The Reliability of Networks with Special Structures

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ABSTRACT

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In network reliability analysis, an important problem is to determine the probability that a specified subset of vertices in an undirected graph are connected. It is well known that, using the factoring theorem, the reliability of a graph can be expressed in terms of the reliabilities of a graph with one fewer vertex, and another with one fewer edge. The theorem can be applied recursively on the reduced graphs. The computations involved in this recursion can be represented by a binary structure such that its leaves correspond to reduced graphs whose reliabilities can be readily evaluated. In general, as the recursion progresses, series and parallel edges are created which can be reduced through degree-2 and parallel rules of reliability, assuming edges fail independently of each other. The computational complexity is a function of the number of leaves in the binary structure.

In this thesis, a technique for evaluating the exact overall reliability of ladders, wheels, star-ladders, and star-wheels is presented- overall reliability is the probability that there exists successful communication among all the vertices of the network. The approach involves applying the factoring theorem to a wheel, thereby generating at each level of the binary structure two subgraphs, one of which is a ladder, whose overall reliability can be computed readily, and the other a wheel of one less vertex. Finally, an algorithm is presented for evaluating the exact overall reliability of star-ladders, and star-wheels.

DEDICATION

This thesis is dedicated to the loving memory of my parents, Arcadio and Carmel Santana; to a very beautiful person, Beverly, without whose love, concern, and exceptional qualities, this thesis would not have been written; and especially to my mentor, Dr. Zohel Khalil, whose friendship, wisdom, knowledge, patience, and guidance made this thesis all possible.

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CHAPTER 1

1. Introduction and Notation

Many engineering systems, such as computer networks, communication networks, and power transmission systems, can be represented by probabilistic graphs in which vertices and edges represent the relevant components of the system. Vertices and edges may fail independently, or may have a joint failure probability distribution. Any two vertices communicate if they are both functional, and if there exists a path containing functional vertices and edges between them.

In recent years, there has been growing interest in the reliability analysis of such systems. This thesis will be devoted to one measure of system reliability, namely the overall reliability.

The overall reliability is the probability that successful communication exists between every vertex-pair of the network.

The networks dealt with in this thesis will have edges which fail independently with edge reliabilities p_e, the probability that edge e functions. Vertices can not fail.

In this chapter, an introduction into overall reliability is given; it's definition and origins are briefly explained in order to prepare the reader for a more detailed analysis. A list of basic definitions and notations is also given.

In Chapter 2, a concise historical background is presented on the overall reliability problem. Several examples are developed to acquaint the reader with the pitfalls and virtues of the different methods used in the computation of overall reliability.

A brief explanation of the research of Wing and Fu is at the commencement of the chapter.

The chapter then proceeds to introduce, define, and explain the inclusion-exclusion principle. Two examples are developed to demonstrate the intractability of this method.

Based on this principle, the concept of the domination of a graph, theorem 1, is developed. A somewhat detailed analysis of this concept is explained through corolary 1, and lemmas 1 and 2. Example 3 gives a lengthy but detailed account of how through the use of domination theory, the overall reliability of a specific structure could be derived.

Using the concept of domination of a graph as a stepping stone, the edge factoring theorem is now derived. The works of several researchers are mentioned to outline the various means of applications for the edge factoring theorem. Examples 4 and 5 derive the overall reliability for two specific structures, and in the process show some of the complexities from the use of this theorem.

The vertex factoring theorem, is similar to the edge factoring theorem in that a graph is decomposed by factoring on a vertex instead of an edge, which is then introduced through theorem 2, and through example 6, where the overall reliability of a specific structure is derived.

The overall reliability of a biconnected graph, a special case of a connected graph, is now computed through the application of the vertex factoring theorem. The overall reliability of any biconnected graph could be derived through theorem 3. An application of this theorem is given in example 7. Example 8 considers the structure of example 7 to be a special case of a biconnected graph, the overall reliability of this structure is again computed through the application of theorem 4.

An in depth analysis into a major piece of work follows. The method, termed "polygon to chain" reductions, is based on a series of seven transformations which reduce certain subgraphs to edges through reliability preserving reductions. These reductions are termed "polygon to chain" reductions. Definition 1 defines a polygon. Through theorem 5, the K-terminal reliability of any structure which has undergone a "polygon to chain" reduction could be obtained. Property 1 and property 2 present two important ideas pertinent to this type of reduction. In figure 9, a table of the seven transformations is presented.

A piece of work similar to that of "polygon to chain" reductions is that of Δ-Y reductions. Definition 2 defines an IFCF-graph, definition 3 a Δ-Y reduction, definition 4 a window, and definition 5 an ICF-graph. Along with property 3 and property 4, examples 9 and 10 present a close study of the concept of Δ-Y reductions.

A short introduction of backtrack fusion is subsequently presented, along with a mention of the works of two well known researchers of this method.

The last method presented is that of modules. A mention of the various works in this area is given, along with a somewhat explicit discussion of this technique.

The fact that the computation of overall reliaility is NP-hard for a general graph is restated at the commencement of chapter 3. It then proceeds to give definitions of several special structures, for example, ladders, wheels, star-ladders, star-wheels.

Through examples 11 and 12, the overall reliability of ladders of orders 4 and 5 are respectively computed. These examples form part of the proof of lemma 3, the overall reliability of a ladder of order n.

Example 13 computes the overall reliability of a star-ladder of order 8 by first reducing the structure to a ladder of order 5 and consequently applying lemma 3 to determine it's overall reliability. A general formula is also given for star-ladders of order n.

The overall reliability of wheels of orders 4 and 5 are computed in examples 14 and 15 respectively. These examples are the basis for the proof of Lemma 4, the overall reliability of wheel structures of order n.

In a method similar to that used in example 13, example 16 computes the overall reliability of a star-wheel of order 7 by first reducing the structure to a wheel of order 4, and consequently applying lemma 4. A general formula is then derived for star-wheels of order n.

In section 3.5 an overall reliability approximation formula is derived for wheels of order n when the probability of an edge failing is negligible.

Section 3.6 presents a comparison of the computational times for the overall reliability of wheels of order 4 through 8, using the edge factoring theorem and the recursive equations derived, lemma 3 and lemma 4. Table 1 lists the time using the edge factoring theorem as well as the number of spanning trees and the total number of subgraphs generated for five orders of wheels (orders 4 through 8).

In the final section of the chapter, a simple algorithm is presented to determine the overall reliability of a graph which is reducible to a wheel or a ladder.

In chapter 4 several concluding remarks are made, in addition to a brief comparison of the methods discussed. A concise mention and list of several related works is given.

1.1 Preliminaries

In this thesis we will be concerned only with undirected graphs. Standard terminology can be found in such texts as Harary [5], Swamy and Thulasiraman [6], and Deo [7]. However a few definitions need to be introduced.

Connected Graph: A graph in which there exists at least one path between every pair of vertices. If no path exists, the graph is said to be disconnected.

Simple Graph: A graph which has neither self-loops nor parallel links.

Spanning Subgraph: A subgraph of the graph containing all the vertices of the graph.

Tree: A connected graph with no cycles.

Spanning Tree: A tree of the graph containing all the vertices of the graph.

Nullity of a Graph: For a connected graph with b edges, n vertices, the nullity is equal to (b-n + 1). Nullity is the rank of the circuit matrix of the graph.

Formation of a Graph: A nonempty subset of spanning trees of a graph whose union yeilds the graph. The formation is termed odd, if it consists of an odd number of trees, and even otherwise.

Overall Reliability: The probability that there exists at least one path between every vertex-pair of the graph. In other words, that there exists communication between every vertex-pair.

Domination: The number of odd formations minus the number of even formations of a graph.

Chain Progression: An alternating sequence of vertices and edges, starting with vertex n and ending with vertex v, such that the degree of all internal vertices is 2. The length of a chain is simply the number of edges it contains.

Circuit Progression: A chain progression in which the starting vertex is the same as the end vertex.

Degree of a Vertex: The number of edges incident on the vertex.

1.1.1 Notation

G: The given probabilistic graph of the system whose overall reliability is sought.

G_i: The i-th spanning subgraph of G.

R(G): Overall reliability of graph G.

d(G): Domination of graph G.

D(G): $|d(G)|$.

$e \sim (u, v)$: Edge e having end vertices u and v.

p(e), p_e: Probability that edge e functions.

q: $q = (1 - p)$

n: The number of vertices of graph G.

b: The number of edges of graph G.

CHAPTER 2

2. Historical Review

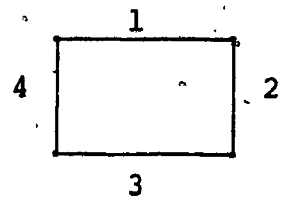
In this chapter a review of the literature on overall reliability in presented.

Wing and Demitriou [8], have considered the overall reliability problem by calculating the 2-terminal reliability for each vertex-pair, e.g. they solve the problem by considering $\frac{n(n-2)}{2}$ subproblems. This approach becomes unwieldly even for moderate size networks. Fu [9], approximated overall reliability thru the application of topological electrical network theory.

Since the overall reliability is the probability that at least one spanning tree of the network has all its edges and vertices functional, the overall reliability can be calculated by use of the well known Inclusion-Exclusion Principle, which is a direct expansion of the union of events. In a network having m spanning trees the probability expression yields (2^m-1) terms. Thus, computing overall reliability using direct expansion is not tractable even for moderate size networks. For the graph of figure 2, m=8, the number of terms in the probability expression would be $2^8-1=255$.

For networks consisting of more than one spanning tree, the actual number of terms, β, in the explicit reliability expression is substantially less than (2^m-1). Hagstrom, Prabhakar, Satyanarayana [10], [11], [12], [13], [14], have studied this problem. For the example graph of figure 1, $\beta=7$, which is less than half the total number of terms, $2^4-1=15$.

<u>Example 1.</u>



<u>Figure 1.</u>

The spanning trees are: $f_1 = \{1,2,3\}$

$f_2 = \{1,3,4\}$

$f_3 = \{1,2,4\}$

$f_4 = \{2,3,4\}$

Using the Inclusion-Exclusion Principle;

$$R(G) = \sum_i P(f_i) - \sum_{i<j} P(f_i,f_j) + \sum_{i<j<k} P(f_i,f_j,f_k) \cdots\cdots \qquad (2.0)$$

$$R(G) = P(f_1 \cup f_2 \cup f_3 \cup f_4)$$

$$\begin{aligned} R(G) = {} & P(f_1) + P(f_2) + P(f_3) + P(f_4) - P(f_1,f_2) - P(f_1,f_3) \\ & - P(f_1,f_4) - P(f_2,f_3) - P(f_2,f_4) - P(f_3,f_4) \\ & + P(f_1,f_2,f_3) + P(f_1,f_2,f_4) + P(f_1,f_3,f_4) \\ & + P(f_2,f_3,f_4) - P(f_1,f_2,f_3,f_4) \end{aligned} \qquad (2.1)$$

$$\begin{aligned} R(G) = {} & P(1,2,3) + P(1,3,4) + P(1,2,4) + P(2,3,4) \\ & - 6P(1,2,3,4) + 4P(1,2,3,4) - P(1,2,3,4) \end{aligned}$$

$$\begin{aligned} R(G) = {} & P(1,2,3) + P(1,3,4) + P(1,2,4) + P(2,3,4) \\ & - 3P(1,2,3,4) \end{aligned} \qquad (2.2)$$

where $P(i,j,k)$ is the probability that edges (i,j,k) function.

Example 2. (Inclusion-Exclusion Principle)

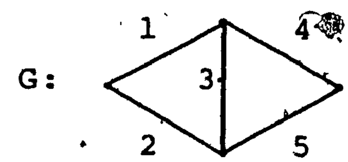


Figure 2.

Spanning trees are: $f_1 = \{1,2,4\}$

$f_2 = \{1,2,5\}$

$f_3 = \{1,3,4\}$

$f_4 = \{1,3,5\}$

$f_5 = \{1,4,5\}$

$f_6 = \{2,3,4\}$

$f_7 = \{2,3,5\}$

$f_8 = \{2,4,5\}$

applying the Inclusion-Exclusion Principle (2.0):

$$R(G) = P(f_1 \cup f_2 \cup f_3 \cup f_4 \cup f_5 \cup f_6 \cup f_7 \cup f_8)$$

$$R(G) = \left[\sum_i P(f_i)\right] - \left[\sum_{i<j} P(f_i,f_j)\right] + \left[\sum_{i<j<k} P(f_i,f_j,f_k)\right] - \ldots \quad (2.3)$$

Taking intersection of spanning trees in (2.3), yields not necessarily unique spanning subgraphs. Hence,

$$R(G) = [P(f_1) + P(f_2) + P(f_3) + P(f_4) + P(f_5) + P(f_6)$$
$$+ P(f_7) + P(f_8)] - [6P(f_1,f_2) + 3P(f_1,f_3)$$

$$+ 10P(f_2,f_3) + 3P(f_2,f_4) + 3P(f_3,f_4) + 3P(f_6,f_7)]$$

$$+ [48P(f_1,f_2,f_3) + 4P(f_1,f_2,f_5) + P(f_1,f_3,f_6)$$

$$+ P(f_2,f_4,f_7) + P(f_3,f_4,f_5) + P(f_6,f_7,f_8)]$$

$$- [69P(f_1,f_2,f_3,f_4) + P(f_1,f_2,f_5,f_8)]$$

$$+ [56P(f_1,f_2,f_3,f_4,f_5)] - [28P(f_1,f_2,f_3,f_4,f_5,f_6)]$$

$$+ [8P(f_1,f_2,f_3,f_4,f_5,f_6,f_7)] - [P(f_1,f_2,f_3,f_4,f_5,$$

$$f_6,f_7,f_8)].$$

or,

$$R(G) = [P(1,2,4) + P(1,2,5) + P(1,3,4) + P(1,3,5)$$

$$+ P(1,4,5) + P(2,3,4) + P(2,3,5) + P(2,4,5)]$$

$$- [6P(1,2,4,5) + 3P(1,2,3,4) + 10P(1,2,3,4,5)$$

$$+ 3P(1,2,3,5) + 3P(1,3,4,5) + 3P(2,3,4,5)]$$

$$+ [48P(1,2,3,4,5) + 4P(1,2,4,5) + P(1,2,3,4)$$

$$+ P(1,2,3,5) + P(1,3,4,5) + P(2,3,4,5)]$$

$$- [69P(1,2,3,4,5) + P(1,2,4,5)] + [56P(1,2,3,4,5)]$$

$$- [28P(1,2,3,4,5)] + [8P(1,2,3,4,5)]$$

$$- [P(1,2,3,4,5)]. \qquad (2.4)$$

Upon collection of like terms, and simplification, one obtains:

$$R(G) = P(1,2,4) + P(1,2,5) + P(1,3,4) + P(1,3,5)$$
$$+ P(1,4,5) + P(2,3,4) + P(2,3,5) + P(2,4,5)$$
$$- 3P(1,2,4,5) - 2P(1,2,3,4) - 2P(1,2,3,5)$$
$$- 2P(1,3,4,5) - 2P(2,3,4,5) + 4P(1,2,3,4,5) \quad (2.5)$$

or simply,

$$R(G) = p_1p_2p_4 + p_1p_2p_5 + p_1p_3p_4 + p_1p_3p_5 + p_1p_4p_5$$
$$+ p_2p_3p_4 + p_2p_3p_5 + p_2p_4p_5 - 3p_1p_2p_4p_5$$
$$- 2p_1p_2p_3p_4 - 2p_1p_2p_3p_5 - 2p_1p_3p_4p_5 - 2p_2p_3p_4p_5$$
$$+ 4p_1p_2p_3p_4p_5.$$

The infeasibility of using the Inclusion-Exclusion Principle also becomes apparent in this example, since equation (2.4) contains 255 terms, many of which cancel, resulting in only 23 terms in (2.5).

Examples 1 and 2, and equation (2.0) demonstrate the complexity of using the Inclusion-Exclusion Principle to compute the overall reliability.

Referring to equations (2.2) and (2.3), it follows that R(G) may be written in the form;

$$R(G) = \sum_k d(\bar{G}_k)P(\bar{G}_k) \quad (2.6)$$

Satyanarayana [1], has proven that $d(\bar{G}_k)$ is simply the number of odd formations minus the number of even formations, which, by definition, is the domination of graph $\bar{G}_k$.

Expressing R(G) using equation (2.6), rather than using equation (2.0), has one major advantage: only non-cancelling terms would be generated.

The problem now is in the computation of the domination of a graph, which is the main theorem in Satyanarayana's paper.

<u>Theorem 1.</u>

For any graph G, the domination $d(G) = d(G_e) - d(G\text{-}e)$.

The domination of a graph G can therefore be computed as the domination of G_e (a subgraph of G obtained by coalescing the vertices of edge e) minus the domination of (G-e) (a subgraph of G obtained by deleting edge e). The domination of each subgraph would then have to be recursively computed in a similar manner. This process would continue until all subgraphs have been reduced to trees, whose dominations are equal to +1.

For some special structures, the domination has been derived as a closed formula, for example, the domination of a complete graph on n vertices (Fact 1).

In comparing the Inclusion-Exclusion Principle and the domination approach, it becomes evident that the domination approach is more efficient in terms of computational time, since the domination approach generates 2^b terms, where b is the number of edges of the graph, whereas the Inclusion-Exclusion Principle generates (2^m-1) terms, where m is the number of spanning trees of the graph. Despite the fact that both computational times are of exponential order, the domination approach is conceivably better.

Corollary 1.

$$d(G) = (-1)^{(b-n+1)} D(G-e) - (-1)^{(b-n)} D(G_e)$$

where $D(G-e)$, $D(G_e)$ are the absolute values of $d(G-e)$ and $d(G_e)$ respectively, and where $(b-n+1)$ is the nullity of the graph G.

Fact 1.

Let G be a complete graph on n vertices.

$$d(G) = (-1)^{(n-1)(n-2)/2}(n-1)!$$

The domination of some graphs with special structures can be easily computed thru application of the following lemmas and corollary.

Lemma 1.

Let G be a graph containing a cut vertex, i, (a cut vertex, is a vertex whose deletion will disconnect the graph, see [6]). Let G_1 and G_2 be the two components of G with vertex i in common, then;

$$d(G) = d(G_1)d(G_2).$$

Lemma 2.

Let G be a graph such that two of it's components G_1 and G_2 together form G ($G_1 \cup G_2 \equiv G$), and ($G_1 \cap G_2$) is a complete graph on m vertices.

$$d(G) = \frac{d(G_1)d(G_2)}{(m-1)!}$$

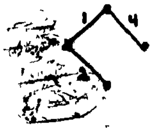
Satyanarayana [1] developed an algorithm to obtain the noncancelling terms in the reliability expression R(G), equation (2.5). The algorithm grows a rooted directed tree in which the vertices represent the spanning connected subgraphs of the given graph G, the root vertex being the graph G. An edge, directed from vertex i, to vertex j of the tree, is weighted with the label of the specific edge, ray e_1 which was deleted from graph $\bar{G}_{(k,i)}$ to obtain $\bar{G}_{(k,j)}$, where $\bar{G}_{(k,i)}$, $\bar{G}_{(k,j)}$ are the i th. and j th. subgraphs obtained from G. The weight associated with each vertex, i, is the domination of graph $\bar{G}_{(k,i)}$.

The algorithm then derives a symbolic expression in factored form for R(G), as a function of the individual edge reliabilities.

Example 3. (Domination Theory approach)

Consider figure 2.

All spanning subgraphs of G are:

G_1 : 

G_2 : 

G_3 : 

G_4 : 

G_5 : 

G_6 : 

G_7 : 

G_8 : 

G_9 : 

G_{10} : 

G_{11} : 

G_{12} : 

G_{13} : 

G_{14} : 

Figure 3.

From equation (2.6), and from the definition of domination of a graph:

$$R(G) = \sum_k d(G_k)P(G_k) = \sum_k F_0(G_k) - F_E(G_k))P(G_k)$$

where $F_O(G_k)$: number of odd formations of G_k.

$F_E(G_k)$: number of even formations of G_k.

$d(G_k)$: is the domination of G_k.

Referring to (2.4), one can see, for example, that;

$F_o(G_2) = 1,\ F_E(G_2) = 0;\ F_o(G_{10}) = 1,\ F_E(G_{10}) = 3;$

$F_o(G_{14}) = 112,\ F_E(G_{14}) = 108.$

Hence,

$$
\begin{aligned}
R(G) = {} & ((1)-(0))P(G_1) + ((1)-(0))P(G_2) + ((1)-(0))P(G_3) \\
& + ((1)-(0))\ P(G_4) + ((1)-(0))\ P(G_5) \\
& + ((1)-(0))P(G_6) + ((1)-(0))P(G_7) \\
& + ((1)-(0))P(G_8) + ((4)-(7))P(G_9) \\
& + ((1)-(3))P(G_{10}) + ((1)-(3))P(G_{11}) \\
& + ((1)-(3))P(G_{12}) + ((1)-(3))P(G_{13}) \\
& + ((112)-(108))P(G_{14})
\end{aligned}
$$

$$
\begin{aligned}
R(G) = {} & P(G_1) + P(G_2) + P(G_3) + P(G_4) + P(G_5) + P(G_6) \\
& + P(G_7) + P(G_8) - 3P(G_9) - 2P(G_{10}) - 2P(G_{11}) \\
& - 2P(G_{12}) - 2P(G_{13}) + 4P(G_{15})
\end{aligned}
$$

$$
\begin{aligned}
R(G) = {} & p_1p_2p_4 + p_1p_2p_5 + p_1p_3p_4 + p_1p_3p_5 + p_1p_4p_5 \\
& + p_2p_3p_4 + p_2p_3p_5 + p_2p_4p_5 - 3p_1p_2p_4p_5 - 2p_1p_2p_3p_4 \\
& - 2p_1p_2p_3p_5 - 2p_1p_3p_4p_5 - 2p_2p_3p_4p_5 + 4p_1p_2p_3p_4p_5 .
\end{aligned}
$$

The advantage of using (2.6) over (2.0) is that only non-cancelling terms are generated, as was seen above.

Another method proposed by Satyanarayana and Chang [2] is based on the same concept of "domination of a graph". This method, referred to as the Factoring Theorem, is based on an elementary event, the success or failure of edge e. Letting p(e) be the probability that edge e functions, R(G) may now be expressed as;

R(G) = p(e)R(G|e is functional) + (1-p(e))R(G|e is not functional) or, more simply, using notation already introduced:

$$R(G) = p_e R(G_e) + q_e R(G-e) \qquad (2.7)$$

Equation (2.7) is referred to as the pivotal decomposition [15], [16].

The use of (2.7) for reliability evaluation was first introduced by Moskowitz [17], for the reliability computation of 2-terminal networks. In his paper, Moskowitz gave a simple, but significant, topological interpretation for the two terms in (2.7).

The computation involved in the recursive application of (2.7) on each subgraph can be represented by a binary tree. Under exhaustive use of (2.7), the binary tree contains 2^b branches, and is equivalent to the enumeration of all possible subgraphs of G. Brown [18], Mine [19], Wing and Demitriou [8], have used (2.7) to compute $R_k(G)$ (source to K-terminal reliability).

By use of the Factoring Theorem, R(G) may be expressed as a function of the reliabilities of a graph with one less vertex, and of another graph with one less edge. Significant computational savings may be obtained if notice is made that graphs G_e and (G-e) may contain parallel and series edges respectively, which can be reduced through degree-2 and parallel reductions. Through such reductions, the size of the binary tree could be reduced significantly. Partaking of this clue, Misra [21], used the Factoring Theorem to decompose a given graph into series-parallel graphs for computing the source to terminal reliability. Hansler [22], Ball [23], and Johnson [24], subsequently followed the same method, with Ball and Johnson going on to prove that the number of branches in the binary tree is at most (n-1)!.

The problem of computing the overall reliability, or for that matter, k-terminal reliability, has been proven to be NP-hard (NP: not polynomial time) by Rosenthal [25], Valiant [26], and Ball and Provan [27]. If edge selections and series-parallel reductions performed on each graph in the binary tree is polynomially bounded as a function of the size of G, the computation involved in generating the binary tree would be proportional to the number of branches in the binary tree. The optimal binary tree obtained by the recursive application of (2.7) is the one with the minimal number of branches.

In their paper, Satyanarayana and Chang [2], develop an algorithm, FACT, which guarantees an optimal binary tree to compute $R_k(G)$, of which the overall reliability is a special case.

Example 4. (Use of the Factoring Theorem)

$$R(G) = p_e R(G_e) + (1-p_e) R(G-e)$$

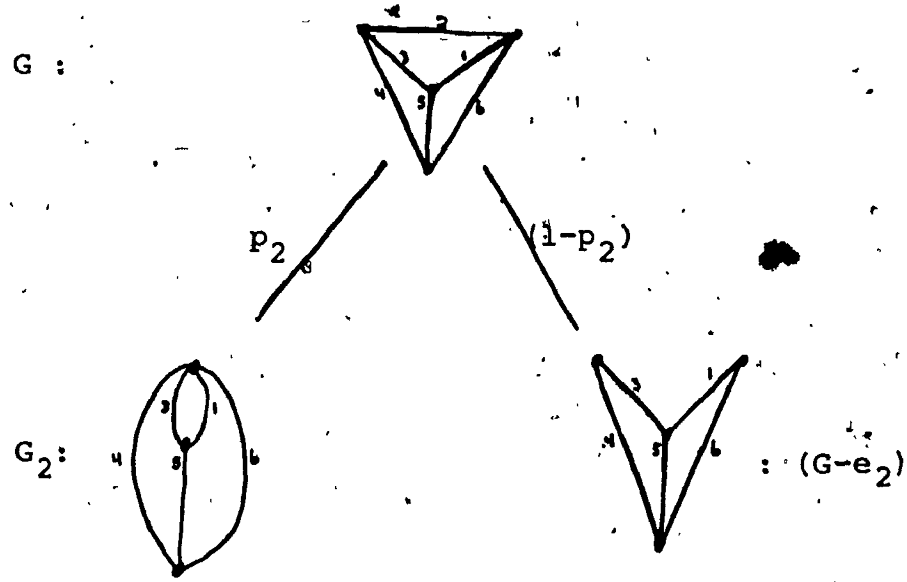


Figure 4.

$$R(G) = p_2 R(G_2) + q_2 R(G-e_2) \quad (2.8)$$

Example 5.

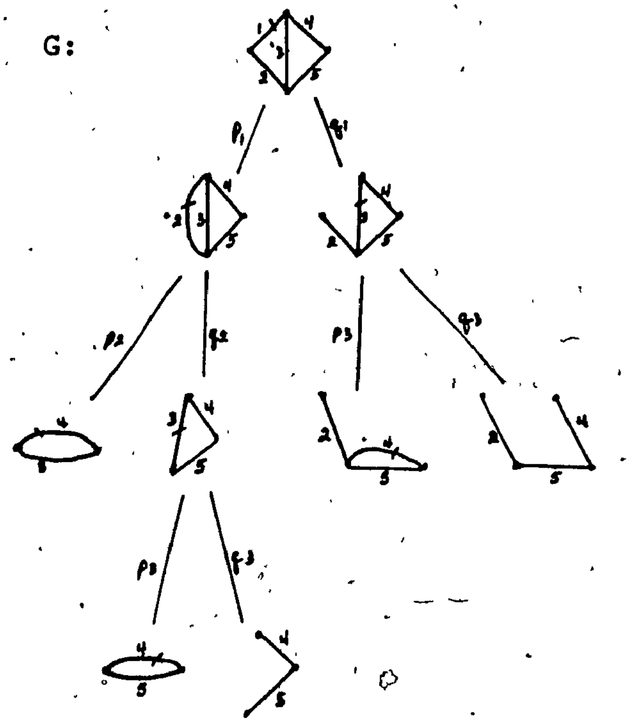


Figure 5.

$$R(G) = p_1(p_2(p_4+q_4p_5) + q_2(p_3(p_4+q_4p_5) + q_3p_4p_5))$$
$$+q_1(p_3(p_4p_2+q_4p_2p_5) + q_3(p_2p_4p_5))$$

In the last two examples, use of the Factoring Theorem was made without use of series-parallel reductions. If in example 4, the Factoring Theorem was repeatedly applied to each subgraph, as was performed in example 5, the total number of graphs generated would be on the order of $2^b = 2^6 = 64$. But, as will be shown in chapter 3, use of degree-2 and parallel reductions will greatly reduce the number of graphs generated.

Satyanarayana, Chang, Khalil [28], in their paper, derive a recursive equation for overall reliability based on a new concept, similar to that of the Factoring Theorem. Unlike the Factoring Theorem, where factoring (decomposition) is performed on edges (examples 4 and 5), this new concept performs the factoring on vertices, hence it's name, Vertex Factoring Theorem.

The main theorem;

<u>Theorem 2.</u>

$$R(G) = \left\{ \sum_{i=1}^{k} p_i \prod_{j \neq i} (1-p_j) \right\} R(G-m) + \sum_{i<j} p_i p_j \prod_{i \neq q < j} (1-p_q) R(\hat{G}_{|i,j|}) \tag{2.19}$$

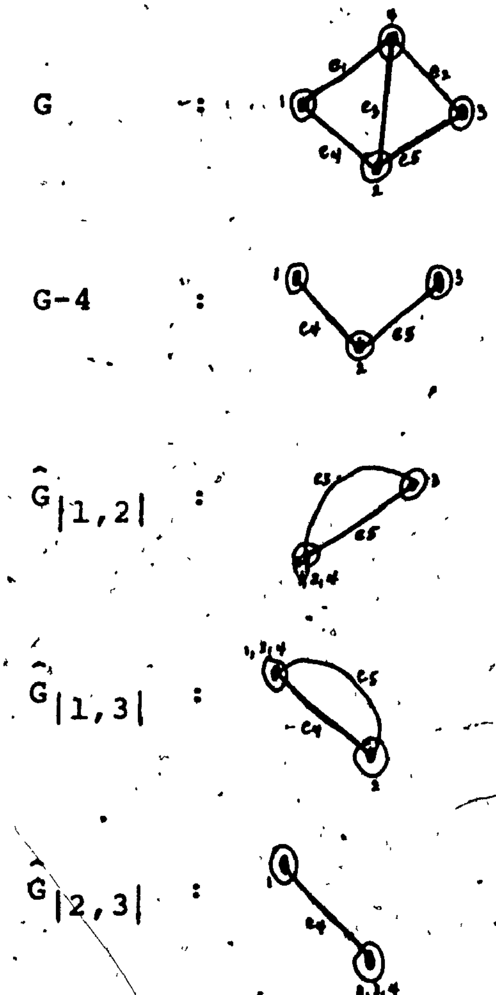
where m : a vertex of degree k>1 in G.

(G-m) : subgraph of G with vertex m and its incident edges deleted.

$\hat{G}_{|i,j|}$: subgraph of G obtained by contracting edges $e_i, e_j (i<j)$ and deleting edges $e_q (i \neq q<j)$.

By use of this theorem, the number of graphs that need to be considered after each decomposition is $(\binom{k}{2}+1)$.

<u>Example 6.</u> (Vertex Factoring Theorem)



<u>Figure 6.</u>

By equation (2.8):

$$R(G) = \{p_1q_2q_3 + p_2q_1q_3 + p_3q_1q_2\}\ R(G\text{-}e_4) + p_1p_2R(\hat{G}_{|1,2|})$$

$$+ p_1p_3q_2R(\hat{G}_{|1,3|}) + p_2p_3q_1R(\hat{G}_{|2,3|}).$$

$$R(G) = p_4p_5\ \{p_1q_2q_3 + p_2q_1q_3 + p_3q_1q_2\}$$

$$+ (1-q_3q_5)p_1p_2 + (1-q_4q_5)p_1p_3q_2 + p_2p_3p_4q_1.$$

As a special case of the connected graph, Satyanarayana, Chang, Khalil [28], consider the overall reliability of the biconnected graph, $R(G_o)$.

A connected graph G, contains a cut-vertex, v, if removing v partitions G into two or more parts. Let G_1, G_2, G_3, ..., G_k be subgraphs of G which partition the edges, and in addition have a cut-vertex in common with them. The overall reliability of G could then be expressed as a product of the overall reliabilities of G_1, G_2, G_3, ..., G_k.

A connected graph, G_o, is said to be biconnected if it contains no cut-vertices, but there exists two vertices v_1, v_2, whose deletion partition (disconnects) G_o.

<u>Theorem 3.</u>

$$R(G_o) = R(G_1)\, R(\bar{G}_2) + \{R(\bar{G}_1) - R(G_1)\}\, R(G_2)$$

where $(G_1 \cup G_2)$: G.

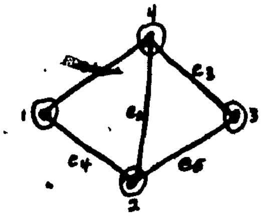
$(G_1 \cap G_2)$ contains only two vertices i,j.

G_1, G_2 : subgraphs of G_o.

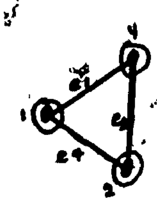
$\bar{G}_1, \bar{G}_2$: graphs obtained by coalescing vertices i,j, in G_1, G_2 respectively.

<u>Example 7.</u>

G_0 :

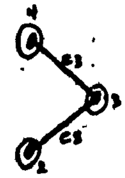


G_1 :



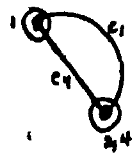
$R(G_1) = p_1p_2+p_1p_4+p_2p_4-2p_1p_2p_4$

G_2 :



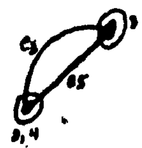
$R(G_2) = p_3p_5$

$\bar{G}_1$:



$R(\bar{G}_1) = p_1+p_4-p_1p_4$

$\bar{G}_2$:



$R(\bar{G}_2) = p_3+p_5-p_3p_5$

Figure 7.

$$R(G_0) = (p_1p_2+p_1p_4+p_2p_4-2p_1p_2p_4)(p_3+p_5-p_3p_5)$$

$$+ \{(p_1+p_4-p_1p_4) - (p_1p_2+p_1p_4+p_2p_4-2p_1p_2p_4)\}\ p_3p_5$$

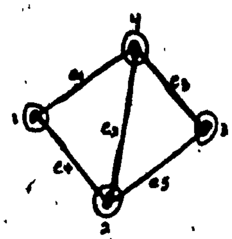
By noting that graph G_0 is a special case of a biconnected graph, by virtue of the fact that $(G_1 \cap G_2)$ contain an edge, e_2, between vertices 2 and 4, $R(G_0)$ may now be computed using the following theorem, again due to Satyanarayana, Chang, Khalil [28].

Theorem 4.

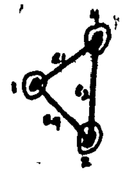
$$R(G_0) = \frac{1}{p_e}\{R(G_1)R(G_2) - (1-p_e)\ R(G_1-e)\ R(G_2-e)\}.$$

Example 8.

G_0 :

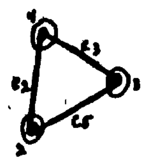


G_1 :



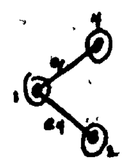
$R(G_1) = p_1p_2+p_1p_4+p_2p_4-2p_1p_2p_4$

G_2 :



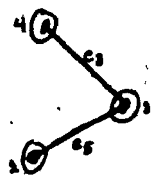
$R(G_2) = p_2p_3+p_2p_5+p_3p_5$
$-2p_2p_3p_5$

G_1-e_2 :



$R(G_1-e_2) = p_1p_4$

G_2-e_2 :



$R(G_2-e_2) = p_3p_5$

Figure 8.

$$R(G_0) = \frac{1}{p_2}\{(p_1p_2+p_1p_4+p_2p_4-2p_1p_2p_4)(p_2p_3+p_2p_5+p_3p_5-2p_2p_3p_5)$$

$$-(1-p_2)(p_1p_4)(p_3p_5)\}.$$

In considering for instance, example 8, as long as the subgraphs G_1, G_2, $\bar{G}_1$, $\bar{G}_2$, (G_1-e_2), (G_2-e_2) possess certain properties, such as being biconnected, S-P reducible, or containing series-parallel edges, one of the above methods for computing overall reliability could be applied.

The vexing problem is that unless the subgraphs G_1, G_2, $\bar{G}_1$, $\bar{G}_2$, (G_1-e_2), $G(_2-e_2)$ possess these properties, no significant savings in computational time can be achieved when computing their overall reliability.

An important piece of work on network reliability was done by Satyanarayana and Wood [3]. They first begin by recalling that computing $R_k(G)$ (K-terminal reliability) in general is NP-hard [29], and then proceed to develop a series of seven "Polygon to Chain" reductions, for general applicability. Using these reductions the K-terminal reliability, $R_k(G)$, may be computed in polynomial time.

The basic requirement they impose on G is that it be a series-parallel graph (either S-P reducible, or S-P complex). In their paper, Satyanarayana and Wood define a "Polygon" as:

<u>Definition 1:</u> Polygon; Let C_1 and C_2 be two chains of lengths L_1 and L_2 respectively, and whose common end vertices are u and v. Then, $C_1 \cup C_2$ is said to be a polygon of length (L_1+L_2).

The great ability to reduce the computational time is due to the "Polygon to Chain" transformations. These transformations are used to create reliability-preserving "polygon to chain" reductions. This method can be used on any S-P graph, but the emphasis is on S-P complex graphs, since K-terminal reliability computations for S-P reducible graphs have been proven to be of polynomial time, for $|K|>2$, as well as for $|K|=2$ [21], [30].

Let $\hat{G}$ be a S-P graph containing a polygon Δ. Let $\hat{G}_\Delta$ be the graph obtained from $\hat{G}$, by replacing polygon Δ with chain C, hence;

Theorem 5.

$$R(\hat{G}) = \Omega_\Delta R(\hat{G}_\Delta)$$

where Ω_Δ: is a corresponding multiplication factor.

It should be noted here that upon performing a "Polygon to Chain" reduction, the reliabilities of each edge of the Chain would be a function of the particular transformation used.

Two interesting properties help prove the non-NP-hard attribute of this method.

Property 1.

Let $\hat{G}_\Delta$ be a graph obtained from $\hat{G}$ by applying a "Polygon to Chain" reduction on $\hat{G}$. Then $\hat{G}_\Delta$ is a S-P graph if and only if $\hat{G}$ is.

Property 2.

Let $\hat{G}$ be a S-P complex graph. Then $\hat{G}$ must admit either a simple reduction (parallel, degree-2) or a "Polygon to Chain" reduction (one of the seven types listed in figure 9).

In conclusion, Satyanarayana and Wood develop an efficient $O(|E|)$ algorithm combining all types of reductions,

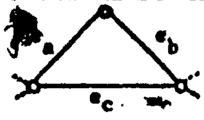
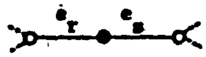
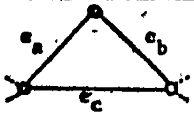
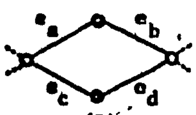
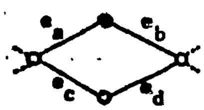
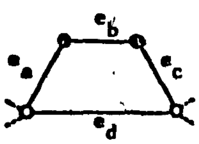
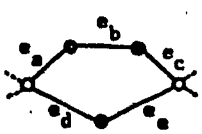
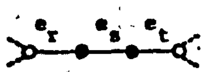
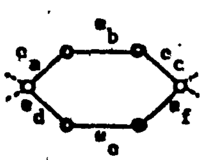
Polygon Type	Chain Type	Reduction Formulas	New Edge Reliabilities
 (1)		$\alpha = q_a p_b q_c$ $\beta = p_a q_b q_c$ $= p_a p_b p_c \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b}\right)$	
 (2)		$\alpha = q_a p_b q_c$ $\beta = p_a q_b q_c$ $\delta = p_a p_b p_c \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b}\right)$	$p_r = \frac{\delta}{\alpha + \delta}$ $p_s = \frac{\delta}{\beta + \delta}$ $\Omega = \frac{(\alpha+\delta)(\beta+\delta)}{\delta}$
 (3)		$\alpha = p_a q_b q_c p_d + q_a p_b p_c q_d$ $\beta = p_a q_b p_c q_d$ $\delta = p_a p_b p_c p_d \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d}\right)$	
 (4)		$\alpha = q_a q_b p_c p_d$ $\beta = p_a q_b q_c p_d + q_a p_b p_c q_d$ $\delta = p_a p_b q_c q_d$ $\gamma = p_a p_b p_c \left(1 + p_d\left(\frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c}\right)\right)$	
 (5)		$\alpha = q_a p_b p_c q_d$ $\beta = p_a q_b p_c q_d$ $\delta = p_a p_b q_c q_d$ $\gamma = abcd\left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d}\right)$	$p_r = \frac{\gamma}{\alpha + \gamma}$
 (6)		$\alpha = q_a p_b p_c q_d$ $\beta = p_a q_b p_c (p_d q_e + q_d p_e)$ $+ p_b (q_a p_c p_d q_e + p_a q_c q_d p_e)$ $\delta = p_a p_b q_c p_d q_e$ $\gamma = p_a p_b p_c p_d p_e \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d} + \frac{q_e}{p_e}\right)$	$p_s = \frac{\gamma}{\beta + \gamma}$ $p_t = \frac{\gamma}{\beta + \gamma}$ $\Omega = \frac{(\alpha+\gamma)(\beta+\gamma)(\delta+\gamma)}{\gamma^2}$
 (7)		$\alpha = q_a p_b p_c q_d p_e p_f$ $\beta = p_a q_b p_c (q_d p_e p_f + p_d q_e p_f + p_d p_e q_f)$ $+ p_a p_b q_c p_f (p_d q_e + q_d p_e)$ $+ q_a p_b p_c q_d (q_e p_f + p_f q_e)$ $\delta = p_a p_b q_c p_d p_e q_f$ $\gamma = p_a p_b p_c p_d p_e p_f \left(1 + \frac{q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_d}{p_d} + \frac{q_e}{p_e} + \frac{q_f}{p_f}\right)$	

Figure 9.

which will compute $R_k(\hat{G})$ in linear time for a S-P graph, where E is the number of edges of $\hat{G}$.

Another major piece of work very similar to that of "Polygon to Chain" reductions, is that of Politof and Satyanarayana [14].

In their paper, they consider the K-terminal reliability of Inner-Four-Cycle-Free-graphs (IFCI-graphs). Through recursive application of degree-2, parallel, and Δ-Y reductions, they show how an IFCF-graph is reducible to a single edge.

Politof and Satyanarayana define IFCF-graph and Δ-Y reduction as:

<u>Definition 2</u>: IFCF-graph; a planar graph G, with no series or parallel edges, is said to be an IFCF-graph, if there exists a face g in G so that every cycle of G, with four or more edges, contains at least one vertex of g.

<u>Definition 3</u>: Δ-Y reduction; consider a planar graph G, fig. 10A, and suppose that g is a face of G, bounded by exactly three edges. Such a face is called a delta (denoted by Δ). Replacing these three edges of the Δ by a star (denoted by Y), fig. 10B, is referred to as a Δ-Y reduction.

Based on the property that IFCF-graphs are reducible to edges, Politof and Satyanarayana developed an $O(|V|^2)$ algorithm to compute the probability that a IFCF-graph is connected, where $|V|$ is the number of vertices of the graph.

The efficiency of this algorithm is attributable to the following two properties:

Definition 4: Window; if a planar graph G is IFCR (ICF) with respect to some face g, then g is defined as a window of G.

Definition 5: ICF-graph; a planar graph G, with no series or parallel edges, is said to be an Inner-Cycle-Free-graph (ICF-graph), if there exists a face g in G so that every cycle of G contains at least one vertex of g.

Property 3.

Let G be an IFCF-graph and w a window of G. If g_1 is an inner Δ (with respect to w) in G, and G' is the graph obtained from G by replacing g_1 with a Y then G' also is an IFCF-graph with respect to w.

Property 4.

Let G be a planar ICF-graph with no series or parallel edges. If w is a window of G, then:

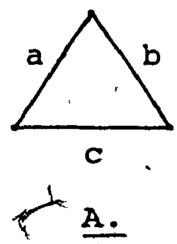
A) G contains a $\Delta \neq w$, and

B) any Δ-Y reduction, $\Delta \neq w$, in G, yields a graph which again is ICF.

As with the "Polygon to Chain" reduction, the reliabilities of the edges in the Y are functions of the reliabilities of the edges of the Δ.

The next two examples will be useful for an understanding of Δ-Y reductions.

Example 9. (Overall reliability is assumed)



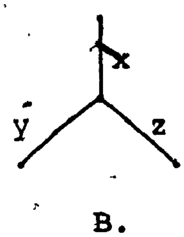


Figure 10.

$$p_x = \frac{\delta}{\delta+\gamma_1} \; ; \; p_y = \frac{\delta}{\delta+\gamma_2} \; ; \; p_z = \frac{\delta}{\delta+\gamma_3}$$

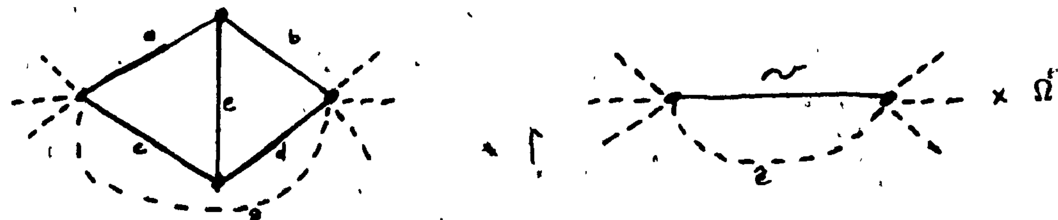
where $\delta = p_a p_b p_c \left(1 + \frac{q_a}{p_a} + \frac{q_c}{p_c}\right)$

$$\gamma_1 = p_b p_c + p_a p_b p_c \left(\frac{1}{p_a} + \frac{2q_b}{p_b} + \frac{q_c}{p_c} + \frac{q_a}{p_a}\right)$$

$$\gamma_2 = p_a p_c + p_a p_b p_c \left(\frac{1}{p_b} + \frac{2q_a}{p_a} + \frac{q_b}{p_b} + \frac{q_c}{p_c}\right)$$

$$\gamma_3 = p_a p_b + p_a p_b p_c \left(\frac{1}{p_c} + \frac{2q_c}{p_c} + \frac{q_a}{p_a} + \frac{q_b}{p_b}\right).$$

Example 10



A. B.

Figure 11.

Though Δ -Y reductions, fig. 11A reduces to fig. 11B.

$$P_\psi = \frac{\delta}{\delta+\gamma} \quad ; \quad \Omega = \delta + \gamma$$

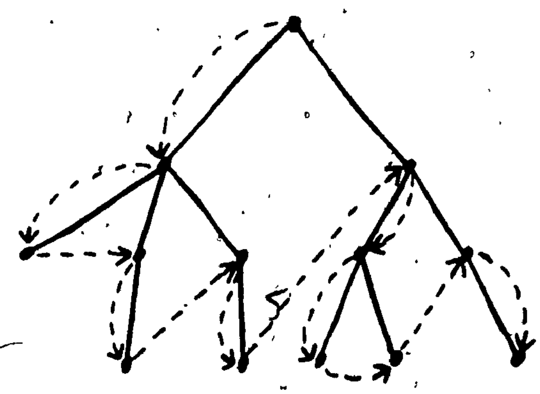
where
$$\delta = p_a p_b p_e + p_c p_d + q_a p_c (q_b p_d + p_b q_d p_e + p_b q_d q_e) + p_a q_c (p_b q_d p_e + q_b p_d p_e + p_b p_d q_c) + q_e p_a p_e (p_b + q_b p_d)$$

$$\gamma = p_e (p_a q_b + q_a p_b) + (p_c q_d + q_c p_d) + q_a q_c [q_b p_d + p_b q_d p_e + p_b p_d p_e (1 + \frac{q_e}{p_e})] + q_a p_c p_d (q_b + p_b q_e) + p_a q_b [q_c (q_d p_e + p_d q_e) + p_c q_d p_e (1 + \frac{q_e}{p_e})]$$

Ω is a weighting factor, and ê is an assumed edge.

Backtrack fusion is the creation, recognition, and merging of isomorphic substructures in a backtrack computational structure to avoid redundancy of computation, Reingold, et al. [11]. Series, parallel, and degree-2 reductions are simple backtrack fusion techniques with vertex decomposition being another.

Backtracking is a general technique for organizing exhaustive searches, say for paths, spanning trees, or spanning subgraphs. The backtracking technique continually tries to extend a partial solution. At each stage of the search, if an extension of the current partial solution is not possible, a "backtrack" is performed to a shorter partial solution, and another attempt is made. This process is represented as a "backtrack structure", an example of which is illustrated in figure 12, Chang [20].



1st choices

2nd choices, given 1st choice

3rd choices, given 1st and 2nd choices

Figure 12.

The search structure for K-trees of the graph of figure 2. The order in which backtrack generates the search structure is shown by dashed lines.

The search procedure is shortened if one:

A) avoids extension of a partial solution which produces no meaningful results;

B) avoids repetition of stages, i.e., avoid generating isomophoric substructures in the backtrack structure.

These backtrack refinements are termed "preclusion (or branch pruning)" or "fusion (or branch merging)" respectively, by Chang.

Although the concept of backtrack is often associated with enumeration problems, it is useful in several additional areas as well, for example, in the computation of K-terminal reliability.

The concept of modules for K-terminal reliability has appeared in many papers, Birnbaum and Esary [34], Barlow and Proschan [15], Murchland [31], Rosenthal [35], Buzacott and Chang [32], Hagstrom [33], Johnson [35], and Satyanarayana et al. [28].

The application of modules varies from author to author, not only with respect to the reliability measure being considered, but in the way it molds the overall solution procedures.

Some authors address the terminal-pair reliability problem, others are concerned with overall reliability. Some solution techniques highlight the modular transformation aspect, while in others the modules are left submerged in some partitioning-decomposition scheme.

Two classes of modules exist for the K-terminal reliability problem. A significant order of computational savings is obtained when modular representation is used in addition to edge decomposition with series and parallel reductions.

It is sometimes possible to replace a connected subgraph $\bar{G}$, of graph G, with an edge e, having appropriate reliability such that the K-terminal reliability of the reduced network $R_{k'}(G')$, equals $R_k(G)$. Such $\bar{G}$'s may be viewed as modules.

The single edge representation requires that the boundary set, B, separating $\bar{G}$ from its complement, $\bar{G}^c$ ($\bar{G} \cap \bar{G}^c = B$, $\bar{G} \cup \bar{G}^c = G$), consist of one or two vertices. The corresponding $\bar{G}$'s will be called 1-subnet and 2-subnet respectively, as per Rosenthal's terminology [25].

Such $\bar{G}$'s satisfy $(\bar{G} \cap K) \subseteq B$. In other words, no vertices of K are contained in $\bar{G}$ unless they are boundary vertices (fig. 13).

A 1-subnet is replaced by a perfect self-loop, and a 2-subnet is replaced by an edge having reliability $R_B(\bar{G})$ (i.e. degree-2 and parallel reductions). The 1-subnet is redundant here, since it is a special case of a graph with a cut vertex, which has been discussed earlier.

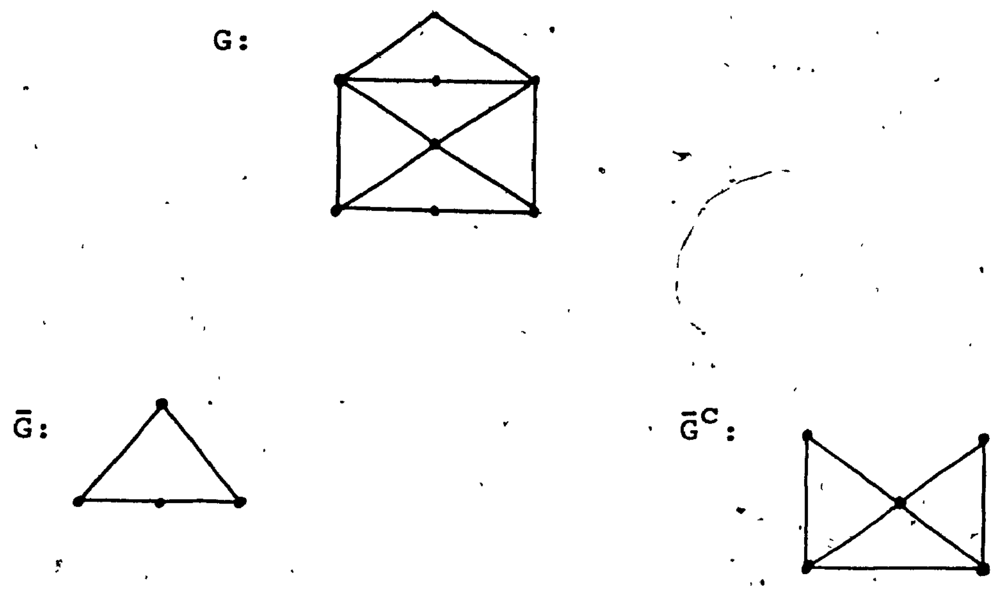


Figure 13.

For the above classes of modules, system success does not restrict the states of $\bar{G}$. Situations exist, however, where system success precludes certain states of a subgraph $\bar{G}$. Conditioned on the event A of the nonoccurrence of states of $\bar{G}$ which cause system failure, it is desired to replace $\bar{G}$ by an edge with appropriate reliability such that the K-terminal reliability of the reduced networks $G', R_{k'}(G')$, equals the conditional reliability of $G, R_k(G|A)$. In other words, $R_k(G) = P(A)R_{k'}(G')$. Such $\bar{G}$'s are called conditional modules.

These $\bar{G}$'s satisfy $B \subset (\bar{G} \cap K)$. Vertices of B are in K, and $\bar{G}$ contains at least one additional K-vertex which is not in B. In these cases, the 1-subnet is replaced by a perfect self-loop, and $P(A) = R_{(\bar{G} \cap K)}(\bar{G})$. The 2-subnet is replaced by an edge having reliability $R_{(\bar{G} \cap K)}(\bar{G})/P(A)$, where $P(A)$ is the probability that each vertex of $(\bar{G}\cap K)$ is connected to a vertex of B (i.e. degree-2 reduction). From a topological viewpoint, it is apparent that $P(A) = R_{(\bar{G}\cap K)_B}(\bar{G}_B)$, where the subscript implies that elements of the subgraph B are identified. In short, B is to be considered as a single "super-vertex".

Summary:

A) $(\bar{G}\cap K)\subseteq B$

1-subnet: $R_k(G) = R_k(\bar{G}^c)$.

2-subnet: $R_k(G) \doteq R_k(\bar{G}^c+e)$ where $p_e = R_B(\bar{G})$.

B) $B\subset(\bar{G}\cap K)$

1-subnet: $R_k(G) = R_{(\bar{G}\cap K)}(\bar{G})R_{(\bar{G}^c\cap K)}(\bar{G}^c)$.

2-subnet: $R_k(G) = R_{(\bar{G}\cap K)_B}(\bar{G}_B)R_{(\bar{G}^c\cap K)}(\bar{G}^c+e)$

where $p_e = R_{(\bar{G}\cap K)}(\bar{G})/R_{(\bar{G}\cap K)_B}(\bar{G}_B)$.

Chang [20], presents a table which compares the amount of polynomial-time computations, expressed in domination, required under algorithm RELY, with and without modular representation.

All the networks reliability computational methods presented here attest to the following:

If a graph is of a S-P type, then polynomial-time algorithms exist. For a general graph, or for a graph possessing properties other than being of a S-P type, for example, the graph may be biconnected, the algorithms existing are NP-hard; exponential-time, factorial time, almost exponential time, ...

CHAPTER 3

3. Overall Reliability of Wheels and Ladders

In the previous chapter, it was shown that the computation of overall realiability is NP-hard unless the graph in question posesses some special properties, for example, biconnectedness, S-P reducibility, etc......

Despite the disadvantages of using the Factoring Theorem (outlined in chapter 2), the theorem still maintains a beauty of its own in that when applied to certain structures, the computational time becomes linear. The application of the Factoring Theorem to a certain graph G, will yield two subgraphs of G, (G-e) obtained from G by deleting edge e, and G_e, obtained from G by coalescing the two vertices of edge e. If one of these subgraphs, say (G-e), is S-P reducible then R(G-e) may be computed readily.

Application of the Factoring Theorem need now only be applied to G_e. After successive application of the Factoring Theorem, if both of the resulting subgraphs, (G-e), and G_e, are S-P reducible, the computational time of R(G) would be linear.

In this chapter, the approach is to derive recursive relationships using the Factoring Theorem, parallel, and degree-2 reductions. Unfortunately, this approach limits the class of networks which can be analysed.

Considered here are four types of networks, namely ladders, star-ladders, wheels, and star-wheels. Recursive relations are derived to compute the overall reliability of such networks, and an algorithm is presented to show the way to compute the overall reliability of a larger class of networks reducible to the above four types.

3.1 Additional Definitions

S-P Reducible (Series-Parallel Reducible)

An S-P reducible graph, is a graph which can be reduced to a tree by successive degree-2 and parallel reductions.

Degree-2 Reduction:

Suppose $e_1 \sim (u, v)$ and $e_2 \sim (v, w)$. A degree-2 reduction replaces e_1 and e_2 with a single edge, $e_3 \sim (u,w)$ such that,

$$p_3 = p_1 p_2/(1 - q_1 q_2), \text{ [4] and}$$

$$R(G) = (1 - q_1 q_2)\, R(G - e_1 - e_2 + e_3) \qquad (3.1)$$

Parallel Reduction:

Suppose $e_1 \sim (u, v)$ and $e_2 \sim (u, v)$. A parallel reduction replaces edges e_1, e_2 with a single edge $e_3 \sim (u,v)$.

$$p_3 = (1 - q_1 q_2) \qquad (3.2)$$

E_n: The extra term generated using the Factoring Theorem to decompose a wheel of order n. Graph E_n would result after (n-3) decompositions.

Ladder:

Consider a chain progression of length n. Add a new vertex by connecting it to every vertex of the chain, hence, forming a ladder of (n + 2) vertices. (See figure 14).

Wheel:

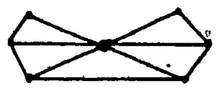
Consider a circuit progression of length n. Add a new vertex by connecting it to every other vertex, hence, forming a wheel of (n + 1) vertices. (See figure 16).

Star-Ladder:

Consider a chain progression of length 2m edges. Connect vertices $v_{(2p-2)}$ and $v_{(2p)}$, p = 1, 2, ..., m. Add a new vertex by connecting it to vertices connected previously, hence, forming a star-ladder of (2m + 2) vertices. (See figure 15).

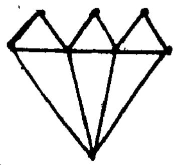
Star-Wheel:

Consider a circuit progression of length 2m edges. Connect vertex $v_{(2p-2)}$ and $v_{(2p)}$, p = 1, 2, ..., m. Add a new vertex by connecting it to vertices connected previously, hence, forming a star-wheel of (2m + 1) vertices. (See figure 17).



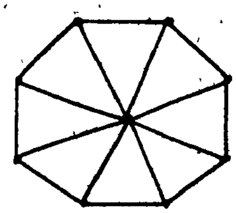
Ladder of order 7.

Figure 14.



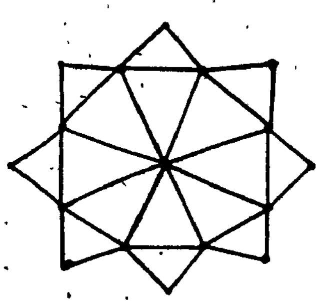
Star-Ladder of order 8.

Figure 15.



Wheel of order 9.

Figure 16.



Star-Wheel of order 17.

Figure 17.

3.2 Overall Reliability of Ladder Structures

Presented in this section are two examples which compute the overall reliability of ladders of 4 and 5 vertices respectively.

Since the ladder is an S-P reducible graph, it is reducible to a single edge by successive degree-2 and parallel reductions.

Example 11

Consider a ladder structure of order 4, figure 18.

L_4 :

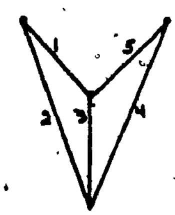


Figure 18.

Edges 1 and 2 are in series, therefore, a degree-2 reduction is performed in order to reduce the size of the graph, yielding edge A. The reliability of edge A is:

$$R(A) = \frac{p_1 p_2}{p_2 + q_2 p_1}$$

A

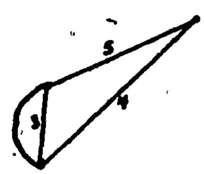


Figure 19.

A parallel reduction is performed on the two parallel edges, edges A and 3, resulting in edge B. The reliability of edge B is:

$$r_1 = R(B) = p_3+q_3R(A)$$

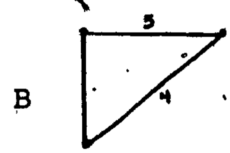


Figure 20.

Another degree-2 reduction is performed on edges B and 4, yielding edge C, followed by a parallel reduction on edges C and 5, resulting in edge D.

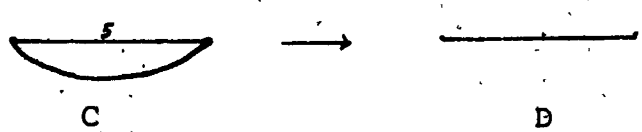


Figure 21.

$$R(C) = \frac{p_4R(B)}{p_4+q_4R(B)} \quad ; \quad R(D) = p_5+q_5R(C)$$

Therefore, using equation (3.1), the overall reliability of the entire structure is:

$$R(L_4) = [\,p_5+q_5R(C)\,]\,(p_2+q_2p_1)(p_4+q_4R(B))$$

or

$$R(L_4) = \left(p_5 + \frac{q_5 p_4 r_1}{p_4 + q_4 r_1}\right) \prod_{k=1}^{2} (p_{(2k)} + q_{(2k)} r_{(k-1)})$$

where $r_0 = p_1$

$$r_1 = p_3 + \frac{q_3 p_2 r_0}{p_2 + q_2 r_0}$$

Example 12.

Consider now a ladder structure of order 5. As in the previous example, this ladder is reducible to a single edge by successive degree-2 and parallel reductions, figure 22.

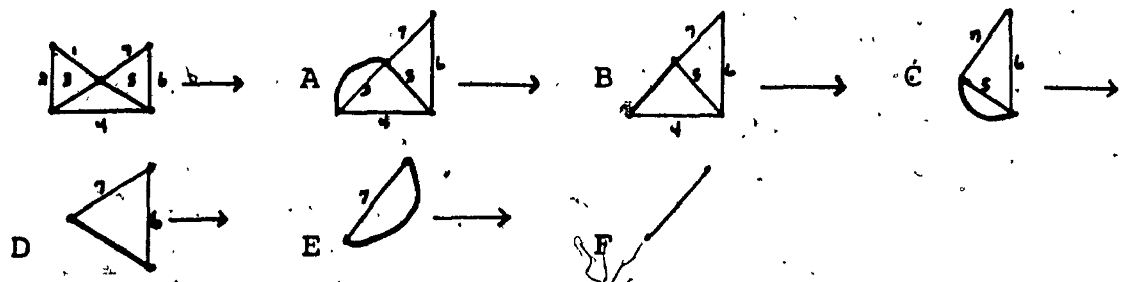


Figure 22.

The respective edge reliabilities are:

$$R(A) = \frac{p_1 p_2}{p_2 + q_2 p_1}$$

$$r_1 = R(B) = p_3 + q_3 R(A)$$

$$R(E) = \frac{p_6 R(D)}{p_6 + q_6 R(D)}$$

$$R(C) = \frac{p_4 R(B)}{p_4 + q_4 R(B)}$$

$$R(F) = p_7 + q_7 R(E)$$

$$r_2 = R(D) = p_5 + q_5 R(C)$$

Applying equation (3.1) successively, the overall reliability of this entire structure will be:

$$R(L_5) = [p_7+q_7R(E)]\,(p_2+q_2p_1)(p_4+q_4R(B))(p_6+q_6R(D))$$

or

$$R(L_5) = \left(p_7+\frac{q_7p_6r_2}{p_6+q_6r_2}\right)\prod_{k=1}^{3}(p_{(2k)}+q_{(2k)}r_{(k-1)})$$

where $r_0 = p_1$

$$r_1 = p_3+\frac{q_3p_2r_0}{p_2+q_2r_0}$$

$$r_2 = p_5+\frac{q_5p_4r_1}{p_4+q_4r_1}$$

Lemma 3.

The overall reliability of a ladder of n nodes is:

$$R(L_n) = \left(p_{(2n-3)}+\frac{q_{(2n-3)}p_{(2n-4)}r_{(n-3)}}{p_{(2n-4)}+q_{(2n-4)}r_{(n-3)}}\right)\left[\prod_{j=1}^{(n-2)}(p_{(2j)}+q_{(2j)}r_{(j-1)})\right]$$

$$r_{(k)} = p_{(2k+1)}+\frac{q_{(2k+1)}p_{(2k)}r_{(k-1)}}{p_{(2k)}+q_{(2k)}r_{(k-1)}} \qquad k\neq 0, \quad r_0 = p_1$$

and where the ladder is labelled lexicographically as shown in figure 23A.

Proof:

By induction.

Examples 11 and 12 prove the reliability equation for $n = 4$, and $n = 5$, respectively.

Therefore, assuming $R(L_n)$ true for $n = r$, what is left to prove is that $R(L_n)$ is true for $n = (r+1)$.

A ladder of $(r+1)$ vertices would be as in figure.23B.

The subgraph with edges, 1,2,....., (2r-3), would constitute a ladder of r vertices.

Therefore,

$$R(L_r) = \bar{R}(L_r) \cdot \prod_{j=1}^{(r-2)} (p_{(2j)}+q_{(2j)}r_{(j-1)})$$

$\bar{R}(L_r)$ is the reliability of edge (2r-3), after (r-2) successive degree-2 and parallel reductions, starting from edge 1 to edge (2r-3).

$$\bar{R}(L_r) = p_{(2r-3)}+\frac{q_{(2r-3)}p_{(2r-4)}r_{(r-3)}}{p_{(2r-4)}+q_{(2r-4)}r_{(r-3)}}$$

The reliability of edge (2r-1) after another degree-2 and parallel reduction is:

$$R' = p_{(2r-1)}+\frac{q_{(2r-1)}p_{(2r-2)}\bar{R}(L_r)}{p_{(2r-2)}+q_{(2r-2)}\bar{R}(L_r)}$$

Therefore the reliability of the entire ladder of (r+1) vertices is:

$$R'' = \left(p_{(2r-1)} + q_{(2r-1)} \frac{p_{(2r-2)}\bar{R}(L_r)}{p_{(2r-2)} + q_{(2r-2)}\bar{R}(L_r)}\right)\left(p_{(2r-2)} + q_{(2r-2)}\bar{R}(L_r)\right)$$

$$\prod_{j=1}^{(r-2)} \left(p_{(2j)} + q_{(2j)} r_{(j-1)}\right)$$

$$R'' = \left(p_{(2r-1)} + q_{(2r-1)} \frac{p_{(2r-2)} r_{(r-2)}}{p_{(2r-2)} + q_{(2r-2)} r_{(r-2)}}\right)\left(p_{(2r-2)} + q_{(2r-2)} r_{(r-2)}\right)$$

$$\prod_{j=1}^{(r-2)} \left(p_{(2j)} + q_{(2j)} r_{(j-1)}\right)$$

$$R'' = \left(p_{(2r-1)} + q_{(2r-1)} \frac{p_{(2r-2)} r_{(r-2)}}{p_{(2r-2)} + q_{(2r-2)} r_{(r-2)}}\right)\left(p_{(2r-2)} + q_{(2r-2)} r_{((r-1)-1)}\right)$$

$$\prod_{j=1}^{(r-2)} \left(p_{(2j)} + q_{(2j)} r_{(j-1)}\right)$$

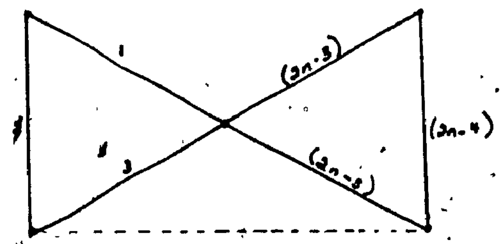


Figure 23A.

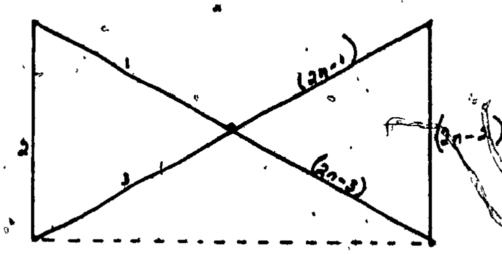


Figure 23B.

$$R'' = \left(p_{(2r-1)} + q_{(2r-2)} \frac{p_{(2r-2)} r_{(r-2)}}{p_{(2r-2)} + q_{(2r-2)} r_{(r-2)}}\right) \prod_{j=1}^{((r+1)-2)} (p_{(2j)} + q_{(2j)} r_{(j-1)}) = R(L_{(r+1)}).$$

Q.E.D.

3.3 Overall Reliability of Star-Ladders.

Example 13.

Consider the star-ladder of figure 24.

S_8:

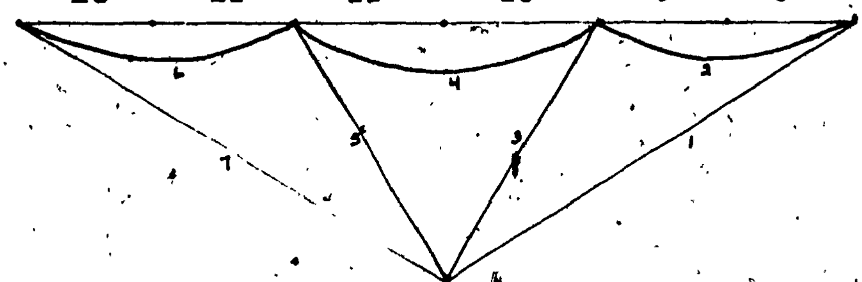


Figure 24.

By performing degree-2 and parallel reductions on edges 2, 8, 9; 4, 10, 11; 6, 12, 13; a standard ladder is obtained with updated edge reliabilities, $\bar{p}$, equal to:

$$\bar{p}_{(2m+2)} = p_{(2m+2)} + \frac{q_{(2m+2)} p_{(\bar{n}+2m)} p_{(\bar{n}+2m+1)}}{(1 - q_{(\bar{n}+2m)} q_{(\bar{n}+2m+1)})}$$

$m = 0, 1, 2, \ldots, \frac{(\bar{n}-4)}{2}$.

where $\bar{n}$: number of nodes of star-ladder.

The reliability of the star-ladder is therefore given by:

$$R(S_{\bar{n}}) = R(L_{\frac{(\bar{n}+2)}{2}}) \prod_{m=0}^{\frac{(\bar{n}-4)}{2}} (1-q_{(\bar{n}+2m)}q_{(\bar{n}+2m+1)})$$

where $S_{\bar{n}}$: star-ladder of order $\bar{n}$.

L_e : ladder of order e (Standard ladder).

In determining $R(S_{\bar{n}})$, the updated edge reliabilities $\bar{p}$, $\bar{q} = (1-\bar{p})$, would be used. $R(L_e')$ is computed using Lemma 3.

3.4 Overall Reliability of Wheel Structures.

In this section, the Factoring Theorem will be used to derive a recursive relationship for the overall reliability of wheel structures.

Example 14.

Consider a wheel of order 4, figure 25.

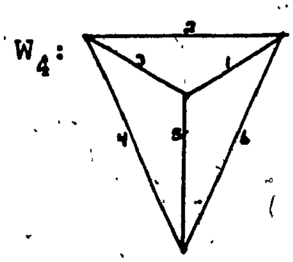


Figure 25.

By equation (2.7), $R(W_4) = p_2R(G_{e_2}) + (1-p_2)R(G-e_2)$

where W_4 : wheel of order 4.

From figure 26, it is evident that $R(W_4)$ may be computed as the weighted sum of the reliabilities of a ladder of order 4, and that of an extra term E_4.

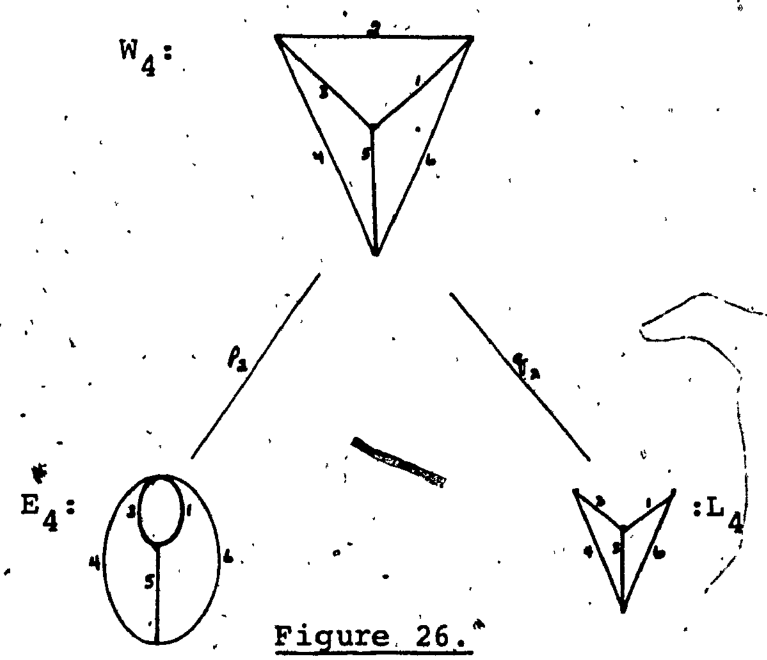


Figure 26.

In other words, $R(W_4) = p_2R(E_4) + q_2R(L_4)$.

$R(L_4)$ is computed using Lemma 3.

$R(E_n)$ can be computed readily by successive parallel and degree-2 reductions:

$$R(E_n) = \left[1-q_4q_6\left(1-\frac{p_5(1-q_1q_3)}{p_5+q_5(1-q_1q_3)}\right)\right](p_5+q_5(1-q_1q_3))$$

Example 15.

Consider a wheel of order 5, figure 27.

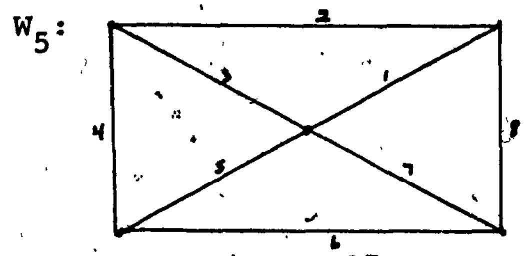


Figure 27.

The overall reliability of this structure is obtained as in the previous example. The use of the Factoring Theorem twice will decompose W_5 into two ladders, one of order 5, the other of order 4, and an extra term E_5.

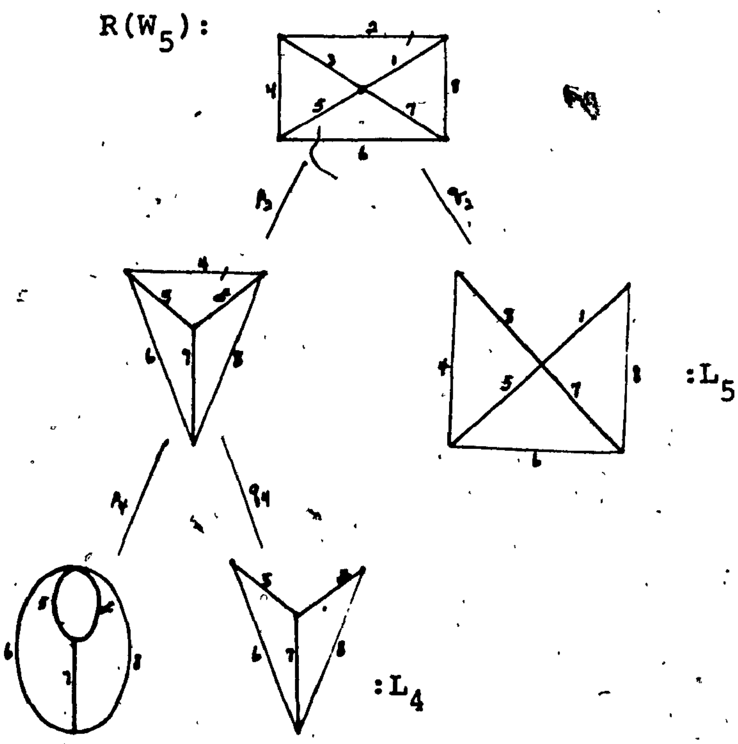


Figure 28.

$$R(W_5) = p_2p_4(R(E_5) + q_2R(L_5) + p_2q_4R(L_4))$$

where $R(L_5)$ and $R(L_4)$ can be computed using lemma 3, after appropriate edge relabelling. Edge α is the result of a parallel reduction of edges 1 and 3. Hence, the reliability of edge α is:

$$p_\alpha = (1-q_1q_2).$$

Derivation of $R(E_5)$ would be analogous to the derivation of $R(E_4)$ of the previous example; in this case, however, one of the edges, α, is the result of i parallel reductions; $i = 1$ in the W_5 case, $i = 0$ in the W_4 case.

$$R(E_5) = \left[1-q_6q_8\left(1-\frac{p_7(1-q_1q_3q_5)}{p_7+q_7(1-q_1q_3q_5)}\right)\right](p_7+q_7(1-q_1q_3q_5))$$

It is evident, that for wheels of larger number of nodes, each successive ladder will have the edge α, the result of an increasing number of parallel reductions.

Let i: be the number of parallel reductions of which edge α is composed of, for a particular ladder, $L_{\bar{n}}$.

Therefore;

$$R(L_{\bar{n}}^{(i)}) = \bar{R}(L_{\bar{n}}^{(i)}) \prod_{k=i}^{(i+\bar{n}-3)} (p_{(2k+4)}+q_{(2k+4)}r(k-i)) \qquad (3.3)$$

$$\bar{R}(L_{\bar{n}}^{(i)}) = \left(1-\prod_{k=0}^{i} q_{(2k+1)}\right) + \frac{\left(\prod_{k=0}^{i} q_{(2k+1)}\right) p_{(2n-2)}r(\bar{n}-3)}{p_{(2n-2)}+q_{(2n-2)}r(\bar{n}-3)} \qquad (3.3)$$

$$r_j = p_{(2j+2i+3)} + \frac{q_{(2j+2i+3)} p_{(2j+2i+2)} r_{(j-1)}}{p_{(2j+2i+2)} + q_{(2j+2i+2)} r_{(j-1)}} \qquad (3.3)$$

$j \neq 0$

where $r_0 : p_{(2i+3)}$

$\bar{n}$: number of vertices of current ladder.

n : number of vertices of original wheel.

Let E_n: be the extra term graph derived from a wheel of n vertices after $(n-3)$ decompositions, using the Factoring Theorem.

It is easily derived, through successive degree-2 and parallel reductions, see figure 22, that:

$$R(E_n) = \left[1 - q_{(2n-4)} q_{(2n-2)} \left(1 - \frac{p_{(2n-3)} \left(1 - \prod_{k=1}^{(n-2)} q_{(2k-1)} \right)}{p_{(2n-3)} + q_{(2n-3)} \left(1 - \prod_{k=1}^{(n-2)} q_{(2k-1)} \right)} \right) \right] \left[p_{(2n-3)} + q_{(2n-3)} \left(1 - \prod_{k=1}^{(n-2)} q_{(2k-1)} \right) \right] . \qquad (3.4)$$

Lemma 4.

The overall reliability of a wheel of n vertices is:

$$R(W_n) = \left[\prod_{i=1}^{(n-3)} p_{(2i)} \right] R(E_n) + \sum_{i=0}^{(n-4)} \left[\prod_{k=0}^{i} p_{(2k)} \right] q_{2(i+1)} R\left(L_{(n-i)}^{(i)} \right) \qquad (3.5)$$

where $p_0 = 1$.

Proof:

By induction.

Examples 14 and 15 prove the reliability equation for $n = 4$ and $n = 5$ respectively.

$R(W_n)$ can be assumed true for $n = v$. (3.6)

By the Factoring Theorem;

$R(W_{(n+1)}) = p_2 R(W_n) + q_2 R(L^{(0)}_{(n+1)})$

or, from the induction assumption (3.6):

$$R(W_{(v+1)}) = p_2 R(W_v) + q_2 R(L^{(0)}_{(v+1)})$$

$$R(W_{(v+1)}) = p_2 \left\{ \left[\prod_{i=1}^{(v-3)} p_{(2i)} \right] R(E_v) + \sum_{i=0}^{(v-4)} \left[\prod_{k=0}^{i} p_{(2k)} \right] q_{2(i+1)} \; R(L^{(i)}_{(v-i)}) \right\} + q_2 R(L^{(0)}_{(v+1)}) \quad (3.7)$$

As seen in figure 29, W_n contains one edge, α which is the result of a parallel reduction of edges 1 and 3. Hence, i is replaced by (i+1) in (3.7), where necessary, to denote this initial composition.

$$R(W_{(v+1)}) = p_2 \left\{ \left[\prod_{i=1}^{(v-3)} p_{2(i+1)} \right] R(E_v) + \sum_{i=0}^{(v-4)} \left[\prod_{k=0}^{i} p_{(2k)} \right] q_{2(i+2)} \; R(L^{(i+1)}_{(v-i)}) \right\} + q_2 \; R(L^{(0)}_{(v+1)}) \; .$$

or,

$$R(W_{v+1}) = p_2 \left\{ \left[\prod_{i=1}^{(v-3)} p_{2(i+1)} \right] R(E_v) + \sum_{i=1}^{(v-4)} \left[\prod_{k=0}^{i} p_{(2k)} \right] \right.$$

$$\left. q_{2(i+2)} \quad R(L^{(i+1)}_{(v-i)}) + p_0 q_4 \; R(L^{(1)}_{(v)}) \right\} + q_2 R(L^{(0)}_{(v+1)}). \qquad (3.8)$$

As seen in (3.8), the terms p_0 and p_2, generated by $\left[\prod_{k=0}^{i} p_{(2k)} \right]$ are redundant, because $p_0 = 1$, and p_2 no longer exists in W_V; therefore performing the appropriate changes, one observes that $\left[\prod_{k=0}^{i} p_{(2k)} \right]$ may be written as $\left[\prod_{k=2}^{(i+1)} p_{(2k)} \right]$

The main equation in the brackets, (3.8), may now be multiplied through by the p_2 term, and the appropriate changes performed in the subscripting limits.

$$R(W_{(v+1)}) = \left[\prod_{i=0}^{(v-3)} p_{2(i+1)} \right] R(E_v) + \sum_{i=1}^{(v-4)} \left[\prod_{k=1}^{(i+1)} p_{(2k)} \right] q_{2(i+2)}$$

$$R(L^{(i+1)}_{(v-1)}) + p_0 p_2 q_4 \; R(L^{(1)}_{(v)}) + q_2 R(L^{(0)}_{(v+1)}).$$

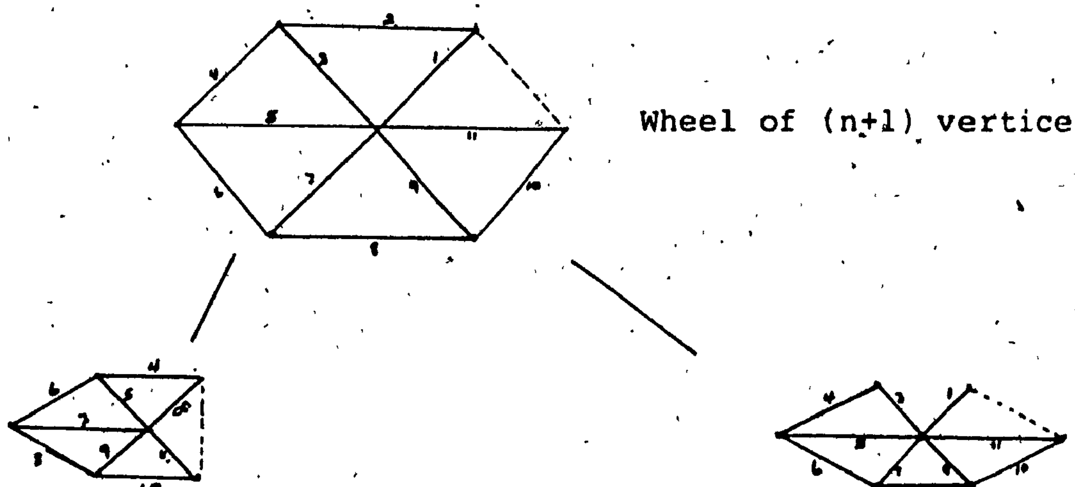
In order to combine the third and fourth terms into the second term, the term $\left[\prod_{k=1}^{(i+1)} p_{(2k)} \right]$ must be re-written as $\left[\prod_{k=0}^{(i+1)} p_{(2k)} \right]$, to compensate for the p_0 in the third term; consequently,

$$R(W_{v+1}) = \left[\prod_{i=0}^{(v-3)} p_{2(i+1)}\right] R(E_v) + \sum_{i=-1}^{(v-4)} \left[\prod_{k=0}^{(i+1)} p_{(2k)}\right] q_{2(i+2)} R(L^{(i+1)}_{(v-i)}) .$$

Letting i=(i-1), one obtains, without loss of generality,

$$R(W_{(v+1)}) = \left[\prod_{i=1}^{((v+1)-3)} p_{(2i)}\right] \cdot R(E_{(v+1)}) + \sum_{i=0}^{((v+1)-4)} \left[\prod_{k=0}^{i} p_{(2k)}\right] q_{2(i+1)} R(L^{(i)}_{(v+1)-i}) = R(W_{(n+1)})$$

Q.E.D.



Wheel of (n+1) vertices.

Wheel of n vertices.

Ladder of (n+1) vertices.

Figure 29.

3.5 Overall Reliability of Star-Wheels

Example 16

Consider a Star-Wheel of order 7, figure 30. The Star-Wheel is reducible to a standard wheel, a wheel of order 4, in the figure 30 case.

T_7:

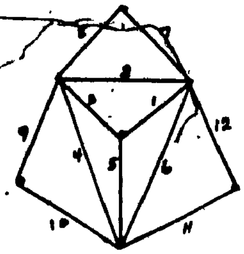


Figure 30.

Through successive degree-2 and parallel reductions on edges 2,7,8; 4,9,10; 6,11,12; the star-wheel, T_7 is reduced to a wheel of order 4, where the reliabilities of the new edges are given by:

$$\bar{p}_{(2m)} = p_{(2m)} + \frac{q_{(2m)}p_{(2m+\bar{n}-2)}p_{(2m+\bar{n}-1)}}{(1-q_{(2m+\bar{n}-2)}q_{(2m+\bar{n}-1)})}$$

$m = 1,2,3, \ldots, (n-1)$.

where $\bar{p}$: updated edge reliabilities.

n : number of vertices of standard (inscribed) wheel.

$\bar{n}$: number of vertices of Star-Wheel.

Let $R(W_n)$ be the overall reliability of the order n inscribed wheel. $R(W_n)$ is computed using lemma 4, along with the updated edge reliabilities, $\bar{p}$, $\bar{q} = (1-\bar{p})$.

The reliability of the Star-Wheel $R(T_{\bar{n}})$ is therefore given by:

$$R(T_{\bar{n}}) = R(W_n) \prod_{m=1}^{(m-1)} (1-q_{(2m+\bar{n}-2)}q_{(2m+\bar{n}-1)}).$$

3.6 Reliability Approximation

For many systems it is quite natural to assume that edges are highly reliable, E.G $q_i \approx 0$, hence it is possible to reduce the reliability equation of the wheel, equation (3.5).

Consider the extra term E_n. Since $q_i \approx 0$, it follows that $\prod_{i=1}^{n} q_i \approx 0$, and hence, each such product can be neglected.

Thus, $R(E_n)$, equation (3.4) reduces to:

$$R(E_n) = 1-q_{(2n-2)}q_{(2n-3)}q_{(2n-4)}.$$

Similarly, $R(L_{\bar{n}}^{(i)})$, equation (3.3) reduces to:

$$R(L_{\bar{n}}^{(i)}) = \prod_{k=i}^{(i+\bar{n}-3)} (p_{(2k+4)} + q_{(2k+4)}r_{(k-i)})$$

where r_j is as in equation (3.3).

Hence, equation (3.5) takes the following form:

$$R(W_n) = \left[\prod_{i=1}^{(n-3)} p_{(2i)}\right] (1-q_{(2n-2)}q_{(2n-3)}q_{(2n-4)})$$

$$+ \sum_{i=0}^{(n-4)} \left[\prod_{k=0}^{i} p_{(2k)}\right] \cdot q_{2(i+1)} \left[\prod_{k=i}^{(i+\bar{n}-3)} (p_{(2k+4)} + q_{(2k+4)}r_{(k-i)})\right]$$

3.7 Comparison of Computational Times.

In this section we present a table listing the run times for a wheel of order 4 to order 8 using the Factoring Theorem, as well as the number of subgraphs generated using this method. In addition, the number of spanning trees corresponding to each order of Wheel (order 4 thru 8) is listed to indicate the order of the number of terms which would be in the overall reliability expression when using the inclusion-exclusion formula. The exponential growth in time from use of the Factoring Theorem is observable from this table.

n	run time	T_S	G_T
4	1 sec.	16	30
5	6 sec.	[illegible]5	88
6	44 sec.	121	240
7	281 sec.	320	638
8	2055 sec.	841	1680

Table 1.

where T_S : number of spanning trees.

G_T : total number of subgraphs generated.

The computational time for the wheel of order 4 thru 8 using the recursive equations (3.3) - (3.5) is .038 seconds.

3.8 General Algorithm for Graphs with Embedded Wheels or Ladders

The following algorithm could be used to determine the overall reliability of graphs, which are embedded either with wheels or ladders. The graphs must be connected, and must be reducible by parallel, and/or degree-2 reductions, to the embedded wheel or ladder.

Algorithm 1.

1. Reduce the given graph by use of parallel, degree-2 reductions to the embedded Wheel or Ladder.
2. Determine the reliability of the resulting edge after each parallel, degree-2 reduction from step 1.
3. Determine the overall reliability of the embedded Wheel or Ladder from step 2 by the appropriate use of equations (3.1), (3.3), (3.4), (3.5).

CHAPTER IV

4. Conclusion

In this thesis, the analysis of the computational complexity of several overall reliability algorithms were analysed. These approaches included the use of: inclusion-exclusion principle, domination theory approach, edge factoring theorem, vertex factoring theorem, polygon to chain reduction approach, Δ-Y reduction approach, backtrack fusion, and the modular approach.

Through the domination concept, the inefficiency of the inclusion-exclusion principle was emphasized. Using an extension of the series reduction, called degree-2 reduction, the overall reliability problem is made computationally equivalent to the terminal-pair problem defined on the same graph [20].

The common approach adopted in all the above algorithms was to decompose the graph into subgraphs for which efficient reduction techniques are applicable. The simplest such decomposition-reduction scheme is the creation of subgraphs containing series and parallel edges by conditioning on the state of an edge. This edge factoring approach, when applied to the overall reliability problem, has been shown to be "exponentially worst" than the vertex decomposition approach, when each is used for the purpose of creating subgraphs with degree-2 vertices. Algorithms based on Boolean algebra offer alternatives to the graph-oriented approaches outlined in Chapter 1, Abraham [36].

Between the simple backtracking decomposition used in Chang's algorithm RELY [2[illegible]], and the intricate recursive composition algorithms of Rosenthal [25], and Buzacott [37], there are those which incorporate backtrack fusion.

In Chapter 2, recursive equations were derived for the overall reliability of ladders and wheels. Using these equations, the overall reliability of star-ladders and star-wheels was then derived.

An algorithm was subsequently introduced to show how the overall reliability of graphs reducible through degree-2 and parallel reductions to ladders and wheels, could be computed. A table of computational times for wheels of order 4 through order 8, using the edge factoring theorem is illustrated. Included in this table are the number of spanning trees corresponding to each order of wheel, and the total number of subgraphs generated for each wheel. A comparison of the computational times using the edge factoring theorem and the derived recursive equations is sufficient to show the linear time of these equations.

It is hoped that the theoretical results of this thesis will be helpful in the formulation of better methods for computing network reliability, of which the main problem still remains: determine the overall reliability of a general graph in polynomial time.

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