THE EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR A CLASS OF PIECEWISE MONOTONIC TRANSFORMATIONS

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ABSTRACT

THE EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR A CLASS OF PIECEWISE MONOTONIC TRANSFORMATIONS

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Let $\tau:[0,1]\to[0,1]$ be a piecewise C^2 transformation with the property that $|\frac{d\tau}{dx}|>1$ where the slope exists. We define the Frobenius-Perron operator $P_\tau:L_1\to L_1$ by the formula

$$P_\tau f(x) = \frac{d}{dx}\int_{\tau^{-1}[0,x]} f(y)dy.$$

It is shown that the variation of $P_\tau^k f$ is bounded for all k. This is the crucial fact in establishing the main result of the thesis: τ admits a (not necessarily unique) absolutely continuous invariant measure.

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CHAPTER I

INTRODUCTION

1.1 ABSOLUTELY CONTINUOUS INVARIANT MEASURES

Let $J = [0,1]$ and let τ be a transformation from J into J, not necessarily one-to-one. For $A \subset [0,1]$, we let $\tau^{-1}(A) = \{x \in J : \tau(x) \in A\}$. We consider the average amount of time the orbit $\{\tau^n(x)\}$ spends in a set $S \subset J$, where τ^n denotes the nth iterate of τ and x is any point in J. The number of times $\{\tau^n(x)\}$ is in S for n between 0 and N is

$$\sum_{n=0}^{N} \chi_S(\tau^n(x)),$$

where χ_S is the characteristic function of the set S, i.e., $\chi_S(y) = 1$ if $y \in S$ and 0 otherwise. The average time spent in S may be defined to be

$$(1.1.1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_S(\tau^n(x)),$$

when this limit exists. We will say f is a density of x for τ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_S(\tau^n(x)) \text{ exists and is equal to } \int_S f(x)dx$$

for every measurable set S in J. Since $\chi_J(\tau^n(x)) = 1$ for all n, it follows that

$$(1.1.2) \qquad \int_0^1 f(x)\,dx = 1.$$

Frequently f is "almost" independent of x, i.e., f is the density of x for τ for almost all x. The Birkhoff Ergodic Theorem [7, p. 18] gives a condition for f to be independent of x. First we recall some definitions.

A measure μ is an absolutely continuous invariant measure (with respect to Lebesque measure) if and only if there exists a function $f: J \to [0,\infty)$, $f \in L_1 = L_1(J)$, the space of integrable functions on J, such that

$$\mu(S) = \int_S f(x)\,dx$$

for every Lebesque measurable set $S \subset J$. The density in (1.1.2) or the corresponding measure μ is said to be invariant under τ if $\mu(\tau^{-1}(A)) = \mu(A)$ for every measurable set A.

The Birkhoff Ergodic Theorem says that if there exists an invariant density and the density is unique, then the limit (1.1.1) exists for almost all x and furthermore

$$(1.1.3) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N=1} g(\tau^n(x)) = \int_0^1 g(x)f(x)\,dx, \quad \text{a.e.}\,\mu$$

where g is an integrable function. In other words, except for x in a set A, $\mu(A) = 0$, the time mean $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} g(\tau^n(x))$ is equal to the space mean

$$\int_0^1 g(x)f(x)\,dx.$$

For $g(x) = \chi_S(x)$, (1.1.3) takes on the familiar form

$$(1.1.4) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_S(\tau^n(x)) = \int_S f(x)\,dx, \qquad \text{a.e.}\mu.$$

We now make the following two important observations as a result of the absolute continuity of μ:

1) time averages are equal to space averages, and
2) the sets of points x for which (1.1.1) does not converge constitute a set of Lebesque measure 0. This gives a feel for the size of the set on which convergence does not occur.

Our interest in absolutely continuous invariant measures is inspired by the foregoing two observations. An important question in mathematics and which is of great interest in the physical and biological sciences is to find verifiable conditions on a transformation which will ensure that it admits an absolutely continuous invariant measure. Once this is done, the next step is to prove that it is unique, which would imply ergodicity. This thesis will deal exclusively with the problem of existence.

1.2 SCOPE OF THESIS

In Chapter II, we introduce the Frobenius Perron operator and develop some of its properties. This operator is the tool needed to establish the existence of absolutely continuous invariant measures for a large class of transformations τ, namely those which are piecewise C^2 and satisfy the condition that $|\frac{d\tau}{dx}|>1$, where the derivative exists.

In Chapter III we present and prove a number of well-known results which are needed in the existence theorem: Helly's Selection Principle, Mazur's Theorem and the Kakutani-Yoshida Theorem.

In Chapter IV, we prove in full detail the existence theorem of Lasota and Yorke [1]. This theorem gives sufficient conditions for a transformation to admit a finite absolutely continuous invariant measure. In Section 4.2, an example is given which shows that the slope condition is essential.

Chapter V deals with recent results of Pianigiani [4], which employs the foregoing existence theorem to establish the existence of absolutely continuous invariant measures for transformations which have slope less than or equal to one.

Even if it is known that there exists a unique absolutely continuous invariant measure, finding it may be a tedious chore. In Chapter VI we discuss a way of approximating the fixed point of the Frobenius-Perron Operator with fixed points of matrices.

In Chapter VII we show that for certain transformations, the fixed point of the Frobenius-Perron Operator is practically a constant. Thus for such transformations, the Birkhoff Ergodic Theorem implies that the average time an orbit spends in a set A is approximately the Lebesque measure of the set A. This is important in the theory of pseudo random number generation.

CHAPTER II

THE FROBENIUS-PERRON OPERATOR

2.1 INTRODUCTION

Let us suppose we have a random variable X on [0,1] with

$$\text{Prob}\,\{X \in A\} = \int_A f\,dm$$

where m is Lebesque measure. We would like to know the probability that X is in A after being transformed by τ. Thus we write,

$$\text{Prob}\,\{\tau(X) \in A\} = \text{Prob}\,\{X \in \tau^{-1}(A)\}$$

$$= \int_{\tau^{-1}(A)} f\,dm.$$

Further, we would like to know if there exists a function ϕ such that

$$\text{Prob}\{\tau(X) \in A\} = \int_A \phi\,dm.$$

Such a function ϕ will obviously depend on f and τ. We refer to it as the Frobenius-Perron Operator acting on f. With this in mind we proceed.

Let $\tau:[0,1]\to[0,1]$ be a measurable transformation such that $m(\tau^{-1}(A)) = 0$ if $m(A) = 0$ for A a measurable subset of [0,1], and define a measure μ where

$$\mu(A) = \int_{\tau^{-1}(A)} f\,dm$$

where $f \in L_1[0,1]$ and A is an arbitrary measurable set.

We see that $m(A) = 0 \Rightarrow m(\tau^{-1}(A)) = 0 \Rightarrow \mu(A) = 0$, that is $\mu \ll m$. Then by the Radon-Nikodym Theorem there exists $\phi \in L_1[0,1]$ such that for all measurable sets A

$$\mu(A) = \int_A \phi dm$$

and ϕ is unique a.e.

We define the Frobenius-Perron Operator for τ by setting $P_\tau f = \phi$. Thus, for all measurable sets $A \subset [0,1]$

$$\int_A P_\tau f dm = \int_{\tau^{-1}(A)} f dm$$

from which it follows that

$$\int_0^x P_\tau f dm = \int_{\tau^{-1}([0,x])} f dm,$$

and so

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f dm.$$

As an example, let us consider the transformation

$$\tau(x) = \begin{cases} 2x & , \ 0 \le x \le \frac{1}{2} \\ -\frac{4}{3}x + \frac{5}{3} & , \ \frac{1}{2} < x \le 1 \end{cases}.$$

We see that

$$\tau^{-1}([0,x]) = [0,\tfrac{1}{2}x] \cup [\tfrac{5}{4} - \tfrac{3}{4}x, 1].$$

If $x \geq \frac{1}{3}$, and

$$\tau^{-1}([0,x]) = [0,\tfrac{1}{2}x].$$

If $x<\frac{1}{3}$. Hence for any $f \in L_1[0,1]$ and $B = [\frac{1}{2},1]$

$$\int_{\tau^{-1}([0,x])} f(s)ds = \int_0^{\frac{x}{2}} f(s)ds + \int_{\frac{5}{4}-\frac{3}{4}x}^{1} f(s)\chi_B(s)ds.$$

Therefore

$$P_\tau f(x) = \frac{1}{2}f(\frac{x}{2}) + \frac{3}{4}f(\frac{5}{4}-\frac{3}{4}x)\chi_B(\frac{5}{4}-\frac{3}{4}x).$$

$$= \frac{1}{2}f(\frac{x}{2}) + \frac{3}{4}f(\frac{5}{4}-\frac{3}{4}x)\chi_J(x)$$

where $J = \tau(B) = [\frac{1}{3},1]$. Later we shall generalize this type of representation for $P_\tau f$.

We shall find the Frobenius-Perron Operator useful in establishing the existence of absolutely continuous invariant measures for a large class of transformations. Before we do this, we shall study some of the properties of the Frobenius-Perron Operator.

Everywhere in this paper, integration is with respect to Lebesque measure "m" unless specified otherwise.

2.2 PROPERTIES

We first notice that P_τ is linear: for $f,g \in L_1[0,1]$ and a,b scalars, we have

$$\int_A P_\tau(af+bg) = \int_{\tau^{-1}(A)} (af+bg)$$

$$= a\int_{\tau^{-1}(A)} f + b\int_{\tau^{-1}(A)} g$$

$$= a\int_A P_\tau f + b\int_A P_\tau g$$

$$= \int_A (aP_\tau f + bP_\tau g).$$

Since A is arbitrary, we have

$$P_\tau(af+bg) = aP_\tau f + bP_\tau g.$$

In Lemma 2.3, we will show that $||P_\tau(f-g)|| \leq ||f-g||$ and therefore it follows that P_τ is continuous since

$$||P_\tau f - P_\tau g|| = ||P_\tau(f-g)|| \leq ||f-g|| \to 0$$

as $||f-g|| \to 0$.

<u>Lemma 2.1</u>. $f \in L_1[0,1]$ and $f \geq 0 \Rightarrow P_\tau f \geq 0$

<u>Proof</u>. For $A \subset [0,1]$

$$\int_A P_\tau f = \int_{\tau^{-1}(A)} f \geq 0$$

Hence $P_\tau f \geq 0$ since A is abritrary. Q.E.D.

<u>Lemma 2.2</u>. P_τ preserves integrals.

<u>Proof</u>. Since $\tau^{-1}([0,1)] = [0,1]$,

$$\int_0^1 P_\tau f = \int_{\tau^{-1}([0,1])} f = \int_0^1 f \qquad \text{Q.E.D.}$$

<u>Lemma 2.3</u>. $||P_\tau f|| \leq ||f||$

<u>Proof</u>. Let $f \in L_1[0,1]$. Let $f^+ = \max(f,0)$ and $f^- = -\min(0,f)$. Then $f^+, f^- \in L_1[0,1]$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$

$$P_\tau f = P_\tau (f^+ - f^-)$$

$$= P_\tau f^+ - P_\tau f^-$$

$$|P_\tau f| \leq |P_\tau f^+| + |P_\tau f^-|$$

$$= P_\tau f^+ + P_\tau f^-$$

by virtue of Lemma 2.1. Then by using Lemma 2.2 we have

$$\int_0^1 |P_\tau f| \leq \int_0^1 P_\tau f^+ + \int_0^1 P_\tau f^-$$

$$= \int_0^1 f^+ + \int_0^1 f^-$$

$$= \int_0^1 |f| .$$

Hence

$$||P_\tau f|| \leq ||f|| \qquad \text{Q.E.D.}$$

<u>Lemma 2.4</u>. Let τ $[0,1]\to[0,1]$, $\sigma:[0,1]\to[0,1]$ with $m(\tau^{-1}(A))=0$ and $m(\sigma^{-1}(A))=0$ if $m(A)=0$, then $P_{\tau\circ\sigma}f = P_\tau P_\sigma f$ and consequently $P_{\tau^n}f = P_\tau^n f$. (τ^n denotes the nth iterate of τ).

<u>Proof</u>. Define $\gamma(A) = \int_{(\tau\circ\sigma)^{-1}(A)} f$, $f\in L_1[0,1]$.

$$\begin{aligned} m(A) = 0 &\Rightarrow m(\tau^{-1}(A)) = 0 \\ &\Rightarrow m(\sigma^{-1}(\tau^{-1}(A))) = 0 \\ &\Rightarrow \gamma \ll m. \end{aligned}$$

Therefore $\gamma(A) = \int_A P_{\tau\circ\sigma}f$ is well defined.

Now

$$\begin{aligned} \int_A P_{\tau\circ\sigma}f &= \int_{(\tau\circ\sigma)^{-1}(A)} f \\ &= \int_{\sigma^{-1}(\tau^{-1}(A))} f \\ &= \int_{\tau^{-1}(A)} P_\sigma f \\ &= \int_A P_\tau P_\sigma f \end{aligned}$$

Since A is arbitrary $P_{\tau\circ\sigma}f = P_\tau P_\sigma f$ a.e.

If we assume $P_{\tau^n}f = P_\tau^n f$, then

$$P_{\tau^{n+1}}f = P_{\tau\circ\tau^n}f$$

$$= P_\tau P_{\tau^n}f$$

$$= P_\tau P_\tau^n f$$

$$= P_\tau^{n+1}f$$

and so inductively we must have

$$P_{\tau^n}f = P_\tau^n f. \quad \text{Q.E.D.}$$

<u>Lemma 2.5</u>. $P_\tau f = f \Longleftrightarrow \mu(\tau^{-1}(A)) = \mu(A)\forall A,\ d\mu = fdm.$

<u>Proof</u>. Assume $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable set A.

Then

$$\int_{\tau^{-1}(A)} fdm = \int_A fdm,$$

and therefore

$$\int_A P_\tau fdm = \int_A fdm.$$

Since A is arbitrary

$$P_\tau f = f.$$

Assume $P_\tau f = f$, then

$$\int_A P_\tau f dm = \int_A f dm.$$

By definition

$$\int_A P_\tau f dm = \int_{\tau^{-1}(A)} f dm$$

and so

$$\mu(\tau^{-1}(A)) = \mu(A). \qquad \text{Q.E.D.}$$

<u>Theorem 2.1</u>. Let $\phi_i \in C^1[b_{i-1}, b_i]$ and monotone where $0 = b_0 < b_1 < b_2 < \ldots < b_q = 1$. Assume also, for the sake of convenience that each ϕ_i can be extended as a monotone C^1 function on $[0,1]$. Then for

$$\phi = \sum_{i=1}^{q} \phi_i \chi_{B_i}, \quad \text{where } B_i = [b_{i-1}, b_i],$$

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_{J_i}(x),$$

where $\psi_i = \phi_i^{-1}$, $\sigma_i = |\psi_i'|$, $J_i = \phi_i(B_i)$ and χ_A is the characteristic function of the set A.

<u>Proof.</u> Set $A_i(x) = \phi_i^{-1}([0,x]) \cap B_i = \psi_i([0,x]) \cap B_i$

then

$$\int_{A_i(x)} f(s)ds = \pm\int_{\psi_i(0)}^{\psi_i(x)} f(s)\chi_{B_i}(s)ds$$

We want $\int_{A_i(x)} f \geq 0$ when $f \geq 0$.

Since ϕ_i is monotone, ψ_i is monotone and ϕ_i and ψ_i are either both increasing or both decreasing. Therefore

$$\frac{\psi_i'(x)}{|\psi_i'(x)|} = \frac{\psi_i'(y)}{|\psi_i'(y)|} \qquad \forall x,y \in [0,1].$$

We use this to set the sign. Thus,

$$\int_{A_i(x)} f(s)ds = \frac{\psi_i'(x)}{|\psi_i'(x)|} \int_{\psi_i(0)}^{\psi_i(x)} f(s)\chi_{B_i}(s)ds$$

$$\frac{d}{dx}\int_{A_i(x)} f(s)ds = \frac{\psi_i'(x)}{|\psi_i'(x)|} \frac{d}{dx} \int_{\psi_i(0)}^{\psi_i(x)} f(s)\chi_{B_i}(s)ds$$

$$= \frac{\psi_i'(x)}{|\psi_i'(x)|} f(\psi_i(x))\chi_{B_i}(\psi_i(x))\psi_i'(x)$$

$$= \frac{|\psi_i'(x)|^2}{|\psi_i'(x)|} f(\psi_i(x))\chi_{B_i}(\psi_i(x))$$

$$= f(\psi_i(x))\sigma_i(x)\chi_{B_i}(\psi_i(x)).$$

We note that

$$\chi_{B_i}(\psi_i(x)) = 1 \iff \psi_i(x) \in B_i$$

$$\iff x \in \phi_i(B_i) = J_i$$

$$\iff \chi_{J_i}(x) = 1.$$

Therefore $\chi_{J_i}(x) = \chi_{B_i}(\psi_i(x))$ and we obtain

$$\frac{d}{dx} \int_{A_i(x)} f(s)ds = f(\psi_i(x))\sigma_i(x)\chi_{J_i}(x).$$

For $\phi = \sum_{i=1}^{q} \phi_i \chi_{B_i}$,

$$\phi^{-1}([0,x]) = \bigcup_{i=1}^{q} A_i(x),$$

where $A_i(x)$'s are disjoint since B_i's are disjoint. Thus

$$P_\phi f(x) = \frac{d}{dx} \int_{\phi^{-1}([0,x])} f(s)ds$$

$$= \frac{d}{dx} \sum_{i=1}^{q} \int_{A_i(x)} f(s)ds.$$

Therefore

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_{J_i}(x). \qquad \text{Q.E.D.}$$

CHAPTER III

PRELIMINARY RESULTS

Many steps in the main Theorem of [1] are left out. In this chapter, we fill in a number of steps that will make the proof of the main Theorem easier to follow.

3.1. THE LEMMAS.

<u>Lemma 3.1</u>. If $f \in C^1[a,b]$ with $|f'|>0$, then f is monotone on $[a,b]$.

<u>Proof</u>. $f \in C^1[a,b] \Rightarrow f' \in C[a,b]$.

$$|f'|>0 \Rightarrow \text{(i)} \quad -\infty<f'<0 \quad \text{or}$$
$$\text{(ii)} \quad 0<f'<\infty .$$

Since f' is continuous, it is only possible to have $f'<0 \ \forall x \in [a,b]$ or $f'>0 \ \forall x \in [a,b]$.

In either case f is monotone on $[a,b]$. Q.E.D.

<u>Definition 3.1</u>. Let P be a partition of $[a,b]$ where $a = x_0<x_1<x_2<\ldots<x_n = b$, then we define $\overset{b}{\underset{a}{V}}(f,P)$ the variation of f with respect to P to be

$$\overset{b}{\underset{a}{V}}(f,P) = \sum_{i=1}^{n} |f(x_i)- f(x_{i-1})| \quad \text{and} \quad \overset{b}{\underset{a}{V}}f = \sup_P \overset{b}{\underset{a}{V}}(f,P),$$

the variation of f on [a,b].

Lemma 3.2. $\overset{b}{\underset{a}{V}}(f+g) \le \overset{b}{\underset{a}{V}}f + \overset{b}{\underset{a}{V}}g$ and thus $\overset{b}{\underset{a}{V}}\left(\sum_{k=1}^{n} f_k\right) \le \sum_{k=1}^{n} \overset{b}{\underset{a}{V}}f_k$.

Proof. Take any partition, P, of [a,b] with

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b$$

then

$$\overset{b}{\underset{a}{V}}(f+g\,,\,P) = \sum_{k=1}^{m} |f(x_k)+g(x_k)-f(x_{k-1})-g(x_{k-1})|$$

$$\le \sum_{k=1}^{m} |f(x_k)-f(x_{k-1})| + \sum_{k=1}^{m} |g(x_k)-g(x_{k-1})|$$

$$\le \overset{b}{\underset{a}{V}}f + \overset{b}{\underset{a}{V}}g$$

Since P is arbitrary

$$\overset{b}{\underset{a}{V}}(f+g) \le \overset{b}{\underset{a}{V}}f + \overset{b}{\underset{a}{V}}g. \quad \text{Q.E.D.}$$

Lemma 3.3. $x \in [a,b] \Rightarrow |f(a)| + |f(b)| \le \overset{b}{\underset{a}{V}}f + 2|f(x)|$.

Proof. For $a \le x \le b$, $\{a,x,b\}$ is a partition of [a,b], call it P.

Then $$\overset{b}{\underset{a}{V}}f \geq \overset{b}{\underset{a}{V}}(f,P)$$

$$= |f(x)-f(a)| + |f(b)-f(x)|$$

$$\geq |f(a)|-|f(x)|+|f(b)|-|f(x)| .$$

and so $$|f(a)|+|f(b)| \leq \overset{b}{\underset{a}{V}}f + 2|f(x)|. \quad \text{Q.E.D.}$$

<u>Lemma 3.4</u>. Let f_i be defined on $[\alpha_i,\beta_i]\subset[a,b]$ and

$$\chi_i(x) = \begin{cases} 1, & x\in[\alpha_i,\beta_i] \\ 0, & \text{otherwise} . \end{cases}$$

Then for $$f = \sum_{i=1}^{n} f_i\chi_i ,$$

$$\overset{b}{\underset{a}{V}}f \leq \sum_{i=1}^{n} \overset{\beta_i}{\underset{\alpha_i}{V}} f_i + \sum_{i=1}^{n} (|f_i(\alpha_i)|+|f_i(\beta_i)|).$$

<u>Proof</u>. For $f_i(x)\chi_i(x)$,

$$\overset{b}{\underset{a}{V}}f_i\chi_i \leq \overset{\beta_i}{\underset{\alpha_i}{V}} f_i + |f_i(\alpha_i)|+|f_i(\beta_i)|.$$

Equality holds if $\alpha_i \neq a$ and $\beta_i \neq b$. (See Figure 1)

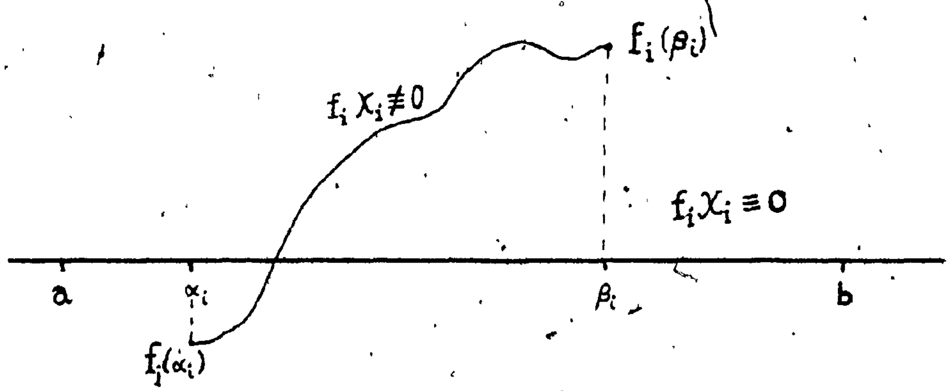

Figure 1

Then from Lemma 3.2

$$\overset{b}{\underset{a}{V}} f = \overset{b}{\underset{a}{V}}\left(\sum_{i=1}^{n} f_i\chi_i\right)$$

$$\le \sum_{i=1}^{n} \overset{b}{\underset{a}{V}} f_i\chi_i$$

$$\le \sum_{i=1}^{n} \left(\overset{\beta_i}{\underset{\alpha_i}{V}} f_i + |f_i(\alpha_i)| + |f_i(\beta_i)|\right). \qquad \text{Q.E.D.}$$

Lemma 3.5. Let f be differentiable and one to one, and let $g=f^{-1}$. If $|f'|\geq\alpha$ then $|g'|\leq\frac{1}{\alpha}$

Proof.
$$f(g(x)) = x$$
$$f'(g(x))g'(x) = 1$$
$$|f'(g(x))| = \frac{1}{|g'(x)|} \geq \alpha$$

therefore $|g'(x)| \leq \frac{1}{\alpha}$. Q.E.D.

Lemma 3.6. If $f\in C^2[a,b]$ is monotone on $[a,b]$, then $\frac{d}{dx}|f'|$ exists everywhere on $[a,b]$.

Proof. f monotone $\Rightarrow f'>0 \ \forall x\in[a,b]$ or $f'<0 \ \forall x\in[a,b]$

$\Rightarrow |f'|=f' \ \forall x\in[a,b]$ or $|f'|= -f' \ \forall x\in[a,b]$

$f'\in C^1[a,b] \Rightarrow -f'\in C^1[a,b]$. Hence $|f'|$ is differentiable everywhere on $[a,b]$. Q.E.D.

Lemma 3.7. If $\overset{1}{\underset{0}{V}}f\leq a$ and $||f||\leq b$, where $||f||=\int_0^1|f|$, then $|f(x)| \leq a+b \ \forall x$.

Proof. $||f||\leq b \Rightarrow \exists\alpha$ such that $|f(\alpha)|\leq b$. If not, then $|f(x)|>b \ \forall x\in[0,1]$ and therefore $\int_0^1|f| > \int_0^1 b = b$ and we have a contradiction.

Also $|f(x)-f(\alpha)| \le \overset{1}{\underset{0}{V}} f \le a$. Therefore

$$\big||f(x)|-|f(\alpha)|\big| \le a$$

and

$$\begin{aligned}|f(x)| &\le a+|f(\alpha)| \\ &\le a+b. \qquad \text{Q.E.D.}\end{aligned}$$

Lemma 3.8. For $f \in L_1[0,1]$ and $\varepsilon>0$, there exists $r=r(\varepsilon)$

such that $\int_0^1 (f^+-r)^+ + \int_0^1 (f^--r)^+ \le \varepsilon$,

where $f^+ = \max(f,0)$, $f^- = -\min(0,f)$.

Proof. Let ϕ be a simple bounded function such that $\phi \ge 0$ and let $M = \max \phi(x)$. Then $\phi-r \le M-r$ and

$$(\phi-r)^+ \le (M-r)^+ = \begin{cases} M-r, & \text{if } M>r \\ 0, & \text{if } M \le r \end{cases}$$

Given $\varepsilon>0$, we set r such that $r \ge M-\varepsilon$. Then

$$M-r \le \varepsilon.$$

Therefore $(M-r)^+ \le \varepsilon$

and so $\int_0^1 (M-r)^+ = (M-r)^+ \le \varepsilon$.

Since $(M-r)^+ \ge (\phi-r)^+$ we have

$$\int_0^1 (\phi-r)^+ \le \varepsilon.$$

Thus the Lemma is proved for ϕ simple and bounded.

The set of bounded simple functions is dense in $L_1[0,1]$ so for $f \in L_1[0,1]$ and $\varepsilon > 0$, we can choose ϕ and ψ simple and bounded such that

$$||f^+ - \phi|| \le \frac{\varepsilon}{4} \quad \text{and} \quad ||f^- - \psi|| \le \frac{\varepsilon}{4}, \quad \phi, \psi \ge 0.$$

For ϕ, ψ we can choose r_1, r_2 such that

$$\int_0^1 (\phi - r_1)^+ \le \frac{\varepsilon}{4} \quad \text{and} \quad \int_0^1 (\psi - r_2)^+ \le \frac{\varepsilon}{4}.$$

Then for $r \ge \max(r_1, r_2)$,

$$\int_0^1 (f^+ - r)^+ + \int_0^1 (f^- - r)^+ = \int_0^1 (f^+ - \phi + \phi - r)^+ + \int_0^1 (f^- - \psi + \psi - r)^+$$

$$\le \int_0^1 (f^+ - \phi)^+ + \int_0^1 (\phi - r)^+ + \int_0^1 (f^- - \psi)^+ + \int_0^1 (\psi - r)^+$$

$$\le \int_0^1 |f^+ - \phi| + \int_0^1 (\phi - r)^+ + \int_0^1 |f^- - \psi| + \int_0^1 (\psi - r)^+$$

$$\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \qquad \text{Q.E.D.}$$

3.2. HELLY'S SELECTION PRINCIPLE

<u>Theorem 3.1</u>. If F is a family of functions such that

$f \in F \Rightarrow \overset{b}{\underset{a}{V}} f \le \alpha$ and $|f(x)| \le \beta$. Then $\exists$ a sequence $\{f_n\} \subset F$ such that $f_n \to f^*$ $\forall x \in [a,b]$ and $f^* \in BV[a,b]$.

$$\left(BV[a,b] = \{\phi \mid \int_a^b |d\phi| = \overset{b}{\underset{a}{V}} \phi < \infty\}\right).$$

<u>Proof</u>. Assume first that F is a family of non decreasing functions. Let $\{r_i\}_{i=1}^{\infty}$ be all the rationals in $[a,b]$.

For all $f \in F$, $|f(r_1)| \le \beta$, so there exists a sequence $\{f_{1,n}\}$ that converges at r_1. Also $|f_{1,n}(r_2)| \le \beta$ and so $\exists$ a subsequence $\{f_{2,n}\} \subset \{f_{1,n}\}$ that converges at r_1 and r_2. In this manner $\{f_{3,n}\} \subset \{f_{2,n}\}$ converges at r_1, r_2 and r_3. Continuing this procedure we get the diagonal sequence $\{\psi_n\} = \{f_{n,n}\}$ converging $\forall r_i$. Thus, $\forall r_i$

$$\psi_n \to \psi$$

where ψ is non decreasing.

Let x be a continuity point of ψ on $[a,b]$, then for $\varepsilon > 0$ $\exists \delta > 0$ such that

$$|\psi(x) - \psi(y)| < \frac{\varepsilon}{6}$$

whenever $|x-y| < \delta$.

Choose r and s rational such that $x-\delta<r<x<s<x+\delta$ and choose n such that

$$|\psi_n(r)-\psi(r)|<\frac{\varepsilon}{6} \quad \text{and} \quad |\psi_n(s) - \psi(s)|<\frac{\varepsilon}{6} .$$

Then

$$\begin{aligned}|\psi_n(r)-\psi_n(s)| &= |\psi_n(r)-\psi(r)+\psi(r)-\psi(x)+\psi(x)-\psi(s)+\psi(s)-\psi_n(s)| \\ &\le |\psi_n(r)-\psi(r)|+|\psi(r)-\psi(x)|+|\psi(x)-\psi(s)| \\ &\quad +|\psi(s)-\psi_n(s)| \\ &< \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{2\varepsilon}{3} .\end{aligned}$$

Since ψ_n is non decreasing

$$\psi_n(r) \le \psi_n(x) \le \psi_n(s) .$$

Hence

$$|\psi_n(r) - \psi_n(x)| < \frac{2\varepsilon}{3} .$$

It therefore follows that

$$\begin{aligned}|\psi(x)-\psi_n(x)| &\le |\psi(x)-\psi(r)|+|\psi(r)-\psi_n(r)|+|\psi_n(r)-\psi_n(x)| \\ &< \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{2\varepsilon}{3}=\varepsilon .\end{aligned}$$

Thus $\{\psi_n\}$ converges at all continuity points and rational points.

Since ψ is non-decreasing, ψ has at most a countable number of discontinuities. Let $\{z_i\}_{i=1}^{\infty}$ be the discontinuities. Note that $|\psi_n(z_1)| \le \beta$, therefore $\exists$ a subsequence $\{\psi_{1,n}\} \subset \{\psi_n\}$ that converges at z_1. Also, $|\psi_{1,n}(z_2)| \le \beta$. Therefore, $\exists$ a subsequence $\{\psi_{2,n}\} \subset \{\psi_{1,n}\}$ that converges at z_1 and z_2. We continue the process and extract the diagonal subsequence as before. Hence it follows that the sequence $\{\phi_n\} = \{\psi_{n,n}\}$ converges $\forall x \in [a,b]$ to ϕ non-decreasing.

For the family F with $f \in F$ not necessarily non-decreasing but $\overset{b}{\underset{a}{V}} f \le \alpha$, we have

$$f = \pi - \nu \quad \forall f \in F$$

such that $\pi(x) = \overset{x}{\underset{a}{V}} f$ and $\nu(x) = \pi(x) - f(x)$, where π and ν are non-decreasing. We see that

$$|\pi(x)| = \overset{x}{\underset{a}{V}} f \le \overset{b}{\underset{a}{V}} f \le \alpha$$

and

$$|\nu(x)| \le |\pi(x)| + |f(x)| \le \alpha + \beta \ .$$

Therefore $\exists$ a sequence $\{f_n\} \subset F$ such that

$$\pi_n(x) = \overset{x}{\underset{a}{V}} f_n + \pi^*(x) \quad \forall x \in [a,b].$$

Also $\exists$ a sequence $\{f_{n_k}\} \subset \{f_n\}$ such that

$$\nu_{n_k}(x) = \pi_{n_k}(x) - f_{n_k}(x) \to \nu^*(x) \quad \forall x \in [a,b].$$

Therefore

$$f_{n_k} = \pi_{n_k} - \nu_{n_k} \to \pi^* - \nu^* = f^*$$

and $f^* \in BV[a,b]$ since π^* and ν^* are non-decreasing. Q.E.D.

3.3. MAZUR'S THEOREM

<u>Definition 3.1.</u> $A \subset X$, a metric space, is 'totally bounded' if for every $\varepsilon>0$ $\exists\{a_1,a_2,\ldots,a_n\}\subset A$ such that $A\subset S(\{a_1,a_2,\ldots a_n\},\varepsilon)$,

where

$$S(\{a_1,a_2,\ldots,a_n\},\varepsilon) = \bigcup_{i=1}^{n} \{a \mid d(a,a_i)<\varepsilon\}$$

and $d(x,y)$ is the metric on X.

For the proof of Mazur's Theorem, we follow Dunford and Schwartz [2].

<u>Theorem 3.2.</u> Let X be a Banach Space with $A\subset X$ relatively compact, then $\overline{co}(A)$ is compact, where $co(A)$ is the convex hull of A and $\overline{co}(A)$ denotes it's closure with respect to the metric topology.

<u>Proof.</u> $\overline{co}(A)$ is complete since $co(A)\subset X$. Therefore it only remains to show that $co(A)$ is totally bounded.

Choose $\varepsilon>0$. A is totally bounded since $\bar{A}$ is compact. Therefore there exists a finite subset $\{\phi_1,\phi_2,\ldots,\phi_n\}\subset A$ such that

$$A \subset S(\{\phi_1,\phi_2,\ldots,\phi_n\}, \tfrac{\varepsilon}{4}).$$

Let $B = co\ \{\phi_1, \phi_2, \ldots, \phi_n\})$ and note that

$$\overline{co}(A) \subset S(co(A), \tfrac{\varepsilon}{4}).$$

Let $\nu : A \to \{1, 2, \ldots, n\}$ be such that

$$g \in A \Rightarrow ||g - \phi_{\nu(g)}|| < \frac{\varepsilon}{4}.$$

For $\psi \in co(A)$

$$\psi = \sum_{i=1}^{m} a_i \psi_i$$

where $\psi_i \in A$, $a_i \geq 0$ and $\sum_{i=1}^{m} a_i = 1$.

Then

$$||\psi - \sum_{i=1}^{m} a_i \phi_{\nu(\psi_i)}|| = || \sum_{i=1}^{m} a_i (\psi_i - \phi_{\nu(\psi_i)}) ||$$

$$\leq \sum_{i=1}^{m} |a_i| \, ||\psi_i - \phi_{\nu(\psi_i)}||$$

$$< \frac{\varepsilon}{4}.$$

That is for $\psi \in co(A)$

$$d(\psi, B) < \frac{\varepsilon}{4},$$

Thus $co(A) \subset S(B, \frac{\varepsilon}{4}) \Rightarrow S(co(A), \frac{\varepsilon}{4}) \subset S(B, \frac{\varepsilon}{2})$, and

hence $\overline{co}(A) \subset S(B, \frac{\varepsilon}{2})$.

Now, the mapping

$$\mu:(a_1,a_2,\ldots,a_n,\phi_1,\phi_2,\ldots,\phi_n) \to \sum_{i=1}^{n} a_i\phi_i$$

is a continuous mapping of the compact set

$$[0,1]\mathrm{x}[0,1]\mathrm{x}\ldots\mathrm{x}[0,1]\mathrm{x} \mathop{\mathrm{X}}_{i=1}^{n} \phi_i \text{ onto B.}$$

Thus B is compact and therefore totally bounded, and so $\exists\{\beta_1,\beta_2,\ldots\beta_k\}\subset B$ such that

$$B \subset S(\{\beta_1,\beta_2,\ldots,\beta_k\},\ \tfrac{\varepsilon}{2}).$$

Since $\overline{co}(A)\subset S(B,\frac{\varepsilon}{2})\subset S(\{\beta_1,\beta_2,\ldots,\beta_k\},\varepsilon)$, $\overline{co}(A)$ is totally bounded and hence compact. Q.E.D.

3.4. KAKUTANI YOSIDA THEOREM

Theorem 3.3. Let T be a bounded linear operator which maps a Banach space X into itself, and such that $\exists c>0$ such that $||T^n|| \le c$ $\forall n=1,2,3,\ldots$. Furthermore, if for any $x \in A \subset X$ the sequence $\{x_n\}$, where $x_n = \frac{1}{n} \sum_{k=1}^{n} T^k(x)$, contains a subsequence $\{x_{n_k}\}$ such that x_{n_k} converges in X, then for any $x \in \bar{A}$, $\frac{1}{n} \sum_{k=1}^{n} T^k(x)$ converges to some $x^* \in X$ and $T(x^*) = x^*$. $(T^{n+1} = T \circ T^n)$.

Proof. Let $x_{n_k} \to x^*$ for some $x_0 \in A$, then

$$||T(x_{n_k}) - x_{n_k}|| = ||\frac{1}{n_k} \sum_{k=1}^{n_k} T^{k+1}(x_0) - \frac{1}{n_k} \sum_{k=1}^{n_k} T^k(x_0)||$$

$$= \frac{1}{n_k} ||T^{n_k+1}(x_0) - T(x_0)||$$

$$\le \frac{2c||x_0||}{n_k} \to 0.$$

Therefore $T(x^*) = x^*$. Now, let

$$R_n = \text{Range } (I - \frac{1}{n}[T+T^2+\ldots+T^n])$$

$$= \text{Range } ([I-T][I+\frac{n-1}{n} T+\frac{n-2}{n} T^2+\ldots+\frac{1}{n} T^{n-1}]).$$

So $y \in R_n \Rightarrow y \in \text{Range}(I-T) = R$ and $\exists z \in X$ such that

$$y = z - T(z).$$

Thus, $$\left\|\frac{1}{n}\sum_{k=1}^{n} T^k(y)\right\| = \frac{1}{n}\left\|\sum_{k=1}^{n}(T^k(z)-T^{k+1}(z))\right\|$$

$$= \frac{1}{n}\|T(z)-T^{n+1}(z)\|$$

$$\leq \frac{2c\|z\|}{n} \to 0.$$

We have,

$$x_o = x^* + (x_o - x^*)$$

$$x_n = \frac{1}{n}\sum_{k=1}^{n} T^k(x_o)$$

$$= x^* + \frac{1}{n}\sum_{k=1}^{n} T^k(x_o - x^*).$$

It follows that

$$x_o - x^* = x_o - \lim_{n_k\to\infty}\left(\frac{1}{n_k}\sum_{k=1}^{n_k} T^k(x_o)\right)$$

and hence $x_o - x^* \in R$, therefore we have

$$x_n \to x^*.$$

For $\varepsilon>0$ and $x\in\bar{A}$ such that

$$||x-x_o||<\varepsilon,$$

we write $x-x^* = (x-x_o)+(x_o-x^*)$. Then

$$\begin{aligned}
\left|\left|\frac{1}{n}\sum_{k=1}^{n}T^k(x)-x^*\right|\right| &= \left|\left|\frac{1}{n}\sum_{k=1}^{n}T^k(x-x_o)+\frac{1}{n}\sum_{k=1}^{n}T^k(x_o-x^*)\right|\right| \\
&\le \frac{1}{n}\sum_{k=1}^{n}||T^k(x-x_o)||+\left|\left|\frac{1}{n}\sum_{k=1}^{n}T^k(x_o-x^*)\right|\right| \\
&< c\varepsilon + \left|\left|\frac{1}{n}\sum_{k=1}^{n}T^k(x_o-x^*)\right|\right| \to c\varepsilon.
\end{aligned}$$

Thus,

$$\frac{1}{n}\sum_{k=1}^{n}T^k(x) \text{ converges } \forall x\in\bar{A}. \qquad \text{Q.E.D.}$$

CHAPTER IV

EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR PIECEWISE MOTONONIC TRANSFORMATIONS

4.1. LASOTA AND YORKE THEOREM

In this chapter we shall prove that a transformation $\tau \in C^2$ (piecewise), with $\inf|\tau'|>1$, admits an absolutely continuous invariant measure.

<u>Definition 4.1.</u> A set $A \subset X$, a Banach Space, is relatively compact if every infinite subset of A contains a sequence that converges to a point in X.

<u>Definition 4.2.</u> A transformation $\tau:[0,1] \to R$ is piecewise C^2 if there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ such that $\forall_i = 1,2,\ldots,p$, the restriction τ_i of τ on (a_{i-1},a_i) is C^2 and can be extended as a C^2 function on $[a_{i-1},a_i]$. τ need not be continuous at the points a_i.

<u>Theorem 4.1.</u> Let $\tau:[0,1]\to[0,1]$ be a piecewise C^2 transformation such that $\inf|\tau'|>1$. Then for $f \in L_1[0,1]$ the sequence $\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f$ is convergent in norm to $f^* \in L_1[0,1]$ with the following properties:

1) $f \geq 0 \Rightarrow f^* \geq 0$

2) $\int_0^1 f^* = \int_0^1 f$

3) $P_\tau f^* = f^*$ and therefore $d\mu^* = f^* dm$ is invariant under τ

4) $f^* \in BV[0,1]$ and moreover $\exists c$ independent of initial choice of f such that

$$\bigvee_0^1 f^* \leq c\|f\|_1 .$$

<u>Proof</u>: Let $s = \inf |\tau'|$ and choose N such that $s^N > 2$. Let $\phi = \tau^N$, where $\tau^N = \tau \circ \tau^{N-1}$. ϕ is also piecewise C^2. Set $0 = b_0 < b_1 < b_2 < \ldots < b_q = 1$ as the corresponding partition of ϕ with ϕ_i as the corresponding C^2 functions. By the chain rule $|\phi_i'(x)| \geq s^N$ $\forall x \in [b_{i-1}, b_i]$.

Let $\psi_i = \phi_i^{-1}$ and $\sigma_i = |\psi_i'|$. We see that $\sigma_i(x) \leq \frac{1}{s^N}$. From the fact that $|\phi_i'| \geq s^N$ and Lemma 3.1, we see that ϕ_i is monotone. Therefore by Theorem 2.1

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x), \tag{4.1.1}$$

where $\chi_i(x) = \begin{cases} 1, & x \in J_i = \phi_i([b_{i-1}, b_i]) \\ 0, & \text{otherwise} \end{cases}$

Let $f \in BV[0,1]$. Then $f(\psi_i(x))\sigma_i(x)\chi_i(x) = 0$ for $x \notin [\phi_i(b_{i-1}), \phi_i(b_i)]$, and so by Lemma 3.4,

$$(4.1.2) \qquad \underset{0}{\overset{1}{VP_\phi}} f \le \sum_{i=1}^{q} \left(\underset{J_i}{V} (f \circ \psi_i)\sigma_i + |f(b_{i-1})\sigma_i(\phi_i(b_{i-1}))| + |f(b_i)\sigma_i(\phi_i(b_i))| \right).$$

Since $\sigma_i(x) \le s^{-N}$ we have

$$(4.1.3) \qquad \underset{0}{\overset{1}{VP_\phi}} f \le \sum_{i=1}^{q} \underset{J_i}{V} (f \circ \psi_i)\sigma_i + s^{-N} \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|).$$

Now,

$$\begin{aligned} \underset{J_i}{V} (f \circ \psi_i)\sigma_i &= \int_{J_i} |d((f \circ \psi_i)\sigma_i)| \\ &\le \int_{J_i} |f \circ \psi_i| |\sigma_i'| dm + \int_{J_i} \sigma_i |d(f \circ \psi_i)| \\ &\le K \int_{J_i} |f \circ \psi_i| \sigma_i dm + s^{-N} \int_{J_i} |d(f \circ \psi_i)| \end{aligned}$$

where $K = \dfrac{\max_{i,x} |\sigma_i'(x)|}{\min_{i,x} (\sigma_i(x))}$. By a change of variables we get

$$(4.1.4) \qquad \underset{J_i}{V} (f \circ \psi_i)\sigma_i \le K \int_{b_{i-1}}^{b_i} |f| dm + s^{-N} \int_{b_{i-1}}^{b_i} |df|.$$

Letting $d_i = \inf \{|f(x)| \mid x \in [b_{i-1}, b_i]\}$ and by using Lemma 3.3 we have

$$|f(b_{i-1})| + |f(b_i)| \leq \overset{b_i}{\underset{b_{i-1}}{V}} f + 2d_i$$

Letting $h = \min_i (b_i - b_{i-1})$ we see that

$$d_i h \leq \int_{b_{i-1}}^{b_i} |f| dm.$$

Then,

$$(4.1.5) \qquad \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|) \leq \sum_{i=1}^{q} \overset{b_i}{\underset{b_{i-1}}{V}} f + 2h^{-1} \sum_{i=1}^{q} \int_{b_{i-1}}^{b_i} |f| dm$$

$$= \overset{1}{\underset{0}{V}} f + 2h^{-1} ||f||.$$

By putting (4.1.4) and (4.1.5) into (4.1.3) we get

$$(4.1.6) \qquad \overset{1}{\underset{0}{V}} P_\phi f \leq K||f|| + s^{-N} \overset{1}{\underset{0}{V}} f + s^{-N} \overset{1}{\underset{0}{V}} f + 2s^{-N} h^{-1} ||f||.$$

Letting $\alpha = K + 2s^{-N} h^{-1}$ and $\beta = 2s^{-N} < 1$, we write

$$(4.1.7) \qquad \overset{1}{\underset{0}{V}} P_\phi f \leq \alpha ||f|| + \beta \overset{1}{\underset{0}{V}} f.$$

For the same f, let $f_k = P_\tau{}^k f$. Then

$$f_{Nk} = P_\tau^{Nk} f$$

$$= P_\tau^N P_\tau^{N(k-1)} f$$

$$= P_\phi f_{N(k-1)} .$$

Therefore,

$$\overset{1}{\underset{0}{V}} f_{Nk} = \overset{1}{\underset{0}{V}} P_\phi f_{N(k-1)}$$

$$\leq \alpha ||f_{N(k-1)}|| + \beta \overset{1}{\underset{0}{V}} f_{N(k-1)} .$$

By Lemma 2.3, $||f_m|| \leq ||f|| \quad \forall m = 1,2,3,\ldots$. Thus,

$$\overset{1}{\underset{0}{V}} f_{Nk} \leq \alpha ||f|| + \beta(\alpha ||f|| + \beta \overset{1}{\underset{0}{V}} f_{N(k-2)})$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\leq \sum_{n=0}^{k-1} \alpha\beta^n ||f|| + \beta^k \overset{1}{\underset{0}{V}} f_o .$$

Consequently, noting that $f_o = f$, we get

(4.1.8) $$\overline{\lim_{k\to\infty}} \overset{1}{\underset{0}{V}} f_{Nk} \leq \frac{\alpha ||f||}{1-\beta}$$

By (4.1.8) and by Lemma 3.7 we have $\forall k$,

$$|f_{Nk}(x)| \leq \frac{\alpha ||f||}{1-\beta} + ||f|| .$$

With this and (4.1.8) again, and using Theorem 3.1, we have that every infinite subset of $C = \{f_{Nk}\}_{k=0}^{\infty}$ contains a subsequence which converges in $L_1[0,1]$. So C is relatively compact in $L_1[0,1]$.

Since P_τ is continuous, $P_\tau^k C$ is also relatively compact. Since

$$\{f_k\}_{k=0}^{\infty} \subset \bigcup_{k=0}^{N-1} P_\tau^k C,$$

we have that $\{f_k\}_{k=0}^{\infty}$ is also relatively compact.

By Theorem 3.2,

$$\mathrm{co}(C) = \{\sum_{k=0}^{n-1} a_k f_k \mid a_k \geq 0, \sum_{k=0}^{n-1} a_k = 1\}$$

is relatively compact. It then follows that

$$\{\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f\} \subset \mathrm{co}(C)$$

is also relatively compact. Finally, by Theorem 3.3,

$$\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f \to f^* \in L_1[0,1]$$

where

$$P_\tau f^* = f^*.$$

Properties 1), 2) and 3) of the Theorem follow from Lemmas 2.1, 2.2 and 2.5 respectively. We therefore proceed to prove property 4). So far, we know that $f^* \in L_1[0,1]$. We will now show that $f^* \in BV[0,1]$ and that $\exists c \ni \bigvee_0^1 f^* \leq c||f||$.

We set $\alpha_i = \tau_i^{-1}$, $\beta_i = |\alpha_i'|$ and redefine

$$\chi_i(x) = \begin{cases} 1, & x \in I_i = \tau_i([a_{i-1}, a_i]) \\ 0, & \text{otherwise} \end{cases}$$

Invoking Theorem 2.1, we get

$$P_\tau f(x) = \sum_{i=1}^{p} f(\alpha_i(x))\beta_i(x)\chi_i(x).$$

By Lemma 3.4 and the same procedure as before, we get

$$\bigvee_0^1 P_\tau f \leq \sum_{i=1}^{p} \bigvee_{I_i} (f \circ \alpha_i)\beta_i + \frac{1}{s}\sum_{i=1}^{p} (|f(a_{i-1})|+|f(a_i)|)$$

$$\leq c_1 \bigvee_0^1 f + c_2||f||,$$

and

$$\bigvee_0^1 f_{Nk+1} = \bigvee_0^1 P_\tau f_{Nk}$$

$$\leq c_1 \bigvee_0^1 f_{Nk} + c_2||f||$$

for some $c_1, c_2>0$. Also

$$\overset{1}{\underset{0}{V}} f_{Nk+2} \le c_1 \overset{1}{\underset{0}{V}} f_{Nk+1} + c_2||f||$$

$$\le c_1(c_1 \overset{1}{\underset{0}{V}} f_{Nk} + c_2||f||)+ c_2||f||$$

$$= c_1^2 \overset{1}{\underset{0}{V}} f_{Nk} + (c_1c_2+c_2)||f||.$$

Consequently, for $m = 1,2,\ldots,N-1$,

$$\overset{1}{\underset{0}{V}} f_{Nk+m} \le c_1^m \overset{1}{\underset{0}{V}} f_{Nk} + c_2 \sum_{j=0}^{m-1} c_1^j||f||$$

$$\le c_1^m \overset{1}{\underset{0}{V}} f_{Nk} + c_2 \frac{1-c_1^N}{1-c_1} ||f|| ,$$

and

$$\overline{\lim_{k\to\infty}} \overset{1}{\underset{0}{V}} f_{Nk+m} \le c_1^m \overline{\lim_{k\to\infty}} \overset{1}{\underset{0}{V}} f_{Nk} + c_2 \frac{1-c_1^N}{1-c_1}||f||.$$

With this and (4.1.8), we have

$$\overline{\lim_{k\to\infty}} \overset{1}{\underset{0}{V}} P_\tau^k f \le c||f||, \quad c>0$$

for $f\in BV[0,1]$, where c is independent of f.

For such an f and for any n,

$$\overset{1}{\underset{0}{V}}\left(\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f\right) \le \frac{1}{n}\sum_{k=0}^{n-1} \overset{1}{\underset{0}{V}} P_\tau^k f$$

$$\le \frac{1}{n}\sum_{k=0}^{n-1} c||f||$$

$$= c||f||.$$

Let

$$T_n = \frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k$$

Then

(4.1.9) $$\overset{1}{\underset{0}{V}} T_n f \le c||f||, \quad f \in BV[0,1].$$

Now,

$$||T_n f|| \le \frac{1}{n}\sum_{k=0}^{n-1} ||P_\tau^k f||$$

$$\le \frac{1}{n}\sum_{k=0}^{n-1} ||f||$$

$$= ||f|| .$$

This, with equation (4.1.9) and Lemma 3.7, yields:

(4.1.10) $$|T_n f(x)| \le (c+1)||f||.$$

From (4.1.9) and (4.1.10) and Theorem 3.1, it follows that there exists a subsequence $\{T_{n_k} f\}$ which converges everywhere on [0,1] to a function of bounded variation. But $T_n f$ converges strongly, thus

$$\bigvee_0^1 Tf \le c\|f\|, \tag{4.1.11}$$

where

$$T = \lim_{n\to\infty} T_n.$$

For $\phi,\psi\in L_1[0,1]$,

$$T(\phi+\psi) = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k(\phi+\psi)\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k \phi\right) + \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k \psi\right)$$

$$= T\phi + T\psi$$

so T is linear.

We note that $L_1[0,1]\subset\overline{BV}[0,1]$ and therefore for any $f\in L_1[0,1]$ there exist a sequence $\{\phi_n\}\subset BV[0,1]$ such that $\phi_n \to f$. By (4.1.11)

$$\bigvee_0^1 T\phi_n \le c\|\phi_n\| .$$

Since

$$||T\phi_n|| \leq ||\phi_n||$$

it follows by Lemma 3.7, that

$$|T\phi_n(x)| \leq (c+1)||\phi_n|| .$$

For $\varepsilon>0$, $\exists N$ such that $\forall n>N$

$$||f-\phi_n|| \leq \varepsilon.$$

Therefore, $\forall n>N$

$$||\phi_n|| \leq \varepsilon+||f||.$$

Thus, $\forall n>N$,

$$\overset{1}{\underset{0}{V}} T\phi_n \leq c(\varepsilon+||f||)$$

and

$$|T\phi_n(x)| \leq (c+1)(\varepsilon+||f||)$$

By using Theorem 3.1 once more, we have the existence of a subsequence $\{\phi_{n_k}\} \subset \{\phi_n\}$ with $n_k>N$, such that

$$T\phi_{n_k} \to f^*$$

for some f* of bounded variation. But also $T\phi_n \to Tf$ and $\{T\phi_{n_k}\} \subset \{T\phi_n\}$, since T is continuous, thus

$$Tf = f^* \in BV[0,1]$$

for $f \in L_1[0,1]$.

This completes the proof. Q.E.D.

4.2 A COUNTER EXAMPLE

In this section we shall show that the condition $|\tau'| \geq \alpha > 1$ is essential. In the example, we shall use a transformation, τ, where $|\tau'(0)| = 1$ but $|\tau'(x)| > 1$ for all other x in [0,1]. This example will prove that for

$$\tau(x) = \begin{cases} \frac{x}{1-x} ; & 0 \leq x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

there exists no finite absolutely continuous invariant measure.

Set $\quad \tau_1(x) = \frac{x}{1-x} \quad , \quad 0 \leq x \leq \frac{1}{2}$

$$\tau_2(x) = 2x-1 \quad , \quad \frac{1}{2} \leq x \leq 1$$

$$J_1 = \tau_1([0,\tfrac{1}{2}]) = [0,1]$$

$$J_2 = \tau_2([\tfrac{1}{2},1]) = [0,1].$$

Define, for $x \in [0,1]$

$$\psi_1(x) = \tau_1^{-1}(x)$$

$$= \frac{x}{1+x} \quad ,$$

and

$$\psi_2(x) = \tau_2^{-1}(x)$$

$$= \frac{x+1}{2} .$$

Also, define

$$\sigma_1(x) = |\psi_1'(x)|$$

$$= \frac{1}{(1+x)^2}$$

and

$$\sigma_2(x) = |\psi_2'(x)|$$

$$= \frac{1}{2} .$$

By Theorem 2.1,

$$P_\tau f(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac{1}{2} f\left(\frac{x+1}{2}\right).$$

Let $f_o \equiv 1$, set $f_{n+1} = P_\tau f_n$, and define

$$g_n(x) = xf_n(x) .$$

Then,

$$g_{n+1}(x) = xf_{n+1}(x)$$

$$= xP_\tau f_n(x)$$

$$= x[\frac{1}{(1+x)^2} f_n(\frac{x}{1+x}) + \frac{1}{2}f_n(\frac{x+1}{2})]$$

$$= \frac{1}{1+x}\frac{x}{1+x} f_n(\frac{x}{1+x}) + \frac{x}{1+x}\frac{x+1}{2} f_n(\frac{x+1}{2}) .$$

Thus,

$$= \frac{1}{1+x} g_n(\frac{x}{1+x}) + \frac{x}{1+x} g_n(\frac{x+1}{2}) . \qquad (4.2.1)$$

For $g_n'(x) \geq 0$ we have,

$$g_{n+1}'(x) = \frac{1}{(1+x)^3} g_n'(\frac{x}{1+x}) + \frac{x}{4(x+1)} g_n'(\frac{x+1}{2})$$

$$+ \frac{1}{(1+x)^2}[g_n(\frac{x+1}{2}) - g_n(\frac{x}{1+x})]$$

We note that

$$(x+1)^2 \geq 2x \Rightarrow \frac{x+1}{2} \geq \frac{x}{1+x}$$

$$\Rightarrow g_n(\frac{x+1}{2}) \geq g_n(\frac{x}{1+x})$$

since g_n non decreasing. Therefore, $\forall x \in [0,1]$

$$g_n'(x) \geq 0 \Rightarrow g_{n+1}'(x) \geq 0.$$

$$g_o(x) = xf_o(x)$$

$$= x$$

and

$$g_o'(x) = 1 \geq 0$$

Thus $\forall n$,

$$g_n'(x) \geq 0.$$

Hence

$$g_{n+1}(1) = \frac{1}{2} g_n(\tfrac{1}{2}) + \frac{1}{2} g_n(1)$$

$$\leq \frac{1}{2} g_n(1) + \frac{1}{2} g_n(1)$$

$$= g_n(1),$$

and so $g_n(1)$ converges to some number α.

Now set $x_{k+1} = \dfrac{x_k}{1+x_k}$ and $x_o = 1$. Assume $\lim_{n\to\infty} g_n(x) = \alpha$ $\forall x \in [x_k, 1]$.

From (4.2.1), we have

$$g_{n+1}(x_k) = \frac{1}{1+x_k} g_n\left(\frac{x_k}{1+x_k}\right) + \frac{x_k}{1+x_k} g_n\left(\frac{1+x_k}{2}\right)$$

$$= \frac{1}{1+x_k} g_n(x_{k+1}) + \frac{x_k}{1+x_k} g_n\left(\frac{1+x_k}{2}\right).$$

It is clear that $\frac{1+x_k}{2} \in [x_k, 1]$. Thus by taking limits on both sides we get

$$\alpha = \frac{1}{1+x_k} \lim_{n\to\infty} g_n(x_{k+1}) + \frac{x_k}{1+x_k} \alpha.$$

Hence,

$$\lim_{n\to\infty} g_n(x_{k+1}) = \alpha.$$

Since $g_n(x)$ is increasing $\forall n$, $\lim_{n\to\infty} g_n(x)$ converges uniformly to α on $[x_{k+1}, 1]$. It follows that

$$g_n(x) \to \alpha \quad \forall x \in [0,1].$$

Hence, $\forall x \in [0,1]$

$$f_n(x) \to \frac{\alpha}{x}.$$

Assume $\alpha > 0$. Then $\exists \varepsilon > 0$ such that

$$\int_\varepsilon^1 \frac{\alpha}{x}\, dx > 1,$$

that is

$$\lim_{n\to\infty} \int_{\varepsilon}^{1} f_n(x)\,dx > 1.$$

But we have $f_o \geq 0$ and so by Lemma 2.1

$$f_n \geq 0 \quad \forall n.$$

By Lemma 2.2, $\forall n$

$$\int_0^1 f_n = \int_0^1 f_o.$$

Therefore

$$\int_{\varepsilon}^{1} f_n \leq \int_0^1 f_n$$

$$= \int_0^1 f_o$$

$$= 1.$$

Thus $\alpha = 0$.

Now assume $f_n' \leq 0$,

$$f_{n+1}'(x) = -\frac{2}{(1+x)^3} f_n\left(\frac{x}{1+x}\right) + \frac{1}{(1+x)^4} f_n'\left(\frac{x}{1+x}\right) + \frac{1}{4} f_n'\left(\frac{x+1}{2}\right)$$

$$\leq 0.$$

Also, $f_0'(x) \le 0$ $\forall x \in [0,1]$ and so f_n is non-increasing $\forall n$. Therefore $f_n \to 0$ uniformly on $[\varepsilon,1]$, for ε arbitrary.

For $f \in L_1[0,1]$ with

$$f^+ = \max(f,0), \quad f^- = -\min(0,f).$$

and given $\varepsilon>0$, we can, by Lemma 3.8 choose r such that

$$\int_0^1 (f^+-r)^+ dm + \int_0^1 (f^- - r)^+ dm \le \varepsilon$$

Since

$$|P_\tau^n f| \le P_\tau^n f^+ + P_\tau^n f^-,$$

we have

$$\int_\varepsilon^1 |P_\tau^n f| dm \le \int_\varepsilon^1 P_\tau^n f^+ dm + \int_\varepsilon^1 P_\tau^n f^- dm$$

$$= 2\int_\varepsilon^1 P_\tau^n r dm + \int_\varepsilon^1 P_\tau^n (f^+-r) dm + \int_\varepsilon^1 P_\tau^n (f^- -r) dm$$

$$\le 2r\int_\varepsilon^1 P_\tau^n(1) dm + \int_0^1 (f^+-r)^+ dm + \int_0^1 (f^- -r)^+ dm$$

$$\le 2r\int_\varepsilon^1 P_\tau^n(1) dm + \varepsilon.$$

Since $f_n \to 0$ uniformly on $[\varepsilon,1]$, by the Dominated Convergence Thoerem we have,

$$\lim_{n\to\infty} \int_{\varepsilon}^{1} |P_\tau^n(1)| \le 0 + \varepsilon.$$

Hence,

$$P_\tau^n f \to 0 \quad \text{a.e.} \quad \forall f \in L_1[0,1]$$

for $x \in [\varepsilon, 1]$. But ε can be arbitrarily small. Thus we have convergence to 0 a.e. on $[0,1]$.

CHAPTER V

EXISTENCE OF INVARIANT MEASURES FOR TRANSFORMATIONS WITH SLOPE LESS THAN OR EQUAL TO ONE

5.1. PIANIGIANI'S THEOREM

In this chapter we shall consider transformations where $|\tau'|$ is not necessarily greater than one. In [4] it is shown that under certain conditions such a transformation will admit finite absolutely continuous invariant measures.

<u>Theorem 5.1.</u> Let τ be a piecewise C^2 transformation from $[0,1]$ into itself. If there exists a function h, piecewise C^1, such that:

a) $h(x) > 0$ a.e. and $\int_0^1 h = 1$

b) $\alpha = \inf \dfrac{|\tau'(x)|\ h(\tau(x))}{h(x)} > 1$,

then there exists a finite absolutely continuous invariant measure for τ.

<u>Proof</u>. Set $g(x) = \int_0^x h$.

We see that $g:[0,1] \to [0,1]$ is monotonically increasing and therefore g^{-1} is monotonically increasing.

Define $T = g \circ \tau \circ g^{-1}$, T is piecewise C^2.

$$T(g(x)) = g(\tau(x))$$

(Note that $g(x)$ maps $[0,1]$ onto $[0,1]$ and hence $T_\circ g:[0,1] \to [0,1]$.)

Thus,

$$T'(g(x))g'(x) = g'(\tau(x))\tau'(x)$$

and
$$|T'(g(x))| = \left|\frac{g'(\tau(x))\tau'(x)}{g'(x)}\right|$$

$$= \frac{|\tau'(x)|\ h(\tau(x))}{h(x)}$$

$$\geq \alpha.$$

Then by Theorem 4.1 there exists a measure invariant under T. That is there exists $f^* \in L_1[0,1]$ such that

$$\int_{T^{-1}(A)} f^* = \int_A f^* \quad \forall A \subset [0,1].$$

By Lemma 2.4, we have the following:

$$\int_{T^{-1}(A)} f^* = \int_A P_T f^*$$

$$= \int_A P_g P_\tau P_{g^{-1}} f^*$$

$$= \int_{g^{-1}(A)} P_\tau (P_{g^{-1}} f^*)$$

$$\int_A f^* = \int_{g(g^{-1}(A))} f^*$$

$$= \int_{g^{-1}(A)} P_{g^{-1}} f^* .$$

Therefore

$$\int_{g^{-1}(A)} P_\tau (P_{g^{-1}} f^*) = \int_{g^{-1}(A)} P_{g^{-1}} f^* .$$

Set $B = g^{-1}(A)$ and $\psi^* = P_{g^{-1}} f^*$.

Then,

$$\int_{\tau^{-1}(B)} \psi^* = \int_B P_\tau \psi^*$$

$$= \int_B \psi^*$$

and so there exists a finite absolutely continuous invariant measure under τ. Q.E.D.

<u>Corollary 5.1</u>. Let τ be defined as in Theorem 5.1 and let $\psi_i = \tau_i^{-1}$. If there exists a function h, piecewise C^1 such that:

a) $h(x) > 0$ a.e. and $\int_0^1 h = 1$.

b) $\lambda = \sup \frac{|\psi_i'| \; h(\psi_i(x))}{h(x)} < 1 \quad \forall i$, then τ admits a finite absolutely continuous invariant measure.

<u>Proof</u>. Set $g(x) = \int_0^x h$

and define

$$T_i = g \circ \tau_i \circ g^{-1}.$$

Let

$$\Psi_i = T_i^{-1} .$$

Then

$$\Psi_i = g \circ \tau_i^{-1} \circ g^{-1}$$

$$= g \circ \psi_i \circ g^{-1}.$$

If follows that,

$$\Psi_i(g(x)) = g(\psi_i(x))$$

and

$$\Psi_i'(g(x))g'(x) = g'(\psi_i(x))\psi_i'(x).$$

Hence

$$|\Psi_i'| = \frac{|\psi_i'(x)| \; h(\psi_i(x))}{h(x)} .$$

If $\sup \dfrac{|\psi_i'(x)| \; h(\psi_i(x))}{h(x)} \le \lambda < 1 \quad \forall i,$ then

$$|\Psi_i'| \le \lambda \quad \forall i$$

and

$$|T_i'| \geq \frac{1}{\lambda} > 1 \quad \forall i$$

and so τ, by Theorem 5.1, admits a finite absolutely continuous invariant measure. Q.E.D.

5.2. AN EXAMPLE

If we consider the transformation $\tau(x) = \sin \pi x$ we note that $\tau'(x) \le 1$ for some $x \in [0,1]$. So Theorem 4.1 is of no help to us here, but we still can show that there exists an absolutely continuous invariant measure.

Let

$$\tau_1(x) = \sin \pi x \quad \text{for } 0 \le x \le \tfrac{1}{2}$$

and

$$\tau_2(x) = \sin \pi x \quad \text{for } \tfrac{1}{2} < x \le 1 .$$

Then

$$\psi_1(x) = \tau_1^{-1}(x) = \frac{1}{\pi} \text{Arcsin} x .$$

and

$$\psi_2(x) = \tau_2^{-1}(x) = 1 - \frac{1}{\pi} \text{Arcsin} x .$$

Set

$$h(x) = \frac{1}{\sqrt{x(1-x)}} .$$

We see that $h(x) > 0$ for all $x \in [0,1]$ and that

$$\int_0^1 h(x)\,dx = 1.$$

We then have

$$\frac{|\psi_1'(x)|\ h(\psi_1(x))}{h(x)} = \frac{\sqrt{x(1-x)}}{\pi\sqrt{1-x^2}\ \sqrt{\frac{1}{\pi}\text{Arcsinx}(1-\frac{1}{\pi}\text{Arcsinx})}}$$

$$= \frac{\sqrt{x(1-x)}}{\sqrt{1-x^2}\ \sqrt{\text{Arcsinx}\ (\pi - \text{Arcsinx})}}$$

$$= \frac{\sqrt{1-x}}{\sqrt{1-x^2}} \cdot \frac{\sqrt{x}}{\sqrt{\text{Arcsinx}}} \cdot \frac{1}{\sqrt{\pi - \text{Arcsinx}}}$$

$$\le 1\ .\ 1\ .\ \sqrt{\frac{2}{\pi}} < 1.$$

With ψ_2 in the place of ψ_1 we get

$$\frac{|\psi_2'(x)|\ h(\psi_2(x))}{h(x)} = \frac{\sqrt{x(1-x)}}{\pi\sqrt{1-x^2}\ \sqrt{(1-\frac{1}{\pi}\text{Arcsinx})\ .\ \frac{1}{\pi}\text{Arcsinx}}}$$

$$= \frac{|\psi_1'(x)|\ h(\psi_1(x))}{h(x)}$$

$$\le \sqrt{\frac{2}{\pi}} < 1.$$

Therefore by Corollary 5.1, there exists an absolutely continuous invariant measure under $f(x) = \sin \pi x$.

CHAPTER VI

APPROXIMATION OF INVARIANT MEASURES

In this chapter we shall study fixed points of simple operators which converge to the fixed point of P_τ.

Let $[0,1]$ be divided into n equal subintervals $I_1, I_2, \ldots, I_n$ with $I_i = [a_{i-1}, a_i]$ and $m(I_i) = \frac{1}{n} = \ell$ $\forall i$. We define P_{ij} as the fraction of I_i which is mapped into interval I_j by τ.

Let

$$A_{ij} = \{x \in I_i \,|\, \tau(x) \in I_j\} .$$

Then,

$$A_{ij} = I_i \cap \tau^{-1}(I_j) .$$

We see that $\tau(A_{ij}) = \tau(I_i \cap \tau^{-1}(I_j)) \subset \tau(I_i) \cap I_j \subset I_j$.

Therefore

$$P_{ij} = \frac{m(A_{ij})}{m(I_i)} = \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_i)} .$$

Let Δ_n be the field generated by $\{\chi_i\}_{i=1}^{n}$, where χ_i is the characteristic function of I_i, and define $P_n(\tau): \Delta_n \to \Delta_n$ as a linear operator such that

$$P_n(\tau)\chi_i = \sum_{j=1}^{n} P_{ij}\chi_j.$$

(We will use P_n for $P_n(\tau)$ wherever possible).

Ulam's conjecture states that the sequence f_n where $P_n f_n = f_n$ should converge to a fixed point of P_τ whenever P_τ has a unique fixed point. We shall prove a very interesting result about the fixed points $\{f_n\}$ but first we shall look at some lemmas.

<u>Lemma 6.1</u>. Let $\Delta_n' = \{\sum_{i=1}^{n} a_i\chi_i \mid a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1\}$. Then

$$P_n\colon \Delta_n' \to \Delta_n'.$$

<u>Proof</u>. Let $f \in \Delta_n'$, then

$$P_n f = P_n(\sum_{i=1}^{n} a_i\chi_i) = \sum_{i=1}^{n} a_i P_n \chi_i.$$

Since

$$P_n\chi_i = \sum_{j=1}^{n} P_{ij}\chi_i$$

we have

$$P_n f = \sum_{i=1}^{n} a_i(\sum_{j=1}^{n} P_{ij}\chi_j)$$

$$= \sum_{j=i}^{n} \sum_{i=1}^{n} a_i P_{ij}\chi_j.$$

We have that $a_i P_{ij} \geq 0 \quad \forall i,j$. Therefore it remains to show that

$$\sum_{j=1}^{n} b_j = 1 \quad \text{with } b_j = \sum_{i=1}^{n} a_i P_{ij}.$$

We have

$$\sum_{j=1}^{n} b_j = \sum_{i=1}^{n} (a_i \sum_{j=1}^{n} P_{ij}) = \sum_{i=1}^{n} a_i = 1.$$

And so $P_n f \in \Delta_n'$. Q.E.D.

Since $P_n(\Delta_n') \subset \Delta_n'$, by the Brouwer Fixed Point Theorem [8], there exists a point $g_n \in \Delta_n'$ such that $P_n g_n = g_n$. Set $f_n = n g_n$. Then $f \in \Delta_n$ and

$$\begin{aligned}
||f_n|| = ||ng_n|| &= n|| \sum_{i=1}^{n} a_i \chi_i || \\
&= n\int_0^1 | \sum_{i=1}^{n} a_i \chi_i | \\
&= n\int_0^1 \sum_{i=1}^{n} a_i \chi_i \\
&= n \sum_{i=1}^{n} a_i \int_0^1 \chi_i \\
&= n . \frac{1}{n} \sum_{i=1}^{n} a_i = 1.
\end{aligned}$$

Definition 6.1. For $f \in L_1$ and for every positive integer n, we define $Q_n : L_1 \to \Delta_n$ by

$$Q_n f = \sum_{i=1}^{n} c_i \chi_i \quad \text{where} \quad c_i = \frac{1}{m(I_i)} \int_{I_i} f(s)\,ds.$$

We see that $f \geq 0 \Rightarrow Q_n f \geq 0$ and that $Q_n(af+bg) = aQ_n f + bQ_n g$. Hence $Q_n f = Q_n(f^+ - f^-)$ and $|Q_n f| \leq Q_n f^+ + Q_n f^-$.

Lemma 6.2. For $f \in L_1$, the sequence $Q_n f$ converges in L_1 to f.

Proof. $f \in L_1 \Rightarrow$ for $\varepsilon > 0$ there exists $g \in C$ (the space of continuous functions) such that $\|f-g\| < \frac{\varepsilon}{3}$. g is uniformly continuous on $[0,1]$ and so we can choose N such that $n > N \Rightarrow |g(x_1) - g(x_2)| < \frac{\varepsilon}{3}$ $\forall x_1, x_2 \in I_i$ $\forall_i$. It follows that

$$\int_{I_i} |(Q_n g)(s) - g(s)|\,ds = \int_{I_i} \left| \sum_{j=1}^{n} \left(\frac{1}{m(I_j)} \int_{I_j} g(t)\,dt\right) \chi_j(s) - g(s) \right| ds$$

$$= \int_{I_i} \left| \frac{1}{m(I_i)} \int_{I_i} g(t)\,dt\,\chi_i(s) - g(s) \right| ds$$

since $s \in I_i$. Therefore,

$$\int_{I_i} |(Q_n g)(s)-g(s)|ds \le \int_{I_i} \frac{1}{m(I_i)} \int_{I_i} |g(t)-g(s)|dtds$$

$$< \frac{1}{m(I_i)} \int_{I_i} \int_{I_i} \frac{\varepsilon}{3} \; dtds$$

$$= m(I_i) \; \frac{\varepsilon}{3} .$$

Hence,

$$||Q_n g-g|| = \int_0^1 |Q_n g-g|$$

$$= \sum_{i=1}^{n} \int_{I_i} |Q_n g-g|$$

$$< \sum_{i=1}^{n} m(I_i) \; \frac{\varepsilon}{3} = \frac{\varepsilon}{3} .$$

And for $\phi \in L_1$,

$$\int_0^1 Q_n\phi = \int_0^1 \sum_{i=1}^{n} \left(\frac{1}{m(I_i)} \int_{I_i} \phi(t)dt\right)\chi_i(s)ds$$

$$= \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} \phi(t) \int_0^1 \chi_i(s)dsdt$$

$$= \sum_{i=1}^{n} \frac{1}{m(I_i)} \int_{I_i} m(I_i)\phi(t)dt$$

$$= \sum_{i=1}^{n} \int_{I_i} \phi(t)dt = \int_0^1 \phi .$$

Then,

$$||Q_n\phi|| \le \int_0^1 Q_n\phi^+ + \int_0^1 Q_n\phi^-$$

$$= \int_0^1 \phi^+ + \int_0^1 \phi^- = ||\phi||.$$

Hence,

$$||Q_n(f-g)|| \le ||f-g||$$

Thus,

$$||Q_n f-f|| \le ||Q_n f-Q_n g|| + ||Q_n g-g|| + ||g-f||$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \text{Q.E.D.}$$

<u>Lemma 6.3</u>. $\quad f\in\Delta_n \Rightarrow P_n f = Q_n P_\tau f$

<u>Proof</u>. $\quad P_\tau\chi_i(x) = \dfrac{d}{dx}\displaystyle\int_{\tau^{-1}([0,x])} \chi_i(s)\,ds.$

$$\int_{I_j} P_\tau\chi_i = \int_{a_{j-1}}^{a_j} d\left(\int_{\tau^{-1}([0,x])} \chi_i(s)\,ds\right)$$

$$= \int_{\tau^{-1}([0,a_j])} \chi_i(s)\,ds - \int_{\tau^{-1}([0,a_{j-1}])} \chi_i(s)\,ds$$

$$= \int_{\tau^{-1}([0,a_{j-1}])} \chi_i(s)\,ds + \int_{\tau^{-1}([a_{j-1},a_j])} \chi_i(s)\,ds - \int_{\tau^{-1}([0,a_{j-1}])} \chi_i(s)\,ds$$

$$= \int_{\tau^{-1}(I_j)} \chi_i(s)\,ds.$$

Therefore,

$$Q_n(P_\tau \chi_i) = \sum_{j=1}^{n} \left[\frac{1}{m(I_j)} \int_{I_j} (P_\tau \chi_i)(x)\,dx \right] \chi_j$$

$$= \sum_{j=1}^{n} \left[\frac{1}{m(I_j)} \int_{\tau^{-1}(I_j)} \chi_i(s)\,ds \right] \chi_j .$$

Since $m(I_i) = m(I_j) = \frac{1}{n}$ $\forall i,j$, we have,

$$Q_n(P_\tau \chi_i) = \sum_{j=1}^{n} \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_i)} \chi_j$$

$$= \sum_{j=1}^{n} P_{ij}\chi_j = P_n \chi_i .$$

And so for $f = \sum_{k=1}^{n} c_k \chi_k$, we have

$$Q_n P_\tau f = Q_n P_\tau \left(\sum_{k=1}^{n} c_k \chi_k \right)$$

$$= \sum_{k=1}^{n} c_k Q_n P_\tau \chi_k$$

$$= \sum_{k=1}^{n} c_k P_n \chi_k$$

$$= P_n \left(\sum_{k=1}^{n} c_k \chi_k \right)$$

$$= P_n f. \qquad \text{Q.E.D.}$$

Lemma 6.4. $f \in \Delta_n \Rightarrow P_n f \to P_\tau f$.

Proof. By lemma 6.3, $f \in \Delta_n \Rightarrow P_n f = Q_n P_\tau f$.

By lemma 6.2, $Q_n P_\tau f \to P_\tau f$. Q.E.D.

Lemma 6.5. If $f \in L_1$, then $\bigvee_0^1 Q_n f \le \bigvee_0^1 f$.

Proof. $Q_n f = \sum_{i=1}^{n} c_i \chi_i$ where $c_i = \frac{1}{\ell} \int_{I_i} f$.

We see that $Q_n f$ is a step function. We can choose points $x_i, y_i \in I_i$ such that

$$\ell f(x_i) \le \int_{I_i} f \le \ell f(y_i).$$

We see that $|c_i - c_{i-1}| \le \max\{|f(x_i) - f(y_{i-1})|, |f(y_i) - f(x_{i-1})|\}$ $\forall i = 1, 2, \ldots, n$.

Let $r_{2i-1} = \min(x_i, y_i)$ and $r_{2i} = \max(x_i, y_i)$.

Then

$$\bigvee_0^1 Q_n f \le \sum_{i=2}^{2n} |f(r_i) - f(r_{i-1})| \le \bigvee_0^1 f. \quad \text{Q.E.D.}$$

Lemma 6.6. If τ is piecewise C^2 with partition $\{b_0, b_1, \ldots, b_q\}$ and $s = \inf|\tau'| > 2$, then $\{\bigvee_0^1 f_n\}$ is bounded.

Proof. By lemma 6.3 $f_n = P_n f_n = Q_n P_\tau f_n \ \forall n$. By lemma 6.5

$$\overset{1}{\underset{0}{V}} Q_n P_\tau f_n \le \overset{1}{\underset{0}{V}} P_\tau f_n .$$

By Theorem 4.1

$$\overset{1}{\underset{0}{V}} P_\tau f_n \le (K+2s^{-1}h^{-1})||f_n|| + \beta \overset{1}{\underset{0}{V}} f_n$$

with $K = \dfrac{\max_{i,x} |\sigma_i'(x)|}{\min_{i,x} (\sigma_i(x))}$, $\sigma_i = |(\tau_i^{-1})'|$, $h = \min_i (b_i - b_{i-1})$

and $\beta = 2s^{-1} < 1$.

Since $||f_n|| = 1$, we have

$$\overset{1}{\underset{0}{V}} f_n \le (K+h^{-1}) + \beta \overset{1}{\underset{0}{V}} f_n .$$

Since $f_n \in \Delta_n$, $\overset{1}{\underset{0}{V}} f_n < \infty$. Hence,

$$(1-\beta) \overset{1}{\underset{0}{V}} f_n \le K+h^{-1}$$

and

$$\overset{1}{\underset{0}{V}} f_n \le \frac{K+h^{-1}}{1-\beta} . \quad \text{Q.E.D.}$$

Theorem 6.1. Let $\tau:[0,1]\to[0,1]$ be a piecewise C^2 function with $s = \inf|\tau'|>2$ and suppose P_τ has a unique fixed point. Then, for any positive integer n, P_n has a fixed point $f_n \in \Delta_n$ with $||f_n||=1$ and $\{f_n\}$ converges to the fixed point of P_τ.

Proof. From Lemmas 6.6 and 3.7, and from Theorem 3.1, the set $\{f_n\}$ is relatively compact. Let $\{f_{n_k}\} \subset \{f_n\}$ be a convergent subsequence and let $f = \lim_{k\to\infty} f_{n_k}$. Then,

$$||f-P_\tau f|| \le ||f-f_{n_k}||+||f_{n_k}-Q_{n_k}P_\tau f_{n_k}||$$

$$+ ||Q_{n_k}P_\tau f_{n_k} - Q_{n_k}P_\tau f||+||Q_{n_k}P_\tau f-P_\tau f||.$$

By Lemma 6.3, $||f_{n_k}-Q_{n_k}P_\tau f_{n_k}||=||P_{n_k}f_{n_k}-Q_{n_k}P_\tau f_{n_k}|| = 0.$

Also, $||Q_{n_k}P_\tau(f_{n_k}-f)||\le||Q_{n_k}||\;||P_\tau||\;||f_{n_k}-f|| \to 0$ as $f_{n_k}\to f$, and by Lemma 6.2, $Q_{n_k}P_\tau f \to P_\tau f$. Hence $P_\tau f=f$.

Any convergent subsequence of $\{f_n\}$ converges to a fixed point of P_τ. By assumption, P_τ has a unique fixed point and so we must have $f_n\to f$. Q.E.D.

<u>Corollary 6.1</u>. (Of Theorem 6.1) Assume $1<s\le 2$ and P_τ has a unique fixed point. Choose N such that $s^N>2$. Set $\phi = \tau^N$ and f_n is a fixed point of $P_n(\phi)$, (P_n is defined at the beginning of this chapter). Let

$$g_n = \frac{1}{N}\sum_{i=0}^{N-1} P_{\tau^i} f_n$$

Then $\{g_n\}$ converges in L_1 to the fixed point of P_τ.

<u>Proof</u>. $\cancel{P_{\tau^i}} = P_\tau{}^i$ is continuous $\forall i$. Therefore we have

$$f_n \to f \Rightarrow \frac{1}{N}\sum_{i=0}^{N-1} P_\tau^i f_n \to \frac{1}{N}\sum_{i=0}^{N-1} P_\tau^i f = g$$

or $g_n \to g$.

$$P_\tau g = \frac{1}{N}\sum_{i=0}^{N-1} P_\tau^{i+1} f$$

$$= \frac{1}{N}\sum_{i=1}^{N-1} P_\tau^i f + \frac{P_\tau^N f}{N}$$

$$= \frac{1}{N}\sum_{i=1}^{N-1} P_\tau^i f + \frac{P_\phi f}{N}$$

since $\tau^N = \phi$. And since $P_\phi f = f$ we have

$$P_\tau g = \frac{f}{N}+\frac{1}{N}\sum_{i=1}^{N-1} P_\tau^i f$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} P_{\tau^i} f = g. \quad \text{Q.E.D.}$$

CHAPTER VII

FIXED POINTS OF SMALL VARIATION

In this chapter, we shall show that under certain conditions the variation of the fixed point of P_τ is small.

For $x\in[0,1]$, we define for the C^2 function g:

$$s = \inf_x |g'(x)|$$

$$\beta_o = \sup_x \left|\frac{g''(x)}{g'(x)}\right|$$

$$\beta = \max(1,\beta_o) \ .$$

<u>Theorem 7.1</u>. Let $g:[0,1] \to R$ be a C^2 function and $g\geq 0$. Let $\tau = g \pmod 1$ and let f be a fixed point of P_τ, such that $||f|| = 1$. Assume also that $s>3$. Then

$$\bigvee_0^1 f \leq \frac{3\beta}{s-3} \ .$$

<u>Proof</u>. Let $0 = b_0<b_1<\ldots<b_n = 1$ be the partition of $[0,1]$ where τ is piecewise C^2 on $[b_{i-1},b_i]$. With $\tau_i = \tau$ on $[b_{i-1},b_i]$ we have $|\tau_i'|\geq s$.

By Theorem 2.1, we have

$$P_\tau f(x) = \sum_{i=1}^{n} f(\psi_i(x))\sigma_i(x)\chi_i(x)$$

where $\psi_i = \tau_i^{-1}$, $\sigma_i = |\psi_i'|$, and χ_i is the characteristic function of $J_i = \tau_i([b_{i-1}, b_i])$. We note that for $i = 2,3,\ldots,n-1$, $J_i = [0,1]$, and hence

$$\bigvee_0^1 P_\tau f \le \sum_{i=1}^n \bigvee_0^1 f(\psi_i(x))\sigma_i(x)\chi_i(x)$$

$$= \bigvee_0^1 f(\psi_1(x))\sigma_1(x)\chi_1(x) + \sum_{i=2}^{n-1} \bigvee_0^1 (f\circ\psi_i)\sigma_i$$

$$+ \bigvee_0^1 f(\psi_n(x))\sigma_n(x)\chi_n(x) .$$

Since $J_1 = [\tau(0),1]$ or $J_1 = [0,\tau(0)]$, by Lemma 3.4

$$\bigvee_0^1 (f\circ\psi_1)\sigma_1\chi_1 = \bigvee_{J_1} (f\circ\psi_1)\sigma_1 + |f(\psi_1(\tau(0)))\sigma_1(\tau(0))|$$

Similarly, since $J_n = [0,\tau(1)]$ or $J_n = [\tau(1),1]$, we have

$$\bigvee_0^1 (f\circ\psi_n)\sigma_n\chi_n = \bigvee_{J_n} (f\circ\psi_n)\sigma_n + |f(\psi_n(\tau(1)))\sigma_n(\tau(1))|$$

We have $\sigma_i(x) \le \frac{1}{s}$ $\forall i$, and hence

$$\bigvee_0^1 P_\tau f \le \bigvee_{J_1} (f\circ\psi_1)\sigma_1 + \frac{|f(0)|}{s} + \sum_{i=2}^{n-1} \bigvee_{J_i} (f\circ\psi_i)\sigma_i + \bigvee_{J_n}(f\circ\psi_n)\sigma_n + \frac{|f(1)|}{s}$$

$$= \sum_{i=1}^n \bigvee_{J_i} (f\circ\psi_i)\sigma_i + \frac{|f(0)|+|f(1)|}{s} .$$

From Lemma 3.5

$$\sigma_i(x) = |\psi_i'(x)| = \frac{1}{|\tau_i'(\psi_i(x))|}$$

Hence,

$$\sigma_i'(x) = \frac{\tau_i''(\psi_i(x))\,|\psi_i'(x)|}{[\tau_i'(\psi_i(x))]^2}$$

and

$$\frac{|\sigma_i'(x)|}{\sigma_i(x)} = \frac{|\tau''(\psi_i(x))|}{|\tau_i'(\psi_i(x))|} \cdot \frac{1}{|\tau_i'(\psi_i(x))|} \le \frac{\beta}{s} .$$

By Theorem 4.1,

$$\underset{J_i}{V} (f_\circ\psi_i)\sigma_i \le \sup \frac{|\sigma_i'|}{\sigma_i} \int_{J_i} |f_\circ\psi_i|\sigma_i + \frac{1}{s}\int_{J_i} |d(f_\circ\psi_i)|$$

$$= \frac{\beta}{s}\int_{b_{i-1}}^{b_i} |f| + \frac{1}{s}\int_{b_{i-1}}^{b_i} |df| .$$

Therefore,

$$\overset{1}{\underset{0}{V}} P_\tau f \le \frac{\beta}{s} \, ||f|| + \frac{1}{s} \overset{1}{\underset{0}{V}} f + \frac{|f(0)|+|f(1)|}{s}$$

Since $||f|| = 1$, by Lemma 3.7 we get

$$|f(x)| \le 1 + \overset{1}{\underset{0}{V}} f.$$

With this and the fact that $P_\tau f = f$ we have,

$$\overset{1}{\underset{0}{V}}f \le \frac{\beta}{s} + \frac{1}{s}\overset{1}{\underset{0}{V}}f + \frac{1}{s}(1 + \overset{1}{\underset{0}{V}}f + 1 + \overset{1}{\underset{0}{V}}f)$$

$$= \frac{1}{s}(\beta+2) + \frac{3}{s}\overset{1}{\underset{0}{V}}f .$$

Since $\beta \ge 1$,

$$(1 - \frac{3}{s})\overset{1}{\underset{0}{V}}f \le \frac{3}{s}\beta$$

and

$$\overset{1}{\underset{0}{V}}f \le \frac{3\beta}{s-3} \qquad \text{Q.E.D.}$$

Note also that if $g(0)$ and $g(1)$ are integers, then $J_1 = [0,1]$ and $J_n = [0,1]$, thus

$$\overset{1}{\underset{0}{V}}f \le \frac{\beta}{s-1} .$$

If $||f|| = 1$, then there exist $s,t \in [0,1]$ such that

$$f(s) \le 1 \le f(t)$$

and

$$|f(s)-1+1-f(t)| = |f(s)-f(t)| \le \overset{1}{\underset{0}{V}}f$$

Therefore, $\forall x \in [0,1]$

$$|f(x)-1| \le \overset{1}{\underset{0}{V}}f.$$

Hence, with $\alpha = \overset{1}{\underset{0}{V}} f$,

$$1-\alpha \le f(x) \le 1+\alpha .$$

For $A \subset [0,1]$

$$(1-\alpha) m(A) \le \int_A f \le (1+\alpha) m(A)$$

where m is Lebesque measure.

If f with $||f|| = 1$ is a unique fixed point of P_τ, then for $x \in A$,

$$\mu(A) = \int_A f = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} \chi_A(\tau^k(x))\right).$$

If we choose $g(x) = 10^N(1+x)^2$ and set $\tau(x) = g(x) \pmod 1$, we have

$$\beta_o = \sup\left|\frac{g''(x)}{g'(x)}\right|$$

$$= \sup \frac{2(10^N)}{2(10^N)(1+x)} = 1$$

and since $g(0)$ and $g(1)$ are integers

$$\overset{1}{\underset{0}{V}} f \le \frac{1}{2(10^N)-1}$$

for f the fixed point P_τ. By [9, Theorem 1], this f is unique and so for $A\subset[0,1]$ and $x\in A$

$$m(A)\left(1-\frac{1}{2(10^N)-1}\right) \le \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\chi_A(\tau^k(x))\right)$$

$$\le m(A)\left(1+\frac{1}{2(10^N)-1}\right).$$

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