The Performance Analysis of a Multicast Packet Switch with Random Selection Policy

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ABSTRACT

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Broadband ISDN will provide diverse services to the users. Among these services, there will be a requirement for multipoint communications. An essential component of such a system will be a multicast switch which will transmit copies of a packet to different destinations. A number of service disciplines may be considered when the packets conflict with each other in a multicast switch. This thesis studies a one-shot discipline with random packet selection policy, i.e., a packet is chosen randomly from the contending packets. An input packet generates a fixed number of primary copies plus a random number of secondary copies. Two cases of input packets are considered. The first case is that all the packets at the inputs are fresh packets which are generated at the beginning of a slot. The second case is that the packets at the inputs are the mix of old and fresh packets. The old packets are the ones which were not switched in the previous slot and left to the next slot. Assuming a constant number of contending packets during a slot, the system is modeled as a renewal process, where renewal points correspond to the successful choice of the packets. From here, the distribution of the number of packets chosen in a slot is derived and the packet and copy throughput are determined. Then, this result is extended to the steady-state distribution through an embedded Markov chain analysis. The packet and copy throughput for the mixed input packets have been determined. Finally, asymptotic results have also been derived for very large switches under saturation. Theoretical results are in agreement with those of simulations.

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Chapter 1

Introduction

1.1 Broadband ISDN

The Integrated Services Digital Network (ISDN) has been heralded as the mechanism that will usher in the Information Age. ISDN, it is said, is a network that provides end-to-end digital connectivity to support a wide range of services, including voice and non-voice services, to which users access by a limited set of standard multi-purpose user-network interfaces. Such an ISDN standard interface was defined and called basic access, comprising two 64kbit/s channels and a 16kbit/s signaling channel. But, the only integration possible is at the level of physical packaging and it provides a limited data communications capability. It has become clear that the circuit switched approach and the limited bandwidth of present ISDN cannot meet the long term requirements of multimedia and multirate communications. To satisfy the future needs of communication services, broadband ISDN has been proposed.

Service	Bit Rate (Mbit/s)	Burstiness
Data transmission (connection-oriented)	1.5 ... 130	1 - 50
Data transmission (connectionless)	1.5 ... 130	1
Document transfer/retrival	1.5 ... 45	1 - 20
Video conference/video-telephony	1.5 ... 130	1 - 5
Broadband videotex/video retrival	1.5 ... 130	1 - 20
TV distribution	30 .. 130	1
HDTV distribution	130	1

Table 1.1: Characteristics of broadband services

Advances, especially in fibre optics and micro electronics, have made it possible to consider an extension of the current narrowband ISDN into a broadband ISDN, featuring much higher transmission rates. In B-ISDN, at least about 150Mbit/s will be offered to the user across the broadband user-network interface. In addition, a second interface type with at least 600Mbit/s in the direction from the network to the user is also foreseen. Such a network would have the capability of transmitting not only the traffic types that may appear in the narrowband ISDN, but also high bandwidth traffic types, emanating from services like video, bulk data transfer and future image services. Estimates of traffic characteristics for some services are given in table 1.1 [1].

The two key design issues for supporting this wide range of services are bandwidth availability and flexibility. To satisfy these requirements, asynchronous transfer mode (ATM) has been recognized as the transmission and switching format of choice for the future integrated broadband network.

Header 5 octets	Information field 48 octets

Figure 1.1: ATM cell structure

1.2 Asynchronous transfer mode

Recently, asynchronous transfer mode (ATM) has been recommended as the basic transport mechanism for the development of future broadband ISDN by the international standards body, CCITT. ATM is defined based on fast packet switching technologies, which has also been known as ATDM (asynchronous time-division multiplexing) or DTDM (dynamic time-division multiplexing). In ATM networks, all information such as voice, video, and data is packed into fixed-size slots called cells. These cells have a 48 octet information field and a 5 octet header shown in figure 1.1.

ATM differs from Synchronous Time Division Multiplexing (STDM) in that the time slots are assigned dynamically rather than on a per-call basis. This dynamic bandwidth allocation implies that a particular connection can no longer be identified by the position of the time slot within a frame structure but by a label contained in the header field of each cell. The header field can also be used for media access control, error control and for priority if necessary.

ATM is a circuit-oriented, hardware-controlled, low-overhead concept of virtual channels which have no flow control or error recovery. The implementation of these virtual channels is done by fixed-size (relatively short) cells and provides the basis for both switching and multiplexed transmission. Thus, ATM networks have the flexibility to support various services and introduce new services easily, and are efficient due to the high utilization of network resources. Furthermore, the use of short cells in ATM and the high transfer rates involved result in small transfer delays.

However, the ATM requires many new problems to be solved. For example, the impact of possible cell loss, cell transmission delay and cell delay variation on service quality need to be determined.

1.3 Multicast packet switch

In B-ISDN, the services will be very diverse. Such diversity of services demands not only point-to-point communications but also multipoint communications, such as teleconferencing, entertainment video, LAN bridging and distributed data processing. Therefore, ATM switching networks, the essential components in the network, should be able to provide the multicast function.

The function of a multicast packet switch is to transmit an arriving multicast packet to a set of destinations. Furthermore, it must be flexible enough to support unicast and broadcast connections as special cases. Like unicast switches, the $N \times N$ multicast switch architecture has N input ports where the traffic arrives and N output ports, where the traffic leaves, as is shown in Figure 1.2.

In recent years, various switch architectures have appeared in the literature[2-5]. A typical implementation of a multicast packet switch consists of a serial combination of a copy network and a non-blocking point-to-point switch as illustrated in Fig.1.3. The copy network replicates input packets from various sources simul-

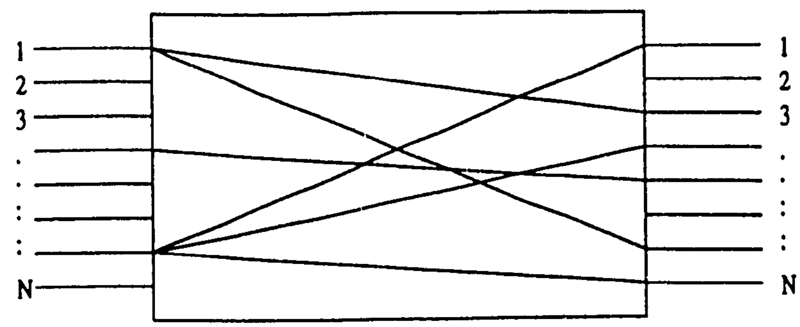


Figure 1.2: A multicast switch

taneously, and then copies of multicast packets are routed to their final destination by the point-to-point switch. A likely implementation of the point-to-point switch is the Batcher-Banyan switch[3,5,6], a cascade combination of Batcher's bitonic sorting network and a Banyan network.

In a nonblocking switch, the packets will not be blocked within the switch if the destination of all input packets are distinct. However, blocking will occur if there is output contention, i.e., the requests for output ports are not distinct. This effect is present in point-to-point switches and is compounded in multicast switches.

For a unicast input queued packet switch, the packets at the Head-Of-Line (HOL) contend for the output servers per slot time. Only one among the conflicting HOL requests (namely those HOL packets with the same output address) is served during the slot time. The rest are blocked, queued at the input, and may retry in the next slot time. This blockage, termed HOL blocking, reduces the throughput of an input queued packet switch to at most 0.586 [7, 8]. This upper bound is also termed the saturation throughput. To reduce HOL blocking, the internal speed of the packet switch may be increased relative to the input/output speed. Consequently, the input time slots may have more than 0.586 occupancy, and queuing is shifted from the

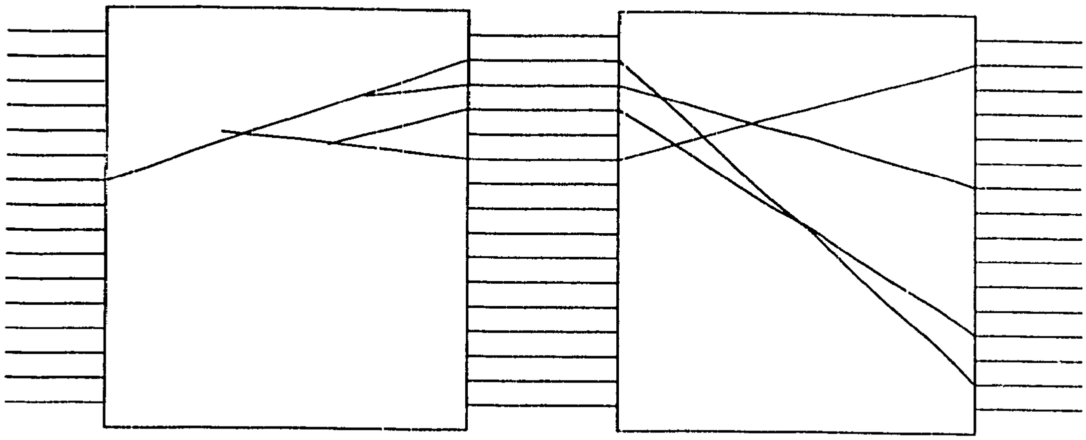


Figure 1.3: A multicast packet switch consists of a copy network and a point-to-point switch

inputs to the outputs.

For multicast switching, a HOL packet may request delivery to multiple destinations. This multiplicity is termed fanout. Hence there are more requests per output per slot than unicast switching. The selection among conflicting HOL packets requests may use different queueing disciplines. A number of service disciplines were proposed and summarized in [9]. Of these the most interesting are the following three [10]. One-shot scheduling requires all the copies of the same packet to be transmitted in the same time slot. Strict-sense (SS) call splitting and wide-sense (WS) call splitting allow the transmission of the packet to be split over several time slots. SS specifies that each packet can send at most one copy to the destination per time slot; WS does not carry this restriction. In [10], these disciplines and their implementation considerations are discussed. Their performances are compared by simulation and some are determined by analysis. The conclusion is that taking into account the complexity of implementation as well as the performance, none of the categories seems to have distinctive superiority over the others. There is a trade-off between the performance and switch structure complexity.

1.4 Review of some work on the multicast switch

So far several analyses of the multicast service discipline have been published. In [11], the performance analysis of a multicast switch for WS call splitting has been determined, where the system is modeled as an independent set of M/G/1 queues. A key assumption in this analysis is that output port contention is settled by random selection among the contending input ports. The same protocol also has been analyzed in [9] under a different set of assumptions. In [12], the analysis of a multicast switch has been given with input ports having rotating priority in each slot for the one-shot discipline.

A lot of work on the multicast switch had been done by Xing Chen in his Ph.D thesis[10]. It discussed various aspects of multicast packet switching with emphases on the issues of call scheduling disciplines, contention resolution algorithms, mathematical analysis and switch architecture. The thesis is composed of four parts: multicast scheduling, contention resolution algorithms, modeling and mathematical analysis and switch architecture. In the part of multicast scheduling, it classified and discussed four call-scheduling disciplines and their implementation considerations. In the part of contention resolution algorithms, it discussed several implementation schemes and has shown the potential improvement of delay-throughput performance by introducing an optimal algorithm. In the part of modeling and mathematical analysis, it presented some analytic tools from traffic theory for the input queueing of the multicast switching system. It discussed the mathematical analysis of both the random selection policy and the cyclic priority input access scheme for the one-shot discipline. Then, it introduced a general unified mathematical model for both the one-shot and the WS call-splitting input-access disciplines, by using matrix-geometric techniques in an effort to measure the effect of HOL blocking. These results could serve to model the multicast packet switch and to predict the onset of congestion in the system. In the part on switch architecture, it summarized some multicast packet switches proposed in the literature, including a banyan-based space-division switch, a knockout switch, a shift switch and a shared-buffer memory switch, Then, it proposed a shared-buffer memory switch structure with maximum queue and minimum allocation, where a shared buffer pool and a reserved buffer pool are properly handled for switching and buffering the packets. The proposed switch possesses multicasting, modularity and priority functions to meet the needs of a wide range of communications applications.

1.5 Outline

The contents of this thesis are organized as follows:

Chapter 2 presents the analysis of a multicast switch under the random selection policy. The copy replication model considered is modified Binomial copy generation process, which overcomes the drawback of Binomial copy generation process. In this chapter, two cases of input packets are considered. The first case is that all the packets at the inputs are fresh packets which are generated at the beginning of a slot, i.e., the packets which were not switched during a slot will be discarded. The second case is that the packets at the inputs are the mix of old and fresh packets. The old packets are the ones which were not switched in the previous slot and left to the next slot. The analysis is divided into two parts, determining the probability that a contending packet will not interfere with the chosen packets, and modeling the system as a renewal process. First, we discuss two models of the random selection policy, the original random selection policy and the modified random selection policy. Then, we determine the probability that a contending packet will not interfere with the chosen packets by using the probability generation function and a property of z transforms. This probability is used as a parameter in the mathematical model of this system. Next, we model this system as a renewal process, where renewal points correspond to the successful choice of the packets. From here, the distribution of the number of packets chosen, given α contending packets at the inputs during a slot, is derived. Then, this result is extended to steady-state. Clearly, the number of packets contending at the inputs will not be a constant, but a mix of old and new packets. A Markov chain analysis is applied to determine the probability distribution of the number of old packets at the beginning of a slot, which will result in the probability distribution of the number of contending packets for mixed packets at the inputs. From this steady-state distribution, the number of packets chosen during a slot is determined. Simulation results are also given, which are in agreement with the analytical results.

Because of the numerical difficulties in calculating the distribution of the number of packets chosen, asymptotic analysis is also given in chapter 3. In chapter 3, first, the average number of packets chosen is derived using weak law of large numbers. Then a simple lower bound of the distribution of the number of packets chosen in a slot is given. The exact distribution of the number of packets chosen is determined using results from renewal theory. More, simulation results are presented, which are also in agreement with the analytical results.

Finally, chapter 4 concludes the thesis with a summary of the results.

1.6 Contribution

The major contributions of this thesis can be summarized as following:

1) The system is analyzed under the modified Binomial copy generation process, which overcomes the drawback of Binomial copy generation process that a packet may not generate any copies at all.

In the pure Binomial copy generation process, each input with a packet will generate a copy to each of the outputs with probability p according to an independent Bernoulli process. Thus the number of copies generated by a packet is Binomially distributed. Clearly, this process has the drawback that a packet may not generate any copies at all. The probability of this happening is $(1-p)^n$.

The modified Binomial copy generation process overcomes this drawback. In this process, a packet generates two types of copies: the primary and secondary copies. First, a fixed number of primary copies ζ are generated and uniformly distributed to the outputs without replacement. Then, secondary copies are generated to each of the outputs which are not selected by the primary copies with probability p. Clearly, $\zeta = 0$ corresponds to pure Binomial copy generation process. Thus, Binomial copy generation process is a special case of modified Binomial copy generation process.

2) The system is modeled as a renewal process, where renewal points correspond to the successful choice of the packets. Using this model, the distribution of the number of packets chosen, given α contending packets at the inputs during a slot,

is determined. Thus, the performance analysis of the multicast switch with random selection policy is given.

3) A Markov chain is applied to model the number of old packets in the system, which provides an analysis of the system with mixed input packets.

4) Due to the numerical difficulties, when the switch size becomes large, it is impossible to calculate the distribution of the number of packets chosen. However, in broadband ISDN, because of the high transmission rates and the diverse services, the switch sizes are expected to be large. Thus, asymptotic results have been derived for very large switches under saturation. The mathematical model used here is also the renewal process. The packet and copy throughput, as well as a lower bound of packet and copy throughput are given.

Chapter 2

Random Selection Policy

This chapter presents the analysis of a multicast switch under random selection policy. As described in the introduction, a multicast switch consists of a copy network and a non-blocking point-to-point switch. The function of the copy network is to replicate input packets. Thus, a packet reaching the head of an input queue generates a random number of copies which are switched to outputs. Only one copy of a packet may be generated to each output. In this chapter, the results are developed under the modified Binomial copy generation process. The system is assumed to operate in discrete time intervals called slots. A slot is the time required to switch a packet from input to output. Accordingly, the duration of events in the analysis are measured in slot times.

If two or more input packets have copies destined to the same output during the same time slot, blocking will occur among these packets. It is assumed that blocking are resolved by random selection policy, i.e., a packet is chosen randomly from the contending packets and then the ones which conflict with the chosen packet are discarded or contend in the next slot.

In this chapter, we give the performance analysis of the random selection policy under the modified Binomial copy generation process. The distribution of the number of packets chosen in a slot is determined for only fresh packets and then for fresh plus old packets at the inputs. The packet and copy throughput are given as a function of the copy generation probability.

2.1 Two models of random selection policy

In this section, we describe, in detail, the random selection policy under consideration. Two variations of the policy will be described and will be referred as the original and the modified random selection policy respectively.

Before the policies are explained, we will define two key phrases which will be useful in the following.

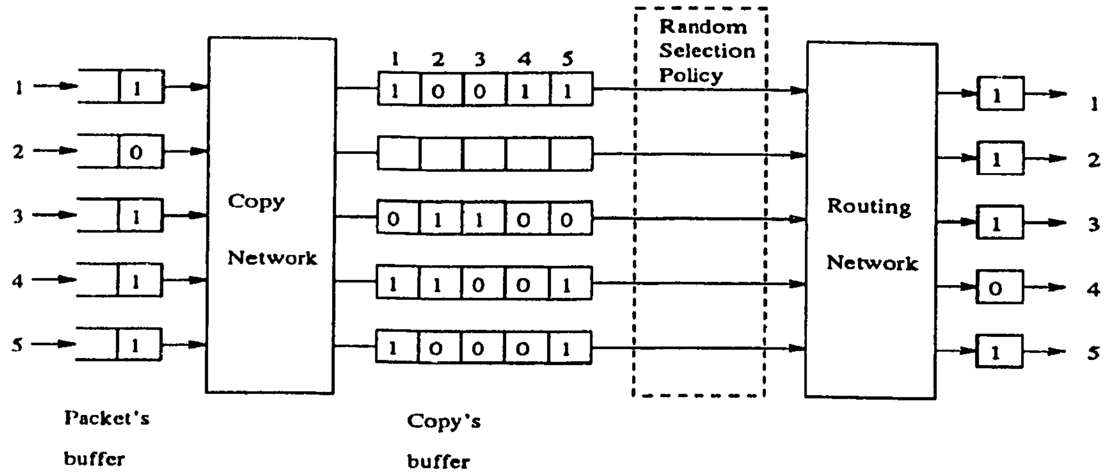
Choosing a packet: This will mean that a packet is chosen from among contending packets at the inputs, and will be switched to the output.

Picking up a packet: This will mean that a packet is picked up from contending packets at the inputs, and may or may not be switched to the output.

During each slot, the original random selection policy will consist of a number of rounds during which the packets to be transmitted will be chosen. In the first round, a packet is chosen randomly from among the contending packets and other packets interfering with the chosen packet are discarded. In the second round, another packet will be chosen randomly from among the remaining contending packets and the packets interfering with the chosen packet will again be discarded. Then the remaining non-interfering packets will compete in the following round. This process will continue on until no more non-interfering packets are left and then the chosen packets will be transmitted during that slot.

Figure 2.1 (a) and (b) illustrate this process with a 5×5 multicast switch. In the figure, there are 4 contending packets at the inputs 1,3,4 and 5. The copies generated by each packet is shown at the outputs of the copy network. For example, the packet at input 1 has copies destined to outputs 1, 4, and 5. In the first round, the packet at input 3 is chosen randomly. As may be seen, the packet at input 4 is discarded because it interferes with the chosen packet at output 2. The packets at inputs 1 and 5 are the remaining non-interfering packets and they compete in the second round. In this round, the packet at input 5 is randomly chosen and then the packet at input 1 is discarded because it interferes with the chosen packet. At this point, there is no non-interfering packet remaining at the inputs. Thus, two packets are chosen and transmitted, while the other two packets are discarded in this slot.

Next we describe the modified random selection policy. This policy may be modeled as many Bernoulli trails as the number of contending packets at the inputs. In the first trial, a packet is picked up randomly from among the contending packets and it is kept as a chosen packet. In the second trial, another packet is picked up randomly. However, this packet is kept only if it does not interfere with the first chosen packet and otherwise discarded. In the third trial, a packet is again picked up from among the remaining contending packets and it is kept only if it does not interfere with the already chosen packets and otherwise discarded. This process will continue on until the contending packets are exhausted and then the chosen packets



(a) 5x5 multicast switch with random selection policy

	First round	Second round
Chosen packets	3	5
Discarded packets	4	1
Remaining packets	1, 5	0

(b) Original random selection ploicy

	First trial	Second trial	Third trial	Forth trial
Picked packets	3	4	5	1
Chosen packets	3		5	
Discarded packets		4		1
Remaining packets	1, 4, 5	1, 5	1	0

(c) Modified random selection policy

Fig. 2.1 An example of multicast switch with original and modified random selection policy are transmitted during that slot. Figure 2.1 (a) and (c) illustrate this process. In the first trial, the packet 3 is picked up randomly and it is kept as a chosen packet. In the second trial, the packet at input 4 is picked up randomly and it is discarded because it interferes with the chosen packet 3. In the third trial, the packet at input 5 is picked up randomly and it is kept as a chosen packet because it does not interfere with the chosen packet. In the forth trial, the packet at input 1 is picked up and it is discarded because it interferes with the chosen packets. Thus packets at input 3 and 5 are transmitted in this slot.

Clearly, both the original and the modified random selection policy describe the same protocol and will give the same results. In the paper [13], both the original and the modified random selection policy under the Binomial copy generation process were analyzed. The results show that both policies are identical. The difference between the two is the way in which they discard the conflicting packets.

In the following analysis, we will use the modified random selection policy model because it gives us a nice closed form expression for the distribution of the number of packets chosen, while the original random selection model does not. However, it is understood that the results will apply to the latter also.

2.2 Copy replication model

In this section, two copy replication models will be described in detail. One of them is Binomial copy generation process and the other is the modified Binomial copy generation process.

In the pure Binomial copy generation process, each input with a packet will generate a copy to each of the outputs with probability p according to an independent Bernoulli process. Thus the number of copies generated by a packet is Binomially distributed. Clearly, this process has the drawback that a packet may not generate any copies at all. The probability of this is $(1-p)^n$.

The modified Binomial copy generation process overcomes this drawback. In this process, a packet generates two types of copies: the primary copies and the secondary copies. First, a fixed number of primary copies, ζ, are generated and uniformly distributed to the outputs. For example, assume that there are three primary copies to be distributed to the outputs. The first one will be equally likely to be sent to any of the outputs. The second one will be equally likely to be sent to any of the outputs except the one chosen by the first primary copy. Then, the third primary copy will be send to any of the outputs except the outputs chosen by the first and second primary copies. After primary copies are distributed to outputs, secondary copies are generated to each of the outputs unselected by the primary copies according to an independent Bernoulli process with probability p. Clearly, $\zeta = 0$ corresponds to pure Binomial copy generation process. Thus, Binomial copy generation process is a special case of modified Binomial copy generation process. If we let $\zeta = 1$ and $p \to 0$, each packet will generate a single copy. Then, this reduces the process to that of unicast switching.

In the following analysis, the copy replication model assumed is the modified Binomial copy generation process.

2.3 Basic Assumptions

The analyses in this chapter are under the following assumptions:

- The switch has n input ports and n outputs ports.

- Each input is busy with probability ρ and this is independent from input to input.

- All copies of a packet should be transmitted in the same slot and they can not be transmitted separately over several slots. Thus, if all the copies of a packet can not be transmitted in the same slot, that packet will be discarded or left to the next slot.

- We assume random selection of packets at the input ports.

- The copy replication model assumed is modified Binomial copy generation process, described in section 2.2.

2.4 Analysis

In this section the packet throughput and copy throughput per port per slot for the multicast switch will be determined. The analysis consists of two parts: The first part will determine the probability that a contending packet will not interfere with the first j chosen packets, and the second part will determine the distribution of the number of packets chosen for a given number of contending packets during a slot.

In the following, we assume that all the contending packets are new (fresh) packets, i.e., packets which can not be transmitted in a slot are discarded and will not be left to the next slot.

2.4.1 Probability P_j

First, the PGF (probability generation function) of the number of unselected outputs after the choice of the j'th packet will be determined. This will lead to the probability that a contending packet will not interfere with the first j chosen packets, P_j.

As explained earlier, a packet generates a secondary copy to each of the outputs which are not chosen by the primary copies with probability p according to an independent Bernoulli process. Let the random variable $\tilde{x}_i$ denote the outcome of this process for the output i,

$$\tilde{x}_i = \begin{cases} 1 & \text{with probability p} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Clearly $\tilde{x}_i$ (i=1,2,...,n) are independent and identically distributed. Its PGF is given by,

$$\tilde{X}(z) = \tilde{X}_i(z) = E[z^{\tilde{x}_i}] = pz + 1 - p \tag{2.2}$$

Let us also define

$$x_i = 1 - \tilde{x}_i$$

Its PGF is given by

$$X(z) = X_i(z) = (1-p)z + p \tag{2.3}$$

In the following, either X(z) or $\tilde{X}(z)$, whichever is more appropriate, will be used.

At the beginning of the selection policy, the number of available outputs and its PGF are given by,

$$r_0 = n$$

$$R_0(z) = E[z^{r_0}] = z^n \tag{2.4}$$

The remaining number of unselected outputs following the first choice will be given by,

$$r_1 = \sum_{i=0}^{r_0-\zeta} x_i$$

The minus ζ in the upper limit accounts for the compulsory primary copies. Since $(r_0 - \zeta)$ is independent of x_i,

$$R_1(z) = E[z^{r_1}] = E[z^{r_0-\zeta}]\ |_{z=X(z)} = \left\{z^{-\zeta}E[z^{r_0}]\right\}\ |_{z=X(z)} \tag{2.5}$$

$$R_1(z) = \frac{[1-(1-p)(1-z)]^n}{[1-(1-p)(1-z)]^\zeta} \tag{2.6}$$

Similarly, the outputs available for the j'th choice are those still unselected following the $(j-1)$'th choice. The j'th packet will generate ζ primary copies plus a number of secondary copies. Thus, the remaining number of unselected outputs following the j'th choice is related to that of the $(j-1)$'th choice by,

$$r_j = \sum_{i=1}^{r_{j-1}-\zeta} x_i$$

Since r_{j-1} and x_i are independent,

$$R_j(z) = E[z^{r_{j-1}-\zeta}]\Big|_{z=X(z)} = \left\{z^{-\zeta}E[z^{r_{j-1}}]\right\}\Big|_{z=X(z)}$$

$$R_j(z) = [z^{-\zeta}R_{j-1}(z)]\Big|_{z=X(z)} \tag{2.7}$$

The repeated application of the above equation with equation (2.4) as the initial condition results in,

$$R_j(z) = \frac{[1-(1-p)^j(1-z)]^n}{\prod_{i=1}^{j}[1-(1-p)^i(1-z)]^\zeta} \tag{2.8}$$

Next, we determine the number of outputs selected by the j'th packet, which is given by,

$$n_j = \zeta + \sum_{i=0}^{r_{j-1}-\zeta} \tilde{x}_i$$

Where r_{j-1} is the total number of outputs available for selection by the j'th packet. In the above ζ denotes the primary copies and the following summation denotes the secondary copies generated by the j'th chosen packet. Taking the PGF of the both sides of the above equation results in,

$$N_j(z) = E[z^{n_j}] = z^{\zeta} \left[z^{-\zeta} R_{j-1}(z)\right]\Big|_{z=\tilde{X}(z)} \tag{2.9}$$

Substituting $R_{j-1}(z)$ from equation (2.8),

$$N_j(z) = \frac{z^{\zeta}[1 - p(1-p)^{j-1}(1-z)]^n}{[1 - p(1-z)]^{\zeta} \prod_{i=1}^{j-1} [1 - p(1-p)^i(1-z)]^{\zeta}} \tag{2.10}$$

From above, the average number of outputs selected by the j'th chosen packet is,

$$\bar{n}_j = \left.\frac{dN_j(z)}{dz}\right|_{z=1}$$

$$\bar{n}_j = [\zeta + p(n-\zeta)](1-p)^{j-1} = \mu(1-p)^{j-1} \tag{2.11}$$

Where $\mu = \zeta + p(n-\zeta)$ is the average number of copies generated by a contending packet. We can see that only the first chosen packet has the same average number of copies as a contending packet and other chosen packets have smaller average number of copies than that.

In the following, the probability P_j that a contending packet will not interfere with any of the earlier chosen packets will be determined. Let us define,

m_j = the total number of outputs selected by the first j chosen packets. Then

$$m_j = \sum_{i=0}^{j} n_i = n - r_j$$

where r_j is the number of outputs available after the j'th choice,

$$M_j(z) = E[z^{m_j}] = E[z^{n-r_j}] = z^n E[z^{-r_j}]$$

$$M_j(z) = z^n R_j(z^{-1}) \tag{2.12}$$

Substituting for $R_j(z)$ from equation (2.8), then

$$M_j(z) = \frac{z^{j\zeta}[z-(1-p)^j(z-1)]^n}{\prod_{i=1}^{j}[z-(1-p)^i(z-1)]^\zeta} \tag{2.13}$$

The average number of outputs chosen is given by,

$$\bar{m}_j = \left.\frac{dM_j(z)}{dz}\right|_{z=1}$$

$$\bar{m}_j = n[1-(1-p)^j] + \frac{\zeta[(1-p)-(1-p)^{j+1}]}{p} \tag{2.14}$$

A contending packet will interfere with the first j chosen packets if it generates copies to the outputs which are selected by these packets. Let,

U_j=the total number of copies that a contending packet has been interfering with the first j chosen packets.

u = the number of primary copies that a contending packet has been interfering with the j chosen packets. Then U_j is given by,

$$U_j = u + \sum_{i=1}^{m_j-u} \tilde{x}_i$$

The PGF of U_j conditioned on u and m_j is given by,

$$U_j(z/u, m_j) = z^u[\tilde{X}(z)]^{m_j-u}$$

Next we determine the probability distribution of u. The n outputs can be considered as consisting of two types of objects, ζ of them which have a primary copy of the contending packet for them and the $(n-\zeta)$ of them which do not have. Let us define success if a primary copy is destined to one of the m_j outputs selected by the j chosen packets. Then u is equal to the number of successes chosen in a random sample of size m_j without replacement from the n outputs. There are $\begin{pmatrix} \zeta \\ u \end{pmatrix}$ ways of selecting u successes from the ζ that are available and for each of these ways we can choose the $m_j - u$ failures in $\begin{pmatrix} n-\zeta \\ m_j - u \end{pmatrix}$ ways. Thus the total number of favorable samples among the $\begin{pmatrix} n \\ m_j \end{pmatrix}$ possible samples is given by $\begin{pmatrix} \zeta \\ u \end{pmatrix}\begin{pmatrix} n-\zeta \\ m_j - u \end{pmatrix}$. Thus, the random variable u has the hypergeometric distribution given by [14],

$$Pr(u/m_j) = \frac{\begin{pmatrix} \zeta \\ u \end{pmatrix}\begin{pmatrix} n-\zeta \\ m_j - u \end{pmatrix}}{\begin{pmatrix} n \\ m_j \end{pmatrix}} \tag{2.15}$$

Unconditioning with respect to both u and m_j, we get the PGF of U_j,

$$U_j(z) = \sum_{m_j} \sum_{u=0}^{\zeta} z^u [\tilde{X}(z)]^{m_j - u} Pr(u/m_j) Pr(m_j) \tag{2.16}$$

Substitution of $z = 0$ gives the probability that a contending packet will not interfere with the first j chosen packets, P_j. Clearly in the inner summation, only the value of $u = 0$ will contribute to this probability. Thus,

$$P_j = U_j(z)\big|_{u=z=0}$$

$$P_j = \left\{ \sum_{m_j} [\tilde{X}(z)]^{m_j} Pr(u=0/m_j) Pr(m_j) \right\}\Bigg|_{z=0}$$

$$P_j = \left\{ \frac{(n-\zeta)!}{n!} \sum_{m_j} \frac{(n-m_j)!}{(n-m_j-\zeta)!} [\tilde{X}(z)]^{m_j} Pr(m_j) \right\}\Bigg|_{z=0} \tag{2.17}$$

It has not been possible to find a closed form expression for the above, therefore a number of specific cases will be given. But first, a z transform property is introduced which will be used in the following.

z transform property: If we have the z transform pair, $f(n) \Longleftrightarrow F(z)$, where $F(z) = \sum_{n=0}^{\infty} f(n)z^n$, then the following are also,

$$\sum_{n=0}^{\infty} nf(n)z^n \Longleftrightarrow z\frac{dF(z)}{dz} \tag{2.18}$$

$$\sum_{n=0}^{\infty} n^2 f(n)z^n \Longleftrightarrow z^2\frac{d^2F(z)}{dz^2} + z\frac{dF(z)}{dz} \tag{2.19}$$

$$\sum_{n=0}^{\infty} n^3 f(n)z^n \Longleftrightarrow z^3\frac{d^3F(z)}{dz^3} + 3z^2\frac{d^2F(z)}{dz^2} + z\frac{dF(z)}{dz} \tag{2.20}$$

Next we are going to use the above results to determine P_j for specific values of ζ.

- $\zeta = 0$

Substituting $\zeta = 0$ into equation (2.17), we have

$$\begin{aligned} P_j &= \left\{ \sum_{m_j} [\tilde{X}(z)]^{m_j} Pr(m_j) \right\} \Bigg|_{z=0} \\ &= \left\{ M_j(z)\big|_{z=\tilde{X}(z)} \right\} \Big|_{z=0} \end{aligned}$$

Substituting for $M_j(z)$ from equation (2.13) and letting $\zeta = 0$, results in,

$$P_j = [1 - p + p(1-p)^j]^n \tag{2.21}$$

- ζ=1

Substituting $\zeta = 1$ into equation (2.17), we have

$$P_j = \left\{ \frac{1}{n} \sum_{m_j} (n - m_j)[\tilde{X}(z)]^{m_j} Pr(m_j) \right\} \Bigg|_{z=0}$$

$$= \left\{ \frac{1}{n} \left[\sum_{m_j} n[\tilde{X}(z)]^{m_j} Pr(m_j) - \sum_{m_j} m_j [\tilde{X}(z)]^{m_j} Pr(m_j) \right] \right\} \Big|_{z=0}$$

Using z transform property in equation (2.18),

$$P_j = \left\{ \frac{1}{n} [nM_j(z) - z\frac{dM_j(z)}{dz}] \Big|_{z=\tilde{X}(z)} \right\} \Big|_{z=0}$$

Substituting for $M_j(z)$ from equation (2.13) and letting $\zeta = 1$, results in,

$$\begin{aligned} P_j &= \frac{[1-p+p(1-p)^j]^n}{\prod_{i=1}^{j}[1-p+p(1-p)^i]} \left\{ (1-p)^j(1-\frac{j}{n}) \right. \\ &\left. +(1-p)^{j+1}[\frac{1}{n}\sum_{i=1}^{j} \frac{1-(1-p)^i}{[1-p+p(1-p)^i]} - \frac{1-(1-p)^j}{1-p+p(1-p)^j}] \right\} \end{aligned} \quad (2.22)$$

- $\zeta = 2$

Substituting $\zeta = 2$ into equation (2.17), we have

$$\begin{aligned} P_j &= \left\{ \frac{1}{n(n-1)} \sum_{m_j} (n-m_j)(n-m_j-1)[\tilde{X}(z)]^{m_j} Pr(m_j) \right\} \Big|_{z=0} \\ &= \left\{ \frac{1}{n(n-1)} \sum_{m_j} [n^2 - n - (2n-1)m_j + m_j^2][\tilde{X}(z)]^{m_j} Pr(m_j) \right\} \Big|_{z=0} \\ &= \left\{ \frac{1}{n(n-1)} \left[n(n-1) \sum_{m_j} [\tilde{X}(z)]^{m_j} Pr(m_j) \right. \right. \\ &\left. \left. -(2n-1) \sum_{m_j} m_j [\tilde{X}(z)]^{m_j} Pr(m_j) + \sum_{m_j} m_j^2 [\tilde{X}(z)]^{m_j} Pr(m_j) \right] \right\} \Big|_{z=0} \end{aligned}$$

Using z transform properties in equations (2.18) and (2.19),

$$P_j = \left\{ \frac{1}{n(n-1)} [n(n-1)M_j(z) - (2n-1)z\frac{dM_j(z)}{dz} + z^2\frac{d^2M_j(z)}{dz^2}] \Big|_{z=\tilde{X}(z)} \right\} \Big|_{z=0}$$

Substituting for $M_j(z)$ from equation (2.13) and letting $\zeta = 2$, results in,

$$P_j = \frac{(1-p)^{j\zeta}[1-p+p(1-p)^j]^n}{\prod_{i=1}^{j}[1-p+p(1-p)i]^{\zeta}} \left\{ (1-\frac{j\zeta}{n})(1-\frac{j\zeta}{n-1}) \right.$$

$$-2(1-\frac{j\zeta}{n-1})\frac{(1-p)[(1-(1-p)^j]}{1-p+(1-p)^j}+\frac{(1-p)^2[1-(1-p)^j]^2}{[1-p+p(1-p)^j]^2}$$
$$+2\zeta(1-p)\left\{\frac{1}{n}-\frac{j\zeta}{n(n-1)}-\frac{(1-p)[1-(1-p)^j]}{(n-1)[1-p+(1-p)^j]}\right\}\sum_{i=1}^{j}\frac{1-(1-p)^i}{1-p+p(1-p)^i}$$
$$+\frac{(1-p)^2\zeta}{n(n-1)}\sum_{i=1}^{j}[\frac{1-(1-p)^i}{1-p+p(1-p)^i}]$$
$$+\frac{(1-p)^2\zeta^2}{n(n-1)}[\sum_{i=1}^{j}\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^2\Bigg\} \qquad (2.23)$$

- $\zeta = 3$

Substituting $\zeta = 3$ into equation (2.17), we have

$$P_j = \left\{\frac{1}{n(n-1)(n-2)}\sum_{m_j}[n-m_j)(n-m_j-1)(n-m_j-2)[\tilde{X}(z)]^{m_j}Pr(m_j)\right\}\Bigg|_{z=0}$$
$$= \Bigg\{\frac{1}{n(n-1)(n-2)}\sum_{m_j}[n(n-1)(n-2)-(3n^2-6n+2)m_j$$
$$+3(n-1)m_j^2-m_j^3][\tilde{X}(z)]^{m_j}Pr(m_j)\Bigg\}\Bigg|_{z=0}$$
$$= \Bigg\{\frac{1}{n(n-1)(n-2)}\Bigg[n(n-1)(n-2)\sum_{m_j}[\tilde{X}(z)]^{m_j}Pr(m_j)$$
$$-(3n^2-6n+2)\sum_{m_j}m_j[\tilde{X}(z)]^{m_j}Pr(m_j)+3(n-1)\sum_{m_j}m_j^2[\tilde{X}(z)]^{m_j}Pr(m_j)$$
$$-\sum_{m_j}m_j^3[\tilde{X}(z)]^{m_j}Pr(m_j)\Bigg]\Bigg\}\Bigg|_{z=0}$$

Using z transform properties in equations (2.18), (2.19) and (2.20),

$$P_j = \Bigg\{\frac{1}{n(n-1)(n-2)}[n(n-1)(n-2)M_j(z)-(3n-1)(n-2)z\frac{dM_j(z)}{dz}$$
$$+3(n-1)z^2\frac{d^2M_j(z)}{dz^2}-z^3\frac{d^3M_j(z)}{dz^3}]\Bigg|_{z=\tilde{X}(z)}\Bigg\}\Bigg|_{z=0}$$

Substituting for $M_j(z)$ from equation (2.13) and letting $\zeta = 3$, results in,

$$P_j = \frac{(1-p)^{3j}[1-p+p(1-p)^j]^n}{\prod_{i=1}^{j}[1-p+p(1-p)^i]^3}\Bigg\{(1-\frac{3j}{n})(1-\frac{3j}{n-1})(1-\frac{3j}{n-2})$$

$$
\begin{aligned}
&-3(1-\frac{3j}{n-1})(1-\frac{3j}{n-2})\frac{(1-p)[1-(1-p)^j]}{1-p+p(1-p)^j}+3(1-\frac{3j}{n-2})\frac{(1-p)^2[1-(1-p)^j]^2}{[1-p+p(1-p)^j]^2} \\
&-\frac{(1-p)^3[1-(1-p)^j]^3}{[1-p+p(1-p)^j]^3}+3(1-p)\left\{\frac{3}{n}(1-\frac{3j}{n-1})(1-\frac{3j}{n-2})\right. \\
&-2\frac{3}{n-1}(1-\frac{3j}{n-2})\frac{(1-p)[1-(1-p)^j]}{[1-p+p(1-p)^j]} \\
&\left.+\frac{3}{n-2}\frac{(1-p)^2[1-(1-p)^j]^2}{[1-p+p(1-p)^j]^2}\right\}\sum_{i=1}^{j}\frac{1-(1-p)^i}{1-p+p(1-p)^i} \\
&+3(1-p)^2\left\{\frac{1}{n(n-1)}(1-\frac{3j}{n-2})-\frac{(1-p)[1-(1-p)^j]}{(n-1)(n-2)[1-p+p(1-p)^j]}\right\} \\
&+\left\{\sum_{i=1}^{j}3[\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^2+[\sum_{i=1}^{j}3\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^2\right\} \\
&+\frac{3(1-p)^3}{n(n-1)(n-2)}\sum_{i=1}^{j}[\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^3+\frac{9(1-p)^3}{n(n-1)(n-2)}[\sum_{i=1}^{j}\frac{1-(1-p)^i}{1-p+p(1-p)^i}] \\
&+\left\{\sum_{i=1}^{j}3[\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^2\right\}+\frac{3(1-p)^3}{n(n-1)(n-2)}[\sum_{i=1}^{j}\frac{1-(1-p)^i}{1-p+p(1-p)^i}]^3\Bigg\}
\end{aligned}
\qquad (2.24)
$$

If we let $p \to 0$, each packet will generate only ζ primary copies and m_j is simply a constant $m_j = j\zeta$. Then from equation (2.17) P_j becomes,

$$P_j = \frac{(n-\zeta)!(n-j\zeta)!}{n!(n-j\zeta-j)!}$$

or,

$$P_j = \prod_{i=0}^{\zeta-1}(1-\frac{j\zeta}{n-1})$$

Finally, letting ζ=1,

$$P_j = 1 - \frac{j}{n}$$

which corresponds to the case that each packet generates a single copy that is equally likely to go to any output. Clearly this is exactly the unicast switching case.

2.4.2 The distribution of the number of chosen packets

In the above, the probability that a contending packet will not interfere with the first j chosen packets (P_j) has been determined. In this section, the PGF of the distribution of the number of chosen packets given α contending packets at the inputs during a slot will be derived.

There has been other work to determine the distribution of the number of chosen packets for the modified random selection policy. In [13], the analysis of the modified random selection policy is given, in which a discrete time birth process is used to model the system. In this section, we will give a alternative modeling of the modified random selection policy using renewal process. It will be seen that the analysis using a renewal process and the analysis using birth process give identical results.

As described before, the modified random selection policy performs α Bernoulli trials where α is the number of contending packets. In each trial we pick up a packet randomly either keeping it or throwing it away. We always keep the packet picked up in the first trial. Then, we discard the packets picked up in subsequent trials which interfere with the first chosen packet. The probability that a packet picked up will not interfere with the first chosen packet is P_1. When we pick up a non-interfering packet with the first chosen one, this packet becomes the second chosen packet. Clearly, the number of trials from first to second choice will be geometrically distributed with parameter P_1. After the choice of the second packet, we continue to pick up and discard interfering packets until we have a non-interfering packet with the first and second chosen ones, and this becomes the third chosen packet. The number of trials from the second choice to the third choice will be geometrically distributed with pa-

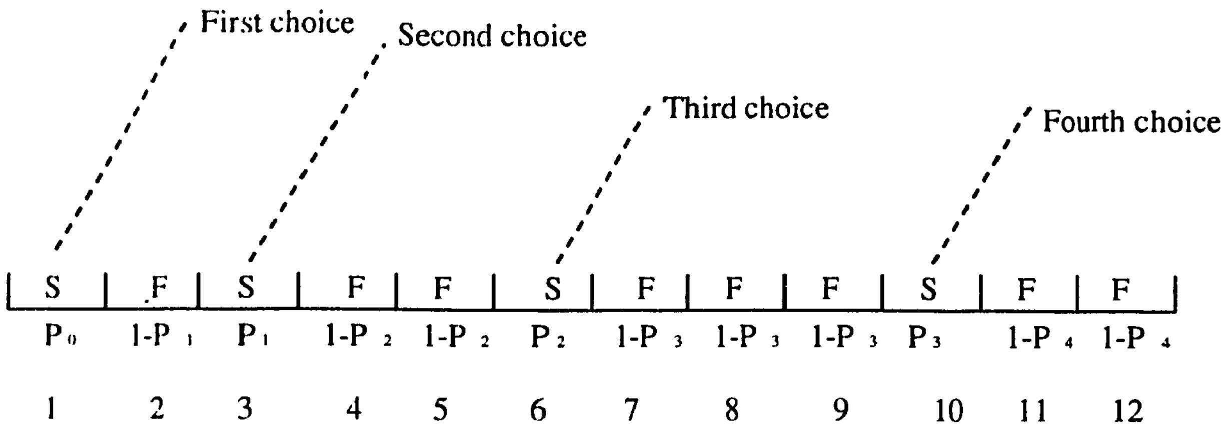


Figure 2.2: An example of choosing packets by the modified random selection policy

rameter P_2. The process continues until we run out of contending packets. Figure 2.2 is an example of choosing packets by the modified random selection policy. In this example, four packets are chosen from $\alpha(\alpha = 12)$ contending packets.

From above, the number of trials leading to each successful choice of a packet is geometrically distributed with a different parameter. Clearly, this process may be modeled as a renewal process with the renewal points taken as the trials that packets are chosen. In the following, a renewal will correspond to the successful choice of a packet.

Let us define X_i as,

X_i = the number of trials for the i'th choice after the (i-1)'th choice, Then, the distribution of X_i is geometric with parameter P_{i-1},

$$P(X_i = k) = (1 - P_{i-1})^{k-1} P_{i-1} \qquad i \geq 1 \tag{2.25}$$

Where we assume that $P_0 = 1$, since the first trial always ends up in a success. The

PGF of X_i is given,

$$X_i(z) = \frac{P_{i-1}z}{1-(1-P_{i-1})z} \qquad (i = 1, 2, 3, ..., j) \tag{2.26}$$

Thus, the renewal process under consideration has independent renewal times, which are geometrically distributed with different parameters.

Next we shall determine the PGF of the distribution of the packets chosen. This analysis follows the one in [18, 17], which gives the analysis for the case of independent identically distributed renewal times.

Let, S_j = the total number of trials required until we choose the j'th packet. Then,

$$S_j = X_1 + X_2 + ... + X_i + ... + X_j \tag{2.27}$$

Where X_i (i=1,2,...,j) are independent but not identically distributed random variables.

Let N_α denote the number of renewals occurring in $[0, \alpha]$. Clearly, we have

$$P(N_\alpha \geq j) = P(S_j \leq \alpha) \tag{2.28}$$

Then, the probability that j and only j packets chosen from α contending packets is

$$P(N_\alpha = j) = P(N_\alpha \geq j) - P(N_\alpha \geq j+1) \tag{2.29}$$

Substituting from (2.28)

$$\begin{aligned} P(N_\alpha = j) &= P(S_j \leq \alpha) - P(S_{j+1} \leq \alpha) \\ &= \sum_{r=1}^{\alpha} [P(S_j = r) - P(S_{j+1} = r)] \end{aligned}$$

Let $f_r^j = P(S_j = r)$, then

$$P(N_\alpha = j) = \sum_{r=1}^{\alpha} [f_r^j - f_r^{j+1}]$$

The PGF of the probability distribution of N_α is given by,

$$\begin{aligned} P_j(z) &= \sum_{\alpha=0}^{\infty} P(N_\alpha = j) z^\alpha \\ &= \sum_{\alpha=1}^{\infty} z^\alpha \sum_{r=1}^{\alpha} [f_r^j - f_r^{j+1}] \end{aligned}$$

Interchanging the order of the summations,

$$P_j(z) = \sum_{r=1}^{\infty} [f_r^j - f_r^{j+1}] \sum_{\alpha=r}^{\infty} z^\alpha$$

Substitute $\sum_{\alpha=r}^{\infty} z^\alpha = \frac{z^r}{1-z}$ into above,

$$P_j(z) = \frac{1}{1-z}\left[\sum_{r=1}^{\infty} f_r^j z^r - \sum_{r=1}^{\infty} f_r^{j+1} z^r\right] \tag{2.30}$$

In the above equation, $\sum_{r=1}^{\infty} f_r^j z^r$ is the PGF of S_j where S_j is the total number of trials required for j successes and its PGF is given by,

$$S_j(z) = \sum_{r=1}^{\infty} P(S_j = r) z^r = \sum_{r=1}^{\infty} f_r^j z^r$$

$$S_j(z) = E[z^{S_j}] = E[z^{X_1+X_2+\ldots+X_j}]$$

Since X_i (i=1,2,...j) are independent variables,

$$S_j(z) = \prod_{i=1}^{j} X_i(z)$$

Substituting $S_j(z)$ into equation (2.30),

$$P_j(z) = \frac{1}{1-z}[\prod_{i=1}^{j} X_i(z) - \prod_{i=1}^{j+1} X_i(z)]$$

Substituting for $X_i(z)$ from equation (2.26),

$$P_j(z) = \frac{1}{1-z}\left[\prod_{i=1}^{j} \frac{P_{i-1} z}{1-(1-P_{i-1})z} - \prod_{i=1}^{j+1} \frac{P_{i-1} z}{1-(1-P_{i-1})z}\right]$$

$$
\begin{aligned}
&= \frac{1}{1-z}\prod_{i=1}^{j}\frac{P_{i-1}z}{1-(1-P_{i-1})z}[1-\frac{p_j z}{1-(1-P_j)z}] \\
&= \frac{1}{1-(1-P_j)z}\prod_{i=1}^{j}\frac{P_{i-1}z}{1-(1-P_{i-1})z} \\
&= \frac{1}{P_j z}\prod_{i=1}^{j+1}\frac{P_{i-1}z}{1-(1-P_{i-1})z}
\end{aligned}
$$

Now, separating $i = 1$ from the rest of the product and substituting $P_0 = 1$,

$$P_j(z) \;=\; \frac{1}{P_j}\prod_{i=2}^{j+1}\frac{P_{i-1}z}{1-(1-P_{i-1})z}$$

Defining i=i-1,

$$P_j(z) = \frac{z^j}{P_j}\prod_{i=1}^{j}\frac{P_i}{1-(1-P_i)z} \tag{2.31}$$

Thus, the PGF of the number of packets chosen in a slot given α contending packets at the inputs is determined. This is exactly the same as the result in [13].

Let

$$P_j(z) = z^j P'(z)$$

Where

$$P_j'(z) = \frac{1}{P_j}\prod_{i=1}^{j}\frac{P_i}{1-(1-P_i)z}$$

Using partial fraction expansion, the above product expression can be changed to a summation,

$$P_j'(z) = \sum_{k=1}^{j}\frac{A_k}{1-(1-P_k)z}$$

where

$$
\begin{aligned}
A_k &= [1-(1-P_k)z]P_j'(z)\Big|_{z=\frac{1}{1-P_k}} \\
&= \frac{P_k}{P_j}\prod_{x=1,x\neq k}^{j}\frac{P_k}{[1-(1-P_x)z]}\Bigg|_{z=\frac{1}{1-P_k}} \\
&= \frac{P_k}{P_j}\prod_{x=1,x\neq k}^{j}\frac{(1-P_k)}{1-\frac{P_k}{P_x}}
\end{aligned}
$$

The term $A_k/(1-(1-P_k)z)$ can be inverted to give,

$$\frac{A_k}{1-(1-P_k)z} \Longleftrightarrow A_k(1-P_k)^{\alpha}$$

Thus, $P_j'(z)$ is inverted to give,

$$P_j'(z) \Longleftrightarrow \sum_{k=1}^{j} A_k(1-P_k)^{\alpha}$$

$P_j(z) = z^j P_j'(z)$ is inverted to give,

$$P_j(z) \Longleftrightarrow \sum_{k=1}^{j} A_k(1-P_k)^{\alpha-j}$$

The above equation gives,

$$P(N_\alpha = j) = \frac{1}{P_j}\sum_{k=1}^{j} P_k(1-P_k)^{\alpha-j} \prod_{x=1,x\neq k}^{j} \frac{(1-P_k)}{1-\frac{P_k}{P_x}} \qquad 1 \le j \le \alpha \tag{2.32}$$

Now letting $P_j(\alpha)$ denote $P(N_\alpha = j)$,

$$P_j(\alpha) = \frac{1}{P_j}\sum_{k=1}^{j} P_k(1-P_k)^{\alpha-j} \prod_{x=1,x\neq k}^{j} \frac{(1-P_k)}{1-\frac{P_k}{P_x}} \qquad 1 \le j \le \alpha \tag{2.33}$$

This gives the distribution of the number of packets chosen given α contending packets at the inputs during a slot for the modified random selection policy. Let $C(\alpha)$ denote the probability that there are α contending packets at the inputs. Since it has been assumed that packets not chosen in a slot are discarded and each input generates a new packet at the beginning of the next slot with probability ρ,

$$C(\alpha) = \binom{n}{\alpha} \rho^{\alpha}(1-\rho)^{n-\alpha}$$

Then, the packet throughput defined as the average number of packets transmitted per port per slot is given by,

$$T_p = \sum_{\alpha=1}^{n}[\frac{1}{n}\sum_{j=1}^{\alpha} jP_j(\alpha)]C(\alpha) \tag{2.34}$$

The copy throughput defined as the average number of copies transmitted per port per slot is given by,

$$T_c = \sum_{\alpha=1}^{n}[\frac{1}{n}\sum_{j=1}^{\alpha}\bar{m}_j P_j(\alpha)]C(\alpha) \tag{2.35}$$

Where $\bar{m}_j$ gives the average number of copies generated by the j chosen packets. Substituting for $\bar{m}_j$ from equation (2.14),

$$T_c = \sum_{\alpha=1}^{n}[\sum_{j=1}^{n}\left\{1-(1-p)^j + \frac{\zeta[(1-p)-(1-p)^{j+1}]}{np}\right\} P_j(\alpha)]C(\alpha) \tag{2.36}$$

2.5 The analysis of multicast switch with mixed input packets

In the analysis above, we assumed that all the contending packets are fresh packets which are generated at the beginning of a slot. That means packets which can not be transmitted in a slot are discarded and will not be left to the next slot. In this section, this assumption is dropped and from here on the contending packets will be the mix of two types of packets. As before, there will be fresh packets which are generated at the beginning of a slot. The second type of packets will be the ones which were not chosen in the previous slot and left to the next slot. These will be referred to as old packets. As a result of the old packets, the number of packets contending in a slot will not be independent of that in the previous slots. The objective of the following analysis is mainly to determine the steady-state distribution of the number of packets chosen in a slot.

2.5.1 Copy generation probability p

As is said above, the contending packets in a slot are the mix of fresh and old packets. The average number of copies generated by the j'th chosen packet determined in equation (2.11) has shown that the average size of chosen packets is smaller than the average size of the contending packets. As a result of this, the packets left behind (old packets) will be larger packets and their copy generation probability to each output will be higher. Let us assume that the fresh and the old packets generate a secondary copy to each output with probabilities p_f and p_o respectively. Then, we have $p_f < p_o$. Let us further assume that each of the mixed packets will generate a secondary copy to an output with probability p. Obviously, $p_f < p < p_o$. In the following, we will use the mixed copy generation probability p in the analysis. Then, a relationship between p and p_f will be derived.

2.5.2 The steady-state distribution of the number of chosen packets

Determining the steady-state distribution of the number of packets chosen is the core of the analysis in this section. Obviously, the number of contending packets will vary as a function of the slot. When the system is in steady-state, the distribution of the number of packets chosen should be same in each slot, i.e., the distribution of the number of old packets should be same in each slot. As a preliminary, we shall determine the latter distribution. A Markov chain analysis will be applied to find this distribution with the embedded points at the end of the slots (Fig.2.3). The packets at an embedded point are those not chosen in the previous slot and left for the next one, which are called old packets. The largest number of old packets at an embedded

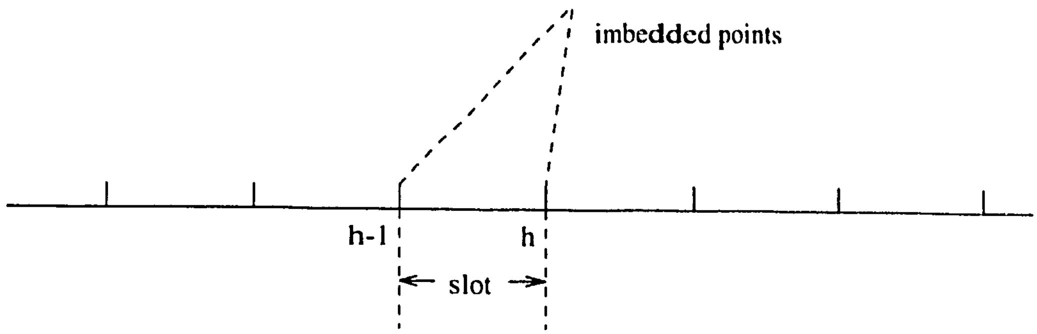


Figure 2.3: The Markov chain model of the number of old packets

point is $n-1$. That is because at least one packet will be transmitted in any slot if there are any contending packets at the inputs. It will be assumed that any input without a packet at an embedded point may generate a fresh packet with probability ρ at the beginning of the next slot. The sum of the old and fresh packets will contend for random selection in that slot, and the chosen packets will be transmitted in the same slot. As explained the unchosen ones become old packets for the following slot.

Let us assume that there are t old packets at the imbedded point h (see Fig. 2.3) and there are m old packets at the previous point (h-1). Then we introduce the following additional notation,

P_{mt}: transition probability of having t old packets at an embedded point given m old packets at the previous embedded point.

v(f/m): probability of f new packets arriving at the beginning of a slot given m old packets at the preceding embedded point.

$\tau(i/m)$: probability that i packets will be contending during a slot given m old packets at the preceding embedded point. This corresponds to (i-m) new arrivals.

π_k: probability of having k old packets at an embedded point at the steady-state.

$C(i)$: probability that there are i contending packets at the inputs at the beginning of a slot.

W_j: probability that j packets will be chosen in a slot at the steady-state.

Next the transition probabilities, P_{mt}, will be determined, but first let us determine,

$$v(f/m) = Pr(f \; new \; arrivals/m \; old \; packets)$$

Since there are m old packets, there will be (n-m) idle inputs , each of which may generate a fresh packet with probability ρ at the beginning of the next slot, Thus

$$v(f/m) = \begin{pmatrix} n-m \\ f \end{pmatrix} \rho^f (1-\rho)^{n-m-f} \tag{2.37}$$

τ(i/m) = Pr(i packets contending / m old packets). Clearly, i packets will be contending during a slot given m old packets if there are (i-m) new arrivals, thus

$$\tau(i/m) = \begin{cases} v(i-m/m) & i \geq m \\ 0 & i \leq m \end{cases}$$

Substituting from equation (2.37),

$$\tau(i/m) = \begin{pmatrix} n-m \\ i-m \end{pmatrix} \rho^{i-m}(1-\rho)^{n-i} \qquad i \geq m \tag{2.38}$$

Next, we give the transition probability matrix.

$$P = \begin{vmatrix} P_{00} & P_{01} & P_{02} & \cdots & \cdots & \cdots & P_{0,n-1} \\ P_{10} & P_{11} & P_{12} & \cdots & \cdots & \cdots & P_{1,n-1} \\ \vdots & \vdots & \vdots & \cdots & P_{mt} & \cdots & \vdots \\ P_{n-1,0} & P_{n-1,1} & P_{n-1,2} & \cdots & \cdots & \cdots & P_{n-1,n-1} \end{vmatrix}$$

Where, P_{mt} is the transition probability of having t old packets at an embedded point given m old packets at the previous embedded point. There are two cases to be considered. The first case is m=t=0. In that case, if there are i contending packets ($i \geq 0$), all these contending packets should be transmitted and there are no old packets left. So that there should be i new arrivals and i chosen packets. The second case is that both m and t are not zero at the same time. Then the number of contending packets should be larger than zero. Thus at least one packet is chosen. In that case, the total number of packets contending between the two embedded points should be equal or larger than the maximum of m and t+1. Assuming a total of i contending packets, there should be (i-m) new arrivals and (i-t) chosen packets. Thus, after considering these two cases, we have

$$P_{mt} = \begin{cases} \sum_{i=0}^{n} P_i(i)\tau(i/m) & \text{if m=t=0} \\ \sum_{i=max(m,t+1)}^{n} P_{i-t}(i)\tau(i/m) & \text{otherwise} \end{cases} \tag{2.39}$$

Where P_{i-t}(i) is the probability that (i-t) packets will be chosen from a total of i contending packets in a slot and is given by equation (2.33), which is repeated here for convenience,

$$P_j(\alpha) = \frac{1}{P_j} \sum_{k=1}^{j} P_k(1-P_k)^{\alpha-j} \prod_{x=1,x\neq k}^{j} \frac{(1-P_k)}{1-\frac{P_k}{P_x}} \qquad 1 \leq j \leq \alpha$$

obviously, $P_0(0) = 1$.

After determining the transition probability matrix, the steady-state distribution of the number of packets at an embedded point can be solved by,

$$\pi = \pi P \tag{2.40}$$

Where π is the probability vector, $\pi = [\pi_0 \pi_1 ... \pi_k ... \pi_{n-1}]$

The above equation gives the probability that k old packets will contend in a slot. Next, we determine the distribution of the total number of contending packets in a slot. The probability that there are i contending packets at the inputs at the beginning of a slot is given by,

$$C(i) = \sum_{k=0}^{i} \tau(i/k)\pi_k \tag{2.41}$$

Substituting for $\tau(i/k)$ from equation (2.38)

$$C(i) = \sum_{k=0}^{min(i,n-1)} \begin{pmatrix} n-k \\ i-k \end{pmatrix} \rho^{i-k}(1-\rho)^{n-i}\pi_k \tag{2.42}$$

Now, we are ready to determine the steady-state distribution of the number of packets chosen during a slot, W_j. Clearly, $P_j(i)$ gives probability that j packets are chosen given i contending packets during a slot. Thus

$$W_j = \sum_{i=j}^{n} P_j(i)C(i)$$

Substituting for $C(i)$ from equation (2.42),

$$W_j = \begin{cases} (1-\rho)^n \pi_0 & j = 0 \\ \sum_{k=0}^{n-1} \sum_{i=max(j,k)}^{n} P_j(i) \begin{pmatrix} n-k \\ i-k \end{pmatrix} \rho^{i-k}(1-\rho)^{n-i}\pi_k & 1 \le j \le n-1 \\ \sum_{k=0}^{n-1} P_n(n)\rho^{n-k}\pi_k & j = n \end{cases} \tag{2.43}$$

The steady-state packet and copy throughput defined as the average number of packets and copies transmitted per port per slot respectively are given:

$$T_p = \frac{1}{n}\sum_{j=1}^{n} jW_j \tag{2.44}$$

$$T_c = \frac{1}{n}\sum_{j=1}^{n} \bar{m}_j W_j \tag{2.45}$$

Where $\bar{m}_j$ is the total number of copies in j chosen packets and is given by equation (2.14). Substituting for $\bar{m}_j$,

$$T_c = \sum_{j=1}^{n} \left\{1 - (1-p)^j + \frac{\zeta[(1-p) - (1-p)^{j+1}]}{np}\right\} W_j \tag{2.46}$$

The steady-state distribution and the throughput results determined above are as a function of the mixed packet copy generation probability p. As described before, contending packets in each slot are the mix of fresh and old packets. Next, we express these results as a function of fresh packet copy generation probability p_f. When system is at the equilibrium, the average number of copies that a fresh packet generates must be equal to the average number of copies of a departing packet at the output, since every arriving packet will eventually depart from the system. The former average is given by $\mu_f = \zeta + (n-\zeta)p_f$, and the latter by,

$$\mu_0 = \sum_{j=1}^{n} \frac{\bar{m}_j}{j}\frac{W_j}{1 - W_0}$$

Where above $\frac{\bar{m}_j}{j}$ is the average number of copies in a chosen packet given that j packets have been chosen. Substituting for $\bar{m}_j$ from equation (2.14) and solving for $\mu_f = \mu_0$ gives,

$$p_f = \frac{1}{n-\zeta}\left\{\sum_{j=1}^{n}\left\{n[1-(1-p)^j] + \frac{\zeta[(1-p)-(1-p)^{j+1}]}{p}\right\}\frac{W_j}{j} - \zeta\right\} \tag{2.47}$$

The equation (2.43) determines the steady-state distribution of the number of packets chosen in a slot as a function of p and the above equation expresses it in terms of the fresh packet copy generation probability. Thus, first W_j is determined as a function of p, and then is expressed as a function of p_f from the above.

2.6 Numerical results

This section presents the numerical results computed from the analysis and the simulation programs. Fig.2.4 - Fig.2.9 present the performance of the modified Binomial copy generation process ($\zeta = 0, 1, 2, 3$) when input packets are the fresh packets only. Fig.2.4 - Fig.2.6 give the packet and copy throughput per port per slot as a function of copy generation probability for the modified Binomial copy generation process with different values of load (ρ), switch size 16×16 and for primary copies $\zeta = 1, 2, 3$.

Fig.2.7 - Fig.2.9 give the performance comparison for different values of ζ, $\zeta = 0, 1, 2$ and 3, for both switch sizes of 8×8 and 16×16. From these figures we can see that as the number of primary copies increase, the packet throughput drops. On the other hand, the copy throughput drops in the middle region and increase at the two ends with the increase of the number of primary copies.

Fig.2.10 - Fig.2.13 give the performance of the modified Binomial copy generation process ($\zeta = 0, 1, 2, 3$) when input packets are the mix of old and fresh packets. The performance results are presented by plotting the steady-state packet and copy throughput per port per slot as a function of the fresh copy generation probability for different values of load, $\rho = 0.2, 0.4, 1.0$, and switch size of 16×16.

Fig.2.14 - Fig.2.16 are the packet and copy throughput per port per slot as a function of the fresh packet copy generation probability for different number of primary copies ζ and the switch sizes $n = 8, 16$. Fig. 2.14 is for load $\rho = 0.2$, Fig. 2.15 is for load $\rho = 0.4$ and Fig. 2.16 is for load $\rho = 1.0$.

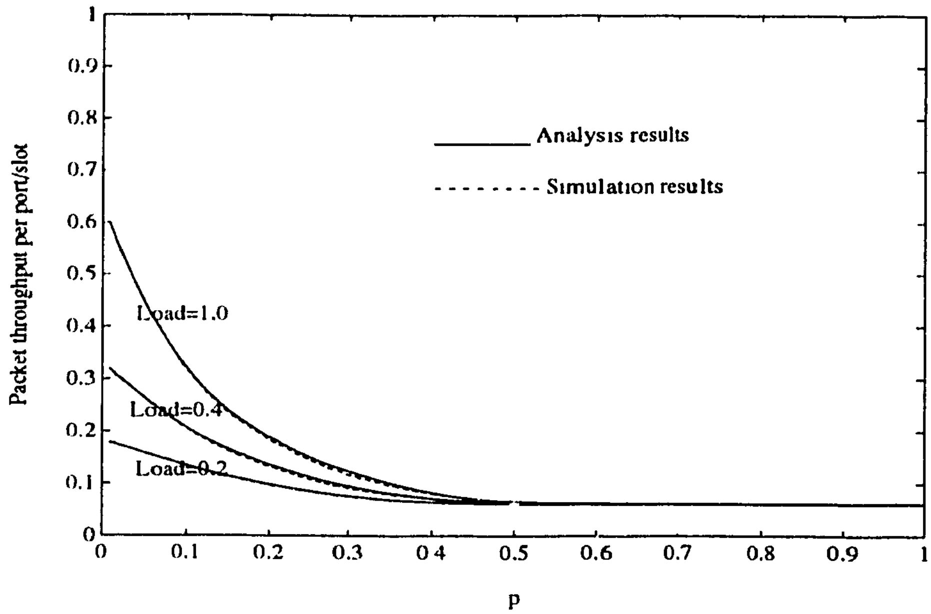
As in the case of only fresh input packets, we can see that as the number cf primary copies increase, the packet throughput drops. On the other hand, the copy throughput drops in the middle region and increase at the two ends with the increase of the number of primary copies.

From all of the figures discussed above, we can find that all the curves have same shape. In all the cases, the packet throughput decrease with the increase of the copy generation probability p and approaches to 1/n. That is because when p increases, the number of copies generated by a packet will increase and that results the increasing interference among the packets. When probability p is large, only one packet can be transmitted. Copy throughput starts with a low value due to very small packet sizes. But it increases with the increase of probability p to a maximum point. After that it drops slightly, then rises linearly with p and approaches to one. From the figures, we can also see that both packet and copy throughput increase with the increase of the load. It appears that the best operating point will be the maximum point in the middle, since both the packets and copy throughput are significantly high.

As it is shown in the figures, analysis results are in agreement with simulation results.Fig.2.17 gives the error between the analysis results and simulation results for for the steady-state packet throughputs per port/slot as a function of the fresh packet copy generation probability for the load $\rho = 0.8$, no. of primary copies $\zeta = 1$, and the switch sizes of $n = 16$. The error is calculated by following formula,

$$error = |Theoretical \;\; results - Simulation \;\; results| * 100 \qquad (2.48)$$

As it is shown in the figure, the error is very small and it is decrease with the increase of the fresh packet copy generation probability p_f.



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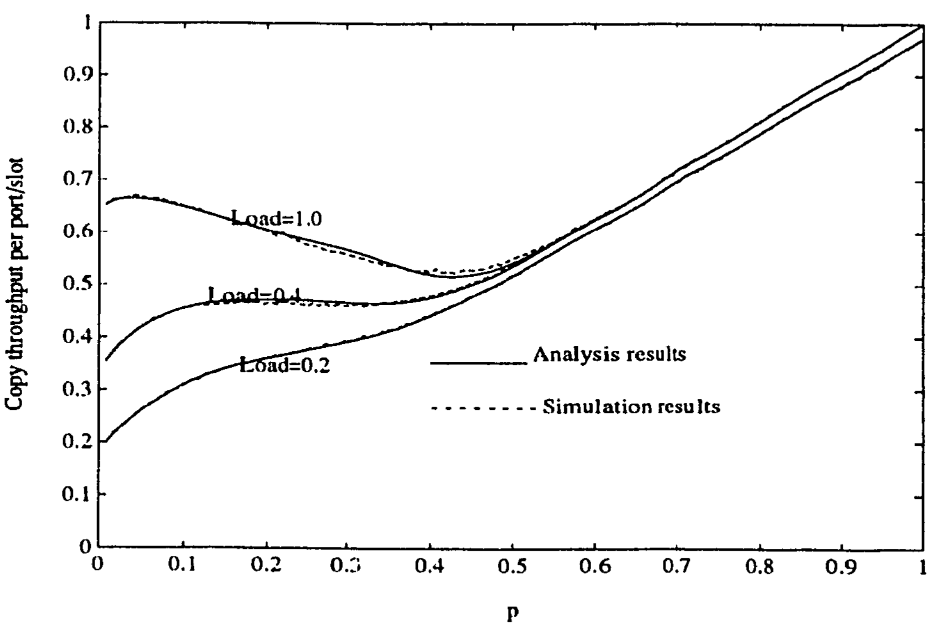

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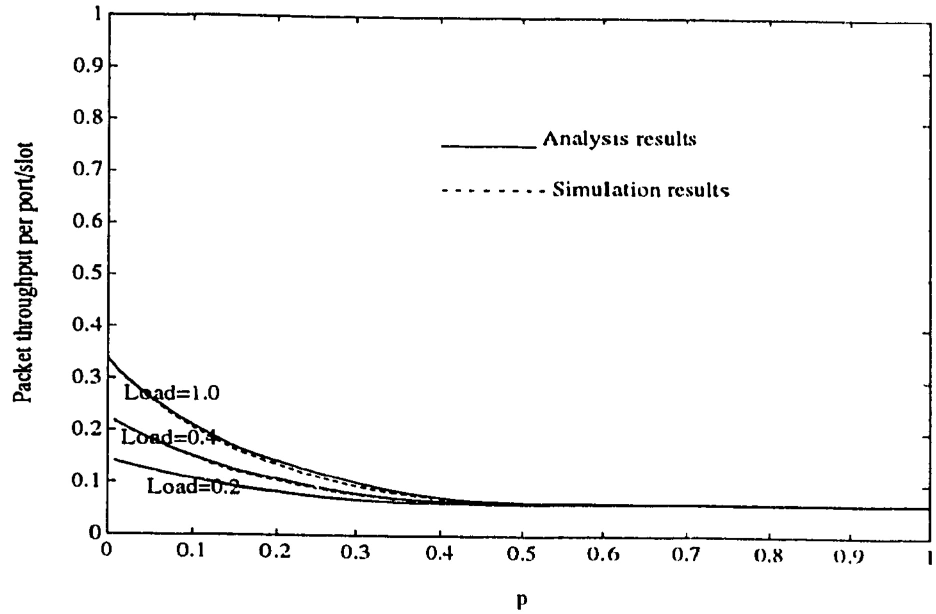
Figure 2.4: The throughputs in packets per port per slot and in copies per port per slot as a function of copy generation probability for modified Binomial copy generation process with $\zeta = 1$, switch size $n = 16$ and different values of load.



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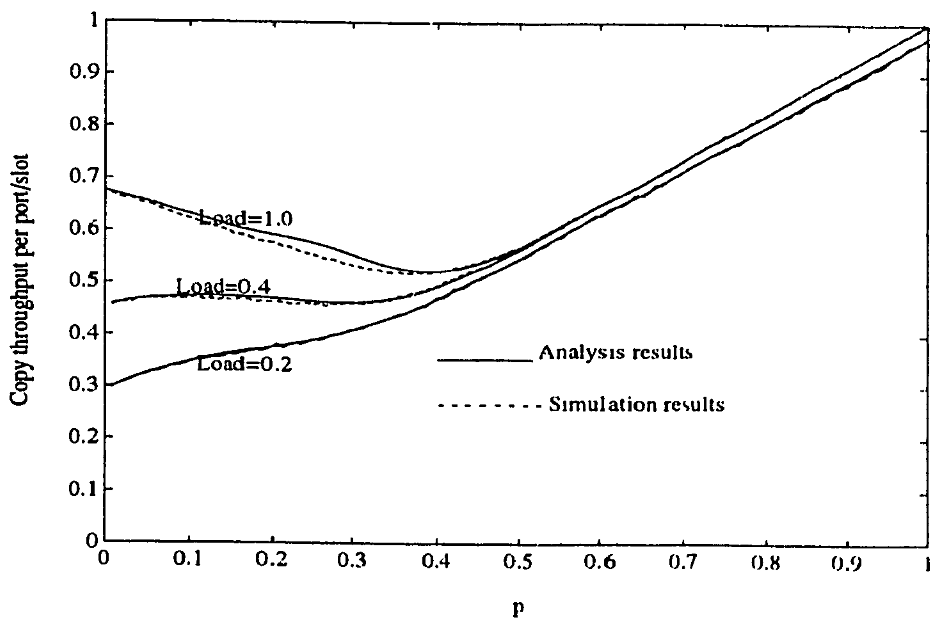

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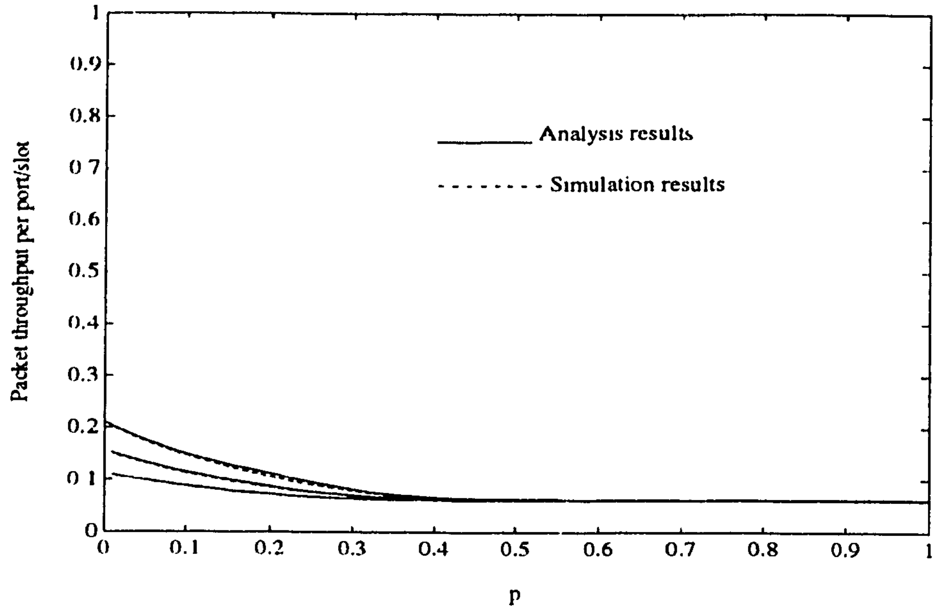
Figure 2.5: The throughputs in packets per port per slot and in copies per port per slot as a function of copy generation probability for modified Binomial copy generation process with $\zeta = 2$, switch size $n = 16$ and different values of load.



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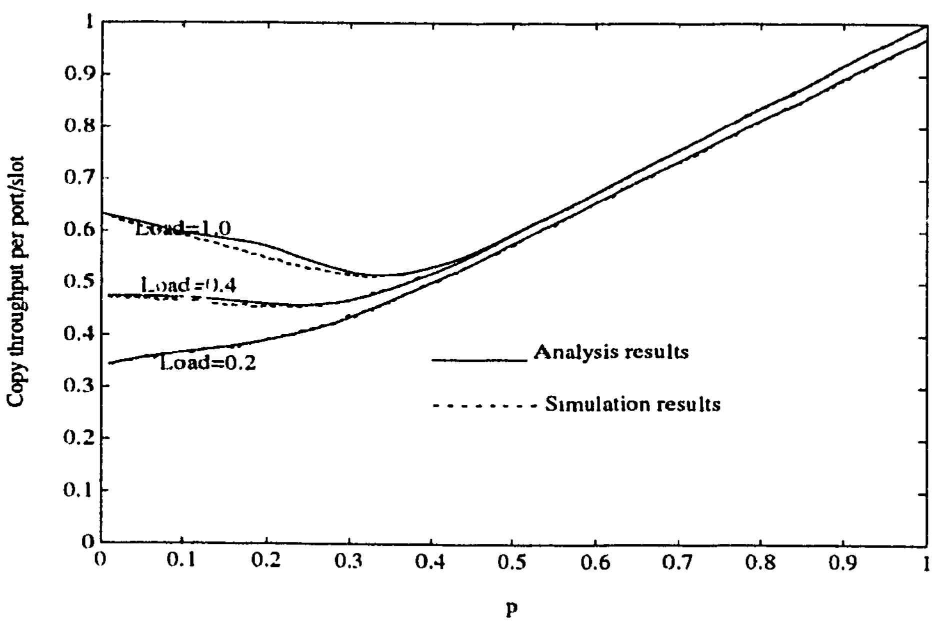

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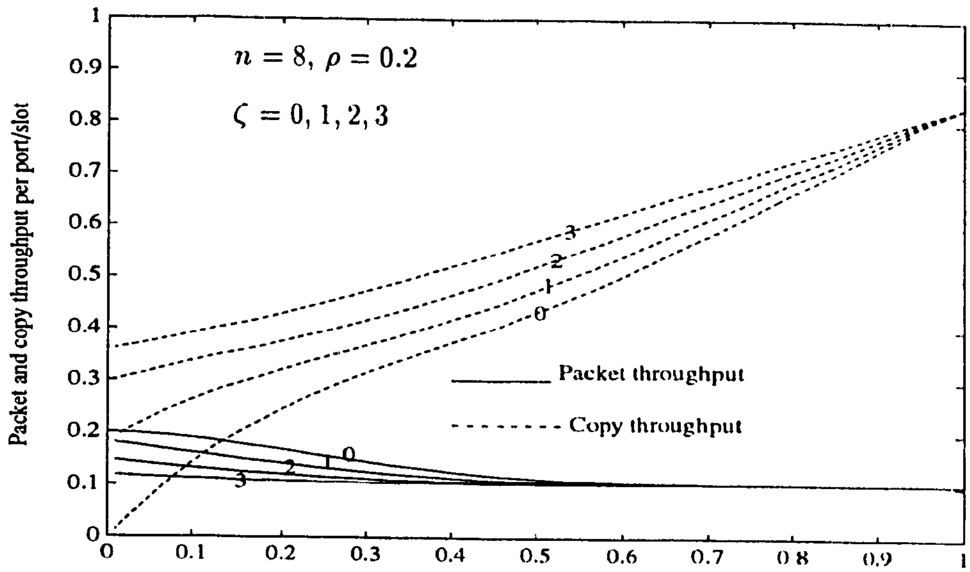
Figure 2.6: The throughputs in packets per port per slot and in copies per port per slot as a function of copy generation probability for modified Binomial copy generation process with $\zeta = 3$, switch size $n = 16$ and different values of load.



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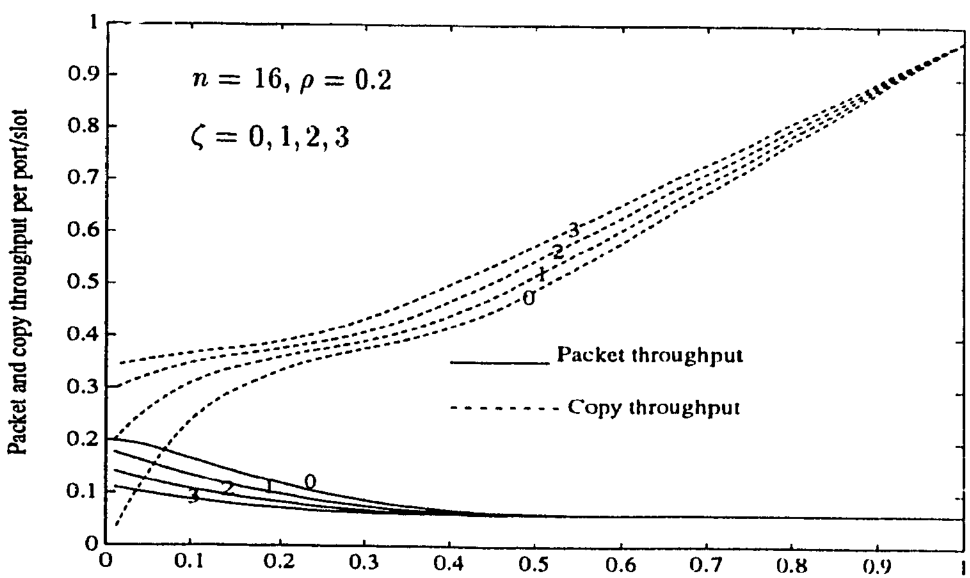

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Figure 2.7: Packet and copy throughputs per port/slot as a function of a packet copy generation probability p for the load $\rho = 0.2$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$.

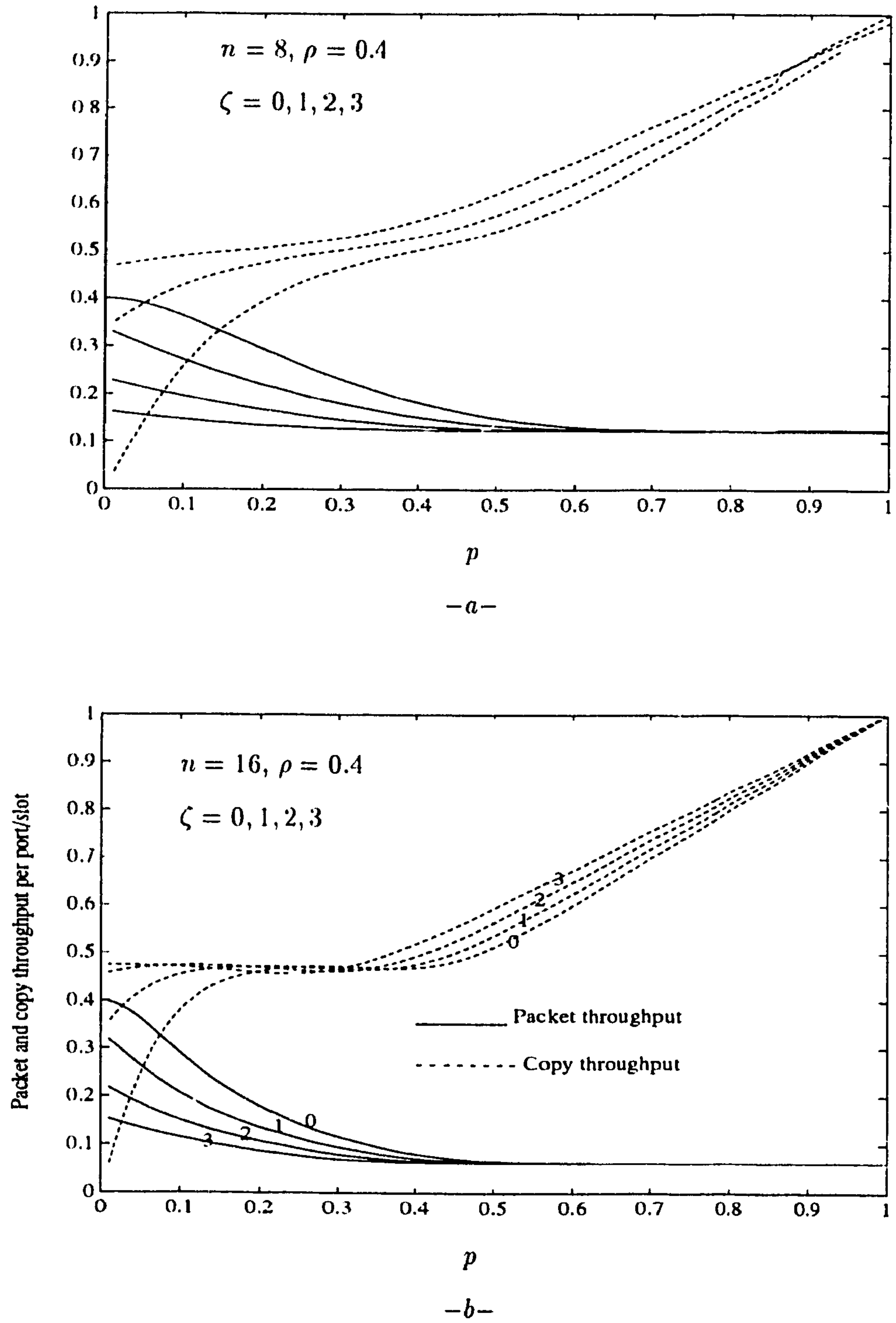
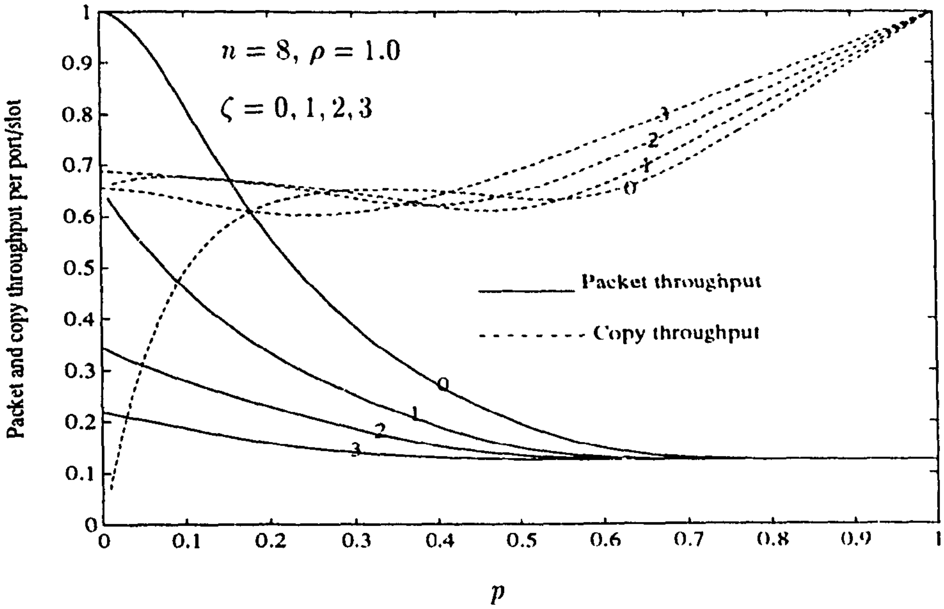


Figure 2.8: Packet and copy throughputs per port/slot as a function of a packet copy generation probability p for the load $\rho = 0.4$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$



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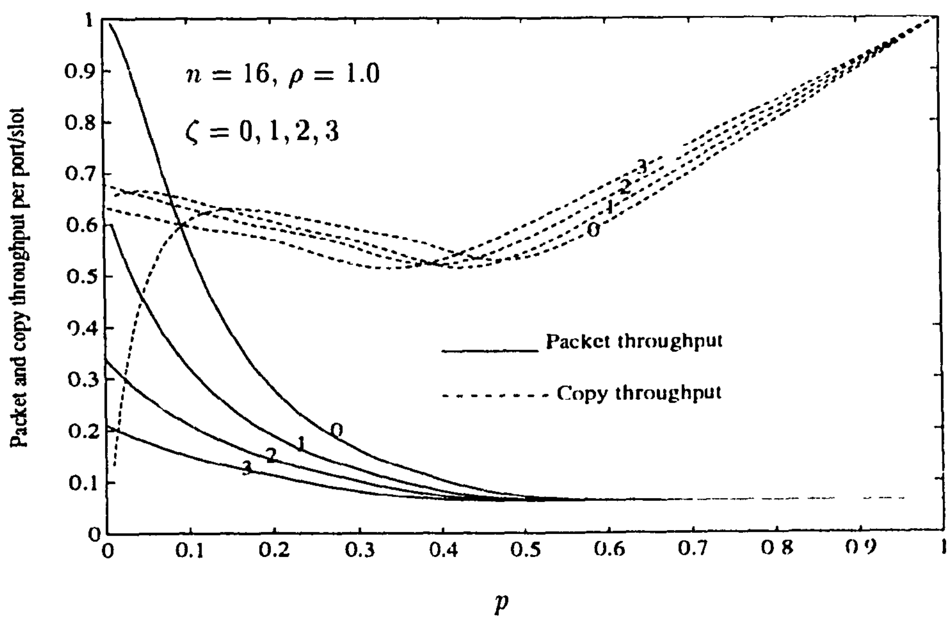

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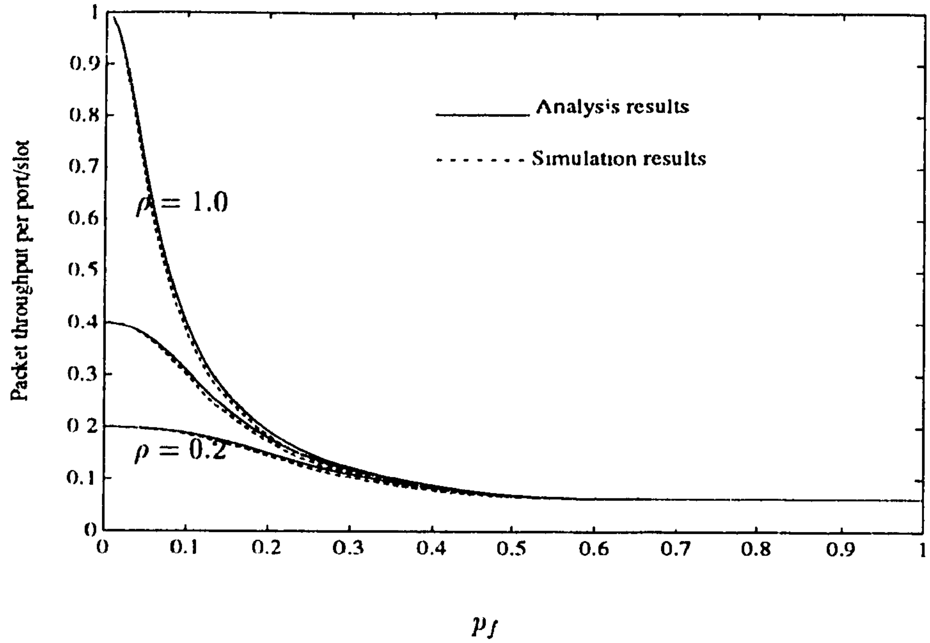
Figure 2.9: Packet and copy throughputs per port/slot as a function of a packet copy generation probability p for the load $\rho = 1.0$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$.



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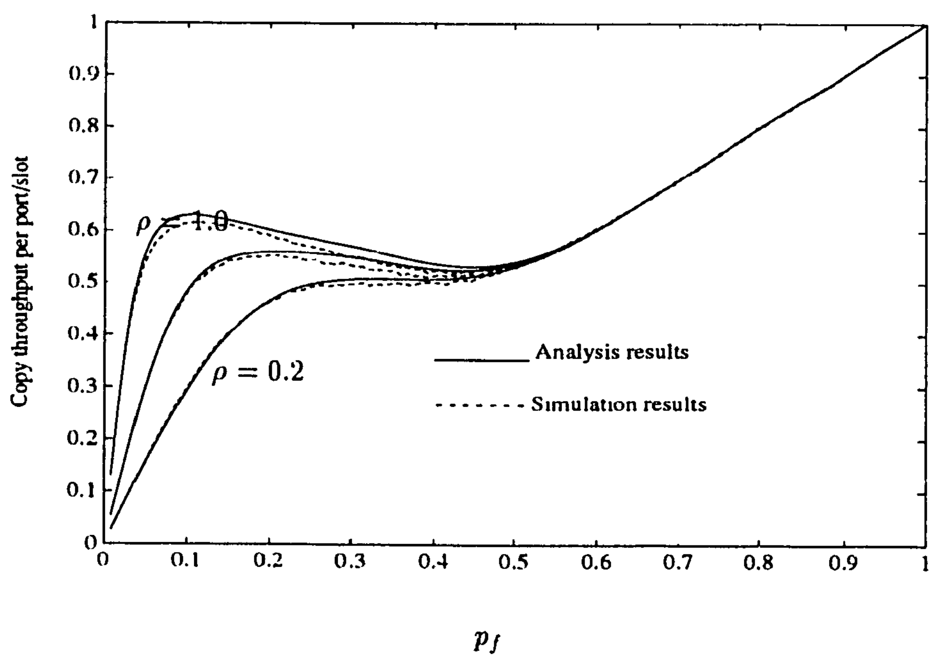

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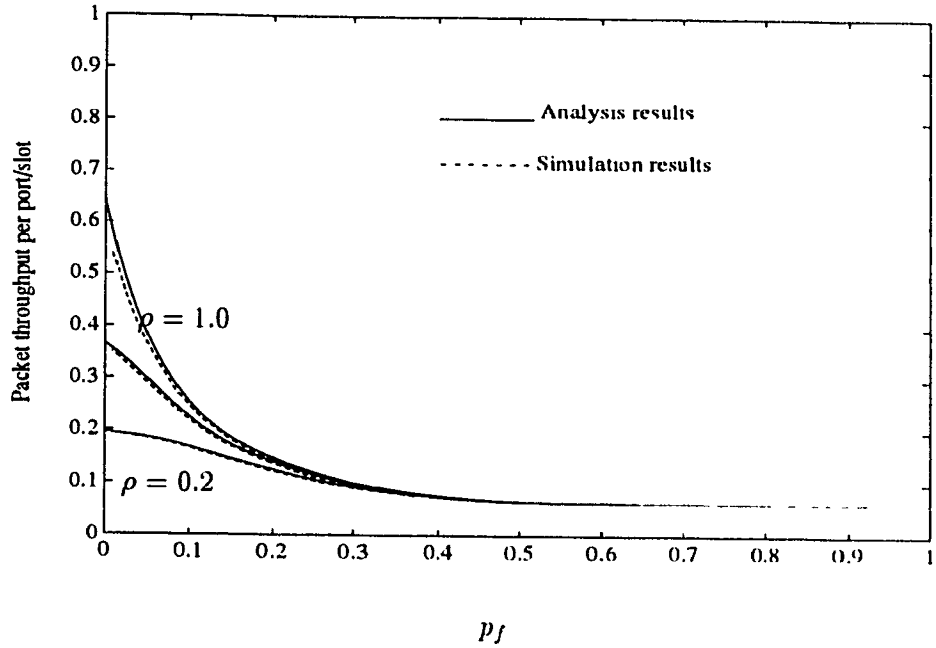
Figure 2.10: The steady-state throughputs in packets per port per slot and in copies per port per slot as a function of the fresh copy generation probability for Binomial copy generation process ($\zeta = 0$) with switch size $n = 16$ and different values of load, $\rho = 0.2, 0.4, 1.0$.



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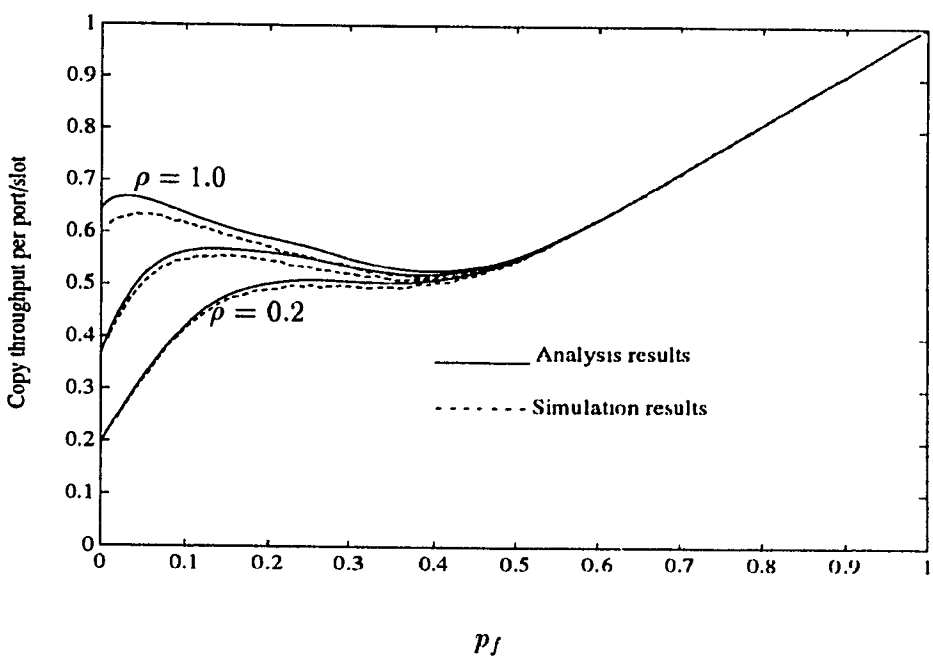

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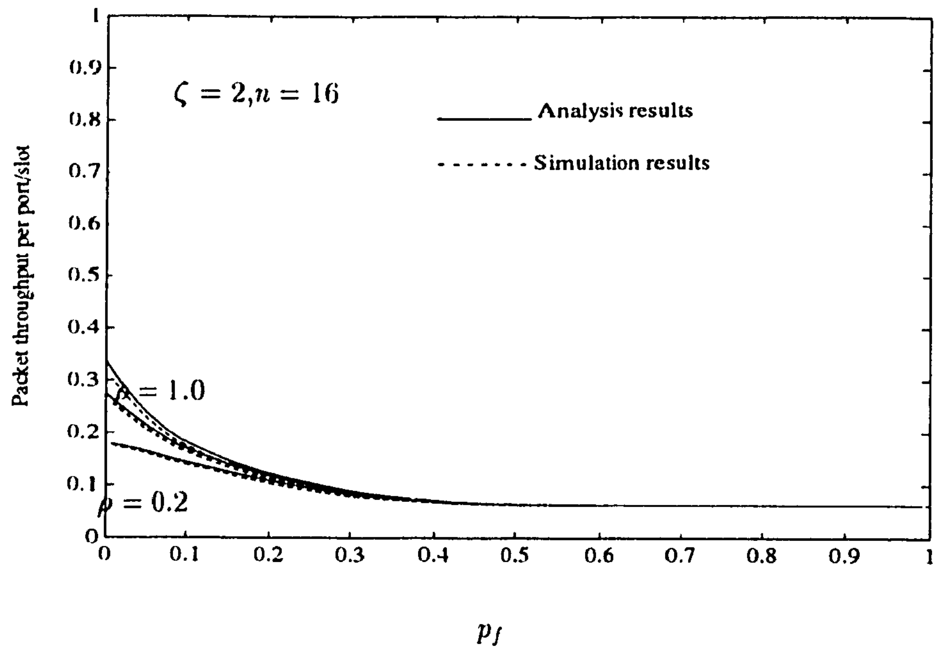
Figure 2.11: The steady-state throughputs in packets per port per slot and in copies per port per slot as a function of the fresh copy generation probability for modify Binomial copy generation process with $\zeta = 1$, switch size $n = 16$ and different values of load, $\rho = 0.2, 0.4, 1.0$.



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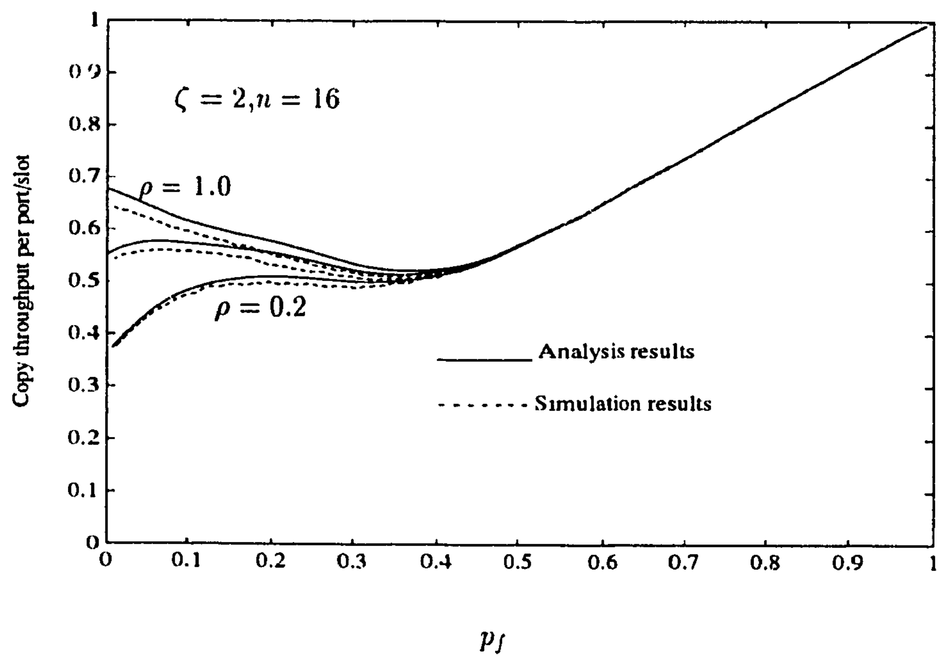

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Figure 2.12: The steady-state throughputs in packets per port per slot and in copies per port per slot as a function of the fresh copy generation probability for modify Binomial copy generation process with $\zeta = 2$, switch size $n = 16$ and different values of load, $\rho = 0.2, 0.4, 1.0$.

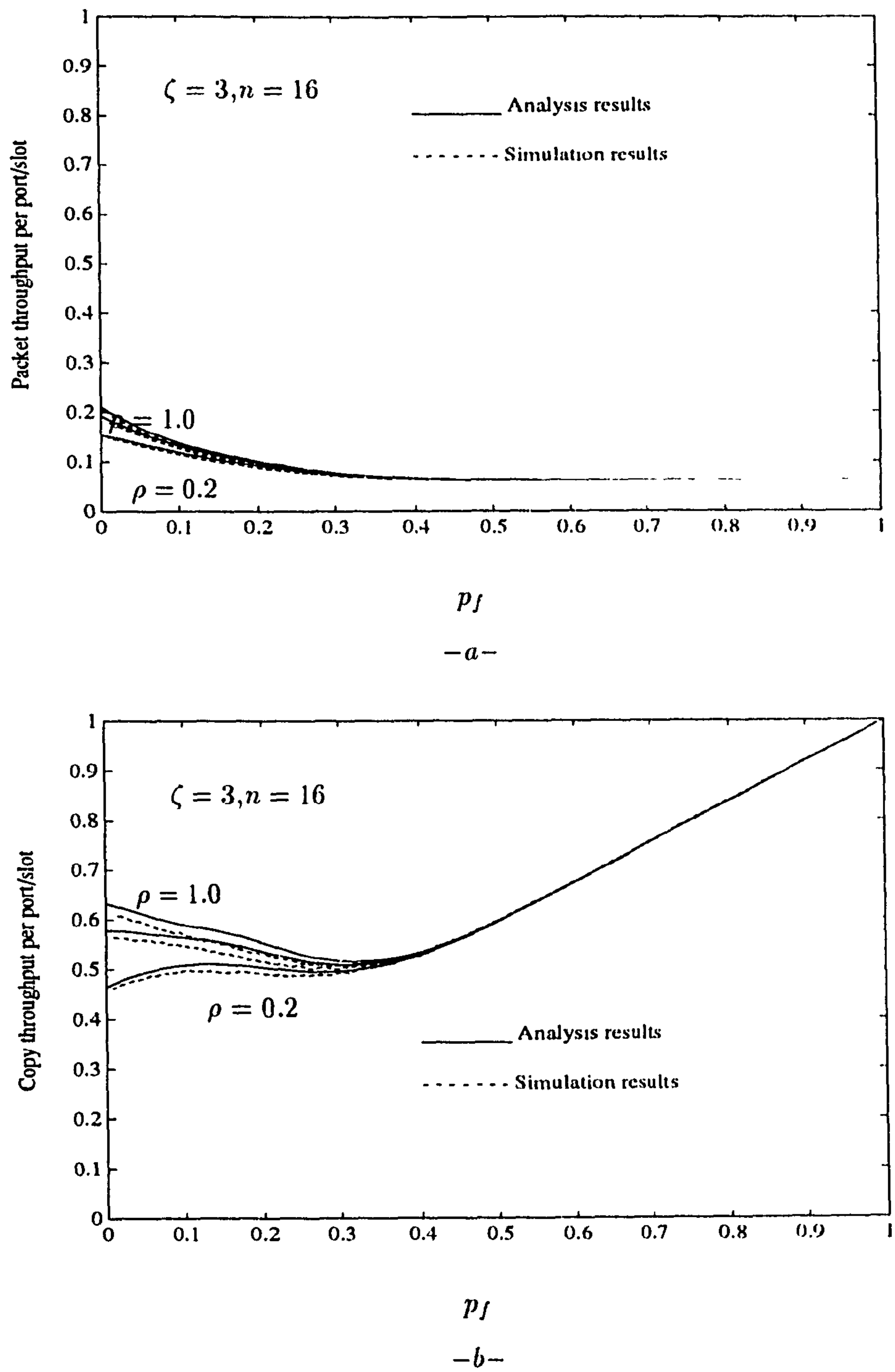
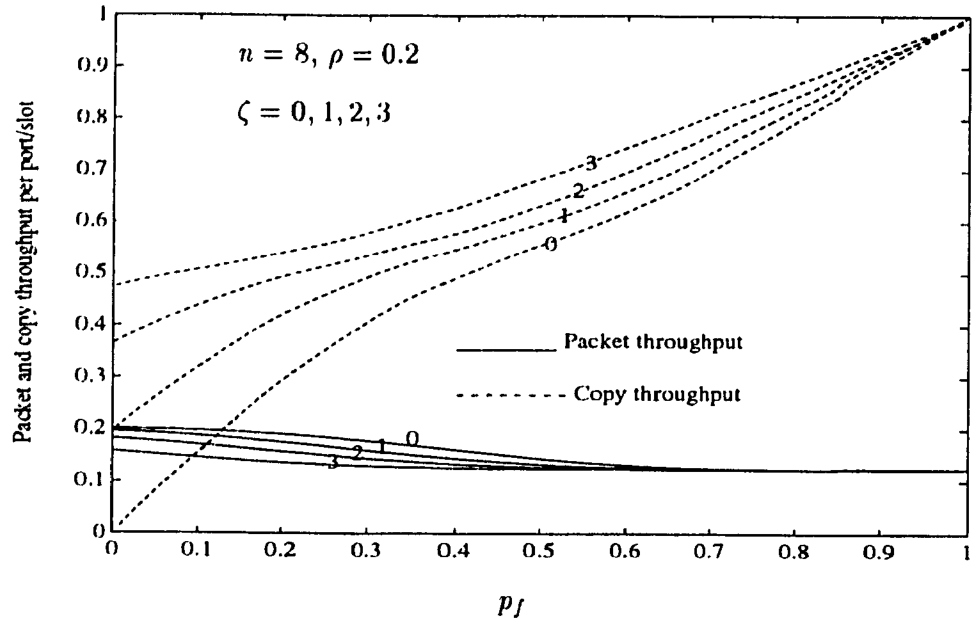


Figure 2.13: The steady-state throughputs in packets per port per slot and in copies per port per slot as a function of the fresh copy generation probability for modify Binomial copy generation process with $\zeta = 3$, switch size $n = 16$ and different values of load, $\rho = 0.2, 0.4, 1.0$.



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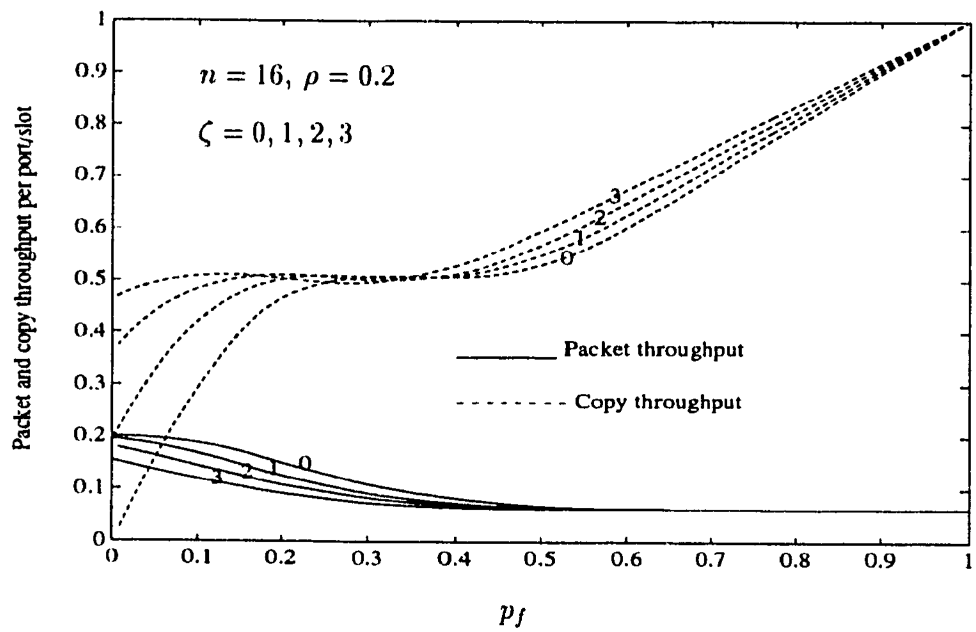

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Figure 2.14: The steady-state packet and copy throughputs per port/slot as a function of the fresh packet copy generation probability for the load $\rho = 0.2$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$.

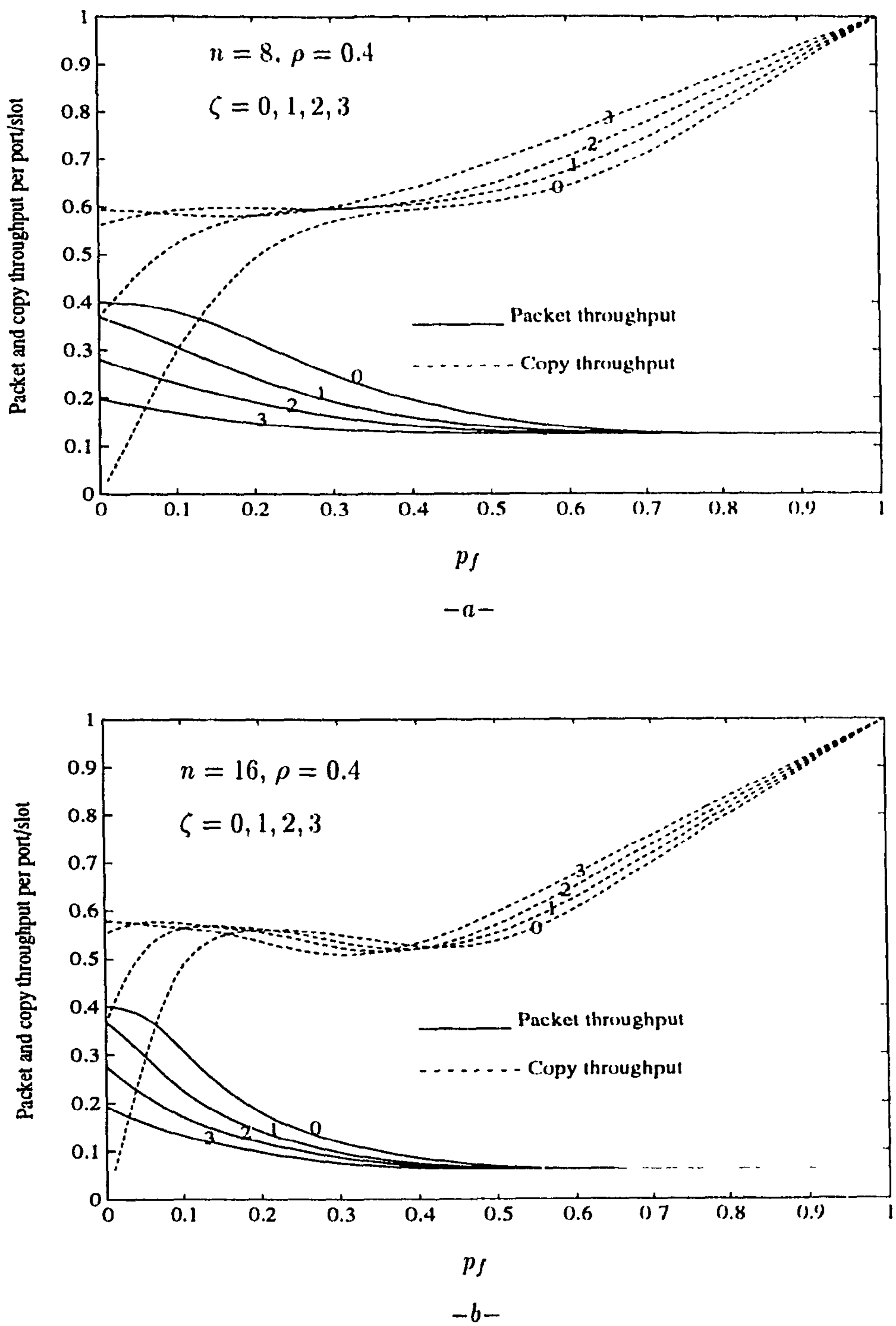
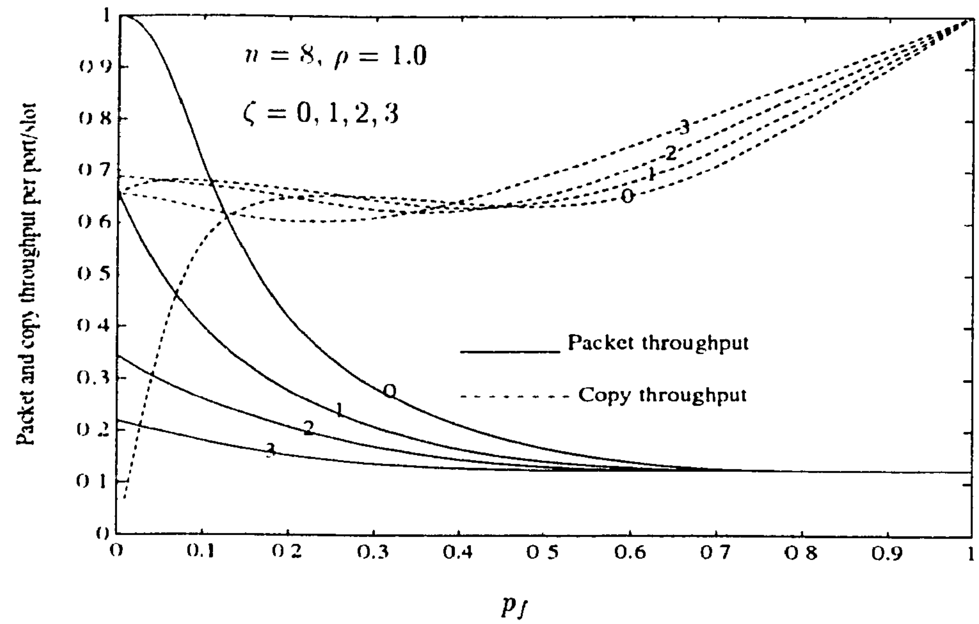


Figure 2.15: The steady-state packet and copy throughputs per port/slot as a function of the fresh packet copy generation probability for the load $\rho = 0.4$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$.



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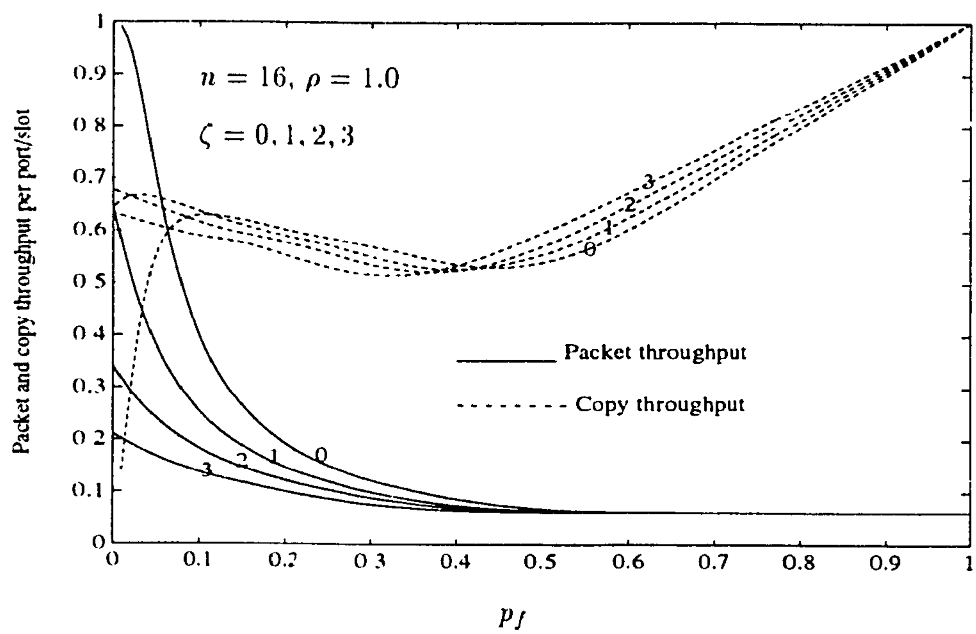

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Figure 2.16: The steady-state packet and copy throughputs per port/slot as a function of the fresh packet copy generation probability for the load $\rho = 1.0$, different no. of primary copies ζ, and the switch sizes of $n = 8, 16$.

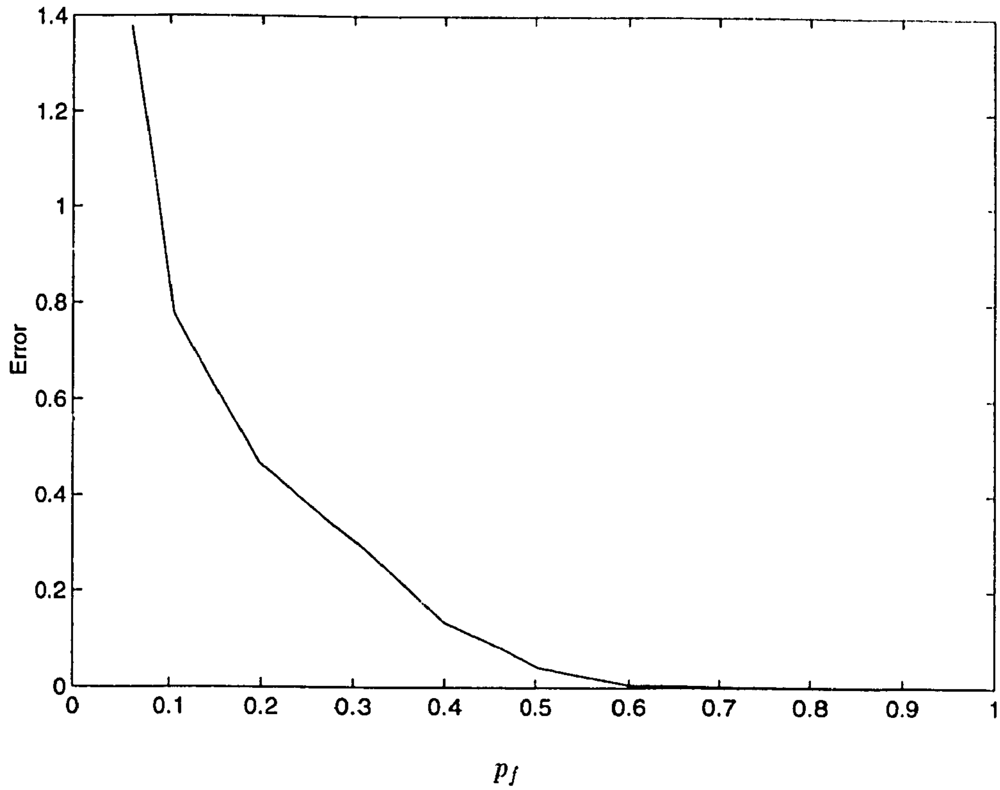


Figure 2.17: The error between analysis results and simulation results for the steady-state packet throughputs per port/slot as a function of the fresh packet copy generation probability for the load $\rho = 0.8$, no. of primary copies $\zeta = 1$, and the switch sizes of $n = 16$.

Chapter 3

Asymptotic Analysis

In chapter two, the distribution of the number of packets chosen was determined for the random packet selection policy for the multicast switching. However, when the switch size becomes large, it is impossible to calculate the distribution of the number of packets chosen due to numerical difficulties. However, in the broadband ISDN, because of the high transmission rates and the diverse services, the switch sizes are expected to be large. Thus the objective in this chapter is to determine the distribution of the number of packets chosen in a slot for very large switch sizes under saturation. The packet and copy throughput per port per slot are also given.

In the following, the average number of packets chosen is derived using the weak law of large numbers. Then a simple lower bound on the distribution of the number of packets chosen in a slot is given. Finally, the exact distribution of the number of packets chosen is determined using results from renewal theory.

3.1 Analysis

3.1.1 Average number of packets chosen in a slot

As described in the section 2.3, random selection policy may be modeled as a renewal process with the renewal points taken as the trials that packets are chosen. Since the inputs are under saturation, there will be n contending packets in a slot. The objective of this section is to determine the average number of chosen packets given that there are n packets at the inputs. Assuming that j and only j packets are chosen in a slot, then the renewal process will have j renewal points occurring during the interval [0,n]. Let us repeat some of the definitions given in section 2.4.2 for convenience,

X_i=interarrival time of the i'th renewal

S_j= the number of trials required for the occurrence of j renewals

Then we have

$$S_j = X_1 + X_2 + ... + X_j \tag{3.1}$$

We notice that X_i (i=1,2,...,j) are independent but not identically distributed random variables and the distribution of X_i is

$$P\{X_i = k\} = (1 - P_{i-1})^{k-1} P_{i-1} \tag{3.2}$$

The mean and variance of X_i are,

$$E(X_i) = \beta_i = \frac{1}{P_{i-1}} \tag{3.3}$$

$$var(X_i) = \sigma_i^2 = \frac{1 - P_{i-1}}{P_{i-1}^2} \tag{3.4}$$

Then the mean of S_j is given by,

$$E(S_j) = E[\sum_{i=1}^{j} X_i] \tag{3.5}$$

Where j is a random variable.

Next we shall give a theorem which will be useful in the following section.

Weak law of large Numbers [15, 17]: Assume a sequence of independent but not identically distributed random variables X_i, with mean and variance β_i and σ_i^2 respectively, then

$$Pr\left\{\frac{|S_j - E[S_j]|}{j} > \epsilon\right\} \rightarrow 0$$

Thus for very large j, we could replace S_j by $E[S_j]$.

Next, we will use this theorem in the analysis.

Let us define, N_n= the number of renewals that occur in n trials, then

$$N_n = max\{j|S_j \leq n\} \tag{3.6}$$

Clearly, $N_n = j$, if and only if $S_j \leq n$ and $S_{j+1} > n$. This is equivalent to,

$$S_{N_n} \leq n < S_{N_n+1}$$

Letting $N_n \rightarrow \infty$ and replacing S_{N_n} with $E[S_{N_n}]$ as discussed above,

$$E[S_{N_n}] \leq n < E[S_{N_n+1}] \tag{3.7}$$

Defining $\bar{N}_n$ as the average number of packets chosen in n trials, the mean of $S_{\bar{N}_n}$ is given by,

$$\begin{aligned} E[S_{\bar{N}_n}] &= E[\sum_{i=1}^{\bar{N}_n} X_i] \\ &= \sum_{i=1}^{\bar{N}_n} E[X_i] \end{aligned} \tag{3.8}$$

Then, from the equation (3.7), we have

$$E[S_{\tilde{N}_n}] \leq n < E[S_{\tilde{N}_n+1}] \tag{3.9}$$

Thus we determine $\tilde{N}_n$ from above equation.

3.1.2 Lower Bound analysis for the distribution of the number of packets chosen in a slot

In this section, we shall give a simple lower bound for the distribution of the number of packets chosen in a slot. Since the system is under saturation, every input will have a contending packet, thus the number of Bernoulli trials in the random packet selection policy will be equal to n. Clearly the maximum number of packets chosen up to i'th trial will be less than or equal to the trial number. Success probability changes after each choice of a packet, and it is given by P_j following the j'th choice. Clearly, the maximum number of successes in i trials will be i, thus the success probability on the i'th trial will be one of the followings,

$$P_1, P_2, ...P_j, ...P_i$$

In section 2.3, it was shown that success probability P_j are monotonically decreasing function of j. Thus the lowest success probability on the i'th trial will be P_i.

Now let us assume that on each trial, the success probability is independent of the number of packets chosen previously and is given by P_i on the i'th trial. The throughput derived based on this success probability will be a lower bound. Let us define the Bernoulli random variable Y_i to denote the outcome of the i'th trial,

$$Y_i = \begin{cases} 1 & with\ probability\ P_i \\ 0 & with\ probability\ 1 - P_i \end{cases}$$

Then

$$N_n = Y_1 + Y_2 + ... + Y_n$$

Thus above N_n denote the number of packets chosen in n trials. Since success probability on each trial is less than or equal to the success probability of the random selection policy, N_n will provide a lower bound. Next we determine the probability distribution of N_n. The mean and variance of N_n are given by,

$$E(N_n) = \beta = \sum_{i=1}^{n} P_{i-1} \tag{3.10}$$

and variance

$$var(N_n) = \sigma^2 = \sum_{i=1}^{n} (P_{i-1} - P_{i-1}^2) \tag{3.11}$$

Clearly, Y_i are independent but not identically distributed random variables. However since Y_i are bounded ($Y_i \leq 1$), the central limit theorem[15] still holds, and the distribution of N_n is given by the normal distribution. Let $f(x)$ denote the pdf of N_n, then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\beta}{\sigma})^2} \tag{3.12}$$

Where β and σ are defined above.

Let W_j denote this asymptotic probability distribution of N_n,

$$W_j = \int_{j-1/2}^{j+1/2} f(x)dx \tag{3.13}$$

Thus the lower bound of packet and copy throughput are

$$T_p = \frac{1}{n} \sum_{j=0}^{n} jW_j$$

$$T_c = \frac{1}{n}\sum_{j=0}^{n} \bar{m}_j W_j$$

Where $\bar{m}_j$ is the average number of copies generated by the j chosen packets. Next we shall apply the above results to following cases:

- Modified Binomial copy generation process

Substituting equations (2.21),(2.22), (2.23) and (2.24) into equation (3.10) and (3.11) respectively, we can get the lower bound of the distribution of packets chosen in a slot. Thus, the lower bound of packet and copy throughput can be determined.

- The case with mixed input packets

Using the way described above, we can get the lower bound of packet and copy throughput against mix packet copy generation probability p. Using equation (2.47), the lower bound of packet and copy throughput against fresh packet copy generation probability p_f can be determined under modified Binomial copy generation process for different number of primary copies ($\zeta = 0, 1, 2, 3$).

3.1.3 Exact distribution of the number of packets chosen in a slot

It has been shown that the distribution of the number of renewals with independent identically distributed renewal times approach to a normal distribution as the number of trials go to infinity [15 - 20]. As explained previously, in the random packet selection policy under consideration, the renewal times are independent but not identically distributed. The above results has been extended to this case also, and the results is given below[16].

Theorem: As the number of trials go to infinity, the distribution of the number of packets chosen, $\lim_{n\to\infty} P_j(n)$, approaches to a normal distribution with mean $\beta_N = \frac{n}{\beta}$ and variance $\sigma_N^2 = \frac{\sigma^2 n}{\beta^3}$. Where

$$\beta = \lim_{j\to\infty} \frac{1}{j}\sum_{i=1}^{j}\beta_i \tag{3.14}$$

$$\sigma^2 = \lim_{j\to\infty} \frac{1}{j}\sum_{i=1}^{j}\sigma_i^2 \tag{3.15}$$

and β_i and σ_i^2 are finite which are given by equations (3.3) and (3.4).

The application of the above theorem presents a problem. The theorem requires that both the number of renewals and the number of trials approach to infinity. However, the number of renewals is dependent on the number of trials, and it has to be less than the number of trials. Because of the dependency between the trials and renewals, it is not clear how to do this. In order to apply the theorem, we need the average of the mean and variance of the interarrival times as the number of renewals approach to infinity. As a result of this complication, we have used the average number of successes in large number of trials derived in section 3.1.1 and the corresponding mean and variance as a substitute for β, and σ^2. This will be justified through simulation results. Thus, the density function of the distribution of packets chosen is,

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{1}{2}(\frac{x-\beta_N}{\sigma_N})^2} \tag{3.16}$$

Let W_j denote this asymptotic probability distribution,

$$W_j = \lim_{n\to\infty} P_j(n) = \int_{j-1/2}^{j+1/2} f_N(x)dx \tag{3.17}$$

Thus, the packet and copy throughput are given by,

$$T_p = \frac{1}{n}\sum_{j=0}^{n} jW_j \tag{3.18}$$

$$T_c = \frac{1}{n}\sum_{j=0}^{n} \bar{m}_j W_j \tag{3.19}$$

Again, we consider following cases for the application of the above results:

- Modified binomial copy generation process

Substituting equation (2.21), (2.22) (2.23) and (2.24) into equation (3.3) and equation (3.4) respectively for $\zeta = 0, 1, 2, 3$, the mean and variance of i'th renewal time can be found. Using equation (3.8) and equation (3.9), the average number of packets chosen in a slot can be determined. Then the distribution of the number of packets chosen can be determined using equation (3.14), (3.15) and (3.17). Thus using equation (3.18) and (3.19) packet and copy throughput can be determined.

- The case with mixed input packets

As described before, input packets are the mix of old and fresh packets. The old packets generate copies with probability p_o, while the fresh packets generate copies with probability p_f. In that case, the throughputs calculated by equations (3.18) and (3.19) are the throughputs against mixed packet copy generation probability p. Using equation (2.47), we can get the packet throughput and copy throughput against fresh packet copy generation probability p_f under the modified Binomial copy generation process.

3.2 Numerical results

This section presents the numerical results computed from the asymptotic analysis and the simulation programs. All the curves show the packet throughput per port per slot or copy throughput per port per slot as a function of the average number of copies generated by a fresh packet, with mixed (old + fresh) input packets.

Fig.3.1 and Fig.3.2 are the packet and copy throughput for the pure Binomial copy generation process ($\zeta = 0$). We can see from the figure that all the curves have same shape. In all the cases, the packet throughput decrease with the increase of the number of copies generated by a packet(μ_f). That is because the increase of μ_f results in the increasing interference among the packets. Copy throughput starts with a low value due to very small packet sizes. But it increases with the increase of μ_f to a maximum point. Then it drops. We can also see that analysis results are in good agreement with simulation results.

Fig.3.3 is the packet and copy throughput as well as the lower bound of packet and copy throughput for the pure Binomial copy generation process ($\zeta = 0$). The lower bound of packet throughput is not far from the packet throughput. But the lower bound of copy throughput is different. When μ_f is less than 1, the lower bound of copy throughput is good. However, when μ_f is greater than 1, it is far from the copy throughput. So this lower bound is good for the packet throughput, but it is not good enough for the copy throughput.

Fig. 3.4 is the packet and copy throughput for the modified Binomial copy generation process ($\zeta = 1$). In this case, the packet throughput decrease with the increase of μ_f, and the copy throughput rises slightly and then decrease with the increase of μ_f. Again, analysis results are in good agreement with simulation results.

It is interesting to compare our result with that of unicast switch. In the case of unicast switching at saturation, the throughput is 0.632 if we assume independence of HOL occupancies as well as HOL addresses. When the number of primary copies ζ is equal to 1 and the copy generation probability $p_f = 0$, Then each packet generates a single copy. Clearly, for this case, the packet and copy throughput are equal and corresponds to that of unicast switching, 0.632 (see Figure 3.4).

Fig.3.5 is the packet and copy throughput for the modified Binomial copy generation process ($\zeta = 0, 1, 2, 3$). The packet throughput decrease with the increase of the number of primary copies ζ, while the copy throughput is different. When $\mu_f > 2$, the copy throughput decrease with the increase of the number of primary copies ζ.

Fig.3.6 presents the packet and copy throughput for the cases that input packets are fresh packets only, and input packets are the mixed packets (old packets plus fresh packets). We can see that the system with mixed input packets has a lower packet throughput than that with fresh input packets. For the copy throughput, when $\mu_f < 2$, the system with mixed input packets has a higher value. That is because the copy generation probability of a mixed input packet is greater than that of a fresh input packet.

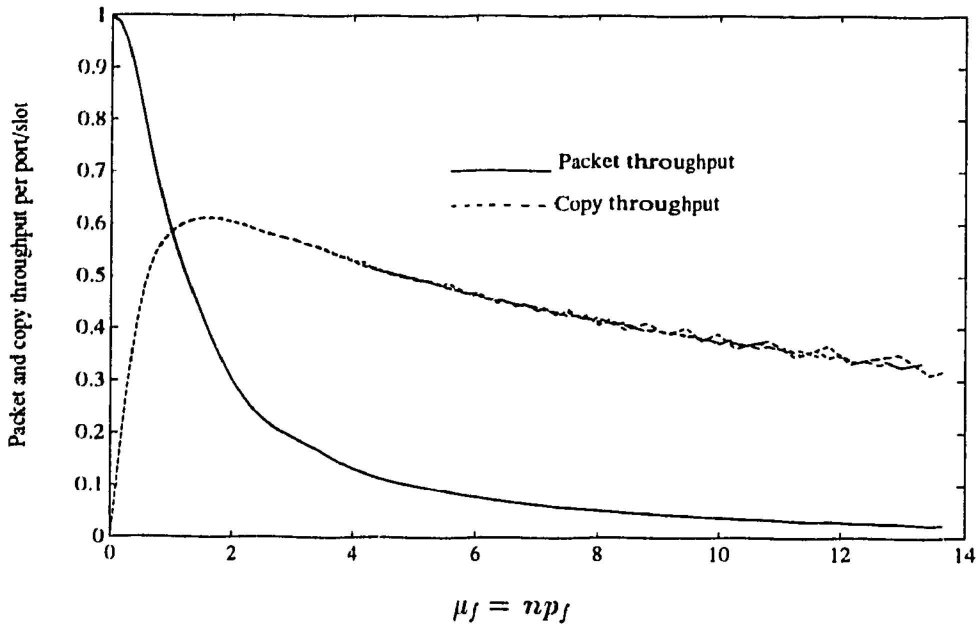


Figure 3.1: Asymptotic packet and copy throughput per port/slot against the average number of copies generated by a fresh packet for the switch sizes of $n = 256, 512, 1024$.

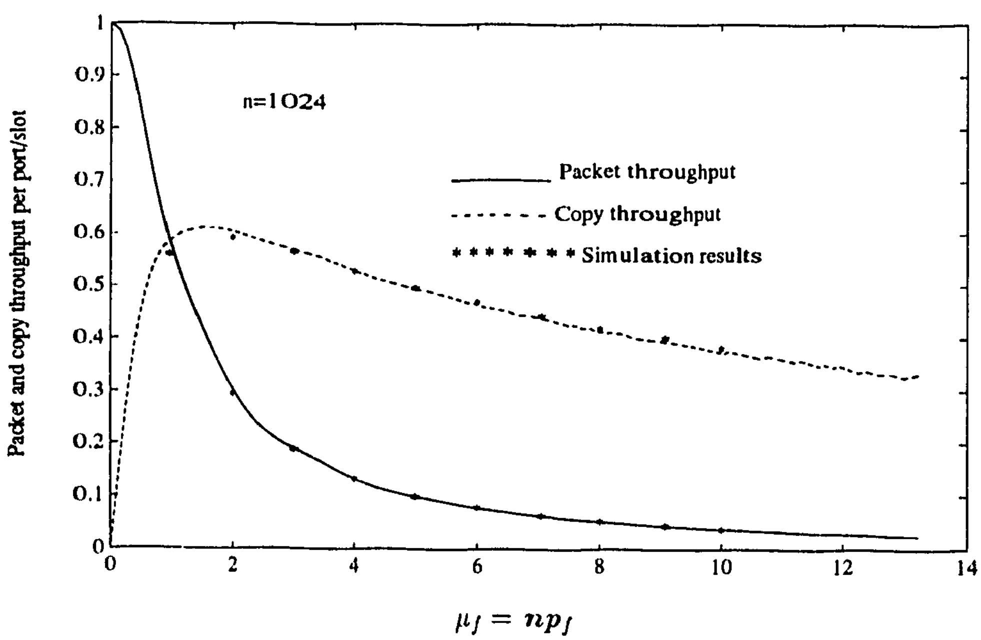


Figure 3.2: Asymptotic packet and copy throughput per port/slot against the average number of copies generated by a fresh packet and simulation results for the switch size of $n = 1024$.

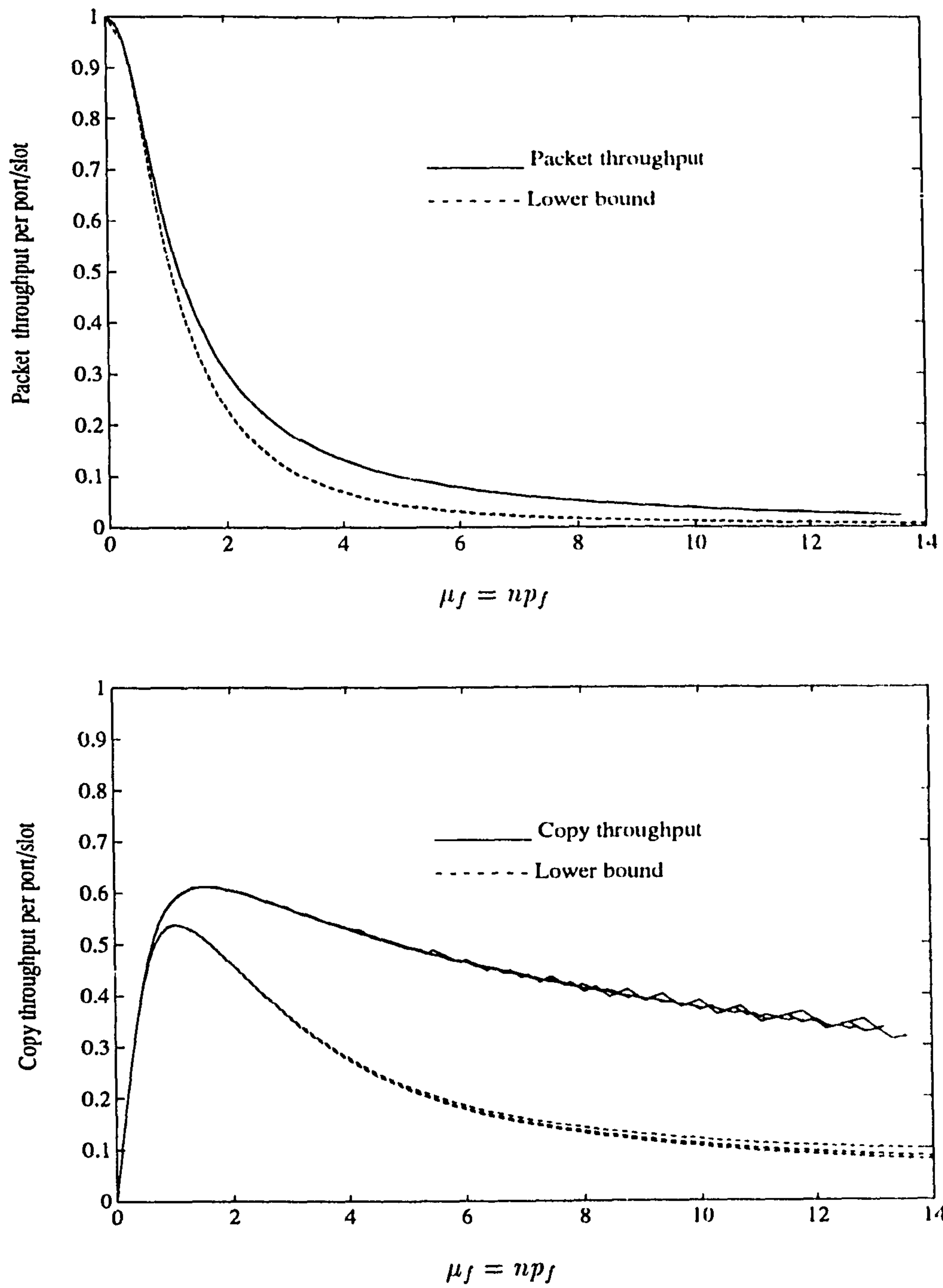
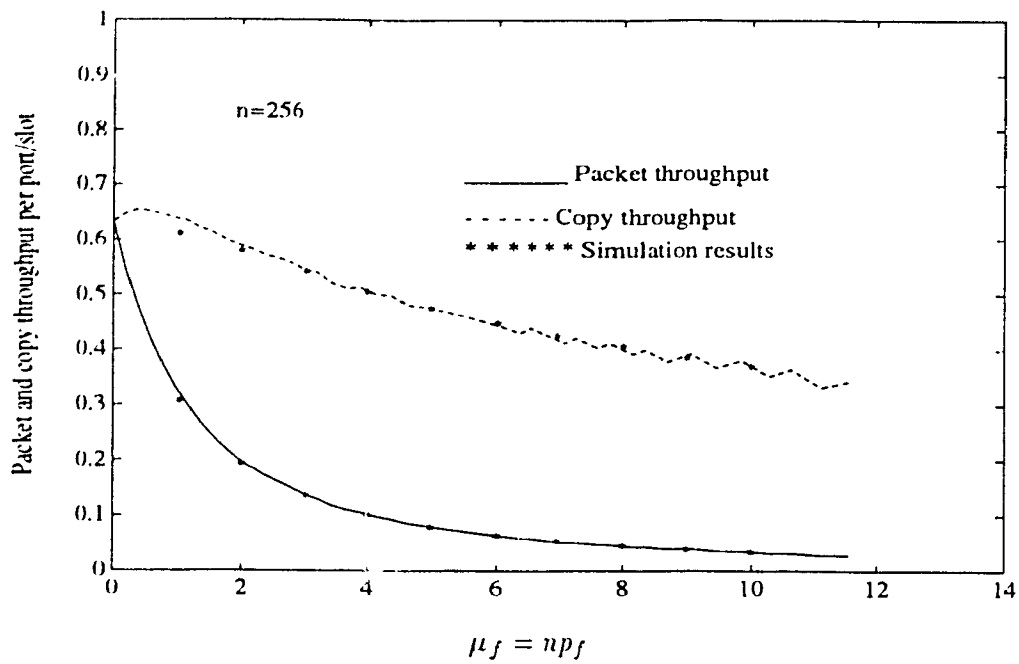


Figure 3.3: Asymptotic packet and copy throughput per port/slot and the lower bound of packet and copy throughput per port/slot against the average number of copies generated by a fresh packet for the switch sizes of $n = 256, 512, 1024$.



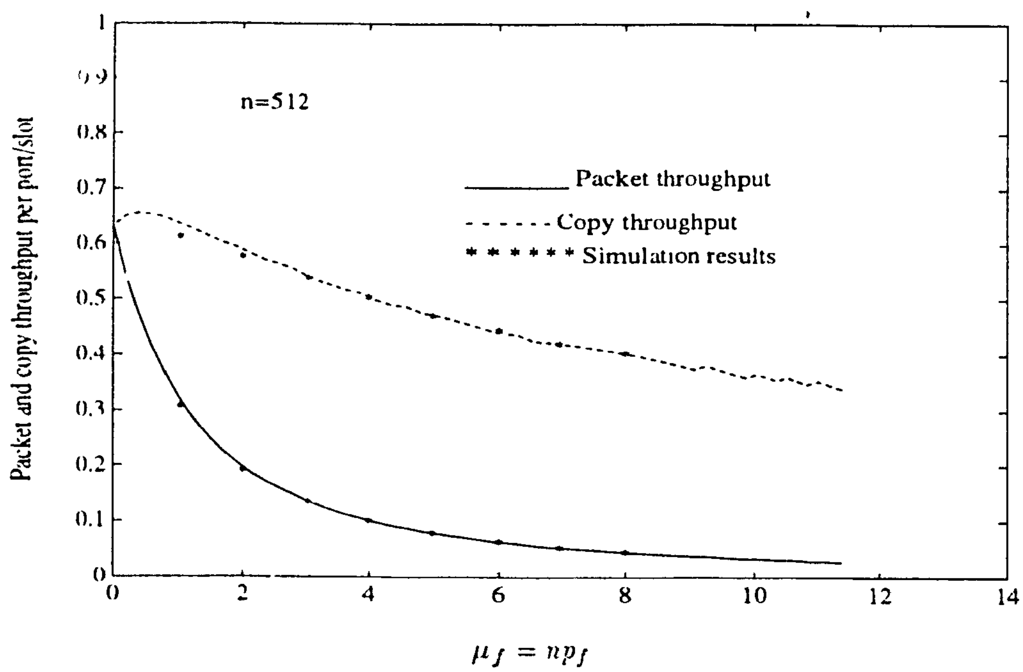


Figure 3.1: Asymptotic packet throughput and copy throughput per port/slot against the average number of copies generated by a fresh packet for primary copy $\zeta = 1$ and the switch sizes of $n = 256, 512$.

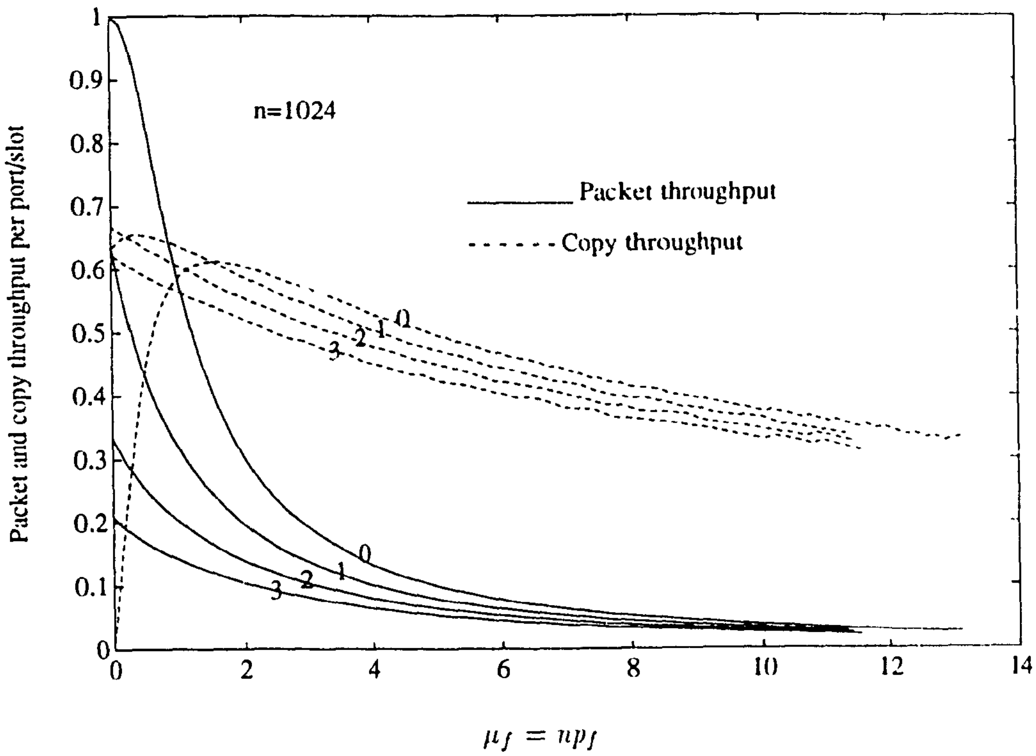


Figure 3.5: Asymptotic packet and copy throughput per port/slot against the average number of copies generated by a fresh packet for the primary copy $\zeta = 0, 1, 2, 3$ and the switch sizes of $n = 1024$.

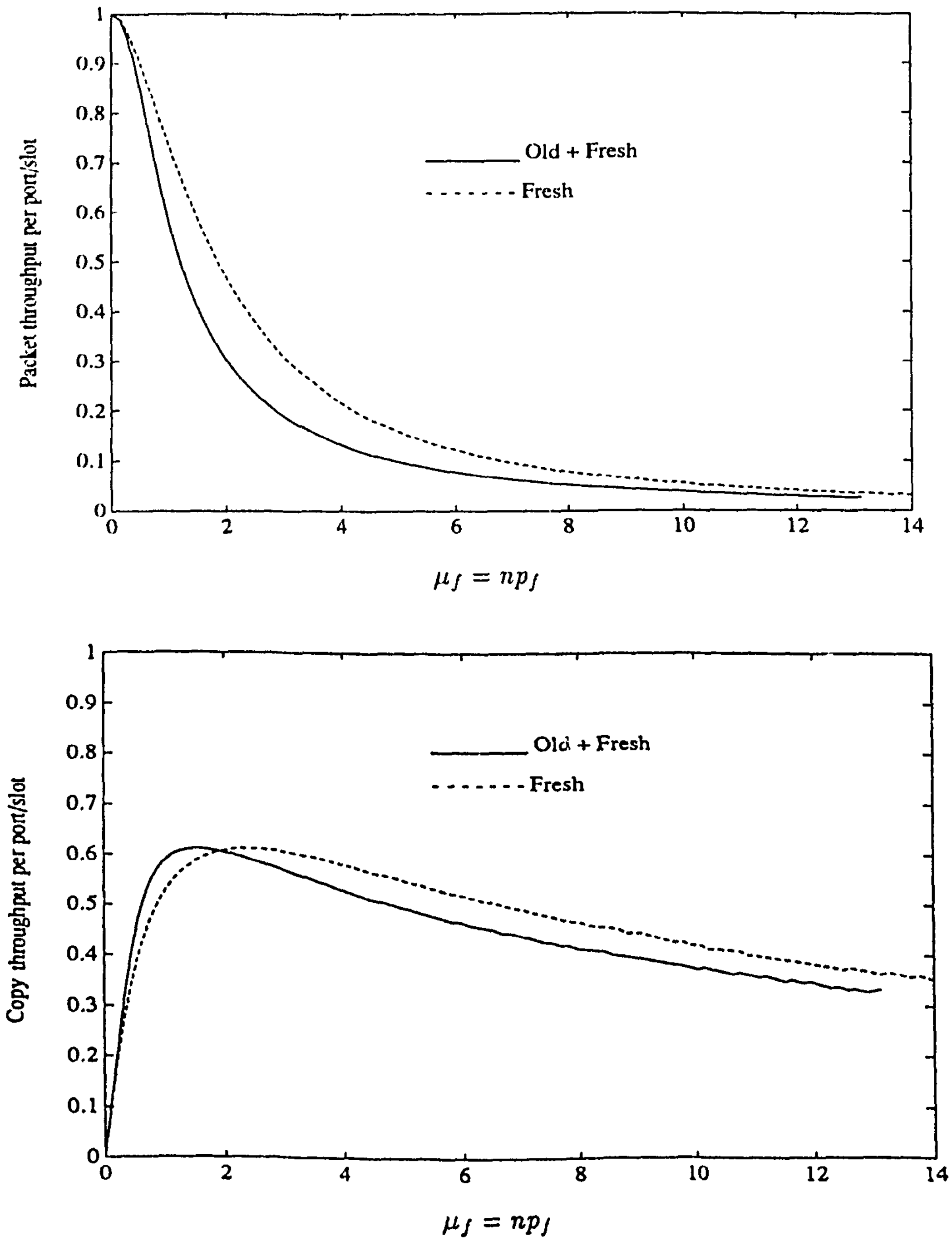


Figure 3.6: Asymptotic packet and copy throughput per port/slot against the average number of copies generated by a fresh packet with mixed input packets and fresh input packets for the switch sizes of $n = 1024$.

Chapter 4

Conclusion

Broadband ISDN will provide diverse services to the users. Among these services, there will be a requirement for multipoint communications. An essential component of such a system will be a multicast switch which will transmit copies of a packet to different destinations.

A multicast switch consists of a copy network and a nonblocking routing network. With a nonblocking routing network, the packets will not be blocked within the switch if the destination of all input packets are distinct. However, blocking will occur if there is output contention, i.e., the request for output ports are not distinct.

A number of service disciplines may be considered where the packets conflict with each other in a multicast switch. This thesis has studied the one shot discipline with random packet selection policy, i.e., a packet is chosen randomly from among the contending packets. This policy is modeled as a renewal process, where renewal points corresponds to the successful choice of the packets. The copy replication model considered is modified Binomial copy generation process, which overcomes the drawback of Binomial copy generation process. In the analysis, two cases of input packets are considered. The first case is that all the packets at the inputs are fresh packets which are generated at the beginning of a slot, i.e., the packets which were not switched during a slot will be discarded. The second case is that the packets at the inputs are the mix of old and fresh packets. The old packets are the ones which were not switched in the previous slot and left to the next slot. The analysis is divided into two parts, determining the probability that a contending packet will not interfere with the chosen packets, and modeling the system as a renewal process. From here, the distribution of the number of packets chosen in a slot is derived and the packet and copy throughput are determined.

Due to the numerical difficulties encountered, when the switch size becomes large, it is impossible to calculate the distribution of the number of packets chosen. However, in the broadband ISDN, because of the high transmission rates and the diverse services, the switch sizes are expected to be large. Thus, the asymptotic results have been derived for very large switches under saturation. The mathematical model used here is also the renewal process. The packet and copy throughput are given, as well as a lower bound of packet and copy throughput are given.

All the theoretical results are in good agreement with the simulation results.

Extension from this thesis are possible, we briefly introduce some of these possibilities.

- In the thesis, we assumed that all the input packets have same copy generation process. However, in practical, copy generation processes may not be same for all input packets. An analysis of other copy generation processes would be a natural extension.

• In this thesis, the output contention is resolved by random selection among the contending packets. However, the output contention may also be resolved by selecting packets based on their sizes. For instance, we could select the smallest or the largest packet among the contending ones. That is if more than one packet send copies to the same output, the packet with smallest size will be chosen (smallest selection) or the packets with largest size will be chosen (largest selection).

APPENDIX A: SIMULATION MODEL

Simulation of the multicast packet switch with random selection policy under modified Binomial copy generation process for fresh input packets as well as for mixed input packets, is implemented through a software package SIMU which was written in C and running at Sun Sparc workstation. The block diagram of SIMU is shown in figure 4.1.

In the simulation, time is segmented into fixed size slots, and a slot is the minimum duration of any event. In the beginning of a slot, a packet will be generated at each empty input with probability ρ. Then, for each newly generated packet at the inputs, copies are generated to output ports according to modified Binomial copy generation process with primary copy $\zeta = 0, 1, 2, 3$. First, ζ primary copies are distributed to n outputs; then, a copy will be generated to each unselected output with probability p. The method used for this distribution is explained in section 2.2. The packets contend for outputs based on random selection policy. The blocked packets in current slot are held in buffers for future contention in following slots or discarded according to the option choose of input packets. In order to keep track of all events during the simulation and to facilitate the manipulation data, we use well defined data structures. The simulation was made over a specified number of slots for each specified configuration. The simulation yields packet and copy throughput.

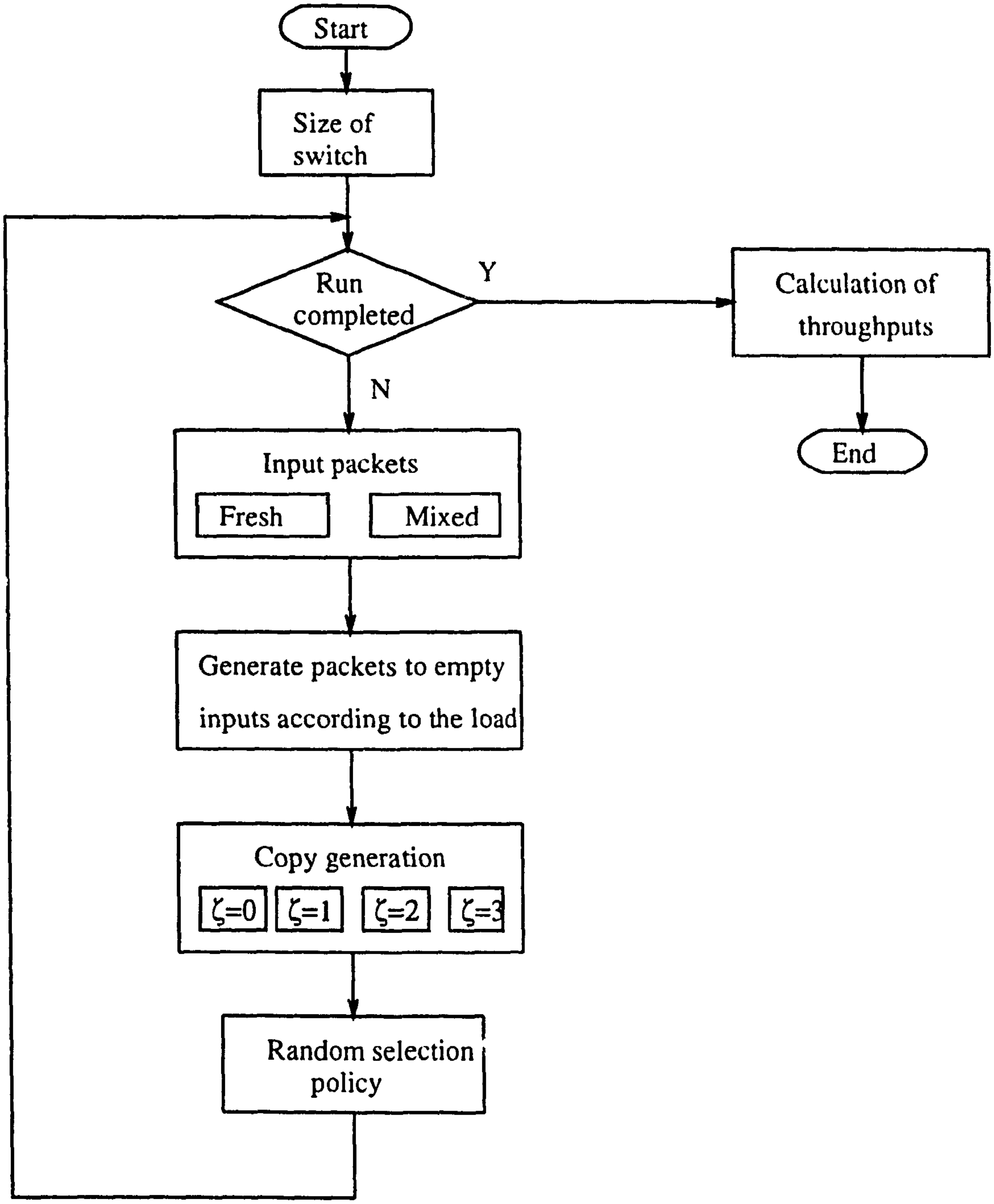


Figure 4.1: A flowchart of program SIMU

APPENDIX B: SYMBOLS

m_j	The total number of outputs selected by the first chosen j packets
$M_j(z)$	PGF of m_j
n	The number of the input and output ports
n_j	The number of outputs selected by the j'th packet
$N_j(z)$	PGF of n_j
N_α	The number of renewals occurring in $[0,\alpha]$
p	Probability that a packet generates a secondary copy to an output
p_f	Probability that a fresh packet generates a secondary copy to an output
p_o	Probability that an old packet generates a secondary copy to an output
P_j	Probability that a contending packet will not interfere with the first j chosen packets
$P_j(z)$	PGF of the probability that j packets are chosen from α contending packets
P_{mt}	Transition probability of having t old packets at an embedded point given m old packets at the previous embedded point
$C(i)$	probability that there are i contending packets at the inputs at the beginning of a slot
r_j	The remaining number of unselected outputs following the j'th choice
$R_j(z)$	PGF of r_j
S_j	The total number of trials required until we choose the j'th packet
$S_j(z)$	PGF of S_j
T_c	Copy throughput
T_p	Packet throughput

u	The number of primary copies that a contending packet has interfered with the j chosen packets
U_j	The total number of copies that a contending packet has interfered with the first j chosen packets
W_j	probability that j packets will be chosen in a slot at the steady-state
$\tilde{x}_i$	Output of an Bernoulli process
X_i	The number of trials for the i'th choice after the (i-1)'th choice
ζ	number of primary copies
μ	The average number of copies generated by a contending packet
μ_f	The average number of copies generated by a fresh packet
μ_o	The average number of copies generated by an old packet
α	The number of contending packets
ρ	Probability that an input is busy
$v(f/m)$	probability of f new packets arriving at the beginning of a slot given m old packets at the preceding embedded point
$\tau(i/m)$	probability that i packets will be contending during a slot given m old packets at the preceding embedded point. This corresponds to (i-m) new arrivals
π_k	probability of having k old packets at an embedded point at the steady-state

Bibliography

[1] William Stallings, "ISDN and Broadband ISDN," Second Edition, MacMillan Publishing Company, New York, 1992

[2] J. S. Turner, "Design of an integrated service network," *IEEE J. Selected Areas Commun.*, VOl. SAC-4, Nov., 1986.

[3] T. T. Lee, R. Boorstyn and E. Arthurs, "The architecture of a multicast broadband packet switch," *Proc. IEEE INFOCOM'88,*, pp. 1-8, New Orleans, LA, 1988

[4] T. T. Lee, "Non-blocking copy networks for multicast packet switching," *IEEE J. Selected Areas Commun.,* VOl. SAC-6, pp.1455-1467, Dec. 1988.

[5] J. S. Turner, "Design of a broadcast packet switching network," *Proc. IEEE INFOCOM'86,* pp. 667-675

[6] A. Huang and S. Knauer, "Starlite: a wideband digital switch," *Proc. of Globecom'84,* pp. 121-125

[7] J. Y. Hui and E. Arthurs, "A broadband packet switch for integrated transport," *IEEE J. Select. Areas Commun.,* Vol. 5, No.8, Oct. 1987, pp. 1264-1273

[8] M. J. Karol, M. G. Hluchyj, S. P. Morgan, "Input versus output queueing on a space-division packet switch," *IEEE Trans. on Comm.*, vol-35, pp. 1347-1356, Dec. 1987

[9] J. Y. Hui and T. Renner, "Queueing strategies for multicast packet switching," *Proc. of Globecom'90,* pp. 1431-1437, San Diego, CA, Dec. 1990

[10] Xing Chen, "Multicast Packet Switching — Scheduling, Contention Resolution, Modeling, and Architecture" *Ph.D Thesis, Department of Electrical and Computer Engineering, Concordia University,* May 1992

[11] J. F. Hayes, R. Breault and M. K. Mehmet Ali, "Performance analysis of a multicast switch," *IEEE Trans. on Commun.*, Vol. 39, No. 4, pp. 581-587

[12] J. F. Hayes, X. Chen and M. K. Mehmet Ali, "Performance comparison of two input access methods for a multicast switch," *IEEE Trans. on Commun.*, Vol.42, No.5, pp.2174 - 2178, May 1994

[13] M. Mehmet Ali, "The performance analysis of a random packet selection policy for multicast switching," *IEEE INFOCOM'93,* Vol. 3, pp. 1117-1126, San Francisco, CA, March 1993

[14] Gunnar Blom, "Probability and Statistics," Springer-Verlag, 1989

[15] Daniel P. Heyman and Matthew J. Sobel, "Stochastic Models in operations research," Volume 1, 1982.

[16] D. R. Cox, "Renewal theory," Methuen, London, 1962.

[17] W. Feller, "An introduction to probability theory and its applications," Volume 1, Wiley, New York, 1971.

[18] N.U. Prabhu, "Stochastic Processes: basic theory and its application," 1965.

[19] A.Gut, "Stopped random walks limit theory and applications," University of Uppsala,1986.

[20] Y. S. Chow and H. E. Robbins, "A renewal theorem for random variables which are dependent or non-identically distributed," *Math. Statist.,* Vol.34,pp.390-401, 1963.