The Notions of Linear Independence/Dependence: A Conceptual Analysis and Students' Difficulties

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ABSTRACT

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The notions of linear independence/dependence are considered to be difficult for students. This thesis aims to identify some of the reasons for this difficulty by exploring these notions from several points of view.

We begin our research by tracing the origins and the historical development of the notions of linear independence/dependence. It is revealed that these notions had their genesis in the settings of systems of linear equations and in geometry, specifically in the move to generalize coordinate geometry to *n* dimensions. We continue with a conceptual and didactic analysis of the concepts; different ways of introducing the concepts are considered together with their underlying rationales, and then consequential difficulties for students are discussed. This is followed by a report on our 'continuous observations' of three students learning linear algebra from textbooks, each with the help of a tutor. These observations give us further insight into the epistemological difficulties associated with the concepts and they allow us to draw some conclusions regarding the kinds of understanding that students can have and the activities that lead to such understanding. Due to the exploratory nature of this research, most conclusions are in the form of recommendations for further research and teaching experiments.

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INTRODUCTION

My own first encounter with the concept of *linear independence* was a confusing experience. I was introduced to the notion through the following formal definition:

> The elements $\mathbf{v}_1,\dots,\mathbf{v}_r \in \mathbf{F}^n$ are said to be *linearly independent* over $\mathbf{F}$ if $a_1\mathbf{v}_1+\dots+a_r\mathbf{v}_r = \mathbf{0}$ only if each of $a_1,\dots,a_r \in \mathbf{F}$ is 0 (where $\mathbf{F}$ is the field of reals, $\mathbf{R}$, or complexes, $\mathbf{C}$).

This definition appeared seemingly out of nowhere, its sole motivation found in a preceding remark that the only linear combination of the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ which gives the zero vector is the trivial linear combination $0\mathbf{e}_1+0\mathbf{e}_2+0\mathbf{e}_3$. The elements $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ were then said to be linearly independent in the sense of this definition.

The concept was not internalized by either myself or most of my classmates. Although I was able to use the definition to some extent, to answer some standard exercise problems which followed, I remember mulling over it repeatedly, trying to extract its meaning and significance. My mathematical immaturity combined with the "poor" presentation, however, made this a largely futile endeavor. The same can be said of the concepts of *spanning set* and *basis* which were presented much the same way shortly thereafter.

It would be easy to dismiss the difficulties as a mere product of an excessively abstract and poorly motivated presentation, but it has been documented (Dorier, 1990, 1991) that the concept of linear independence together with the related notions of spanning set, basis, and dimension can be difficult for students, even when presented in a less stark setting. It therefore seems warranted to undertake an analysis, in some detail, of the notion of linear independence.

Students can encounter many difficulties when trying to understand the concept of linear independence. For instance, understanding the above formal definition on the level of logic alone can be formidable. Indeed, a preliminary investigation of this definition reveals that it is logically complicated; it is a compound conditional statement which contains a quantifier:

$$V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \text{ is said to be a linearly independent set over } \mathbf{F}$$

$$\Leftrightarrow \left[\sum_{i=1}^{r} a_i\mathbf{v}_i = \mathbf{0} \wedge a_i \in \mathbf{F} \Rightarrow \overset{r}{\underset{i=1}{\forall}} \{a_i = 0\} \right].$$

Typical students enrolled in a first linear algebra course are at best concurrently taking a first course in logic. Consequently, many don't feel at ease with this definition - it proves to be difficult to decipher so that the comprehension of the statement is weak from the start. Also, inexperience with logical negation, in particular the inability to relate a statement to its logical negation, can be the source of difficulty in understanding the relationship between linear *independence* and *dependence*, for example.

While it may seem plausible to believe that such difficulties can be easily remedied by lessons in basic logic and mathematical reasoning, it would be simplistic to think of this as a panacea. This is because understanding the logic and the statement of a formal definition in mathematics need not imply an understanding of the underlying concept. Indeed, the understanding of the two can remain mutually exclusive for the learner.

We may take our investigation of the difficulties with the concept of linear independence into a somewhat deeper level by subjecting it to an epistemological analysis. This seems necessary if there is to be any hope of attaining a deep understanding[1] of the notion and its significance. Part of such an analysis will be an investigation into the history of its development from its origins to its present guise. Indeed, it will become evident that the definition of linear independence, in its most abstract form, is the culmination of over one hundred and fifty years of mathematical developments and refinements. What students are usually presented with is the finished product, de-contextualized from its historical development. Hence, it offers little, if any, insight into its reason for being and does not beckon to the fundamental notions which gave rise to it.

Even if one does not ascribe to the biogenetic paradigm that psychogenesis recapitulates phylogenesis, at least not in any direct way, it does seem reasonable to expect an historico-epistemological study of the notions of linear independence/dependence to provide some insight into the cognitive and epistemological obstacles that students potentially face. Such a study may reveal information about the conditions which prevailed when the concepts emerged and evolved from one stage to the next. This in turn holds the potential to provide clues which could ultimately lead to a curriculum

[1] Our idea of a deep understanding, or "good" understanding, of linear independence/dependence is specified in chapter II.

designed to ensure a deeper understanding of the concept. Perhaps this point is best articulated by the following quotation:

> "The analysis of historical genesis is a promontory when studying a given teaching process or a basis when recreating, for the teaching, an artificial genesis" (Artigue, 1989)

We begin our research into the concepts of linear independence/dependence with the above-mentioned historical investigation by tracing their origins in the settings of systems of linear equations and analytic geometry. In chapter two we undertake a conceptual and didactic analysis of the concepts through a consideration of several different ways of introducing them. Chapter three represents the 'field work' of our research; here we report on our observations of students trying to learn the concepts from textbooks with the aid of tutors.

Our hope is to pull together these three points of view - the historical, the conceptual-didactic, and the empirical - to form a richer, dare we say 'three-dimensional', perspective of the notions of linear independence/dependence together with their attendant subtleties and epistemological difficulties.

CHAPTER I

LINEAR INDEPENDENCE/DEPENDENCE AND THE NOTIONS OF BASIS, DIMENSION, AND RANK: AN HISTORICO-EPISTEMOLOGICAL OVERVIEW

Systems of linear equations and geometry are two settings in which the fundamental concepts of linear algebra emerged and developed. In the former setting are found the origins of the basic concepts as well as the development of the first theoretical results (linear independence and rank, for example). The geometric setting played an important role in the generalization of these results; ideas and language born in $\mathbf{R}^2$ and $\mathbf{R}^3$ were carried over first to $\mathbf{R}^n$, and then to abstract vector spaces.

We begin with an overview of the history of systems of linear equations.

Linear Systems

As is often the case in the study of linear algebra, linear systems appear relatively early in the history of mathematics. Their study, which undoubtedly grew out of efforts to solve them, gave rise to fundamental concepts of linear algebra.

In antiquity the problems of day to day life (commerce and measurement, for example) gave rise to linear systems with numerical coefficients. The existence of such systems, as well as solution methods, is traced back to the period of the Han Dynasty (206 B.C.-A.D. 220). The mathematical classic of this era, the *Chui ch'ang Suan-shu* (*Nine Chapters on the Mathematical Art*), specified a method for solving simultaneous equations which is equivalent to the modern day method of row reduction of the augmented matrix of the system. At that time the Chinese had a distinctive place-value number system which used counting-rods arranged in columns from left to right. Arithmetic operations with counting-rods were carried out as one would on an abacus; the rods were repositioned column by column, on a table or a 'counting-board', according to whether the numbers were added or subtracted. Eventually the positions of the rods came to represent algebraic symbols, and operations with the rods represented algebraic operations. Chapter eight of this work is entitled *Fang chen* (Method of Tables) and deals specifically with solving simultaneous linear equations, it explains how counting-rods can be set up for column operations which would yield a solution. Let us look at an example of a problem and its solution taken from this work:

> 5 large containers and 1 small container have a total capacity of 3 *hu*.
> 1 large container and 5 small containers have a capacity of 2 *hu*.
> Find the capacities of 1 large container and 1 small container.

The solution by the method of tables begins by first setting up the information given in the problem in the form of a table:

large containers	1	5
small containers	5	1
total capacity	2	3

Step 1 is to multiply the first column by 5 and to then subtract the second column from the result. This result then becomes the first column of the next table:

large containers	0	5
small containers	24	1
total capacity	7	3

Step 2 is to multiply the second column by 24 and to then subtract the first column from the result. This result then becomes the second column of the next table:

large containers	0	120
small containers	24	0
total capacity	7	65

Thus a small container has a capacity of 7/24 *hu*, and a large container has a capacity of 65/120 *hu* or 13/24 *hu*. (Joseph, 1991, pp.172)

Evidently this method is identical to solving the linear system

$$x+5y = 2$$
$$5x+y = 3$$

by column reducing the transpose of the augmented matrix $\begin{pmatrix} 1 & 5 \\ 5 & 1 \\ 2 & 3 \end{pmatrix}$

to the echelon form $\begin{pmatrix} 0 & 120 \\ 24 & 0 \\ 7 & 65 \end{pmatrix}$, and then using division to obtain the capacity of each container.

There was no proof or justification given for the method of tables, and it has been pointed out that the absence of such a solution method in any other tradition before the advent of modern mathematics forces us to conclude that the method must have been a logical outcome of rod numeral computational techniques (Joseph, 1991, pp. 176).

Linear systems in early times generally contained as many unknowns as equations and in most cases the equations were linearly independent. Such systems were also known in ancient Babylon, Greece, Egypt, and India. However, the Chinese method of tables is not likely to have been known to these civilizations, for the unique solutions were generally found by more heuristic methods such as successive guesses of the unknowns, substitution and addition. These methods were informal and rhetorical.

Systems with multiple solutions were seen as early as the third century B.C. Diophantus is thought to have been one of the first to study their methods of solution; in his famous work

Arithmetica (A.D. 250) he presents such so called underdetermined systems and finds several solutions by using different informal techniques. It was not the complete solution which was of interest, rather it sufficed to find a few solutions empirically using various techniques. The question of a general solution was never considered. Furthermore, the practical problems which gave rise to these systems usually restricted the solutions to a finite set of values (often integral values). This together with the lack of any theoretical basis for the study of linear systems makes it unlikely that a notion of infinitely-many solutions existed at the time. Until the fifteenth century such problems were regarded as mere curiosities, and it was not uncommon to discard systems for which the available methods yielded no solution; these were thought to be the result of badly posed questions.

Questions pertaining to solutions and methods of determining these solutions hardly evolved before the seventeenth century. Mathematicians before this time continued to use informal methods to find a few solutions and the study of linear systems and their solutions lacked any theoretical basis

The advent of symbolic algebra beginning with Viète in the late sixteenth century and culminating with Descartes in the following century provided the conditions for the first advance in the treatment of linear systems. Equations could now be expressed in general and this made possible a change in perspective from an operational to a more structural point of view. Prior to the time of Viète equations were largely regarded as tools to solve specific problems, algebraic notation made it possible to symbolically represent both known and unknown quantities in an equation in general. This resulted not only in their applicability to a wide class of problems but more importantly it led to a systematic procedure for manipulating these symbols and for solving equations. Hence, the focus shifted from solving certain problems to studying the structure and properties of the tools used to solve them (Sfard, 1991).

These developments provided the conditions which made it potentially possible to interpret an equation as an algebraic object, to describe not only the solution set of a linear system in terms of the undetermined coefficients, but also the linear dependence relations between equations.

The systematic study of a system of linear equations was initiated before 1678 by Leibniz. By the end of the seventeenth century Leibniz used a double index notation for the coefficients of a system of three equations in two unknowns x and y (Kline, 1972, pp. 606). He then developed a method of eliminating the unknowns, this method involved a determinant whose evanescence was a sufficient condition for the system to be consistent. The algorithm for constructing this determinant was given rhetorically by Leibniz, it was apparently vague and non-rigorous and reflects how little familiarity with the notion of linear dependence there was at the time (Dorier, 1991, pp. 53)

It is speculated that the solution of linear systems in two, three, and four unknowns by the method of determinants was created by Maclaurin, probably in 1729, and published in his posthumous *Treatise of Algebra* (1748). It wasn't until 1750 that this type of notation was improved and generalized in the work of the Swiss mathematician Gabriel Cramer. This innovation made it possible to consider general linear systems with undetermined coefficients and to give a more thorough and consistent treatment of the method of their solution.

It is Cramer who is credited with having introduced the first calculations with determinants, he also gave what we now call Cramer's rule in connection with determining the coefficients of the general conic, $A + By + Cx + Dy^2 + Exy + x^2 = 0$, passing through five given points. In 1764 the French mathematician Bézout gave a systematic procedure for determining the signs of the terms of a determinant and showed that a square homogeneous system has non-trivial solutions if the determinant of the coefficient matrix vanishes (Kline, pp.606). Since its inception the determinant was always associated with solutions of linear systems; it was advances in the theory of determinants which drove the progress in the methods of solution. It was not until the theory of determinants reached a high point, with Cauchy's 1812 treatment of the subject, that the two subjects began to diverge.

Many developments of eighteenth century mathematics were the outcome of attempts to solve physical problems. The exploration of such problems led inevitably to the search for more knowledge about curves and surfaces, because, for instance, the paths of moving objects are described by curves. Mathematicians used the powerful methods of the calculus and analytical geometry to tackle geometrical problems. It was in this broad setting that a notion of linear independence/dependence first emerged in the work of Euler and Cramer.

The problem of determining the intersection of two curves received much attention; in 1717 Stirling proposed that an algebraic curve (in x and y) of degree n is uniquely determined by $\frac{n(n+3)}{2}$ of its points because it has that number of essential coefficients. We illustrate this with an example:

An arbitrary algebraic plane curve of degree 3 is represented by the general equation

$$a_0 y^3 + (a_1 + a_2 x)y^2 + (a_3 + a_4 x + a_5 x^2)y + (a_6 + a_7 x + a_8 x^2 + a_9 x^3) = 0,$$

where a_0, a_9 are not both zero (a non-degenerate case). Note that there are $1+2+3+4 = 10$ coefficients.

Assuming that $a_0 \neq 0$, we get the equivalent equation $y^3 + (b_1 + b_2 x)y^2 + (b_3 + b_4 x + b_5 x^2)y + (b_6 + b_7 x + b_8 x^2 + b_9 x^3) = 0$ having 10-1 =9 undetermined coefficients $b_i = \frac{a_i}{a_0}$, $i = 1,\ldots 9$. Hence, according to Stirling 9 points should completely determine a unique 3rd degree curve. In general, an arbitrary algebraic curve of degree n has $1+2+3+\ldots+n+(n+1)-1 = \frac{n(n+1)}{2}+(n+1)-1 = \frac{n(n+3)}{2}$ undetermined coefficients, so by Stirling's proposition the curve should be uniquely determined by as many of its points.

Based on the enumeration of points of intersection for special cases of curves, Maclaurin conjectured that two distinct algebraic curves of degree m and n may intersect in at most mn points. In the treatise of 1750, *Introduction à l'analyse des courbes algébraiques* Cramer tackled the paradox which now bears his name: if n = 3, the curve should be uniquely determined by 9 points. But since two third degree curves may intersect in 9 points, these 9 points do not determine a *unique* third degree curve.

For example, the two curves of degree 3 with equations $y^3 - y - 5x + 5x^3 = 0$, and $y^3 - y + 5x - 5x^3 = 0$ intersect at the nine points

(1)	(-1,1)	(4)	(-1,0)	(7)	(-1,-1)
(2)	(0,1)	(5)	(0,0)	(8)	(0,-1)
(3)	(1,1)	(6)	(1,0)	(9)	(1,-1)

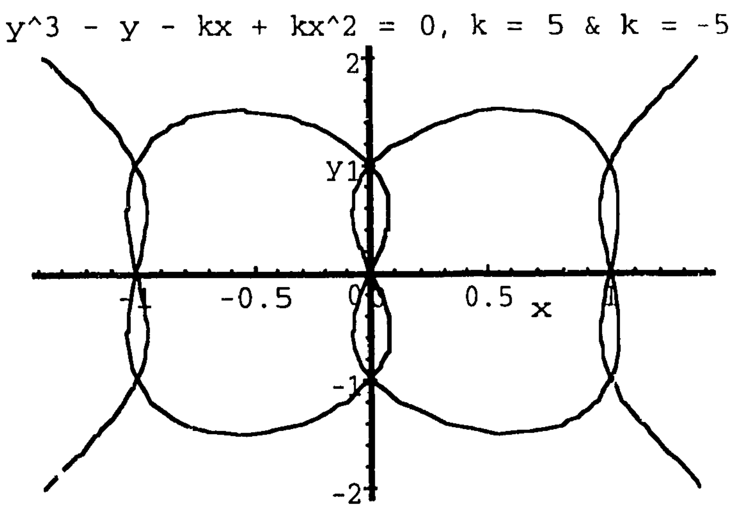

Figure 1.1. Two plane algebraic curves intersecting at nine points.

Hence, these points do not determine a unique curve of the same degree. Similar paradoxes arise when $n = 4$, and 5. Cramer's explanation was that the n^2 equations that determine the n^2 intersection points are not independent.

Let us summarize Cramer's paradox:

although neither was proven at the time, the following two propositions were thought to be true in the beginning of the 18th century;

Stirling's proposition:

$\frac{n(n+3)}{2}$ points are necessary and sufficient to uniquely determine a plane algebraic curve of degree n.

MacLaurin's proposition:

Two distinct plane algebraic curves of degree m and n may have no more than mn points in common.

For $n > 2$ we get $\frac{n(n+3)}{2} \leq n^2$, so it seems that two plane algebraic curves (both of degree n) may have more points (n^2) in common than is sufficient to determine each of them ($\frac{n(n+3)}{2}$), this contradiction constitutes the paradox.

In his work of 1750, *Sur une contradiction apparente dans la doctrine des lignes courbes*, Euler supplied a detailed analysis of the paradox when n = 3, 4, and 5. He concluded that Stirling's proposition might not be true in some cases because n equations are sometimes insufficient to determine as many unknowns and so $n(n+3)/2$ points may not suffice to determine a unique algebraic curve of degree n. Euler provided examples to show how one equation could be "comprised" ("comprise" in French) in one or several others. This text may be the earliest in which a notion of linear dependence emerged, and though the nature of his analysis is qualitative and exploratory it shows that Euler had an informal notion of linear dependence of equations. It is worth going into a few of these examples in some detail in order to understand Euler's thinking. He begins by considering the example of two equations, $3x-2y = 5$ and $4y = 6x-10$, of which he remarks:

"it is not possible to determine the two unknowns x and y as while eliminating x, the other disappears and an identical equations remains, of which nothing can be deduced. The reason for such an incident is of course at first quite obvious, as the second equation can be changed into 6x-4y =10, which, being nothing but the double of the first 3x-2y = 5, does not differ from it at all." (Euler, in Dorier, 1995, pp.3)

He solves the system by elimination and substitution in order to prove his assertion that one unknown is not determined and to highlight the "incident".

He then considers the case of three equations by looking at an example containing two similar equations, and another example in which one equation is twice the sum of the other two. In these there is no trial for solving the systems. He concludes:

"Thus, when one says that to determine three unknowns, it is sufficient to have three equations, the restriction needs to be added that these three equations are so different that none is already comprised in the others."
(Euler, in Dorier, 1995, pp. 3)

In the case of four equations Euler remarks that sometimes two unknowns may not be determined and he provides the following example:

"
$$
\begin{aligned}
5x+7y-4z+3v-24 &= 0\\
2x-3y+5z-6v-20 &= 0\\
x+13y-14z+15v+16 &= 0\\
3x+10y-9z+9v-4 &= 0
\end{aligned}
$$

they are only worth two, as after extracting from the third the value of

$$x = -13y+14z-15v-16,$$

and after its substitution in the second, one gets:

$$y = \frac{33z-3v-52}{29} \quad and \quad x = \frac{-23z+33v+212}{29};$$

> the substitution of these two values of *x* and *y* in the first and fourth equations leads to identical equations, therefore the quantities *z* and *υ* will remain undetermined." (Euler, in Dorier 1995, pp. 4)

Here again his proof is by elimination and substitution, and Euler makes no mention of linear relations between the equations, though they are obvious ((1)-(2) = (4) and (1) -2x(2) = (3), for example, if the equations are labeled from (1) through (4) in descending order). After these examples, he concludes with a general statement:

> "When one says that to determine *n* unknown quantities, it is sufficient to have *n* equations giving their mutual relations, the restriction must be added that they are all different or that none is enclosed (enfermée) in the others." (Euler, in Dorier, 1995, pp. 4)

The terms "comprised" and "enclosed" are not clearly defined. A modern interpretation of these terms might be that they indicate a linear relation between the equations, but Dorier asserts that this is not exactly the meaning given by Euler. Instead he uses these terms to refer to the " "incident" in the final process of elimination and substitution that results in one or several unknowns remaining undetermined " (Dorier, 1995, pp. 4).

We see that Euler does note linear relations between equations, albeit in an informal and unsystematic way, and, as Dorier points out further, because Euler's proofs never rely on such linear relations, his was a notion of *inclusive dependence* rather than linear dependence. Although the two notions coincide when applied to linear equations, Dorier's distinction serves to accentuate the point that dependence in the context of linear equations cannot simply be transferred to other linear situations (like n-tuples).

Euler's notion of dependence was a very intuitive one which suited his needs. The arguments he developed when treating the case $n = 4$ (last example) almost suggest that he had an empirical intuition of the notion of rank, this can be seen again at the end of the text when he considers Cramer's paradox:

> "When two curves of fourth order meet in 16 points, as 14 points, when they lead to different equations, are sufficient to determine one curve of this order, these 16 points will always be such that three or more of the equations are already comprised in the others. In this way the 16 points do not determine more than if there were 13 or 12 or even less points and in order to determine the curve entirely, one must add to these 16 points one or two others." (Euler, in Dorier 1995, pp. 5)

From this work Euler concluded that for n^2 points of intersection of two algebraic curves of degree n there corresponds a set of n^2 linear equations which are necessarily linearly dependent if $n \geq 3$. Furthermore, he attempted to enumerate these dependent equations and made the astute observation that their number increased as n increased. In so doing Euler may have been the first to implicitly research the concept of rank.

A comprehensive and rigorous treatment of these ideas had to wait at least one hundred years. This was partly due to the fact that the determinant was the main tool used to solve linear systems, consequently it became the central object of focus. There was a move to catalogue all possible cases of solutions of linear systems as a function of the number of variables and equations, here different determinants associated with the system were introduced and developed. It is speculated that the sophistication and highly technical nature of determinants may have obscured and suppressed the explicit emergence of the more fundamental concepts of linear independence and rank (Dorier, 1991). However, between 1840 and 1879 these notions did *implicitly* play a central role in the description of linear systems. This is because Euler's concept of inclusive dependence became connected to the vanishing condition of the main determinant of a square linear system, and because the notion of a *minor* (another determinant) was related to the maximal number of independent equations. Specifically, if the number r denotes the maximal order of non-vanishing minors of a system of p linear equations in n unknowns, then the number, n-r, of free variables which describe the solution set represents the maximal number of linearly independent equations in the system. All these notions came into play in the standard solution method of the time (see Dorier, 1991,1995 for details).

It was Frobenius who succeeded in developing clear and simple formulations of the fundamental notions, some of which were uncluttered by explicit reference to determinants. In a paper of 1875, *Über das Pfaffsche Problem*, he connected the notions of independent equations and independent n-tuples by giving them a common definition - the modern definition still used today. Referring to a homogeneous system of m "independent" linear equations,

$$a_1^{(\mu)}u_1+\ldots+a_n^{(\mu)}u_n = 0, \quad (\mu=1, \ldots m) \qquad (10),$$

he writes:

"If $\mathbf{A}_1,\ldots,\mathbf{A}_n$ and $\mathbf{B}_1,\ldots,\mathbf{B}_n$ are two particular solutions of the system of equations (10), then $a\mathbf{A}_1+b\mathbf{B}_1,\ldots,a\mathbf{A}_n+b\mathbf{B}_n$ are again solutions.
A set of particular solutions $\mathbf{A}_1^{\beta},\ldots,\mathbf{A}_n^{\beta}$, $(\beta=1,\ldots,k)$, is called *independent* or *different* when $c_1\mathbf{A}_\alpha^{(1)}+\ldots+c_k\mathbf{A}_\alpha^{(k)}$ does not vanish for $\alpha=1,\ldots,n$ except in the case where $c_1,\ldots,c_k$ are simultaneously zero, in other words when the k linear forms $\mathbf{A}_1^{(\beta)}u_1+\ldots+\mathbf{A}_n^{(\beta)}u_n$ are independent."
(Frobenius, 1875)

In the same paper Frobenius defines the notion of an "associate" system to a given homogeneous system, the definition involves the idea of the coefficients of equations as constituting a basis of the solution set of the original system. Here he considers equations and n-tuples as similar objects which can be seen from different perspectives. He also shows the existence of a maximum of p-r independent solutions to any given system of n linear equations in p unknowns having non-vanishing minor of maximal order r. He then easily states and relates all the invariants attached to this number r.

Frobenius also gave a precise and rigorous definit'on of the concept of rank in a paper of 1879, *Über homogene totale Differentialgleichungen*:

"When in a determinant, all minors of order $m+1$ vanish, but those of order m are not all zero, I call *rank* (*Rang*) of the determinant the value of m." (Dorier, 1995, pp. 7)

This induced a kind of unification of the fundamental concepts; the rank was seen as the invariant property of the linear system which represented the maximal number of linearly independent equations, the order of what was called the principal determinant, or equivalently the number of dependent variables in the solution. Indeed, in his paper of 1905, *Zur Theorie der* *linearen Gleichungen*, Frobenius gave a complete structured exposé on the theoretical results in the study of linear systems.

These developments represent a high point in the theory of solutions of linear systems. Within the first two decades of the twentieth century their study regained importance in the field of functional analysis; attempts to solve differential equations whose coefficients were given by infinite series gave rise to linear systems with infinitely many equations and unknowns. Mathematicians including Poincaré, Hilbert, Toeplitz, and Hadamard figure prominently in such research.

It is interesting to note that the gap between the finite and the infinite was bridged by a 'marriage' of determinants and convergence; results on finite dimensional determinants (of functions) combined with limiting processes based on convergence theorems allowed the passage to the infinite. This led to an extension of the notions to the infinite case, even the idea of an infinitely countable basis emerged, though it was not rigorously defined. These extensions marked the beginnings of countable infinite-dimensional linear algebra. Its flavor, however, was to change somewhat, as attempts to generalize results through the principal tool of finite systems, the determinant, proved to be difficult and cumbersome. This was partly what necessitated a change in approach, and eventually determinants were supplanted by an axiomatic approach in the early part of our century. Here the work of Banach figures prominently (Banach, 1932).

With the move to axiomatize and to unify mathematics in the beginning of this century the fundamental concepts of linear independence, rank, and basis were finally emancipated from the setting of linear systems. These notions were generalized to abstract vector spaces and took on the their modern guise. We refer the reader to Dorier's research (Dorier, 1995) for a detailed account and analysis of these developments.

Analytic Geometry

This analysis so far reveals that the fundamental notions in question arose out of attempts to solve certain geometrical problems, so in an indirect way they were born out of geometry. But these concepts also emerged directly in a geometric setting, as a result of the move to extend the ideas of space geometry (i.e two and three dimensional geometry) to *n*-dimensions.

The notion of a geometric vector as a directed line segment was known in Aristotle's time. Aristotle knew that forces can be represented by vectors and that the combined action of forces can be obtained by the parallelogram law to give a resultant force. Later scientist, among them Stevin, Galileo, and Newton used this law in their work in statics and dynamics. Galileo stated the law explicitly.

As early as 1679 Leibniz criticized the analytic method of Descartes and Fermat as unsuitable for describing position, he wrote:

> "I am still not satisfied with algebra because it does not give the shortest method or the most beautiful constructions in geometry. This is why I believe that, as far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation directly as algebra expresses magnitudes. [...] Algebra is the

characteristic for undetermined numbers or magnitudes only, but it does not express situation, angles and motion directly. hence it is often difficult to analyze the properties of a figure by calculation, and still more difficult to find very convenient geometrical demonstrations and constructions, even when the algebraic calculation is completed."(Leibniz, in Dorier, 1995, pp.8)

Leibniz attempted, unsuccessfully, to create an intrinsic geometric analysis which he called the "geometry of situation". From the beginning of the 19th century, and on the basis of the above criticisms, several mathematicians embarked on a search for such an intrinsic geometrical analysis. The geometric representation of complex numbers was, indirectly, an initial answer to this problem because it provided a model for a two dimensional geometrical analysis.

One of the earliest attempts to formulate some sort of analytic representation of geometric vectors and operations on them was due to the amateur mathematician Wessel (1745-1818). In a paper entitled *On the Analytical Representation of Direction; an Attempt* (1799) Wessel essentially thinks of the complex number $a+bi$ as the vector with coordinates (a, b), where the real numbers a and b are plotted on the real and imaginary axes respectively. He then defines the operations with vectors by defining the operations with complex numbers in geometrical terms. His definitions of the four operations are practically the ones we learn today. Unfortunately, Wessel's paper went unnoticed until it was translated into French in 1897. Another amateur mathematician, Argand, also regarded the complex number $a+bi$ as symbolizing the geometric sum of a and bi. He too showed how to combine complex numbers geometrically. Gauss, by 1831, had similar ideas; he not only gives the representation of $a + bi$ as a point (not a vector) in the complex plane, but also describes the geometrical addition and multiplication of complex numbers (Kline, 1972, pp. 628-631).

This use of complex numbers to represent vectors in a plane became widely known and accepted in mathematics by 1830.

During this same period the mathematicians Möbius and Bellavitis developed two different systems of geometrical analysis in two and three dimensions, these laid the basis for vector geometry.

In his work of 1827, *Barycentrische Calcul*, Möbius introduces the notion of a directed line segment. Here he denotes the line segment from a point A to a point B as AB and states that AB = -BA, he also defines the addition of collinear line segments as well as their linear combinations. The addition of non-collinear line segments and their multiplication by negative numbers are defined in his work of 1843, *Elemente der Mechanic des Himmels*. Finally, in 1887 Möbius published *Über geometrische Addition und Multiplication* in which he defined addition of non-collinear line segments, their multiplication by any number, and two kinds of products of line segments. Möbius provided an algebra of points, but his intention was not to present an algebraic structure. Rather, he was driven to his creation by the need for a practical and efficient method for solving geometrical and physical problems. Although Möbius did point out some fundamental aspects of vector geometry, his theory was based on the physical perception of space, and so it never offered the possibility of extension to a more general concept of vector space (Crowe, 1967, Dorier, 1995)

It was Giusto Bellavitas who defined the addition of vectors in space in his *Calcolo delle Equipollenze* in 1888. He also defined the multiplication of coplanar directed line segments. It has been pointed out that the calculus of equipollences[1] offered no more possibilities than complex numbers, and Bellavitas developed his theory as an alternative to the geometric representation of complex numbers, which he refused to accept as part of mathematics. His presentation was original because (i) unlike complex numbers the objects on which his calculus was created were purely geometric entities, and (ii) because the first part of the calculus can be applied in space geometry (Crowe, 1967, Dorier, 1995, pp. 10)

The need to treat problems involving bodies under the action of forces in space and the desire to describe such problems and their solutions analytically fueled the search for a three-dimensional analog of complex numbers. The first successful development here was actually a four-dimensional quantity called the Quaternion; the English mathematician Hamilton announced his creation in 1843. It is also interesting to note that Hamilton defined a non-commutative multiplication of quaternions. These properties are necessary in order for the quaternion to be analogous in every way (other than in number of components) to complex numbers in the plane. Hamilton spent fifteen years creating the quaternions and their usual significance is that they were the first example of 'numbers' whose multiplication was non-commutative - a revolutionary idea at the time. From the point of view of this study,

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[1] Bellavitis called two line segments *equipollent* if they are equal, parallel and directed in the same sense.

however, their significance lies more in that they represent the first attempt to generalize vectors to three-dimensional space and also that they provided the impetus for further generalizations to n dimensions. Indeed, Hamilton himself began work on n-tuples, or *hypernumbers*, as they were then called.

In 1845 Arthur Cayley gave a generalization of quaternions; he introduced the "unit elements" $1, e_1, e_2, \ldots, e_7$ together with an algebra for their products and then he defined the general *octonion* by $x = x_0 + x_1e_1 + x_2e_2 + \cdots + x_7e_7$, where the x_i are real numbers. The norm of x he defined as $N(x) = \sqrt{x_0^2 + x_1^2 + \cdots + x_7^2}$.

The most ambitious generalization of complex numbers was developed by Hermann Grassmann (1809-77). In 1844 Grassmann published the first version of his *Die Lineale Ausdehnungslehre* (literally "*linear theory of extension*"), it was introduced by Grassmann as the first part of a general theory, *Die Ausdehnungslehre*, which he never completed. Grassmann claimed that he had created a new abstract theory which could be applied to geometry, mechanics, and other scientific disciplines. Because geometry refers to reality for validation, Grassmann believed that it should be distinct from mathematics. To him geometry was a science outside mathematics and the theory of extension was the mathematical model to be applied to it:

> ". . . geometry can in no way be viewed, like arithmetic or the theory of combinations, as a branch of mathematics; instead, geometry relates to something already given in nature, namely, space. I also had realized that there must be a branch of mathematics which yields in a purely abstract way laws similar to those of geometry, which is limited to space. By means of the new analysis it is possible to form such a purely abstract branch of mathematics; indeed this new

> analysis, developed without assuming any principles established outside its domain and proceeding purely by abstraction, was itself this science."
> (trans. in Fearnley-Sander, 1982)

Because Grassmann's theory was essentially self-contained there were many definitions and it introduced many new notions using new terminology. Also, in his presentation there was an overwhelming tendency to mix up mathematical results with obscure philosophical considerations. All of this resulted in a lack of clarity and made it very difficult to follow. Consequently, his work drew some harsh criticism (Dorier, 1995, pp.18). In 1862 Grassmann published a completely revised version of the *Ausdehnungslehre* which had a more traditional mathematical presentation, and from which most of the philosophical considerations had been deleted. But readers were still discouraged because it did not allow for a partial reading of the theory, as one had to read it from the very beginning in order to understand the meaning of any concept. Due to these difficulties Grassmann's work was little known for years.

Still, Grassman's theory introduced fundamental concepts such as linear independence, basis, and dimension accurately and in a very general context. As such it contained the essential components for a unified theory of linearity. We will examine some of the main ideas of his work.

Grassmann was concerned with n-dimensional geometry, and because he was led to his results by studying the geometric interpretation of negative quantities and the addition and multiplication of directed line segments in 2 and 3 dimensions.

Consequently, his exposition was almost inextricably bound up with geometric ideas (Boyer & Merzbach, 1989, pp. 654). His basic notion, called an *extensive quantity* (*extensive Grösse*), is defined differently in the two works mentioned above (this is due to the drastically different theoretical frameworks used in both papers). Let's begin with the 1844 *Ausdehnungslehre:*

here are specified certain rules for the construction and comparison of new entities by connection with others. Generation is a central concept in this work. Entities are not given a priori, instead of being defined according to the properties of their operations they are created through the "evolution" or the connection of other entities. When introducing the concept of *extensive quantity* he speaks of a given element generating

"a system of first order" by the "continuous action of the same fundamental evolution"; then another "evolution", applied to each element of the system of first order will generate a system of second order, and so on, with no limitation on the number of orders. The concept of "evolution" corresponds in geometry to a movement along a straight line, but Grassman's meaning of it is more general, it refers to "the fundamental intuition of space and time", which is given "a priori," and is "originally inherent to us like the body is to the soul" (Dorier, 1995, pp. 19).

If nothing else, the reader can understand why Grassman's text was considered to be vague and difficult to follow.

He speaks of a system of *n*-th order being generated by *n* fundamental methods of evolution which are given as independent - meaning that none is included in a system generated by some of the others. The order of a system, which is the "natural" dimension, is intrinsically related to the concepts of generation and dependence; it represents the measure of extension (Dorier, 1995, pp.19). He also says:

> "(A) system of *m*-th order is generable by any *m* methods of evolution belonging to it that are mutually independent".

We see that he has a notion equivalent to the modern concept of basis and he gives the value *m* a general meaning close to the concept of dimension. Although there is no mention of *m* being the minimal number of methods of evolution required to generate the system, this is implied by a result he gave:

> First I will show that if the system is generated by *m* methods of evolution whatever, I can replace any given one of them by a new method of evolution (*p*) belonging to the same system of *m*-th order and independent of the remaining (*m-1*), and, using this in combination with the other (*m-1*), generate the given system. (Dorier,1995, pp. 19)

This is in great contrast to the *Ausdehnungslehre* of 1862; here an *extensive quantity* is an *n*-tuple, α, which is a linear combination of the *system of units* $e_1, e_2, \ldots, e_n$. That is $\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$, where the α_i are real numbers and the units $e_1, e_2, \ldots, e_n$ are linearly independent magnitudes. Addition (subtraction) of extensive quantities and their multiplication (division) by a scalar are defined as in the usual sense in $\mathbf{R}^n$, and the properties of these operations are listed - these are almost the same as the modern day vector space axioms.

For n = 3 these *primary units* were represented geometrically by line segments of unit length each directed along one of three mutually orthogonal axes. The $\alpha_i e_i$ are multiples of the primary units and are represented by lengths $|\alpha_i|$ along the respective axes, while α is represented by a directed line segment in space whose projections onto the axes are the lengths $|\alpha_i|$.

Grassmann developed in detail, essentially as it is done today, the theory of basis and dimension for finite-dimensional linear spaces. The space of extensive quantities he called a *region* and a basis for it was the system of units (Fearnley-Sander, pp.164). He defines a system of order *m* as the system of all linear combinations of the units (i.e. spanning set). He discusses subspaces, their unions and intersections, and he states and proves theorems that are equivalent to common results on subspaces, such as the dimension theorem dim(S+T) = dim(S)+dim(T)-dim(S$\cap$T), where S and T are subspaces of a vector space V (Boyer & Merzbach, 1989, pp.655).

In the same work Grassmann defines different products of extensive quantities. The inner product is among these, it is denoted by $\alpha|\beta$ and given by $\alpha|\beta = \sum_{i=1}^{n} \alpha_i \beta_i$, the *magnitude* of α is defined as $\sqrt{\alpha|\alpha} = \sqrt{\sum_{i=1}^{n} \alpha_i^2}$.

If Grassmann's work had not gone largely unnoticed he might have been credited as the main contributor to the creation of linear algebra. Though his ideas were based on geometry his work was clearly a great extension and abstraction of these. Despite Grassmann's comprehensive work, it was Cayley and Hamilton who were considered the figureheads in the move to algebraicize geometry and who were credited as the precursors, if not the creators, of linear algebra. Nonetheless, as Dorier points out, Grassmann's work can be seen as a kind of 'a posteriori' of linear structure, and in this sense it played an important role in the discovery of the axiomatic theory. Indeed, some of the first axiomatic approaches which followed were inspired by Grassmann's work; for instance, Peano's *Calcolo geometrico* of 1888, which gives the first axiomatic definition of a vector space (in modern terms) is considered to be a condensed version of his own reading of Grassmann's *Ausdehnungslehre.* However, most of Grassmann's concepts were reestablished independently of his work. A notable example is the work of Dedekind in the theory of fields, his was a very general and modern approach to linear structure in which generation played a central role. The reader is referred to Dorier's paper for the details (Dorier, 1995).

This account of the evolution of fundamental notions of linear algebra in the setting of analytic geometry allows us to take a rather broad perspective; we can say that linear algebra developed in two stages that correspond to two processes. The first process was the *arithmetization* of space, as it ocurred in the passage from synthetic geometry to analytic geometry in $\mathbf{R}^n$. The second process was the de-arithmetization of space or its *structuralization,* whereby vectors lost the coordinates that anchored them to the domain of real numbers and became abstract elements whose behavior is defined axiomatically. This is an admittedly simplistic view of things, but it helps us distinguish different modes of thinking in linear algebra. These distinctions in turn help us understand what is involved in constructing the concepts of linear algebra by students (Sierpinska et al., 1995a).

CHAPTER II

A CONCEPTUAL AND DIDACTIC ANALYSIS OF THE NOTIONS OF LINEAR INDEPENDENCE/DEPENDENCE

In this chapter we will look at different ways of introducing the concepts of linear independence/dependence, we will consider the rationale underlying each of these, and then we shall discuss some possible difficulties for students which arise from these different starting points. Each of these introductions can lead to a different understanding of the concepts, so our analysis will allow us to identify possible different ways of understanding. An awareness of these is necessary for us to make informed assessments of students' understanding of the concepts.

There are several ways of understanding these concepts; linear independence/dependence can be seen as a property of a set of vectors, as a relation between vectors, it can also be seen geometrically. Having a "good" understanding might consist of being able to see the concepts in different ways, to grasp the relationships between these, and to coordinate them so as to arrive at a synthesis of all this knowledge. Now, in teaching one must choose a starting point and then build on it, so one's understanding of a concept is built on a particular way of introducing it (see *Chronogenèse des Saviors* in Chevallard, 1985).

Introducing the concepts of linear independence/dependence In terms of solutions to a homogeneous equation

A perusal of several introductory linear algebra textbooks used in North America reveals that students' first exposure to the concepts of linear independence/dependence usually occurs through formal definitions.[1] These are given by statement 1 below:

> A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ in a vector space $\mathbf{V}$ over a field $\mathbf{K}$ is *linearly independent* $\Leftrightarrow$ $[c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k=\mathbf{0} \Rightarrow c_1=\ldots=c_k=0]$, where $c_j \in \mathbf{K}$.
>
> A set of vectors is *linearly dependent* $\Leftrightarrow$ it is not linear independent.

The way in which the vectors $\mathbf{v}_i$ are defined depends on the nature of the vector space $\mathbf{V}$; if this is an abstract structure then the vectors are defined by the vector space axioms as elements satisfying certain properties and having a certain function. One the other hand, if $\mathbf{V}$ is the space $\mathbf{R}^n$ then these vectors may be defined by construction as n-tuples.

Linear Independence

Globally, the above definition of linear independence has the form of a compound two way implication $p \Leftrightarrow q$, where p is the statement

> [A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ in a vector space $\mathbf{V}$ over a field $\mathbf{K}$ is *linearly independent*]

[1] This is based on the reasonable assumption that most students learn linear algebra more or less by following lesson plans set out in textbooks.

and q is the statement

$$[c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k=\mathbf{0} \Rightarrow c_1=\ldots=c_k=0], \text{ where } c_i \in \mathbf{K}.$$

The compound nature of $p \Leftrightarrow q$ is due to the fact that statement q is itself an implication.

On a more local level we see that the subject of the definition (i.e. of the independence) is a set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$. That linear independence is a property of a set of vectors is communicated clearly in statement p. The topic of the equivalence $p \Leftrightarrow q$ is both the sufficiency and the necessity of the condition given by statement q - namely that the homogeneous vector equation $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k=\mathbf{0}$ have uniquely the zero solution for the elements $c_1,\ldots,c_k \in \mathbf{K}$. The communication of this uniqueness is a rather subtle affair because it involves not only the explicit implication given by statement q but also the implicit understanding that such an equation always has at least the zero solution. This definition states that whenever this condition holds then we say that the set of vectors is linearly independent. Conversely, if we say that the set of vectors is linearly independent then it is necessary that this condition hold, for this is precisely what it means (in the sense of the definition) to say so. This definition introduces the notion of a set of vectors satisfying the condition of statement q and gives it a special name, thus it sets up a correspondence between such a set of vectors and the words *linearly independent*. This correspondence, so typical of mathematical definitions, establishes the equivalence between the concept and the concept name so that the two are always associated with each other.

Linear Dependence

The notion of linear dependence may be defined with reference to linear independence as its negation. This highlights the fact that any set of vectors must satisfy one of these two mutually exclusive conditions, i.e. a set of vectors must be either dependent or independent - it can never be both.

This introduction of the concepts presents independence as the central notion. Dependence, being defined in terms of independence, appears almost as an auxiliary notion. Thus, the notion of independence is stressed over dependence even though they may be equally important and are logically equivalent through negation.

The rationale for introducing the concepts through these formal statements

For this analysis a pertinent question is "what is the rationale for defining linear independence/dependence in this way ?". In fact there are several reasons :

- These statements are terse; they offer conciseness and economy of exposition.

- We may say that these definitions are "definitive" (Méray, 1901); the statements are sufficiently abstract and general to be applicable in all cases and settings, it is not necessary to give one definition to deal with systems of linear equations and a different definition to study bases and dimension, for instance.

- These statements offer consistency and unification of language in a set-theoretic approach to building the theory of linear algebra.

- These definitions are operational; in $\mathbf{R}^n$ (and isomorphic spaces) they provide a useful test of independence/dependence by reducing the question of dependence to one of existence of non-trivial solutions for a certain homogeneous system of equations.

The first three of these are in accordance with a structuralist perspective, the last is a reductionist motive.

A structure is characterized by three central ideas: wholeness, transformation, and self-regulation. Elements of a structure are subordinate to transformational laws, and it is in terms of these laws that the structure defines the whole system. A relational perspective of the property of wholeness gives primacy not to the elements or the whole itself, but rather to the relations which determine the whole. Self-regulation entails self-maintenance and closure in the sense that "the transformations inherent in a structure never lead beyond the system but always engender elements that belong to it and preserve it's laws" (Piaget, 1970, pp. 14). From a structuralist perspective mathematics is seen as a unified whole in which the meaning and significance of every part is determined by its function in this whole. Here, the work of synthesis, the bringing together and organization of results becomes very important. The drive to have a "complete picture" helps us understand the rationale behind general definitions like the one of linear independence which includes the extreme case of the singleton set.

Possible sources of difficulty for students

Introducing the concepts this way has consequences for the learner. We are concerned in particular with possible difficulties that this approach poses:

(a) Statement 1 uses the word "dependence" in an unnatural way. Webster's College Dictionary defines dependence as "the state of being conditional or contingent on something". Thus in the vernacular one thing cannot be dependent (or independent) without reference to something else, rather dependence must involve at least two things. Since in linear dependence we speak of a set - a single object - being dependent, there is an apparent incompatibility in the use of the word "dependent" in the two senses. This immediately sets the stage for a kind of cognitive conflict which can lead to confusion in understanding the concept (Tall & Vinner, 1981, Tall, 1983, Vinner, 1991). As an example of such confusion, we observed a student who was presented with a set of vectors which was linearly dependent. After having determined that the vectors were all pair-wise independent he was asked if this implied that the whole set was linearly independent, his response was: "... I would say yes because we've determined that they're all independent of each other ..."

The set-theoretic approach uses language counter-intuitively in other domains as well, for example in geometry (intuitively a point "lies" on a line, it does not "belong" to a line). Here, by intuition we mean images conveyed by the use of words in natural language.

(b) The logical construction of the definition of independence is complicated, it involves a compound implication as noted above. In general, conditional statements are difficult to use and apply because of their non factual, speculative, and abstract nature (Girotto, 1989). One type of difficulty concerns conditional statements in general; some students tend to fixate on either the premise or the conclusion, sometimes adding a general quantifier. We cite an example of a student who thought that "linearly independent vectors are always zero" and who accordingly substituted **0** for the linear combination, $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k$, of these vectors. This student may have read the definition as : " for any scalars $c_1,\ldots,c_k$, $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k = \mathbf{0}$ ", thus completely disregarding the conditional nature of the statement (Sierpinska, 1994a).

Another impediment to understanding the concept of linear dependence/independence through these definitions is the difficulty of negating logical implications, especially statements like these which involve quantifiers. Our own survey indicates that students have considerable difficulty with logical negation: a class of thirty three undergraduate linear algebra students at Concordia University was given the definition of linear dependence and was presented with the statement that a set of vectors is called linearly independent in the case that it is not linearly dependent. The group was then asked to give the definition of linear independence without reference to linear dependence. Only two respondents

(6%) were able to produce a statement that was a correct logical negation of the original statement.

(c) In a structural approach the meaning of an element of the theory is determined by its function in the whole structure, and not by reference to external things such as visualization and applications. Structural meaning eludes the novice because he/she cannot yet see the whole structure. In fact, in a structural approach to the teaching of a theory, understanding must be suspended for quite a while, at least until one has seen several elements of the structure and has made some of the necessary connections between them. Then, one may begin to understand the function of each element, and to view the structure as a cohesive whole.

(d) The operational usefulness of the definitions (as a text) is not apparent unless one is familiar with vectorial representations of systems of equations; one must realize that the vector equation $c_1\mathbf{v}_1+\ldots+c_k\mathbf{v}_k=\mathbf{0}$ is equivalent to the system of linear equations $M\mathbf{X}=\mathbf{0}$, where the matrix $M=[\mathbf{v}_1|\ldots|\mathbf{v}_k]$ and the vector $\mathbf{X}=\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$.

Then, the problem of determining whether the set of vectors is linearly independent reduces to determining whether a homogeneous linear system has only the trivial solution. Students do have difficulty with these multiple representations in linear algebra, this may be a problem of not having seen enough of the elements of the structure and not having established the connections between them. In the same survey previously mentioned students were also asked how they would determine if a set of vectors was linearly dependent or independent. The success rate for this question was not high; very few responses (12%) showed knowledge of a general method to determine whether a set of vectors in $\mathbf{R}^n$ is linearly independent. This suggests that students are not always able to draw out the operational value from the text on their own.

(e) There can be difficulties related to the tacit conventions in the use of variables in formal definitions. These conventions reserve the last letters of the alphabet, *s* through *z*, to denote variables, and the first few letters usually denote constants. The formal statements given here don't abide by these conventions because the variables in the vector equation $c_1\mathbf{v}_1+...+c_k\mathbf{v}_k = \mathbf{0}$ are denoted by c's. This may result in some ambiguity about the status of these scalars in the equation. But there are creative ways of using suggestive notation designed to avoid such ambiguities; in his book, Linear Algebra and It's Applications, David Lay presents the concepts through the following less formal and more explicit statements:

> A set of vectors $\{\mathbf{v}_1,...,\mathbf{v}_k\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation
>
> $$x_1\mathbf{v}_1+x_2\mathbf{v}_2+...+x_k\mathbf{v}_k=\mathbf{0}$$
>
> has only the trivial solution.
>
> The set $\{\mathbf{v}_1,...,\mathbf{v}_k\}$ is said to be **linearly dependent** if there exist weights $c_1,...,c_k$, not all zero, such that
>
> $$c_1\mathbf{v}_1+c_2\mathbf{v}_2+...+c_k\mathbf{v}_k=\mathbf{0}$$ (Lay, 1994, pp. 64).

In defining linear independence this author refers to a vector equation and a solution of this equation, so the reader might think that the vectors are the unknowns. However, the use of x's to denote scalars in this equation suggests that these are the variables rather than the vectors. In formulating the notion of dependence the words *equation* and *solution* are not explicitly mentioned, instead the notion is equated with the existence of scalars, called weights, satisfying a certain relation. Since dependence establishes the existence of these scalars and one is no longer looking for a solution of an equation, these scalars are no longer denoted by letters reserved for unknowns.

Perhaps, explicitly stating what the unknowns are in this definition is a more simple and direct way to remove any ambiguity about the status of the objects; one might say '*if the vector equation* $x_1\mathbf{v}_1+x_2\mathbf{v}_2+...+x_k\mathbf{v}_k=\mathbf{0}$ *has only the trivial solution for the unknowns* $x_1, x_2,...,x_k$ ', for example.

Introducing the concepts of linear dependence/independence in terms of a linear relation between vectors

An important equivalent characterization of linear dependence is given by the following statement 2:

> A set of (two or more) vectors $\{\mathbf{v}_1,...,\mathbf{v}_k\}$ in a vector space $\mathbf{V}$ is *linearly dependent* $\Leftrightarrow$ one of the vectors is expressible as a linear combination of the others.

If we start with this statement as a definition then we make dependence the more basic notion. Independence can then be defined in terms of this by logical negation:

A set of vectors is *linearly independent* ⇔ it is not linearly dependent.

i.e.

A set of vectors is *linearly independent* ⇔ <u>none</u> of the vectors in the set is expressible as a linear combination of the others.

This definition above equates the notion of linear dependence of an abstract set of vectors with the possibility of expressing one of the vectors in terms of the others (in the sense of linear combination). The possibility of so expressing a vector in the set is communicated by the existential quantifier "if one of the vectors", this is equivalent to saying "there exists a vector ...". The existence of such a vector in the set is a property of that set which is then said to be linearly dependent. The notion of independence is then equated with the non-existence of such a vector in the set, i.e. with the impossibility of so expressing any vector in the set.

The rationale

There are several reason for choosing to introduce the notion of dependence/independence by way of statement 2

- In statement 1 (page 29) the use of the words *dependent* and *independent* seem abstract and their significance is not easily grasped. This is not the case in statement 2 above because the word *dependent* here is used in a more natural sense; it is compatible with its vernacular use since, if, say, $\mathbf{v}_2$ is a linear combination of the other vectors, we write $\mathbf{v}_2 = b_1\mathbf{v}_1+b_3\mathbf{v}_3+...+b_k\mathbf{v}_k$

then one can "see" the dependence, i.e. the reliance of $\mathbf{v}_2$ on the other vectors. Questions concerning the use and significance of the words *independent* and *dependent* may therefore be less likely to arise.

- This formulation of linear dependence given in statement 2 equates the notion of linear dependence with that of linear combination - the most fundamental and pervasive construction in linear algebra. Thus, it anchors knowledge around this most basic notion; a newer and more sophisticated notion (linear dependence) is defined in terms of the more basic and previously defined notion of linear combination, it builds on, and preserves links with, more basic knowledge. Thus, it serves to unify and strengthen ties between concepts.

- The definition of linear dependence in statement 2 can be seen as a generalization of the idea of a scalar multiple of one vector. In the case of the geometric spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ this would build on concrete geometric understanding and more experiential knowledge.

- The logical construction of statement 2 is less complicated than that of statement 1 (page 29) because it is a simple, rather than a compound, bi-conditional statement. It is thus easier to understand.

Possible sources of difficulty

(a) Although statement 2 appears to be logically simple, it does contain an existential quantifier which some students have difficulty interpreting. The word *one* in this context need not imply uniqueness, as it usually does in the vernacular, rather it allows for the possibility of one or more vectors and so must be interpreted as *at least one*. A vernacular interpretation of the quantifier can cause difficulties for students trying to understand linear dependence. Other times students assume too much and interpret the statement as meaning that every vector may be so expressed. A similar misinterpretation of the word *others* can also lead to difficulty in understanding, we illustrate this with an example of a student who believed that a set of two vectors could never be dependent:

> "Can a set of two vectors be linearly dependent ? No, it's impossible, impossible! No! [...] No. You know why they can never be dependent? [...] Because ... listen: 'a system of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others'. Right ? So you need at least three vectors." (see student C in Sierpinska, 1995d, pp. 10)

This misconception is due to a vernacular interpretation of the word *others*; in this mathematical context the plural form need not imply the existence of more than two vectors in the set. One wonders how replacing the word *others* with *the rest* in statement 2 would affect its comprehension.

(b) As a definition, statement 2 is less definitive than statement 1 because it does not account for the extreme case of the singleton set; one cannot use this statement to determine whether a set consisting of a single vector is dependent or independent. One could make this into a complete definition by adding provisions for the singleton set, but the simplicity and elegance of the original statement would be destroyed.

(c) The formulation of linear dependence given in statement 2 is very useful in the case of two, and possibly three, vectors because it offers a method of determining linear dependence/independence by inspection. However, the statement has a limited operational value because this method does not generalize easily to a set of more vectors. In fact it may well be a hindrance to try and use this statement as a test of dependence in general because one would end up having to proceed by trial and error checking to see if each vector in a set is a linear combination of the others.

Introducing the concepts through geometry

The first two approaches to introducing the concepts of linear independence/dependence were purely algebraic, but the concepts can be presented as geometric properties of vectors in the plane and in Euclidean space. We have statements 3 and 4:

> 2 vectors in the plane are linearly dependent $\Leftrightarrow$ they are collinear.
>
> 3 vectors in space are linearly dependent $\Leftrightarrow$ they are coplanar.

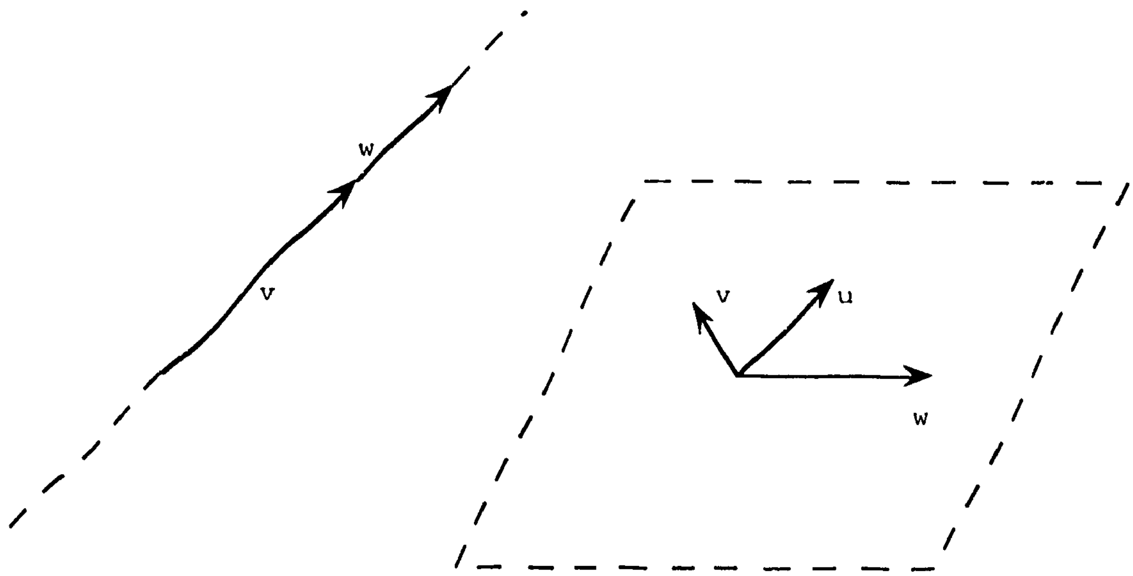

Figure 2.1 Two collinear vectors and three coplanar vectors.

A set of vectors is linearly independent if and only if it is *not* linearly dependent, so by negation we have:

> 2 vectors in the plane are linearly independent ⇔ they do not have the same direction.
>
> 3 vectors in space are linearly independent ⇔ they are not coplanar.

This approach can be appropriate for students having some knowledge of synthetic geometry and the concept of free vectors. These definitions speak of nonzero vectors, of course, since otherwise the notion of direction is meaningless.

In Griffel's book *Linear Algebra and its Applications* (Griffel, 1989) the concept of basis is presented at the very beginning of the course in the context of geometry. The concept is defined in coordinate-free language as in the case of linear independence above; a basis for vectors in the plane is defined simply as a pair of nonzero vectors in different directions, and a basis for vectors in space is a set of three nonzero vectors not all lying in the same plane.

The transition to analytic geometry in $\mathbf{R}^2$ and $\mathbf{R}^3$ is made by introducing a basis in each of these spaces and then building a coordinate system on these. The algebraic representation of vectors is always relative to some basis. This can be stressed from the beginning by introducing the concept of basis as a set of generators of the space and the notion of decomposing (by linear combination) any vector in the space in terms of the basis vectors. The name "basis" is a very natural one, implying that the set has certain basic properties (especially in the case of a normalized basis) and that it plays a fundamental role in describing the whole space - this is the *completeness* property of a basis. These vectors form a "skeleton" on which everything in the space is anchored, they are indeed basic vectors. This leads to the question of the minimum number of vectors necessary to describe the whole space, and from this the notion of dimension emerges to make precise the intuitive idea of a plane as 2-dimensional and space as 3-dimensional.

The definition of coordinates of a vector can incorporate this: the coordinates of a vector $\mathbf{x}$ with respect to a basis $\{ \mathbf{u}, \mathbf{v} \}$, for $\mathbf{R}^2$, are the numbers a, b such that $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$. The coordinates of a vector in $\mathbf{R}^3$ are defined analogously. The property of the uniqueness of these coordinates is a subtle matter because there are many different choices of bases and a given vector has different coordinates with respect to each of these. It must be stressed that this uniqueness is only with respect to a given basis. The canonical bases for $\mathbf{R}^2$ and $\mathbf{R}^3$ can be introduced as particularly easy and natural to work with, one then develops the notation

$$\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ and } \mathbf{y} = \begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

as a shorthand for representing the vectors $\mathbf{x}$ and $\mathbf{y}$ in terms of their coordinates relative to these special bases. Then the geometric operations of stretching (or shrinking) vectors, rotation of vectors by 180^o, and vector addition are formulated algebraically and properties of vector algebra are developed.

All of these considerations pave the way for the transition to the space $\mathbf{R}^n$. This transition is marked by a generalization of all these geometric ideas and a carrying-over of geometric language into a non-spatial setting: the elements of $\mathbf{R}^n$ are defined by construction as n-tuples denoted by $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ with the tacit understanding that the numbers $x_1,\ldots,x_n$ are the coordinates relative to the "canonical basis" for $\mathbf{R}^n$; a basis is seen as a "generalized coordinate system" and its "vectors" "generate" the whole space; linear independence can be seen as a property of a basis and must be reformulated more precisely in algebraic terms (using special cases of statements 1 or 2). We speak of $\mathbf{R}^n$ as "n-dimensional space" and we begin to see that these abstractions offer a powerful kind of reduction in that they allow us to describe some fundamental properties of the whole space by studying only small finite subsets of elements.

Finally, the passage from $\mathbf{R}^n$ to general vector spaces is a move towards the structuralization of the notion of space, and all these ideas are made fully abstract and formal. The elements of the space are defined not by construction, but rather by axioms and properties. All characterizations, in their final form, are algebraic, but the language of space geometry continues to be used even though references to geometry reflect an attempt to appeal to our intuitions and are there only to help ground abstract notions. The space $\mathbf{R}^n$ is seen as only one particular example of a general vector space. Students encounter other familiar objects such as polynomials and matrices which must now be viewed from a completely different perspective. The concept of linear independence is seen very much as a property of a minimal set of generators of a vector space.

The rationale for a geometric introduction

This order of introducing and developing the concepts, and related notions, tries to follow the logic of a historical development of linear algebra. Thus we may refer to it as a genetic-geometric approach in order to distinguish it from other genetic approaches which take non-geometric settings as their starting points. For instance, a genetic-algebraic approach might begin with the study of systems of linear equations by introducing the notion of dependent equations, as suggested by the historical analysis. In this context it is natural to introduce the notion of the rank of a (coefficient) matrix and to develop its relation to the number of solutions of the system and the number of 'redundant' equations. The transition to $\mathbf{R}^n$ can then be made by representing each equation as an $(n+1)$-tuple according to the mapping $(a_1x_1+a_2x_2+\ldots+a_nx_n= b) \mapsto (a_1, a_2,\ldots,a_n, b)$. In any case this kind of chronological recapitulation of the historical development of concepts is regarded as good in view of the belief that psychogenesis recapitulates phylogenesis.

A genetic-geometric approach builds on concrete visual understanding and, driven by successive generalizations, moves towards abstract understanding. Each new stage in this development builds on understanding acquired in the preceding stages and is marked by a further generalization of previously encountered notions. Thus, it serves to both reinforce and expand the concepts.

Finally, a genetic approach can offer a dynamic perspective of the concepts and can help one understand the significance of, and reasons for having, the very formal and abstract definition of linear independence/dependence.

$$\begin{matrix}\text{synthetic}\\ \text{geometry}\end{matrix} \xrightarrow{1} \begin{matrix}\text{analytic}\\ \text{geometry}\\ \text{in } \Re^2 \,\&\, \Re^3\end{matrix} \xrightarrow{2} \begin{matrix}n-\text{dimensional}\\ \text{geometry, } \Re^n\end{matrix} \xrightarrow{3} \begin{matrix}\text{general}\\ \text{vector spaces}\end{matrix}$$

Figure 2.2. An overview of the genetic-geometric approach.

Possible sources of difficulties

As shown in Figure 2.2 the genetic-geometric approach is marked by transitions between different settings, each of these transitions involve making generalizations. The mental act of generalizing can be difficult for students partly because it is an operation which acts on *something*, be it a concept or a problem. Thus, to be able to make a generalization requires that one first identify that *something* as an object (Sierpinska, 1994, pp. 59). For instance, if one doesn't understand the algebraic formulation of the concept of linear dependence in $\mathbf{R}^2$ and $\mathbf{R}^3$, than how can one understand it in $\mathbf{R}^n$? Furthermore, there are different types of generalizations to be made, some causing more cognitive strain than others. For example, generalizing the vector sum and scalar multiples from $\mathbf{R}^3$ to $\mathbf{R}^n$ consists essentially of applying the same techniques to each coordinate in a broader system. Algebraically, this process expands an already existing schema (operations on 3-tuples) without reconstructing it. Geometrically, the process involves a modification of geometric ideas in three dimensional space to a mental image of n-dimensional space. This type of generalization requires a reconstruction of an existing schema in order to widen its applicability range. Harel and Tall refer to these two as *expansive* and *reconstructive* generalizations, respectively (Harel & Tall, 1991).

One resource needed in abundance to effectively implement a curriculum based on a genetic approach is time - there must be enough time for the concepts introduced at each stage to become firmly grounded so that there is something to generalize. Otherwise, students will surely have serious difficulties in understanding the concepts. Let us now mention some more specific difficulties:

(a) There are difficulties in the transitions 2 and 3 (referring to Fig.2.2) from the geometric spaces to n-dimensional space and to general vector spaces. These difficulties can be related to the use of geometric language to describe these spaces whose characterization is completely algebraic. There is a tacit understanding that the geometric language is used for the purpose of analogy only, yet it causes us to revert to geometric thinking when trying to understand a concept and may instead hinder our understanding.

(b) There are difficulties in transition 3 related to the drastically different ways of defining the elements of these spaces. Consider, as an example, the zero vector; in $\mathbf{R}^n$ this vector is defined as the n-tuple $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, but in a general vector space $\mathbf{V}$ the zero vector is defined as the neutral element. i.e. that element, denoted $\mathbf{0}$, having the property that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u} \in \mathbf{V}$. The previous identification of the zero vector as an n-tuple of zeros is so deeply ingrained in students that at first it is almost impossible for them to conceive of it being defined differently. An excerpt of a student (C) learning linear algebra with the help of a tutor (T) illustrates these points:

> *C first reads the definition of a vector space.*
> T: What is a zero vector?
> C: One that is composed of zeros only, that's what it is!
> T: No, listen this is a general theory and a zero vector is not necessarily an n-tuple of zeros. $\mathbf{R}^n$ is but a particular example of a vector space. . . . even functions can be vectors, do you understand?
> C: You mean you can write a function in the form of a vector?
> T: No, no, this is an arbitrary thing . . . you can add and multiply by scalars . . . and they satisfy these 10 conditions, this is a vector space, see that?
> C: So what?

T: So a zero does not have to be an n-tuple of zeros.

.

.

C: How come they cannot have all zeros? That's incredible! We are speaking about the zero vector all the time . . . Explain it to me, how can there be a zero that is not composed of zeros? How can such a thing be at all possible? . . . How would it look like? Give me an example. If you can't explain this to me then I don't think I'll be able to go on.
T: A function that assigns zero to any number, is this "composed of zeros"? Can you say that?
T then defines the operations in the space of functions and claims that the zero function is the zero vector of this space
C: So what do you get? A vector whose elements are not (at) all zeros, right? Is such a vector a zero vector?
T: This is the zero of this space.
C: This is the zero of this space, but is the zero vector of this space the zero vector? Is this something I have to accept into my mind? . . .(sighs) . . . I have just discovered that my world has been built as a house of cards, and someone has just blown it away . . . I must say that this has really put me down. Realizing that all that you've been taught and you thought you have understood . . .. (Sierpinska, 1995c).

This is a case of "good understanding" in $\mathbf{R}^n$ acting as a serious obstacle to understanding the next level of abstraction. When student C finally understood what the zero vector is in an abstract vector space, the realization came in a flash and was induced by the tutor's words "zero as a number, zero as a property". The student latched on to the phrase "zero as a property" and this helped him distinguish the zero vector from zero as a number or zero as an n-tuple. So it seems that the continued use of geometric language in a non-spatial setting was causing a great deal of confusion. Indeed, student C subsequently went into a long tirade about how this confusion was caused by the tutor's "use of jargon instead of precise mathematical language".

(c) In $\mathbf{R}^n$, for $n > 3$, the concepts can no longer be visualized spatially and must be formulated algebraically. Thus, there is no longer any recourse to geometric intuition and the concepts may suddenly appear to be distant from the original (geometric) notions. This can leave one feeling that the concepts are no longer grounded in reality; things can appear very abstract and not real.

In another session, tutor T and student C are discussing the space $\mathbf{R}^n$ and hyperplanes. At the end of the session student C says:

> " ... if I speak of an n-dimensional space that has more than 4 dimension then it is like speaking about things that either do not exist at all or we don't know if they exist. . . . It seems to me that when I speak of the n-dimensional space I start speaking of things that are completely abstract. This becomes, I don't know, a game, it is not serious anymore, it has no practical use ... "

(d) An understanding of the independence/dependence of the set {**0**} and the notion of dimension zero cannot make appeals to geometry because the geometric formulation of linear independence/dependence excludes the case of the zero vector.

Introducing the concepts of linear independence/dependence through examples and problems[2]

Yet another approach to introducing the concepts is exemplified in Fletcher's book *Linear Algebra through its applications*. Here, linear independence is introduced in a completely informal way through use of the term in context. The first chapter is entitled '*Linear Spaces*' and the first two sections introduce the notion of closure under linear combinations by exploring what results from adding arithmetic progressions term by term and multiplying each term by the same scalar, and then doing the same with magic squares. The definition of a linear space is given informally:

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2 Here we mean examples and problems of a non-standard kind in linear algebra.

" any mathematical objects which may be (i) added together and (ii) multiplied by scalars, may be called vectors. The set of vectors together with the two operations is called a vector space or linear space" (Fletcher, pp.5).

The concepts of basis and dimension are introduced informally in the context of arithmetic progressions:

" any arithmetical progression may be expressed in terms of two special progressions,

$$\mathbf{x} = 1, 1, 1, 1, \ldots$$

and

$$\mathbf{y} = 0, 1, 2, 3, \ldots$$

[...] Thus the sequence

6, 11, 16, 21, 26, . . .

is expressible as $6\mathbf{x} + 5\mathbf{y}$; that is it is expressible linearly in terms of **x** and **y**.

The two **x** and **y** may be called a *basis* for the space.

[...] Since any other arithmetical progression may be expressed in terms of a special *two*, we say that the space of arithmetical progressions is *two-dimensional*." (pp. 2)

In the context of semi-magic squares the idea of a minimal set of generators is approached by asking thought - provoking questions such as: "Can we find some especially simple semi-magic squares with a view to using them as base vectors?". To answer the question a set of seven semi-magic squares, {**a**, **b**, **c**, **d**, **e**, **f**, **g**}, is given and then he poses the question: "How long is it before we start finding that some are not 'really new' because they can be expressed linearly in terms of squares which we already have?" He remarks that **a** = **d** + **f** + **c** = **e** + **g** + **b** from inspection and then says; "This shows that **a** is redundant and that any one of the remaining six can be expressed in terms of the other five." This then leads to more thought-provoking questions: "Can we express *any* other semi-magic squares in terms of the ones we have found so far?" and "Are there any further relations which lead us to say that fewer than five can be taken as basic, because one of these can be expressed in terms of the others?" (pp.4). In this way the idea of redundancy prepares the reader for the concept of linear independence.

The word *independent* surfaces for the first time when referring to a given set of semi-magic squares, {**u**, **v**, **w**, **x**, **y**}, he remarks: "It is easy to see that **y** = **u** + **v** + **w** + **x**, so **y** is not an independent semi-magic square if we have already selected **u**, **v**, **w**, and **x**." (pp.5). The reader is then invited to undertake his/her own investigations by attempting the exercises which ask questions like "Can any of **a**, **u**, **v**, **w**, and **x** be expressed linearly in terms of the others?" and "Show that the space of magic squares is of dimension three."

Some thirty five pages later, at the very end of chapter one, the formal definition of a basis is given but nowhere is the notion of linear independence made precise.

The characterization of a linear space as a set closed under linear combinations is illustrated throughout the rest of the chapter with a multitude of examples which include polynomials, sequences, solutions to linear differential equations, binary codes, and networks. This veritable tour of linearity is preceded by an interesting comment:

> " 'What *is* a vector?' is not really a proper question. The proper question is 'How does a vector behave?' Mathematical definitions nowadays answer the second type of question far more often than answer the first." (Fletcher, pp.5)

This statement reflects some similarity between Fletcher's approach and the structural approach, both of which ask not what a vector *is* but rather how a vector *behaves*. The similarity ends here however, because the structural approach answers the question explicitly by specifying a set of axioms, whereas Fletcher attempts to show how vectors behave in different contexts.

The rationale

The purpose of such a presentation is not to build a theory, but rather to build up a set of tools by introducing concepts as useful for solving certain problems:

> " The aim is to approach the theories by way of the problems which the theories can solve, and by way of the situations which they illumine . . . our concern is not to prove theorems but to discuss situations from which the theoretical ideas emerge." (Fletcher, preface)

This usefulness and the contexts in which they are presented is what gives meaning to the concepts. The words *dimension* and *independent* are not defined precisely, instead their meaning is given by their use in expressions, sentences, and contexts of the problems (Wittgenstein, 1965). Thus, the reader is expected to develop an intuitive understanding of the concepts.

The reader is exposed to a wide variety of vectors and vector spaces, the author hopes that this will bring one closer to the concept of a general vector space by impressing upon him/her that the property of closure under linear combinations takes primacy over the vectors themselves. Also, one will have many examples to draw on, and seeing the same basic notions surface again and again in different settings serves to reinforce their importance. All of this is meant to help lay the groundwork for a more formal treatment of linear algebra.

Possible sources of difficulty for students

Fletcher's approach to linear algebra is undeniably an uncommon one. Following this approach, one could face several hardships:

(a) Perhaps one of its most distinctive features is the lack of formal definitions of the basic concepts which emerge repeatedly throughout the book. But this can be rather unnerving because nothing is precisely defined and the notions seem fuzzy. All the intellectual work of determining characteristic properties of concepts is left to students, this may be overwhelming for the novice. If the student is presented with, say, the formal definition of linear independence at a later time it may appear to be completely unfamiliar and to have no ties to previous intuitive understandings.

(b) Concepts are introduced through non-trivial applications, some of which require solving large systems of equations. There is the risk of getting bogged down by technical difficulties, and these may in the end overshadow the importance of the concepts.

(c) The exercises often consist of problems related to those through which the concepts were introduced, and they make challenging demands on the student to construct proofs and to make generalizations. This can be especially difficult and time consuming for a student having only an intuitive idea of the concepts and no theoretical tools (i.e. definitions and theorems) that could facilitate the solution of such problems.

DISCUSSION

Our analysis outlines four different approaches to introducing the concept of linear independence/dependence and some related notions. It describes the rationale underlying each approach and lists difficulties which we expect students will experience as a result of following them. But we can say more in the way of comparing these approaches, there are several features which distinguish them, and there are also certain commonalties between them. Our first aim here is to bring these to light.

Although the concepts of linear independence and dependence may be regarded as two sides of the same coin, by virtue of their being the logical negation of each other, the approaches given by statements 1 and 2 reflect the usefulness of regarding them as distinct. Recall that statement 1 presents independence as the primary notion and dependence as secondary, and statement 2 does the opposite. So to begin with these two approaches differ in the primacy that they give to either concept.

The historical analysis shows that the notion of dependence is genetically more primitive, but one can argue that linear independence is more fundamental because it is an abstraction of the primal notion of two non-parallel lines (vectors) in the plane as necessary for describing all vectors in the plane - an idea which underlies the most basic notions of linear algebra, those of basis and dimension. This point of view seems to be in accordance with a genetic-geometric perspective.

Conceptually, linear dependence is related to the notion of generators of a space, and through the idea of redundancy of generators it ties into the notion of independence. From an algebraic point of view it seems natural to speak first of a set of generators and then of a minimal such set, and this makes linear dependence the more fundamental notion. Thus, how the concepts are distinguished can depend very much on one's point of view.

We also see that statement 1 offers a practical general test of linear independence/dependence, whereas statement 2 does not. On the other hand statement 2 has a simple logical construction and uses language in a natural way, whereas statement 1 does not. So these two approaches also differ in their concern for practicality and naturalness, and this reflects a kind of tension in mathematics education between the desire to have definitions that are practical and definitions that are natural. Because of this it seems that either approach would have to make use of both statements; if we begin with statement 2 as the definition of linear dependence, then statement 1 would have to be introduced (for its practical purpose) as a theorem giving a general test of dependence/independence. Combining this reason with the fact that the relation between the two statements is not so obvious as to be trivial provides a strong rationale for proving this theorem. Actually, the common proof of this theorem is direct and constructive, it is a *proof that explains* (Hanna, 1989) and so could potentially serve as students' first experience in building links between different ways of seeing the concepts of linear dependence/independence. On the other hand, if we start with statement 1 to define the concepts then we have a general test of dependence/independence. Statement 2 could then be introduced as an equivalent, and perhaps more natural, characterization of dependence. One would then look at the proof as a way to understand this equivalence.

On page 33 we pointed out some features of the first approach to introducing the concepts (statement 1) that make it consistent with a structuralist perspective of mathematics. But approach 2 may also be identified as a structural one because it too introduces the concepts through a formal definition - one which shares some of the same features as statement 1; it is also a terse statement and so offers compactness and economy of expression, and in speaking of dependence as a property of a set of vectors it also offers consistency and unification of language in a set-theoretic approach to building the theory. Although we have described only how the concepts may be introduced, a structural approach then typically develops the concepts in an order that follows the logic of the theory. From the definitions other statements follow by logical inference, and from these follow yet more results. These results are often interspersed with illustrative examples meant to help concretize the concepts. In this approach the focus is not only on the collection of results but also on how they are logically related.

These results together with the relationships between them are to be coordinated, synthesized and integrated to form a whole. For students, however, this holistic picture is seldom the one that emerges. Rather, as the number of results becomes numerous the chains of logical inference between them become too long to manage and they often feel lost in a sea of meaningless and disconnected statements. It seems that the structural approach to the teaching of a theory presupposes some belief that learning consists largely of the accumulation of knowledge. Indeed, the very first step towards attaining any sort of holistic picture would have to be the acquisition of the facts, these then have to be transformed into knowledge. For the learner the construction of this knowledge is bound up with making the logical connections between the facts; because a structural approach makes few appeals to external representations, it is these connections that first give meaning to the facts - their significance is relative to the existing body of facts (see Connectionist models). For instance, understanding the significance of statement 2 as an equivalent characterization of linear dependence may well be the same as understanding its relation to statement 1, this is especially true given the constructive nature of the proof of statement 2 (as a theorem).

From the teacher's perspective statements 1 and 2 are good starting points; with an expert's hindsight he/she already sees the whole picture and knows that these are basic statements that will generate a series of results which build the theory. From the learner's perspective, which lacks such a global view, however, statements 1 or 2 can seem arbitrary and their significance is not easily grasped. This last point illustrates an important distinction between the structural and genetic-geometric approaches. In the latter, the notion of independence is presented first in the concrete setting of geometry. Then, in analytic geometry, the concept is seen again when the definition is reformulated algebraically. This definition is then generalized to n-dimensional geometry. By the time the student encounters statement 1 in the setting of abstract vector spaces it has some significance and can be seen as the end result of a long and gradual process of generalization. The genetic-geometric approach assumes that learning is more of a *sequential* process (Piaget & Garcia, 1989) than a cumulative one. In this model each new formulation of a concept must be integrated into the individual's existing mental schema, sometimes this requires no change in the schema (assimilation), but usually it requires some reorganization, if not a reconstruction, of these schemas. Fletcher's book also tries to offset this lack of meaning and significance which can result from trying to learn the concept of linear independence from the formal definition. His strategy is to introduce the concept, and concept name, through its use in solving a certain problem - the intent is to make it immediately meaningful and significant. This book covers all the basic notions of linear algebra in this same way, but because it falls short of making most of these precise it may not be appropriate as a textbook. It is, however, a rich source of non-trivial problems with which to motivate and introduce the fundamental concepts.

We have described three classes of approaches to introducing the concept of linear independence/dependence. The book by Fletcher exemplifies an applied approach, and Griffel's book gives a more or less genetic-geometric[3] development of the concepts. There are many books which exemplify the structural approach and we should provide an example of one. Paul Halmos' book, *Finite-Dimensional Vector Spaces*, is a paradigmatic example of a structural approach to linear algebra; here the final characterization of the notion of dimension is as an 'isomorphism invariant', i.e. as a property of vector spaces which is invariant under one-to one linear transformations on these spaces - a highly structural point of view (Halmos, 1987, pp.14).

Each approach has something to offer the learner, and the reality of educational practice is that most approaches borrow features from each of these classes we've described. Thus, it is difficult to make definitive statements about the way linear algebra is taught in North America. Still, the basic approach favored by us is the genetic-geometric one because of the focus it places on the notion of linear independence by building concepts around it, and because at every step of the way it attempts to strengthen ties between linear independence and the idea of a coordinate system.

The goal of this chapter was to identify possible different ways of understanding the concepts of linear independence/dependence

3 Between analytic geometry and the space $\mathbf{R}^n$ there is a detour into the world of matrices.

through a consideration of several ways of introducing the concepts.

To summarize this work we list these different ways of "seeing" the concepts:

- One can understand the concepts by associating them with the test in the formal definition: i.e. to say that a set of vectors $\{\mathbf{v}_1,...,\mathbf{v}_k\}$ is linearly independent means that the homogeneous equation $c_1\mathbf{v}_1+...+c_k\mathbf{v}_k= \mathbf{0}$ has only the zero solution. To say that the set is linearly dependent means that this equation has other solutions. An operational understanding can result from seeing this as a practical test, this require: i) understanding the conditional statements in the definition, and ii) understanding how the above vector equation can be represented by a system of linear equations.

- One can understand linear dependence as a linear relation between vectors in a set (statement 2). This can be seen as a generalization of a scalar multiple of a vector.

- In the context of systems of linear equations one can understand linear dependence in terms of redundant equations, rank, and the number of solutions of a system.

- One can have a geometric understanding of the concepts of linear independence/dependence as the (non)co-linearity of two vectors in a plane and the (non)co-planarity of three vectors in space. One may then see the formal definitions (statements 1 & 2) as algebraic formulations of these geometric notions, and linear independence/dependence in general as abstractions of these.

- One can see linear independence as the property of a set of generators which makes it a basis - a minimal generating set (or a maximal linearly independent set). A generating set which is not minimal is linearly dependent, it contains redundant information in the sense that some vector(s) in the set are expressible as linear combinations of the others. A basis can be seen as a generalized coordinate system.

- One can have an intuitive understanding of the notions of linear independence/dependence by seeing how they arise naturally in certain applications and how they may help to solve certain problems.

All of these ways of "seeing" the concepts constitute an awful lot of knowledge and we hope to have given the reader some sense of the difficulties that the learner can face when trying to construct these ways of knowing.

Having these different understandings would indeed be an accomplishment, but we believe that the ability to coordinate and synthesize all of this knowledge is the mark of a deeper understanding of the concepts. We should not underestimate how difficult it can be for students to attain such an understanding. Doing so involves not only a certain degree of motivation and tenacity, but it also requires that one engage in particular kinds of activities that engender the mental actions which can lead to it. In the next chapter it will become apparent what some of these activities can be.

CHAPTER III

EXAMPLES OF STUDENTS' UNDERSTANDINGS OF, AND DIFFICULTIES WITH, THE CONCEPTS OF LINEAR INDEPENDENCE/DEPENDENCE

The last chapter dealt with different understandings of the concepts of linear independence/dependence that could result from different ways of introducing them. Although we documented some difficulties that students experienced while trying to understand these concepts, our analysis was of a more theoretical nature - we scrutinized each way of introducing the concepts and then based on this we predicted what difficulties students might experience and what understandings of the concepts they might arrive at.

The aim of the present chapter is to supplement the work of chapter two with examples of students' actual difficulties and understandings of the concepts. We have no intention of providing a complete empirical counterpart of the previous chapter here, i.e. we will not try the various approaches already mentioned on a test group of students and then measure their understanding of the concepts relative to a control group. Our research is simply not yet at this stage, and we would like to emphasize that the analysis in chapter two in particular *is* the result of our research - one which included much reflection on the nature of the concepts and their historical evolution, on the way that they are presented and developed in textbooks, and on our own experiences learning and teaching linear algebra.

Some of the examples considered here confirm the predicted difficulties (and understandings), while others bring to light difficulties which were unforeseen and which appear to be distantly related to the concepts at hand. Our ultimate goal is to get a sense of some of the actual understandings that students can have of the concepts and how they arrive at these understandings. The observations we make here will hopefully enrich our perspective of the notions of linear independence/dependence.

Our research methodology is perhaps unique in that it consists, in part, of what we call 'continuous' long-term observations of students learning elementary linear algebra from a textbook with the aid of a tutor. These observations are continuous in the sense that we could observe the learning process every step of the way; the novice students studied the subject for five hours every week during several sessions per week and spanning a period of 13 weeks, they were not permitted to take the textbook home in between learning sessions and were asked not to study linear algebra during this time. The tutors (two graduate students and one mathematics professor) also did not prepare for these sessions, their role was not to give lectures or lessons but rather to guide the students in understanding the material presented in the textbooks by answering and asking questions and by discussing ideas with them. The tutors are referred to as T1, T2, and T3, and the students as S, P, and C. Since gender is not a factor in our investigations we refer to all participants in the masculine. Two different textbooks were used; *Problems in Mathematics. A textbook for Economics Studies* by Bazanska et al, and *Linear*

Algebra and it's Applications by David C. Lay. We will sometimes refer to these as Text 1 and Text 2, respectively. The interaction within each of the triads S-T1-Text 2, P-T2-Text 2, and C-T3-Text 1 was audio recorded and some of it was subsequently transcribed. This was our main source of data, but other data were collected from questionnaires and tests given to students enrolled in a first year linear algebra course taught at Concordia University.

About Text 2

We mentioned in the previous chapter that many linear algebra textbooks designed for the North American audience introduce the concepts of linear independence/dependence through formal definitions (statement 1). Text 2 is no exception and the concepts are motivated by the following opening remark:

> The homogeneous equations of section 2.3 can be studied from a different perspective by writing them as vector equations. In this way the focus shifts from the unknown solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations [...] This equation has a trivial solution, of course, [...] As in section 2.3, the main issue is whether the trivial solution is the *only one.* (Lay, pp. 63)

To help the reader put this comment into context we begin by giving a brief overview of things that lead up to it. In Text 2 the section on linear independence is preceded by an entire chapter devoted to the study of systems of linear equations and the first three sections of chapter two deal with vector and matrix equations. In the first chapter emphasis is placed on the Gaussian reduction technique applied to the augmented matrix of a linear system, and in describing echelon forms of these matrices the language of *pivot positions* and *pivot columns* is used. The author also speaks of *basic variables* and *free variables* to describe the general solution of a system. The existence of a solution of a system is reduced to a test on an echelon form of the augmented matrix: "A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column ..." (pp. 22). The uniqueness, or non-uniqueness, of a solution is stated in terms of the non-existence, or existence, of free variables. In the second chapter the sections leading up to linear independence emphasize the three different representations of a linear system as:

1) an augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n | \mathbf{b}]$

2) a matrix equation $A\mathbf{x} = \mathbf{b}$

3) a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$

Solutions to systems in three variables are interpreted geometrically, and the question of the existence of nontrivial solutions for homogeneous systems is reduced to the following test: " ... $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable" (pp. 58). The student arrives at the section on linear independence well prepared to solve homogeneous systems and to say, by glancing at an echelon form of the coefficient matrix, whether there exist nontrivial solutions or not. The understanding of the concepts of linear independence/dependence, as presented in this textbook, depends very much on this knowledge and skill.

Student P's understanding

Below is an excerpt from the session where student P first encounters the concepts of linear independence/dependence. The concepts are introduced through a formal definition which is a variation of statement 1:

> A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation
>
> $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$$
>
> has only the trivial solution. The set $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ is said to be **linearly dependent** if there exist weights $c_1,\ldots,c_k$, not all zero, such that
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$
>
> (Lay, pp. 64)

This definition poses no problems for student P, he reads it and finds it to be a "straightforward" statement:

2. P: Yes, linear independence, (*reads opening remark and definition*) [...] O.k., so if I understand that definition properly it's linearly independent only if the only solution is a trivial solution.
3. T: Right.
4. P: And if it's not then it's linearly dependent, right?
5. T: Yeah.
6. P: So that's pretty straight forward.

The definition is stated in language that student P is familiar with, and in the setting of $\mathbf{R}^n$ the concepts of linear independence/dependence are reduced to the nonexistence/existence of nontrivial solutions for a homogeneous vector equation, or, equivalently, a homogeneous linear system. Student P had no difficulty solving such systems and so he felt quite comfortable with the concepts of linear independence and dependence as they were presented. At this point the concepts were not really new for him because they amounted to little more than giving a special name to a set of vectors satisfying a certain property which was familiar to him. Indeed, the opening remark suggests that the concepts of linear independence/dependence simply offer a new perspective of familiar ideas, and this may well reflect the author's intention of unifying notions and of ensuring a smooth learning experience.

Student P was subsequently able to understand how this definition could be used in concrete settings, i.e. to determine whether a given set of vectors in $\mathbf{R}^n$ was linearly independent or dependent. For him the concept of dependence was very much equivalent to the existence of free variables in the homogeneous system because this is in turn equivalent to the existence of nontrivial solutions, this understanding was heavily reinforced by the textbook and the tutor T2. Following the definition is an example which asks the reader to determine whether a set, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, of given vectors in $\mathbf{R}^3$ is linearly independent. Student P reads the solution which consists of reducing an augmented matrix to an echelon form:

16. P: O.K.. (*reads*) " Example 1 [...] Clearly, x_1 and x_2 are basic variables and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (1) (*i.e. the homogeneous vector equation*). Hence v_1, v_2, v_3 are linearly dependent and not linearly independent."
17. T: Do you understand how the solution works?
18. P: Yeah well they determine through the row operations that there was a free variable
19. T: Yeah
20. P: so since there's a free variable there's infinitely many solutions,
21. T: Right
22. P: whatever you give to the value of x_3 it's gonna satisfy the equations.
23. T: So is it clear that you can just reduce the matrix to echelon form and use that theorem that says a homogeneous system has non-trivial solutions if and only if there's at least one free variable?
24. P: Hmm hmm
25. T: So you can see right away that there are non-trivial solutions.
26. P: Yeah o.k.
27. T: So the vectors are linearly dependent.

For the student there is nothing new here and it is the tutor who tries not only to summarize a method which already seems obvious to him (student P) but also to emphasize the new naming of a familiar property. Student P went on to solve similar problems on his own without difficulty, and each time he equated the concepts of linear independence/dependence with the nonexistence/existence of free variables in the homogeneous system - something which can be determined, in general, after a series of calculations or operations are performed on the augmented matrix. Accordingly, we will refer to this type of understanding as an *operational* understanding of the concepts (Sfard, 1991). This type of understanding is due to a fixation on the operational value of the definition. In Text 1 both the material leading up to the definition and the examples that follow it reinforce this understanding of the concepts. An operational understanding is obviously an important one, but it can be difficult to break out of. In abstract settings, which require more thinking than calculating, this type of understanding may actually act as an obstacle. As an example we consider student P trying to understand the proof of the fact that any set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent. Text 2 labels this result as Theorem 7 and gives the following proof: "By renumbering the vectors, we may suppose that $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1+0\mathbf{v}_2+\ldots+0\mathbf{v}_p= \mathbf{0}$ shows that S is linearly dependent."

348. P: [...] Why do they go "$1\mathbf{v}_1+0\mathbf{v}_2$ and $+0\mathbf{v}_p$" ? O.K..
349. T: How would you uhh, why don't you try thinking about how you would show that a set of p vectors is linearly dependent? What would you have to show? Say you were to go by the definition.
350. P: That one is a linear combination of the uhh
351. T: That's a theorem actually, that's not the definition it's a result of the definition.
352. P: O.K..
353. T: Let's say you were to apply the definition of linear independence.
354. P: O.K..

355. T: How would you try to prove this statement?
356. P: That uhh, uhh that the system has more than - uhh has a nontrivial solution.
357. T: Right.
358. P: Then it would be dependent.
359. T: Right. [...]

In trying to understand the proof, student P seems perplexed by the vector equation $1\mathbf{v}_1+0\mathbf{v}_2+\ldots+0\mathbf{v}_p=\mathbf{0}$ (line 348). At first he doesn't see the purpose of it. The tutor tries to help by asking student P how he would show that the set is dependent by using the definition (lines 349-355). Student P's first instinct (line 350) is to use the equivalent characterization of linear dependence (i.e. one of the vectors is a linear combination of the others), his response suggests that this is a more natural formulation of dependence for him and that he may not be thinking operationally. This natural tendency, however, is quickly thwarted by the tutor who points out that the student is using a theorem and not the definition, he then insists that the student use the definition. Thus, student P is forced into using a definition which causes him to think operationally, this is apparent in line 356 where he speaks of the *system* having a nontrivial solution. Here, however, his operational understanding of dependence doesn't help him because the vectors are arbitrary and so there is no system of equations to solve. This is a kind of impasse for the student because his understanding of the proof requires that he use the definition in a very different way; his first understanding of dependence which focused on the operational value of the definition may be obscuring the conditional nature of the statement "if there exist weights $c_1,\ldots,c_k$, not all zero, such that $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k=\mathbf{0}$".

Student P is also confused by the seemingly arbitrary choice of $\mathbf{v}_1$ as the zero vector in the set, and when the tutor questions him about this he decides to read the definition of linear independence/dependence which he thinks may explicitly account for the case of a set containing the zero vector (lines 363-370). In questioning student P's motives for believing this we were led to the textbook; there, shortly after the definition of linear independence/dependence, we find the justification of the claim that the singleton set $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$. The argument contains the equation $x_1\mathbf{v} = \mathbf{0}$, and because student P is reflecting on the equation $1\mathbf{v}_1 = \mathbf{0}$ (line 371 below) we believe that it was this claim and not the definition that he was thinking of. When this fails to help he thinks again about the significance of the equation $1\mathbf{v}_1+0\mathbf{v}_2+\ldots+0\mathbf{v}_p = \mathbf{0}$ and this leads to a confusing interchange (lines 371-382) brought about by the fact that the tutor is speaking about coefficients whereas the student is referring to vectors.

363. T: [...] If a set contains the zero vector, then we have to show that it's linearly dependent, so we have to assume that one of these is the zero vector.
364. P: Hmm hmm, then we do it here, we say let $\mathbf{v}_1 = \mathbf{0}$.
365. T: Do you think there's any problem with assuming that the first one is the zero vector? Do you think it matters which one we choose as the zero vector?
366. P: (*silence*) Hmm, I'm going to have to go back here
367. T: Sure, although you were on the right track before when you said that we'd have to show that there is a nontrivial solution
368. P: Hmm hmm
369. T: to that vector or matrix equation.
370. P: Yeah but I'm sure that in the definition, well not sure but I'm thinking probably in the definition that they're saying that if there is a zero vector then you can assume something. That's what I'm looking to see, but o.k. (*re-reads the proof*) o.k. $1\mathbf{v}_1+0\mathbf{v}_2+\ldots+0\mathbf{v}_p$ is equal to zero, hmm hmm (*thinks*).
371. P: I know that $1\mathbf{v}_1 = \mathbf{0}$
372. T: So is there a nontrivial solution to that equation?
373. P: No
374. T: The system as it's shown here has no unknowns now
375. P: That's right
376. T: Are they all zero? (*referring to the coefficients*)
377. P: Hmm hmm

378. T: They are?
379. P: Well $1\mathbf{v}_1$ is still the zero vector
380. T: No not zero vector, not zero vector
381. P: $\mathbf{v}_1$ which is equal to zero
382. T: Uhh it is zero, excuse me (*both laugh*). Yeah o.k., but I mean - let's look at the definition again

The tutor then decides to take more control and on his suggestion student P again reads the definition of linear independence. Then, by using a sequence of very specific questions[1] (lines 384-401) the tutor manages to help the student understand just how the equation $1\mathbf{v}_1+0\mathbf{v}_2+\ldots+0\mathbf{v}_p=\mathbf{0}$ implies the dependence of the set:

383. P: (*reads*) " A set of vectors $\mathbf{v}_1$ through to $\mathbf{v}_k$ in $\mathbf{R}^n$ is said to be linearly independent if the vector equation $x_1\mathbf{v}_1+\ldots+x_k\mathbf{v}_k=\mathbf{0}$ has only the trivial solution."
384. T: Let's stop there.
385. P: Hmm hmm
386. T: So once you have such an equation, to say that it has only the trivial solution to you means what for the x_1 through to x_k ?
387. P: They're all zero.
388. T: Right. Now, we have such an equation but are the x_i's all zero?
389. P: No.
390. T: So? The solution is nontrivial.
391. P: O.K.. (*thinks*) O.k. in other words
392. T: Hmm hmm
393. P: this x_1 is free
394. T: Uhh
395. P: because this is the zero vector
396. T: Sure, yeah
397. P: this x_1 is free
398. T: Yeah, it doesn't matter
399. P: what we use
400. T: Yeah
401. P: O.k.

Student P finally 'sees the light' when he says "O.k., in other words ... this x_1 is free". This language suggests that he is still thinking operationally and we see how persistent a mode of thinking this can be.

The section on linear independence in text 2 is confined to $\mathbf{R}^n$ and it includes the geometric interpretation of

[1] This is a typical example of what has been appropriately called the *funneling* pattern of interaction (Wood, 1992).

independence/dependence for n=2 and n=3. While there are some good exercises of the "justify or give a counterexample" type which really test the understanding of the material presented, the purpose of the section seems to be to provide the reader with the tools necessary to determine, perhaps by inspection, if a given set of vectors in $\mathbf{R}^n$ is independent or dependent. Indeed, the type of understanding required to follow the proofs of some of the theorems was different and seemed almost incompatible with the book's implicit agenda. Sometimes student P became so engrossed in trying to understand the details of a proof, on the tutor's insistence, that he would lose sight of the greater picture and almost forget what had been proven. This occurred with the above Theorem 7; shortly following it student P was trying to determine if a given set of vectors containing the zero vector was dependent but he only vaguely recalled "one theorem or definition when there was a zero vector". When the tutor suggested that he use Theorem 7, the student was utterly surprised and acted as if he had never seen the theorem before:

"Wow ! ... That's very simple, eh? ... (*laughs*) I'm always looking for the more complicated solution."

Questions concerning the significance or usefulness of the concepts never arise in this section of the textbook, nor were any posed by the tutor during the session. Accordingly, student P really had little reason to change his operational way of understanding. A more conceptual development of linear independence/dependence is given some one hundred and fifty pages later in the book when the notion of independence is revisited in the context of abstract vector spaces. However, student P never got that far and his operational understanding stayed with him because conditions which would have perturbed this way of thinking were not that prevalent.

This is not to say that student P was not inclined to think non-operationally about linear independence/dependence. In fact we've seen that quite the contrary is true and there were other instances, when he was not forced to use a definition which induced him to think operationally, in which student P displayed a much more conceptual understanding of linear dependence. This occurred mostly in proving situations (of which there were few) which demanded that student P reason and argue mathematically. In one such instance student P was asked to justify or give a counterexample to the following statement:

> If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbf{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

This was his response:

114. P: [...] Well that has to do with our Theorem 5, the Characterization theorem, where they're saying that vectors1. . the set vector1, vector2, vector3 are linearly dependent. So that means. . we can also say that at least two vectors in this set are comb. . are linear combinations of each other.
115. T: At least one.
116. P: At least one is a linear combination of the others.
117. T: Right.
118. P: It's written . . since they're telling us this by saying that they are . . the set is linearly dependent.
119. T: Right.
120. P: Now the new set. . vector. . is the linear combination in the set vector1, vector2, vector3, that same linear combination exists in the new set of vector1, vector2, vector3, and vector4.
121. T: mm . . mm
122. P: So it's true that that set is also linearly dependent.

Despite some initial confusion about the statement of Theorem 5 (this theorem is equivalent to our statement 2 of chapter two), the student correctly applies this result by noting that if the original set is linearly dependent than one of the vectors must be a linear combination of the others. He has an intuitive sense that this fact is not changed by adding a new vector to the set, so the enlarged set must also be linearly dependent. Although the argument is not formal, it is very clear and simple and the reasoning suggests that the association between dependence and linear combinations is very strong for student P.

The operational understanding is certainly not restricted to the concept of linear independence/dependence (or even linear algebra). We see evidence of it in students' attempts to answer questions such as 'define the notion of a basis' or 'define the notions of Eigenvector and Eigenvalue'. We give an example of one student's responses:

> "*Basis is found by solving a matrix ... the nonzero rows are the basis of the matrix*"
> "*Eigenvalues are the roots of the characteristic polynomial when the determinant is found from the matrix det(A-tI)*" and "*Eigenvectors are the eigenvalues subtracted from the diagonal of the matrix and then solving for the matrix will get you the eigenvectors*"

This is followed by a specific example in which the student actually calculates the eigenvalues of a matrix.

These excerpts suggest that the notions of basis, eigenvector, and eigenvalue as abstractly defined objects having certain properties are meaningless for the student. Instead, their status as objects is only by virtue of their being products of calculations - for the student this is really what gives meaning to the concepts.

Examples like these are not so unusual, in fact we believe that the operational mode of thinking is most common. But this is quite understandable if we ascribe to Piaget's view of the construction of knowledge as being intimately linked to performing transformations or mental operations on objects or concepts (Forman & Kuschner, 1977). We may also get a sense of why operational understanding is most common if we consider the mathematical backgrounds of students entering university and if we consider more metamathematical questions such as 'what is the role or purpose of a mathematics education in North American society'. We will not enter into such a discussion here, suffice it to say that stressing only the utilitarian value of mathematics may be "unpropitious for the flowering of mathematical vocations" (Dieudonné, 1992, pp. 9).

Student S's understanding

Student S followed text 2, so his introduction to linear independence/dependence also occurred through a formal definition, namely:

> A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation
>
> $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$$
>
> has only the trivial solution. The set $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ is said to be **linearly dependent** if there exist weights $c_1,\ldots,c_k$, not all zero, such that
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$
>
> (Lay, pp. 64)

Student S's first difficulties with this definition were related to the use of different letters to denote the scalars in the cases of independence and dependence:

1219. S: (*reads definition of linear independence/dependence and then re-reads certain parts for her own understanding*) Let me read this thing again (*re-turns to definition and reads first part of it*) I'm confusing the x's with the c's.
1220. T: They're all the same weights. I mean, they're weights, right?
1221. S: They're weights. (*re-reads the definition*) But they both end up equaling zero.
1222. T: Here is the trivial solution, but you can always have $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x}$ has (is) a non-trivial solution, right?
1223. S: Right.
1224. T: These are the cases where the weights are not zero. Where the parameters are not zero, right?
1225. S: What happened to my x's in this equation?
1226. T: It's the same, but they changed it so that - o.k., read here then (*turns back to opening paragraph*)
1227. S: (*re-reads opening paragraph*)
1228. T: Right?
1229. S: Yeah (*very doubtfully*)
1230. T: When can it have a nor trivial solution? When x_1, x_2, x_3 are not all zero. Right?
1231. S: So then you call them c?
1232. T: Yeah. No big deal, what you call them.

From the very beginning student S says that he is "confusing the x's with the c's" in the vector equation. The choice of letters to denote the scalars is certainly not central to the concept and the tutor downplays this distinction by saying "They're all the same weights. I mean, they're weights, right?". But the distinction seems important to the student and he notes that the vector equation is equal to the zero vector in both the case when the weights are denoted by x's and in the case when they are denoted by c's (line 1221). The tutor tries to focus his attention on the crux of the definition - the existence of non-trivial solutions to the homogeneous equation (line 1222-1224), and explains that dependence corresponds to the case when the "weights are not (all) zero. Where the parameters are not zero, right?" Student S's subsequent question, "What happened to my x's in this equation?", suggests that the important point for him is not the existence of non-trivial solutions, but rather that in the case of dependence different letters denote the scalars in the vector equation. The tutor cannot answer his question and decides that an example might help the student understand (line 1226). Accordingly, he then asks the student to look at an example found in the introduction to this section. The tutor tries to make the point that the vector equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$ has a non-trivial solution when x_1, x_2, x_3 are not (all) zero. Student S's response, "So then you call them c?", again suggests that for him the important part of the definition is that in the case of dependent vectors the scalars are to be denoted by subscripted letter c's.

So for student S we get the sense of his first 'understanding' of the concepts as they are presented in the definition; for independence x's are used in the vector equation and for dependence c's are used. One may well argue that this is no understanding at all, but rather an obstacle to understanding. In any case the student is very sensitive to this distinction in the notation and perhaps because the point of the definition (i.e. the existence of non-trivial solutions to a homogeneous equation) is not really new for him, this is what appears to distinguish the concepts of independence and dependence. This fixation on one aspect of the *form* of the definition is in effect obscuring its content, and thus acts as an obstacle. From the expert's point of view this kind of obstacle can be difficult to understand and we can see this in the behavior of tutor 2. In chapter two we offered a short analysis of the author's rationale for presenting the definition in this particular form and we concluded that it was designed to offset any ambiguity about the status of the scalars as the variables in the vector equation. However, in the case of student S this strategy may have actually induced another problem, one which may be viewed as an exaggerated response to the author's intent. Another difficulty of student S's, one of a more conceptual nature, was a very shaky understanding of the notion of a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Let us see some examples of this. In the excerpts below the discussion is about the column vectors of a matrix A:

108. S: [...] (*reads*) The columns of a matrix A are linearly independent if and only if the equation **Ax** equals **0** has *only* the trivial solution." (*re-reads*) So these will equal zero, "if and only if the equation **Ax** equals **0** has only the trivial solution." I hate the way that's worded. I feel like I'm going around in circles. (*re-reads*) These are the columns, right?
109. T: mm . . mm
110. S: And they're linearly independent if the equation **Ax** equals **0**. So A times this vector must equal zero.
[...]
177. T: O.K. You forgot the definition. The columns of a matrix A are linearly independent if and only if. .
178. S: If there's only one. . if there's only . . if it only <u>has</u> the trivial solution.
179. T: Which means it has?
180. S: **Ax** equals **0**.

Student S is quite vocal about how confused he is by the statement "if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution". The statements in lines 110 and 180 are indicative of the students' lack of understanding of the concept of a solution (and perhaps of the notion of an algebraic equation); for him linear independence of the columns of matrix A or the existence of only the trivial solution is the same as saying that A times **x** gives the zero vector. The student makes no mention of a vector **x** satisfying the constraint $A\mathbf{x} = \mathbf{0}$, instead he seems to think that the trivial solution is denoted by the symbol "**0**" and is the result of the calculation **Ax** (lines 178-180).

This kind of thinking stayed with student S for some time before tutor T2 confronted him about it:

350. T: What does it mean Ax = 0? You're always saying, "When Ax = 0". Ax = 0 is the homogeneous system. You have to do something for it.
351. S: When are the columns of the matrix linear independence? When there is only the trivial solution.
352. T: When there is only the trivial solution. Phrase it better. If and only if, *something* has the trivial solution? What's the thing?
353. S: If and only if, what thing has the trivial solution?. .
354. T: The system, right?
355. S: Which is then, Ax equals 0. That's Ax equals 0.
356. T: So you have to do something with Ax = 0. It's nothing. You're saying they are linear independence if and only if Ax = 0. What does it mean Ax = 0? If and only if Ax = 0 has only the trivial solution. O.K.
357. S: (no response)
358. T: I mean you have to solve Ax = 0 . .
359. S: Right.
360. T: . . to find.
361. S: Right. The nontrivial solution.
362. T: The trivial solution.
363. S: Trivial. . independent. . trivial

The tutor tries to make the point that in this context it is meaningless to speak of the equation **Ax** = **0** without referring to its solution. When he says "*Ax = 0 is the homogeneous system. You have to do something for it*". He is trying to impress upon the student that there is a solution to be found and that finding the solution involves some work or calculation (line 350). This makes little impression on student S who responds by saying that the columns of A are linearly independent "*when there is only the trivial solution*" without referring to an equation. The tutor is not aware that there is some deep misunderstanding on the student's part and responds by "treating the symptoms", i.e. he tries to correct student S's ungrammatical use of the mathematical language and says: "*Phrase it better. If and only if something has (only) the trivial solution? What's the thing?*" The students' response "*...What thing has the trivial solution?*" (line 353) and a subsequent lack of response (line 357) to a similar line of questioning suggests that he does not understand what the tutor is making such a big fuss about and what the point is.

The actual root of the problem is never addressed and the student learns that he must use the 'proper' language and say the 'correct' things even if he does not quite understand what it all means, i.e. he learns to say the things that the tutor wants to hear. This became apparent sometime later when the tutor tested the student on the concepts that they had covered:

582. T: [...] The columns of A are linearly independent when?
583. S: When, umm .. God! I'm going to say it wrong! They're linearly independent
584. T: mm ..mm. When Ax = 0.
585. S: I was going to say that, but you hate it when I say that. When Ax = 0 has only the trivial solution.
586. T: Yeah. You're not memorizing are you!
587. S: No. No. (*laughing*)

The student is clearly apprehensive about responding correctly; his concern is that his response should fulfill the tutor's expectations. When the tutor anticipates his response and says "*When* $A\mathbf{x} = \mathbf{0}$" (this may have been a test!), the student's first reaction is not a correction but rather a confirmation that this was the response he was contemplating but that "*you hate it when I say that*". Then he gives the "correct" response and conforms to the tutor's expectations, but the tutor begins to suspect that the response may be feigned (line 586).

This difficulty of student S's has been documented by Sierpinska & Defence (Sierpinska & Defence, 1994) and is referred to as an "arithmetic obstacle" because of the way in which the zero vector is thought of as the result or "solution" of the arithmetic operation $A\mathbf{x}$. This is very much reminiscent of the way that grade school children interpret arithmetic equations: the number to the right side of the equal sign is seen as the result or the 'solution' of the operation specified on the left side. It is interesting that this mode of thinking in student S was not detected before the use of the notation A$\mathbf{x}$ = $\mathbf{0}$ to denote a linear system. But this is not very surprising because one of the aims of using such notation is to have a simple and efficient way of speaking about such a complicated thing as a system of constraints (i.e. a system of linear equations). The notation A$\mathbf{x}$ = $\mathbf{0}$ represents not only a decomposition of the system into its constituent parts - coefficients, unknowns, and right hand sides, but it is also a synthesis of all these into a single matrix equation. It offers a compact and global view of the structure of a system and the concept of a solution in a way which is uncluttered by the details of coefficients and variables. The focus is no longer on the these details, but rather on the *relational* aspect of the system. This is very much a structural representation, and for the novice this perspective of a solution may appear remote from the idea of a solution as something which is determined through calculation. We are thus reminded of the importance of stressing the distinction between a solution of an equation and the techniques of finding a solution.

Let us see how student S understood the concepts of linear independence/dependence before he was exposed to the notation A$\mathbf{x}$ = $\mathbf{0}$. In the following excerpt the student is trying to determine whether a set of three given vectors in $\mathbf{R}^3$ is linearly dependent, he has already transformed the augmented matrix of the system to the reduced echelon form:

25. T: From your matrix, what can you tell?
26. S: There's a free variable.
27. T: Which is. . ?
28. S: Which is the x_3.
29. T: O.K.
30. S: So x_3 can be anything, so it's . . dependent.
31. T: O.K. You're right. It's dependent but. .
32. S: I skipped logic? (laughing)
33. T: Yeah. If you find the general solution . . you didn't finish writing your general solution.
34. S: Right.
35. T: How do you write it?
36. S: I write it as. .
37. T: Say . . Say it.
38. S: x_1. .
39. T: No. No. I mean, what's the logic in it? How do you write. . ?
40. S: Writing all the variables in terms . .
41. T: Which variables?
42. S: All the basic variables in terms of the free variable.
43. T: O.K. Good.
44. S: You want me to do it? (laughing) $x_1 = 2x_3$.

and $x_2 = -x_3$.

and x_3 is free

So my free variable means that x_3 can be anything, so it's dependent (*in a sing-song voice*).
45. T: Because whatever value you give to x_3, what will happen to x_1 and x_2?
46. S: They'll change.
47. T: O.K. That's why it's dependent.
48. S: Yes, but that's what free variable means, right? Free variable means it changes the other variables.
49. T: Yeah. But we didn't say in terms of dependent, that's all.
50. S: Oh, O.K.

For student S the concept of dependence has been reduced to the existence of free variables in the system of equations - much the same way (and for the same reasons) it had been for student P. The tutor seems to be unsatisfied with this way of understanding and requests that the student write the general solution to the system and then asks "*what's the logic in it?*" (line 39). Now this might seem like a nonsensical question given the fact that the only logic to follow is that given by the definition; since there exist free variables the vectors are dependent - there is nothing more to say! But as we see in the rest of the excerpt, what the tutor wants is for the student to express the other (basic) variables in terms of the free variable and to see that there is also a dependence of variables (lines 45-47) "*That's why it's dependent*" he exclaims. The tutor may have perceived that student S's understanding was mechanical so he sought to give more meaning to the concept of linear independence of vectors by appealing to the more natural, and familiar, notion of dependence of variables. The student seems to have accepted and retained this view as is demonstrated in an explanation he gives at a later time:

164. S: [...] What was the question?
165. T: Determine if the columns of A are linear independence? And tell me your logic.
166. S: Independent. . ? Yeah, because each variable is equal to the. . equal to zero. There's no free variable so. . the basics aren't. . the basic variables aren't dependent on any free variable. Does that explain it enough, because here I have like $x_1 = 0$, and $x_2 = 0$, and $x_3 = 0$. No free variables so that's all it equals to. So Ax equals 0.

Although student S appears to have some operational understanding of the concepts he seems to have a wrong understanding of the notion of a solution of a system. This student also never saw a more conceptual development of the notions of linear independence/dependence, and his understanding of these notions remained at a very mechanical level.

Student C's understanding

Let us precede our investigation into student C's understanding of linear independence/dependence by first giving a brief description of Text 1. This book was published in Poland and was designed for students of economics. It is not a linear algebra textbook per se; most of the volume is devoted to calculus but it includes a chapter on analytic geometry and another one entitled '*Elementary Linear*

Algebra'. This latter chapter does not develop the theory of linear algebra in any great detail, a few theorems are stated without proofs and most of the exercises are of a computational nature. The chapter begins with the space $\mathbf{R}^n$ (denoted as V_n) in which vectors are defined not axiomatically but purely analytically as n-tuples - no recourse to geometry is made. The usual algebraic structure of $\mathbf{R}^n$ is established by defining addition and scalar multiplication of n-tuples. Also defined is the dot product of two vectors, norm, cosine and Euclidean distance between two vectors. Immediately after the notion of linear combination is introduced the concepts of linear dependence/independence of vectors in V_n are defined as follows:

> A system of vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ in V_n is called *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, not all equal to zero simultaneously such that
>
> $$\sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{0}.$$
>
> A system of vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ in V_n is called *linearly independent* if
>
> $$\sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{0} \text{ only when } \lambda_1 = \lambda_2 = \ldots = \lambda_k = 0.$$

This is followed by two pertinent theorems and a definition:

> PROPOSITION 1. *A system of vectors* $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ *is linearly dependent if and only if at least one of these vectors is a linear combination of the others.*
>
> PROPOSITION 2. *Every system of n+1 vectors of the space* V_n *is linearly dependent.*
>
> A basis of a vector space V_n is any system of n linearly independent vectors of this space.

Linear dependence/dependence is not mentioned again until ten pages later when the notion of rank is defined as the maximal number of linearly independent rows or columns of a matrix, it seems that this is the intended use of the concept. The concept then re-emerges thirty four pages later when consistent systems of linear equations are classified as independent or dependent (i.e. having one or infinitely many solutions, respectively). Let us have a look at student C's first reaction to the definition of linear dependence/independence:

133. C: (*reads*) " A system ($\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$) of vectors in V_n is called linearly dependent if there exist numbers lambda, okay, not all equal zero, such that the sum from i equal 1 to k of lambda i a i is equal zero. Is this zero or o?
134. T: Zero vector.
135. C: Equal zero vector. Does this mean that the sum of these vectors must be equal zero vector?
136. T: With some coefficients.
137. C: Aha. Not all equal zero. So it is linearly dependent if it is so?
138. T: Uhuh.
139. C: And it is linearly independent if all lambdas are equal zero. If you multiply a vector by zero then you get the zero vector, right?
140. T: What do they write?
141. C: No, but it's not true for all vectors, it seems to me that it can happen only in very special cases, very special cases.
142. T: That what?
143. C: So that the sum of some vectors multiplied by some numbers gives zero. It can only happen exceptionally [...]

Some ambiguity about the symbol "**0**" and the meaning of the equation $\sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{0}$ in the definition of linear dependence is clarified by the tutor. When the student understands the meaning of the whole statement in the definition of linear dependence, i.e. that there can be vectors whose non-trivial linear combination adds up to the zero vector, he has an immediate intuitive sense that such sets of vectors are rare. This followed just after he understood that $0\mathbf{v} = \mathbf{0}$, for then he conceived that it could be difficult to make the linear combination $\sum_{i=1}^{k} \lambda_i \mathbf{a}_i$ equal to the zero vector if the coefficients are not all zero. This seems like a rather subtle thing to deduce from such a formal statement, and it is interesting that a novice would come to this realization so quickly. In fact the belief that sets of n dependent vectors in n-dimensional space are the more rare case may be mathematically justified. Also historically, in the context of dependent equations, discussions related to Cramer's paradox show that the discovery that systems of n equations in n unknowns could contain redundant equations was received as something unexpected.

At first student C has no questions about the connotation of the word *dependent* in the definition, he seems to accept that this is just a special name given to a set of vectors satisfying a certain condition (line 137). It is only after the tutor asks him to construct an example of a dependent set of vectors that we get a glimpse of the confusion that can arise from the clash between different connotations of the word:

156. T: Could you think of an example of two linearly dependent vectors in two dimensions, for example?
157. C: I don't know! I don't know! I can look for it. Suppose I take a vector, two vectors, one two, and three four (*writes* $a_1 = (1, 2)$ *and* $a_2 = (3,4)$)
158. T: Yeah. Are they linearly dependent?
159. C: They depend on this lambda as we multiply them, right?
160. T: They are dependent as a system, linearly.
161. C: What do you mean?
162. T: Because we speak of vectors here. A system of vectors is either dependent or independent.
163. C: Yes, but all this, it seems to me, depends on this lambda. It is linearly dependent if lambda is such, and linearly independent if lambda is different.

We see that student C makes an almost arbitrary choice of two vectors in $\mathbf{R}^2$, he is not using the definition in a way that would insure that the vectors will be dependent. When asked if the vectors are dependent he says "*They are dependent on this lambda as we multiply them, right?*" The tutor then responds by emphasizing that one speaks of a dependent system of vectors, but student C does not really understand what this means. He seems to think (lines 159, 163) that whether a set of vectors is dependent or independent *depends* on the coefficients in the linear combination $\sum_{i=1}^{k}\lambda_i \mathbf{a}_i$. He does not realize that a set of vectors is a priori dependent or independent, instead his choice of vectors together with his comments almost suggest an understanding of dependence or independence as something which comes after the fact, i.e. as properties which can be imposed on a set of vectors by a suitable choice of coefficients. It may be that the student's confusion between dependence in the sense of the definition and his sense of dependence on the coefficients is interfering with his understanding the conditional nature of the definition.

This example also illustrates that drawing the operational value out of this definition may not be easy for students. Indeed, as we mentioned in chapter II, the ability to do so usually requires some familiarity with the different representations of systems of equations - something that student P did not possess at this point. Perhaps it is more accurate to say that statement 1 (of which this definition is a variation) has a *potential* operational value.

These seemingly serious difficulties faded into the background as student C finally went on to show that his choice of vectors was independent, he did this by rewriting the vector equation $L_1(1, 2) + L_2(3, 4) = \mathbf{0}$ in the form $L_1 1 = -L_2 3$, $L_1 2 = -L_2 4$ and then solving this system by the substitution method. At the end of it all he announces quite triumphantly "*L_2 must be equal to 0. This means that L_1 must also be zero, haha! Now YOU find a set* *so that lambdas are not zero! I don't think it is going to be very easy!* " When a set of dependent vectors is finally found, student C begins to realize that such sets are not as rare as he had believed.

Student C also had difficulty with the conditional part of the definition of linear independence, and it is interesting to see how he comes to understand the meaning of the statement:

239. C: That's what they say here (*reads*) "Only if $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$."
240. T: Then this is equal to zero. Such a combination.
241. C: When they're independent.
242. T: .. is equal to zero only if all the coefficients are zero.
243. C: Yes, but this combination is a condition for something to be linearly independent (*reads*) "We call .. linearly independent if the sum (of all this) is equal ..
244. T: Only if .. is equal to zero, only if all the coefficients are equal to zero. And where do you have ..?
[...]
247. C: Independent .. (*laughs*) .. It's impossible exclusively when this happens. So it's independent exclusively when λ's are equal to zero.
248. T: Something is not right here .. That's how I understand it.

The tutor senses that the student may not understand that there are two conditions in the definition. The student considers each condition in isolation and does not make the necessary connection between them that could lead to an understanding of the whole statement. This is evidenced by the utterances of lines 243 and 247 which suggest that he associates independence with either condition and not both.

250. T: A system is linearly independent if the linear combination is equal to zero only if all coefficients are equal to zero.
251. C: You see how you throw in this other sentence that's unrelated.
252. T: No!
253. C: Say it once more.
254. T: We looked at this ..
255. C: No. Say once more what you said.
256. T: The system of vectors ($\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_k$) is linearly independent
257. C: O.k. Let's stop here. For it to be linearly independent ..
258. T: ..when this equation is possible only with coefficients equal to zero.
259. C: ..when this equation is possible only with .. Aha! Only for that, right?
260. T: Only when all coefficients are equal to zero.
261. C: mm . mm
[...]

272. T: A system of vectors is linearly independent when the only linear combination that gives zero is such a trivial combination, namely when all coefficients are zero. In other words . . Do you want it in different words, once more?
273. C: Hold on. No. I have to write it myself, summarize it. (writes) Is linearly independent only . .

The student does not see why the statement "*only if all the coefficients are equal to zero*" should be included after the statement "*if the linear combination is equal to zero*", he accuses the tutor of "*throwing in this other sentence that's unrelated*" (line 251). The statement has to be heard several times before he can determine that the difficulty is with the word *only*. The verbalization of the mathematical conditions "if $\sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{o}$ only when $\lambda_1 = \lambda_2 = \ldots = \lambda_k = o$" is a linguistically complicated statement: "*if the linear combination is equal to zero only if all coefficients are equal to zero*". This compound statement can be difficult to understand perhaps because of the way that each part begins with the word *if*. If the entire statement is said without pausing after the word *zero* (in the first part), so as to accentuate the demarcation between the two parts, the whole can sound like a confusing run-on sentence. The tutor is aware that as a spoken statement this is confusing for the student so he offers to say the same thing in different ways, but what the student needs is to write these statements down on paper.

After an arduous session full of experiments, mistakes, and negotiations with the tutor, it seemed that student C had some understanding of the notion of linear dependence. But two days later, in the next session when the tutor came back to the subject, it first appeared otherwise:

1. T: Do you remember when a system of vectors is linearly dependent?
2. C: Hem? Well, a system of vectors is linearly INdependent when all the lambdas are zero [...]
5. T: According to your definition (1, 2), (-1, -2) are linearly independent.
6 C: What is your point? This is not MY definition, it is a definition from the book.
7. T: Do you agree that (1, 2), (-1, -2) are linearly independent?
8. C: If you had different lambdas, it would be dependent. They are opposites of each other.
9. T: Does it depend upon the lambdas whether the system is dependent or independent? [...]
10. T: Can you give an example of two linearly independent vectors?
11. C: *writes 0(1, 2) + 0(3, 4)*)
12. T: What is linearly independent?
13. C: This.
14. T: What? This linear combination?
15. C: I guess so.
16. T: A linear combination is linearly independent?
17. C: This (points to *0(1, 2) + 0(-1, -2)*) is a system of linearly dependent, and this (points to *0(1, 2) + 0(3, 4)*) is a system of linearly independent.
18. T: What do you mean by "this"?
19. C: This equation.
(Sierpinska, 1992)

When student C says "*a system of vectors is linearly INdependent when all the lambdas are zero*" it seems that he remembers only one of the conditions (perhaps the most striking one for him) which form part of the definition - he does not immediately have a picture of the definition as a whole compound proposition. Even in situations where students are free to review the definition of linear independence as often as they wish they often understand only fragments of statements and ignore the conditional nature of the whole definition (see example in chapter II). For the novice it can be quite difficult to extract the meaning of a concept from its analytical representation; in the case of this definition of linear dependence/independence the preponderance of detail in the notation (e.g. subscripts, the sigma sign, etc.) can easily obscure the logic and the intended meaning of the statement (Sierpinska, 1992). When the tutor presents him with the vectors (1, 2), (-1, -2) as a counterexample to his 'definition' we see student C reverting to his way of thinking (one which we thought he had shed) that dependence/independence depends on the coefficients (line 8).

Student C is also having difficulty with the grammar of the terms dependent/independent when he refers to linear combinations and equations being dependent and independent (lines 17 and 19). After this a discussion ensues in which the student and tutor establish that the object of linear dependence or independence is not a linear combination or an equation but a system of vectors. Then student C suddenly remembers everything as he had first seen it:

C: Oh, I see! This is the same old joke! A system of vectors is linearly independent IF, right? the sum, etceteras, ONLY when the lambdas, etceteras! (*silence*) O.k., so the first is linearly independent because the lambdas do not have to be zero for this to give the zero vector. And the second is linearly independent because lambdas MUST be zero for this to give the zero vector.

This is our lasting impression of the student's understanding of the concepts which resulted from reading the formal definitions. He was able to overcome the logical, linguistic, and notational complexities of the formal definition to arrive at what appears to be a very clear and natural interpretation of its statement. This was a long way to come for the student and his understanding may still be a tentative one. We note that Text 1 does not reduce the concepts of linear independence/dependence to the question of whether a homogenous systems of equations has non-trivial solutions. Rather, the concepts are unrelated to most of the preceding material so they appears as something completely new - they grabbed the reader's attention. Also, immediately following the formal definition, the text gives a sole example of a set of linearly dependent vectors. The lack of a detailed treatment of the concepts in Text 1 essentially forced student C to fend for himself (with the help of the tutor), and through some hard work and thinking he arrived at a more conceptual understanding than did students P and S. Student C realized that all the experimenting and discussing that he engaged in was valuable in helping him achieve a certain understanding of the notions of linear dependence/independence. He also had a sense that the definitions he had worked so hard to understand were more than just a mere curiosity and that they should have some important applications:

C: To think that if I hadn't done all this here, hadn't dirtied all this paper, then I still wouldn't know ... I mean I wouldn't suspect ... I am not saying I am fully convinced, now but let's say I suppose at this point that this can have some application, deeper than just curiosity.

In subsequent sessions when student C studied the concept of the rank of a matrix he did in fact use the concept of linear dependence to argue that if the rank of a square matrix is less than it's dimension then the matrix is singular. A lot of time was spent discussing and justifying this proposition, let us have a glimpse of his understanding and reasoning:

C: Because it is dependent, so one of these .. these .. these rows is a linear combination of the other rows .. so this means that all the others can be added, divided, calculated, multiplied by a minus, and we'll have a row of zero [...] so when you multiply these, you know [in calculating the determinant] you get zero.
T: O.k. but won't the determinant change when you do these operations on the rows?
C: Well, yeah, it will change but these are normal elementary operations ... You are not changing the rank here.

Although the argument is not complete, the reasoning is sound and we get a sense that student C has an understanding of the usefulness of row operations and their relations to the notions of rank and linear dependence. This understanding is backed not by logical reasons but rather by more pragmatic ones such as the feeling of the usefulness of the concepts. For instance, the understanding that the rank is an invariant of elementary operations is more of a practical knowledge rather than a theoretical knowledge for student C. This understanding is reinforced by Text 1 in which the question of the logical validity of the assertion is never posed. Yet in the approach of Text 1 this concept plays a central part in solving systems of equations where the question of the uniqueness of a solution is related to the concepts of linear dependence and rank. Even the terminology is suggestive and is meant to reinforce the connection; a consistent system is either dependent or independent and in the solution process it is not variables that are "eliminated" (as in Text 2), but rather redundant equations - those that are linear combinations of the others.

After student C read the definitions of dependent and independent systems the tutor asks him if he sees any link between these and the concepts of linear dependence/independence he had previously encountered. The student responds by referring to the vectorial representation of a linear system $\mathbf{A}_1 x_1 + ... + \mathbf{A}_n x_n = \mathbf{b}$:

C: Yes, in a sense, yes because these here .. columns are vectors .. the solution is also in the form of a vector.

Upon the tutor's request he produces an example of a dependent system of equations

$$4x+6y = 4$$
$$-6x - 9y = -6.$$

and when asked to explain why the system is dependent, the student responds:

C: Because the rank of this matrix (*referring to the coefficient matrix*) is less than it's dimension [...] Because it's the same, this row is the same as the other one .. multiplied by something [...] Because, in sum, this is the same thing, in a sense, o.k. I have already learned to understand it in a practical way ... (Sierpinska et al, 1995).

In the end student C had acquired a practical knowledge of the notion of rank, it was viewed as something that would allow him to guess the number of solutions of a system, and in this context it was intimately tied to the concepts of linear dependence/independence.

Overview

This report on our observations of three students trying to understand the concepts of linear independence/dependence is really a collage of snapshots of the learning process and all its complexities. Our task is the difficult one of making sense of what we have witnessed and seeing what whole picture emerges from all these bits and pieces. We have tried to bring to light difficulties that students can face when they try to learn the concepts. Many of these seem to be related to *different* aspects of mathematics; there are difficulties with logic, difficulties with symbolism and notation, difficulties with multiple representations, etc. Globally these can be seen as difficulties in deciphering meaning from analytical representations of concepts. While we can document such difficulties it seems almost futile to try to explain the reasons why a student experiences any one of them, at this point the best we can do is to speculate. But this is not even the most important thing for us, rather what is more important, and what emerges from our investigations, is how different kinds of activities can lead to different kinds of understandings of the concepts.

This point can be seen clearly if we compare the two extreme cases of students S and C. In the later case we get a sense of how the student's understanding of linear dependence/independence grows; it begins with his tentative first steps of trying to extract the meaning of the concepts from a formal definition which seems to appear out of nowhere, then his understanding is questioned by the tutor who gives him counterexamples and asks him to produce sets of dependent and independent vectors. Student S's activities, on the other hand, are largely restricted to checking whether given sets of vectors are dependent or independent and he is never asked to construct his own sets of dependent or independent vectors - such an activity requires quite a different thinking about the definition and this could potentially have led to questions and to a different understanding of the concepts.

Student C is a very active participant in the discussions with his tutor, he is always asking questions and he challenges the tutor's statements and knowledge. Through this interaction we see how the student's understanding is always under revision and his cognitive structure undergoes many re-organizations. Of course Text 1's development of the concepts is partly responsible for this; the concepts emerge several times in different contexts and in different guises, thus forcing the student to coordinate and synthesize them.

We saw that student C engaged in proving activity as well (e.g. $\text{rank}(A) < m \Rightarrow A_{mxm}$ is singular) and it was through the mathematical reasoning and arguing involved here that his understanding seemed to grow even more. This is where he developed a more meaningful understanding of linear dependence/independence because now the concepts had a certain usefulness for answering questions about solutions to systems of equations - there was a purpose to learning the concepts after all! These are the kinds of activities and mental actions that drove the growth of student C's understanding.

This is all in sharp contrast to the experience of student S. To begin with, here the roles of the student and the tutor were quite asymmetric; the tutor was the 'master' and the student was expected to play the part of the passive and compliant learner who trusted the tutor and did not question his knowledge too much. Also, the kinds of questions asked of the student never deviated from the agenda dictated by Text 2 so that he acquired only a very limited and mechanical understanding of linear independence/dependence. We never get a sense that this understanding grew or even changed, and there was little hope of understanding the usefulness or purpose of the concept because this came only much later in the text. Finally, there were none of those activities such as conjecturing, refuting, and proving that are characteristic of mathematical creativity and which are essential to bring about change and growth in one's understanding.

Student P is somewhere in between these two extremes. He begins by acquiring an operational understanding of the notions and though we don't get much of a sense that his understanding really evolved, we do see glimpses of more conceptual thinking and the ability and desire to reason and argue mathematically.

Although it has not been the focus of our analysis, it is clear that the type of student-tutor interaction can have a significant impact on what kind of understanding the student arrives at. We refer the reader to the study by Sierpinska & Defence (1995) which gives a detailed analysis of the types of interaction within each of our student-tutor-text triads and which considers the effects of these interactions on the students' linear algebra knowledge.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

The last ten years or so have seen an increase in research in the learning and teaching of linear algebra. We are not, however, aware of any published work which focuses specifically on students' difficulties with the notions of linear independence/dependence. As far as we know ours may be a first effort in this direction.

Our research method, as described in chapter three, did not allow the tutors to act in the capacity of researchers; all interactions were spontaneous, no two students even attempted all the same problems, and there were certainly no pre-planned questions or problems set for students because we had no clear idea of what we were looking for. In fact, part of the raison d'être for these 'continuous observations' was to provide data whose analysis would then point us in more specific directions. These considerations will hopefully help the reader put the exploratory nature of this research into perspective.

So what reasonable conclusions can be drawn from such exploratory research? Our investigations and analyses highlight several points which we believe are worth pursuing as avenues for further research:

- At the risk of appearing pompous we assert that this work can be a valuable preliminary reading for anyone interested in researching students' understandings of the notions considered here. It may be particularly useful as a prelude to a comparative study of students' understandings that can arise from different ways of introducing the concepts.

- Given the reality of educational practice we know that students will continue to have to learn mathematical concepts from their formal definitions, and considering the difficulties that student C experienced with the definitions of linear independence/dependence, we make two recommendations concerning these:

i) We should try to determine whether some of the difficulties that we observed our students having with the formal definitions are generalizable to a larger student population. Most notably, difficulty with the conditional nature of the formal definition of linear independence/dependence seems to be a major obstacle to deciphering the meaning of the statements. In fact, statistics from some of our ongoing research already confirm that this difficulty is more widespread; a sample of thirty students enrolled in a first year linear algebra course at Concordia University was asked the following question on a class test:

> "What exactly is wrong with the following statement:
> Vectors $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are linearly independent if $a_1\mathbf{v}_1+\ldots+a_p\mathbf{v}_p = \mathbf{0}$ for $a_1 = a_2 = \ldots = a_p = 0$"?

Only nine respondents (30%) gave answers that could be considered 'correct'; while all of these did quote the correct definition of linear independence, none explained the difference between the two statements or showed any awareness that the tautology $a_1\mathbf{v}_1+\ldots+a_p\mathbf{v}_p = \mathbf{0}$ *for* $a_1 = a_2 = \ldots = a_p = 0$ makes <u>any</u> set of vectors linearly independent according to the above statement.

ii) We would like to see questions and activities designed to help students test their understanding of the formal definitions of linear independence/dependence. For example, the following might be a worthwhile assignment: sometime after their introduction to the formal definitions students would be presented with several statements which resemble these in some way (as in the example in item i) above). They would be asked to compare each of these to the definitions and to describe the differences in detail, and to give examples of sets of vectors which satisfy the conditions of each statement.

- Historically, the notions of linear independence/dependence did not simply spring into existence as though they were someone's brilliant spontaneous creation. Rather, it was certain contexts which gave rise to them and this implies that they were somehow useful notions. For instance, in the context of linear equations these concepts are useful for determining the number of solutions, and in the search for a geometrical calculus they were intimately related to the fundamental notions of generation and minimal sets of generators. Their implicit usefulness and purpose in such contexts is what gives them meaning.

Unfortunately the formal definitions of linear independence/dependence alone cannot give these concepts deep meaning. This is because formal definitions, in general, represent abstractions of the concepts, and as such they transport the concepts out of the contexts which gave them meaning. It is one of the ironies of modern mathematics that statements which are meant to capture the essence of a concept do so by stripping it of its meaning in another sense. These sentiments are supported by our observations of our students, especially student C who really began to have a deeper understanding of linear independence/dependence once he was able to see its use. Prior to this he had achieved an understanding of the concepts from the definition, but they had no meaning for him.

Towards a more conceptual understanding of the notions of linear independence/dependence we make a few more recommendations:

iii) The comparison of student S's and student C's understanding of the notions of linear independence/dependence suggests that certain contexts may enhance conceptual understanding. With a larger group of students we would like to try out the contexts of argumentation and proving, conjecturing and constructing counterexamples. We would also like to see students produce their own sets of dependent and independent vectors rather than just checking whether given sets are dependent or independent.

iv) Concerning the understanding of linear dependence, we have seen that different kinds of dependencies come into play; there are dependent vectors, dependent equations, and dependent variables. We would like to design activities which would let students explore and develop the relationships between these and the notion of rank. The historical analysis together with our observations of student C suggest that the context of linear systems may be a natural one for this.

v) We would like to research the design of an experimental teaching sequence which has the notion of linear combination as a central theme. The linear combination would be presented as the fundamental operation of linear algebra - as a basic method of generation. Following Fletcher's approach, the fundamental concepts of linear independence/dependence, generators, basis, and dimension could be introduced in this context through their application and use. They could then be developed more formally but in a way that reinforces ties to linear combinations.

One of the implicit aims of this research was to impress upon the reader that the concepts of linear independence/dependence are rather deep and fundamental. Also, each of the perspectives we took reveal that these are subtle and epistemologically difficult notions. Accordingly, we should seriously question the value of any approach which tries to reduce the notions to merely new names for familiar concepts, as the procedural approach of text 2 does. Though this approach may succeed in bypassing those difficulties related to the logical nuances of the formal definitions, it risks leaving the student with what we feel is an overly simplistic view of the concepts - a view which is devoid of any sense of their usefulness or purpose.

Of course one can argue that an operational understanding makes for a good first understanding, and this may well be the rationale underlying text 2's approach. But if a more conceptual re-development of the notions, one which at least gives the student a sense of their purpose, does not follow shortly thereafter, the operational understanding is the only one that students may ever have. On the other hand it can be argued that the availability of mathematics software makes the procedural approach rather obsolete and that students should focus on the concepts only.

This dialectic between the procedural and the conceptual reminds us of the tool-object dialectic of mathematical concepts (Douady, 1986) and, on another level, it reflects a certain tension between a teacher's *obligation* to give students working definitions and his/her *desire* that they acquire a conceptual understanding of the notions.

REFERENCES

ANTON, H. (1991): *Elementary linear algebra* , 6^{th} edition. New York: Wiley & Sons, Inc.

ARTIGUE, M. (1989): Epistémologie et didactique. *Cahiers DIDIREM n°3. IREM de Paris VII.*

ARTIGUE, M. (1989): L'ingénierie didactique. *Vème école d'été de Didactique des Mathématiques*, edition coordonnée par R. Gras, Institut de Mathématiques de Rennes et IRESTE de Nantes, Plestin-les-Grèves.

BANACH, S. (1932): *Theories des opérations linéaires.* Warszawa: Z subwencji Funduszu Kultury Narodowej.

BANCHOFF,T. & WERMER, J. (1992): *Linear algebra through geometry.* New York: Springer-Verlag.

BAZANSKA. T., KARWACKA, I., NYKOWSKA, M. (1976): *Zadania z matematyki. Podrecznik dla studiow ekonomicznych.* Warszawa: Panstwowe Wydawnictwo Naukowe.

BOYER, C. B. , MERZBACH, U. C. (1989): *A history of mathematics* , 2^{nd} edition. New York: Wiley & Sons, Inc.

BYERS, V. (1980): What does it mean to understand mathematics? *International Journal of Mathematical Education in Science and Technology 11(1)*, 1-10.

CAJORI, F. (1919): *A history of mathematics* , 2^{nd} edition, revised and enlarged. New York: The Macmillan Company.

CHEVALLARD, Y. (1985): *La transposition didactique: du savior savant au savior enseigné.* Grenoble: La Penseé Sauvage Éditions.

COOLIDGE, J. L. (1963): *A history of geometrical methods.* New York: Dover Publications, Inc.

CROWE, M. (1967); *A history of vector analysis -The evolution of the idea of a vectorial system.* Notre-Dame (USA): University of Notre-Dame Press.

DIEUDONNÉ, J. (1992): *Mathematics - the music of reason.* Heidelberg, Germany: Springer-Verlag.

DORIER, J. L. (1990): *Contribution a l'étude de l'enseignement à l'université des premiers concepts d'algèbre linéaire. Approches historique et didactique.* Unpublished doctoral dissertation. France: Université Joseph Fourier.

DORIER, J. L. (1991): L'enseignement des concepts élémentaires d'algèbre linéaire a l'université. *Recherches en Didactique des Mathématiques 11(23)*, 325-364.

DORIER, J. L. (1992): Notes from a presentation on the teaching of linear algebra. Concordia University (unpublished).

DORIER, J. L., ROBERT, A., ROBINET, J., ROGALSKI, M. (1994): The teaching of linear algebra in first year of French science University: Epistemological difficulties, use of the "metalever", long term organization. In *Proceedings of the XVIIIth International Conference for the Psychology of Mathematics Education, July 1994, Lisbon, Portugal*, Vol. IV, 137-144.

DORIER, J. L. (1995): A general outline of the genesis of vector space theory, *Historia Mathematica* 22(4).

DOUADY, R. (1986): Jeu de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7(2), 5-31.

EULER, L. (1750): Sur une contradiction apparente dans la doctrine des lignes courbes. *Mémoires de l'Académie des Sciences de Berlin.*

FEARNLEY-SANDER, D. (1982): Hermann Grassmann and the prehistory of universal algebra. *The American Mathematical Monthly 89(3).*

FLETCHER, T. J. (1972): *Linear algebra through its applications.* London: Van Nostrand Reinhold Company.

FORMAN, G. E. & KUSCHNER, D. S. (1977): *The child's construction of knowledge: Piaget for teaching children.* Monterey, California: Brooks/Cole Publishing Company.

FROBENIUS, F. G. (1875): *Gesammelte Abhandlungen.* Berlin, Heidelberg, New York: Springer.

GIROTTO, V. (1989): Logique mentale, obstacles dans le raisonnement naturel et schémas pragmatiques. In N. Bednarz & C. Garnier (Eds), *Construction des Savoirs. Obstacles et Conflits.* Ottawa: Agence d'Arc, Inc.

GRIFFEL, D. H. (1989): *Linear algebra and its applications, Volume 1. A first course.* England: Ellis Horwood Limited.

HALMOS, P, R. (1987): *Finite-dimensional vector spaces.* New York: Springer-Verlag.

HANNA, G. (1989): Proofs that prove and proofs that explain. In *Proceedings of the XIIIth International Conference for the Psychology of Mathematics Education*, Vol. II, 45-51,

HAREL, G. (1985): *Teaching linear algebra in high school.* Unpublished doctoral dissertation. Isreal: Ben-Gurion University of Negev, Beer Sheva.

HAREL, G. (1986): A comparison between two approaches to embodying mathematical models in the abstract system of linear algebra. *In Proceedings of the VIIIth Annual Conference of the PME-NA.*

HAREL, G. (1987): Variations in linear algebra content presentations. *For the Learning of Mathematics 7(3).*

HAREL, G. (1990): Learning and teaching linear algebra: difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics 11(1-2)*, 139-148.

HAREL, G. & TALL, D. (1991): The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics 11(1)*, 38-42.

HERSTEIN, I. N. , WINTER, D. J. (1988): *Matrix theory and linear algebra.* New York: Macmillan Publishing Company,

HILLEL, J. & SIERPINSKA, A. (1994): On one persistent mistake in linear algebra. In *Proceedings of the XVIIIth International Conference for the Psychology of Mathematics Education, July 1994, Lisbon, Portugal*, vol. III, 65-72.

JACOBSON, R. (1980): A few remarks on Peirce, a pathfinder in the science of language. *In The Framework of Language, Michigan Studies of Humanities, no. 1.*

JOHNSON, L. W. , RIESS, R. D. , ARNOLD, J. T. (1993): *Introduction to linear algebra*, 3rd edition. Addison-Wesley Publishing Company, Inc.

JOSEPH, G. G. (1991): *The crest of the peacock. Non-european roots of mathematics.* Middlesex, England: Penguin Books.

KLINE, M. (1972): *Mathematical thought from ancient to modern times.* New York: Oxford University Press.

LAY, S. (1994): *Linear algebra and its applications.* New York: Addison-Wesley Publishing Company

LERON, U. (1989): Structuring mathematical proofs. *American Mathematical Monthly, 90(3)*, 174-185.

MARTIN, A. D. & MIZEL, V. J. (1966): *Introduction to linear algebra.* New York: McGraw- Hill, Inc.

MÉRAY, C. (1901): L'enseignement des mathématiques, *L'Enseignement Mathématique 3*, 172-194.

MUIR, T. (1960): *The theory of determinants in the historical order of development.* New York: Dover Publications, Inc.

PEDOE, D. (1963): *A geometric introduction to linear algebra.* New York: Wiley & Sons, Inc.

PIAGET, J. (1970): *Structuralism.* New York: Basic Books, Inc.

PIAGET, J. & GARCIA, R. (1989): *Psychogenesis and the history of science.* New York: Columbia University Press.

ROBINET, J. (1986): Esquisse d'une genèse des notions d'algèbre linéaire enseignées en DEUG. *Cahiers de Didactique des Mathématiques n°29*, éd. IREM de Paris VII.

ROBERT, A. & ROBINET, J. (1989): Quelques résultats sur l'enseignement de l'algebre linéaire. *Cahiers de Didactique des Mathématiques n°53*, IREM de Paris VII.

SFARD, A. (1991): On the dual nature of mathematical conceptions: reifications on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, Vol. 22, 1- 36. Dordecht: Kluwer Academic Publishers.

SKEMP, R.K. (1978): Relational understanding and instrumental understanding. *Arithmetic Teacher, 26(3)*, 9-15.

SIERPINSKA, A., DEFENCE A., KHATCHERIAN, T., SALDANHA, L. (1995a): Synthetic geometric, analytic and structural arguments in linear algebra. To appear in J. Kilpatrick, C. Hoyles, O. Skovsmose (Eds), *Communication and Meaning in the Mathematics Classroom*, BACOMET 4.

SIERPINSKA, A., DEFENCE, A. (1995b): Formats d'interaction e' les saviors des étudiants en algèbre linéaire (submitted for publication).

SIERPINSKA, A. (1995c): Mathematics: "in context", "pure", or "with applications"? A contribution to the question of transfer in mathematics. *For the Learning of Mathematics 15(1)*, 2-15.

SIERPINSKA, A. (1995d): The diachronic dimension in research on understanding in mathematics - usefulness and limitations of the concept of epistemological obstacle. In M. Otte, N. Knoche, and H. N. Jahnke (Eds), *Interaction between History and Mathematics Learning.* Göttingen: Vandenhoeck & Ruprecht (in press).

SIERPINSKA, A. (1994a): *Understanding In Mathematics.* London: The Falmer Press.

SIERPINSKA, A. (1994b): *A "local models" approach to studying teacher-student interactions. Examples from a study of a student learning linear algebra with the help of a tutor.* Manuscript of a paper presented at BACOMET 4 meeting in Melle, Germany.

SIERPINSKA, A., LERMAN, S., ARTIGUE, M. *Epistemology of mathematics and epistemology of mathematics education* (in progress)

STRUIK, D. J. (1968): *A concise history of mathematics*, 3rd revised edition New York: Dover Publications, Inc.

TALL, D. O. (1993): The transition to advanced mathematical thinking: functions, limits, infinity, and proof. In D. Grouws (Ed), *Handbook on Research in Mathematics Education*. Reston, Virginia: NCTM, Inc.

TALL, D. O. & VINNER, S. (1981): Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics 12 (2)*, 151-169.

VINNER, S. (1991): The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed), *Advanced Mathematical Thinking*. Dordecht: Kluwer Academic Publishers.

WILDER, R. (1944): The nature of mathematical proof. *American Mathematical Monthly* Vol. 51, 309-323.

WITTGENSTEIN, L. (1958): *Philosophical Investigations*. Oxford: Basil Blackwell.

WOOD, T. (1992): *Funneling or focusing? Alternative patterns of communication in mathematics class*. Paper presented at the ICME6, Quebec.