Unique Factorization of Ideals
In
Fully Idempotent Rings

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ABSTRACT

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This thesis is a study of the structure of fully idempotent factorization rings. Factorization rings which are von Neumann regular are given special attention. A number of examples and some algorithms for generating examples are presented. New results are presented in the afterword.

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DEDICATION

This thesis is dedicated to my parents, who have had an enormous influence on my education, and without whose help this thesis would not have been written.

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INTRODUCTION

In 1968, R.C. Courter [2] proved that every ideal of a ring is idempotent if and only if finite intersections and finite products of ideals coincide. Such rings are called fully idempotent. It is easily seen that a ring is a commutative fully idempotent ring if and only if it is a commutative Von Neumann regular ring. Among the non-commutative fully idempotent rings are regular rings, biregular rings, and rings all of whose simple right modules are injective (called V-rings - see [15]).

A factorization ring is one in which every ideal can be written as a finite product of primes. In [18], Professor R. Raphael showed that in a fully idempotent factorization ring, each ideal has a unique prime factorization of minimal length (i.e. using the fewest number of prime ideals), where the uniqueness is taken to mean "unique up to order". A number of interesting consequences on the structure of such a ring arise from Levy [14].

This thesis is based mainly on [18], and in small part on [14]. In Chapter 1 the necessary machinery is developed that is used to prove the structural results of section 2.1. In section 2.2, Levy's results on finite subdirect-products of prime rings [14] are used to examine the consequences of chain conditions in fully idempotent factorization rings.

In section 2.3, examples are constructed, with special attention paid to regular factorization rings. Finally in the afterword, partial answers are given to questions 1 and 3 of [18], and Proposition 2.2.8 (ii) is strengthened.

Throughout this thesis, all rings shall be associative rings with identity.

Chapter 1 Preliminaries

1. Central Idempotents and Boolean Rings

Definition 1.1: Let R be a ring with identity. Then R is called a boolean ring if $x^2 = x$ for each $x \in R$.

If I is an ideal of a boolean ring, then $x \in I \Rightarrow x^2 \in I$ shows that $I \subseteq I^2$. Since $I^2 \subseteq I$ always, $I^2 = I$ and so boolean rings are fully idempotent rings.

Lemma 1.2: The following are true in any boolean ring R:
(i) $x = -x$ for all $x \in R$. (ii) R is commutative.

Proof: (i) $(x + 1)^2 = x + 1 = x^2 + 2x + 1 \Rightarrow 2x = 0$
$\Rightarrow x = -x \quad \forall x \in R$.

(ii) $(x - y) = (x - y)^2 = x^2 - xy - yx + y^2$
$= x + xy - yx - y$ by (i) $\Rightarrow 0 = xy - yx \Rightarrow yx = xy$ for all $x, y \in R$. □

Proposition 1.3: Given the ring $\langle R, +, \cdot \rangle$, if $B(R)$ is the set of central idempotents of R then $\langle B(R), *, \cdot \rangle$ is a boolean ring, where $a * b = a + b - 2ab$.

Proof: Proof is trivial, and involves only the verification of the ring axioms. It is worth noting however that the identity element of * is the additive identity of R. □

B(R) is a complemented distributive lattice as follows: first define the operations $\vee$ and $\wedge$ and B(R) by $e_1 \vee e_2 = e_1 + e_2 - e_1 e_2$ and $e_1 \wedge e_2 = e_1 e_2$. Define also $f \leq e$ whenever $ef = f$. Clearly $0 \leq e$ for each $e \in B(R)$ and $e \leq 1$ for each $e \in B(R)$.

Define the complement of $e \in B(R)$ by $e' = 1 - e$. Note that $ee' = e(1 - e) = e - e^2 = 0$ so that $e \wedge e' = 0$ and $e \vee e' = e + (1 - e) - ee' = 1$.

Furthermore, if $a, b, c \in B(R)$ then: $a \wedge (b \vee c) =$ $a(b + c - bc) = ab + ac - abc = ab + ac - a^2 b c =$ $(ab) \vee (ac) = (a \wedge b) \vee (a \wedge c)$.

These definitions make B(R) into a complemented distributive lattice if we note, finally, that $f \leq g \Rightarrow f \vee g = f + g - fg = f + g - f = g$.

Lemma 1.4: If P is a prime ideal of a boolean ring R, then R/P is a copy of the two element field.

Proof: Suppose $x + P \neq 0$. Then $x \in P$. Now $(1 - x)x = x - x^2 = 0 \in P$ as x is idempotent. Since P is prime, $1 - x \in P$. $\therefore 1 + P = x + P$. Thus R/P is a ring containing only 0 and 1, hence it is the two element field. □

Corollary 1.5: In a boolean ring, every prime ideal is maximal. □

Definition 1.6: Let I be some indexing set. A set of idempotents $\{e_i\}_{i\in I}$ is said to be orthogonal if $e_i e_j = 0$ whenever $i \neq j$.

Proposition 1.7: The following are equivalent:

(i) R is isomorphic to a finite direct product of rings R_i, $i = 1, 2, \ldots, n$.

(ii) There are central orthogonal idempotents $e_i \in R$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^{n} e_i = 1$ and $e_i R \cong R_i$.

Proof: (i) → (ii) Regard $R \cong \bigoplus_{i=1}^{n} R_i$ as n-tuples whose i^{th} co-ordinate contains arbitrary members of R_i. Define e_i to be the n-tuple containing the multiplicative identity of R_i in the i^{th} co-ordinate, and zeroes elsewhere. Clearly each e_i is idempotent and $\sum_{i=1}^{n} e_i = 1$. Also $e_i R = \{(a_1, \ldots, a_n) \mid a_j = 0 \text{ for } j \neq i, a_i \in R_i\}$ which is isomorphic to R_i trivially, the isomorphism being the canonical projection to the i^{th} co-ordinate. Easily $\{e_i\}_{i\in I}$ is orthogonal.

(ii) → (i) Let $e_i r \in e_i R$. Then $e_i(e_i r) = e_i r = (e_i r)e_i$ since e_i is a central idempotent. Thus $e_i R$ is a ring with identity e_i . Since by hypothesis $1 = \sum_{i=1}^{n} e_i \in \sum_{i=1}^{n} e_i R$, we have $\sum_{i=1}^{n} e_i R = R$. Now suppose $e_i r = \sum_{\substack{k=1 \\ jk \neq i}}^{m} e_{jk} r_k$. Then $e_i r = e_i^2 r = \sum_{k=1}^{m} e_i e_{jk} r_k = 0$ by orthogonality. Thus $R = \sum_{i=1}^{n} e_i R = \bigoplus_{i=1}^{n} e_i R$. Setting $e_i R = R_i$ completes the proof. □

<u>Definition 1.8</u>: Let B be a boolean ring. An atom of B is a minimal non-zero element.

Note that this definition assumes an ordering on B; the ordering is the one given earlier, namely $f \leq g \leftrightarrow gf = f$.

<u>Lemma 1.9</u>: If $e \in B(R)$, then eR is indecomposable (as a direct sum of modules) if and only if e is an atom of B(R).

<u>Proof: ($\rightarrow$)</u> If e is not an atom of B(R), there is $0 \neq f \in B(R)$ with $f \leq e$, $f \neq e$. Since $f(e - f) = fe - f^2 = f - f = 0$, f and (e − f) are orthogonal idempotents. Now consider eR as a ring with identity e . Clearly $f = ef \in eR$ and $e - f = e^2 - ef = e(e - f) \in eR$. Moreover $f + (e - f) = e$ the identity of eR . By Proposition 1.7 above, $eR = f(eR) \oplus (e - f)(eR) = fR \oplus (e - f)R$ which shows that if eR is indecomposable then e must be an atom of B(R) .

<u>($\leftarrow$)</u> Suppose eR is not indecomposable. Again by Proposition 1.7, there are orthogonal central idempotents

in eR, call them f and g, with $eR = feR \oplus geR$ and $f + g = e$. Now neither f nor g is 0 since if so the above direct sum decomposition of eR would be trivial and we are assuming that it is not. $\therefore f = e - g \Rightarrow f^2 = f = ef - gf = ef$ so $f \le e$ and $f \ne e$ since if $f = e$ then $g = 0$ which is a contradiction. Thus e is not an atom. We have shown that if e is an atom, eR is indecomposable. □

<u>Lemma 1.10</u>: Any set of atoms of B(R) is orthogonal.

<u>Proof</u>: Let e_1, e_2 be distinct atoms. Then $e_1e_2 = e_1 \wedge e_2 = \text{g.l.b}(e_1, e_2) \le e_1$ and similarly, $e_1e_2 \le e_2$. Note that $e_1e_2 \ne e_2$ since if $e_1e_2 = e_2$ then $e_2 \le e_1$ which is impossible since $e_2 \ne e_1$ by assumption and $e_2 \ne 0$ by definition of an atom. Similarly, $e_1e_2 \ne e_1$. $\therefore e_1e_2 = 0$ by the minimality of e_1 and e_2. □

<u>Lemma 1.11</u>: If B(R) is finite with atoms $e_1, \ldots, e_n$ then B(R) is lattice isomorphic to the lattice of subsets of an n-element set.

<u>Proof</u>: Define $f: B(R) \to \mathcal{P}(\{e_1, \ldots, e_n\})$ by $f(a) = \{x \in \{e_1, \ldots, e_n\} \mid x \le a\}$. Clearly $f(1) = \{e_1, \ldots, e_n\}$ and $f(0) = \{\ \}$. Now suppose $e_i \le a$ and $e_i \le b$, that is $e_i \in f(a) \cap f(b)$. Then $e_ia = e_i$ and $e_ib = e_i$. $\therefore$ $e_iab = e_i$, which means $e_i \in f(a \wedge b)$.

$\therefore\ f(a) \cap f(b) \subseteq f(a \wedge b)$. Also if $e_i \in f(a \wedge b)$, $e_i \le ab$. Since $ab \le a$ and $ab \le b$, $e_i \le a$ and $e_i \le b$ by the transitive property of the partial order. $\therefore\ e_i \in f(a) \cap f(b)$; so $f(a \wedge b) \subseteq f(a) \cap f(b)$ from which $f(a \wedge b) = f(a) \cap f(b)$. Thus f preserves the meet.

If $e_i \in f(a \vee b)$, $e_i \le a \vee b$. That is $e_i(a \vee b) = e_i a + e_i b - e_i ab = e_i$. If $e_i \not\le a$ then by minimality $e_i a = 0$ which implies $e_i b = e_i$, that is $e_i \le b$. Similarly $e_i \not\le b \Rightarrow e_i \le a$. Therefore $f(a \vee b) \subseteq f(a) \cup f(b)$. If $e_i \in f(a) \cup f(b)$ then $e_i \le a$ or $e_i \le b$. Now $e_i \le a \Rightarrow e_i(a \vee b) = e_i a + e_i b - e_i ab = e_i + e_i b - e_i b = e_i$ that is $e_i \le (a \vee b)$ so that $e_i \in f(a \vee b)$. Similarly if $e_i \le b$, $e_i \in f(a \vee b)$. Thus $f(a) \cup f(b) \subseteq f(a \vee b)$ so we have $f(a) \cup f(b) = f(a \vee b)$, and f preserves join.

Now $e \in f(a') \Leftrightarrow e \in f(1 - a) \Leftrightarrow e(1 - a) = e - ea = e$ (since $e \le 1 - a$) $\Leftrightarrow ea = 0 \Leftrightarrow e \not\le a$ $\therefore\ e \in f(a)' \Leftrightarrow e \in f(a')$, so that f preserves complement.

We have shown that f is a lattice homomorphism. Moreover, f is onto since if $\{e_{i_1}, \ldots, e_{i_k}\} \subseteq \{e_1, \ldots, e_n\}$ then its preimage is $\bigwedge_{j=1}^{k} e_{i_j}$.

We now show f is a monomorphism. Suppose $a \ne 0$. Then there is an atom in B(R), call it e , with $e \le a$. Then $e \in f(a)$ so $f(a) \ne \{\ \}$. Now if $a \ne b$, then $a \wedge b' \ne 0$ or $a' \wedge b \ne 0$. If $a \wedge b' \ne 0$ then $f(a \wedge b') \ne \{\ \}$ so

$f(a) \cap f(b)' \neq \{\ \}$ so $f(a) \neq f(b)$. Similarly $a' \wedge b \neq 0$ $\Rightarrow f(a) \neq f(b)$. Thus f is mono, and hence is a lattice isomorphism. □

2. Regular Rings

Regular rings were invented by John Von Neumann to simplify certain aspects of his work on algebras of operators on Hilbert spaces. His set of lecture notes [19] contains an excellent introduction to the topic. A general reference for this section is Goodearl's Von Neumann Regular Rings [6].

Definition 2.1: A ring R is regular if $x \in R \Rightarrow \exists\, y \in R$ with $x = xyx$.

An immediate consequence of this definition is that every principal right (left) ideal is generated by an idempotent. For, if $x \in R$, a regular ring, then $x = xyx \Rightarrow xy = xyxy$. That is, xy is an idempotent. Now since $xy \in xR$, $xyR \subseteq xR$. But $xR = xyxR \subseteq xyR$, so $xR = xyR$, establishing the claim. Proof for left ideals is essentially identical.

Lemma 2.2: In a regular ring R, every finitely generated (left, right) ideal is principal.

Proof: It is sufficient to show that ideals generated by two elements are principal. The more general result follows by induction on the number of generators.

Let $I = xR + yR$. Let e be an idempotent such that $eR = xR$. Since $y - ey \in eR + yR$, $eR + (y - ey)R \subseteq eR + yR$. Moreover $er_1 + yr_2 = er_1 + eyr_2 + yr_2 - eyr_2 = e(r_1 + yr_2) + (y - ey)r_2 \in eR + (y - ey)R$. $\therefore$ $eR + yR \subseteq eR + (y - ey)R$. Thus $eR + yR = eR + (y - ey)R$. Now, since $e(y - ey) = ey - e^2y = ey - ey = 0$, if $\xi \in (y - ey)R$ then $e\xi = 0$. In particular if f is an idempotent generating $(y - ey)R$, then $ef = 0$. Now set $g = f - fe$. We easily see the following:

(i) $g^2 = f^2 - fef - f^2e + fe\,fe = f - fe = g$

i.e. g is an idempotent.

(ii) $eg = ef - efe = 0$, and $ge = fe - fe^2 = 0$

$\therefore$ g is orthogonal to e.

(iii) $fg = f^2 - f^2e = f - fe = g$ and $fg = f^2 - fef = f^2 = f$.

Thus $gR = fR = (y - ey)R$ from which we have $I = xR + yR = eR + gR$. Now consider a typical element $er_1 + gr_2$ of I. Since $eg = ge = 0$, $er_1 + gr_2 = e^2r_1 + egr_2 + ger_1 + g^2r_2 = (e + g)(er_1 + gr_2)$ and so $eR + gR \subseteq (e + g)R$. Clearly $(e+g)r = er + gr \in eR + gR$, so the reverse inclusion also holds, yielding that $I = (e + g)R$ is principal, as required. □

We note in passing that if every principal (right, left) ideal of R is generated by an idempotent, then R is regular, for the following reasons. If $\alpha \in R$, $\alpha R = eR$ for an idempotent e. Since e is a left identity for eR, and $\alpha \in eR$, $\alpha = e\alpha$. Also $e = \alpha\alpha'$ for some $\alpha' \in R$. Thus $\alpha = \alpha\alpha'\alpha$, and since α was arbitrary, R is regular.

We now proceed to prove:

<u>Lemma 2.3</u>: Let $J \subseteq K$ be two-sided ideals of a regular ring R. Then K is regular if and only if J and K/J are both regular.

<u>Proof</u>: <u>(⇒)</u> Suppose K is regular. It is immediate that K/J is regular. Now let $x \in J$. $x = xyx$ for some $y \in K$. Thus $x = xyx = xyxyx$, and since $yxy \in J$, J is regular.

<u>(⇐)</u> Suppose J, K/J are both regular. Then if $x \in K$, $x + J = xyx + J$ for some $y \in K$. That is, $x - xyx \in J$. Since J is regular, there is $z \in J$ with $x - xyx = (x - xyx)\,z\,(x - xyx)$. Expanding and transposing terms yields: $x = x(z - yxz - zxy + yxzxy + y)x$. Now, the term in the brackets is clearly in K, from which K is regular. □

Definition 2.4: A ring R is called a subdirect product of the rings S_i ($i \in I$, some indexing set) if there is a monomorphism $\mu : R \to \prod_{i \in I} S_i$ so that $\pi_i \circ \mu$ is epi for each projection π_i.

Lemma 2.5: R is a subdirect product of the rings S_i, $i \in I$, if and only if $S_i \cong R/K_i$ where K_i is an ideal of R and $\bigcap_{i \in I} I_i = 0$.

Proof: ($\Rightarrow$) If R is a subdirect product of the rings S_i, $i \in I$, $\pi_i \circ \mu : R \to S_i$ is epi by definition. Thus $S_i \cong R/K_i$ where $K_i = \ker(\pi_i \circ \mu) = \{r \in R \mid \mu(r) \in \ker \pi_i\}$.

Now $\ker \pi_i \cong \prod_{\substack{j \in I \\ j \neq i}} S_j$, so $\bigcap_{i \in I} \ker \pi_i = 0$

$\therefore \bigcap_{i \in I} K_i = \bigcap_{i \in I} \{r \in R \mid \mu(r) \in \ker \pi_i\} =$
$\{r \in R \mid \mu(r) \in \bigcap_{i \in I} \ker \pi_i\} = \{r \in R \mid \mu(r) = 0\} = 0$ since μ is mono.

($\Leftarrow$) Assume $S_i \cong R/K_i$, $\bigcap_{i \in I} K_i = 0$. Define $\mu : R \to \prod_{i \in I} R/K_i$ by $\mu(r) = (r + K_i)_{i \in I}$. Then $\ker \mu = \{r \in R \mid r \in K_i \ \forall\, i \in I\} = \bigcap_{i \in I} K_i = 0$. That is, μ is mono. Moreover, $(\pi_j \circ \mu)(r) = \pi_j((r + K_i)_{i \in I}) = r + K_j$, so it is evident that $\pi_j \circ \mu$ is a surjection, hence epi, for all $j \in I$. Thus R is a subdirect product of R/K_i, $i \in I$ whenever $\bigcap_{i \in I} K_i = 0$. $\square$

The following is due to Fisher:

Proposition 2.6: Any finite subdirect product of regular rings is regular.

Proof: It is sufficient to show the result for subdirect products of two rings, the more general result following by induction.

If R is a subdirect product of two regular rings, these rings are isomorphic to R/J, R/K where J and K are ideals of R with $J \cap K = 0$ (by the preceding lemma).

Now $K \subset J + K$ so $(J + K)/K$ is an ideal of R/K. By Lemma 2.3, it is regular. Moreover since $J \cap K = 0$ and $K/K = 0$, $(J + K)/K \cong J$. Thus J is regular. Since R/J is also regular by Lemma 2.3 so is R. □

The following provide prototypical examples of regular rings:

Lemma 2.7: Let R be the ring of $n \times n$ matrices over a field. R is regular.

Proof: For $0 \in R$, $0 = 0 A 0$ for any $A \in R$. Now suppose $X \in R$ is non-zero. If rank $X = r \leq n$, there are elementary matrices $E_1, \ldots, E_k$ such that $X \cdot E_1 \cdot \ldots \cdot E_k$ $= \left[\begin{array}{c|c} I_r & * \\ \hline 0 & 0 \end{array}\right]$, which is an idempotent. Since the elementary matrices are invertible, the ideal generated by X is equal to the ideal generated by the idempotent. Now by the remarks preceeding Lemma 2.3, R is regular. □

We note that since $X = X(E_1 \cdot \ldots \cdot E_k)X$, and since $E_1 \cdot \ldots \cdot E_k$ is a unit, R is unit-regular.

<u>Lemma 2.8</u>: Let V_k be an infinite dimensional vector space, and let I be the ideal of End(V) containing the linear transformations of finite rank. I is a regular ideal.

<u>Proof</u>: Let $\{e_i\}$ be a basis for V and let $T:V \to V$ have finite rank. That is im T has basis $\{v_1, \ldots, v_k\}$ where $v_i = \sum_{j=1}^{n_i} \xi_{i_j} e_{i_j}$. A typical element of im T has the form $\sum_{i=1}^{k} \lambda_i V_i = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_i \xi_{i_j} e_{i_j}$ which is a linear combination of finitely many elements of $\{e_i\}$, call them $e_{i_1}, \ldots, e_{i_m}$ where $m \leq \sum_{i=1}^{k} n_i$.

Now T can be regarded as being the zero map on all but an m-dimensional subspace of V, call it V_m. Write V as $V \cong V_m \oplus N$ where N has basis $\{e_i\} - \{e_{i_1}, \ldots, e_{i_m}\}$. By Proposition 2.7, there is a linear transformation $A:V_m \to V_m$ with $T|_{V_m} = T|_{V_m} A\, T|_{V_m}$. Now define $\bar{A}: V \to V$ by $\bar{A} = (A, 0)$. Then $T = T\bar{A}T$. □

<u>Corollary 2.9</u>: Let V and I be as above.

Let $C = \{kI_V \mid k \in K\}$. Then the subring of End(V) generated by I and C is regular.

Proof: Since the rank of kI_V is the dimension of V, $C \cap I = (0)$. Thus $I + C/F \cong C/I \cap C \cong C$ which is regular as every element in C is invertible. By Lemma 2.3, I + C is regular. □

More generally we have:

Proposition 2.10: If V is a vector space over a field, End(V) is regular.

Proof: Let $f \in \mathrm{End}(V)$, $f \neq 0$. We may write $V \cong \ker f \oplus U \cong \mathrm{im}\, f \oplus W$. Define $\bar{f}: U \to \mathrm{im}\, f$ by $\bar{f} = f|_u$. Since $\ker \bar{f} = \ker f \cap U = 0$, $\bar{f}$ is injective and since it is clearly surjective, it is an isomorphism. Let $\bar{g} = \bar{f}^{-1}$ Define $g: V \to V$ by $g|_{\mathrm{im}\, f} = \bar{g}$ and $g|_W = 0$. Then, as required $f g f = f$.

We end this chapter with the following:

Definition 2.11: A ring is called biregular if every two sided principal ideal is generated by a central idempotent.

We note that if $x \in R$, a biregular ring, $x = er$ for $e \in B(R)$, $r \in R$. Thus $x = ex = (e \cdot 1)(e \cdot r)$, so that

$(x) \subseteq (x)^2$. Clearly every ideal of a biregular ring is idempotent.

Chapter 2 Fully Idempotent Factorization Rings

1. Elementary Definitions and Main Results

Definition 1.1: A ring is called semiprime if no non-zero ideal is nilpotent.

Definition 1.2: A ring is called fully idempotent if for each ideal I, $I^2 = I$.

Lemma 1.3: Let R be a ring. The following are equivalent:

(i) $IJ = I \cap J$ for all ideals I, J of R.

(ii) $I^2 = I$ for each ideal I of R.

(iii) R/I is semiprime for each ideal I of R.

Proof: (i) $\Rightarrow$ (ii) $IJ = I \cap J \Rightarrow II = I^2 = I \cap I = I$.

(ii) $\Rightarrow$ (iii) Let J_I be an ideal in R/I and let J be its preimage in R under the canonical homomorphism $\sigma: R \to R/I$. Since R is fully idempotent, $J^2 = J$, so that each element $x \in J$ has the form $x = \sum_{i=1}^{n} a_i b_i$ for some $a_i, b_i \in J$, $i = 1, 2, \ldots, n$. Then $\sigma(x) = \sum_{i=1}^{n} \sigma(a_i)\, \sigma(b_i)$. Thus $J_I \subseteq J_I^2$ and since always $J_I^2 \subseteq J_I$, we have $J_I = J_I^2$. Thus if $J_I \neq (0)$, $J_I^n = J_I \neq 0$ for any $n \in \mathbb{N}$, so R/I has no non-zero nilpotent ideals. $\therefore$ R/I is semiprime.

(iii) $\Rightarrow$ (ii). We prove the contrapositive. Suppose R

is not fully idempotent. Then there is an ideal X of R with $X^2 \subsetneq X$. Consider R/X^2 . Since the above containment is strict, $\sigma(X) \neq (0)$ in R/X^2 , where σ is the canonical homomorphism $R \to R/X^2$. In fact $\sigma(X) = X / X^2$ so $(\sigma(X))^2 = (X / X^2)^2 = X^2 / X^2 = (0)$. We have displayed a non-zero nilpotent ideal, so R/X^2 is not semiprime.

$\therefore$ If R/I is semiprime for each ideal I of R, then R is fully idempotent.

<u>(ii) $\Rightarrow$ (i)</u> Clearly $IJ \subseteq I \cap J$. Also,
$(I \cap J)^2 = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in I \cap J, b_i \in I \cap J\} \subseteq$
$\{\sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J\} = IJ$. Thus $(I \cap J)^2 \subseteq IJ \subseteq I \cap J$.
Since R is fully idempotent, we have a sandwich, and
$(I \cap J)^2 = IJ = I \cap J$. □

<u>Definition 1.4:</u> Let R be a ring. R is a factorization ring if every ideal of R can be written as a finite product of prime ideals.

<u>Definition 1.5:</u> Let R be a fully idempotent factorization ring, and let I be an ideal of R . A representation for I is a finite product of prime ideals equalling I . A reduction of a representation for I is a representation using a subfamily of the ideals used in the original representation. A reduction is proper if the subfamily is proper. A representation is irredundant if it has

no proper reductions.

<u>Proposition 1.6</u>: Let R be a fully idempotent factorization ring. Then :

<u>(i)</u> Any ideal has an irredundant representation, which is unique up to order.

<u>(ii)</u> Any representation for an ideal can be reduced to an irredundant one, which will have smaller length unless the original representation was irredundant.

<u>Proof</u>: <u>(i)</u> Let I be an ideal of R, with $I = \prod_{i=1}^{n} P_i$. If non of the P_i's can be removed without changing their product, then there is no proper reduction of this representation and so it is irredundant. If, on the other hand, some prime ideal P_{i_1} can be removed from the list without changing the product, we have a proper reduction of the representation : $I = \prod_{\substack{i=1 \\ i \neq i_1}}^{n} P_i$. This process of removing prime ideals can continue at most $n-1$ times. Whenever it ends, the result is an irredundant representation.

<u>(ii)</u> Suppose $I = P_1 \cdot \ldots \cdot P_n = Q_1 \cdot \ldots \cdot Q_m$ are two distinct irredundant representations of I . Since R is fully idempotent, finite products of ideals coincide with finite intersections, so products of finitely many ideals

commute. Thus, without loss of generality, we may let the common prime ideals come first in each representation. That is, $P_i = Q_i$ for $i = 1, 2, \ldots, r$ with $r \leq n$, $r \leq m$. Note that r cannot equal n for if so $r \underline{<} m$ by distinctness of the representations. Then $I = P_1 \cdot \ldots \cdot P_r = Q_1 \cdot \ldots \cdot Q_r = Q_1 \cdot \ldots \cdot Q_m$ which contradicts the irredundancy of $Q_1 \cdot \ldots \cdot Q_m$. By a symmetric argument, $r \neq m$.

Now since $\bigcap_{i=1}^{n} P_i = \prod_{i=1}^{n} P_i = \prod_{i=1}^{m} Q_i = \bigcap_{i=1}^{m} Q_i$, $\prod_{i=1}^{m} Q_i \subset P_{r+1}$. Since P_{r+1} is prime, $Q_i \subset P_{r+1}$ for some i, $1 \leq i \leq n$. Similarly $\prod_{i=1}^{n} P_i \subset Q_i$, and Q_i is prime so $P_j \subset Q_i$ for some j, $1 \leq j \leq n$. $\therefore P_j \subset P_{r+1}$. This containment cannot be proper, since if so, we have $P_1 \cdot \ldots \cdot P_n = P_1 \cdot \ldots \cdot P_r \cdot P_{r+2} \cdot \ldots \cdot P_n$ which conradicts the irredundancy of the representation. $\therefore P_j = P_{r+1}$. But since $P_j \subset Q_i \subset P_{r+1}$, $Q_i = P_{r+1}$ which is a contradiction as P_1 through P_r were the only common primes. $\therefore$ The representations cannot be distinct, which completes the proof. □

<u>Definition 1.7:</u> Let R be a ring. Denote by $B(R)$ the set of central idempotents of R.

$B(R)$ can be made into a ring with the same multiplication as R, but in general with a different addition (see, Proposition 1.1.3). We now prove two lemmas concerning $B(R)$.

Lemma 1.8: (i) Suppose I is prime ideal of R. Then $I \cap B(R)$ is a prime ideal of $B(R)$.

(ii) If J is any ideal of $B(R)$, then $JR \cap B(R) = J$.

Proof: (i) It is trivial to verify that $I \cap B(R)$ is a group under the binary operation $a * b = a + b - 2ab$. Now suppose $a \in I \cap B(R)$. Then if $x \in B(R)$, $ax \in I$ and $ax \in B(R)$ so $I \cap B(R)$ is closed under the ring action from $B(R)$. $\therefore$ $I \cap B(R)$ is an ideal. Let $a, b \in B(R)$ with $ab \in I \cap B(R)$. Then $ab \in I \Rightarrow a \in I$ or $b \in I$ since I is prime in R. This implies $a \in I \cap B(R)$ or $b \in I \cap B(R)$. Since $B(R)$ is commutative, this means I is prime.

(ii) Since $J \subset JR$, $J \subset JR \cap B(R)$. To see the reverse inclusion, we need to show that any element of JR has the form er, $e \in J$, $r \in R$. Following Pierce [17] we do it this way: Since $e_i \cdot (e_1 \vee e_2 \vee \ldots \vee e_n) = e_i$ for $1 \leq i \leq n$, we have $\sum_{i=1}^{n} e_i r_i = \sum_{i=1}^{n} (e_1 \vee e_2 \vee \ldots \vee e_n) e_i r_i = (e_1 \vee e_2 \vee \ldots \vee e_n) \sum_{i=1}^{n} e_i r_i$. Since J is closed under $\vee$ (see section 1.1), every element of JR has the required form.

Let $er \in JR \cap B(R)$. Since $e \in J$ which is an ideal of $B(R)$, $er = e^2 r = e(er) \in J$.

$\therefore$ $JR \cap B(R) \subset J$ which completes the proof. $\square$

Lemma 1.9: If R is a fully idempotent factorization ring, then $B(R)$ is a factorization ring, and hence a finite boolean ring.

<u>Proof</u>: Let I be an ideal of $B(R)$. By Lemma 1.8 (ii), and since R is a factorization ring, $I = IR \cap B(R) = (P_1 \cap \ldots \cap P_n) \cap B(R) = (P_1 \cap B(R)) \cap \ldots \cap (P_n \cap B(R))$, with P_1 through P_n prime ideals of R. In view of Lemma 1.8 (i), $P_i \cap B(R)$ is also prime in $B(R)$ for $i = 1, 2, \ldots, n$. Thus $B(R)$ is a factorization ring.

In particular, $e\,B(R)$ and $(1 - e)\,B(R)$ can be written as $M_1 \cap \ldots \cap M_n$ and $P_1 \cap \ldots \cap P_k$ respectively. Suppose $ex = (1 - e)y$ for some $x, y \in B(R)$. Then $y = e(x + y)$ which implies that $ex = (1 - e)\,e\,(x + y) = 0$. Thus $M_1 \cap \ldots \cap M_n \cap P_1 \cap \ldots \cap P_k = (0)$ so by Lemma 1.2.5 $B(R)$ is a subdirect product of the rings $B(R)/M_i$, $B(R)/P_j$ with $1 \le i \le n$ and $1 \le j \le k$. Since $M_1, \ldots, M_n, P_1, \ldots, P_k$ are prime, by Lemma 1.1.4 these rings are in fact all copies of the two-element field. Now $B(R)/M_1 \oplus \ldots \oplus B(R)/M_n \oplus B(R)/P_1 \oplus \ldots \oplus B(R)/P_k$ is finite and $B(R)$ is isomorphic to a subring of this ring, and so $B(R)$ is itself finite. □

<u>Proposition 1.10</u>: A finite direct product of fully idempotent factorization rings if a fully idempotent factorization ring.

Proof: It is sufficient to show the result for a ring R which is a direct product of two fully idempotent factorization rings, the more general result following by induction on the number of summands. We first show that if $R = R_1 \oplus R_2$, R_1, R_2 fully idempotent, then R is also fully idempotent.

Let I be an ideal of $R = R_1 \oplus R_2$ and set $I_1 = I \cap (R_1 \oplus (0))$ and $I_2 = I \cap ((0) \oplus R_2)$. It is clear that $I = I_1 + I_2$ and $I_1 \cap I_2 = (0)$. Therefore $I = I_1 \oplus I_2$. Notice also that I_1 is isomorphic to an ideal of R_1, namely $\pi_1(I)$, while I_2 is isomorphic to an ideal of R_2, namely $\pi_2(I)$, where π_1 and π_2 are the canonical projections. $\therefore$ $I_1^2 = I_1$ and $I_2^2 = I_2$. Now $I^2 = (I_1 \oplus I_2)^2 = I_1^2 \oplus I_2^2 = I_1 \oplus I_2 = I$. Therefore R is a fully idempotent ring.

We claim now that the prime ideals of $R_1 \oplus R_2$ are those of the form $P_1 \oplus R_2$ or $R_1\ P_2$ where P_i is prime in R_i for $i = 1, 2$. For suppose $I_1 \subset R_1$ and $I_2 \subset R_2$ are ideals. If either I_1 or I_2 is not prime, it is clear that neither is $I_1 \oplus I_2$ in $R_1 \oplus R_2$. Moreover if both are prime and proper ideals of R_1 and R_2 respectively, then we note that $I_1 \oplus R_2 \not\subset I_1 \oplus I_2$ and $R_1 \oplus I_2 \not\subset I_1 \oplus I_2$ yet $(I_1 \oplus R_2)(R_1 \oplus I_2) = I_1 R_1 \oplus R_2 I_2 = I_1 \oplus I_2$ so $I_1 \oplus I_2$ is not prime. Conversely, ideals of the required form are trivially prime, which proves the claim.

Now if $I = I_1 \oplus I_2$ is an ideal of $R = R_1 \oplus R_2$, regarding I_i as an ideal of R_i for $i = 1, 2$ yields: $I_1 = P_1 \cdot \ldots \cdot P_n$ and $I_2 = Q_1 \cdot \ldots \cdot Q_m$, from which: $I = (P_1 \oplus R_2) \cdot \ldots \cdot (P_n \oplus R_2) \cdot (R_1 \oplus Q_1) \cdot \ldots \cdot (R_1 \oplus Q_m)$, which is a product of prime ideals, so R is a factorization ring as well. □

Proposition 1.11: A finite subdirect product of regular factorization rings is a regular factorization ring.

Proof: By Proposition 1.2.6, finite subdirect products of regular rings are regular. Let R be a subdirect product of the rings R/J_i, $i = 1, 2, \ldots, n$ each of which is a regular factorization ring. By 1.2.5 $\cap J_i = 0$, and since R is regular (and so fully idempotent) $J_1 \ldots J_n = 0$ as well. For each i, $K + J_i$ contains J_i and so $(K + J_i)/J_i$ is an ideal of R/J_i and so factors into a product of primes by hypothesis. Thus $K + J_i$ factors into a product of primes in R. Let $I = (K + J_1) \cdot \ldots \cdot (K + J_n)$. Clearly I factors into a product of primes, and $K = K^n \subseteq I$. Since $J_1 \cdot \ldots \cdot J_n = 0$, also $I \subseteq K$, so K factors, which completes the proof. □

Proposition 1.12: A homomorphic image of a fully idempotent factorization ring is a fully idempotent factorization ring.

Proof: If $f:R \to S$ is epi, $S \cong R/K$ where $K = \ker f$. Let $\bar{J}$ be an ideal of $\bar{R}$ where $\bar{R} = R/K \cong S$, and suppose that R is a fully idempotent factorization ring. If J is the canonical pre-image of $\bar{J}$ in R, then $J^2 = J$ and $J = P_1 \cdot \ldots \cdot P_n$ where P_i $i = 1, 2, \ldots, n$ are all prime ideals of R, each containing J and so necessarily containing K since J does. $J^2 = J \Rightarrow (\bar{J})^2 = (J/K)^2 = J^2/K = J/K = \bar{J}$, so $R/K \cong S$ is fully idempotent. Moreover $\bar{J} = (P_1 \cdot \ldots \cdot P_n)/K = P_1/K \cdot \ldots \cdot P_n/K$ is a product of prime ideals since each P_i contains K. Thus S is a factorization ring as well. $\square$

Proposition 1.13: A fully idempotent factorization ring is a finite direct product of indecomposable fully idempotent factorization rings.

Proof: Let R be a fully idempotent factorization ring. By Lemma 1.9, $B(R)$ is finite, and so contains atoms $e_1, \ldots, e_n$. By Lemma 1.1.9, $e_1 r, \ldots, e_n R$ are all indecomposable rings ($e_i R$ having identity e_i). By Lemma 1.1.10 the set $\{e_1, \ldots, e_n\}$ is orthogonal. Moreover, this means that $e_1 \vee \ldots \vee e_n = \sum_{i=1}^{n} e_i$. Referring to Lemma 1.1.11, we see that as a lattice $B(R)$ is isomorphic to the lattice of subsets of $\{e_1, \ldots, e_n\}$. Since the isomorphism preserves minimal non-zero elements, $f(e_1 \vee \ldots \vee e_n) = \{e_1\} \cup \ldots \cup \{e_n\} = \{e_1, \ldots, e_n\} = f(1)$. As f is an isomorphism $e_1 \vee \ldots \vee e_n = \sum_{i=1}^{n} e_i = 1$.

Now by proposition 1.1.7, $R = \bigoplus_{i=1}^{n} e_i R$, and as noted above, each $e_i R$ is indecomposable. Moreover, since $e_i R = \pi_i(R)$ where π_i is the canonical projection to the $i^{\underline{th}}$ coordinate, proposition 1.12 shows that each $e_i R$ is a fully idempotent factorization ring, which completes the proof. □

We now remark on notation: 1. Denote by R_n the ring of $n \times n$ matrices with entries from R. 2. Denote by E_{ij} the $n \times n$ matrix with 1 in the i-j position and 0 elsewhere. 3. Denote by ${}_{ij}P$ the permutation matrix interchanging rows i and j and by P_{ij} the permutation matrix interchanging columns i and j.

<u>Lemma 1.14</u>: Every ideal of R_n is of the form U_n where U is an ideal of R.

<u>Proof</u>: Let I be an ideal of R_n, and suppose $[a_{ij}] \in I$. Simple computation shows that $E_{ij}\,[a_{ij}]\,E_{ij} = a_{ij}\,E_{ij} \in I$. Now $E_{ij}\, I\, E_{ij}$ is the set of elements of I with 0 in each $k-\ell$ entry for $k \neq i$, $\ell \neq j$. For fixed i, j, $E_{ij}\, I\, E_{ij}$ is an R-module and in fact isomorphic to an ideal of R, the isomorphism given by $E_{ij}\,[a_{ij}]\,E_{ij} \to a_{ij}$. Write $E_{ij}\, I\, E_{ij}$ as $I_{ij} \subseteq R$ ideal. Then $I = [I_{ij}]$, that is the projection to R of each coordinate of I is an ideal. These ideals are all equal since ${}_{ik}P\; a_{ij}\, E_{ij}\, P_{j\ell} = a_{ij}\, E_{k\ell} \in E_{k\ell}\, I\, E_{k\ell}$, which completes the proof. □

Lemma 1.15: If R is fully idempotent, so is R_n .

Proof: (This proof is essentially that of Courter [2]). Any ideal of R_n has the form U_n with U an ideal of R by the previous lemma. Consider $a_{ij} E_{ij} \in U_n$. Since $a_{ij} \in U = U^2$, $a_{ij} = \sum_{i=1}^{k} u_i v_i$ where u_i and v_i are elements of U . Thus $a_{ij} E_{ij} = u_1 v_1 E_{ij} + \dots + u_k v_k E_{ij} = (u_1 E_{ij})(v_1 E_{ij}) + \dots + (u_k E_{ij})(v_k E_{ij}) \in U_n^2$. Inasmuch as $[a_{ij}] = \sum_{1 \le i,j \le n} a_{ij} E_{ij}$, this shows that $U_n \subseteq U_n^2$. Since always $U_n^2 \subseteq U_n$, R_n is fully idempotent. □

Note that this correspondence between the ideals of R_n and R preserves intersection of ideals, that is $U_n \cap I_n = (U \cap I)_n$ for ideal U, I of R . Since in a fully idempotent ring finite products coincide with intersections, primeness is also preserved. This shows that:

Proposition 1.16: If R is a fully idempotent factorization ring, so is R_n .

Proof: I is an ideal of $R_n \Rightarrow I = U_n$, U an ideal of R. $U = P_1 \cap \dots \cap P_k \Rightarrow I = (P_1)_n \cap \dots \cap (P_k)_n$. Since each P_i is prime, so is each $(P_i)_n$, whence the result. □

Lemma 1.17: If R is a fully idempotent factorization ring and $e = e^2 \in R$, then eRe is also a fully idempotent factorization ring.

Proof: Let I be an ideal of eRe . Then RIR is an ideal of R, so $RIR = (RIR)^2 = RIRIR$. Clearly $I \subseteq RIRIR$ and since $eIe = I$ (because $e^2 = e$) we have $I \subseteq eRIRIRe$. For the same reason. $I \subseteq eR(eIe)R(eIe)Re = (eRe)I(eRe)I(eRe) \subseteq I^2$. As always $I^2 \subseteq I$ so eRe is a fully idempotent ring. Since R is a factorization ring, $RIR = P_1 \cdot \ldots \cdot P_n$ for P_i prime ideal of R . Now $I = (P_1 \cap \ldots \cap P_n) \cap eRe = (P_1 \cap eRe) \cap \ldots \cap (P_n \cap eRe)$.
Now suppose $(er_1e)(eRe)(er_2e) \subseteq P_i \cap (eRe)$. Then $er_1e \in P_i$ or $er_2e \in P_i$ since P_i is prime in R . Since both these elements are in eRe, at least one is in the intersection, so $P_i \cap eRe$ is prime for $i = 1, \ldots, n$.

$\therefore$ eRe is a factorization ring. $\square$

Lemma 1.18: Let $e = e^2 \in \mathrm{End}({}_RM)$, M a left R-module. Then there is a ring isomorphism $\phi: e\mathrm{End}({}_RM)e \to \mathrm{End}({}_RMe)$ given by $(xe)\,\phi\,(ese) = xese$.

Proof: 1. $(xe)\,\phi\,(es_1e + es_2e) = (xe)\,\phi\,(e(s_1+s_2)e)$ $= xe(s_1+s_2)e = xes_1e + xes_2e = (xe)\,\phi\,(es_1e) + (xe)\,\phi\,(es_2e)$ $= (xe)(\phi(es_1e) + \phi(es_2e))$ $\therefore$ ϕ is additive

2. $(xe)\,\phi\,(e) = (xe)\,\phi\,(e1e) = xe1e = xe$

$\therefore$ $\phi(e) = 1 \in \mathrm{End}({}_RM)$ $\therefore$ ϕ preserves identity.

3. $(xe)\,\phi\,(eseete) = xeseete = xesete =$
$((xe)\,\phi\,(ese))\,\phi\,(ete) = (xe)(\phi(ese)\cdot\phi(ete))$
$\therefore\ \phi$ is a ring homomorphism.

4. $\phi(ese) = 0 \Rightarrow xese = 0 \quad \forall\, x \in M \Rightarrow ese = 0$.
$\therefore\ \phi$ is a mono.

5. Consider the diagram:

$$\begin{tikzcd}
Me \arrow[r, "e"] \arrow[d, "g"'] & M \arrow[dl, dashed, "\bar{g}"] \\
Me &
\end{tikzcd}$$

Now for $x \in M$, $((xe)e)e = xe$, so the map $e{:}Me \to M$ is split (and trivially mono). Thus g factors through e, that is there is $\bar{g}{:}M \to M_e$ with $g = e\bar{g}$. Considering $\bar{g}$ as a map $M \to M$, we note that since its image is contained in Me, $\bar{g}e = \bar{g}$. Thus, $(xe)g = (xe)e\bar{g} = (xe)e\bar{g}e = xe\bar{g}e$, that is, for any $g \in \mathrm{End}({}_R Me)$, there is $\bar{g} \in \mathrm{End}({}_R M)$ with $\phi(e\bar{g}e) = g$. Thus ϕ is epi and consequently an isomorphism. $\square$

<u>Proposition 1.19:</u> If R is a fully idempotent factorization ring and P is a finitely generated projective R-module, then $\mathrm{End}({}_R P)$ is a fully idempotent factorization ring.

<u>Proof:</u> Suppose $m_1, \ldots, m_n$ are the generators of P, and let $R_i = R$ for $i = 1, \ldots, n$. Then $\phi{:}\ \bigoplus_{i=1}^{n} R_i \to P$ defined

by $\phi(r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i m_i$ is an epimorphism. The sequence $0 \to \ker\phi \to \bigoplus_{i=1}^{n} R_i \xrightarrow{\phi} P \to 0$ is obviously exact and splits since P is projective. Thus $\bigoplus_{i=1}^{n} R_i \cong P \oplus \ker\phi$. Consider $e^2 = e \in \mathrm{End}({}_R \bigoplus_{i=1}^{n} R_i)$ defined by $e = (\phi, 0)$. Now $\mathrm{im}\, e = \mathrm{im}(\phi, 0) = P \oplus (0) \cong P \cong (\bigoplus_{i=1}^{n} R_i)e$. By lemma 1.18, $e\,\mathrm{End}({}_R \bigoplus_{i=1}^{n} R_i)e \cong \mathrm{End}({}_R(\bigoplus_{i=1}^{n} R_i)e) \cong \mathrm{End}({}_R P)$. However $\mathrm{End}({}_R \bigoplus_{i=1}^{n} R_i)$ is isomorphic to the $n \times n$ matrix ring over R, which is a fully idempotent factorization ring by proposition 1.16. Lemma 1.17 then asserts that $e\,\mathrm{End}({}_R \bigoplus_{i=1}^{n} R_i)e$ is a fully idempotent factorization ring, which completes the proof. □

<u>Proposition 1.20</u>: If R and S are Morita equivalent rings and R is a fully idempotent factorization ring, then so is S.

<u>Proof</u>: By [1 , Cor. 22.4] there is a finitely generated projective generator ${}_R P$ with $S \cong \mathrm{End}({}_R P)$. By proposition 1.19, $\mathrm{End}({}_R P)$ is a fully idempotent factorization ring since R is. Thus so is S . □

<u>Lemma 1.21</u>: The center of a fully idempotent ring is regular.

Proof: The following proof is due to Michler and Villamayor [15 , 2.3 d] . Let $a \in Z(R)$. Then aR is a two sided ideal, so $aR = (aR)^2 = a^2R = aRa$. $\therefore$ There is $x \in R$ with $a = axa$. Now $(ax)^2 = axax = ax$ is an idempotent, so let $e = ax = xa$. Note also that $ea = ae = a$. We have: $a(be - eb) = abe - aeb = abe - bae = (ab - ba)e = 0$ since a is central. That is for any $b \in R$, $be - eb \in Ann_r(a)$. We see easily that:

$$1) \quad a\lambda = 0 \Rightarrow xa\lambda = 0 \Rightarrow e\lambda = 0$$

$\therefore$ $Ann_r(a) \subseteq Ann_r(e)$

$$\text{and } 2) \quad e\lambda = 0 \Rightarrow xa\lambda = 0 \Rightarrow axa\lambda = 0 \Rightarrow a\lambda = 0$$

$\therefore$ $Ann_r(e) \subseteq Ann_r(a)$. $\therefore$ $be - eb \in Ann_r(e) = (1 - e)R$, that is $be - eb = (1 - e)r$ for some $r \in R$.
$\therefore$ $ebe - e^2b = ebe - eb = 0$. Also $be^2 - ebe = (1 - e)re = (1 - e)rax = a(1 - e)rx = 0$ since $Ann_r(e) = Ann_r(a)$. That is we also have $be = ebe$. Thus, $eb = be$ for any $b \in R$, i.e., $e \in Z(R)$.

Now $aZ(R) = axa\,Z(R) = ea\,Z(R) \subseteq e\,Z(R)$ and $e\,Z(R) = ax\,Z(R) \subseteq a\,Z(R)$, so $e\,Z(R) = a\,Z(R)$.

Therefore $a = ae$ and $e = ac = ca$ for some $c \in Z(R)$. $\therefore$ $a = aca$ with $c \in Z(R)$. $\therefore$ $Z(R)$ is regular. □

Proposition 1.22: The center of a fully idempotent factorization ring is a finite direct product of fields.

Proof: Let R be a fully idempotent factorization ring. Since B(R) is finite (Lemma 1.9) there are atoms $e_1, \ldots, e_n$ with $e_i e_j = 0$ for $i \neq j$ and $\sum_{i=1}^{n} e_i = 1$. (Lemma 1.1.10 and the proof of proposition 1.13). $\therefore$ $Z(R) = \bigoplus_{i=1}^{n} e_i Z(R)$ (since $B(Z(R)) = B(R)$). By Lemma 1.1.9, $e_i Z(R)$ $i = 1, \ldots, n$ are indecomposable rings, so they contain no non-zero idempotents except e_i. Since (Lemma 1.21) Z(R) is regular $e_i r = e_i rxr$ for some $x \in Z(R)$. In fact $e_i r = e_i^3 r = e_i r e_i x e_i r$ so $e_i Z(R)$ is regular. Thus if $e_i r \neq 0$, $e_i r e_i x$ is a non-zero idempotent of $e_i Z(R)$, so $e_i r e_i x = e_i$ and similarly $e_i = e_i x e_i r$. $\therefore$ $e_i Z(R)$ is a commutative regular division ring, and so is a field, which establishes the result. □

Proposition 1.23: A commutative ring is a fully idempotent factorization ring if and only if it is a finite direct product of fields.

Proof: (⇒) If R is a commutative fully idempotent factorization ring $R = Z(R)$ is a finite direct product of fields by proposition 1.22.

(⇐) Fields are trivially fully idempotent factorization rings, so this assertion is immediate from Proposition 1.10. □

Proposition 1.24: The center of a fully idempotent factorization ring is a regular factorization ring.

Proof: By Lemma 1.21 the center of a fully idempotent ring is regular. By proposition 1.22, this center is a finite direct product of fields, and since it is commutative, proposition 1.23 shows that it is a factorization ring. □

Proposition 1.25: A biregular ring is a factorization ring if and only if it is a finite direct product of simple rings.

Proof: **(⇒)** Suppose R is a biregular factorization ring. Since biregular rings are fully idempotent, R is a finite direct product of indecomposable rings of the form e_iR, $i = 1, \dots, n$ where $e_i \in B(R)$. (See proposition 1.13.) Let I be a non-zero ideal of e_iR, and let $0 \neq e_ia \in I$. Define $I' = (e_iR)(e_ia)(e_iR)$. I' is a non-zero ideal of e_iR and we may write $I' = e_iRaR = e_iRe'R = e_ie'R$ where $e' \in B(R)$, since R is biregular. We now have $e_ie' \in I' \subseteq I \subseteq e_iR$. Also since $I' \neq 0$ (it contains e_ia), $e_ie' \neq 0$. But e_ie' is an idempotent contained in e_iR which is indecomposable. $\therefore$ $e_ie' = e_i$. Thus $I' = I = e_iR$, which shows that each e_iR is simple.

(⇐) Suppose R is biregular and is a finite direct sum of simple rings. Since simple rings are trivially fully idempotent factorization rings, the result follows from proposition 1.10. □

<u>Lemma 1.26</u> In a semiprime ring, left and right annihilator ideals coincide.

<u>Proof:</u> Let H be an ideal of R, a semiprime ring and let $k \in Ann_L(H)$, $h \in H$. Then $(hk)^2 = RhkRhkR = (0)$ since $RhkR \subseteq H$. $\therefore$ $hk = 0$ since R is semiprime, that is $k \in Ann_R(H)$. $\therefore$ $Ann_L(H) \subseteq Ann_R(H)$, and by a symmetrical argument we get the reverse inclusion as well. □

<u>Definition 1.27:</u> If $N_R \subseteq M_R$ are right R-modules such that for any non-zero submodule K of M, $K \cap N \neq 0$, then N_R is called an essential submodule of M_R, and M_R is called an essential extension of N_R.

<u>Definition 1.28:</u> If M_R is any right R-module, a submodule N_R is closed in M_R in case N_R has no proper essential extension contained in M_R.

Now, the set of essential extensions of N_R in M_R is not empty since it contains N_R itself, and any chain in this set has a maximal element in the chain (namely, the union of the elements of the chain), so applying Zorn's lemma, we see that this set has a maximal element. This suggests that a maximal essential extension of N_R in M_R may be a rather special object, and indeed it is, when M_R is injective:

<u>Lemma 1.29</u>: Let E be a maximal essential extension of N_R in an injective module M_R . Then E is injective.

<u>Proof</u>: See [9 , Theorem 1.9.2] . □

<u>Definition 1.30</u>: Denote by $Z(M_R)$ the set: $\{x \in M_R \mid Ann_R(x)$ is an essential right ideal of R $\}$. This set is a submodule of M_R (see Faith, [3 , Chapter 4, Proposition 4]), and is called the singular submodule of M_R . M_R is called nonsingular in case $Z(M_R) = (0)$.

<u>Lemma 1.31</u> If K is an ideal of a fully idempotent ring R , then $Z(R/K) = 0$.

<u>Proof</u>: $Z(R/K) = \{x + K \mid Ann(x + K)$ is essential in R$\}$ = $\{x + K \mid (K:x)$ is essential in R $\}$. Clearly $0 \in Z(R/K)$ since $(K:0) = R$ which is essential in R . Now suppose $(K:x)$ is essential in R. Then if $A = Ann(K:x)$, $0 = A(K:x) = A \cap (K:x)$ so $A = 0$ since $(K:x)$ is essential. ∴ $Ann(A) = Ann(0) = R = (K:x)$ (look ahead to the proof Lemma 2.1, part 2). Thus $xR \subset K$ so $x \in K$.

$\therefore\ x + K = 0 + K$ so $Z(R/K) = 0$. □

The following four results are due to Goodearl [5] .

<u>Lemma 1.32</u>: Let H be an ideal of R, a regular self-injective ring, so that H is closed as a right ideal. Then $R = H \oplus K$, where $K = \mathrm{Ann}\ (H)$.

<u>Proof</u>: Regular rings are fully idempotent so R/K is non-singular by 1.31. That is $(K:x)$ is an essential extension of K in R only if $x \in K$. Clearly, this means there are no proper essential extensions of K in R, that is, K is closed in R . Since H and K are their own maximal essential extensions in R and R is injective, by 1.29 so are H and K . Note that $H \cap K = HK = (0)$ so that $H \oplus K = H + K$ is also injective and so the imbedding $H \oplus K \to R$ splits, showing that $H \oplus K$ is a direct summand of R . Suppose $R = H \oplus K \oplus I$ for some right ideal $I \neq 0$. Since $I \not\subseteq K$, $IH \neq 0$ so $I \cap H \neq 0$ which is a contradiction, so $I = (0)$. □

<u>Proposition 1.33</u>: Any prime ideal of a regular self-injective ring R is either essential in R or closed in R .

<u>Proof</u>: If H is the maximal essential extension of P in R

and $K = \mathrm{Ann}(H)$ then by 1.32 above $R = H \oplus K$. Since $HK = 0 \subseteq P$, either $H \subseteq P$ or $K \subseteq P$. If $H \subseteq P$, then $H = P$ so P is closed in R . If $K \subseteq P$, $K \cap H = K$ but since $K \cap H = KH = 0$ this means $K = 0$ so $H = R$ so R is an essential extension of P . That is P is essential in R . □

<u>Definition 1.34</u>: A ring is prime if $AB = 0$ implies $A = 0$ or $B = 0$ for two-sided ideals A and B .

Note that every prime ring is semiprime.

<u>Proposition 1.35</u>: A regular self-injective ring is prime if and only if it is indecomposable as a ring.

<u>Proof</u>: If $R = R_1 \oplus R_2$ then $R_1 \oplus (0)$ and $(0) \oplus R_2$ are non-zero ideals of R whose product is zero. That is, prime rings are indecomposable.

Now suppose R is regular, self-injective, and indecomposable. Since any closed ideal of R would, by Lemma 1.32, be a direct summand of R, no proper non-zero ideal of R is closed in R . In particular, the maximal essential extension in R of any non-zero ideal (which must be closed) cannot be proper, and so must be R . That is, every non-zero ideal of R is essential in R . Now suppose A and B are non-zero ideals of R . Since R is regular,

$AB = A \cap B$ which is non-zero as B is essential and A is non-zero. Hence, R is prime. $\square$

The following proposition of Goodearl [5] we state without proof:

<u>Proposition 1.36</u>: A regular self-injective ring is prime if and only if it is primitive. $\square$

<u>Lemma 1.37</u>: A direct summand of a self-injective ring is self-injective.

<u>Proof</u>: Suppose $R \cong R_1 \oplus R_2$ is a self-injective ring. Then $R_1 \cong R/R_2$, so ideals of R_1 correspond to ideals of R containing a copy of R_2. Let I/R_2 be an ideal of R_1 and consider the diagram:

$$\begin{tikzcd} I/R_2 \arrow[r, "f"] \arrow[d, "i"'] & R_1 \\ R_1 & \end{tikzcd}$$

Since directs summands of injective modules are injective, R_1 is injective as an R-module, so there is a map $\bar{f}: R_1 \to R_1$ extending f. Now by Baer's criterion, (see [10, p. 157]) R_1 is injective as an R_1-module, that is, R_1 is self-injective. $\square$

<u>Proposition 1.38</u>: A right self-injective regular

factorization ring is a finite direct product of primitive right self-injective regular rings.

Proof: By Proposition 1.13, a regular factorization ring is a finite direct product of indecomposable fully idempotent factorization rings, each of which must be regular (since multiplication is done co-ordinatewise) and self-injective (by the previous lemma). Now Proposition 1.35 shows that these summands are prime, from which Proposition 1.36 gives us their primitivity. □

2. Chain Conditions

Lemma 2.1: Let $\mathcal{N}$ be the set of annihilator ideals of a semiprime ring R. Define $f: \mathcal{N} \to \mathcal{N}$ by $f(A) = \text{Ann}(A)$. Then

1. f in containment reversing
2. f is onto
3. f is 1 - 1
4. $f^{-1} = f$

Proof: 1. Let $A \subseteq B$; if $x \in \text{Ann}(B)$ $xB = 0$ so $xA = 0$ so $x \in \text{Ann}(A)$ $\therefore$ $\text{Ann}(B) \subseteq \text{Ann}(A)$. i.e. for annihilator ideals $N_1 \subseteq N_2$ we have $f(N_2) \subseteq f(N_1)$.

2. We show that every annihilator ideal is in fact the annihilator of some annihilator ideal. Let K be an ideal of R, and $N = \text{Ann}(K)$. $N \in \mathcal{N}$, and $NK = KN = 0$.

$\therefore\quad K \subseteq \mathrm{Ann}\ (N) = N'$.

Now by 1, $N'' = \mathrm{Ann}\ (N') \subseteq \mathrm{Ann}\ (K) = N$. But also $NN' = N'N = 0 \Rightarrow N \subseteq N''$ $\therefore$, $N = N'' = \mathrm{Ann}\ (N')$.

$\therefore\quad N = f(N')$ so f is onto.

3. From the above we have $N = \mathrm{Ann}\ (\mathrm{Ann}\ (N))$. Now suppose $f(N_1) = f(N_2)$ <u>i.e.</u> $\mathrm{Ann}\ (N_1) = \mathrm{Ann}\ (N_2)$ for $N_i \in \mathcal{N}$, $i = 1, 2$. Then $\mathrm{Ann}\ (\mathrm{Ann}\ (N_1)) = \mathrm{Ann}\ (\mathrm{Ann}\ (N_2))$ so $N_1 = N_2$ and f is $1-1$.

4. Has already been shown in 2, above. □

<u>Lemma 2.2</u>: Let R be semiprime. Then R has A.C.C. on annihilator ideals if and only if R has D.C.C. on annihilator ideals.

<u>Proof</u>: Suppose R has D.C.C. on annihilator ideals. Let $A_1 \subseteq A_2 \subseteq \ldots$ be an ascending chain in $\mathcal{N}$, the set of annihilator ideals of R . By the previous lemma, $\mathrm{Ann}\ (A_1) \supseteq \mathrm{Ann}\ (A_2) \supseteq \ldots$ is a descending chain and so is stationary at, say, $\mathrm{Ann}\ (A_k)$. That is $\mathrm{Ann}\ (A_k) = \mathrm{Ann}\ (A_{k+j})$ for $j = 0, 1, 2, \ldots$. Since $f(A) = \mathrm{Ann}\ (A)$ is a $1-1$ map by the previous lemma, $A_k = A_{k+j}$ for $j = 0, 1, 2, \ldots$ i.e. the ascending chain above is also stationary. A.C.C. $\Rightarrow$ D.C.C. is proven in an entirely analogous fashion. □

Lemma 2.3: A ring R is a sub-direct product of (finitely many) prime rings if and only if (0) is the intersection of a (finite) set of prime ideals. Moreover, such a ring is semiprime.

Proof: If R is a sub-direct product of prime rings S_α, $\alpha \in A$, then each $S_\alpha \cong R/P_\alpha$ and $\bigcap_{\alpha\in A} P_\alpha = 0$ by Lemma 1.2.5. The isomorphism shows that R/P_α is a prime ring for $\alpha \in A$. Now suppose $a, b \in R$ with $aRb \subset P_\alpha$. Then $aRb + P_\alpha = 0$ which implies $Ra\,Rb + P_\alpha = 0$ i.e. $(Ra + P_\alpha)(Rb + P_\alpha) = 0$. Since R/P_α is prime $Ra + P_\alpha = 0$ or $Rb + P_\alpha = 0$, that is $Ra \subset P_\alpha$ or $Rb \subset P_\alpha$. In particular, $a \in P_\alpha$ or $b \in P_\alpha$ so P_α is prime. Thus (0) is the intersection of prime ideals. Now if $(0) = \bigcap_{\alpha\in A} P_\alpha$ for P_α prime, Lemma 1.2.5 assures us that R is a subdirect products of the ring R/P_α, $\alpha \in A$, each of which is clearly prime. Let I be a nilpotent ideal of R where R is a subdirect product of R/P_α, $\alpha \in A$. Then in the following sequence

$$R \xrightarrow{\mu} \prod_{\alpha\in A} R/P_\alpha \xrightarrow{\pi_\alpha} R/P_\alpha$$

μ is mono, π_α is the canonical projection and $\pi_\alpha\mu$ is epi. If $I^n = 0$, $\pi_\alpha\mu(I^n) = 0 = (\pi_\alpha\mu(I))^n$. Since μ and π_α are canonical, $\pi_\alpha\mu(I)$ is an ideal of R/P_α and so must be (0) (since R/P_α is a prime ring). $\therefore$ $\mu(I) \subseteq \ker \pi_\alpha$ for each $\alpha \in A$. $\therefore$ $\mu(I) \subseteq \bigcap_{\alpha\in A} \ker \pi_\alpha = \bigcap_{\alpha\in A} P_\alpha = 0$. $\therefore$ $\mu(I) = 0$ and since μ is mono, $I = 0$.

Thus R is semiprime as claimed. $\square$

<u>Definition 2.4</u>: A ring R is an irredundant subdirect product of rings S_i, $i \in I$ if it is a subdirect product of the rings S_i and not a subdirect product of any subfamily of the rings S_i.

Recall that Lemma 1.2.5 asserts that $S_i \cong R/K_i$ with $\bigcap_{i\in I} K_i = 0$, so the above is equivalent to saying that no subfamily of the K_i's has zero intersection.

We now prove two lemmas which allow us to establish the equivalence of the conditions of Lemmas 2.2 and 2.3 for semiprime rings. They are essentially due to L. Levy [14]

<u>Lemma 2.5</u>: Let R be an irredundant subdirect product of the prime rings R/P_α, $\alpha \in A$. Then the annihilator ideals of R are in bijective inclusion reversing correspondence with the power set of A.

<u>Proof</u>: Define ϕ : $P(A) \to \mathcal{N}$ by:

$$\phi(S) = \begin{cases} \bigcap_{\alpha\in S} P_\alpha & \text{if} \quad S \neq \phi \\ R & \text{if} \quad S = \phi \end{cases}$$

We note in passing that $\phi(A) = \bigcap_{\alpha\in A} P_\alpha = (0)$ which is the annihilator ideal of R since R is semiprime (see Lemma 2.3).

We now show that $\phi(S) = \mathrm{Ann}(\bigcap_{\alpha\notin S} P_\alpha)$ and so is in $\mathcal{N}$: Suppose $x \in \mathrm{Ann}(\bigcap_{\alpha\notin S} P_\alpha)$. Then $x \bigcap_{\alpha\notin S} P_\alpha = 0 \in \bigcap_{\beta\in S} P_\beta$. By irredundancy, $\bigcap_{\alpha\notin S} P_\alpha \not\subset \bigcap_{\beta\in S} P_\beta$ so by the primeness of P_β's, $x \in \bigcap_{\beta\in S} P_\beta$. That is $\mathrm{Ann}(\bigcap_{\alpha\notin S} P_\alpha) \subset \bigcap_{\beta\in S} P_\beta$.

Now if $x \in \bigcap_{\beta\in S} P_\beta$ then $x \bigcap_{\alpha\notin S} P_\alpha \subseteq \bigcap_{\lambda\in A} P_\lambda = 0$, which proves the reverse inclusion. $\therefore$ $\phi(S) = \bigcap_{\beta\in S} P_\beta = \mathrm{Ann}(\bigcap_{\alpha\notin S} P_\alpha) \in \mathcal{N}$ as claimed.

That ϕ is order-reversing is immediate from its definition. We now show ϕ is $1-1$: Suppose $S_1 \neq S_2$ are subsets of A. $\phi(S_1) = \bigcap_{\alpha\in S_1} P_\alpha$ and $\phi(S_2) = \bigcap_{\alpha\in S_2} P_\alpha$. Assume without loss of generality that $\alpha_1 \in S_1$, $\alpha_1 \notin S_2$. Now if $\bigcap_{\alpha\in S_1} P_\alpha = \bigcap_{\alpha\in S_2} P_\alpha$ then $\bigcap_{\alpha\in S_2} P_\alpha \subset P_{\alpha_1}$.
$\therefore$ $\bigcap_{\alpha\in S_2} P_\alpha \cap P_{\alpha_1} = \bigcap_{\alpha\in S_2} P_\alpha$ which implies $\bigcap_{\substack{\alpha\neq\alpha_1 \\ \alpha\in A}} P_\alpha = \bigcap_{\alpha\in A} P_\alpha = 0$ which is a contradiction by irredundancy. $\therefore$ $\phi(S_1) \neq \phi(S_2)$.

To see that ϕ is onto, let $N \in \mathcal{N}$ be the annihilator of some ideal K. Let $S = \{\alpha \in A \mid K \not\subset P_\alpha\}$. Now $(\bigcap_{\alpha\in S} P_\alpha)K \subseteq \bigcap_{\alpha\in S} P_\alpha$ and $(\bigcap_{\alpha\in S} P_\alpha)K \subseteq K \subseteq \bigcap_{\alpha\notin S} P_\alpha$ so $(\bigcap_{\alpha\in S} P_\alpha)K \subseteq \bigcap_{\alpha\in A} P_\alpha = 0$ that is $\bigcap_{\alpha\in S} P_\alpha \subseteq \mathrm{Ann}(K) = N$. Now let $S' = \{\alpha \in A \mid N \not\subset P_\alpha\}$. Arguing similarly, we get $\bigcap_{\alpha\in S'} P_\alpha \subseteq \mathrm{Ann}(N)$ so by Lemma 2.1, $\mathrm{Ann}(\mathrm{Ann}(N)) = N \subseteq \bigcap_{\alpha\notin S'} P_\alpha = \mathrm{Ann}(\bigcap_{\alpha\in S'} P_\alpha)$. Combining these inclusions yields, $\bigcap_{\alpha\in S} P_\alpha \subseteq N \subseteq \bigcap_{\alpha\notin S'} P_\alpha$ which cannot be strict by irredundancy. $\therefore$ $N = \bigcap_{\alpha\in S} P_\alpha = \phi(S)$, so ϕ is onto as claimed. $\square$

In particular, if A is finite then so are the number of annihilator ideals which trivially yields A.C.C. and D.C.C. on annihilator ideals. In fact, the reverse implication is also true:

<u>Lemma 2.6</u>: If R is a semiprime ring with A.C.C. on annihilator ideals then R is a subdirect product of finitely many prime rings.

<u>Proof</u>: R semiprime $\Rightarrow \text{Ann}(R) = (0) \Rightarrow$ there are proper maximal annihilator ideals. If (0) is a maximal annihilator ideal then it is the only annihilator ideal and then R is a prime ring. So suppose there are non-zero maximal annihilator ideals P_α, $\alpha \in A$, annihilating minimal non-zero annihilator ideals L_α, $\alpha \in A$. Now $|A| \neq 1$ since if so $L_\alpha \subseteq P_\alpha$ which would imply $L_\alpha^2 = 0$ so $L_\alpha = 0$ since R is semiprime. But $L_\alpha \neq 0$ so $|A| > 1$. Let P_α, P_β respectively annihilate L_α, L_β, $\alpha \neq \beta$. Then $P_\alpha + P_\beta \subseteq \text{Ann}(L_\alpha \cap L_\beta)$ so by the maximality of P_α and P_β, $\text{Ann}(L_\alpha \cap L_\beta) = R$. By Lemma 2.1 $\text{Ann}(R) = L_\alpha \cap L_\beta = (0)$. Now $\bigcap_{\alpha \in A} P_\alpha = \bigcap_{\alpha \in A} \text{Ann}(L_\alpha) = \text{Ann}(\sum_{\alpha \in A} L_\alpha)$ Since A.C.C. $\Rightarrow$ D.C.C. on annihilator ideals in semiprime rings (Lemma 2.2) if $\bigcap_{\alpha \in A} P_\alpha \neq 0$ then it contains a minimal annihilator ideal $L_\beta \neq 0$. $\therefore$ $L_\beta \subseteq \text{Ann}(\sum_{\alpha \in A} L_\alpha)$ which implies $(0) = L_\beta \sum_{\alpha \in A} L_\alpha = L_\beta^2$ since if $\alpha \neq \beta$ $L_\alpha L_\beta \subseteq L_\beta \cap L_\alpha = (0)$. Now $L_\beta^2 = 0 \Rightarrow L_\beta = 0$ as R is semiprime, but this is a contradiction. Thus $\bigcap_{\alpha \in A} P_\alpha = 0$. Moreover we shall see that

the P_α's are prime for if A, B are ideals of R with $AB \subseteq P_\alpha$ then $ABL_\alpha = 0$. If $BL_\alpha = 0$ then $B \subseteq \mathrm{Ann}\, L_\alpha = P_\alpha$. If $BL_\alpha \neq 0$ then $BL_\alpha \subseteq L_\alpha \Rightarrow P_\alpha = \mathrm{Ann}(L_\alpha) \subseteq \mathrm{Ann}(BL_\alpha)$. By maximality of P_α's, $P_\alpha = \mathrm{Ann}(BL_\alpha)$ so $A \subseteq P_\alpha$. $\therefore$ Each P_α is prime and $\bigcap_{\alpha \in A} P_\alpha = 0$ so R is a subdirect sum of prime rings R/P_α.

By Lemma 2.5, $\mathcal{P}(A)$ is in bijective correspondence with $\mathcal{N}$, so A.C.C. on $\mathcal{N}$ implies D.C.C. on $\mathcal{P}(A)$. Consider the chain formed in $\mathcal{P}(A)$ by starting with A and removing one element at a time. Since this chain becomes stationary after finitely many steps, A is a finite set, so R is a subdirect sum of finitely many prime rings as claimed. $\square$

We now summarize results 2.1 through 2.6 in the following portmanteau proposition:

<u>Proposition 2.7</u>: If R is a semiprime ring then the following are equivalent:

(i) R has A.C.C. on annihilators

(ii) R has D.C.C. on annihilators

(iii) (0) is the intersections of finitely many prime ideals

(iv) R is a subdirect product of finitely many prime ideals

<u>Proof</u>: Results 2.1 through 2.6. $\square$

This proposition can be used to prove the following one:

Proposition 2.8: If R is a fully idempotent ring then:

(i) R has A.C.C. on ideals $\Rightarrow$ R is a factorization ring.

(ii) R has D.C.C. on ideals $\Rightarrow$ R is a factorization ring with finitely many maximal ideals.

(iii) R has both A.C.C. and D.C.C. on ideals $\Rightarrow$ the number of ideals in R is finite.

(iv) R is a prime factorization ring with a minimal non-zero ideal $\Rightarrow$ R has finitely many non-zero minimal prime ideals.

Proof: (i) Since R is fully idempotent, by Lemma 1.3 R/I is semiprime for any ideal I of R. The ascending chain condition on R passes to R/I, so by Proposition 2.7, $(0) = I/I = \bigcap_{i=1}^{n} P_i/I$, each prime. Thus I is the intersection of the canonical preimages of $P_1/I, \ldots, P_n/I$ each prime, so R is a factorization ring.

(ii) Again, R/I is semiprime so by Proposition 2.7 it is a factorization ring. Suppose $M_1, M_2, M_3, \ldots$ are incomparable prime ideals. $M_i \not\subseteq M_j$ for $j \neq i$, so $M_i \not\supseteq \bigcap_{j=1}^{i-1} M_j = \prod_{j=1}^{i-1} M_j$. Thus $\prod_{j=1}^{i} M_j = \prod_{j=1}^{i-1} M_j \cap M_i \subsetneq \prod_{j=1}^{i-1} M_j$.

That is, the sequence $\{\prod_{j=1}^{i} M_i\}_{i=1,2,\ldots}$ is a strictly decreasing sequence which is impossible by D.C.C.. Thus, there can be only finitely many incomparable prime ideals in R . In particular, this means that there are finitely many maximal ideals in R .

Before proving (iii) we state the following result from graph theory:

<u>Lemma 2.9 (König)</u>: Let G be a connected locally-finite infinite graph. Then for any vertex v of G , there exists an infinite chain with initial vertex v .

Proof is trivial and can be found in [21] .

We now prove <u>(iii)</u>: Consider the lattice of prime ideals of R . This lattice may be thought of as a connected graph, directed edges corresponding to inclusions. As a graph it is locally finite since by (ii) above, the number of incomparable primes of R is finite. Since by the chain conditions there is no infinite chain of prime ideals, König's result above shows that this graph is finite, so there are finitely many prime ideals. Now by (ii), R is a fully idempotent factorization ring, so every ideal has an irredundant representation as an intersection of finitely many prime ideals. Since there are only finitely many such intersections, the assertion is proved.

(iv) Let R be a prime factorization ring with minimal ideal I, having irredundant representation $I = P_1 \cdot \ldots \cdot P_n$, $I \neq 0$. Now if $I \not\subset A$, $I \cap A \neq I$ so $I \cap A = 0$ by the minimality of I. Since $IA \subset I \cap A$, $IA = 0$. Since R is prime and $I \neq 0$, $A = 0$. $\therefore$ Every non-zero ideal of R contains I. In particular, if P is prime $I = P_1 \cdot \ldots \cdot P_n \subset P$ so by the primeness of P, $P_i \subset P$ for some $i \in \{1, 2, \ldots, n\}$. Thus the minimal primes must be from $P_1, \ldots, P_n$. Moreover since the representation $I = P_1 \cdot \ldots \cdot P_n$ is irredundant, if $i \neq j$ then $P_i \not\subset P_j$. That is, each of $P_1, \ldots, P_n$ are minimal primes. □

3. Examples and Ways to Generate Them

Example 1: Let I be an ideal of a regular ring R. Since for every $a \in I$ there is $x \in R$ with $a = a\,x\,a \in I^2$, $I = I^2$ so that regular rings are fully idempotent. K.R Goodearl has shown [5, Cor. 7] that if R is also self injective, then any proper two-sided ideal which contains a prime ideal is itself prime. This shows that R/P is a regular factorization ring, since its ideals are prime (being canonical images of ideals of R containing P).

Example 2: Let K be a field of characteristic zero and let S_∞ be the group of permutations on an infinite set leaving all but finitely many elements fixed. Any finitely generated subgroup of S_∞ also leaves all but finitely many elements of the infinite set fixed, so each such subgoup is contained in S_n for some n. That is, S_∞ is locally finite, so we may apply a theorem independantly discovered by Villamayor and Connell (see Passman [16, pp. 69-70]) to see that the group algebra KS_∞ is regular. Since the set of ideals of KS_∞ is at least as big as the set of normal subgroups of S_∞ (see [13, Lemma 1, p. 153]), KS_∞ has infinitely many ideals. Formanek and Lawrence [4] showed that KS_∞ has A.C.C. on two-sided ideals, so by Proposition 2.2.8 parts (i) and (iii), KS_∞ is an example of a regular factorization ring with A.C.C. but without D.C.C. .

<u>Example 3</u>: The following example shows that if the dimension of a vector space is large enough, then its endomorphism ring may be a factorization ring with D.C.C. but without A.C.C. . For the most part, the background involved in this example is from Jacobson [11], while the example itself is due to Raphael [18].

Let V be an infinite dimensional vector space, with $A, B \in \mathrm{End}(V)$. The following three properties are standard theorems of linear algebra: if $R(A)$ denotes the rank of A, and $N(A)$ its nullity, then 1) $R(A) + N(A) = \dim V$ 2) $R(A+B) \leq R(A) + R(B)$, and, 3) $R(AB) \leq \min(R(A), R(B))$. Now let e be an infinite cardinal with $e \leq \dim V$. Define $L_e = \{A \in \mathrm{End}(V) \mid R(A) < e\}$. Clearly, $0 \in L_e$, and if $A \in L_e$ then so is $-A$ since $R(-A) = R(A)$. Since the sum of two infinite cardinals is the larger of the two, 2) above shows that L_e is closed under addition. Moreover if $A \in L_e$ and $B \in \mathrm{End}(V)$ then AB and BA are both in L_e by 3) above. Thus, we see that L_e is a two-sided ideal of $\mathrm{End}(V)$. We note that if $e < e' \leq \dim V$, then $L_e \subseteq L_{e'}$ for infinite cardinals e, e'. Moreover, this containment is strict, since there is at least one element of $\mathrm{End}(V)$ with rank e, namely the one which is the identity on a subset of the basis of V having cardinality e, and the zero map on the remaining basis elements. This map is in $L_{e'}$ but not L_e; so evidently $L_e \subsetneq L_{e'}$ which shows that the correspondence $e \to L_e$ from the set of infinite cardinals less than or equal to $\dim V$ to the set of proper ideals of $\mathrm{End}(V)$ is one-one and inclusion preserving. To show that this correspondence is onto we require the following lemma:

<u>Lemma 3.1</u>: If $A, B \in \mathrm{End}(V)$ with $R(B) \leq R(A)$ then there are $P, Q \in \mathrm{End}(V)$ with $B = PAQ$.

<u>Proof</u>: $V = V_1 \oplus N(A) = V_2 \oplus N(B)$, where $R(B) = \dim V_2 \leq \dim V_1 = R(A)$. Let $(y_\alpha)_{\alpha\in I}$ be a basis for V_2. Since $\dim V_2 \leq \dim V_1$, the basis of V_1 contains a subset of cardinality equal to that of I, call it $(x_\alpha)_{\alpha\in I}$. Define $P:V \to V$ by $y_\alpha P = x_\alpha$ and $P|_{N(B)} = 0$. Clearly P is a linear transformation. Now since A is monic on V_1, the set $(x_\alpha A)_{\alpha\in I}$ is linearly independant (since the x_α's were) and

so is the set $(y_\alpha B)_{\alpha \in I}$ (for the same reasons). Moreover, they have the same cardinality, so there is a linear transformation Q such that $x_\alpha AQ = y_\alpha B$. That is, $y_\alpha PAQ = y_\alpha B$ so PAQ agrees with B on V_2. If $y \in N(B)$, $yP = 0$ so $yPAQ = 0 = yB$. Thus PAQ agrees with B on all of V, completing the proof. □

In fact this lemma asserts that if an ideal of $\mathrm{End}(V)$ contains an element of rank r, it contains as well every element of $\mathrm{End}(V)$ of rank $\leq r$.

Suppose X is a proper non-zero ideal of $\mathrm{End}(V)$, and let e be the smallest infinite cardinal with $e > R(B)$ for every $B \in X$. Now $e \leq \dim V$, since if not, X would contain an element of rank equal to $\dim V$, and so would not be a proper ideal (by the previous lemma). Thus by definition, $X \subseteq L_e$. On the other hand if $B \in L_e$ with infinite rank, then $R(B) < e$ so by definition of e, there must be $A \in X$ with $R(B) \leq R(A)$. ∴ By the lemma, $B \in X$. We note in passing that X must contain all linear transformations of finite rank. This is because, being non-zero, it contains (by the lemma) all linear transformations of rank one, and every linear transformation of finite rank can be written as a finite sum of linear transformations of rank one. This shows that $L_e \subseteq X$, so in fact $X = L_e$.

We have shown here that the correspondence $e \to L_e$ is a bijective correspondence between infinite cardinals no bigger than $\dim V$ and the proper ideals of End(V) . The correspondence preserves order so the ideals of End(V) are linearly ordered, and since the infinite cardinals are well-ordered, so are the ideals of End(V) . That is, End(V) has D.C.C. and if the dimension of V is large enough (e.g. l.u.b.$\{\aleph_0, \aleph_1, \aleph_2, \ldots\}$) then End(V) does not have A.C.C. (e.g. $L_{\aleph_0} \subsetneq L_{\aleph_1} \subsetneq \ldots$ is an infinite ascending chain). Since by Proposition 1.2.10 End(V) is regular, Proposition 2.2.8 ii) shows that End(V) is a regular factorization ring. Since the ideals of End(V) are linearly ordered, it will have at most one maximal ideal.

<u>Example 4</u>: We now prove two results whose goal it is to provide a means for producing new fully idempotent factorization rings from old. We require the following lemma of Jacobson [12, p. 109] :

<u>Lemma 3.2</u>: Let R be a ring with center F, a field, and let S be a central simple F-algebra. If I is an ideal of R , then $I \otimes_F S$ is an ideal of $R \otimes_F S$ and the correspondence $I \to I \otimes_F S$ is a lattice isomorphism of the lattice of ideals of R onto the lattice of ideals of $R \otimes_F S$

<u>Proof</u>: Clearly $I \otimes_F S$ is an ideal of $R \otimes_F S$ since elements

of $I \otimes_F S$ have the form $\sum_{i=1}^{k} a_i \otimes b_i$, where $a_i \in I$, $b_i \in S$.

Now suppose I_1 and I_2 are ideals of R with $a \in I_1$, $a \notin I_2$. Clearly for any $s \in S$, $a \otimes s \in I_1 \otimes_F S$. Suppose $0 \neq a \otimes s \in I_2 \otimes_F S$ as well. Then $a \otimes s = \sum_{i=1}^{k} a_i \otimes s_i$ where $a_i \in I_2$ and $s_i \in S$ for $i = 1, \ldots, k$, and the summands $a_1 \otimes s_1, \ldots, a_k \otimes s_k$ are linearly independant. It follows easily that $a_1, \ldots, a_k$ are linearly independant (considering R as an F-vector space). Moreover since $a \notin I_2$, $a_1, \ldots, a_k, a$ are linearly independant, so since $a_1 \otimes s_1 + \ldots + a_k \otimes s_k + a \otimes s = 0$, the set $s_1, \ldots, s_k, s$ is not linearly independant, and this is true for any choise of s, in particular $s = s_1$. Thus, $(a_1 + a) \otimes s_1 + \ldots + a_k \otimes S_k = 0$, and inasmuch as $(a_1 + a)$, $a_2, \ldots, a_k$ are linearly independant, $s_1, \ldots, s_k$ cannot be, which contradicts the assumption that $a_1 \otimes s_1, \ldots, a_k \otimes s_k$ are linearly independant. Thus $a \otimes s \in I_2 \otimes_F S$, and we have shown that the correspondence $I \to I \otimes_F S$ is injective and inclusion preserving.

We now show that this correspondence is onto. Let $(0) \neq I$ be an ideal of $R \otimes_F S$ and define $V = \{r \in R \mid r \otimes s \in I \text{ for each } s \in S\}$. V is a subspace of R and clearly $V \otimes_F S \subseteq I$. Suppose $0 \neq a \otimes s \in I$. We claim $a \in V$, and show it as follows. Define $U = \{u \in S \mid a \otimes u \in I\}$. Since $s \in U$, $U \neq 0$, and it is easy to verify that U is an ideal of S, and since S is simple, $U = S$. That is, for any $s \in S$, $a \otimes s \in I$ so

$a \in V$. $\therefore$ $I \subseteq V \otimes_F S$. $\therefore$ $I = V \otimes_F S$. It remains to show that V is an ideal of R. Let $r \in V$. Then $r \otimes s \in I \ \forall \ s \in S$. If $t \in R$, $tr \otimes s = (t \otimes 1)(r \otimes s) \in I$ and $rt \otimes s = (r \otimes s)(t \otimes 1) \in I$ for all $s \in S$ since I is an ideal. $\therefore$ tr and rt are in V, so V is an ideal as claimed. □

<u>Proposition 3.3</u>: If R is a fully idempotent factorization ring with centre F a field, and S is a central simple F algebra, then $R \otimes_F S$ is a fully idempotent factorization ring.

<u>Proof</u>: If $I \otimes_F S$ is an ideal of $R \otimes_F S$, $(I \otimes_F S)^2 = I^2 \otimes_F S^2 = I \otimes_F S$ so $R \otimes_F S$ is fully idempotent. If P is a prime ideal of R, and $(I \otimes_F S)(J \otimes_F S) \subseteq P \otimes_F S$ then $(IJ \otimes_F S) \subseteq P \otimes_F S$ so $IJ \subseteq P$. Since P is prime $I \subseteq P$ or $J \subseteq P$ from which $I \otimes_F S \subseteq P \otimes_F S$ or $J \otimes_F S \subseteq P \otimes_F S$. That is the correspondence $I \to I \otimes_F S$ preserves prime ideals and since it is a lattice isomorphism it preserves inclusion and intersection. Since R and $R \otimes_F S$ are both fully idempotent, finite multiplication of ideals is preserved since it coincides with finite intersection. This shows that $R \otimes_F S$ is a factorization ring. □

Example 5: In this example, the goal is to prove:

Proposition 3.4: Let R be a fully idempotent factorization ring with centre F , a field. If K is an algebraically closed subfield of F , and G is a group whose order is invertible in F , then RG is a fully idempotent factorization ring.

Proof of Proposition 3.4 follows several lemmas.

Lemma 3.5: Let R be a ring, K a field in the centre of R , and G a finite group whose order is a unit in K . The $RG \cong R \otimes_K KG$.

Proof: A typical element of RG has the form $\sum_{i=1}^{n} r_i g_i$ wher $n = |G|$. Since R is a K-vector space, this can be rewritten as $\sum_{i=1}^{n} (\sum_{i=1}^{m} k_{i_j} r_{i_j}) g_i$ where $r_{i_1}, \ldots, r_{i_m}$ are linearly independant for fixed i . By taking a maximal linearly independant subset of $\{r_{i_j}\}$ and re-indexing it as $\{r_j\}_{j=1,\ldots,m}$ (for some new value of m, of course), a typical element of RG can be written as $\sum_{i=1}^{n} (\sum_{j=1}^{m} k_{i_j} r_j) g_i$.

Now a typical element of $R \otimes_K KG$ has the form $\sum_{j=1}^{m} (r_j \otimes \sum_{i=1}^{n} k_{i_j} g_i)$ where $r_1, \ldots, r_m$ are K-linearly independant elements of R . By the properties of tensor arithmetic this is just: $\sum_{j=1}^{m} \sum_{i=1}^{n} (k_{i_j} r_j \otimes g_i) = \sum_{i=1}^{n} (\sum_{j=1}^{m} k_{i_j} r_j) \otimes g_i$.

Let $f: RG \to R \otimes_K KG$ be defined by $f(\sum_{i=1}^{n} (\sum_{j=1}^{m} k_{ij} r_j) g_i) = \sum_{i=1}^{n} (\sum_{j=1}^{m} k_{ij} r_j) \otimes g_i$. Clearly if $\sum_{i=1}^{n} (\sum_{j=1}^{m} k_{ij} r_j) g_i \in \ker f$ then $\sum_{j=1}^{m} k_{ij} r_j$ must be 0 in R for each $i = 1, \ldots, n$. By the linear independance of $r_1, \ldots, r_m$, this means that $\ker f = 0$. Thus f is a K-vector space isomorphism. It remains to show that f preserves multiplication, but this ~~is~~ immediate from the definitions of multiplication in RG and $R \otimes_K KG$. Thus f is the required K-algebra isomorphism. □

We now state without proof Maschke's Theorem.

<u>Lemma 3.6</u>: Let G be a finite group acting on a vector space V over a field K such that $|G|$ is a unit in K. Then every representation of G into GL(V) is completely reducible. □

For the proof of this theorem, and the details of the following discussion, the reader is referred to Jacobson [10, pp. 251-264], or Van der Waerden [19]. We note that in the above case, since G is finite, KG is finite dimensional over K. This means that $KG \cong \bigoplus_{i=1}^{m} M_{n_i}(\Delta_i)$, that is, KG is a direct sum of the simple rings $M_{n_i}(\Delta_i)$ where Δ_i is a finite dimensional central division algebra over K. If K is also algebraically closed then $KG \cong \bigoplus_{i=1}^{m} M_{n_i}(K)$ since in this case the only finite dimensional division algebra over K is K .

Lemma 3.7: Let M and N_i , $i = 1, \ldots, j$ be K-algebras, K a ring. Then $M \otimes_K \bigoplus_{i=1}^{j} N_i \cong \bigoplus_{i=1}^{j} (M \otimes_K N_i)$.

Proof: The functor $M \otimes_K -$ has as right adjoint $\mathrm{Hom}_{\mathbb{Z}}(M, -)$, and functors with right adjoints preserve coproducts [9, p. 110] . □

Lemma 3.8: Let R be a ring, K a field in its centre. Then $R \otimes_K M_n(K) \cong M_n(R \otimes_K K)$.

Proof: Define $f: R \otimes_K M_n(K) \to M_n(R \otimes_K K)$ by $f(r \otimes [a_{ij}]) = [r \otimes a_{ij}]$. This is clearly an algebra isomorphism. □

Lemma 3.9: Let R be a K-algebra such that the action from K commutes with elements of R , that is, for every $r \in R$, $k \in K$, $rk = kr$. Then $R \otimes_K K \cong R$.

Proof: The required K-algebra isomorphism is $f: R \otimes_K K \to R$ defined by $f(\sum_{i=1}^{n} r_i \otimes k_i) = \sum_{i=1}^{n} r_i k_i$. □

We are now ready for

Proof of Proposition 3.4: By Lemma 3.8, $RG \cong R \otimes_K KG$. Since $|G|$ is a unit in F it is a unit in K as well so by the comments following Maschke's Theorem (Lemma 3.6), $R \otimes_K KG \cong R \otimes_K (\prod_{i=1}^{m} M_{n_i}(K))$. By Lemma 3.7 this is isomorphic to $\prod_{i=1}^{m} R \otimes_K M_{n_i}(K)$ and by Lemmas 3.8 and 3.9 we have $\prod_{i=1}^{m} R \otimes_K M_{n_i}(K) \cong \prod_{i=1}^{m} M_{n_i}(R \otimes_K K) \cong \prod_{i=1}^{m} M_{n_i}(K)$. Now a matrix ring over a field is a fully idempotent factorization ring by Proposition 2.1.16 and so by Proposition 2.1.10 so is a finite direct product of these matrix rings, which establishes the result. □

Example 6: The following three results offer a construction which produces a prime fully idempotent factorization subring of a prime fully idempotent factorization ring.

Lemma 3.10: Let R be a subring of a ring S . When considered as an R-module, let S be an essential extension of R . Then:

i) If R is prime, so is S .

ii) If S is prime and R is semiprime then R is prime.

Proof: (i) Suppose A and B are non-zero ideals of S . Since they are also R-modules $R \cap A \neq (0) \neq R \cap B$, because R is essential in S . Since R is prime, $(R \cap A)(R \cap B) \neq 0$. But $(R \cap A)(R \cap B) \subseteq AB$ so $AB \neq 0$, showing that S is prime.

(ii) Let I and J be ideals of R, and suppose $JI = 0$. Considering S as an R-module, ISJ is a submodule of S. Moreover, $(R \cap ISJ)^2 \subseteq (ISJ)^2 = ISJISJ = 0$, and since $R \cap ISJ$ is an ideal of R which is semiprime, $R \cap ISJ = 0$. Inasmuch as R is essential in S, this means that $ISJ = 0$. Now $ISJ = 0 \Rightarrow (SIS)(SJS) = 0 \Rightarrow SIS = 0$ or $SJS = 0$ since S is prime, and since $I \subseteq SIS$ and $J \subseteq SJS$, this means that $I = 0$ or $J = 0$, which shows that R is prime as claimed. □

Let R be an indecomposable regular ring, let I be a proper ideal of R, and F the center of R. Recall (Lemma 1.21) that F is regular, so $0 \neq x \in F \Rightarrow xyx = x$ for $y \in F$, and thus xy and yx are non-zero central idempotents. Since R is indecomposable, its only central idempotents are 0 and 1, so $xy = 1 = yx$, which shows that F is a field. I is also regular, since for $x \in I$, $x = xyx = x(yxy)x$ and $yxy \in I$. Note that since I is proper, it contains no units, so $I \cap F = 0$ $\therefore$ $F = I + F/I$. To summarize, I is regular and so is $I + F/I$, so by Lemma 1.2.3, $I + F$ is regular.

We are now ready to prove:

<u>Lemma 3.11</u>: Let R, I, F and $I + F$ be as above. If P is a prime ideal of R then $P \cap (I + F)$ is a prime ideal of $I + F$.

Proof: $P \cap (I + F)$ is clearly an ideal of $I + F$. We consider two cases:

Case 1: $I \subseteq P$ This implies that $P \cap (I + F) \supseteq I$. Since I contains all the non-units of $I + F$, I is maximal so $P \cap (I + F)$ is either I or $I + F$. But P contains no units as it is proper in R. $\therefore$ $P = I$ and so is maximal (and thus prime).

Case 2: $I \not\subseteq P$ Consider the ring $(I+F)/P \cap (I + F)$. The canonical ring homomorphism $\mu:(I + F)/P \cap (I + F) \to R/P$ is mono so $\mu(I/P \cap (I+F))$ is a non-zero ideal of R/P, and since R/P is prime, this ideal is essential in R/P. In particular, for any $x \notin P$, $\mu(I/P \cap (I + F))(x + P) \neq 0$ so $\mu(I + F/P \cap (I + F))(x + P) \neq 0$, so when considering R/P as an $I + F/P \cap (I + F)$-module, R/P is an essential extension of $(I + F)/P \cap I + F$. As previously mentioned, $I + F$ is regular, and so fully idempotent, so by Lemma 1.3, $(I + F)/P \cap (I + F)$ is semiprime. We can now apply Lemma 3.10 (ii) to see that $(I + F)/P \cap (I + F)$ is in fact prime as well, so $P \cap (I + F)$ is a prime ideal of $I + F$ as claimed. $\square$

Proposition 3.12: Let R, I, F and $I + F$ be as above. If R is a factorization ring, so is $I + F$.

Proof: Let J be an ideal of $I + F$. We examine two cases:

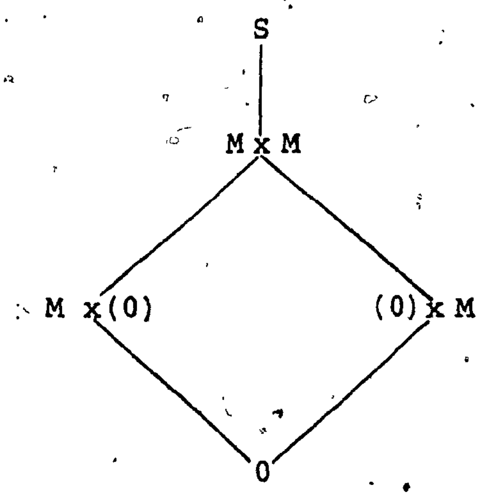
<u>Case 1: $J \subset I$</u> Since I is an ideal of R, $JR \subset I = IR$. $\therefore$ $J^2R \subset JIR$ and since $I + F$ is fully idempotent, $JR \subset JIR$. The reverse inequality being true, we have $JR = JIR = JI = J \cap I = J$. Similarly $RJ = J$, so J is an ideal of R. As R is a fully idempotent factorization ring, $J = P_1 \cap \ldots \cap P_n = [P_1 \cap (I + F)] \cdot \ldots \cdot [P_n \cap (I + F)]$ each of which are prime by the previous lemma.

<u>Case 2: $J \not\subset I$</u> By the maximality of I in $I + F$, $J + I = I + F$, so $(I + F)/J \cong (J + I)/J \cong I/I \cap J$. Since $I \cap J \subset I$, case 1 above shows it is a finite product of prime ideals of $I + F$. By the isomorphism, above, in the ring $I/I \cap J$, 0 is finite product prime ideals. Since this is true also in $(I + F)/J$, J is a finite product of prime ideals of $I + F$, which completes the proof. $\square$

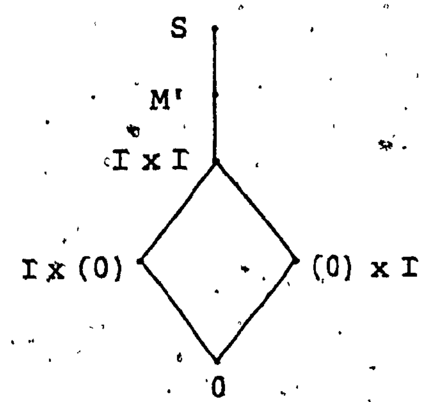
<u>Example 7:</u> By Lemma 3.11 we see that if R is prime so is $I + F$. The following construction does not preserve primeness. Let R be a prime regular factorization ring with a proper ideal $I \neq 0$. Denote by S the smallest subring of $R \times R$ containing $I \times I$ and $\{(r, r) \mid r \in R\}$. Since for any $r \in R$ $\pi(r, r) = r$, S is a subdirect product of $R \times R$ so by Proposition 1.11 S is a regular factorization ring. S is not prime since $I \times (0)$ and $(0) \times I$ are non-zero ideals of S whose product is (0). S is indecomposable however, for the following reason: R is indecomposable since it is prime (see the proof of Proposition 1.35) so its only central

idempotents are 0 and 1 . Thus the only central idempotents of $R \times R$ are (0, 0), (1, 1), (0, 1) and (1, 0) . If either of (0, 1) or (1, 0) were in S, this would contradict the proper containment of I in R . Thus the only central idempotents of S are (0, 0) and (1, 1) , so S is indecomposable.

We now consider some examples of this process. Let R be a regular ring having three ideals (0), M and R . One such ring is End(V) where $\dim V = \aleph_0$ (see Example 3) . Then S has ideal structure

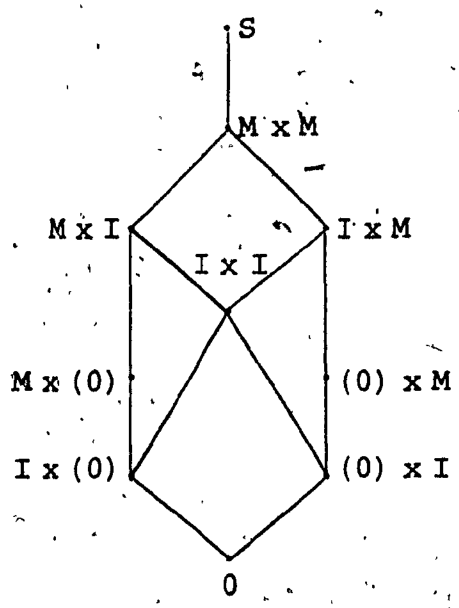


If R is a regular ring with four linearly ordered ideals $(0) \subset I \subset M \subset R$ (e.g. End(V) , where $\dim V = \aleph_1$) , then we can use either I or M to generate S . Using I , the ideal lattice of S is:



where $M' = \{(m + i_1, m + i_2) \mid m \in M,\ i_1, i_2 \in I\}$.

Using M, one obtains the ideal lattice:



If we use $I \times I$ in the last ring constructed to construct S', that is, the subring of $S \times S$ generated by $I \times I \times I \times I$ and $\{(s, s) \mid s \in S\}$, then the new ring will have as ideals:

$$I_1 = \{(m + i_1, m + i_2, m + i_3, m + i_4) \mid m \in M,\ i_j \in I\}$$

which is a maximal ideal, and,

$$I_2 = \{(m + i_1, i_2, m + i_3, i_4) \mid m \in M, i_j \in I\}$$

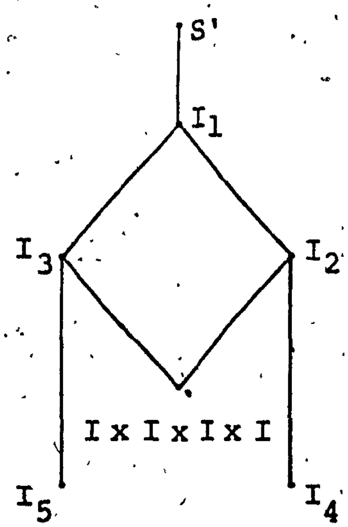
$$I_3 = \{(i_1, m + i_2, i_3, m + i_4) \mid m \in M, i_j \in I\}$$

both of which are contained in I_1 and contain $I \times I \times I \times I$, and

$$I_4 = \{(m + i_1, 0, m + i_2, 0) \mid m \in M, i_j \in I\}$$

$$I_5 = \{(0, m + i_1, 0, m + i_2) \mid m \in M, i_j \in I\}$$

which are contained in I_2 and I_3 respectively, but do not contain nor are contained in $I \times I \times I \times I$. Furthermore, any ideal of $R \times R \times R \times R$ contained in $I \times I \times I \times I$ is an ideal of S', and this completes the list of ideals of S'. A partial lattice diagram for S' is:



(followed by ideals of $R \times R \times R \times R$ contained in $I \times I \times I \times I$.)

<u>Example 8:</u> Not all prime regular rings are factorization rings.

Proof: Let R be the ring of row-finite matrices having the form A + X where A is a row-finite matrix with an $n \times n$ matrix of arbitrary size in its upper left corner, and zeros elsewhere, while X has the form:

$$\begin{bmatrix} c_1 & & & & & & & & & \\ & c_2 & & & & & & & & \\ & & c_1 & & & & 0 & & & \\ & & & c_2 & & & & & & \\ & & & & c_3 & & & & & \\ & & & & & c_1 & & & & \\ & & & & & & c_2 & & & \\ & & 0 & & & & & c_3 & & \\ & & & & & & & & c_4 & \\ & & & & & & & & & \ddots \end{bmatrix}$$

R is clearly a regular ring, and the non-zero ideals are in bijective inclusion reversing correspondence with $\mathcal{P}(\mathbb{N})$, the power set of the naturals. The bijection is given by $S \leftrightarrow \{A + X \mid i \in S \Leftrightarrow c_i = 0\}$. The only prime ideals of R are the maximal ideals, and these correspond to the subsets of $\mathbb{N}$ having precisely one element. Clearly this means that there are infinitely many minimal prime ideals. R has a single minimal ideal, namely the one in which $X = 0$ for each element. R is prime since the product of any two ideals contains a non-zero element of the minimal ideal. Recall (Proposition 2.2 (iv)) that a prime fully idempotent factorization ring with a minimal non-zero has finitely many non-zero minimal prime ideals. Hence R is not a factorization ring. □

AFTERWORD

The following results have been obtained:

1. There are prime regular factorization rings having n maximal ideals for any $n \in \mathbb{N}$.

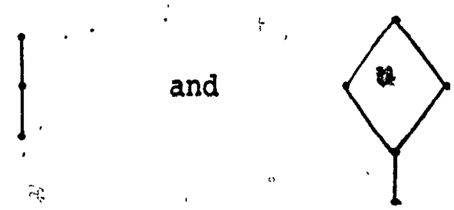
Proof: Let R be the ring of row finite matrices over a field having the form $B + X$ where B has a $k \times k$ matrix of arbitrary size in the upper left corner and zeros elsewhere, and X had the repeating sequence $c_1, \ldots, c_n, c_1, \ldots, c_n, \ldots$ in the diagonal and zeros elsewhere. Let $A = \{c_1, \ldots, c_n\}$. The non-zero ideals of R are in 1-1 inclusion reversing correspondence with the subsets of A. One sees this by defining $f: \mathcal{P}(A) \to L(R)$ by $f(S) = \{X \in R \mid c_i \in S \Rightarrow c_i = 0\}$ for any subset S of A. Every non-zero ideal of R is in the image of f, and f is clearly an injection. Moreover the maximal ideals of R correspond to the subsets of A containing exactly one element, and there are n of these.

R is regular since both the field and any $n \times n$ matrix ring over the field are. More specifically, if A is an $n \times n$ matrix with $A = ABA$, then

$$\begin{bmatrix} A & & & & & & & 0 \\ & c_1 & & & & & & \\ & & \ddots & & & & & \\ & & & c_n & & & & \\ & & & & c_1 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & c_n & \\ & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} B & & & & & & & 0 \\ & c_1^{-1} & & & & & & \\ & & \ddots & & & & & \\ & & & c_n^{-1} & & & & \\ & & & & c_1^{-1} & & & \\ & & & & & \ddots & & \\ & 0 & & & & & c_n^{-1} & \\ & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} A & & & & & & & 0 \\ & c_1 & & & & & & \\ & & \ddots & & & & & \\ & & & c_n & & & & \\ & & & & c_1 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & c_n & \\ & & & & & & & \ddots \end{bmatrix} = \begin{bmatrix} A & & & & & & & 0 \\ & c_1 & & & & & & \\ & & \ddots & & & & & \\ & & & c_n & & & & \\ & & & & c_1 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & c_n & \\ & & & & & & & \ddots \end{bmatrix}$$

R is clearly prime. Because R has finitely many ideals, it has A.C.C. and D.C.C., and so is a factorization ring by Proposition 2.2.8. □

Note that R also serves as an example of a prime regular factorization ring whose ideals are not linearly ordered if $n \geq 2$. For $n = 1$ and $n = 2$ the lattice diagrams are, respectively :



2. Let $C_1 : I_1 \supseteq I_2 \supseteq I_3 \supseteq , \ldots$ and
$C_2 : J_1 \supseteq J_2 \supseteq J_3 \supseteq , \ldots$ be

descending chains of ideals. Call C_1 and C_2 disjoint if $I_m \neq J_n$ for every $n , m \in N$. We claim that if a fully idempotent factorization ring has infinitely many incomparable prime ideals then it has infinitely many pairwise disjoint non-terminating chains of ideals strictly descending from each of the incomparable prime ideals.

<u>Proof</u>: Let P be ~~an~~ element of the infinite collection of incomparable prime ideals in a factorization ring R . Choose a countable subset $S = \{P_1 , P_2 , \ldots\}$ of this collection, such that $P \notin S$. We form the following subsets

of S (using an algorithm based on Cantor's enumeration of the rationals):

$$S_1 = \{P_1, P_2, P_4, P_7, P_{11}, \dots\}$$
$$S_2 = \{P_3, P_5, P_8, P_{12}, \dots\}$$
$$S_3 = \{P_6, P_9, P_{13}, \dots\}$$
$$S_4 = \{P_{10}, P_{14}, \dots\}$$
$$\vdots \qquad \vdots$$

Clearly if $i \neq j$ then $S_i \cap S_j = \emptyset$ and $\bigcup_{i=\mathbb{N}} S_i = S$.

Consider the descending chains:

$$C_1 : P \supset PP_1 \supset PP_1P_2 \supset PP_1P_2P_4 \supset \dots$$
$$C_2 : P \supset PP_3 \supset PP_3P_5 \supset PP_3P_5P_8 \supset \dots$$
$$C_3 : P \supset PP_6 \supset PP_6P_9 \supset PP_6P_9P_{13} \supset \dots$$
$$\vdots \qquad \vdots$$

By the incomparability of the primes, each ideal of each chain is an irredundant represantation and so is unique up to order. Thus each chain is strictly decreasing, and the chains are pairwise disjoint. □

3. Let R be a regular self-injective ring. Then R is a factorization ring if and only if (0) is the intersection of a finite set of prime ideals.

<u>Proof</u>: (⇒) This implication is true by definition.

($\Leftarrow$) If $(0) = P_1 \cap \ldots \cap P_n$, P_i prime, then R is the subdirect product of the prime rings R/P_i $i = 1, \ldots, n$ by Proposition 2.2.7 . R/P_i is a regular factorization ring for each i (see Example 1, Section 2.3), so by Proposition 2.1.11, R is a regular factorization ring. □

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