he uncovered and said 5.

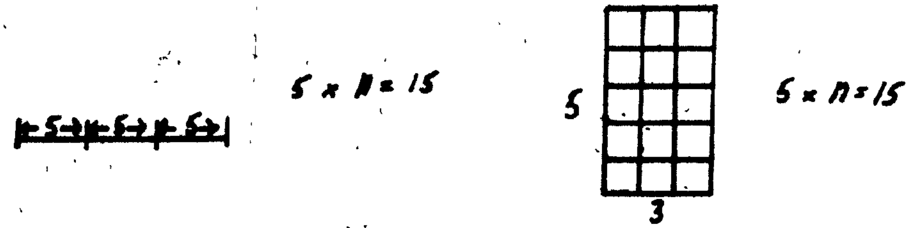
The interviewer then went back to the area problem and reminded Filippo of the problem types in the lesson, drawing a rectangle with height 6 and hiding the base with a piece of cardboard, followed by the question: "...how would you write the area of this rectangle?" Filippo replied, "I would write 6 times something equals, or 6 times a equals N ." He was then reminded that he was not given the area, to which he replied "We just said 6 times a ."

Yvette prepared the following dot problem. She showed 3 rows and covered the number of dots in each row. She had written above the problem, '$3 \times n = 15$, $n = 5$',



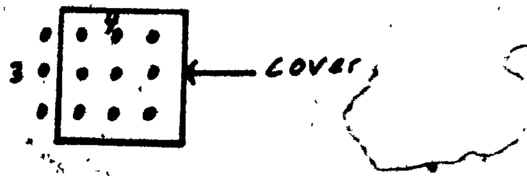
When she was questionned if she could have written '$3 \times a$', she replied "yes".

Further questionning in the interview revealed that there had been a misunderstanding in the instructions. She had read the word 'is' to be the number 15 in the instruction "Draw a dot problem where the total number of dots is '$3 \times a$'." She had prepared a line and area problem indicating the same misunderstanding.



She was then asked to redo the three problems.

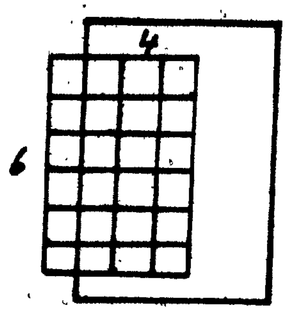
Dot Problem: She presented the following problem,



and wrote '3 × A'.

When she was asked what the A stood for, she uncovered and said 4.

Area Problem: She redrew one to represent the expression '6 × d' as follows:



She hid the base, and wrote '6 × d'.

When she was asked what her letter stood for, she uncovered and said 4.

The homework shows that Lesson 1 was not as easy as it appeared. Three of the six students, Antoinetta, Frankie and Filippo indicated some cognitive difficulty with using letters as representing hidden quantities. Antoinetta did not bother to cover any dimension in the dot problem; Frankie felt he had to ask, "Should I cover it up?"; and

Filippo drew a rectangle showing the height to be d units in length and wrote 'Area = 24', instead of '6 × d', thus avoiding the use of the letter d for the unknown quantity. It should be noted that these responses reflect different kinds of difficulties. For Antoinetta it may be a question of remembering how to set up these problems, for Frankie it could be merely a question of reassurance, and for Filippo it might indicate an avoidance of the use of letters. However, the cognitive difficulties should not be judged out of proportion since they were easily overcome during the course of the review, as indicated by their second and third problems.

It is interesting to note that in the dot problem, 5 out of the 6 subjects reversed the original presentation by having the letter represent the number of dots in each row rather than the number of rows. Also in the area problem, Frankie indicated that the letter represented the height, not the base, as in the problem type that was presented in Lesson 1. Thus, the subjects did not appear tied to specific representations, and had understood that the literal symbol could stand for any hidden quantity.

The presentation of Lesson 1 linked algebraic expressions to "equations", as in the example: 'total number of dots = 5 × a', where the left-hand side expresses in words the quantity or measure that is sought. However, three of the 6 subjects, (Wendy, Gail and Yvette) in presenting their problems wrote the algebraic expression by itself, as the 'answer' to the problem. This could perhaps indicate that they were able to accept the lack of closure of such expressions within the context of such visualizable problems. Even Filippo who was very tied to an equation framework (He wrote an equality next to each problem) at first stated that 'I would write 6 times something equals, or 6 times <u>a</u> equals N ", but finally concluded at the end of the review that if he were not given the area of the rectangle "We just write 6 times <u>a</u> ." This ability of the students to leave expressions such as '6 × a' as answers, not only touches on the acceptance of lack of closure, but also addresses Davis' 'name-process' dilemma. Students were specifically asked to draw problems where the <u>answer</u> would be a particular algebraic expression. For example: "Please make up a problem type where the answer is '3 × a'." At no point did any of the subjects question that the expression was the <u>answer</u> and in most cases presented correct problems to represent these 'answers'.

<u>CONCLUSION</u>

Lesson 1 was developed to prepare the student for the subsequent lessons in the teaching outline. The aim of the teaching outline was to construct meaning for algebraic expressions by presenting them as <u>answers</u> to visualizable problems. Thus the first step was to introduce the three different problem types (total number of dots in an array; length of a line segment; area of a rectangle). The analysis of Lesson 1 indicated that the introduction of these problem types presented little or no difficulty for any of the subjects. Only Filippo had experienced some minor difficulty in drawing a line problem where the answer would be '3 × □'. He had interpreted the box as representing the length of each part in a line segment rather than the number of parts.

A fundamental part of the presentation of the problem types was the understanding that the placeholder, and later the literal symbol, represented some <u>hidden</u> dimension of the problem. The extent to which the subjects had grasped this concept is important to determine. The extent of this understanding can somehow be inferred from the 'reversals', that is, where students had to generate their own problems corresponding to given algebraic expressions. During the course of Lesson 1, the students had very few difficulties in generating these problems, whether the hidden quantity was represented by a placeholder or by a literal symbol. As shown in the interviews the minor difficulties were easily overcome.

However, the homework review had provided evidence that this concept, the representation of the hidden dimension, was not so easily understood or remembered as initially believed. Antoinetta had completely forgotten about hiding a dimension of the problem; Frankie questionned whether or not to hide; and Filippo merely drew a rectangle with height <u>d</u> units, hiding no dimension of the figure. However, these difficulties were easily overcome during the course of the review, and by the end, all students were presenting correct problems, hiding appropriate parts. Whether these difficulties uncovered in the review, are due to understanding or to memory cannot be ascertained.

Another major concern of Lesson 1 was that the students should not experience the cognitive dilemma evidenced in out Initial Pilot Study, "How can you multiply a number by a letter?" Thus, the lesson introduced letters as merely extensions of the familiar placeholder, 'the empty box' - both represented some hidden quantity, both were tied to a numerical referent. The analysis of this lesson indicates that the transition to letters was very easy for all of the subjects. None of the students experienced any cognitive dilemma with seeing and writing numbers multiplying letters. The ease of the students' transition from placeholder to letters can be explained by the fact that the placeholder, and then the literal symbol, was always tied to a numerical referent. The placeholder, itself, based on the students' experience, carried with it the notion of number, in particular the 'filling in' of a missing number. And since the letter was introduced as an extension of the placeholder, this numerical referent was maintained. Also, the presentation of the lesson associated both symbolisms with the representation of some hidden quantity which was continually evaluated after the uncovering. The tie the students had between the letter and a numerical referent cannot be solely attributed to the presentation of the lesson since such a connection was also evident in the pretest which indicated that the students, prior to the lesson, had viewed letters as representing numbers.

A final interesting aspect to note is that some of the subjects were beginning to indicate acceptance of lack of closure, as evidenced by their leaving expressions such as '$3 \times a$' unconnected to equations. The students accepted these expressions as answers and drew appropriate problems. Such results suggest that the presentation in Lesson 1 provides a possible solution to Davis' 'name-process' dilemma experienced by many beginning algebra students, who cannot view an algebraic expression as both an answer and an operation.

CHAPTER VII

ANALYSIS OF LESSON 2

The Literal Symbol as an Unknown Quantity

INTRODUCTION

Lesson 1 introduced the literal symbol as an extension of the placeholder, and both were used to express a hidden dimension of a problem. Assuming that the subjects could use literal symbols without any cognitive dilemma in the context of a hidden quantity, Lesson 2 aimed at introducing the conventional usage of the literal symbol, that of representing an unknown quantity, not just a hidden one. The same three problem types introduced in Lesson 1 were presented again in Lesson 2. Thus the literal symbol used as representing an unknown quantity became an extension of its previous use, that of representing a hidden quantity. In this lesson, where possible, the motivation for substitution was generated by measuring. Of course only the area and length problems provided such a situation. For the dot problem, the notion of hiding had to be maintained.

Concatenation had been delayed until the second part of Lesson 2. This was justified essentially by two pedagogical considerations. First, it was felt that prior to Lesson 3, which dealt with algebraic expressions involving both multiplication and addition, it was essential that this convention be introduced in order to make the expression less cumbersome. The second consideration which prevented the immediate introduction of concatenation was based on the results of the pretest. These results showed that concatenation was a major problem, since to the student, this notation eliminated arithmetic meaning, that of a product. Thus, in order to keep the literal symbols close to their numerical referents, the multiplication sign is essential at the beginning, since it explicitly creates an arithmetical context.

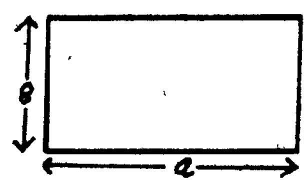
A. <u>The Literal Symbol as an Unknown Quantity</u>

The order in which the problems in this lesson were presented did not coincide with that of Lesson 1. The first problem type to be introduced was the area problem, since it provided the best situation to represent the letter as an unknown dimension. The dot problem, although the first to be presented in Lesson 1, was presented third in this lesson, since it was found in the Initial Pilot Study that a simple convention that we could devise to denote an unknown quantity of dots was not interpreted as such by the subjects. Consequently, this problem type was presented last and the notion of hiding was maintained.

1. <u>Area Problem</u>

i) <u>The Initial Problem</u>

The first area problem was introduced as follows:



Let's look at this area problem. Do you think you could write down the area of this rectangle?

If the student was unable to respond, the interview continued as follows:

What is the length of the height?
What do you think is the length of the base?
What do you think the letter <u>a</u> stands for?
This problem is a bit different from what we used to do. In our other problems we used to cover the base so that you did not know the length of

the base. Do you remember, what you did then to write the area?

Area = 8 × ?

(If the student completes with a letter continue. However, if he uses a box, remind him about what we used after boxes)

So this problem is different in that neither of us knows the length of the base. But the letter <u>a</u> can be used to stand for it. Can you now write the area?

(If the student was able to immediately answer by writing '8 × a', the explanation regarding the difference in the problems in this lesson and the other one was still presented).

The interview with each of the subjects proceeded as follows:

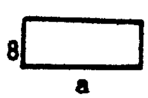
<u>Wendy</u>

I: Do you think you could write down the area of this rectangle?
W: Can I use the ruler to find the measurement of the <u>a</u>?
I: Without the ruler.
W: (writes '8 × <u>a</u> = ') I couldn't, I would have to find what
'8 × _ = ') replaces <u>a</u>.
I: What if I write this: 'Area = '. Complete this equation.
W: (writes '8 × a')
I: You have written 8 times <u>a</u>. Do you understand it better when I write Area equals, rather than if I ask you to write the area?
W: Pardon
I: When I wrote Area equals, you completed the equation easily, but here when I asked you first to write the area, you wrote '8 × a = ', and then you wrote 8 times a line. Why did you write 8 times line?
W: I didn't have to draw the line, I could have just left a space.
I: Why would you have to leave a space?
W: Because you would have to have the measurement of <u>a</u>.
I: So if I say what is the area of this rectangle, you can't just write 8 times <u>a</u>.
W: No, unless you write 8 times <u>a</u> is the area of the figure.

This interview excerpt with Wendy highlights four important problems: 1) the unintended misinterpretation of the question, 2) the transition from a letter representing a hidden dimension to a letter representing an unknown dimension, 3) the strong link of letters with the notion of placeholder and 4) the inability to accept lack of closure as well as the cognitive obstacles involved in the 'name-process' dilemma.

1) Based on Wendy's ability to produce a problem when the question was worded as in her homework "please make an area problem where the answer is $6 \times d$", it was expected that no difficulty would be encountered when she was asked to write down the area of the given rectangle. However, the form of the first question in this interview differed from the area problems in Lesson 1, where each question was started by asking her to complete the equation, 'Area = '. This may partly explain why Wendy felt she had to measure the length of a rather than provide an algebraic expression as the answer.

2) This need to measure cannot be simply attributed to the wording of the question. It may also reflect the change in the connotation of the literal symbol (from hidden to unknown dimension). It may be easier to hold in suspension an algebraic expression such as '$8 \times a$' in the context of a hidden quantity, for the hiding implies a second phase, that of uncovering, which will allow the student to evaluate the expression. On the other hand, in the context of unknown dimension, as in



, the notion of hiding is missing and thus cannot help in holding '$8 \times a$' in suspension. This may also explain Wendy's spontaneous response "can I use a ruler..."

3) The strong link of the letter with the placeholder interpretation is expressed by Wendy writing '$8 \times a =$ '

and '$8 \times \underline{\quad} =$.'

It should be noted that she wrote the equal symbol, thus indicating that either form of expression was still incomplete. And it would remain incomplete until she would be able to evaluate <u>a</u> ("I would have to find what replaces <u>a</u> ").

4) Thus as seen, '$8 \times a$', is incomplete. It is only when Wendy was asked to complete the sentence, 'Area = ' that she seemed to accept '$8 \times a$' without feeling she had to evaluate the <u>a</u> . By at first writing the equal symbol after the expression '$8 \times a$', Wendy clearly indicated that she viewed the expression '$8 \times a$' as incomplete. Her expressed desire to find what replaces <u>a</u> is a clear indication of her inability to accept the lack of closure of this expression. The problem here is not limited to the lack of closure of the expression, but it also involves the name-process dilemma. This is very clearly illustrated by the fact that Wendy had no difficulty in completing the equation 'Area = ' and by her further comment indicating that the expression '$8 \times a$' was acceptable <u>only</u> in this context ("unless you write 8 times <u>a</u> is the area"). A possible explanation here is that in the context of completing the equation 'Area = ' the expression '$8 \times a$' need not play a double role, that of name and that of process. The word "Area" appearing on the left expresses the <u>name</u>, thus '$8 \times a$' can be limited to express the <u>process</u>. It seems that the question "Can you complete

'Area = ?'" might avoid the cognitive problems involving lack of closure and the name-process dilemma.

The interview with Frankie proceeded as follows:

Frankie

I: Do you think you could write down the area of this rectangle?
F: 8 times a, you would have to find the number for a. (Writes '$8 \times a$').
I: You have to find the number for a.
F: You could measure it.
I: If I said (says and writes) 'Area = '.
F: (Completes with) 8 times a (writes $8 \times a$)

Frankie, unlike Wendy, was able to spontaneously write '$8 \times a$'. However, he demonstrated that he sensed it was not the answer by stating "You would have to find the number for a ... you could measure it." Thus, in the sense that the algebraic expression is not the answer, he was unable to accept the lack of closure. As in the case of Wendy, the context of an equation "Area = $8 \times a$" appeared to have assisted him, and he no longer appeared to require the measurement of a. Thus within this context the answer seemed complete, since no desire for measurement was stated.

Antoinetta responded in the following manner:

Antoinetta

I: Do you think you could write down the area of this rectangle?
A: (no response)
I: O.K. What is the length of the height?
A: 8
I: What do you think is the length of the base?
A: (no response)
I: What do you think the letter a stands for?
A:... maybe 10,... well I am just looking at the lines.
I: This problem is a bit different from what we used to do. In our other problems we used to cover the base so that you didn't know the length of the base. Do you remember what you did to write the area?
A: (no response)

I: You could write 'Area = 8 × '
A: Times <u>a</u> (completes with an <u>a</u>.)

Antoinetta was initially totally unable to provide any response at all. And this is quite surprising, after all, the pretest showed she had no problem with area in a numerical context: the area problems presented in Lesson 1 were handled easily with the literal symbols representing hidden quantities; and the first homework assignment was dealt with efficiently. So it was quite surprising to find that Antoinetta could not make the transition from letter used as a hidden quantity to letter used as unknown quantity. In fact her paralysis indicates quite a major cognitive gap, preventing this transition.

When probed about the meaning of the letter <u>a</u>, she responded numerically by estimating the length of the base. Reminding her of her previous work did not elicit any response. She had to be literally fed the answer to the problem by completing the equation "Area = 8 × ".

The interview with Gail proceeded as follows:

<u>Gail</u>

I: Could you write the area of this rectangle?
G: (writes '8 × a = ; 8 × 10 = 80')
I: You wrote 8 times <u>a</u> equals, and 8 times 10 equals 80, why did you write 8 times 10?
G: Because the <u>a</u> stands for the 10 -
I: How do you know it is 10?
G: Because of the length, it is a little longer than that (referring to the height), it's two more, so it would be 10.
I: Do you know for sure it is 10?
G: No.
I: So is the area definitely 80?
G: No.
I: (explains the difference between the problem types of this lesson and those of the previous one, then asked) Do you recall the formula for area of a rectangle?
G: Area?
I: Area = length of height × length of base (says and writes it)
G: (says formula with interviewer)

I: What is the length of the height?
G: 8
I: Write 8 (Gail writes 8 under the words 'length of height') and What is the length of the base?
AREA = LENGTH OF HEIGHT × LENGTH OF BASE
8 × a
G: a (writes '× a')

Gail initially behaved similarly to Wendy, and provided at first the response '8 × a = ', but then substituted the number 10 for the a (an estimate of the length of the base), arriving at a final numerical answer (8 × 10 = 80). Thus as with Wendy, the expression '8 × a' was incomplete to her, as indicated by her writing an equal sign after it, and by her further attempt at providing an estimated answer for the area. This behaviour strongly indicates her inability to accept the lack of closure of the expression '8 × a'. However, within the context of the formula, the numerical and literal substitutions seemed acceptable.

Filippo's interview proceeded as follows:

Filippo

I: Do you think you could write down the area of this rectangle?
F: Like what do you mean, make it up.
I: Can you write its area?
F: No.
I: What is the length of the height?
F: 8
I: What do you think is the length of the base?
F: 12?
I: Do you know for sure it is 12?
F: No.
I: What do you think the letter a stands for?
F: The missing number, the number that signifies the length of the base.
I: (explains the difference between this problem and the ones of the previous lesson) Do you remember what you did then to write the area?
F: First I would have multiplied the two numbers that I had.
I: Let's say I would have covered it, how would you have written the area? Area =

F: says and writes '8 × d'.
I: (continues to explain the difference between the two problems) So we write the area as... Area = 8 ×
F: a (writes 'a')
I: Something seems to be bothering you.
F: Well first of all, if you are trying to find the answer you won't get it.
I: The answer does not have to be the area as a number, I want to find a way to write the area without knowing the length of the base.
F: So you are saying you just have to say what you have to multiply to get the area.

As in the case of Antoinetta, Filippo could not respond to the initial question asking him to write the area of the given rectangle. And the reason appears to be his numerical frame of mind which is indicated by his need to estimate the length of the base.

Within the context of the equation "Area = 8 × ", although he completed it with an a, this correct answer did not suggest to Filippo that he had finally arrived at a response to the original question. He clearly stated, "If you are trying to find the answer you won't get it." Thus the algebraic expression to Filippo is definitely not the answer, even within the context of the equation. To Filippo, the expression is a statement of the operation to be performed to get the answer. ("So you are saying you just have to say what you have to multiply to get the area.") Filippo's difficulty to perceive the algebraic expression as the answer clearly demonstrates the Davis "name-process" dilemma.

The interview proceeded as follows with Yvette:

Yvette

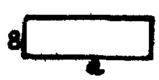
I: Do you think you could write down the area of this rectangle? Area =
Y: Says and writes '8 × A'; that could be a number; so '8 × 2' (writes it under '8 × A'
'8 × 2')

I: So you wrote $8 \times A$ and then you wrote 8×2. Do you know the length of this line? (referring to the base)
Y: No, you don't. It could be any number.
I: If I want you to complete this 'Area = ', is it '$8 \times A$' or '8×2'?
Y: The area is '8×2'.
I: The area is '8×2', why?
Y: Because that means you have to find how long that is (referring to the a)
I: Do you know it is 2?
Y: It could be any number.
I: How do we show it could be any number?
Y: You put the '$8 \times a$'.
I: The a is any number. So if I were to ask you to write the area of this rectangle, without a final number answer...
Y: You would just put 8 times a.

Yvette had the advantage of a variation in the wording of the initial question. The interviewer immediately provided the incomplete statement "Area = ". This can explain why Yvette did not have the initial cognitive problems that the other students experienced. She completed the statement with the expression '$8 \times a$'. However her insistance ~~that~~ '8×2' and not '$8 \times A$' answered the question "to write the area" indicates that she also feels an incompleteness with the algebraic expression, even in the context of an equation. This need to provide a numerical answer indicates her difficulty in accepting lack of closure. She stated however, that if no final numerical answer was required, "you would just put 8 times a". Recalling from the pretest that Yvette was the student who had achieved a very high level interpreting the literal symbol as a 'generalized number' ("x can represent any number") it is interesting to note that this interpretation did not seem to help her in accepting the lack of closure of '$8 \times a$'.

FIRST REMARKS

In Lesson 1 we had asked students to complete equations such as "area (length, number of dots) = ?", while they were shown an appropriate representation of the problem in which one element of the problem was hidden. Within that first lesson, the students were asked verbally "to make up a line problem where the answer (or length) is seven times d ", while only the expression $7 \times d$ was written. Similarly, they were asked verbally "to make up a dot problem where the total number of dots (or the answer) would be nine times c while only '$9 \times c$' appeared in writing. For the area problem, students were free to make up their own problem without being provided with any expression. Furthermore, they were successful in generating such problems for their homework. Thus it seemed that our subjects were beginning to accept the lack of closure of algebraic expressions.

Lesson 2 had started with a review of the homework and therefore these types of questions, in which the algebraic expression appeared by itself, were fresh in the student's mind. Thus we did not anticipate any difficulty with the question "Can you write down the area of this rectangle ()?" since all that was required was for the student to write '$8 \times a$'. However, the responses of each one of our six subjects indicate that they all experienced new difficulties. An analysis of the interview reveals that the explanation for this situation is quite complex.

A first observation is that all students misinterpreted the question in the sense that they felt that they had to provide a numerical answer to the problem. Both Wendy and Frankie expressed the need to measure the length of the base. Antoinetta, Gail and Filippo estimated the length. While Yvette immediately substituted a numerical value in her expression.

This need to provide a numerical answer did not manifest itself in Lesson 1, where the literal symbol represented a hidden quantity. Thus it seems that this transition to letter used as unknown involves greater cognitive obstacles. The conjecture provided in the analysis of Wendy's interview seems to be a reasonable explanation: it may be easier to hold in suspension an algebraic expression such as '$8 \times a$' in the context of a hidden quantity, for the hiding implies a second phase, that of uncovering, which will allow the student to evaluate the expression. On the other hand, in the context of unknown dimension, the notion of hiding is missing and thus cannot help in holding '$8 \times a$' in suspension.

Of course, one can hardly discuss holding an expression in suspension without relating it to the acceptance of the lack of closure. Two of our subjects, Wendy and Gail, indicated that they viewed the expression '$8 \times a$' to be incomplete by writing an equal sign after the expression ('$8 \times a =$ '). Both Frankie and Yvette wrote the expression '$8 \times a$' but immediately made some reference to evaluating the a . Antoinetta and Filippo were not able to write the expression, whereas they had managed to do so in the previous lesson. Thus it appears our assumption based on the results of Lesson 1 were premature and that in fact our subjects did not accept the lack of closure of an algebraic expression.

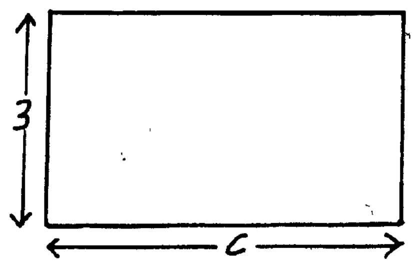
While our subjects gave strong indications that they could not accept the algebraic expression by itself, they also indicated that it was far more acceptable to them in the context of completing the equation "Area = ". Wendy stated that "no, (you can't just write '8 × a), unless you write 8 times <u>a</u> is the area of the figure." Frankie completed the equation without further reference to the need for measuring. Gail substituted '8 × a' in the formula for the area, without using her estimated length for <u>a</u>. Filippo also completed the equation without using his estimated length. A possible explanation of the students' greater acceptances of the expression within the framework of an equation can be sought in terms of the <u>name-process dilemma</u>. By completing the equation "Area = ?", the left hand side clearly expresses the <u>name</u>, thus leaving to the right-hand side the function of expressing the <u>process</u>, 8 × a. Yvette indicated this process interpretation of the expression by responding to the question about writing the area of the rectangle without a final number answer: "You would just put 8 times <u>a</u>". But it is Filippo who expresses this idea most explicitly in commenting on the '8 × a' in the question, 'Area = 8 × a': "So you are saying you just have to say what you have to multiply to get the area."

Filippo perceives the algebraic expression as indicating a process '<u>how to</u> find the area', rather than as an answer to the problem. He clearly stated that in writing '8 × a' for the area of the rectangle "if you are trying to find the answer, you won't get it." Based on his comments, it is interesting to conjecture whether or not a process oriented approach ('how to') in the construction of meaning for algebraic expressions might lead to an easier acceptance of lack of closure.

ii) The Second Area Problem

The second area problem provided another situation for the subjects to write an algebraic expression to represent the area of a rectangle. The problem was presented as follows:

What is the area of this rectangle?



All subjects went about finding an answer to this problem without any of them expressing the need to ask further questions.

Wendy, Antoinetta, Filippo and Yvette wrote 'Area = 3 × c'.

Frankie and Gail wrote just '3 × c' as the answer.

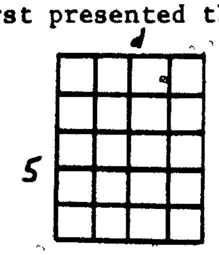
The ease with which all students found the answer to this second problem indicates that the cognitive difficulties uncovered with the first problem seemed to be overcome. None of the students expressed the need to measure or to provide a numerical value for c . Quite obviously, this second area problem was interpreted by the students as intended. The students seemed to have made the transition from letter as a hidden quantity to letter as unknown. It would be too hypothetical to infer about any change in their acceptance of lack of closure, since no further questions were asked. However, it is interesting to note that four of the subjects spontaneously answered within the format of an equation, and that two of them just wrote the expression.

iii) The Reversal

As in the previous Lesson, the succeeding problem provided a situation which required the subjects to generate a problem given the algebraic expression.

The subjects responded as follows:

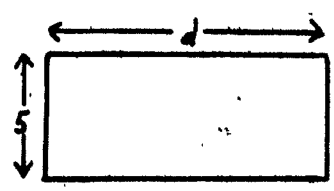
Wendy at first presented the following problem.



and wrote '5×4'.

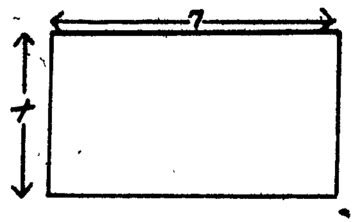
The interviewer then proceeded to remind Wendy of the type of problems in this lesson, that is, a dimension should be unknown to both student and interviewer. The interview proceeded as follows:

I: Why did we use the letter d ?
W: Because that is the letter you gave me.
I: Yes, but why are we using a letter?
W: Because we don't know what it represents.
I: We don't know what it represents, we don't know what number. I had asked you to draw a rectangle where the area is $5 \times d$. I would say that the area of the rectangle you drew is 5×4 , I know the d .
W: (says something inaudible).
I: Why do we have to use a letter for that rectangle?
W: We don't have to.
I: Could you draw a rectangle where you would have to use a letter?
W: No.
I: No, could you draw a rectangle where we both don't know the length of the base?
W: 5 times d ?
I: Uh hum
W: (draws the following)

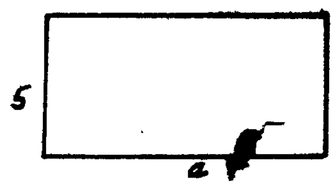


W: That wasn't supposed to be there, the squares.
I: Why weren't the squares supposed to be there?
W: Because we weren't dividing them into parts. We were just given the length and the base.

The interviewer asked Wendy to draw another rectangle where the area would be '7 × t' . She drew the following correct rectangle



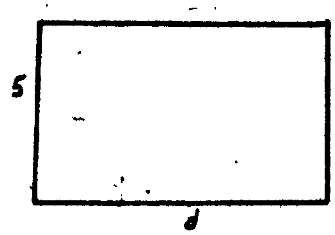
<u>Frankie</u> drew the following rectangle



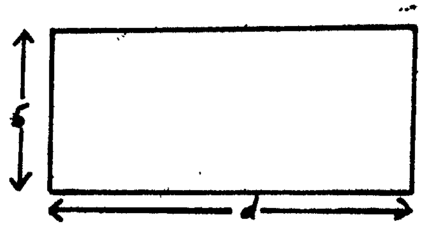
and wrote '5 × a'.

When he was reminded that the area was supposed to be '5 × d' , he changed the <u>a</u> to a <u>d</u> .

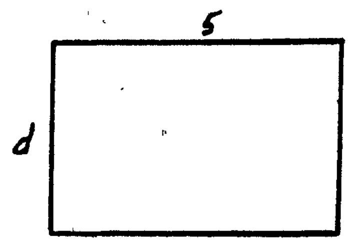
<u>Antoinetta</u> drew the following correct problem.



<u>Gail</u> drew a rectangle as follows:

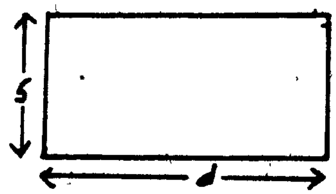


Filippo presented the following drawing:



When he was asked what the d stood for, he replied, "The number that would help you find the area, the height."

Yvette drew the following rectangle:



The problems presented by the six subjects indicate that five of them grasped the concept that the letter represented some unknown quantity, and presented correct rectangles to represent the algebraic expression '$5 \times d$' . Only Wendy initially experienced some difficulty. She drew a problem in which the two dimensions were known. Her grid seems to be a 'throw back' to Lesson 1, but it is not accompanied by any hiding. This indicates that our assumption based on her success with the second area problem was premature, and that she did not yet use the letter to represent an unknown quantity. However, as the interview proceeded, she was able to grasp that the letter was intended to represent some unknown quantity, stating, "That wasn't supposed to be there, the squares." Her succeeding problem (a rectangle whose area was '$7 \times t$') indicates that she (as well as Filippo) perceived that the letter could represent any unknown dimension, not just the base.

Filippo's response, "(d) the number that will help you find the area", further indicates his process interpretation of the algebraic expression.

iv) Substitution

The next problem required the measurement of the length of the unknown dimension in the first area problem. The measurement was followed by the substitution of this numerical value in the algebraic expression representing its area. The aim of this problem was to maintain the numerical referent for the literal symbol that had been established in Lesson 1. Previously, the link was accomplished by uncovering, now it required measurement. The problem was presented in the following manner:

> Here is a ruler. Can you measure the length of the base (referring to the first area problem, height 8, base a).
> Student measures.
> So what does the letter a stand for?
> Can you replace it in the formula, 'Area = $8 \times$ '?

The responses were as follows:

> All six subjects measured the length of a to be 12 cm., stated the a stood for '12' and substituted the number 12 for the letter a by writing '8×12'.

Yvette wrote the entire equality, as follows: 'Area = 8×12'.

Frankie and Gail completed the equation, 'Area = $8 \times$ ', with the number 12, with Gail giving a final numerical answer, 96.

Filippo and Antoinetta wrote '8×12', besides the '$8 \times a$'.

Wendy just wrote '8×12'.

The responses presented by the students indicate that no difficulty was present for any of the subjects. All students perceived the <u>a</u> to be the number 12 and provided appropriate substitutions.

<u>Summary</u>

The first area problem revealed unsuspected cognitive difficulties experienced by the students. Our assumption that they had started to accept the lack of closure of an algebraic expression proved to be premature. None of them viewed '8 × a' as the answer to the area problem as we had expected. They expressed the need to provide a numerical value for <u>a</u>. This might have been due to the wording of our question, problems in the transition from letter used to represent a hidden quantity to letter as an unknown, as well as the inability to accept lack of closure. The expression proved to be far more acceptable to the subjects in the context of completing the equation "Area = ?". This can be explained by the elimination of the name-process dilemma.

The correct responses given in the second area problem by all six subjects, suggest that the cognitive difficulties experienced by the subjects in the first problem had been overcome. No reference to determining a numerical value for the literal symbol was made by any of the subjects, suggesting that they had accepted the notion of unknown quantity. This may be true for five of the six subjects, since the reversal problem showed that Wendy had not made this transition.

The measuring and subsequent substitution, proceeded with no evident cognitive problems.

2.3 Line Problem

A modification in the line problem was necessary due to the change in the orientation of the problem from the letter representing a hidden quantity, to the letter representing an unknown quantity. Previously, the number of parts was hidden, thus, the letter represented the unknown number of parts. Since these new problems did not require any hiding, the number of parts was shown, therefore, the letter now represented the unknown length of each part. This new version of the line problem had caused some confusion for the student in our Exploratory Case Study. Thus, the introduction of the line problems in this lesson required a preamble which explained the change. The presentation was as follows:

> Here is a line problem. It is slightly different from the ones you did before. In this case we know the number of parts but we do not know the length of each part. That is why we use a letter.
> How many parts do you see?

> What does the letter <u>a</u> stand for?
> Can you write the length of the line?
> (If the student is unable to respond, have him recall the formula for determining the length of a line.)

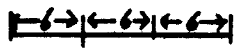
The subjects responded in the following manner:

<u>Wendy</u> The interviewer explained the new version of the line problem, then asked : "What does the letter <u>a</u> stand for?"

W: 4
I: How do you know?
W: Because there is 4 parts.
I: O.K. There are 4 parts. If we know the number of parts do we need to use a letter for the number?
W: No.
I: This time we know the number of parts, but we don't know the

I: length of each part, so we are using the letter a to stand for the length of each part.
W: a times 4 (and then says) 16.
I: Did you say a times 4?
W: a times 4
I: What is a times 4?
W: The letter a times it is 4 parts. We don't know the length.
I: Is it this, the length of the line equals...(writes 'length of line = '). What is the length of this line?
W: 4, no it is divided into 4 parts.
I: Can we write the length of this line with the information we have?
W: no response.

Wendy appeared to be having some difficulty and became rather confused. Thus, the interviewer proceeded to have Wendy recall the formula for determining the length of a line segment, and a purely numerical problem was presented to her as follows.



To which she responded the length was '6 × 3' . When she was referred back to the original problem, she said, "a times 4" and then was instructed to write 'Length = a × 4'.

Frankie was asked to write the length of the line, by completing the equation 'Length of line = '. He said and wrote '4 × a'.

Antoinetta did not wait for the explanation regarding the different versions of the line problem, she immediately stated, "I think I know what to write", then wrote 'Length = 4 × a'.
When she was asked what the letter a stood for, she answered "the part".

Gail responded to "Can you write the length of this line by completing the equation, 'Length of line = '?" by writing 'a × 4' and '4 × a'.

Filippo was accidentally presented with the second line problem,

instead of the line whose length was '4 × a'. The explanation and questioning proceeded in the same manner prepared for the first problem. When he was asked to write the length of the line he wrote, '3 × d'.

Yvette responded to "Can you complete this equation: Length of line = " by saying and writing '4 × a'. She spoke in a questioning tone, thus the interviewer proceeded to inquire what she was questioning. To which she replied, "You could have put an n there', (instead of a) (commenting on the choice of letter).

The responses given by the subjects to the first line problem indicate that five out of six subjects were able to understand the line problem, and showed no cognitive difficulties, as evidenced by the ease with which they wrote an algebraic expression to express the length of the line. Only Wendy had experienced the same confusion that was evidenced in the Exploratory Case Study. That is, she was focusing on the literal symbol as representing the number of parts, not the length of each part, which resulted in some cognitive confusion for her. She appeared to have overcome her difficulty by the end of this section, as indicated by the fact that she was able to provide the correct answer to the problem.

Of interest to note is that none of the students provided a numerical referent for the a, either by referring to 'measuring' or by stating an estimated value for the letter. They appeared to accept the concept that the literal symbol represented an unknown dimension, and that this letter could be used as part of an algebraic expression representing the length of the line. The ease with which Frankie, Gail and Yvette were able to solve this problem, by providing a correct algebraic expression, was likely due to the request to complete the statement, "length (of line) = ". Antoinetta, herself, spontaneously wrote the full equation 'Length = 4 × a'. While Filippo wrote only the expression.

It seems clear that the reason for the subjects' success in this first line problem is that after being presented with the area problems and with the letter representing an unknown dimension, they now understood the nature of the problem in this lesson, and thus were able to respond according to the intended meaning of the question.

A similar line problem, followed the first one. This additional problem was included to ensure that the subjects understood this new version of the line problem, and would be able to provide correct algebraic expressions as answers to the length of the line. The problem was as follows:

Here is another line problem.

Can you write the length of this line?

The responses were as follows:

Wendy, Antoinetta, Gail and Yvette formed an equation. On the left side they wrote either 'Length' or 'Length of Line' and on the right-side completed the equation with the expression 'd × 3' or '3 × d', with Gail writing both expressions.

Frankie just wrote '3 × d'.

Filippo was presented with the first line problem and wrote '4 × a' as the answer.

Both Frankie and Filippo did not write an equation. They wrote the expression by itself as an answer.

The responses to the second problem confirmed our impression that the line problem was well understood by all the students, including Wendy, with four of the six subjects spontaneously forming their own equation, although they were not asked to complete an equation.

As in the previous section, the introduction of the problem type was followed by a reversal. The student was asked to generate a line problem as follows:

> Can you make up a line problem where the length of the line is '$5 \times b$' ?

Antoinetta, Gail, Filippo and Yvette drew correct problems, such as

besides the given expression '$5 \times b$' which was written by the interviewer.

Wendy presented the following line problem.

> The interviewer pointed out that there was no unknown, to which she replied, "Am I supposed to cover it?" When she was asked, "How do you make a problem where I don't know the length of each part?" She replied, "the 5 would be before the b". (She was referring to the expression '$5 \times b$'). The interviewer then discussed the commutative axiom of multiplication, showing that $5 \times b = b \times 5$. She was then asked to draw a line problem which is $5 \times b$ in length and "You and I both don't know one of the numbers." She drew the following line:
>
> 
>
>
> and wrote "$b \times 4$ or $4 \times b$".
> She was then reminded that she was asked to draw a line whose length is '$5 \times b$'. She subsequently added another part to the line b units in length.

Frankie (Accidentally the interviewer forgot to present this problem to him).

Wendy was the only subject who experienced any difficulty with this problem. Once again, she was focusing on the original line problems, where the letter represented the hidden dimension, asking "Am I supposedto cover it?" Also previously in an expression like $5 \times b$, 5 indicated the length of each part, and b was to represent the number of parts. Therefore, she felt that if b was to represent the length of each part, the expression must be written, '$b \times 5$'. The short review of the commutative axiom assisted Wendy in perceiving that the order was not of importance, but that the literal symbol in these problems represented the unknown length of each part, and the 5 the number of parts.. She finally seemed to grasp this notion, by drawing two correct line problems, one which represented '$4 \times b$' (after which she wrote '$4 \times b$ or $b \times 4$') and another for '$5 \times b$'. The remaining four students presented correct line segments, the b representing the unknown length of each part indicating they understood the nature of the problem, and the role of the literal symbol in the algebraic expression.

The final problem in this section required the measurement of one of the parts in order to maintain the link between the letter and a numerical referent within the context of the line problem. The problem was as follows:

> Here is a ruler, can you measure the length of a part
> (referring to the first problem '$4 \times a$')
> What does a stand for?
> Can you replace it in the formula?

The responses were as follows:

All six subjects correctly measured the a to be 2 cm., and stated the a stood for the number 2.

Wendy given the incomplete statement "Length = ", completed it with '2×4'.

Frankie wrote '4×2', under the previously written '$4 \times a$'.

Antoinetta wrote '4×2' beside the original '$4 \times a$'. However, she appeared confused, and then when questioned about what was troubling her, she said, "I get mixed up from the other ones.. Before I was writing how many parts, now I wrote the length of each part".

Gail completed the equation, 'Length of line = $4 \times$ ', first with the number 3 and wrote, '$4 \times 3 = 12$', then changed the 3 to a 2, and wrote '$4 \times 2 = 12$'. (She forgot to change the 12).

Filippo wrote '4 × 2' beside '4 × a', however he added
' = length'. ('4 × 2 = Length')

Yvette completed the equation, 'Length of line = ' with = '4 × A'
= '4 × 2'

All six subjects had no difficulty with these problems. Antoinetta, however, did make some reference to the confusion from the change in orientation of the problem type.

Summary

The line problems did not appear to cause any major difficulties for any of the subjects. The confusion experienced by the student in our Exploratory Case Study (that it, focusing on the letter as representing the unknown number of parts, rather than the unknown length of each part) was experienced only by Wendy who had difficulty adjusting to the new problem type, as evidenced not only in the first line problem, but also in the reversal. Antoinetta, also, in the substitution problem, stated that she felt a bit confused by this new version of the line problem.

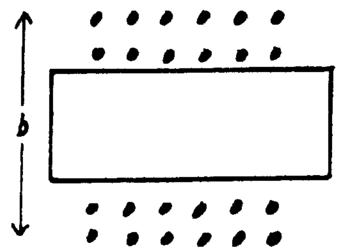
The students' efficient handling of these line problems can be attributed to two reasons. First the initial line problem was presented as a completion one, that is, they had to complete the equation 'Length of line = ?', which again eliminated the name-process dilemma. The second reason appears to be transfer from their experience with the area problems involving an unknown to similar line problems.

3. Dot Problems

As mentioned previously, the notion of hidden quantity had to be maintained in the dot problem since a simple convention acceptable to the students to denote an unknown number of dots in each row, or the unknown number of rows, could not be found. In place of a piece of cardboard, a rectangle was drawn in these problems to represent the cardboard used in the previous lesson.

Prior to having the students write the total number of dots, a sequence of questions was prepared, such that it would ensure that the subjects were clear as to what each factor was in the resultant expression. It was presented as follows:

> Here is a dot problem. How many dots are in each row?
> What do you think the letter <u>b</u> stands for?
> Can you write the total number of dots?



The students responded in the following manner:

<u>Wendy</u> at first stated the <u>b</u> stood for the number of dots, but then changed her response to the "number of rows". She completed the equation, 'Total number of dots = ' with 'b × 6 or 6 × b'.

Wendy's response indicates that she understood what the letter represented (the unknown number of rows) and could write a correct algebraic expression to represent the total number of dots, when given to complete the equation 'Total number of dots = '.

<u>Frankie</u> responded that <u>b</u> stood for the "number of rows" and completed the equation, "Total number of dots = ", by writing '6 × b'. However, he seemed to question his response stating "I thought you had to count all the dots." The interviewer then pointed out that the problems were similar to the other ones in Lesson 1 where the total number of rows was unknown. However, in the present problem, no uncovering was possible, therefore the expression '6 × b' must be left as the total number of dots.

Frankie also exhibited no difficulty with the dot problem and seemed to understand the use of the literal symbol (the unknown "number of rows"). However, he did indicate that he gave some thought to the notion of counting all the dots, which would have allowed him to provide a final numerical response to complete the statement 'total number of dots = '.

Antoinetta appeared to have some difficulty with this problem. At first she stated that the b stood for "The number of things you have under the square" but then corrected herself "no, all the rows". When she was asked to write the total number of dots, she wrote "6 × 8 = 48". (Lined paper was used and there were 4 lines under the rectangle, thus she counted each of these lines and the 4 rows that were shown to arrive at 8 rows.) The interviewer then explained that neither of them knew how many rows of dots were under the rectangle.

I: How do you think we can write the total number of dots even though we don't know the total number of rows?
A: 6 times b, but I don't know the answer.

She was then asked to complete the equation:

'Total number of dots = '.

She then completed it by writing on the right-hand side '6 × b'.

Antoinetta's attempt to provide a numerical value for the letter b (6 × 8 = 48) and her ultimate conclusion "6 times b, but I don't know the answer", highlights a major difference between this problem and the previous ones. Previously, the letter could be evaluated either by uncovering (as in Lesson 1) or by measuring (as in Lesson 2). This dot problem presented no such situation. Possibly the lack of an opportunity to evaluate the unknown, as well as the quantitative nature of the dot problem (How many dots?) are explanations for Antoinetta's search for a numerical answer.

Another reason for her difficulty could be due to the fact that this dot problem was presented to Antoinetta in a similar manner as the first area problem. That is, no equation to be completed, such as 'Total number of dots = ' was initially presented to her. She was merely asked to write the total number of dots.

<u>Gail</u> when asked what the letter <u>b</u> stood for, replied "There are supposed to be dots here, but they are covered (referring to the rectangle)" to which she then added:

G: 48
I: <u>b</u> stands for 48?
G: yes

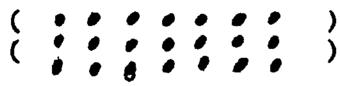
It appeared as though Gail was a little confused by the problem and the first question, (What do you think the letter <u>b</u> stands for?), thus the interviewer had her recall the previous dot problems where the letter stood for the number of rows. She was then asked to complete the statement "Total number of dots = ".
She wrote '$6 \times b = 48$' to complete the statement.

G: I looked at it, and each row was on a line, and there are 8 lines (referring like Antoinetta to the lines on the paper)
I: ...what if I didn't write dots on all the lines, and some were closer together, could you write the total number of dots?
G: Says and writes '$6 \times 8 = 48$' under the '$6 \times b = 48$'.
I: I am saying you do not know the number of rows. Why are we using a letter?
G: Because we don't know the number of rows.
I: Right, you don't know how many rows. You think there are eight if I drew dots on every line, but you don't know. Maybe one line doesn't have dots or there is a row of dots closer together, you don't know.
G: You can't find out.
I: So to write the number of dots...
G: (says "6 times <u>b</u>", but writes '$6 \times b =$ ', that is, the equation was completed in the following way:

'Total number of dots = $6 \times b =$ '

Gail's response '$6 \times b = 48$', and '$6 \times 8 = 48$', like Antoinetta's shows a need to provide a numerical value for <u>b</u> and to provide ultimately a final numerical answer, as she did by writing '$6 \times 8 = 48$'. The incompleteness of the algebraic expression that she felt, was further indicated by her final response, (total number of dots = $6 \times b =$ ') where by placing an equal sign at the end, she seemed to indicate symbolically (not verbally like Antoinetta) that she had not arrived at a final answer.

Filippo had difficulty perceiving that the b stood for the total number of rows and stated "b stands for the missing number of rows." Even when he was referred to his homework review problem



('Total number of dots = 3 × a'), he claimed that the letter was used "To find out what the missing amount is, like a would equal the missing number of dots."

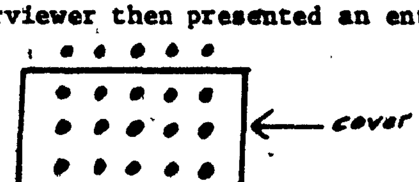
The interviewer then covered the rows, showing only one row of 7 dots.

> I: How would you write the total number of dots in the problem you drew here?
> F: (writes '7 × a')
> I: What does the a stand for?
> F: The amount of dots in the height. (writes and says a = 3)
> I: So a stands for the total number of ...
> F: dots
> I: All the dots?
> F: no response
> I: a stands for the total number of rows.
> F: Oh yes, that is true.
> I: Going back to this problem, what does the letter b stand for?
> F: b stands for the dots, and you have to see if they go in there.
> I: What if I did this (covers the rows, shows only 6 dots in the top row). What does the b stand for?"
> F: The amount of dots...in a row...the missing amount of rows.

The interviewer returned to his review problem showing him that the a was 3, and 3 was the total number of rows, not just the missing rows, however, he still maintained the letter stood for the missing number of rows.

> F: b stands for what is missing. b means how many other rows there are and you have got to find out what it is, like there might be 1,2,3,4,5, it's covering it up, b equals what is under there (referring to the rectangle).

The interviewer then presented an entirely new problem as follows:



(A row of 5 dots was shown; the other rows were covered.)

I: Write the total number of dots using the letter b.
F: (writes '$5 \times B$')
I: Alright, I am uncovering, what does the letter b stand for?
F: b stands for how many rows there are in all.
I: How many rows there are altogether.
F: says and writes, '$B = 4$'.
I: So what does the letter b stand for, the missing rows, or the rows altogether?
F: All the rows together.
I: So let's look at the first dot problem. What does the b stand for?
F: All the rows.
I: Can you write the total number of dots in this picture?
F: writes '$6 \times b$'.
That is all you can do for now, you've got 3 rows, you don't know how many are in there, there could be one.

Filippo did appear at the end of the interview to understand that the b stood for all the rows, but it is difficult to ascertain to what extent he had actually accepted this concept. The future dot problems may perhaps provide a better indication of his understanding of this concept.

He ultimately wrote '$6 \times b$' as the total number of dots and stated "That is all you can really do for now..." This response indicates that he had accepted the algebraic expression as the answer but only to a certain extent. It is an answer which cannot be given a final numerical value, thus he added "for now" suggesting that he did not perceive his answer as complete. It should be recalled that Filippo views algebraic expressions as 'operations', (see previous area problems) and an operation is not complete until one has arrived at a final numerical answer, which is not attainable in the dot problem.

Yvette when asked what the b stood for, replied, "How many rows there are." When asked to write the total number of dots by completing the equation "Number of dots = "she wrote '$6 \times b$'.

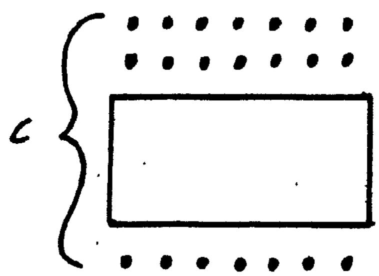
Yvette's response indicates that she had no difficulty in understanding the meaning of the letter within the context of this dot problem, and could easily write an algebraic expression as an answer by completing a given statement such as "number of dots = ".

As in the previous problem types, the students were now asked to make up a dot problem, given an algebraic expression.

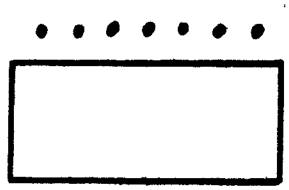
> Can you make up a dot problem where the total number of dots is '$7 \times c$'?
> (The interviewer wrote the expression '$7 \times c$')

The subjects presented the following problems:

<u>Wendy, Frankie, Antoinetta, Gail and Yvette</u> drew a problem such as the following:



<u>Filippo's</u> variation was as follows:



He did not indicate what the <u>c</u> represented.

The subjects did not experience any difficulty with the reversal in this problem type. Antoinetta, Gail and Filippo appeared to have overcome some of the cognitive difficulties they were experiencing in the previous dot problem. However, Filippo's acceptance of the letter as representing the total number of dots is still not evident, since in his drawing he did not indicate what the <u>c</u> represented.

Summary

The first dot problem presented some cognitive difficulties for three students, Antoinetta, Gail and Filippo. Antoinetta and Gail felt that their answer '6 × b' was incomplete, and Filippo had difficulty comprehending what the letter b stood for. He also indicated that the algebraic expression '6 × b' was not a complete answer. A possible explanation for the difficulties could be that in contrast to the previous problems in Lesson 1 and Lesson 2 where the letter could always be evaluated by uncovering or measuring, respectively, this dot problem did not allow for any such evaluation. Also the quantitative nature of the wording of the problem ("How many dots?") must also be considered, since such a question implies a numerical answer.

B. Concatenation - Numerical and Literal Symbols

1. The Introduction of Concatenation

As discussed in the introduction to this chapter, concatenation was not immediately introduced to the subjects since research literature and the results of the pretest suggested that this notation could result in cognitive difficulties for the beginning algebra student. However, it was felt that prior to Lesson 3, which dealt with algebraic expressions involving both multiplication and addition, it was essential that this convention be introduced in order to make the expressions less cumbersome. The justification for concatenation presented to the students was that concatenation eliminates the possible confusion between the letter x as representing an operation, or as representing some unknown number. The introduction began as follows:

How do you write 6 times 7 in arithmetic?
In algebra we can use all the letters in
the alphabet - capitals and small letters.
We often use the letter x . This can get
a little confusing especially if you have
two x's as in

'8 × x' (8 times x)

In order not to have any confusion, in algebra
we can just leave out the multiplication sign
and write:

'8x' for '8 × x'

Whenever you see a letter attached to a number,
there is a hidden multiplication there.
What does 5a mean?
What does 4b mean?

The responses to the presentation were as follows:

Wendy, Frankie, Filippo and Yvette followed the presentation and answered that 5a meant '5 times a' and that 4b meant '4 times b', writing '5 × a', and '4 × b', respectively.

Antoinetta and Gail questioned if it is always multiplication, with Antoinetta ultimately concluding that if it represented other operations "Then you wouldn't know what to replace the sign with...addition, subtraction." They both correctly responded that 5a meant '5 × a' and 4b meant '4 times b'.

Thus, to this point the students appeared to comprehend the new notation, with Antoinetta indicating further understanding in her suggestion that it can only represent one operation in order to avoid possible confusion. The presentation then proceeded to illustrate why this type of notation was only possible in algebra.

Can we do this in arithmetic?
That is, can we remove the multiplication sign
in '6 × 7' and write the numbers next to
each other as in 8x ?

It should be recalled that the statement "and write the numbers next to each other" was included in the teaching outline since the Exploratory Case Study had indicated that the student did not spontaneously write the numbers next to each other, and subsequently began providing possible explanations for the blank space between the numbers. Thus this statement was added in order to eliminate such confusion. It was expected that the students would see that concatenation was not possible in arithmetic since a number sixty-seven (67) would be formed, which is not the product of '6 × 7' .

The students responded as was expected.

Wendy, Frankie, Antoinetta, Gail and Yvette answered that it could not be done in arithmetic since, "It would be sixty-seven".

Filippo provided the same response as the other five subjects. However, he added that he had reservations about this new notation, "Because some people, they might find it difficult. Like in 8x , they might think it is a number with something missing. They might think it is eighty something... They might get confused."

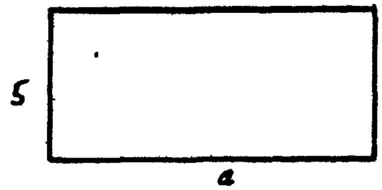
Thus the subjects observed the difference between arithmetic and algebra and accepted the fact that concatenation was not possible in arithmetic. It is interesting to note Filippo's comment on the 'place value' interpretation for concatenation. He still felt that even if the students knew the correct interpretation for concatenation they might still think in terms of a 'place value' one, and consequently get confused.

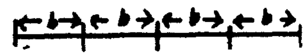
The presentation then proceeded to tie concatenation with the problem types introduced in the teaching outline. The justifications for this link are: 1) it does not leave the introduction of concatenation as an isolated topic, but rather ties this convention explicitly to the problems in the teaching outline, and 2) this convention is a necessary part of the problems presented in the succeeding lesson which involved more complex expressions. The problems were as follows:

Can you make up an area problem where <u>5a</u> is
the answer?
Can you make up a line problem where <u>4b</u> is
the answer?

The responses were as follows:

<u>Wendy, Frankie, Gail, Filippo, and Yvette</u> drew correct problems such as:





<u>Antoinetta</u> drew a correct area problem, as above, however the line problem presented some difficulty for her.

A: Does the 4 stand for the parts or the length of the part?
I: We are not going to cover anything?
A: (drew the following:

I am not too sure what the <u>b</u> stands for?
I: If I were to ask you what is the length of this line...
A: Oh... (erases the 4's and replaces them with <u>b's</u>.

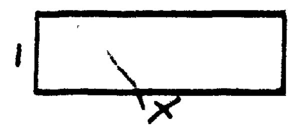
She later explained, "I put 4 before, because I didn't know 4 parts... I thought <u>b</u> was the parts, and I didn't know how long each part is?"

The responses demonstrate that the students in general, had no difficulty linking this new notation to the previous problems. Antoinetta's difficulty wau not due to the new convention, she had forgotten the new orientation of the line problems presented in this lesson, and thus could not initially perceive the need for the use of a letter. Her difficulty was also evident in the last line problem, the one requiring substitution, when she had hinted that she was becoming a bit confused with the two versions of the line problems, and stated, "I get mixed up from the other ones... Before I was writing how many parts, now I wrote the length of each part."

2. The Introduction of Some Conventions in Algebra Involving Concatenation

At this point in the teaching outline two other algebraic conventions, both linked to concatenation, were introduced. The first one, was that of multiplying a letter by the number one, that is 1x is written as x . The presentation was as follows:

> Here is a rectangle, can you tell me
> what the area is? (expected response
> '1 × x' or '1x')
> What do we do with a product
> '1 × 3'? (expected response,
> "we just write 3")
> The same thing in algebra, we
> omit the 1 and just write the x .



The students responded as follows:

Gail and Filippo wrote 'Area = 1x'.

Wendy just wrote '1x'.

Yvette and Frankie wrote first '1x' and then '1 × x'.

Since the presentation of this convention was in the form of an exposition, it cannot be determined to what extent the students had understood or accepted this convention. Only succeeding problems where this convention is present could be used as an indication.

However, it is interesting to note that all six subjects had used the convention of concatenation, shown in the previous section, that is, they all wrote 1x . A further point of interest is that three students (Wendy, Frankie, and Filippo) responded to "can you tell me what the area is", by just writing the expression '1x' and/or '1 × x', unattached to an equation.

The next convention to be presented, was that in algebra, an expression containing a letter and a number is always written with the number preceeding the letter, that is, 'a × 3' is not written as a3 but as 3a. The presentation was as follows:

> One other convention that I want to show you is the following one:
>
> 'a × 3', we also write as 3a .
>
> Does it make any sense? Do you think we should write a3? It does not make much difference since
>
> 'a × 3 = 3 × a'
>
> Is this true for any value of a ,
> Can you check it with some number?
> And another?
> From now on, we write
>
> 'b × 5' as?
> 'c × 6' as?

The responses to this presentation were as follows:

Wendy stated that 'a × 3' or 3a was the "same thing, in algebra you leave out the multiplication sign". She was not asked to substitute different values for the a . When she was asked how to write 'b × 5', she responded by writing b5 or 5b , underlining the 5b , and responded similarly for 'c × 6'.

Frankie at first felt 'a × 3' should be written as a3 . Within the context of the commutative axiom, he was shown that 3a equals a3 and provided appropriate numerical substitutions as follows:

'2 × 3 = 6' '3 × 2 = 6'

'4 × 3 = 12' '3 × 4 = 12'

He correctly wrote that 'b × 5' is 5b and 'c × 6' is 6c .

Antoinetta's first reaction was, "what would happen if you put a3?" The interviewer proceeded to explain that they were both the same. However, the convention in algebra is to put the number before the letter. She provided appropriate numerical substitutions to illustrate that 3 × a = a × 3 , by writing

'4 × 3 = 3 × 4'
'2 × 3 = 3 × 2'

She wrote 5b for 'b × 5' and 6c for 'c × 6'.

Filippo felt 3a was correct since "it still is the missing number times 3," and illustrated this by writing: a × 3

3 × □ 3a

and stated "...something times 3." He saw that 'a × 3 = 3 × a', "as long as you have the same number for a" and wrote '7 × 3 = 3 × 7' and showed that if one of the 7's were a 6 "it wouldn't work out". He then wrote 5b for 'b × 5' and 6c for 'c × 6'.

Gail thought that a3 would be better, but accepted 3a for a 'a × 3'. To show 3 × a = a × 3 she wrote:

3 × 3 = 9
3 × 3 = 3 × 3
6 × 3 = 3 × 6

and wrote 5b for 'b × 5' and 6c for 'c × 6'.

Yvette unlike the other students spontaneously preferred 3a, although she clearly saw that they "can mean the same thing, 3 times a". She illustrated that '3 × a = a × 3' by writing

'2 × 3 = 6'   '3 × 2 = 6'
'4 × 3 = 12   '3 × 4 = 12'

She then wrote 5b for 'b × 5' and 6c for 'c × 6'.

This presentation, as the previous one, was an exposition and thus not one where the students could exhibit any cognitive problems they were having with this convention. Of interest to note is that three subjects, (Gail, Frankie and to some extent, Antoinetta) felt more comfortable with 'a3' rather than '3a'. Thus the convention to write the number before the letter to the beginning algebra student, may not be as obvious or acceptable as some instructors may believe. All six subjects responded correctly to the last questions and wrote 5b for 'b × 5' and 6c for 'c × 6'. However, their actual acceptance of the necessity of this convention is questionable and cannot be determined at this point.

C. Analysis of Homework

As at the end of Lesson 2, problems were assigned to the students, and reviewed at the beginning of Lesson 3, which occurred four to five days later, depending on the student. The problems were as follows:

For our next meeting, please make up a/an:
- area problem where the answer is 6a
- dot problem where the answer is 4d
- line problem where the answer is 6b
- line problem where the answer is 1c or c .

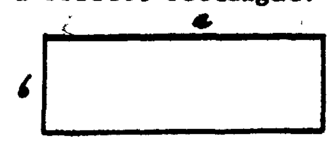
The questions prepared for the review were as follows:

At our last meeting I had asked you to make up 4 problems. First an area problem where the answer is 6a, that is, the area of the rectangle is 6a. And a dot problem where the total number of dots is 4d . And a line problem, where the length of the line is 6b . And another line problem where the length of the line is 1c or c .

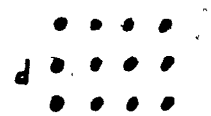
The review of the homework assignment proceeded as follows with each of the subjects:

Wendy did not prepare the homework assignment. Therefore she was asked to do the problems at the beginning of Lesson 3.

Area Problem: She drew a correct rectangle.

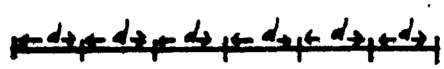


Dot Problem: She drew the following problem:



The interviewer asked: "When do we use a letter?" to which she replied "to represent a number".

Line Problem: She drew a correct line segment to represent the expression '6 × d', as follows:

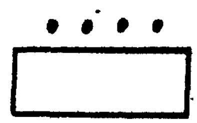


The interviewer then proceeded to explore her difficulties with the dot problem. Wendy was questioned whether she knew what the letter a stood for in her area problem, and the d in the line problem, to which she responded, "no". However, it was pointed out to her that in the dot problem she knew what the letter d stood for.

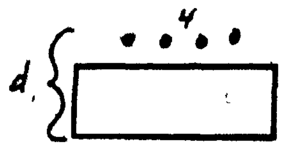
The interviewer proceeded as follows:

I: What did we do in the first lesson when I didn't want you to see the number of rows?

W: You covered it.
I: In the second lesson, I didn't have a cover so I did something else instead. Do you recall what I did?
W: No response.
I: I used a box. I would draw 4 dots, and I would pretend that there is a cover by drawing a box. (Draws the following:)

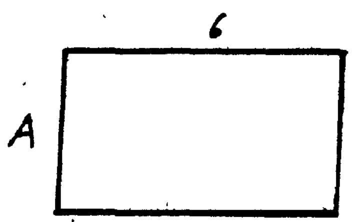


W: Oh Yeh! and that is your <u>d</u> (writes a <u>d</u> next to the number of rows with a brace bracket, and then writes 4 on top of the rows:

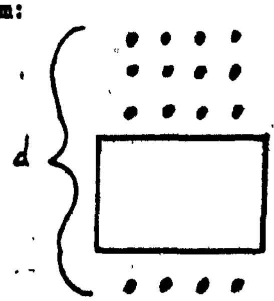


<u>Frankie</u> for his homework had prepared four correct problems.

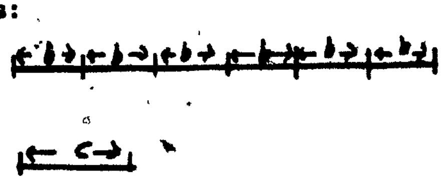
Area Problem:



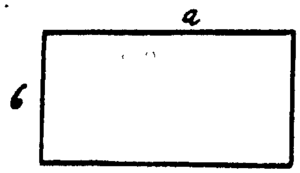
Dot Problem:



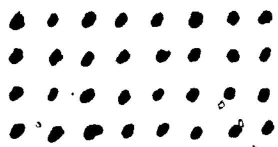
Line Problems:



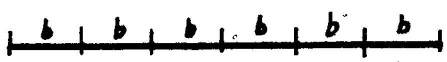
Antoinetta for her homework had drawn the following problems:



and had written: 'Area = 6a'.



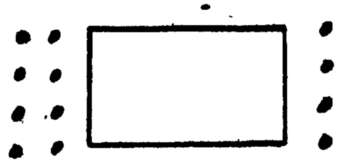
and had written: '4d'.



and had written: 'Length = 6b'

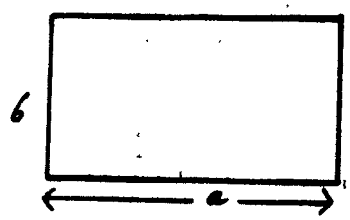
and had written: 'Length = 1c'

The interviewer then pointed out to Antoinetta that in the area and line problems a letter was used because we didn't know the length of the a and b. However, in the dot problem she had presented, this was not the case. As with Wendy, the interviewer reviewed the dot problems presented in Lesson 1 and 2. She was subsequently asked to draw a dot problem "where the answer is 4d ". She drew the following:



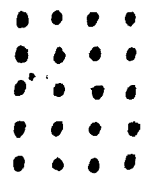
When she was asked what the letter d stood for, she replied, "the number of rows."

Gail presented the following problems for her homework.



and had written: '6 × a'
6a

For the dot problem Gail presented the array as follows:

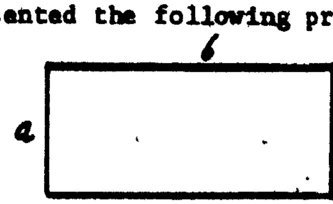


and had written '4 × d, 4d and then asked for a cardboard to be used as a cover. She then showed only one row and hid the other rows.

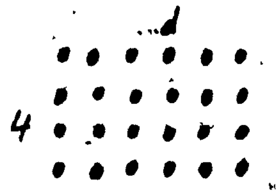
and had written: 6b.

and had written: 1c.

Filippo presented the following problems for his homework.



and had written: '6 × A = 6a'



and had written: $4 \times d = 4d$.

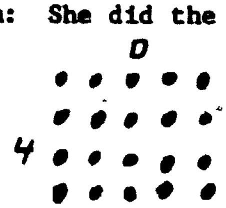
and had written: '$6 \times b = 6b$'

and had written: '$1 \times c = 1c$'

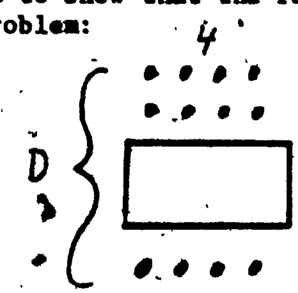
(The error Filippo made with the dot problem was not pursued by the interviewer.)

Yvette had forgotten to do the homework assignment. Therefore, she was asked to do it at the beginning of Lesson 3.

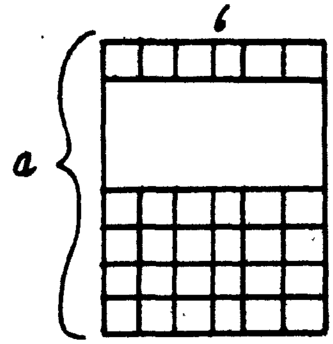
Dot Problem: She did the dot problem first, and drew the following:



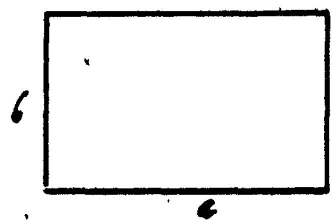
She was then reminded that letters were used to represent a number we didn't know, and in Lesson 1 we had covered the rows, and in Lesson 2, we drew a square to show that the rows were covered. She then drew the following problem:



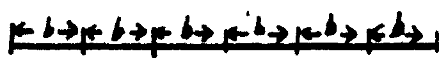
Area Problem: She drew the following problem:



The interviewer then showed her how the area problem had been drawn in the previous lesson.



Line Problem: She drew the following:



The homework review indicates that all the students were able to draw appropriate area and line problems to represent the given algebraic expression. They appeared to have overcome the cognitive difficulties associated with these two problem types that were uncovered in the analysis of Lesson 2. In each case they had no difficulty interpreting the concatenated form of the expression presented to them, that is, they correctly understood, that <u>6a</u> meant '6 × a', and that <u>6b</u> meant '6 × b'. Also in the area and line problems drawn by the subjects, they illustrated that they understood that the literal symbol represented some unknown dimension (length).

However, the dot problem did not reflect the same understanding of the literal symbol within the context of this problem type. Only Frankie presented a dot problem as illustrated in Lesson 2. Although Gail's problem correctly represented the expression <u>4d</u>, she had forgotten the new version of the dot problem and reverted back to hiding the rows with cardboard as in Lesson 1. The four other students presented dot problems showing all the dots, thus no unknown quantity was present, that is, their drawing did not justify the use of a literal symbol. Thus is appears that the convention used in Lesson 2 to suggest an unknown quantity in the dot problem (an empty rectangle) was not understood or remembered by the majority of subjects. Perhaps a new convention to represent an unknown number of dots might be developed or failing that, we could perhaps consider removing the dot problem from the teaching outline at the point where the literal symbol is introduced as an unknown quantity.

Of interest to note is the students' obvious acceptance of concatenated expressions such as <u>6a</u> to represent '6 × a', and their obvious facility in the use of this notation.

CONCLUSION

Lesson 2 introduced the letter as representing an <u>unknown</u> quantity within the context of the three problem types (area, length and total number of dots in an array). Also as part of the lesson, the algebraic convention of concatenation was introduced, along with some other conventions associated with concatenation (<u>a3</u> is written as <u>3a</u>, and <u>1x</u> is written as <u>x</u>).

In this lesson, the letter was used within the context of the same problem types that were presented in Lesson 1, where the letter represented a hidden quantity. It was therefore believed that since the same problem types were used the subjects would view the letter as representing an unknown quantity as merely an extension of it representing a hidden quantity, and thus not experience any major cognitive difficulties. However, the first problem in this lesson (an area problem) proved to the contrary. The transition to letter used as unknown was not spontaneous, with new cognitive difficulties surfacing.

Some of the cognitive obstacles uncovered were due to the change in wording of the question. In Lesson 1, the subjects were always requested to complete an equation such as "Area = ". However, in the first area problem, the incomplete equation was not provided, they were merely asked to "write down the area of the rectangle."

All the subjects appeared to have misinterpreted the intended meaning of the question and sought at first a numerical answer to the problem, by either providing an estimated length for the base of the rectangle, or by the suggestion of measuring. Since the need for a numerical answer did not appear in Lesson 1, possibly the new context of the letter as an unknown and not a hidden quantity, could have been one of the causes for this response. It may be easier to hold in suspension an algebraic expression such as '$8 \times a$' in the context of a hidden quantity, for the hiding implies a second phase, that of uncovering, which will allow the student to evaluate the expression. On the other hand, in the context of unknown dimension the notion of hiding is missing and thus cannot help in holding '$8 \times a$' in suspension.

The inability of the students to hold the expression in suspension appears to suggest that the subjects did not achieve a high level of acceptance of lack of closure. However, although they could not accept the expression by itself, it was far more acceptable to them within the context of the equation 'Area = ', as evidenced by their verbal statements, such as Wendy's "You can't just write '$8 \times a$', unless you write 8 times a is the area of the figure". A possible explanation for the students' greater acceptance of the expression within the framework of an equation could be sought in terms of the name-process dilemma. By completing the equation 'Area = ', the left-hand side clearly expresses the name, thus leaving to the right-hand side the function of expressing the process.

As the lesson proceeded, the students appeared to have overcome the cognitive difficulties they had initially experienced. The questions seemed to be interpreted by the students as they were intended. They no longer expressed any need to determine a numerical value for the literal symbol, and accepted it as an unknown quantity. Thus it seemed that they had made the transition from viewing the letter as a hidden quantity to letter as an unknown quantity.

Of the three problem types, only the dot problem created any significant difficulties for the subjects. Although a new version of the line problem was introduced (the letter represented the unknown length of each part, not the unknown number of parts), only Wendy and to a small extent Antoinetta, appeared to have any difficulty in adjusting to it. The dot problem presented no major difficulties for the subjects during the lesson, however, the homework review indicated that the convention used to denote unknown number of rows was not remembered by five of the six subjects. Thus perhaps a new convention to represent an unknown number of dots might be developed, or failing that, we could perhaps consider removing this problem type from the teaching outline at the point where the literal symbol is introduced as an unknown quantity.

The second aim of Lesson 2 was to introduce concatenation. The interview proceeded in the form of an exposition, thus no cognitive difficulties were uncovered, and all subjects appeared to accept this new notation.

The analysis of the homework provides an indication of the extent of the achievement of the aims of the lesson. All subjects presented correct area and line problems which indicate that they now understood that the letter represented some unknown quantity. (As mentioned previously, the difficulties evidenced in the dot problem could be due to the convention used to represent the unknown quantity.) The cognitive difficulties uncovered in Lesson 2, did not appear in the homework review, giving further evidence that they had been overcome. In the homework assignment, the concatenated form of the expression was used, for example <u>6a</u> was used instead of '6 × a'. None of the subjects expressed any difficulty correctly interpreting the concatenated form of the expression.

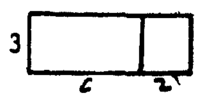
CHAPTER VIII

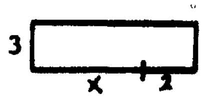
ANALYSIS OF LESSON 3

Algebraic Expressions Involving Both Multiplication and Addition

INTRODUCTION

Lesson 2 introduced the letter as an unknown quantity within the context of three different problem types (area, length and number of dots in an array). All the problems were purely multiplicative in nature. In Lesson 3, more complex problems were presented, involving both multiplication and addition.

In this lesson the area of a rectangle was determined by dividing it into smaller parts (

), in order to make the problem less confusing and easily visualizable. Although it would have been possible to introduce the topic of bracketing in this lesson (e.g. 

, area = $3(x + 2)$) , it was believed that bracketing might cause some confusion at this point and thus it was not included.

The lesson was divided into two sections. The first one introduced expressions containing only one unknown, the second, two unknowns. All three problem types were presented in the first section. However, in the second part, the dot array problem was omitted because the convention to represent an unknown number of rows, did not lend itself easily to represent two unknowns, and consequently could cause too much confusion for the student.

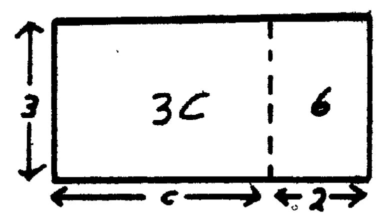
The problem types were introduced in the same sequence as Lesson 2 in the first part (one unknown). However, in the second part (two unknowns) one line problem was introduced before the area problem since it demonstrated more clearly the concept of two unknowns, that is, two different segments of a line each a different length.

A. PART 1 (One Unknown)

1. Area Problem

The first area problem was presented as follows:

Here is a rectangle. The length of the height is 3 units. The length of the base is in two parts, one part is unknown, so I marked c and the other part is 2 units. In order to find the area, of this rectangle, I'm going to simplify the problem. Here I'm drawing a line, splitting it up into two smaller rectangles.
What is the area of the rectangle on the left? (3c)
(Write 3c in that rectangle.)
What is the area of the rectangle on the right? (6)
(Write 6 in that rectangle.)
What is the area of the blue rectangle? That is the rectangle we started with.
(Student was expected to perceive additive nature of areas.)



O.K. We can write:
Area of blue rectangle =

The responses of the students were as follows:

Wendy responded that the area of the rectangle on the left was '3 × c or c × 3'. When reminded about concatenation, she wrote 3c . For the rectangle on the right, she answered as follows:

W: 3 times 2
I: Write down what you have just said
W: (Writes 32)
I: You have written 32 (three-two), what number is that?
W: Three times two
I: Read the number you just wrote
W: Three two in algebra

The interviewer then had Wendy recall what had been discussed in the previous lesson, that is, that concatenation was not possible in arithmetic. She then said that the area was 6. She was then asked to write the area of the rectangle, and after a little confusion (she misunderstood which rectangle) she completed the equation "area of rectangle = " by writing '3c × 2'.

Wendy's responses highlight the cognitive obstacles uncovered in the Exploratory Case Study (See Chapter 4, pp. 83-85). When asked the area of the rectangle, she said "3 times 2" and wrote 32. When questioned about it, she explained her writing by saying "three two in algebra". This is very interesting because it appears to be a first instance of generalization of a convention learned within the context of algebra being transferred to arithmetic. Wendy very logically specifies "three two <u>in algebra</u>", because in arithmetic she obviously knows "three two" (32) is thirty-two. Thus she believes that as long as she is working in algebra, she can use concatenation also for the product of two numbers.

As in the case with Philip, Wendy also indicated that her strong association of area with multiplication prevented her from perceiving the additive nature of the problem presented to her. (She wrote '3c × 2'.) As will be seen further, all the other students, with the exception of Filippo, used multiplication instead of addition. The remedial part of the interviews are dealt with after this first introduction.

<u>Frankie</u> correctly answered that '3 times c' (3c) and 6, were the areas of the rectangles on the left and right, and he wrote 3c and 6 in the appropriate rectangle. When he was asked to complete the equation "Area of blue rectangle (outside rectangle) = " he wrote '3 × c × 2'.

Frankie's expression to represent the area of the blue rectangle resembles that of Wendy. That is, he too was associating area with multiplication. He also multiplied <u>3c</u> by 2, part of the base.

Antoinetta responded that the area of the rectangle on the left was "3 times c or 3c" and the area of the rectangle on the right was 6 (she wrote 3c and 6 in the appropriate rectangles). When she was asked "What is the area of the blue rectangle", by completing "Area = " she first wrote "$3 \times c \times 2$" and questioned whether she could write '$3c \times 2$' , which she ultimately did.

The link between multiplication and area is once again evidenced in this interview with Antoinetta. She also used part of the base as the second factor.

Gail at first responded to "what is the area of the rectangle on the left" by saying "twelve" since she had estimated the c to be 4 units. When her attention was returned to the previous lesson, which included similar rectangles with height 3 and base some unknown length c , she ultimately saw that the area of the rectangle was 3c and wrote 3c in the rectangle. For the rectangle on the right, she wrote 6 inside that rectangle. When she was asked to complete the equation "area of blue rectangle = " she wrote '$3c \times 2$' .

Gail's spontaneous response to estimate the length of c and her subsequent numerical answer (12) , for the area of the rectangle on the left, indicates that she was once again searching for a numerical answer in response to the question "What is the area...?" (She had previously done so in the first area problem, and the first dot problem in Lesson 2.) Although she did ultimately provide the correct response (3c) suggesting her difficulty may have been a question of memory, the fact that she spontaneously thought of a numerical answer rather than an algebraic expression, may suggest that she was not yet comfortable with algebraic expressions as 'answers' to open questions such as "What is the area of ...?" As with the previous students, she also related area to multiplication, since she wrote '$3c \times 2$' .

Filippo responded that the area of the rectangle on the left and right was 3c and 6, respectively, and wrote the areas in the appropriate rectangles. When he was asked to complete the equation "Area of blue rectangle = ", he wrote '$3c \times c + 2$'. When he was asked to explain his answer, he changed the plus sign to multiplication and wrote '$3 \times c \times 2$'. However, when he was asked to explain what he had just written, he said "first you try to figure out this one, so that's 3 times c, and then that one which is 3 times 2 (pointing first to the left rectangle and then to the rectangle on the right). So 3 times c plus 3 times 2". And he wrote '(3 × c) + (3 × 2)'. The interview proceeded with Filippo explaining the use of brackets in this problem. He was then directed to write '3 × c' as 3c and he finally answered by writing '3c + 6').

Filippo was the only student who perceived the additive nature of the problem. Even though at one point he appeared to be changing his response to that of multiplication, his further explanation indicated that it was only a temporary reversal.

Yvette first wrote '3 × c', then '3c', after being reminded about concatenation, for the rectangle on the left. For the area of the rectangle on the right she responded:

Y: ...It is times 2 (writes '3 × 2 , 32')
I: thirty-two?
Y: Oh no, (erases 32 and replaces it by a 6)

When she was asked to write the area of the blue rectangle by completing the equation "Area = " she wrote '6 × 3c'.

Yvette's response indicates that she, like Wendy, at first generalized from algebra to arithmetic by writing '32' for 3 times 2. The interviewer's interjection of the number 'thirty-two', however, led Yvette to quickly perceive her error. Her final response, '6 × 3c', also shows that she links the sum of the areas with multiplication and does not perceive the additive aspect of the problem. However, her response is slightly different, since she multiplied 3c by 6 (the area of the other rectangle) not by 2 (the length of part of the base.)

The responses thus far in the first area problem point out some very interesting aspects. 1) Wendy and Yvette revealed the possibility of students transferring the concatenation convention from algebra to arithmetic, (eg. '3 × 2' written as 32 in algebra). 2) The responses also indicate that although concatenation had been previously presented to the students, three of the six students (Wendy, Filippo and Yvette) did not spontaneously use the convention, and had to be reminded about this new form. 3) One very clear result is that to the novice student, the area concept is so strongly linked with multiplication that it creates a 'mind-set', preventing him from perceiving the additive nature of the problem involving the sum of two areas. This was evidenced by five of the six subjects, with only Filippo perceiving the additive aspect of the problem. In the multiplication of areas, the second factor used by four of the subjects (Wendy, Frankie, Antoinetta and Gail) was not the area of a rectangle, but the base of the second rectangle. That is, they wrote '3c × 2' , not '3c × 6'. This perhaps could be explained in terms of their focusing on the formula for area (Length of height times length of base).

The inability of the subjects to perceive this additive aspect was not anticipated in the preparation of the teaching outline. For some reason we believed that this behaviour observed in the Exploratory Case Study was exceptional. Thus a spontaneous arithmetic situation was presented to the students by the interviewer in order to demonstrate that the correct response required the addition of the areas of two smaller rectangles. A rectangle divided into two parts was drawn, such as $\boxed{12 \mid 40}$, the area of each rectangle written inside each one. The students were then asked to write the area of the larger rectangle, which would be '12 + 40' or 52. Consequently, it was expected that the students would then relate this problem to the one they were originally doing $\boxed{3C \mid 6}$ and subsequently provide the correct answer, '3c + 6'. Of course, this presentation was not required for

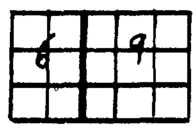
Filippo.

The responses were as follows:

Wendy first responded that the area would be "twelve times forty". The interview then proceded:

I: You are saying 12 times 40 (writes 12 × 40). What is area?
W: Multiplication

The interviewer then proceeded to demonstrate that area meant the number of squares it takes to cover a given surface. Proceeding along this line, Wendy responded that 40 and 12 squares were required to cover each of the smaller rectangles. The interviewer then asked "So how many squares would it take to cover the large rectangle?" Wendy replied, "12 times 40". The interviewer then attempted to divide the rectangle in 12 and 40 squares, and asked how would she determine the total number of squares. She still insisted upon multiplying, 12 and 40. The interviewer subsequently drew a smaller rectangle with 6 and 9 squares.



Although at first she insisted that there were 6 times 9 squares, after counting them, she realized "you add." However, upon relating this arithmetic problem to the original one, she still replied "3c times 6" for the area. When she was reminded about the additive nature of the problem, she wrote '3c + 6'.

Frankie at first responded that the area of the rectangle would be '12 times 40' but then changed his answer to '12 + 40' saying "If you times it you are going to get a bigger answer. And it wouldn't make sense." When his attention was drawn to the original problem and asked to write the area, he wrote '3 × c + 6'. After being reminded about concatenation, he wrote '3c + 6'.

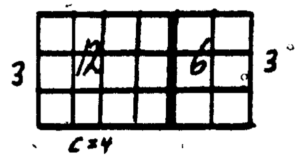
Antoinetta replied that the area of the rectangle would be '52'. When she was asked "How did you get 52?" she replied, "add, you really are supposed to multiply, but you want to know how much it is altogether, so you add." She was again asked the areas of the two smaller rectangles in the original problem, followed by the questions:

I: So altogether what is the area?
A: 9c ?
I: 9c
A: I don't know because I don't know what c represents.

When she was asked how she determined the area of the rectangle which contained the two smaller rectangles (12 and 40) she replied

A: I added them
I: Do you think you can add here (pointing to the original problem)?
A: No
I: Why?
A: Because it is not a full number like 40 or 12.

The interviewer then presented an example which provided a specific value for the c as follows:

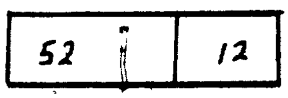


When asked what is the area of the large rectangle, she replied, "18".

I: 12 is 3c ... If I didn't know the length of c, how could I write the area of the rectangle?
A: (writes 'Area = 3c + 6')

Gail responded correctly to the arithmetic problem, and then subsequently provided the correct answer for the original area problem. When she was asked why she had previously written '3c × 2' she answered, "Because I thought of area and I put 3c and then I multiplied by 2, because that is the length of that little piece.

Yvette was presented with the following rectangle,



and when she was asked the area of the larger rectangle she wrote '52 × 12'. The interviewer then drew a smaller rectangle

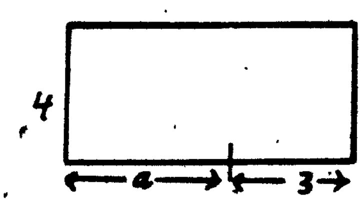
This time Yvette responded that the area would be '12 + 6' and subsequently changed the answer she gave for the first rectangle, to '52 + 12'. Returning to the original problem, she correctly completed the equation "Area = " with '6 + 3c'.

The responses by the subjects given in this section of the interviews once again bring out the strong association between area and multiplication, as evidenced by three of the five subjects using multiplication even within a visualizable numerical (not algebraic) situation. (They wrote 12 × 40). Wendy clearly indicated this strong link when she responded to the question, "What is area?", she replied, "multiply cation". Even when she was asked to count the number of squares, she multiplied. Antoinetta, also, correctly responded, '52'. When asked how she got '52', she responded initially by saying "...You really are supposed to multiply..."

Antoinetta's difficulty, however, was not merely a question of the close link between area and multiplication. She could not add 3c and 6 because 3c to her "...is not a full number like, 40 or 12". That is, 3c to Antoinetta does not represent a number. Her conflict appears to be similar to the one experienced in the Initial Pilot Study ("How can you multiply a number by a letter?") only in this case, it is a question of addition, which includes a concatenated expression, 3c. Her response is quite surprising because in the pretest and in the first and second lessons, she seemed to perceive letters as representing numbers.

A second area problem was then introduced to ensure that the students had grasped the new version of the area problem. The presentation was as follows:

> Here is a rectangle.
> What is the area of this rectangle?
> (If the student was unable to respond, he was to be guided into separating the large rectangle into two smaller ones, and the line of questioning would follow that of the previous problem.)

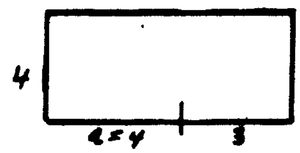


The subjects responded in the following manner:

Wendy divided the rectangle into two smaller ones and wrote inside each rectangle their respective areas, '4a' and '4 × 3 = 12'. Her final answer was written in the form of an equation. 'Area = 4a + 12'.

<u>Frankie</u> initially began measuring the length of <u>a</u>, when he was asked the area of the given rectangle. The interviewer then requested that he not measure, so he wrote '$4 \times a + 3$
$4a + 3$'

He was then asked to divide the rectangle into two smaller ones, and then asked their areas. He replied <u>4a</u> and 12, and wrote the areas in the appropriate rectangles. He then responded to "What is the area of the blue rectangle?" by writing '$4a + 12$'. The interviewer questioned why he had written plus 3 at the beginning, to which he replied, "I was looking at that rectangle here (points to the one at the right) and there is a 3". As the interview proceeded, it became apparent that Frankie did not comprehend why the answer was '$4a + 12$'. He stated, "I don't understand this problem. You times this and add that and you get <u>4a</u> plus 3, what is the difference, if you write '4a times 3'. The interviewer then presented Frankie with a purely numerical version of the same problem.



By substituting the numerical value 4 for <u>a</u>, he was shown that numerically '$4a \times 3$' (16×3) does not equal '$4a + 12$' (the correct answer to the problem) since it would be '$16 + 12$'.

<u>Antoinetta</u> wrote 'Area = $4a + 12$' without dividing the given rectangle into two smaller ones.

<u>Gail</u> wrote 'Area of rectangle = $4a + 12$'. She also did not divide the rectangle, however, she questioned whether she should have.

<u>Filippo</u> first wrote $(4A) + (4 \times 3)$. An attempt by the interviewer to have him write $4A + 12$ caused some confusion for Filippo, he began substituting a numerical value for <u>A</u>. However, he easily returned to his initial answer. He commented on the problem as follows:
"I think a thing like this is very confusing for some student... with the four like this, they won't know what it means...draw a line into separate squares. Like here it would be simple (showing a divided rectangle). When they put it together (no division) they have to multiply it altogether without the lines. (Thus he was stating that the dividing line made the area problem clearer.)

During his discussion, the interviewer managed to have him say that '4×3' was 12, and he rewrote the expression, as $4A + 12$.

<u>Yvette</u> wrote $\overset{4a}{4A}$ and $\overset{4\times3}{12}$ in each of the two smaller rectangles, and then wrote $12 + 4a$ as the answer.

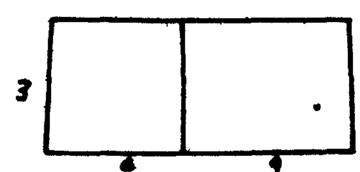
The responses given by the subjects to the second area problem suggest that they had managed to accept the additive aspect of these area problems. However, the interview with Frankie brings into question the extent to which they understand or accept the additive aspect of the problem. Frankie wrote '4a + 12' but further questioning by the interviewer revealed that he did not completely understand why it was not "4a times 3". In the case of the other students, no further questioning had occurred, thus perhaps they also may have been still experiencing some cognitive problems. Filippo clearly stated that he felt the inclusion of the line, dividing the rectangle into two small rectangles was imperative in order for the student to clearly see the additive aspect of the problem.

It is interesting to note that all the students were now using the concatenated form of '4 × a', that is 4a , without any need of a reminder from the interviewer.

The next problem required a reversal. Given an algebraic expression containing two terms, they were asked to generate an area problem similar to the previous ones.

> Now it is your turn. Can you make up an area problem where the area of the rectangle would be '3a + 9'?

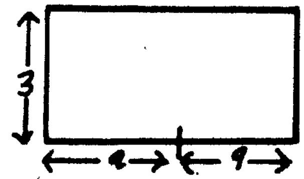
<u>Wendy</u> drew the following rectangle, (the interviewer had written the expression '3a + 9')



The interviewer then asked her to write down the areas inside each of the two smaller rectangles. She wrote 3a and 27 in the appropriate rectangles. Seeing her error, she changed the 9 on the base to a 3 .

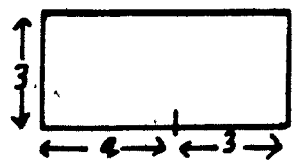
<u>Frankie</u> (The interviewer accidently omitted this problem with Frankie).

Antoinetta drew at first

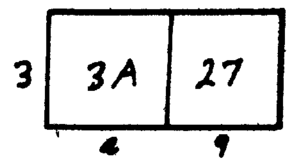


but then changed the nine to a three, without any intervention by the interviewer.

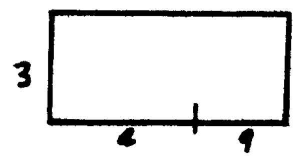
Gail spontaneously presented the following correct rectangle.



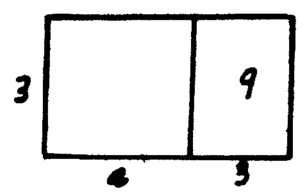
Filippo drew the following rectangle:



and wrote '3A + 27' . When he was again asked to draw a rectangle whose area was '3A + 9' he drew the following:

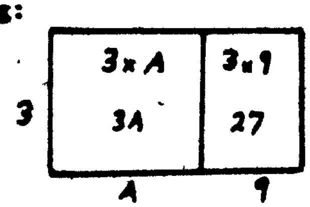


and wrote '3A + 9'. The interviewer then asked him to separate it into two smaller rectangles. He then stated, "Now I understand what you mean..." and drew a new rectangle.



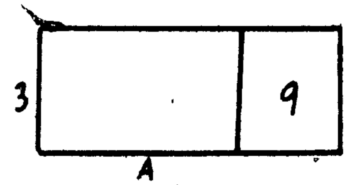
and wrote '3A + 9' .

Yvette drew the following:



and wrote '3A + 27' . She first drew the rectangle on the left then added to it the rectangle on the right. When asked to draw a rectangle with an area of '3a + 9' , she drew a rectangle with dimensions

3 (height) and A (for the base) and attached another rectangle with a 9 written in the centre.



When she was asked the length of the base of the rectangle on the right, she first wrote 5 , then subsequently changed it to a 6 , and then said, "Oh, I was plussing it, it is times) and changed the 6 to a 3.

The reversals suggest that the students now understood the additive aspect of these area problems, overcoming their strong link between area and multiplication that was evidenced in the first area problem. Yvette's drawing in particular, illustrates her interpretation of the additive aspect. She drew one rectangle with area 3c , and then added on another rectangle whose area was 9 . However, 5 out of the six subjects could not spontaneously determine the correct dimensions for the base of a rectangle whose area would be '3a + 9' . They were still focusing on the second term of the expression as being a part of the base, as evidenced by their writing a and 9 as the length of the two segments of the base. Yvette and Filippo perceived immediately that the area of the rectangles they drew to be '3a + 27' . However in each case, once the students were directed into seeing that a correction was necessary, they easily provided the correct dimensions. It is interesting to note that Filippo was unable to provide a correct response until he drew a 'dividing line' in his rectangle. In the previous problem he had suggested that the omission of such a line could create confusion, which it did for him.

Summary

The presentation of the area problems requiring two operations, addition and multiplication, highlighted some new difficulties. The first area problem revealed the strong link students have between area and multiplication, which interfered with their ability to perceive the additive aspect of the problem. It also indicated a transfer by two students of the concatenation convention from algebra to arithmetic, that is, '3 times 2' is '32' <u>in algebra</u>.

As the lesson proceeded to the next area problem, it appeared that the students had managed to overcome the difficulties they had experienced in the first one, that is, they perceived the additive aspect. The reversal provided further evidence of their apparent understanding of the additive aspect of this problem type. However, they experienced some difficulty in determining the dimensions of the rectangle, a difficulty which was easily overcome as soon as the students were brought to view the second term as an area (product) and not just as another part of the base.

2. Line Problems

A line problem was then introduced which required as an answer an expression containing both multiplication and addition.

> Here is a line. What is the length of this line?

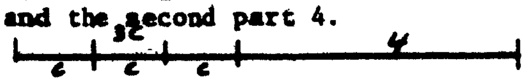
The subjects responded in the following manner.

Wendy completed the equation "Length = " by writing '4 × c or 4c'. The interview then proceeded as follows:

I: (covering the part of line 4 units long) What is the length of the line till there?
W: 3...3c

I: Very good. (Covering the c's) What is the length of this part of the line?
W: 4
I: How would you write the length of this line?
W: 4 times 3
I: Let's draw another line. I will divide the line into two parts.

Let's say the length of the line till here is 12, and the other part is 6. What would you do to find the length of the line?
W: 12 plus 6
I: Now looking at our original problem. What is the length of the line from here to here (Pointing to the 3 c's)
W: 3...3c
I: Right, and from here to there (pointing to the part 4 units long)
W: 4c...no just 4.
I: Yes, it is 4, so what is the length of the line?
W: 4 plus 3
I: 4 plus 3. Is the length of the line from here to here 3?
W: 3c
I: from here to here
W: 4
I: Altogether, what is the length of the line?
W: 4 plus 3
I: (redraws the line, dividing it into two parts and labelling the first part 3c and the second part 4.



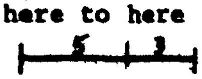
So the length from here to here is...
W: 3c
I: and from here to here
W: 4
I: So what is the length of the line?
W: 3c plus 4
I: 3c plus 4, good write it down.
W: (writes 3c + 4)
I: You got the correct answer when I redrew it and wrote 3c and 4. What was bothering you when you saw the separate c's?
W: I dropped the <u>c</u>. I knew this length was 3c and that was 4, and said 3 plus 4, I forgot all about the <u>c</u>. When it is beside I can see it.

Wendy had a number of difficulties in determining the length of the line. Initially she was concentrating only on the multiplicative aspect of the line problem, thus she answered 4 times <u>c</u>. The line problem in Lesson 2 involved the product of the length of a segment (unknown) times the number of segments, so it is possible that Wendy was trying to fit the new problem into that framework.

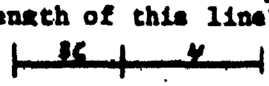
A second difficulty she experienced was that she tended to ignore the letter, as evidenced in her two responses '3 times 4', and '3 plus 4'. It was only when the interviewer wrote 3c on a drawing to represent the length of that segment, that she correctly responded, explaining "I dropped the c. I knew this length was 3c and that was 4, and said 3 plus 4, I forgot all about the c. When it is beside I can see it." These last words of Wendy's bring out the difference in this presentation and that of the area problem. Had the sequence of questions been asked for the line problem, as in the area problem, the first question would have required the student to write '3c' on top of the left-hand side.

Frankie initially wrote '4 × c'. He then changed his answer to '4 + c'. The interviewer then showed Frankie that '4 + c' only represented a part of the line, so he added 2 c's to the expression and wrote '4 + c + c + c'.

I: What is another way of writing it?
F: 3c?
I: That's it, very good. So the length from here to here is...?
F: 3 times c
I: or 3c, and here to here
F: 4
I: Good, so can you write the length of the line?
F: 3 + 4
I: That is 7.
F: 4 times c
I: (Drawing a new line) If I said the length from here to here is 5 and from here to here is 3, what is the length of the whole line?

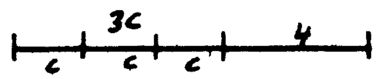


F: 5 plus 3
I: 5 plus 3 (Drawing another line) If I said that the length of the line from here to here is 3c, and from here to here is 4, what is the length of this line?



F: 3c plus 4
I: (Looking at the original problem) From here to here what is the length (Pointing to the c's)?
F: 3c
I: and from here to here (pointing to the part 4 units long)
F: 4
I: So how would you write the length of this line?
F: 3c plus 4

The interviewer then divided 3c into 3 parts, each c units long to show that it was the same problem as the initial one.



Frankie experienced some of the same difficulties as Wendy. He also initially wrote '4 × c' as the length of the line. However, he quickly perceived the additive aspect of the problem as evidenced by his writing '4 + c + c + c', but was unable to write '4 + 3c', (at one point ignoring the letter and writing '3 + 4'), until the interviewer drew a line, which contained two segments, one with length 3c (to represent the 3 c's) and one 4 units long. As in the case of Wendy, he also required a clear separation of the line problem into its two parts, 3c and 4 in order to arrive at the correct answer.

Antoinetta in response to the question "What is the length of this line... length equals" wrote 'Length = 3c + 4'. She claimed that initially she was going to write 'c + c + c', for 3c, but corrected herself after she wrote the first c

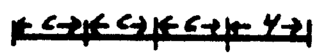
Antoinetta did not experience the difficulties uncovered in the interviews with Wendy and Frankie. She, herself, separated the problem into its two parts, 3c for the 3 c's and 4, that is, she spontaneously perceived both the multiplicative and additive aspects of the problem, as evidenced by her writing 'Length = 3c + 4'.

Gail was erroneously presented with the line

as opposed to the intended one. She responded to the question, "what is the length of this line?" by writing 'Length = d × 6'. She explained "There are four <u>d's</u> and each one takes a space, and probably two of these spaces would equal to the 6". The interviewer remarked that she did not know for sure that the 6 contained two <u>d's</u>. And proceeded as follows:

> I: What is the length of the line till here? (showing only the <u>d's</u>)
> G: Writes 'Length = 4d'
> I: Good, and what about this? (showing the 6)
> G: (continues to complete her equation: 'Length = 4d + 6')

Gail's plausible explanation for her first response and the ease she had in ultimately providing a correct answer, suggests that she was not experiencing any cognitive problems. Her explanation for writing '6 × d' could possibly account for Wendy's and Frankie's initial answer '4 × c'. Since in the problem,

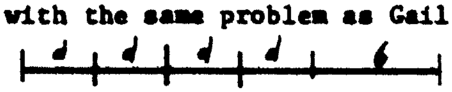


they may have just perceived the 4 to be another <u>c</u> since it was close in length to the <u>c</u>.

<u>Filippo</u> first questioned "is 4 everyone?", to which the interviewer replied, "no, the length of these is <u>c</u> and the length of this is 4". He was then requested to write "Length = ", and he responded by writing "Length = 4 + c", but then put a 3 next to <u>c</u>, "Length = 4 + c3". After being reminded about the convention of writing the number before the letter, he wrote '4 + 3c', under the '4 + c3'.

Filippo had no difficulty in writing the length of the line and perceived the additive and multiplicative nature of the problem. Filippo's first question "is four everyone?", confirms that the line problem appeared to be divided into four equal parts. Thus Frankie's and Wendy's initial incorrect response '4 × c' becomes more justifiable.

<u>Yvette</u> was presented with the same problem as Gail.



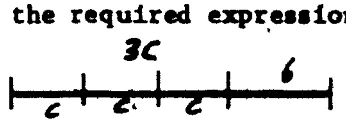
She answered by writing "Length = 4 × D
4D".

The interview then proceeded:

I: What is the length of the line till here (showing only the d's)?
Y: 4
I: The length of the line till here with the <u>d's</u> is 4.
Y: four parts and the <u>d</u> you don't know...there are four parts here and the <u>d</u> you don't know.
I: How can you write the length of the line from here to here, the part just with the <u>d's</u>?
Y: 4 times <u>d</u>
I: Right, so from here to here is 4 times <u>d</u> or <u>4d</u>, how much longer is it?
Y: 6
I: Now can you write the length of the line?
Y: Says and writes '4D + 6'.

Yvette's response, '4 × D, 4D', suggests she was focusing on the multiplicative aspect of the line problem, based on her experience with the previous line problems. She also was able to easily arrive at a correct response, once the problem was simplified by its separation into two parts, <u>4d</u> and 6.

The first line problem illustrates some of the difficulties associated with this problem type. One, the student may focus only on the multiplicative aspect of the problem, based on the line problems presented in the first and second lesson. Two, some students require the separation of the line into two parts, such that each part clearly represents each term of the required expression, that is,



This is not to say that the problem should have been presented as follows:

since Lesson 2 had provided the background for the students to perceive that the length of a line

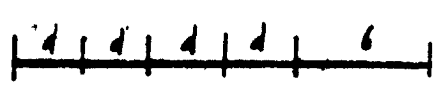
is 3c. (Antoinetta and Filippo managed themselves to make the separation.) However a series of questions appear necessary for some students, such as in the interview, which would assist the student in making this separation, as was done in the first area problem.

Perhaps the need for this separation may also be explained in terms of Collis' Levels of Acceptance of Lack of Closure, whereby initially a student can handle only one open operation. Perhaps the reason the students managed to write the two term expression when each part was clearly indicated, could be due to the fact that no longer were two operations explicitly indicated. That is, the writing of '$3 \times c$', as 3c, might have visually removed the 'multiplication'. Thus the expression may have appeared to the student to contain only one operation; addition.

A third difficulty seems to have been caused by the similarity of the length denoted by a number, and the length denoted by the letter. This seems to have induced many of the students (Wendy, Frankie and Gail) to multiply 4 times c or 6 times d.

A second line problem was presented to the students to ensure that they understood this new version of the line problem. Although it was prepared in particular for only those students who were experiencing difficulty, all subjects were presented with a second line problem.

What is the length of this line?



<u>Wendy</u> said "<u>4d</u> plus 6", and then wrote "Length of line = 4d + 6".

<u>Frankie</u> wrote '6 + d + d + d + d', and then '6 × d', and '6d'.

The interview proceeded as follows:

> I: What is the length of the line from here to here (pointing to the d's)?
> F: 4d
> I: ...and from here to here
> F: 6
> I: So what is the length of the line?
> F: says and writes "4d + 6"

Interviewer presented him with another problem,

to ensure he understood the line problems. He correctly wrote "5a + 4"

<u>Antoinetta and Filippo</u> spontaneously wrote "Length = 4d + 6".

<u>Gail and Yvette</u> were presented with the first line problem:

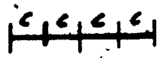
Gail wrote "Length = 3c + 4" and Yvette wrote
"Length = 3 × c
3c   3c + 4
4

The difficulties evidenced in the first line problem, only surfaced in the interview with Frankie, who required again that the problem be explicitly separated into its two terms. The other five students spontaneously wrote the correct expression, each responded by forming an equation such as "Length (of line) = 4d + 6". The ease with which the initial difficulties had been overcome could be explained in terms of familiarity with the problem type.

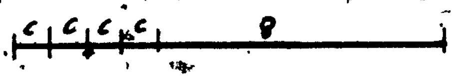
As in all previous problem types, the student now was requested to generate his own line problem.

> Make up a line problem where the length of the line would be '4c + 8'

Yvette first drew



then added on a part 8 units long



and wrote '4 × c
4c + 8'.

Filippo and Gail drew a correct line such as

At no point was the expression '4c + 8' written by either the interviewer or the student.

Antoinetta and Frankie generated the following problem to represent the written expression, '4c + 8':

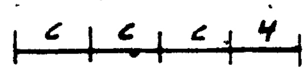
Wendy drew

and wrote '4c + 8'.

The ease with which the correct line problem was generated by each of the subjects confirms that all the subjects had overcome any initial difficulty they were experiencing, and now clearly understood the nature of this type of line problem. Yvette's method of drawing the line problem in two parts, illustrates her obvious need for the separation of the problem.

Summary

The presentation of the first line problem illustrated that some students require the separation of the problem into two explicit parts, each representing a term in the final answer, as was done in the area problems. A possible explanation suggested was that this separation possibly reduces the problem to one operation, addition, for the students. That is, <u>3c</u> may be perceived more as the <u>result</u> of the operation rather than '$3 \times c$'. If this is so the expression '$3c + 4$' may be perceived primarily as one operation, which is easier to accept than an expression with two operations since according to Collis' levels of Acceptance of Lack of Closure, initially a student can only handle one unclosed operation.

The possibility of a link between multiplication and the line problems, a link based on the previous lessons, was suggested as another reason for the difficulty the students experienced in perceiving the additive aspect of the problem. However, it became apparent (in Gail's interview) that the problems, themselves, could have accounted in part for this error. (In the line 

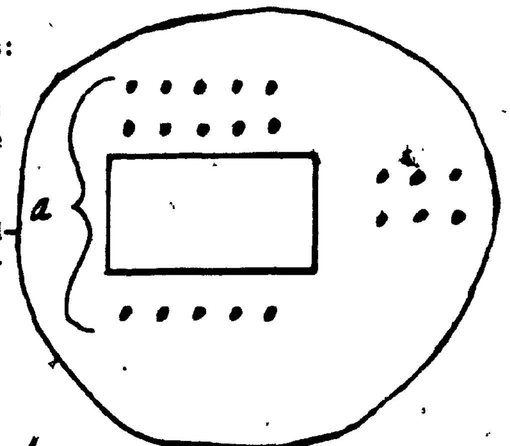
the segment of length 4, appeared to be the same length as the c's, thus the students could have perceived the length to be '$4 \times c$', '4c'.)

In the second line problem only Frankie required the separation suggested above, since he was still operating solely within a multiplicative framework. The other five students spontaneously responded correctly, indicating that they had overcome any initial cognitive difficulty, and now understood the problem type. The reversal further confirmed their understanding.

3. <u>Dot Problem</u>

The inherent difficulties in the dot problem (that is, the difficulty in finding an appropriate convention to represent an unknown quantity of dots) resulted in the inclusion of only one dot problem, with no reversal, in this lesson. This dot problem was added to illustrate another situation where an algebraic expression containing two operations could be presented within the context of a visualizable problem. In the formulation of this problem, we were well aware of the possible difficulties that the subjects may experience. Nonetheless, it was felt that it would be useful to the subjects to present this third context. The problem was presented as follows:

> Here is a dot problem. What is the total number of dots in the circle? (The words 'in the circle' were added in order to ensure that the subjects considered all the dots in their answer).



The students responded as follows:

<u>Wendy</u>:

I: Here is a dot problem, I would like you to write the total number of dots in the circle. What does the <u>a</u> stand for?
W: The number of parts.
I: What do you mean by parts in this problem?
W: How many dots...yes, how many dots in the row.
I: You can see there are 5 dots in each row.
W: No, the rows
I: The number of rows
W: Yes, the number of rows
I: O.K. Can you write the number of dots in this circle?
W: Writes and says '5 × a'
I: or
W: 5a
I: Is that the total number of dots in the circle?
W: Yes...no, there are twenty-one dots that are showing.
I: Twenty-one dots showing. How many dots are here (pointing to the left side)
W: 15
I: There are some dots which are not showing, that is why we have the box here. How would you write the total number of dots here?
W: 5 times <u>a</u>
I: Yes, 5 times <u>a</u> , what about these here?
W: Oh! Writes and says '5a + 6'.

<u>Frankie</u> spontaneously responded by writing '5a + 6'. He responded to the question, "Do you remember what the <u>a</u> stood for?" by saying, "how many rows".

<u>Antoinetta</u> first questioned, "Do I have to write the equation or just the number of dots?" She then wrote "total number of dots = 5a", but questioning, "Do I count these here?" (referring to the other 6 dots) She then completed the equation by writing '5a + 6'.

<u>Gail</u> was first asked, "What does the <u>a</u> stand for?" to which she responded "You don't know what is behind the square." She was then asked, "Can you write the total number of dots in the circle?" After some thought, she wrote '5a + 6'. Further questioning by the interviewer revealed that the inclusion of the word "the circle" assisted her in arriving at the correct response.

<u>Filippo</u> wrote '6 + 10 + 5 = 21 dots' in response to "What is the total number of dots in the circle". The interview then proceeded as follows:

I: Why did I use the letter <u>a</u>?
F: <u>a</u> represents the number of rows.
I: Do you know the number of rows?
F: No
I: Yet you are sure there are 21 dots.
F: No, only 21 dots are showing.
I: Oh, 21 dots showing, I would like you to write the number of dots including the ones that are covered.
F: So what you are saying is I have to give you the total number... so that is 5 times <u>a</u>, all the rows down, plus 6 (writes '5 × a + 6'). After being reminded about concatenation, he wrote '5A + 6'.

<u>Yvette</u> was first asked "Do you remember what the <u>a</u> stood for?" to which she replied "the rows". When she was asked, "Can you tell me what is the total number of dots in this circle?", she said, "5 times a, which is 5a and you have 6, so it is 5a + 6 (writes 5 × A
5A + 6)

The difficulties we had anticipated for the dot problem did not surface to the extent we had expected. Wendy and Filippo, both were focusing on the dots that could be seen (21). However, as soon as they understood the nature of the problem, they experienced no difficulty in providing a correct response. The other four subjects did not show any evidence of any cognitive difficulty with this dot problem, and appeared to have understood what the <u>a</u> stood for — the unknown number of rows.

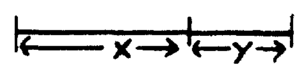
B. PART 2 (Two Unknowns)

In this part of the lesson, the subjects were presented problems containing two unknown quantities. Only the first problems involved one operation, addition, while the others involved two operations, addition and multiplication. As was stated in the introduction to this chapter, only two problem types were presented, the area and line problems. The dot problem was omitted due to the difficulty in illustrating explicitly a dot problem with two unknowns.

The sequence of the presentation of the problem types was altered slightly. At first a line problem was presented since it was felt that it better illustrated the notion of two unknown quantities, that is, two segments of a line, each a different length.

The introductory line problem was as follows:

Suppose I had a line as follows: You notice that it is in two parts. Why do you think I used different letters? What is the length of this line?



The responses were as follows:

Wendy at first responded that two different letters were used, "because you hid the number of parts". When she was asked, "Could I have used an x here and an x there (referring to the two parts). She replied, "Yes, it is complicated in algebra." Although she perceived that the x and y segments were two different lengths, she did not relate that to the use of two different letters. The interviewer essentially had to tell her that two different letters were used because of two different lengths. When she was asked to write the length of the line, she correctly wrote 'x + y'.

Wendy's rationalization for the use of two different letters "because you hid the number of parts", may stem from the fact that all previous line problems consisted of many segments of equal length. Here there was only one segment of each length. Although Wendy was unable to see why two different letters were being used, after the explanation by the interviewer, she perceived the additive nature of the problem and correctly wrote 'x + y' as the length.

Frankie immediately saw that two different letters were used "because one part is longer, and one part is shorter". He responded to, "Can you write the length of the line?", by writing 'x + y'.

Frankie clearly saw that two different letters were used to represent two different lengths (numbers), and had no difficulty in writing the length of the line.

Antoinetta responded that two different letters were used "because they represent different numbers". The interview then proceeded as follows:

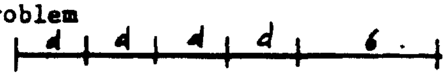
I: Can you write the length of this line?
A: Can you put two numbers together like x times y ?
I: Show me what you mean.
A: Before in numbers you could put 3x which is 3 times x , here can we put xy? (writes 'Length = xy')
I: When we left out a sign, what was the hidden sign?
A: times
I: Times. In algebra when we have two letters together it also means times. So you want to write x times y since you wrote xy ?
A: (Changes her answer, erases the y ,puts a plus sign in and then rewrites the y next to the plus sign)

Antoinetta had no difficulty in realizing that two different letters represented two different numbers. However, she appeared to be focusing on the multiplicative nature of the line problems presented in Lesson 1 and 2. Her response is surprising since in the first line problem in this lesson, she was one of the only students who spontaneously realized the additive aspect of the line problem. Perhaps she was confused by only one segment of each length, whereas in the previous line problems there were many segments of the same length which required multiplication as well as addition. In this problem no multiplication was required.

Gail immediately saw that two different letters were used "because they are different lengths". When she was asked to complete the equation "Length of Line = ", she wrote 'xy'. However, when she was reminded that xy meant x times y, she changed her answer to 'x + y'. She was then asked why she wrote 'x times y' to which she replied, "I was thinking of all the pieces that we did, so I took x and I multiplied it by y because it was a different piece."

Although Gail realized that the two segments were of different lengths, she, like Antoinetta, was confused due to the fact there was only one segment of each length, and she was still thinking in terms of the previous problems, where there were "all the pieces" of equal length which had to be multiplied.

Filippo felt that two different letters were being used to represent the two different parts of the line problem shown in the previous section. That is, in the problem



the d's were one part and the 6 was another. Thus the x would be one part, and the y another.

I: If I would put x and x it would be more d's?
F: It would be like a whole thing of d's.

When he was asked to write the length of the line, he completed the equation "Length = " by writing 'x + y'. The interviewer then attempted to have Filippo see that two different letters were being used because the length of x was different than that of y.

I: What about x and y? Are they the same in length?
F: No, because they are different parts. Just because they are different parts it does not mean they have to be different lengths, but in this drawing the x is larger than the y.
I: If they were the same length would it be necessary for me to use two different numbers?
F: You could have used b and c.
I: Of course in this problem I could have used b and c, but what I am saying ...why did I bother putting an x and a y and not two x's?
F: You are trying to show me that they are two different parts.
I: Not only are they different parts, they are also different...
F: Lengths.

It is interesting to note Filippo's interpretation as to why two different letters were used. He was focusing on the line problems in the first section of this lesson, and thus he associated the two different letters to the two different parts (the letter part and the numerical part). Although he maintained that they could be different lengths, they did not have to be. Thus he did not spontaneously realize that different letters were used to represent different numbers (lengths). Filippo's response is our first experience with a literal symbol used as a label for a geometric part rather than as an unknown length, as has been reported by Kaput (1982) and discussed in Chapter 2.

Yvette did not see why two different letters were being used. Although she realized that the segments were of different lengths, she did not relate this to two different letters. The interviewer had to tell her why two different letters were used. When she was asked to complete the equation "Length of Line = ", she wrote '$x \times y$', and then 'xy'. The interviewer then presented her with a purely numerical problem, a line as follows:

and she responded that its length was '6×8' .
The interview then proceeded as follows:

I: What is the length of this part?
Y: 6
I: And the length of that part?
Y: 8
I: So altogether how long is it?
Y: Oh plus, (and wrote '$6 + 8 = 14$')

When she was directed back to the original line problem, she correctly said and wrote '$x + y$'. An attempt by the interviewer to determine why she had written x times y, did not result in Yvette giving any clear explanation.

As with Antoinetta and Gail, Yvette thought that in order to write the length of the line, she must multiply x times y. Although Yvette did not provide any clear explanation for her answer, the reasons discussed in the analysis of Antoinetta and Gail can provide a possible explanation. That is, the nature of the previous line problems were interfering with her perception of the additive aspect of the problem. Yvette also did not spontaneously realize that different letters were used to represent different lengths (numbers).

Summary

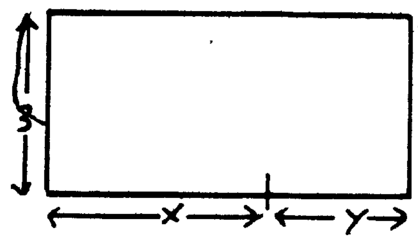
Only three of the six subjects, (Frankie, Antoinetta and Gail) perceived that two different letters were used because the segments were of different lengths (thus different numbers). Filippo was focusing on the previous line problems, and concluded that the different letters represented the two different parts (the letter part and the number part). The problem presented here was a visualizable one. The two segments were obviously of different lengths, yet still three students did not relate the different letters to the different lengths (numbers).

In writing the length of the line, Antoinetta, Gail and Yvette multiplied <u>x</u> and <u>y</u> . This error was due to their viewing the problem within the framework of the previous line problems, where segments of equal length had to be multiplied. In this problem, there was only one segment of each length. However, in all three cases, they easily were directed to see that <u>x</u> and <u>y</u> had to be added.

1. Area Problems

Once the idea that two different letters represent two different lengths was presented, an area problem was then introduced where the base was divided into two unequal segments, both lengths unknown (<u>x</u> and <u>y</u>). The presentation was as follows:

Here is a rectangle. Can you write the area of this rectangle? (If no response, the interviewer was to divide the rectangle as done previously, and then have the student determine the area of each of the smaller rectangles, that is, 3x and 3y.)

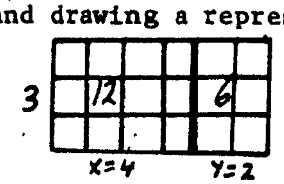


The subjects responded in the following manner:

Wendy at first responded by writing '3x + y'. The interviewer then had Wendy divide the rectangle into two smaller rectangles, and she was directed to write the areas, 3x and 3y in the appropriate rectangles. When she was again asked what the area of the blue outside rectangle was, she said "3x times 3y", then changed her answer to "3x times y". The interviewer had her recall what was area, that is, the number of squares it takes to cover a surface, saying "since 3x and 3y are the number of squares it takes to cover each of the rectangles, so what if you want to know the total number of squares?" Wendy replied, "you add", and then wrote the area of the rectangle was '3x + 3y'.

Wendy's initial response, '3x + y', shows she was keeping in mind both the multiplicative and additive nature of the problem. However, after Wendy determined the area of each of the rectangles, she reverted back to using only multiplication (3x times y, 3x times 3y) illustrating that the additive aspect of this type of problem is very tenuous. The notion of providing an answer which would give the total number of squares, led Wendy to once again perceive the additive nature of the problem, and she then provided the correct response.

Frankie like Wendy wrote that the area was '3x + y'. Even when he had divided the rectangles into two smaller ones and subsequently wrote 3x and 3y in the appropriate rectangles, he still said that the area was '3x + y'. The interviewer then had him look at the previous area problems, where the areas of the smaller rectangles were added to determine the area of the 'blue rectangle'. He then wrote '3x + 3y'. Further questioning revealed that Frankie was not sure of his answer. The interviewer then provided a numerical situation, by assigning values to the x and y and drawing a representative rectangle.



Viewing area in terms of the number of squares it takes to cover a given surface, he was directed to see that 12 and 6 squares were required to cover each of the smaller rectangles. Thus to cover the blue rectangle it was '12 + 6 = 18'. However, he had written '3x + y', which would have been '12 + 2, (14)' and not 18.

Frankie's initial response, like Wendy's indicates that he was aware of the multiplicative and additive aspect of this problem. Although Frankie did finally provide the correct answer, once again (as in the second area problem in the first section) we see that his correct response did not necessarily imply an understanding of the problem.

Antoinetta first wrote 'area = 3x + y". She was then asked to divide the rectangle into two parts, and requested to write their areas inside each rectangle. She wrote 3x and 3y in the appropriate rectangles. She subsequently changed her initial answer saying, "You times it, 3x times 3y". After being reminded about the previous problems, and the definition of area, that is the number of squares it takes to cover a surface, she then changed her answer by writing "Area = 3x + 3y". The interview then proceeded as follows:

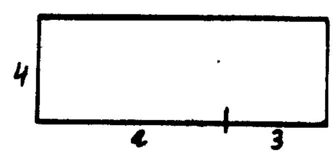
> I: ... you seem to be concerned about something, please tell me...
> A: I knew 3y meant 3 times y. I started getting really confused and started writing 3x times 3y.
> I: Which part confused you?
> A: well...uh...because it was adding and I changed it.
> I: Because you had 3x + y and then you had to change it?
> A: Well...to find the area you always multiply, I just...I got confused because I thought you had to times it.

Antoinetta, like Wendy and Frankie, initially wrote '3x + y' as the area. This constant omission of the 3 in the second term requires some explanation. Perhaps the placement of the 3 on the extreme left of the rectangle could account for the omission. This problem could be overcome perhaps by writing the 3 on the right-hand side as well as the left-hand side of the rectangle.

Also, as in the case of Wendy, Antoinetta reverted back to multiplication, and changed her answer by stating, "You times it, 3x times 3y". It is interesting to note that Antoinetta had no difficulty in the second area problem in the first section of this lesson, and immediately had perceived the additive aspect of that problem. However, as in the case of Wendy, Antoinetta's acceptance of the additive aspect of the problem is very tenuous. Thus the acceptance of addition in an area problem seems rather difficult for a novice who strongly links area <u>only</u> to multiplication and cannot perceive it within any other context, as Antoinetta clearly stated, "to find area you always multiply...I got confused because I thought you had to times it."

<u>Gail</u> at first wrote 'Area of rectangle = 3xy'. She then was instructed to separate the rectangles into two smaller ones, and wrote <u>3x</u> and <u>3y</u> inside the appropriate rectangles. When she was again asked, "What is the area of the blue outside rectangle?", she wrote, '3x3y'. When the interviewer reminded her that '3x3y' meant 3x times 3y, she responded, "No, 3x plus 3y" and wrote a plus sign between the 3x and the 3y . She claimed she had thought it was multiplication but "because it is two pieces, you add."

Although in the second area problem,



it had appeared that Gail had understood the additive aspect of this area problem, her responses (3xy and 3x3y) indicate, like Wendy and Antoinetta, that the strong link she had between area and multiplication was difficult for her to overcome. She appeared at the end of the interview to once again see the reason for the additive aspect of this problem by stating "because it is two pieces, and you add them".

<u>Filippo</u> spontaneously responded by writing 'Area = 3x + 3y', indicating he clearly understood this problem type.

<u>Yvette</u> wrote the area of the rectangle as follows: "Area = 3 + xy". When she divided the rectangle into two parts, she wrote the areas '3x' and '3 × y', meaning multiplication and underneath it she wrote '3y' in the appropriate rectangles. When she was again asked to write the area of the blue outside rectangle, she wrote "Area = 3x × 3y", but

she then changed her answer saying "no plus, (changing the times to a plus sign) I just remembered." An attempt by the interviewer to uncover why she had added resulted in her merely saying that "you have to add them up" without providing any explanation.

Yvette's first response also indicates that she was well aware that two operations were required and wrote '3 + xy' for the area. Like Wendy and Antoinetta, further questioning led Yvette to drop the additive aspect of the problem and she then wrote the area as '3x × 3y' which she spontaneously corrected to read '3x + 3y', explaining, "I just remembered."

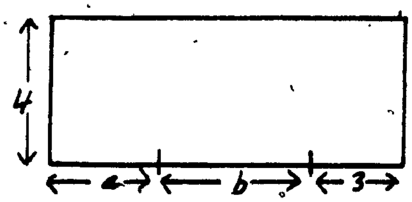
The first area problem with two segments on the base of unknown length, brought back some of the initial cognitive difficulties uncovered in the first area problem of this lesson. Wendy, Antoinetta, Gail and Yvette reverted at some point to using only multiplication in their answers by writing eigher '3xy (3x times y), and/or 3x3y (3x times 3y). Thus the strong link between area and multiplication was once again interfering with their seeing the additive aspect of the problem. It is interesting to note that three of those students, Wendy, Antoinetta and Yvette had initially responded by providing an expression with both operations, however, the additive part of the expression was quickly dropped by them. This showed just how tenuous was their acceptance of the additive aspect of the problem.

Once again in the interview with Frankie, we saw that a correct response did not actually imply understanding.

Also, three students responded initially by writing '3x + y' as the area. It was suggested that the omission of the 3 could be due to the location of the 3, which appears on the extreme left, and that perhaps the writing of a 3 on the right-hand side could possibly overcome this problem.

The second area problem went one step further. The base consisted of three parts, two segments of unknown length and one segment of known length.

> Look at this rectangle. Recalling how we found the area of the previous rectangles, can you write the area of this rectangle?



The responses of the subjects were as follows:

Wendy divided the rectangle into three smaller rectangles, writing '4a, 4b, 4 × 3 = 12', in the appropriate rectangles. She then wrote, 'Area = 4a + 4b + 12'.

Frankie also divided the rectangle into three smaller rectangles, writing '4a, 4b and 12' in the appropriate rectangles. He then completed the equation "Area = ", by writing '4a + 4b + 12'.

Antoinetta just wrote 'Area = 4a + 4b + 12'.

Gail divided the rectangles into three smaller rectangles, but she did not write their areas inside. She just wrote a final answer 'Area of rectangle = 4a + 4b + 12'.

Filippo wrote '4a + 4b + 4 × 3 , then wrote a 12 under the '4 × 3'. He was asked to write his answer in one line, he wrote '4a + 4b + 12'. When he was asked to explain his answer, he said, "First of all it is 4 times a then 4 times b and then 4 times 3, and then you add them.

Yvette divided the rectangle into three smaller rectangles, and wrote '4 × a (4a), 4b, and 4 × 3 = 12' in the appropriate rectangles. Her final answer she wrote in the form of an equation, 'Area = 4a + 4b + 12".

All six subjects responded easily to this problem indicating not only that they had overcome their initial difficulties, but also that they could transfer and extend their knowledge and understanding acquired in the previous area problems to one whose area had to be expressed by writing an algebraic expression containing three, not two, terms. Thus it seems that once the initial difficulties are overcome the area problem provides a visualizable context, which can easily be handled by