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The Stochastic Approach
to
Future Shares

Rodrigo Arias

A Thesis
in
The Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

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Abstract

The Stochastic Approach
to
Future Shares
Rodrigo Arias

The traditional deterministic definitions of actuarial future values and Ramsay’s stochastic definitions are discussed. A modification of Ramsay’s definitions is proposed and an application to life insurance reserves is suggested.
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TO
MY WIFE,
MY DAUGHTERS
AND
MY PARENTS.
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Introduction

In life contingencies there are many situations in which actuaries face the problem of determining the expected future share per survivor of an initial group, each of whose members makes payments into a fund. This problem is especially important in the case of long term life insurance and pension funds.

In the theory of stochastic life contingencies as presented by Bowers et. al. [1] and Gerber [3], definitions of future accumulated values are not stochastic.

Some authors have proposed different approaches. Martin-Löf [8] presents a model that takes into account stochastic interest rates and the random fluctuations in the collective. Frees [2] considers different ways to define random net premiums and life insurance reserves. In this work we are interested in the point of view of Ramsay [11], who proposes an alternative stochastic definition of future shares taking into account the size of the initial group and its random evolution. By two examples, he shows that when the size of the initial group is large enough, the deterministic traditional definitions are quite appropriate.

In Chapter one of this thesis we present both approaches, the deterministic and Ramsay’s stochastic formulations of future shares.
In Chapter two some properties of the first inverse moment of a positive binomial variate are presented and Ramsay’s definitions of future shares are discussed.

In Chapter three we present some modifications of Ramsay’s stochastic future values under the principle of equivalence. These new definitions are used in Chapter four to define retrospective random losses. The thesis ends with some numerical results and conclusions.
Chapter 1

Actuarial Future Shares

This chapter presents our notation and assumptions. Two problems are discussed to which both the deterministic solution and Ramsay's stochastic solution are presented.

1.1 Assumptions and notation

This section describes some assumptions and the notation from Bowers et. al. [1] that will be used throughout this work.

(i) Time is assumed discrete. $(x)$ means a person of age $x$, $\nu p_x$ the probability that $(x)$ survives $t$ years and $\nu q_x = 1 - \nu p_x$ that $(x)$ dies within $t$ years.

(ii) A group of $n = 1, 2, \ldots$ mutually independent lives is considered, all of age fixed, $x = 0, 1, \ldots, w - 1$ where $w$ is the span of the future lifetime of $(0)$.

(iii) The curtate future lifetime of $(x)$, usually denoted by $K$, is assumed to have the same distribution for all $(x)$ in the group. Its probability function is $k|q_x = \nu p_x q_{x+k}$, for $k = 0, 1, \ldots, w - x - 1$. When future lifetime is considered
continuous then it is denoted by $T$. Its probability density function is given by $g(t) = (t q_x)' = t p_x \mu_x t$, where $\mu_x = -s'(x)/s(x)$ is the force of mortality and $s$ is the survival function of (0) given by $s(x) = x p_0$.

(iv) The annual rate of interest is denoted by $i$ and is assumed to be fixed for all times. The discount factor is $v = 1/(1 + i)$, the rate of interest in advance is $d = 1 - v$ and the force of interest is $\delta = \ln(1 + i)$.

1.2 Two problems of life contingencies

The two life contingencies problems considered by Ramsay [11] are presented here. The traditional actuarial solutions to these two problems are described in the next section. In Section 1.4 Ramsay’s stochastic solutions are presented and in Chapters 2 and 3 they are discussed and reformulated.

**Problem 1**: Each member of a group of $n$ independent lives age $x$ makes a single payment of 1 into a fund at time 0. At time $t$ the accumulated value of the fund is divided equally among the survivors, if any. Assuming the fund earns interest at rate $i$ per annum, calculate the expected share of each survivor.

**Problem 2**: Each member of a group of $n$ independent lives age $x$ deposits an amount of 1 into a fund at the start of each year as long as he/she is alive. At time $t$, the accumulated value of the fund is divided equally among the survivors, if any. Assuming the fund earns interest at rate $i$ per annum, calculate the expected share of each survivor.
1.3 The traditional deterministic approach

There is no book on life contingencies that presents stochastic definitions of future values or future shares. In the modern stochastic theory of life contingencies solutions to Problems 1 and 2 are found by considering a single policy $(x)$ and its future random lifetime. In this way, present values are defined stochastically, but future values are defined only as functions of actuarial present values and so they are deterministic (e.g. Bowers et. al. [1]).

**Definition 1.1 Pure endowment.** Consider a unit payment due at the end of $t$ years provided that $(x)$ survives. This benefit is called a $t$-year pure endowment of 1 with respect to $(x)$. Its present value is defined as the random variable:

$$ Z = \begin{cases} 
0 & \text{if } T \leq t, \\
\nu^T & \text{if } T > t. 
\end{cases} $$

The expected value of $Z$, $E[Z]$, is called the net single premium (NSP) or the actuarial present value of the benefit. It is given by:

$$ E[Z] = A_{x+t} = tE_x = \nu^t \cdot t p_x. $$

**Definition 1.2 Actuarial accumulated value.** The actuarial accumulated value at the end of $t$ years of 1 contributed at age $x$ is defined as the amount $t s_x$ such that its actuarial present value is 1. That is, $(tE_x)_t s_x = 1$, or

$$ t s_x = \frac{1}{tE_x} = \frac{(1 + i)^t}{t p_x}. $$

Let $l_x$ be the initial size of the group. Since the number of survivors after $t$ years, denoted by $L(x+t)$, is a binomial random variable with parameters $(l_x, t p_x)$ then the
expected number of survivors at time $t$ is $E[L(x + t)] = l_x t p_x$. Therefore,

\[
\begin{align*}
 ts_x &= \frac{l_x(1 + t)^t}{E[L(x + t)]}, \\
 &= \frac{\text{Future accumulated value of the } l_x \text{ payments of 1 at time 0}}{\text{Future expected number of survivors from the initial group } l_x}, \\
 &= \text{Expected future share per survivor.}
\end{align*}
\]

In this sense $ts_x$ becomes the traditional deterministic solution to Problem 1. That is, the expected future share per survivor at time $t$ is given by $ts_x$ (see Bowers et. al. [1], pp. 101, 140 and 146).

**Definition 1.3** *Temporary life annuity.* A $t$-year temporary life annuity of 1 payable at the beginning of each year while $(x)$ survives has a present value defined as:

\[
Y = \begin{cases} 
\hat{a}_{K+1} & \text{if } K < t, \\
\hat{a}_{t} & \text{if } K \geq t.
\end{cases} \tag{1.1}
\]

The actuarial present value or NSP of this annuity, denoted by $\bar{a}_{x;\overline{t}}$, is given by:

\[
E[Y] = \bar{a}_{x;\overline{t}} = \sum_{k=0}^{t-1} k E_x = \sum_{k=0}^{t-1} v^k k p_x. \tag{1.2}
\]

**Definition 1.4** *Term life insurance.* The present value of a $t$-year term life insurance of 1 payable at the end of the year of death of $(x)$ is defined as:

\[
Z = \begin{cases} 
v^{K+1} & \text{if } K < t, \\
0 & \text{if } K \geq t.
\end{cases} \tag{1.3}
\]

The NSP of this benefit, denoted by $A_{x;\overline{t}}$, is given by:

\[
E[Z] = A_{x;\overline{t}} = \sum_{k=0}^{t-1} v^{k+1} k p_x q_x+k. \tag{1.4}
\]

**Definition 1.5** *Endowment life insurance.* The present value of a $t$-year endowment life insurance of 1 payable at the end of the year of death of $(x)$ or at $t$, whichever occurs first, is defined as:

\[
Z = \begin{cases} 
v^{K+1} & \text{if } K < t, \\
v^t & \text{if } K \geq t.
\end{cases} \tag{1.5}
\]
The NSP, equals to $E(Z)$, is denoted by $A_{x: \bar{r}}$, and is given by:

$$E[Z] = A_{x: \bar{r}} = A_{x: \bar{r}} + iE_x.$$  

(1.6)

The variance of $Z$ is easily seen to be:

$$V(Z) = \alpha^2 A_{x: \bar{r}} - A_{x: \bar{r}}^2,$$  

(1.7)

where $\alpha A_{x: \bar{r}}$ is the NSP of a $t$-year endowment insurance computed at a force of interest of $2\delta$.

Since (1.1) can be written as $Y = (1 - Z)/d$, where $Z$ is given by (1.5), then using (1.7) the variance of this random variable is found to be:

$$V[Y] = \frac{\alpha^2 A_{x: \bar{r}} - A_{x: \bar{r}}^2}{d^2}.$$  

(1.8)

**Definition 1.6** Actuarial accumulated value of a temporary life annuity. The actuarial accumulated value at the end of the term of a $t$-year temporary life annuity of 1 per annum, payable while $(x)$ survives, is denoted by $\bar{s}_{x: \bar{r}}$ and is defined as the amount such that its actuarial present value equals $\bar{a}_{x: \bar{r}}$. In other words, $\bar{s}_{x: \bar{r}} tE_x = \bar{a}_{x: \bar{r}}$, or

$$\bar{s}_{x: \bar{r}} = \frac{\bar{a}_{x: \bar{r}}}{tE_x} = (1 + i)^t \frac{\bar{a}_{x: \bar{r}}}{t \bar{p}_x}.$$  

Assuming that mortality occurs according to the life table then $t \bar{p}_x = l_{x+t}/l_x$. Using (1.2) we get:

$$\bar{s}_{x: \bar{r}} = \frac{\sum_{k=0}^{t-1} l_{x+k} (1 + i)^{t-k}}{l_{x+t}},$$

$$= \frac{\text{Actuarial accumulated value of all payments}}{\text{Expected number of survivors}},$$

$$= \frac{\text{Expected future share per survivor.}}{\text{}}$$

This is why $\bar{s}_{x: \bar{r}}$ is seen as the traditional deterministic solution of Problem 2, i.e., the future share per survivor at time $t$ is given by $\bar{s}_{x: \bar{r}}$ (see Bowers et. al. [1], pp. 144-146).
1.4 Ramsay’s stochastic approach

We present here Ramsay’s stochastic definitions of future shares for Problems 1 and 2 and some preliminary results needed to this (c.f. [11]).

1.4.1 Preliminary results

Definition 1.7 The positive binomial distribution. Let \( X \) be a binomial random variable with parameters \((n, p)\), i.e., its probability function is given by

\[
 f_X(m) = P_{nm}(p) = \binom{n}{m} p^m q^{n-m}, \quad m = 0, 1, \ldots, n. \tag{1.9}
\]

Then a random variable \( Y \) is called a positive binomial distribution with parameters \((n, p)\) if \( f_Y(m) = f_{X|X>0}(m) \), i.e.,

\[
 f_Y(m) = \frac{P_{nm}(p)}{1 - q^n}, \quad m = 1, 2, \ldots, n.
\]

In this case its \( k^{th} \) inverse moment is denoted by \( b_k(n, p) = E[Y^{-k}] \) (c.f. [6]).

Definition 1.8 For \( n \geq 1 \) and \( 0 \leq p \leq 1 \) we define the polynomials:

\[
 B_n(p) = \sum_{m=1}^{n} \frac{n}{m} P_{nm}(p), \tag{1.10}
\]

\[
 Q_n(p) = \sum_{m=1}^{n} \frac{n^2}{m^2} P_{nm}(p). \tag{1.11}
\]

The polynomials \( B_n \) and \( Q_n \) have various applications in actuarial mathematics and statistics. In a more general interpretation they can be identified, respectively, as the Bernstein polynomials of the functions

\[
 b(x) = \begin{cases} 
 \frac{1}{x} & \text{if } 0 < x \leq 1, \\
 0 & \text{if } x = 0,
\end{cases} \quad \text{and} \quad q(x) = \begin{cases} 
 \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\
 0 & \text{if } x = 0.
\end{cases} \tag{1.12}
\]

As a second application, let \( N \) be a binomial random variable with parameters \((n, p)\) and define \( X \) as

\[
 X = \begin{cases} 
 \frac{n}{N} & \text{if } N = 1, 2, \ldots, n, \\
 0 & \text{if } N = 0.
\end{cases} \tag{1.13}
\]
Then
\[ B_n(p) = E[X], \quad Q_n(p) = E[X^2] \quad \text{and} \quad V[X] = Q_n(p) - B_n^2(p). \quad (1.14) \]

A third application of the polynomials \( B_n \) and \( Q_n \) is as a representation of \( b_1(n, p) \) and \( b_2(n, p) \), respectively, up to a factor. In fact, we see that
\[ B_n(p) = n(1 - q^n)b_1(n, p), \quad (1.15) \]
\[ Q_n(p) = n^2(1 - q^n)b_2(n, p). \quad (1.16) \]

Using (1.10) and (1.11) one can compute \( B_n(p) \) and \( Q_n(p) \) directly for any finite \( n \). Unfortunately, for large \( n \), the binomial probabilities are cumbersome to find. In view of this difficulty, recursive formulas are used to accelerate the computations. Alternatively, some approximations are also used.

Grab and Savage [5] provided the recursive equation:
\[ b_1(n + 1, p) = \frac{q(1 - q^n)}{1 - q^{n+1}}b_1(n, p) + \frac{1}{n + 1}, \quad n = 0, 1, \ldots \quad (1.17) \]
with \( b_1(0, p) = 0 \). Hence (1.15), together with (1.17), can be used to compute \( B_n(p) \) recursively.

Govindarajulu [4] provided the following recursive relationship:
\[ b_2(n + 1, p) = \frac{q(1 - q^n)}{1 - q^{n+1}} \left[ \frac{b_1(n, p)}{n + 1} + b_2(n, p) \right] + \frac{1}{(n + 1)^2}, \quad n = 0, 1 \ldots \quad (1.18) \]
with \( b_2(0, p) = 0 \). Then (1.17) and (1.18), together with (1.15) and (1.16), can be used to compute \( B_n(p) \) and \( Q_n(p) \) jointly and recursively.

To approximate \( B_n(p) \) and \( Q_n(p) \) the following formula from Mendenhall and Lehman [10] can be used:
\[ b_k(n, p) \approx \frac{1}{n^k} \prod_{r=1}^{k} \frac{a + b - r}{a - r}, \quad (1.19) \]
where \( a = (n - 1)p \) and \( b = (n - 1)q \). Other approximations to \( b_k(n, p) \) are found in [12] and [13].
When \( k = 1 \) the expression in (1.19) reduces to \( b_1(n, p) \approx \frac{(n-2)}{n(a-1)} \) for \( p > \frac{1}{n-1} \). Replacing this in (1.15) yields

\[
B_n(p) \approx \frac{(1 - q^n)(n - 2)}{a - 1}.
\]

(1.20)

For large values of \( n \) this approximation becomes fairly accurate and may be used to replace (1.10) and (1.17).

Replacing (1.19) in (1.16) when \( k = 2 \) we obtain

\[
Q_n(p) \approx \frac{(1 - q^n)(n - 2)(n - 3)}{(a - 1)(a - 2)}.
\]

For large values of \( n \) this approximation can be used to estimate \( Q_n(p) \) instead of equations (1.11) and (1.18).

1.4.2 Stochastic future shares

Ramsay [11] defines the random future shares for Problems 1 and 2 as follows:

**Definition 1.9 Random future shares.** Let \( S_1 \) and \( S_2 \) be the actual share accumulated in the fund for each survivor at time \( t \), in Problems 1 and 2, respectively. If there are no survivors then \( S_1 = 0 \) and \( S_2 = 0 \). Therefore,

\[
S_1 = \begin{cases} 
\frac{n(1 + i)^t}{N} & \text{if } N = 1, 2, \ldots, n, \\
0 & \text{if } N = 0,
\end{cases}
\]

(1.21)

and

\[
S_2 = \begin{cases} 
\frac{(N \hat{a}_{\overline{1}+t} + \sum_{i=1}^{D} \hat{a}_{K_i+1} (1 + i)^t)}{N} & \text{if } N = 1, 2, \ldots, n, \\
0 & \text{if } N = 0,
\end{cases}
\]

(1.22)

where

\( N = \) number of survivors at age \( x + t \) from the group of \( n \) lives age \( x \),

\( D = n - N \) is the number of deaths between ages \( x \) and \( x + t \),

\( K_i = \) curtate future life time for the \( i^{th} \) life who died.
By assumption, the $n$ lives are mutually independent so $N$ has a binomial distribution with parameters $(n, \tau p_x)$. From (1.21) and (1.22), both $E[S_1]$ and $E[S_2]$ depend on $b_1(n, \tau p_x)$. It should be pointed out that, in addition to depending on $N$, $S_2$ also depends on the random variables $K_i$, for $i = 1, 2, \ldots, D$, so expectations must be taken over $K_i$ as well.

**Proposition 1.1 (Ramsay [11])** The expectations of $S_1$ and $S_2$ defined by (1.21) and (1.22) are given by

$$E[S_1] = (1 + \tau)^t B_n(\tau p_x), \quad (1.23)$$

$$E[S_2] = \tilde{s}_\eta (1 - \tau q_x^n) + [B_n(\tau p_x) - (1 - \tau q_x^n)] \left( \tilde{s}_x \tilde{\eta} - \tilde{s}_\eta \right) \frac{\tau p_x}{\tau q_x}, \quad (1.24)$$

where $B_n(\tau p_x)$ is defined in (1.10).

**Proof.** The proof of (1.23) is immediate. For (1.24) using (1.9) we have:

$$E[S_2] = E[E(S_2|N)] = \sum_{m=1}^{n} E[S_2|N = m] P_{nm}(\tau p_x),$$

$$= (1 + \tau)^t \sum_{m=1}^{n} \left[ m \tilde{a}_\eta + \sum_{i=1}^{n-m} \frac{E(\tilde{a}_{K_{i+1}}|K_i < t)}{m} \right] P_{nm}(\tau p_x).$$

But

$$E[\tilde{a}_{K_{i+1}}|K_i < t] = \sum_{k=0}^{t-1} \frac{k! q_x^{k+1}}{\tau q_x} = \frac{\tilde{a}_x \tilde{\eta} - \tau p_x \tilde{a}_\eta}{\tau q_x}. \quad (1.25)$$

Then

$$E[S_2] = (1 + \tau)^t \sum_{m=1}^{n} \left[ m \tilde{a}_\eta + \frac{(n - m) (\tilde{a}_x \tilde{\eta} - \tau p_x \tilde{a}_\eta)}{m} / \tau q_x \right] P_{nm}(\tau p_x),$$

$$= (1 + \tau)^t \sum_{m=1}^{n} \left[ \tilde{a}_\eta + \frac{(n - m) (\tilde{a}_x \tilde{\eta} - \tau p_x \tilde{a}_\eta)}{m} / \tau q_x \right] P_{nm}(\tau p_x),$$

$$= \sum_{m=1}^{n} \left[ \tilde{s}_\eta + \frac{(n - m) \left( \tilde{s}_x \tilde{\eta} - \tilde{s}_\eta \right)}{m} \frac{\tau p_x}{\tau q_x} \right] P_{nm}(\tau p_x),$$

$$= \tilde{s}_\eta (1 - \tau q_x^n) + [B_n(\tau p_x) - (1 - \tau q_x^n)] \left( \tilde{s}_x \tilde{\eta} - \tilde{s}_\eta \right) \frac{\tau p_x}{\tau q_x}. \quad \diamond$$
To compare both, stochastic and deterministic formulas of future shares, Ramsay defines the following ratios:

$$\rho_1 = \frac{E[S_1]}{t s_z} = B_n(t p_x) t p_x, \quad (1.26)$$

$$\rho_2 = \frac{E[S_2]}{\bar{S}_z n}. \quad (1.27)$$

For numerical evaluation of $\rho_1$ and $\rho_2$ he makes the following assumptions:

(i) The same mortality law used by Bowers et. al. [1], p. 78, i.e.,

$$\mu_x = A + B c^x, \quad (1.28)$$

where $A = 0.0007$, $B = 0.00005$ and $c = 10^{0.04}$.

(ii) Ages $x = 20, 30, 40, 50, 60$ and $x + t = 65$, so that lives must survive to age 65 in order to receive survival benefits.

(iii) A constant annual interest rate $i = 6\%$, since $\rho_1$ does not depend on $i$ and $\rho_2$ appears to be fairly insensitive to interest rates changes.

(iv) Values of $n = 5, 10, 20, 30, 40, 50, 100, 200, 300, 400, 500$.

Ramsay computes the exact values of $B_n(t p_x)$ and also uses (1.20). Here we present only the exact results.

Effectively, as Ramsay points out, for large values of $n$, say $n > 20$, it appears that there is no need to use exact formulas since, in those cases, the traditional actuarial approach provides a very good approximation [see Tables 1.1 and 1.2]. But it is clear that for small values of $n$ the traditional approach and Ramsay's stochastic approach are quite different.

Numerical results in Tables 1.1 and 1.2 seem to indicate that $\rho_1 > 1$ and $\rho_2 > 1$. Ramsay argues for $\rho_1$ that the result is due to Jensen's inequality [see Corollary 2.1]. For $\rho_2$ he simply points out that this makes sense from the point of view of risk and return, but provides no proof of the inequality in general. In both cases the inequalities turn out to be true only for large values of $n_t p_x$, but not in general. This point is further discussed in Chapter 2.
Table 1.1: $\rho_1$ when $(x)$ survives to age 65

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x = 20$</th>
<th>$x = 30$</th>
<th>$x = 40$</th>
<th>$x = 50$</th>
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<tr>
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<td>1.00301</td>
</tr>
<tr>
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<td>1.00679</td>
<td>1.00613</td>
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<tr>
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<td>1.00265</td>
<td>1.00240</td>
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Table 1.2: $\rho_2$ when $(x)$ survives to age 65

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Chapter 2

Further Results on Ramsay’s Definitions

Some additional properties of the polynomials $B_n$ and $Q_n$, given by (1.10) and (1.11), are shown here. Ramsay’s definitions of random future shares presented in Chapter 1 are also further discussed.

2.1 Preliminary technical results

Lemma 2.1 (Jensen’s inequality) Let $X$ be a random variable in $(a,b)$ with $\mu = E[X] < \infty$. Let $f$ be a function such that $f''(x) > 0$, for all $x \in (a,b)$. Then:

$$E[f(X)] \geq f(\mu).$$

Proof. For each $x \in (a,b)$ there exists a $\zeta_x$ between $\mu$ and $x$ such that

$$f(x) = f(\mu) + f'(\mu)(x - \mu) + \frac{1}{2}f''(\zeta_x)(x - \mu)^2 \geq f(\mu) + f'(\mu)(x - \mu).$$

Then $f(X) \geq f(\mu) + f'(\mu)(X - \mu)$. Thus, $E[f(X)] \geq f(\mu) + 0 = f(\mu).$ \hfill \Box

Corollary 2.1 Let $X$ be a positive random variable such that $E[X] < \infty$. Then:

$$E\left[\frac{1}{X}\right] \geq \frac{1}{E[X]}.$$

Proof. Apply Lemma 2.1 to $f(x) = 1/x$ for $x > 0$. \hfill \Box
Lemma 2.2 Let \( X \) be a random variable such that \( X \geq 0, \ 0 < \rho = Pr[X = 0] < 1 \) and \( V[X] < \infty \). For any pair of real numbers \( a \) and \( b \) define the random variable

\[
Y = \begin{cases} 
X + (a - b)I_{X > 0} + b, & \text{if } X > 0, \\
b & \text{if } X = 0,
\end{cases}
\]

where \( I \) is an indicator function. Then \( V[Y] \), as a function of \( a \) and \( b \), is minimal when \( a = b - E[X]/(1 - \rho) \). In this case \( E[Y] = b \) and \( V[Y] = V[X] - E[X]^2 \rho/(1 - \rho) \).

\textbf{Proof.} The variance of \( Y \) is given by

\[
V[Y] = V[X] + V[(a - b)I_{X > 0}] + 2Cov[X, (a - b)I_{X > 0}],
\]

\[
= V[X] + \rho(1 - \rho)(a - b)^2 + 2\rho E[X](a - b). \tag{2.1}
\]

Then \( V[Y] \) is seen to be a second degree polynomial in \((a - b)\), with coefficient \( \rho(1 - \rho) > 0 \). This means that \( V[Y] \) reaches a minimum when \( a - b = -2\rho E[X]/[2\rho(1 - \rho)] = -E[X]/(1 - \rho) \), i.e., \( a = b - E[X]/(1 - \rho) \). Replacing this expression in (2.1) we get \( V[Y] = V[X] - E[X]^2 \rho/(1 - \rho) \). Finally \( E[Y] = E[X] + (a - b)(1 - \rho) + b = E[X] - E[X] + b = b \).  

\( \Diamond \)

Lemma 2.3 Let \( K \) be the curtate future life time of \((x)\) and \( t \) an integer. Then

\[
a_1 = E[\hat{a}_{K+1}^2|K < t] = \frac{\hat{a}_{x:t} - tpx \hat{a}_{t}^2}{tq_x}, \tag{2.2}
\]

\[
v_1 = V[\hat{a}_{K+1}^2|K < t] = \frac{1}{tq_x} \left[ \frac{2A_{x:t} - A_{x:t}^2}{d^2} - (\hat{a}_{x:t} - \hat{a}_{t}^2)^2 \frac{tp_x}{tq_x} \right]. \tag{2.3}
\]

\textbf{Proof.} (2.2) was proved in (1.25). To prove (2.3) we consider the random variable \( Y \) given in (1.1). We see that

\[
\sum_{k=0}^{t-1} k[q_x \hat{a}_{k+1}^2] = E[Y^2] - tpx \hat{a}_{t}^2 = V[Y] + E[Y]^2 - tpx \hat{a}_{t}^2,
\]

\[
= \frac{2A_{x:t} - A_{x:t}^2}{d^2} + \hat{a}_{x:t}^2 - tpx \hat{a}_{t}^2, \ 	ext{from (1.8) and (1.2)}.\]
Therefore,
\[
E[\bar{a}_{K+1}^2|K < t] = \sum_{k=0}^{t-1} \lambda[k]q_x \bar{a}_{k+1}^2 / t q_x = \left[\frac{2A_{x;\bar{a}} - A_{x;\bar{a}^2}}{d^2} + \bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2 \right] / t q_x .
\]
This and (2.2) yield
\[
v_1 = E[\bar{a}_{K+1}^2|K < t] - E[\bar{a}_{K+1}^2|K < t]^2 ,
\]
\[
= \left[\frac{2A_{x;\bar{a}} - A_{x;\bar{a}^2}}{d^2} + \bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2 \right] / t q_x - \left(\frac{\bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2}{t q_x}\right)^2 ,
\]
\[
= \frac{1}{t q_x} \left[\frac{2A_{x;\bar{a}} - A_{x;\bar{a}^2}}{d^2} + \bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2 - \left(\bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2\right)/t q_x \right] ,
\]
\[
= \frac{1}{t q_x} \left[\frac{2A_{x;\bar{a}} - A_{x;\bar{a}^2}}{d^2} - \left(\bar{a}_{x;\bar{a}^2} - t p_x \bar{a}_{x;\bar{a}^2}^2\right)/t q_x \right] . \quad \diamondsuit
\]

Lemma 2.4 Let 0 \leq p \leq 1, n \geq 1 and \( P_{nm}(p) \) as in (1.9). Then
\[
0 \leq z \leq \frac{3}{2} \sqrt{np(1-p)} \Rightarrow \sum_{|m-np| \geq 2z \sqrt{np(1-p)}} P_{nm}(p) \leq 2e^{-z^2} .
\]

Proof. It can be found in Lorentz [7], pp. 18-19. \quad \diamondsuit

Corollary 2.2 Let 0 < a < b < 1 and 0 < \delta \leq \min\{a(1-a), b(1-b)\}. Then
\[
\sum_{\frac{|m-np|}{\delta}} P_{nm}(p) \leq 2e^{-n\delta^2} , \quad \text{for all } p \in [a, b] .
\]

Proof. For any \( p \in [a, b] \) the inequalities \( \delta \leq 3p(1-p) \) and \( 4p(1-p) \leq 1 \) hold. Now
\[
\left|\frac{m}{n} - p \right| \geq \delta \iff |m - np| \geq \delta n = \frac{2\delta \sqrt{n}}{2\sqrt{p(1-p)}} \sqrt{np(1-p)} = 2z \sqrt{np(1-p)} ,
\]
where \( z = \frac{\delta \sqrt{n}}{2\sqrt{p(1-p)}} \). Since \( 0 \leq z \leq \frac{3p(1-p)\sqrt{n}}{2\sqrt{p(1-p)}} = \frac{3}{2} \sqrt{np(1-p)} \), then Lemma 2.4 yields
\[
\sum_{\frac{|m-np|}{\delta} \geq 6} P_{nm}(p) = \sum_{|m-np| \geq 2z \sqrt{np(1-p)}} P_{nm}(p) ,
\]
\[
\leq 2e^{-z^2} = 2e^{\frac{n\delta^2}{4p(1-p)}} \leq 2e^{-n\delta^2} . \quad \diamondsuit
\]
Lemma 2.5 Let \( n \geq 2 \) be an integer.

(i) If \( 0 < p \leq 1 \) then \( (1 + np)(1 - p)^n < 1 \).

(ii) If \( n \geq 3 \) and \( 2/n \leq p \leq 1 \) then \( [2(n - 1)p + 1](1 - p)^{n-1} < 1 \).

(iii) \( r > 0 \) and \( 0 \leq p \leq 1 \Rightarrow p^r(1-p)^n \leq \left(\frac{r}{n + r}\right)^r \left(1 - \frac{r}{n + r}\right)^n < \left(\frac{r}{n}\right)^r e^{-r} \).

(iv) If \( 0 < p \leq 1 \) then \( \left[\frac{n(n-1)}{2}p^2 + (n-1)p + 1\right](1-p)^{n-1} < 1 \).

(v) \( 0 < z < 1 \Rightarrow \frac{z}{\sqrt{1-z}} + \ln(1-z) > 0 \). Also, for \( n \geq 1 \) and \( 0 < x < n \):

\[
\frac{x}{n-x} + \ln\left(1 - \frac{x}{n}\right) > 0. \tag{2.4}
\]

(vi) For \( 0 < q < 1 \), \( p = 1 - q \) and \( k = 1, 2, \ldots \),

\[
q^{-k} \geq 1 + k \left(\frac{p}{q}\right) + \frac{1}{2} k(k-1) \left(\frac{p}{q}\right)^2 \geq 1 + k \left(\frac{p}{q}\right). \tag{2.5}
\]

Proof. In (i)-(iv) we may assume if necessary that \( p < 1 \) since if \( p = 1 \) all these inequalities are trivially true.

(i) If \( f(p) = (1 + np)(1 - p)^n \) for \( 0 \leq p < 1 \), then \( f'(p) < 0 \). Hence \( f(p) < f(0) = 1 \).

(ii) Let \( f(p) = [2(n - 1)p + 1](1 - p)^{n-1} \). Then \( f'(p) < 0 \) for \( \frac{1}{2n} < p \leq 1 \). Since \( 1/(2n) < 2/n \) then \( f(p) \leq f(2/n) = \left(\frac{5n-4}{n-2}\right)(1 - \frac{2}{n})^n \), for \( 2/n \leq p \leq 1 \). From this we see that \( f(p) < 1 \) for \( n = 3, 4 \). If \( n \geq 5 \) then \( f(p) < \left(\frac{5(n-4)}{5-2}\right) e^{-2} < 1 \).

(iii) The first inequality is due to the fact that the function \( u(p) = p^r(1-p)^n \), for \( 0 \leq p \leq 1 \), has a global maximum at \( p = r/(n + r) \). The second inequality follows since the function \( h(x) = (1 - 1/x)^2 \) is increasing for \( x \geq 1 \) with \( \lim_{x \to \infty} h(x) = e^{-1} \).

(iv) Let \( h(p) = \left[\frac{n(n-1)}{2}p^2 + (n-1)p + 1\right](1-p)^{n-1} \) for \( 0 < p < 1 \). Then

\[
h'(p) = (n-1)(1-p)^{n-2} \left\{ (np+1)(1-p) - \left[\frac{n(n-1)p^2}{2} + (n-1)p + 1\right]\right\},
\]

\[
= (n-1)(1-p)^{n-2} \left[ np + 1 - np^2 - p - \frac{n(n-1)p^2}{2} - np + p - 1\right],
\]

\[
= -(n-1)(1-p)^{n-2} \left( n + \frac{n(n-1)}{2}\right) p^2 < 0.
\]
This yields $h(p) < h(0) = 1$ for $0 < p \leq 1$.

(v) The function $f(z) = \frac{x}{\sqrt{1-z}} + \ln(1-z)$ for $0 \leq z < 1$ is such that $f'(z) > 0$ for $0 < z < 1$. Therefore, $f(z) > f(0) = 0$. This proves the first inequality. Using this with $z = x/n$, and since $1 - z < \sqrt{1-z}$ for $0 < z < 1$, we conclude that (2.4) is also true (for more general results see [9]; p. 272; theorem 3.6.15).

(vi) It follows from the identity $q^{-k} = (1 + p/q)^k = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \\ \end{array} \right) \left( \frac{x}{q} \right)^j$.

\[ \Diamond \]

**Corollary 2.3**

(i) $K_n(x) = \left(1 - \frac{x}{n}\right)^{1/x}$ is decreasing in $[0, n]$ for $n \geq 1$.

(ii) $F_n(x) = \left[ \frac{(n-x)(n-x-1)}{n(n-1)} \right]^{1/x}$ is decreasing in $[0, n-1]$ for $n \geq 2$.

**Proof.** (i) If $R_n(x) = \ln[K_n(x)]$ for $0 < x < n$, then from (2.4)

\[ R'_n(x) = \frac{1-x}{x} + \frac{\ln (1-x)}{x^2} < 0. \]

(ii) It is true since $F_n(x) = K_n(x)K_{n-1}(x)$.

\[ \Diamond \]

2.2 Some properties of the $B_n$ and $Q_n$ polynomials

In this section we present some properties of the polynomials $B_n$ and $Q_n$ defined in (1.10) and (1.11). They are related with the first two inverse moments of a positive binomial variate according to (1.15) and (1.16).

2.2.1 Recursive equations for $B_n$ and $Q_n$

**Proposition 2.1** The polynomials $B_n$, given by (1.10), satisfy the recursive equation:

\[ B_{n+1}(p) = q \left( \frac{n+1}{n} \right) B_n(p) + 1 - q^{n+1}, \quad n = 1, 2, \ldots \]

\[ (2.6) \]

**Proof.** From (1.10) and the identity $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$, for $n \geq 1$, $m = 1, \ldots, n$:

\[ B_{n+1}(p) = \sum_{m=1}^{n+1} \frac{(n+1)}{m} \binom{n+1}{m} p^m q^{n+1-m}, \]

\[ 18 \]
\[\begin{align*}
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)}{m} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)}{m} \left[ \binom{n}{m} + \binom{n}{m-1} \right] p^m q^{n+1-m}, \\
&= p^{n+1} + q \left( \frac{n+1}{n} \right) \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} p^m q^{n-m} + \sum_{m=1}^{n} \frac{(n+1)}{m} \binom{n}{m-1} p^m q^{n+1-m}, \\
&= q \left( \frac{n+1}{n} \right) B_n(p) + \sum_{m=1}^{n+1} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= q \left( \frac{n+1}{n} \right) B_n(p) + 1 - q^{n+1}. \quad \diamond
\end{align*}\]

Note that this proposition also follows from (1.15) when used with (1.17). The formula in (2.6), together with \(B_1(p) = p\), can be used to compute \(B_n(p)\) recursively.

**Proposition 2.2** The polynomials \(Q_n\), given by (1.11), satisfy the recursive equation:

\[Q_{n+1} = q \left( \frac{n+1}{n} \right)^2 Q_n + B_{n+1}, \quad n = 1, 2, ...\]

where \(B_n\) is given in (1.10).

**Proof.** Using the relation \(\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}\) we have

\[\begin{align*}
Q_{n+1}(p) &= \sum_{m=1}^{n+1} \frac{(n+1)^2}{m^2} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \left[ \binom{n}{m} + \binom{n}{m-1} \right] p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \binom{n}{m} p^m q^{n+1-m} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \binom{n}{m-1} p^m q^{n+1-m}, \\
&= q \left( \frac{n+1}{n} \right)^2 \sum_{m=1}^{n} \frac{n^2}{m^2} \binom{n}{m} p^m q^{n-m} + \sum_{m=1}^{n} \frac{n+1}{m} \binom{n+1}{m} p^m q^{n+1-m} + p^{n+1}, \\
&= q \left( \frac{n+1}{n} \right)^2 Q_n(p) + \sum_{m=1}^{n+1} \frac{n+1}{m} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= q \left( \frac{n+1}{n} \right)^2 Q_n(p) + B_{n+1}(p). \quad \diamond
\end{align*}\]

This proposition also follows from (1.16) when used with (1.18). It together with (2.6) and \(B_1(p) = Q_1(p) = p\), can be used to compute \(B_n(p)\) and \(Q_n(p)\) jointly and recursively.
2.2.2 An exact formula for $B_n$

Proposition 2.3 The polynomials $B_n$ can be written as powers of $q = 1 - p$:

$$B_n(p) = \sum_{m=0}^{n-1} \frac{n}{n-m} q^m - q^n \sum_{m=1}^{n} \frac{n}{m}.$$ 

Proof. Using $p^m = (1 - q)^m = \sum_{k=0}^{m} \binom{m}{k} (-q)^k$ in (1.10) we have

$$B_n(p) = \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} (1 - q)^m q^{n-m},$$

$$= \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} \left[ \sum_{k=0}^{m} \binom{m}{k} (-q)^k \right] q^{n-m},$$

$$= \sum_{m=1}^{n} \sum_{k=0}^{m} \frac{n}{m} \binom{n}{m} \binom{m}{k} (-1)^k q^{n-m-k}. $$

Rearranging the sums according to terms with common power of $q$ yields

$$B_n(p) = \sum_{m=1}^{n-1} \frac{n}{n-m} \left[ \sum_{k=0}^{m} \binom{m}{k} (-1)^k \right] q^m + \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} (-1)^k q^n.$$

We need to show that $a_m = \frac{n}{n-m}$ and $a_n = -\sum_{m=1}^{n} \frac{n}{m}$. To this

$$a_m = n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{n-m+k},$$

$$= (-1)^{n-m} n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{n-m+k}}{n-m+k},$$

$$= (-1)^{n-m} n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \int_0^{-1} t^{n-m+k-1} dt,$$

$$= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} \left[ \sum_{k=0}^{m} \binom{m}{k} \frac{t^{n-m+k-1}}{k} \right] dt,$$

$$= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} t^{n-m-1} \sum_{k=0}^{m} \binom{m}{k} t^k dt,$$

$$= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} t^{n-m-1} (1 + t)^m dt,$$

$$= (-1)^{2(n-m)} n \binom{n}{m} \int_0^{1} z^{n-m-1} (1 - z)^m dz, \text{ change of variable } z = -t,$$

$$= n \binom{n}{m} \frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)} \int_0^{1} \frac{\Gamma(n+1)}{\Gamma(n-m)\Gamma(m+1)} z^{n-m-1} (1 - z)^{m+1-1} dz,$$
\[ a_n = n \binom{n}{m} \frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)} (1), \text{ since the latter has a beta distribution,} \]
\[ = \frac{n}{n-m}, \text{ because } \Gamma(r+1) = r! \text{ for all integer } r. \]

For \( n \geq 2 \) we have:

\[
a_n = \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n}{k+1} (-1)^{k+1},
\]
\[
= \sum_{k=0}^{n-2} \frac{n}{k+1} \left[ \binom{n-1}{k+1} + \binom{n-1}{k} \right] (-1)^{k+1} + (-1)^n,
\]
\[
= \sum_{k=0}^{n-2} \frac{n}{k+1} \binom{n-1}{k+1} (-1)^{k+1} + \sum_{k=0}^{n-2} \frac{n}{k+1} \binom{n-1}{k} (-1)^{k+1} + (-1)^n,
\]
\[
= \sum_{k=1}^{n-1} \frac{n}{k} \binom{n-1}{k} (-1)^k - \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} (-1)^k + (-1)^{n-1} + (-1)^n,
\]
\[
= \frac{n}{n-1} \sum_{k=1}^{n-1} \frac{n-1}{k} \binom{n-1}{k} (-1)^k + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1},
\]
\[
= \left( \frac{n}{n-1} \right) a_{n-1} + (1-1)^n - 1,
\]
\[
= \left( \frac{n}{n-1} \right) a_{n-1} - 1.
\]

Hence, \( a_n = \left( \frac{n}{n-1} \right) a_{n-1} - 1 \). This recursive relation and \( a_1 = -1 \) imply \( a_n = -\sum_{k=1}^{n} \frac{n}{k} \), for all \( n \geq 1 \).

Formula in Proposition 2.3 is the counterpart of the one cited without proof by Stephan [12] for \( b_1(n,p) \). This also can be proved using (2.6) and used to compute \( B_n(p) \) easily.

### 2.2.3 Limit properties of the \( B_n \) and \( Q_n \) polynomials

As pointed out in Subsection 1.4.1, the polynomials \( B_n \) and \( Q_n \) are the Bernstein polynomials of the functions given in (1.12). But since these functions are not bounded at 0 then we cannot ensure that their Bernstein polynomials converge uniformly in \([0,1]\) nor in \([0,1]\). But they do in any closed interval \([a,1]\) as stated in the following propositions.
Proposition 2.4 For $0 < a < 1$, the polynomials $B_n$ given in (1.10) satisfy:

$$\lim_{n \to \infty} B_n(p) = \frac{1}{p}, \quad \text{uniformly in } [a, 1],$$

$$\lim_{n \to \infty} B_n(p) = \frac{1}{p}, \quad \text{point-wise in } ]0, 1]. \quad (2.7)$$

**Proof.** For simplicity suppose that $a < 1/2$. Let $1/2 < b < 1$ and $\epsilon > 0$. Denote by $c = a/2$ and $d = (1 + b)/2$. Since the function $f(p) = 1/p$ is uniformly continuous in $[c, d]$ there exists a $\delta_0 > 0$ such that

$$\left| \frac{1}{x} - \frac{1}{y} \right| < \epsilon/3, \quad \text{for all } x, y \in [c, d] \text{ and } |x - y| < \delta_0. \quad (2.8)$$

Let $\delta < \min\{\delta_0, c(1-c), d(1-d)\}$. This makes sure that if $p \in [a, b]$ and $|x - p| < \delta$ then $x, p \in [c, d]$ and (2.8) holds for $x$ and $p$. Also, since $\delta < \min\{a(1-a), b(1-b)\}$ then the inequality of Corollary 2.2 holds in $[a, b]$. Besides, there exists $n_0$ such that for all $n \geq n_0$

$$\frac{q^n}{p} < \epsilon/3, \quad \text{for all } p \in [a, b], \quad (2.9)$$

$$2ne^{-n\delta^2} < \epsilon/3. \quad (2.10)$$

Let $p \in [a, b]$ and $n \geq n_0$. It is clear that

$$\left| \frac{1}{m/n} - \frac{1}{p} \right| \leq n, \quad \text{for all } m = 1, 2, \ldots, n. \quad (2.11)$$

Then

$$\left| B_n(p) - \frac{1}{p} \right| = \left| B_n(p) - \frac{1}{p} + \frac{q^n}{p} - \frac{q^n}{p} \right|,$$

$$= \left| B_n(p) - \frac{1}{p} \left(1 - q^n\right) - \frac{q^n}{p} \right|,$$

$$\leq \left| B_n(p) - \frac{1}{p} \left(1 - q^n\right) \right| + \frac{q^n}{p}, \quad \text{by the triangular inequality,}$$

$$< \sum_{m=1}^{n} \frac{n}{m} P_{nm}(p) - \sum_{m=1}^{n} \frac{1}{p} P_{nm}(p) + \epsilon/3, \quad \text{from (1.10) and (2.9),}$$

$$= \left| \sum_{m=1}^{n} \left[ \frac{1}{m/n} - \frac{1}{p} \right] P_{nm}(p) \right| + \epsilon/3,$$
\[ \leq \sum_{m=1}^{n} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \epsilon/3, \text{ by the triangular inequality,} \]
\[ = \sum_{\|\frac{m}{n} - p\| < \delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \sum_{\|\frac{m}{n} - p\| \geq \delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \epsilon/3. \]

For the first term:
\[ \sum_{\|\frac{m}{n} - p\| < \delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) \leq (\epsilon/3) \sum_{\|\frac{m}{n} - p\| < \delta} P_{nm}(p), \text{ by (2.8),} \]
\[ \leq (\epsilon/3) \sum_{m=0}^{n} P_{nm}(p) = \epsilon/3. \]

For the second term:
\[ \sum_{\|\frac{m}{n} - p\| \geq \delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) \leq n \sum_{\|\frac{m}{n} - p\| \geq \delta} P_{nm}(p), \text{ by (2.11),} \]
\[ \leq 2ne^{-n\delta^2}, \text{ by Corollary 2.2,} \]
\[ < \epsilon/3, \text{ by (2.10).} \]

Therefore,
\[ \left| B_n(p) - \frac{1}{p} \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \text{ for all } p \in [a, b], n > n_0. \]

Hence, \( B_n(p) \to \frac{1}{p} \) uniformly in \([a, b]\). Since this holds for any \( b \) as close as possible to 1 and \( B_n(1) = 1 \) continuously, then it holds in \([a, 1]\) as well and so \( B_n(p) \to \frac{1}{p} \) uniformly in \([a, 1]\). This also implies (2.7). \( \diamond \)

We present here an alternative proof of Proposition 2.4.

**Proof.** Let \( 0 < \epsilon < a \). There exists \( n_1 \) such that for all \( n > n_1 \):
\[
\begin{align*}
(1 - a)^n &< \epsilon a, \\
\frac{n}{n} [\ln(n) + 1] (1 - a)^n &< \epsilon, \\
\left( \frac{k}{n-k} \left( \frac{1-q}{1-\epsilon} \right)^k \right) &< \epsilon^2, \quad k = 0, 1, \ldots, n - 1.
\end{align*}
\]

(2.12)

From Proposition 2.3
\[ B_n(p) = \sum_{k=0}^{n-1} \left( 1 + \frac{k}{n-k} \right) q^k - q^n \sum_{k=1}^{n} \frac{n}{k}, \]

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\[
\begin{align*}
\frac{1 - q^n}{p} + \frac{n-1}{n-k} q^{n-k} - q^n \sum_{k=1}^{n} \frac{n}{k}, \\
= \frac{1 - q^n}{p} + \frac{n-1}{n-k} q^{n-k} - q^n \sum_{k=1}^{n} \frac{n}{k}.
\end{align*}
\]

Then for \(a \leq p \leq 1\) and \(n > n_1\),
\[
\left| B_n(p) - \frac{1}{p} \right| = \left| -\frac{q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k - q^n \sum_{k=1}^{n} \frac{n}{k} \right|,
\]
\[
\leq \frac{q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k + q^n \sum_{k=1}^{n} \frac{n}{k}, \quad \text{by the triangular inequality},
\]
\[
\leq \frac{(1-a)^n}{a} + \sum_{k=0}^{n-1} \frac{k}{n-k} (1-a)^k + (1-a)^n \sum_{k=1}^{n} \frac{n}{k},
\]
\[
\leq \frac{(1-a)^n}{a} + \sum_{k=0}^{n-1} \left[ \frac{k}{n-k} \left( \frac{1-a}{1-\epsilon} \right)^k \right] + n[\ln(n) + 1](1-a)^n,
\]
\[
\leq \epsilon + \epsilon^2 \sum_{k=0}^{n-1} (1-\epsilon)^k + \epsilon, \quad \text{from (2.12)},
\]
\[
= 2\epsilon + \epsilon^2 \left[ \frac{1 - (1-\epsilon)^n}{\epsilon} \right] < 3\epsilon.
\]

\textbf{Proposition 2.5} For \(0 < a < 1\), the polynomials \(Q_n\) defined in (1.11) satisfy
\[
\lim_{n \to \infty} Q_n(p) = \frac{1}{p^2}, \quad \text{uniformly in} \ [a, 1],
\]
\[
\lim_{n \to \infty} Q_n(p) = \frac{1}{p^2}, \quad \text{point-wise in} \ [0, 1].
\]

\textbf{Proof.} It can be proved in a similar way to that of Proposition 2.4.

\textbf{2.2.4 Further properties of the \(B_n\) polynomials}

By studying the first and second derivatives of \(B_n\) we find some inequalities as well as some increase and convexity properties for these polynomials.

\textbf{Corollary 2.4} For \(n \geq 1\) the following inequality holds
\[
B_n(p) \geq \frac{(1-q^n)^2}{p}.
\]

\textbf{Proof.} It follows from (1.15) and Corollary 2.1.
Proposition 2.6 For each \( n \geq 2 \) there exists a unique \( 0 < \alpha_n < 1 \) such that

(i) \( B_n(\alpha_n) = \frac{1 - (1 - \alpha_n)^n}{\alpha_n} \).

(ii) \( B_n(p) < B_n(\alpha_n), \forall p \in [0,1], p \neq \alpha_n. \)

(iii) \( B_n \) increases in \([0, \alpha_n]\) and decreases in \([\alpha_n, 1]\).

(iv) \( B_n(p) < \frac{1 - q^n}{p}, \forall p \in ]0, \alpha_n[ \) and \( B_n(p) > \frac{1 - q^n}{p}, \forall p \in ]\alpha_n, 1[. \)

(v) The following inequalities hold for \( n \geq 3: \)

\[
\frac{1}{n} \leq \alpha_n \leq u_n = \frac{\sqrt{3(n-1)(7n-11)} - 3(n-1)}{\sqrt{3(n-1)(7n-11)} + (n-1)(n-5)} \leq \frac{2}{n+1}. \quad (2.15)
\]

(vi) \( \{\alpha_n\} \) is decreasing.

(vii) \( \lim_{n \to \infty} \alpha_n = 0. \)

Proof. For \( 0 < p < 1: \)

\[
B'_n(p) = n \left[ \sum_{k=0}^{n-1} \frac{q^k}{n-k} - q^n \sum_{k=1}^{n} \frac{1}{k} \right], \quad \text{from Proposition 2.3,}
\]

\[
= n \left[ \sum_{k=0}^{n-1} \frac{-kq^{k-1}}{n-k} + nq^{n-1} \sum_{k=1}^{n} \frac{1}{k} \right], \quad \text{since} \ q = 1 - p, \quad (2.16)
\]

\[
= -\frac{n}{q} \left[ B_n(p) - \frac{1 - q^n}{p} \right], \quad \text{by (2.13),}
\]

\[
= -nq^{n-1}f(p),
\]

where

\[
f(p) = \sum_{k=0}^{n-1} \frac{k}{n-k} q^{-(n-k)} - \sum_{k=1}^{n} \frac{n}{k} \quad (2.17)
\]

\[
= \frac{B_n(p) - \frac{1 - q^n}{p}}{q^n}. \quad (2.18)
\]

This function is continuous, strictly increasing in \([0,1[\), \( \lim_{p \to 0^+} f(p) = -n < 0 \) and

\[
\lim_{p \to 1^-} f(p) = +\infty. \quad (2.19)
\]

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Thus, by the mean value theorem there exists a unique \( \alpha_n \in ]0,1[ \) such that

\[
f(p) < f(\alpha_n) = 0 < f(t), \quad (2.20)
\]

for all \( p \in ]0, \alpha_n[ \) and all \( t \in ]\alpha_n, 1[ \). From (2.20) and (2.18) the proof of (i)-(iv) is complete.

(v) We may write (2.17) as

\[
f(p) = \sum_{k=1}^{n} \left[ \frac{(n-k)q^{-k} - n}{k} \right]. \quad (2.21)
\]

Then we get \( B'_n(p) > 0 \) if \( (n-k)q^{-k} - n < 0 \), for all \( k = 1, \ldots, n \). From Corollary 2.3 - (i), we get \( q > \max_{k=1,\ldots,n} \left( \frac{n-k}{n} \right)^{1/k} = 1 - 1/n \). This yields \( p < 1/n \) and so \( 1/n \leq \alpha_n \), which proves the first inequality of (2.15). Use of (2.5) in (2.21) yields

\[
f(p) \geq \sum_{k=1}^{n} \left\{ \frac{(n-k) \left[ 1 + k \left( \frac{p}{q} \right) + \frac{1}{2}k(k-1) \left( \frac{p}{q} \right)^2 \right] - n}{k} \right\},
\]

or

\[
f(p) \geq \sum_{k=1}^{n} \left\{ \frac{(n-k) \left[ 1 + k \left( \frac{p}{q} \right) \right] - n}{k} \right\},
\]

or

\[
f(p) \geq n \left[ \frac{1}{12} (n-1)(n-2) \left( \frac{p}{q} \right)^2 + \frac{1}{2} (n-1) \left( \frac{p}{q} \right)^2 - 1 \right], \quad (2.22)
\]

or

\[
f(p) \geq n \left[ \frac{1}{2} (n-1) \left( \frac{p}{q} \right)^2 - 1 \right].
\]

Solving \( A_2 > 0 \) for \( n \geq 3 \) we get \( f(p) > 0 \), i.e., \( B'(p) < 0 \), for

\[
p > u_n = \frac{\sqrt{3(n-1)(7n-11)} - 3(n-1)}{\sqrt{3(n-1)(7n-11)} + (n-1)(n-5)}.
\]

This proves the second inequality of (2.15). Now solving \( A_1 > 0 \) for \( n \geq 3 \) we get \( B'(p) < 0 \) for \( p > \frac{2}{n+1} \). This means that \( \alpha_n \leq \frac{2}{n+1} \). The last inequality of (2.15) follows since \( A_2 \geq A_1 \).
(vi) By (iv) it is enough to prove that $B_{n+1}(\alpha_n) > \frac{1-(1-\alpha_n)^{n+1}}{\alpha_n}$. To this let $p = \alpha_n$ and $q = 1 - \alpha_n$. Then

\[
B_{n+1}(p) = q \left(\frac{n+1}{n}\right) B_n(p) + 1 - q^{n+1}, \text{ by (2.6)},
\]

\[
= q \left(\frac{n+1}{n}\right) \left(\frac{1-q^n}{p}\right) + 1 - q^{n+1}, \text{ by (i)},
\]

\[
= \left(1 + \frac{1}{n}\right) \left(\frac{q-q^{n+1}}{p}\right) + 1 - q^{n+1},
\]

\[
= \left(1 + \frac{1}{n}\right) \left(\frac{1}{p} - \frac{q^{n+1}}{p} - 1\right) + 1 - q^{n+1},
\]

\[
= \frac{1-q^{n+1}}{p} - 1 + \frac{1}{n} \left(\frac{q-q^{n+1}}{p}\right) + 1 - q^{n+1},
\]

\[
= \frac{1-q^{n+1}}{p} + \frac{q}{np} \left[1 - q^n - npq^n\right],
\]

\[
= \frac{1-q^{n+1}}{p} + \frac{q}{np} \left[1 - (np+1)q^n\right],
\]

\[
> \frac{1-q^{n+1}}{p}, \text{ by Lemma 2.5 - (i)}.
\]

(vii) It follows from (v). \ding{51}

Table 2.1: Values of $\alpha_n$, lower and upper bounds and $1.5/n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$1/n$</th>
<th>$\alpha_n$</th>
<th>$u_n$</th>
<th>$2/(n+1)$</th>
<th>$1.5/n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.20000</td>
<td>0.288677</td>
<td>0.292893</td>
<td>0.333333</td>
<td>0.300000</td>
</tr>
<tr>
<td>10</td>
<td>0.10000</td>
<td>0.147419</td>
<td>0.152067</td>
<td>0.181818</td>
<td>0.150000</td>
</tr>
<tr>
<td>20</td>
<td>0.05000</td>
<td>0.074439</td>
<td>0.077545</td>
<td>0.095238</td>
<td>0.075000</td>
</tr>
<tr>
<td>50</td>
<td>0.02000</td>
<td>0.029946</td>
<td>0.031394</td>
<td>0.039216</td>
<td>0.030000</td>
</tr>
<tr>
<td>100</td>
<td>0.01000</td>
<td>0.015001</td>
<td>0.015761</td>
<td>0.019802</td>
<td>0.015000</td>
</tr>
<tr>
<td>200</td>
<td>0.00500</td>
<td>0.007507</td>
<td>0.007897</td>
<td>0.009950</td>
<td>0.007500</td>
</tr>
<tr>
<td>500</td>
<td>0.00200</td>
<td>0.003005</td>
<td>0.003163</td>
<td>0.003992</td>
<td>0.003000</td>
</tr>
<tr>
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<td>0.00100</td>
<td>0.001503</td>
<td>0.001582</td>
<td>0.001998</td>
<td>0.001500</td>
</tr>
<tr>
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<td>0.000751</td>
<td>0.000791</td>
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<td>0.000750</td>
</tr>
<tr>
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<td>0.00020</td>
<td>0.000301</td>
<td>0.000316</td>
<td>0.000400</td>
<td>0.000300</td>
</tr>
</tbody>
</table>

In our empirical study it was seen that $\alpha_n \approx 1.5/n$. It is shown in Table 2.1 together with $\alpha_n$ and its lower and upper bounds $1/n$, $2/(n + 1)$ and $u_n$. 

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Proposition 2.7 For each \( n \geq 2 \) the equation

\[
B_n(p) = \frac{1}{p},
\]  

has a unique solution \( \beta_n \) in \( \{0, 1\} \) such that

(i) \( \alpha_n < \beta_n < 2.16/n \), where \( \alpha_n \) is given by Proposition 2.6.

(ii) \( B_n(p) < \frac{1}{p}, \forall p \in ]0, \beta_n[ \) and \( B_n(p) > \frac{1}{p}, \forall p \in ]\beta_n, 1[. \)

(iii) \( \{\beta_n\} \) is decreasing.

(iv) \( \lim_{n \to \infty} \beta_n = 0. \)

Proof. Since \( \frac{1}{p} - \frac{n^2}{p} \) it follows from Proposition 2.6 that (2.23) has no solution in \( \{0, \alpha_n\} \). Thus, (2.23) is equivalent to \( g(p) = 0 \), where

\[
g(p) = f(p) - \frac{1}{p},
\]

for \( p \in [\alpha_n, 1[ \) and \( f(p) \) defined by (2.17). This function \( g \) is continuous in \( [\alpha_n, 1[ \) and since \( f(p) \) and \(-1/p\) are strictly increasing then \( g(p) \) is also strictly increasing. Besides from (2.20), \( g(\alpha_n) = f(\alpha_n) - 1/\alpha_n = -1/\alpha_n < 0 \) and

\[
\lim_{p \to 1-} g(p) = +\infty, \text{ by (2.24) and (2.19)}. \]

Therefore, by the mean value theorem there exists a unique \( \beta_n \in [\alpha_n, 1[ \) such that

\[
g(p) < g(\beta_n) = 0 < g(t),
\]

for all \( p \in [\alpha_n, \beta_n[ \) and all \( t \in [\beta_n, 1[ \). This ends the proof of (i)-(ii), except the inequality \( \beta_n \leq 2.16/n \). To prove this, we use the inequality (2.22) in (2.18) to get

\[
B_n(p) \geq \frac{1-q^n}{p} + nq^n \left[ \frac{1}{12} (n-1)(n-2) \left( \frac{p}{q} \right)^2 + \frac{1}{2} (n-1) \left( \frac{p}{q} \right) - 1 \right].
\]  

If \( n = 2 \) then \( \beta_2 < 1 < 2.16/2 \). Thus, from (2.27) it is sufficient to prove that for \( n \geq 3 \) and \( p = 2.16/n \):

\[
\frac{1-q^n}{p} + nq^n \left[ \frac{1}{12} (n-1)(n-2) \left( \frac{p}{q} \right)^2 + \frac{1}{2} (n-1) \left( \frac{p}{q} \right) - 1 \right] > \frac{1}{p}. 
\]

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Equivalently, this can be written as
\[
\frac{1}{12} n(n-1)(n-2) \left( \frac{p}{q} \right)^2 + \frac{1}{2} n(n-1) \left( \frac{p}{q} \right) - n - \frac{1}{p} > 0,
\]
and finally, after replacing \( p = 2.16/n \), as
\[
\frac{0.006n(n + 300.35)(n - 2.12)}{(n - 2.16)^2} > 0,
\]
which is obviously true for \( n \geq 3 \).

(iii) By (ii) it is enough to prove that \( B_{n+1}(\alpha_n) > \frac{1}{\alpha_n} \). To prove this let \( p = \alpha_n \) and \( q = 1 - \alpha_n \). Then
\[
B_{n+1}(p) = q \left( \frac{n+1}{n} \right) B_n(p) + 1 - q^{n+1}, \quad \text{by (2.6)},
\]
\[
= (1 - p) \left( 1 + \frac{1}{n} \right) \frac{1}{p} + 1 - q^{n+1}, \quad \text{by (2.23)},
\]
\[
= \frac{1}{p} + \frac{1}{np} - 1 - \frac{1}{n} + 1 - q^{n+1},
\]
\[
= \frac{1}{p} + \frac{1 - p - npq^{n+1}}{np},
\]
\[
= \frac{1}{p} + \frac{q}{np} (1 - np^q),
\]
\[
> \frac{1}{p}, \quad \text{by Lemma 2.5 - (iii) when } r = 1.
\]

(iv) It follows from (i). \( \diamond \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_n )</th>
<th>( \beta_n )</th>
<th>( 2.16/n )</th>
</tr>
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<tr>
<td>5</td>
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<tr>
<td>5000</td>
<td>0.000301</td>
<td>0.000402</td>
<td>0.000432</td>
</tr>
</tbody>
</table>
In Table 2.2 we illustrate numerically Proposition 2.7 - (i). We see that for large values of $n$ the upper bound $2.16/n$ can be taken as an approximation of $\beta_n$. Figure 2.1 illustrates Propositions 2.4, 2.6 and 2.7.

**Proposition 2.8** For $n \geq 2$ and $\tau > 0$ the equation

$$B_n(p) = \frac{1}{p} + \tau q^n,$$  \hspace{1cm} (2.28)

has a unique solution $\gamma_n$ in $]0, 1[$ such that

(i) $\beta_n < \gamma_n$, where $\beta_n$ is given by Proposition 2.7. If $0 < \tau \leq 3$, then $\gamma_n < 2.25/n$.

(ii) $B_n(p) < \frac{1}{p} + \tau q^n$, $\forall \ p \in ]0, \gamma_n[$ and $B_n(p) > \frac{1}{p} + \tau q^n$, $\forall \ p \in \ ]\gamma_n, 1[$.

(iii) $\{\gamma_n\}$ is decreasing as a function of $n$.

**Proof.** (i) - (ii) The inequality $\frac{1}{p} < \frac{1}{p} + \tau q^n$ and Proposition 2.7 show that (2.28) has no solution in $]0, \beta_n]$. Therefore, (2.28) can be written as $v(p) = 0$, where $v(p) = g(p) - \tau$, for $p \in \ [\beta_n, 1[$ and $g$ is defined by (2.24). This function $v$ is strictly increasing.

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and continuous in \( [\beta_n, 1] \). Besides, (2.26) implies that \( v(\beta_n) = g(\beta_n) - r = -r < 0 \), and from (2.25) it follows that \( \lim_{p \to 1^-} v(p) = +\infty \). Then, by the mean value theorem there exists a unique \( r \gamma_n \in ]\beta_n, 1[ \) such that \( v(p) < v(r \gamma_n) = 0 < v(t) \), for all \( p \in ]\beta_n, r \gamma_n[ \) and all \( t \in ]r \gamma_n, 1[ \). To prove the last inequality of (i) it is enough to prove that \( B_n(p) > \frac{1}{p} + 3q^n \), for \( p = 2.25/n \) and \( n \geq 3 \). From (2.27) it is sufficient to prove that
\[
\frac{1}{12} n(n-1)(n-2) \left( \frac{p}{q} \right)^2 + \frac{1}{2} n(n-1) \left( \frac{p}{q} \right) - n - \frac{1}{p} - 3 > 0.
\]
Replacing \( p = 2.25/n \) the latter becomes approximately
\[
\frac{0.10(n-2.19)(n^2 - 11.59n + 67.77)}{(n - 2.25)^2} > 0,
\]
which is true for \( n \geq 3 \).

(iii) For \( p = r \gamma_n \) and \( q = 1 - r \gamma_n \) we have
\[
B_{n+1}(p) = q \left( \frac{n+1}{n} \right) \left( \frac{1}{p} + rq^n \right) + 1 - q^{n+1}, \quad \text{by (2.6) and (2.28)},
\]
\[
= \frac{1}{p} + rq^{n+1} + \frac{q}{np} (1 + rpq^n - npq^n),
\]
\[
> \frac{1}{p} + rq^{n+1} + \frac{q}{np} (1 - npq^n),
\]
\[
> \frac{1}{p} + rq^{n+1}, \quad \text{by Lemma 2.5 -(iii)}.
\]

This and (ii) complete the proof.

We observe that Propositions 2.7 and 2.8 are special cases of a more general result. In fact, it is clear that equation \( B_n(p) = \frac{1}{p} + \psi(p)q^n \) has a unique solution in \( ]0, 1[ \) for \( \psi \) a bounded, positive and decreasing function in \( ]0, 1[ \).

As an illustration of Proposition 2.8 - (i) we present in Table 2.3 values of \( r \gamma_n \) for the special cases where \( r \) is given by
\[
r = \iota r_x = \left( \frac{\tilde{g}_x}{\tilde{g}_x \eta - \tilde{g}_\eta} \right) \iota p_x - 1,
\]
and \( \iota p_x \) is derived using (1.28) for \( x = 20, 30, 40, 50, 60, t = 65 - x \) and \( i = 0.06 \). For comparison we also include \( \beta_n \) and 2.25/n. We note that since values \( \iota r_x \) are small then values of \( r \gamma_n \) are close to \( \beta_n \).

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Table 2.3: Values of $\beta_n$, $2.25/n$ and $r\gamma_n$ for $r = \sqrt{x}$

<table>
<thead>
<tr>
<th></th>
<th>$x = 20$</th>
<th>$x = 30$</th>
<th>$x = 40$</th>
<th>$x = 50$</th>
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<td>0.808959</td>
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<td>$n$</td>
<td>$\beta_n$</td>
<td>$r\gamma_n$</td>
<td>$2.25/n$</td>
<td></td>
<td></td>
</tr>
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<td>0.004014</td>
<td>0.004014</td>
</tr>
</tbody>
</table>

Proposition 2.9 For each $n \geq 3$ there exists a unique $0 < \delta_n < 1$ such that

(i) $B_n(\delta_n) = \frac{1}{\delta_n} + \frac{(1 - \delta_n) - [2(n - 1)\delta_n + 1](1 - \delta_n)^n}{(n - 1)\delta_n^2}$.

(ii) $B_n(p) < \frac{1}{p} + \frac{q - [2(n - 1)p + 1]q^n}{(n - 1)p^2}$, $\forall p \in ]0, \delta_n[,$

$B_n(p) > \frac{1}{p} + \frac{q - [2(n - 1)p + 1]q^n}{(n - 1)p^2}$, $\forall p \in ]\delta_n, 1[.$

(iii) $B_n$ is concave in $]0, \delta_n[$ and convex in $]\delta_n, 1[.$

(iv) The following inequalities hold for $n \geq 4$:

$$\frac{2}{n} \leq \delta_n \leq v_n = \frac{\sqrt[4]{(n - 2)(13n - 35) - 4(n - 2)}}{\sqrt[4]{(n - 2)(13n - 35) + (n - 2)(n - 7)}} \leq \frac{9}{2n + 5},$$ (2.30)

$$\beta_n < \delta_n.$$ (2.31)

(v) $\{\delta_n\}$ is decreasing.

(vi) $\lim_{n \to \infty} \delta_n = 0$.

Proof. We will prove that $B''_n(p)$ changes sign only once in $]0, 1[$. This is not true for $n < 3$ since $B_n$ is a polynomial of degree $n$. For $0 < p < 1$ we have from (2.16)

$$B''_n(p) = n \left[ \sum_{k=0}^{n-1} \frac{k(k-1)q^{k-2}}{n-k} - n(n-1)q^{n-2} \sum_{k=1}^{n} \frac{1}{k} \right] = nq^{n-2}h(p),$$ (2.32)
where
\[
h(p) = \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} q^{-(n-k)} - n(n-1) \sum_{k=1}^{n} \frac{1}{k}.
\] (2.33)

This function is continuous, strictly increasing in \([0,1]\), \(\lim_{p \to 1^-} h(p) = +\infty\) and
\[
\lim_{p \to 0^+} h(p) = \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} - n(n-1) \sum_{k=1}^{n} \frac{1}{k},
\]
\[
< \sum_{k=0}^{n-1} \frac{n(n-1)}{n-k} - n(n-1) \sum_{k=1}^{n} \frac{1}{k},
\]
\[
= 0.
\]

Therefore, by the mean value theorem there exists a unique \(\delta_n \in \]0,1[\) such that \(h(p) < h(\delta_n) = 0 < h(t)\), for all \(p \in \]0,\delta_n[\) and all \(t \in \]\delta_n,1[\). To prove (i)-(iii) we still need to verify that for \(0 < p < 1\),
\[
B_n''(p) = 0 \iff B_n(p) = \frac{1}{p} + \frac{q - [2(n-1)p + 1]q^n}{(n-1)p^2}.
\] (2.34)

From (2.32) we see that \(B_n''(p) = 0\) if and only if
\[
0 = \sum_{k=1}^{n} \frac{(n-k)(n-k-1)}{k} q^{n-k} - n(n-1)q^n \sum_{k=1}^{n} \frac{1}{k},
\]
\[
= \sum_{k=1}^{n} \frac{n(n-1) + (-2n+1)k + k^2}{k} q^{n-k} - n(n-1)q^n \sum_{k=1}^{n} \frac{1}{k},
\]
\[
= (n-1) \left[ \sum_{k=1}^{n} q^{-k} - \sum_{k=1}^{n} q^k \right] + \sum_{k=1}^{n} \frac{(-2n+1)k}{k} q^{n-k} + \sum_{k=1}^{n} k q^{n-k},
\]
\[
= (n-1)B_n(p) + (-2n+1)\left(\frac{1}{p} - \sum_{k=0}^{n-1} q^k\right),
\]
\[
= (n-1)B_n(p) + (-2n+1)\left(\frac{1}{p} - n\frac{1-q^n}{p}\right) - q \frac{d}{dq} \left(\sum_{k=0}^{n-1} q^k\right),
\]
\[
= (n-1)B_n(p) - (n-1)\left(\frac{1-q^n}{p}\right) - q \frac{d}{dq} \left(\frac{1-q^n}{1-q}\right),
\]
\[
= (n-1) \left[ B_n(p) - \frac{1}{p} \right] + (n-1)\frac{q^n}{p} - q \left(\frac{-npq^{n-1} + 1 - q^n}{p^2}\right),
\]
\[
= (n-1) \left[ B_n(p) - \frac{1}{p} \right] - \frac{(n-1)pq^n - npq^n + q - q^{n+1}}{p^2},
\]

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\[(n - 1) \left[ B_n(p) - \frac{1}{p} \right] - \frac{q - [(n - 1)p + np + q]q^n}{p^2}, \]
\[= (n - 1) \left[ B_n(p) - \frac{1}{p} \right] - \frac{q - [2(n - 1)p + 1]q^n}{p^2}. \]

This yields (2.34).

(iv) We may write (2.33) as
\[h(p) = \sum_{k=1}^{n} \frac{(n-k)(n-k-1)q^{-k} - n(n-1)}{k}. \tag{2.35}\]

We can get \(h(p) < 0\) and thus, \(B''_n(p) < 0\), if the condition
\[(n - k)(n - k - 1)q^{-k} - n(n - 1) < 0, \quad \text{for all } k = 1, \ldots, n\]
is added to (2.35). This condition is equivalent to
\[q > \left[ \frac{(n-k)(n-k-1)}{n(n-1)} \right]^{1/k}, \quad k = 1, \ldots, n - 2.\]

From Corollary 2.3 - (ii), we obtain
\[q > \max_{k=1, \ldots, n} \left[ \frac{(n-k)(n-k-1)}{n(n-1)} \right]^{1/k} = \frac{n-2}{n}, \]
or \(p < 2/n\). This proves the left hand side inequality of (2.30). To prove the other inequalities in (2.30) we use the inequalities of (2.5) in (2.35) to get
\[h(p) \geq \sum_{k=1}^{n} \left[ \frac{(n-k)(n-k-1)}{k} \left[ 1 + k \left( \frac{p}{q} \right) + \frac{1}{2} k(k-1) \left( \frac{p}{q} \right)^2 \right] - n(n-1) \right], \]
\[\geq \sum_{k=1}^{n} \left[ \frac{(n-k)(n-k-1)}{k} \left[ 1 + k \left( \frac{p}{q} \right) \right] - n(n-1) \right], \]
or
\[h(p) \geq n(n-1) \left[ \frac{1}{24} (n-2)(n-3) \left( \frac{p}{q} \right)^2 + \frac{1}{3} (n-2) \left( \frac{p}{q} \right) - \frac{3}{2} \right], \]
\[\geq n(n-1) \left[ \frac{1}{3} (n-2) \left( \frac{p}{q} \right) - \frac{3}{2} \right]. \]
Solving $D_2 > 0$ for $n \geq 4$ we get $h(p) > 0$, i.e., $B''(p) > 0$, for

$$p > u_n = \frac{\sqrt{4(n-2)(13n-35)} - 4(n-2)}{\sqrt{4(n-2)(13n-35)} + (n-2)(n-7)}.$$  

This proves the second inequality of (2.30). Now solving $D_1 > 0$ for $n \geq 4$ we get $B''(p) > 0$ for $p > \frac{9}{2n+5}$. This leads $\delta_n \leq \frac{9}{2n+5}$. The last inequality of (2.30) follows since $D_2 \geq D_1$. Inequality (2.31) follows from (i), the first inequality of (2.30), Lemma 2.5 - (ii) and Proposition 2.7 - (ii).

(v) From (ii) it is enough to prove that for $p = \delta_n$ and $q = 1 - \delta_n$

$$B_{n+1}(p) > \frac{1}{p} + \frac{q}{n p^n} \left[ 1 - (2np + 1)q^n \right].$$

To this, using (2.6) and (i) we have

$$B_{n+1}(p) = \left( \frac{n-1}{n} \right) \left\{ \frac{q}{p} + \frac{q}{(n-1)p^2} \left[ q - (2np + 1)q^n + 2pq^n \right] \right\} + 1 - q^{n+1},$$

$$= \left( \frac{n-1}{n} \right) \left\{ \frac{q}{p} + \frac{q}{(n-1)p^2} \left[ q - (2np + 1)q^n + 2pq^n \right] \right\} + \frac{2q}{n} \left\{ \frac{1}{p} + \frac{q}{(n-1)p^2} \left[ 1 - \{2(n-1)p + 1\}q^{n-1} \right] \right\} + 1 - q^{n+1},$$

$$= \left( \frac{n-1}{n} \right) \frac{q}{p} + \frac{q}{n p^n} \left[ 1 - p - (2np + 1)q^n + 2pq^n \right]$$

$$+ \frac{2q}{n} \left\{ \frac{1}{p} + \frac{q}{(n-1)p^2} \left[ 1 - \{2(n-1)p + 1\}q^{n-1} \right] \right\} + 1 - q^{n+1},$$

$$= L_n(p) - \frac{q}{np} + \frac{q}{np^2} (2pq^n - p)$$

$$+ \frac{2q}{n} \left\{ \frac{1}{p} + \frac{q}{(n-1)p^2} \left[ 1 - \{2(n-1)p + 1\}q^{n-1} \right] \right\} - q^{n+1},$$

$$= L_n(p) + \frac{q}{np} \left\{ (2 - np)q^n + \frac{2q}{(n-1)p} \left[ 1 - \{2(n-1)p + 1\}q^{n-1} \right] \right\},$$

$$= L_n(p) + \frac{q}{n(n-1)p^2} \left\{ (n-1)(2 - np)pq^n + 2q - 2[2(n-1)p + 1]q^n \right\},$$

$$= L_n(p) + \frac{q}{n(n-1)p^2} \left\{ 2q + [(n-1)(2 - np)p - 4np + 4p - 2]q^n \right\},$$

$$= L_n(p) + \frac{q^2}{n(n-1)p^2} \left\{ 2 + [-n(n-1)p^2 - 2(n-1)p - 2]q^{n-1} \right\},$$
\[ \begin{align*}
&= L_n(p) + \frac{2q^2}{n(n-1)p^2} \left\{ 1 - \left[ \frac{n(n-1)}{2} p^2 + (n-1)p + 1 \right] q^{n-1} \right\}, \\
&> L_n(p), \text{ from Lemma 2.5-(iv).}
\end{align*} \]

(vi) It follows from (iv). \(\diamondsuit\)

Table 2.4: Values of \(\delta_n\), lower and upper bounds and \(2.87/n\)

<table>
<thead>
<tr>
<th>n</th>
<th>2/n</th>
<th>(\beta_n)</th>
<th>(\delta_n)</th>
<th>(v_n)</th>
<th>(9/(2n+5))</th>
<th>(2.87/n)</th>
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</table>

Proposition 2.9 - (iv) is illustrated numerically in Table 2.4. It is seen numerically that \(2.87/n\) provides an adequate approximation for \(\delta_n\) when \(n\) is large.

**Proposition 2.10** Let \(n \geq 2\). The function

\[
\varphi_n(p) = \begin{cases} 
\frac{n(1-q^n)}{(n-1)p} - B_n(p) & \text{if } 0 < p \leq 1, \\
\frac{n^2}{n-1} & \text{if } p = 0,
\end{cases}
\]

has the following properties:

(i) For \(n \geq 3\), \(\varphi_n\) is strictly decreasing in \([0, \delta_n[\) and strictly increasing in \(]\delta_n, 1]\).

Therefore, for all \(p \in [0, 1]\), \(p \neq \delta_n\) we have

\[
\varphi_n(p) > \varphi_n(\delta_n) = \frac{1}{(n-1)\delta_n^2} \left\{ 2\delta_n - 1 + [(n-2)\delta_n + 1](1-\delta_n)^n \right\}. \quad (2.36)
\]

(ii) \(\varphi_n(p) > 0\), for all \(0 \leq p \leq 1\) and \(2 \leq n \leq 5\).

(iii) \(\varphi_n(p) > \left(\frac{nq}{n-1}\right)\varphi_{n-1}(p)\), for all \(\frac{n}{2(n-1)} \leq p \leq 1\).
(iv) For $n \geq 6$, $\varphi_n$ has two roots $\epsilon_n$ and $\epsilon_n$ in $]0, 1[$ such that $\frac{1}{n-1} \leq \epsilon_n \leq \beta_n$ (the last inequality when $n \geq 10$) and $1/2 < \epsilon_n < 7/12$, and
\[
0 = \varphi_n(\epsilon_n) < \varphi_n(p) < \varphi_n(0) = \frac{n^2}{n-1}, \quad \text{for all } 0 < p < \epsilon_n,
\]
\[
\varphi_n(\delta_n) < \varphi_n(p) < \varphi_n(\epsilon_n) = 0, \quad \text{for all } \epsilon_n < p < \epsilon_n,
\]
\[
0 = \varphi_n(\epsilon_n) < \varphi_n(p) < \varphi_n(1) = \frac{1}{n-1}, \quad \text{for all } \epsilon_n < p < 1.
\]

Proof. (i) We may write $\varphi_n$ as
\[
\varphi_n(p) = -\left(\frac{n}{n-1}\right) \sum_{k=1}^{n} \frac{(n-k-1)q^{n-k} - (n-1)q^n}{k}.
\]
(2.37)
This and (2.32) yield $\varphi_n'(p) = (\frac{n}{n-1}) B''_n(p)$. Proposition 2.9 ends the proof.

(ii) It is easily seen that the result is true for $n = 2$. For $3 \leq n \leq 5$ we have $\delta_n > 1/2$ or $2\delta_n - 1 > 0$. Then from (2.36) we get $\varphi_n(p) \geq \varphi_n(\delta_n) > 0$, for all $0 \leq p \leq 1$.

(iii) The result is true for $p = 1$. Let $\frac{n}{2(n-1)} \leq p < 1$. Since the function $\frac{n(n-1)p}{n(1-p)}$ is increasing in $]0, 1[$, we get $\frac{n(n-1)p}{nq} \geq \frac{n-1}{n-2}$. From this and (2.6)
\[
\varphi_n(p) = \frac{n(1-q^n)}{(n-1)p} - B_n(p),
\]
\[
= \frac{n(1-q^n)}{(n-1)p} - \left\{ q \left[ \frac{n}{n-1} \right] B_{n-1}(p) + 1 - q^n \right\},
\]
\[
= \frac{q}{n-1} \left\{ \left[ \frac{n}{n-1} \right] B_{n-1}(p) \frac{1-q^n}{p} - B_{n-1}(p) \right\},
\]
\[
> \frac{q}{n-1} \left\{ \left[ \frac{n}{n-1} \right] B_{n-1}(p) \frac{1-q^{n-1}}{p} - B_{n-1}(p) \right\},
\]
\[
\geq \frac{q}{n-1} \left[ \frac{(n-1)(1-q^{n-1})}{(n-2)p} - B_{n-1}(p) \right] = \frac{q}{n-1} \varphi_{n-1}(p).
\]
(2.38)

(iv) Let $0 \leq p \leq \alpha_n$. From Proposition 2.6 - (iv) and since for these values of $p$, $\varphi_n(p) > \frac{1-q^n}{p} - B_n(p) \geq 0$, we get $\varphi_n(p) > 0$. For $n = 6$, by direct computation we can see that if $p > 0.52$ then $\varphi_6(p) > 0$. Since for $n \geq 7$ we have $\frac{n}{2(n-1)} \leq 7/12$, it follows from (iii) that $\varphi_n(p) > 0$, for $7/12 \leq p \leq 1$. Now take $\delta_n < p < 1/2$. Since in this interval $\varphi_n$ is increasing we have $\varphi_n(p) \leq \varphi_n(1/2)$. From (2.38) and Proposition 2.9 - (ii), we get
\[
\varphi_n(p) = \frac{q}{n-1} \left\{ \left[ \frac{n}{n-1} \right] \frac{1-q^n}{p} - B_{n-1}(p) \right\},
\]
(2.39)
\[
\begin{align*}
\leq \frac{nq}{n-1} \left\{ \left[ \frac{n-(n-1)p}{nq} \right] \frac{1}{p} - \frac{1}{p} - \frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p^2} \right\}, \\
= \frac{nq}{(n-1)p} \left\{ \left[ \frac{n-(n-1)p}{nq} \right] - 1 - \frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p} \right\}, \\
= \frac{1}{(n-1)(n-2)p^2} \left\{ -2p^2 + n(2p-1) + n[2(n-2)p+1]q^n \right\}.
\end{align*}
\]

If \( p = 1/2 \) then \( H = -\frac{1}{2} + \frac{n(n-1)}{2n} < 0 \) for \( n \geq 6 \). Hence \( \varphi_n(1/2) < 0 \) and so \( \varphi_n(p) < 0 \) for \( \delta_n \leq p \leq 1/2 \). Now, since \( \varphi_n(0) > 0 \), \( \varphi_n(1/2) < 0 \), \( \varphi_n(1) > 0 \) and \( \varphi_n \) is a continuous function with \( \varphi_n'(p) \) vanishing only at \( \delta_n \), then \( \varphi_n \) has exactly two roots \( \epsilon_n \) and \( \epsilon_n \) in \( ]0,1[ \) such that \( \alpha_n < \epsilon_n < \delta_n \) and \( 1/2 < \epsilon_n < 7/12 \). We need to prove that \( \frac{1}{n-1} \leq \epsilon_n \leq \beta_n \). The first inequality follows from (2.37). In fact from there it is enough to prove that for \( p < \frac{1}{n-1} \)
\[
\sum_{k=1}^{n} \frac{(n-k-1)q^{-k} - (n-1)}{k} < 0.
\]
This is true since by Corollary 2.3 - (i), \( (n-k-1)q^{-k} - (n-1) < 0 \) for \( q > \frac{n-2}{n-1} \) or \( p < \frac{1}{n-1} \). Now we have to prove that \( \epsilon_n \leq \beta_n \) for \( n \geq 10 \). Let \( \beta_n < p < 1 - n^{-1/n} \) (this is why we need \( n \geq 10 \)). Then from Proposition 2.7 - (ii), we have
\[
\varphi_n(p) = \frac{n(1-q^n)}{(n-1)p} - B_n(p) < \frac{n(1-q^n)}{(n-1)p} - \frac{1}{p},
\]
\[
= \frac{1}{(n-1)p} - nq^n < 0, \text{ since } p < 1 - n^{-1/n}.
\]
The inequalities are easily proved using \( \epsilon_n, \epsilon_n \) and (i). \( \diamond \)

We observe that \( \lim_{n \to \infty} \epsilon_n = 0 \). Also, from Proposition 2.10 - (iii) it appears that \( \epsilon_n \) is close to \( \frac{n}{2(n-1)} \). In fact in our empirical studies it was seen that this expression provides a good approximation for \( \epsilon_n \). This is shown in Table 2.5 together with the values of \( \epsilon_n \) and \( \epsilon_n \). We see that \( \epsilon_n \) tends to \( 1/2 \) as \( n \) increases.

In terms of \( B_n \) and \( b_1(n, p) \) the inequalities of Proposition 2.10 - (iv) may be written as in the following corollary. For small values of \( np \) the inequalities of Propositions 2.6 and 2.7 and Corollary 2.4 are better so they have been included instead.
Table 2.5: Values of $\epsilon_n$, $\epsilon_n / n$, $1/(n-1)$ and $n/[2(n-1)]$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$1/(n-1)$</th>
<th>$\epsilon_n$</th>
<th>$\epsilon_n$</th>
<th>$n/[2(n-1)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.200000</td>
<td>0.360000</td>
<td>0.513290</td>
<td>0.600000</td>
</tr>
<tr>
<td>10</td>
<td>0.111111</td>
<td>0.182082</td>
<td>0.558385</td>
<td>0.555556</td>
</tr>
<tr>
<td>20</td>
<td>0.052632</td>
<td>0.081842</td>
<td>0.528293</td>
<td>0.526316</td>
</tr>
<tr>
<td>50</td>
<td>0.020408</td>
<td>0.031047</td>
<td>0.510437</td>
<td>0.510204</td>
</tr>
<tr>
<td>100</td>
<td>0.010101</td>
<td>0.015270</td>
<td>0.505104</td>
<td>0.505051</td>
</tr>
<tr>
<td>200</td>
<td>0.005025</td>
<td>0.007574</td>
<td>0.502526</td>
<td>0.502513</td>
</tr>
<tr>
<td>500</td>
<td>0.002004</td>
<td>0.003015</td>
<td>0.501004</td>
<td>0.501002</td>
</tr>
<tr>
<td>1000</td>
<td>0.001001</td>
<td>0.001505</td>
<td>0.500501</td>
<td>0.500501</td>
</tr>
<tr>
<td>2000</td>
<td>0.000500</td>
<td>0.000752</td>
<td>0.500250</td>
<td>0.500250</td>
</tr>
<tr>
<td>5000</td>
<td>0.000200</td>
<td>0.000301</td>
<td>0.500100</td>
<td>0.500100</td>
</tr>
</tbody>
</table>

**Corollary 2.5** Let $n \geq 2$.

(i) If $0 < p < \alpha_n$ then

\[
\frac{(1 - q^n)^2}{p} < B_n(p) < \frac{1 - q^n}{p},
\]

\[
\frac{1 - q^n}{np} < b_1(n, p) < \frac{1}{np}.
\]

(ii) If $\alpha_n < p < \beta_n$ then

\[
\frac{1 - q^n}{p} < B_n(p) < \frac{1}{p},
\]

\[
\frac{1}{np} < b_1(n, p) < \frac{1}{np(1 - q^n)}.
\]

(iii) For $\epsilon_n < p < \epsilon_n$ and $\varphi_n(\delta_n)$ given by (2.36),

\[
\frac{n}{(n-1)} \left( \frac{1 - q^n}{p} \right) < B_n(p) < \frac{n}{(n-1)} \left( \frac{1 - q^n}{p} \right) - \varphi_n(\delta_n),
\]

\[
\frac{1}{(n-1)p} < b_1(n, p) < \frac{1}{(n-1)p} - \frac{\varphi_n(\delta_n)}{n(1 - q^n)}.
\]

(iv) If $\epsilon_n < p < 1$ then

\[
\frac{n}{(n-1)} \left( \frac{1 - q^n}{p} \right) - \frac{1}{n - 1} < B_n(p) < \frac{n}{(n-1)} \left( \frac{1 - q^n}{p} \right),
\]

\[
\frac{1}{(n-1)p} - \frac{1}{n(n-1)(1 - q^n)} < b_1(n, p) < \frac{1}{(n-1)p}.
\]

**Proof.** Apply Corollary 2.4 and Propositions 2.6, 2.7 and 2.10. \(\diamondsuit\)
When $n$ is large we can say that (i) and (ii) of Corollary 2.5 do not hold since in these cases $\beta_n$ is almost null. Hence we can say that (iii) holds for almost the whole interval $]0, 1/2[$ and (iv) holds for almost the whole interval $]1/2, 1[$.

### 2.3 Results on Ramsay's definitions

In this section some inequalities for the ratios $\rho_1$ and $\rho_2$ from (1.26) and (1.27) are presented. The variances of Ramsay's random future shares are also derived.

#### 2.3.1 Inequalities for $\rho_1$ and $\rho_2$

Some inequalities for $\rho_1$ and $\rho_2$ given by (1.26) and (1.27) are presented here. These inequalities could be improved in some intervals by using the inequalities of Corollary 2.5. It is also proved that the inequalities $\rho_1 > 1$ and $\rho_2 > 1$, conjectured by Ramsay, are not true in general.

**Proposition 2.11** The ratios $\rho_1$ and $\rho_2$ satisfy:

(i) $(1 - t q_x^2)^2 \leq \rho_1 < B_n(t p_x)$, for all $0 < t p_x < 1$.

(ii) $(1 - t q_x^2)^2 \leq \rho_2 \leq \frac{\bar{s}_{x:1}}{\bar{s}_{x:1}} B_n(t p_x) \leq B_n(t p_x)$, for all $0 < t p_x < 1$.

(iii) $\lim_{t p_x \to 0^+} \rho_1 = 0$, $\lim_{t p_x \to 1^-} \rho_1 = 1$, and $\lim_{n \to \infty} \rho_1 = 1$.

(iv) $\lim_{t p_x \to 0^+} \rho_2 = 0$, $\lim_{t p_x \to 1^-} \rho_2 = 1$, and $\lim_{n \to \infty} \rho_2 = 1$.

**Proof.** (i) It follows from Corollary 2.4 and from $\rho_1 = t p_x B_n(t p_x) < B_n(t p_x)$.

(ii) It is easy to see that

\[
\bar{s}_{x:1} \leq \bar{s}_{x:1} \leq \frac{\bar{s}_{x:1}}{t p_x}.
\]

From Corollary 2.4 we have that $B_n(t p_x) \geq 1 - t q_x^2$. Then

\[
\rho_2 = \frac{\bar{s}_{x:1} (1 - t q_x^2) + [B_n(t p_x) - (1 - t q_x^2)] \frac{t p_x}{\bar{s}_{x:1}}}{\frac{\bar{s}_{x:1}}{t q_x}},
\]

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\[ \frac{\tilde{s}_{\tilde{\eta}}(1 - \eta q_{\tilde{\eta}}) + \left[ B_{n}(\eta p_{x}) - (1 - \eta q_{\tilde{\eta}}) \right] \left( \frac{\tilde{s}_{\tilde{\eta}}}{\eta p_{x}} - \tilde{s}_{\tilde{\eta}} \right) \frac{\eta p_{x}}{\eta q_{x}}}{\tilde{s}_{\tilde{\eta}}}, \text{ by (2.39)}, \]

\[ = \frac{\tilde{s}_{\tilde{\eta}}}{\tilde{s}_{\tilde{\eta}}} \left\{ 1 - \eta q_{\tilde{\eta}} + \left[ B_{n}(\eta p_{x}) - (1 - \eta q_{\tilde{\eta}}) \right] \left[ \frac{1}{\eta p_{x}} - 1 \right] \frac{\eta p_{x}}{\eta q_{x}} \right\}, \]

\[ = \frac{\tilde{s}_{\tilde{\eta}}}{\tilde{s}_{\tilde{\eta}}} \left[ 1 - \eta q_{\tilde{\eta}} + B_{n}(\eta p_{x}) - (1 - \eta q_{\tilde{\eta}}) \right], \]

\[ = \frac{\tilde{s}_{\tilde{\eta}}}{\tilde{s}_{\tilde{\eta}}} B_{n}(\eta p_{x}) \leq B_{n}(\eta p_{x}), \text{ by (2.39)}. \]

Again from Corollary 2.4,

\[ \rho_{2} = \frac{\tilde{s}_{\tilde{\eta}}(1 - \eta q_{\tilde{\eta}}) + \left[ B_{n}(\eta p_{x}) - (1 - \eta q_{\tilde{\eta}}) \right] \left( \tilde{s}_{\tilde{\eta}} - \tilde{s}_{\tilde{\eta}} \right) \frac{\eta p_{x}}{\eta q_{x}}}{\tilde{s}_{\tilde{\eta}}}, \]

\[ \geq \frac{\tilde{s}_{\tilde{\eta}}(1 - \eta q_{\tilde{\eta}}) + \left[ (1 - \eta q_{\tilde{\eta}}) \eta p_{x} \eta q_{x} - (1 - \eta q_{\tilde{\eta}}) \right] \left( \tilde{s}_{\tilde{\eta}} - \tilde{s}_{\tilde{\eta}} \right) \frac{\eta p_{x}}{\eta q_{x}}}{\tilde{s}_{\tilde{\eta}}}, \]

\[ = \frac{(1 - \eta q_{\tilde{\eta}}) \left( \tilde{s}_{\tilde{\eta}} + (1 - \eta q_{\tilde{\eta}}) \left( \tilde{s}_{\tilde{\eta}} - \tilde{s}_{\tilde{\eta}} \right) \right)}{\tilde{s}_{\tilde{\eta}}}, \]

\[ = \frac{(1 - \eta q_{\tilde{\eta}}) \left( \eta q_{\tilde{\eta}}^{n-1} \tilde{s}_{\tilde{\eta}} + (1 - \eta q_{\tilde{\eta}}^{n-1}) \tilde{s}_{\tilde{\eta}} \right)}{\tilde{s}_{\tilde{\eta}}}, \]

\[ = \frac{(1 - \eta q_{\tilde{\eta}}) \left[ \eta q_{\tilde{\eta}}^{n-1} \left( \frac{\tilde{s}_{\tilde{\eta}}}{\tilde{s}_{\tilde{\eta}}} \right) + 1 - \eta q_{\tilde{\eta}}^{n-1} \right]}{\tilde{s}_{\tilde{\eta}}}, \]

\[ \geq (1 - \eta q_{\tilde{\eta}}) \left[ \eta q_{\tilde{\eta}}^{n-1}(\eta p_{x}) + 1 - \eta q_{\tilde{\eta}}^{n-1} \right], \text{ by (2.39)}, \]

\[ = (1 - \eta q_{\tilde{\eta}})^{2}. \]

(iii) It follows from (i) and (2.7).

(iv) It follows from (ii) and (2.7). \(\diamond\)

**Proposition 2.12** For \(\rho_{1}\) and \(\rho_{2}\) we have:

(i) If \(n = 1\) then \(\rho_{1} < 1\) and \(\rho_{2} < 1\), for all \(0 < \eta p_{x} < 1\).

(ii) For \(n \geq 2\):

\[ \rho_{1} < 1 \iff \eta p_{x} < \beta_{n}, \]

\[ \rho_{1} = 1 \iff \eta p_{x} = \beta_{n}, \]

\[ \rho_{1} > 1 \iff \eta p_{x} > \beta_{n}. \]

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(iii) For \( n \geq 2 \) and a given \( \iota p_x \) consider \( r \) as in (2.29) and \( \rho_2 = \rho_2(\iota p_x) \). Then

\[
\begin{align*}
\rho_2 < 1 & \iff \iota p_x < r\gamma_n, \\
\rho_2 = 1 & \iff \iota p_x = r\gamma_n, \\
\rho_2 > 1 & \iff \iota p_x > r\gamma_n.
\end{align*}
\]

**Proof.** (i) From (1.26) we see that \( \rho_1 = \iota p_x^2 < 1 \) and from (1.27) and (2.39) it follows that \( \rho_2 = \frac{\iota p_x - \bar{s}}{\bar{s} - \bar{a}} < 1 \).

(ii) From (1.26) it is seen that \( \rho_1 = 1 \iff B_n(\iota p_x) = \frac{1}{\iota p_x} \). Apply now Proposition 2.7.

(iii) For a given \( \iota p_x \) consider \( r \) as in (2.29). Let \( \varepsilon \) such that

\[
B_n(\iota p_x) = \frac{1}{\iota p_x} + (\varepsilon - 1) \iota^q q^n.
\]

Then from (1.27) we may write

\[
E[S_2] - \bar{s} = \iota q^{-1} \left[ \iota p_x \varepsilon (\bar{s} - \bar{q}) - \iota q_n \bar{s} \right].
\]

Therefore,

\[
\begin{align*}
\rho_2 < 1 & \iff E[S_2] - \bar{s} < 0 \iff \varepsilon - 1 < r \iff B_n(\iota p_x) < \frac{1}{\iota p_x} + r q^n, \quad (2.40) \\
\rho_2 = 1 & \iff E[S_2] - \bar{s} = 0 \iff \varepsilon - 1 = r \iff B_n(\iota p_x) = \frac{1}{\iota p_x} + r q^n, \quad (2.41) \\
\rho_2 > 1 & \iff E[S_2] - \bar{s} > 0 \iff \varepsilon - 1 > r \iff B_n(\iota p_x) > \frac{1}{\iota p_x} + r q^n. \quad (2.42)
\end{align*}
\]

Since \( \iota p_x \) is fixed then \( r \) is a constant. Applying Proposition 2.8 to this fixed \( r \) there exists a unique \( r, \gamma_n \) in \( ]0, 1[ \) such that

\[
\begin{align*}
B_n(p) & < \frac{1}{p} + r q^n \iff 0 < p < r \gamma_n, \quad (2.43) \\
B_n(p) & = \frac{1}{p} + r q^n \iff p = r \gamma_n, \quad (2.44) \\
B_n(p) & > \frac{1}{p} + r q^n \iff r \gamma_n < p < 1. \quad (2.45)
\end{align*}
\]

Now, since \( \iota p_x \) is given, the proof is ended by (2.43) - (2.40), (2.44) - (2.41) and (2.45) - (2.42). \( \diamond \)
Since values $r\gamma_n$ depend on $\iota p_x$ and $\beta_n$ does not depend on $\iota p_x$ then (ii) and (iii) of Proposition 2.12 are very different from each other. In fact, (iii) is true for a given $\iota p_x$ while (ii) is true for all $\iota p_x$. The following corollary is clearer.

**Corollary 2.6**  
(i) For $n \geq 2$:
\[
\rho_1 < 1, \text{ for all } 0 < \iota p_x \leq 1/n, \\
\rho_1 > 1, \text{ for all } 2.16/n < \iota p_x < 1.
\]

(ii) Let $n \geq 2$ and $r = \iota r_x$ given by (2.29). Assume that $0 < r \leq 3$ for all $2.25/n < \iota p_x < 1$. Then
\[
\rho_2 < 1, \text{ for all } 0 < \iota p_x \leq \beta_n, \\
\rho_2 < 1, \text{ for all } 0 < \iota p_x \leq 1/n, \\
\rho_2 > 1, \text{ for all } 2.25/n < \iota p_x < 1.
\]

**Proof.** This follows from Propositions 2.12, 2.6, 2.7 and 2.8. \hfill \Diamond

The restriction $r \leq 3$ for all $2.25/n < \iota p_x < 1$ in Corollary 2.6 - (ii) is not too
strong as we may see from Table 2.3.

Proposition 2.12 and Corollary 2.6 make clear the fact that the inequalities \( \rho_1 > 1 \) and \( \rho_2 > 1 \) are not true in general. This is also illustrated in Figures 2.2 and 2.3.

### 2.3.2 Variances of \( S_1 \) and \( S_2 \)

The variances of future random shares given by (1.21) and (1.22) are found here. They are important to estimate the risk involving the use of their means to pay future benefits. These formulas are also used to find the variances of the new random future shares defined in Chapter 3.

**Proposition 2.13** The variances of \( S_1 \) and \( S_2 \) are found to be

1. \( V[S_1] = (1 + i)^2 \left[ Q_n(t, p_x) - B_n^2(t, p_x) \right] \).

2. \( V[S_2] = E[S_2^2] - E[S_2]^2, \quad E[S_2^2] = (1 + i)^2 \left\{ \bar{a}_n^2 B_n(t, p_x) - (\bar{a}_n - a_1)^2 \left[ B_n(t, p_x) - (1 - t q_x^n) \right] + R_n(t, p_x) \left( \frac{v_1}{n} + a_1^2 \right) \right\} \),
where \( B_n(t p_x) \) and \( Q_n(t p_x) \) are defined in (1.10) and (1.11), \( a_1 \) and \( v_1 \) are given by (2.2) and (2.3), \( E[S_2] \) by (1.24) and \( R_n(t p_x) = Q_n(t p_x) - B_n(t p_x) \).

**Proof.** (i) From (1.21)

\[
S_1^2 = \begin{cases} 
\frac{n^2(1 - i)^{2t}}{N^2} & \text{if } N = 1, 2, ..., n, \\
0 & \text{if } N = 0.
\end{cases}
\]

Then

\[
E[S_1^2] = (1 + i)^{2t} \sum_{n=1}^{N} \frac{n^2}{m} P_{nm}(t p_x) = (1 + i)^{2t} \left\{ Q_n(t p_x) - B_n^2(t p_x) \right\}.
\]

Using this and (1.23)

\[
V[S_1] = E[S_1^2] - E[S_1]^2 = (1 + i)^{2t} \left\{ Q_n(t p_x) - B_n^2(t p_x) \right\}.
\]

(ii) From (1.22)

\[
S_2^2 = \begin{cases} 
(1 + i)^{2t} \frac{N^2 \left( \bar{\sigma}_1^2 \right) + X + Y}{N^2} & \text{if } N = 1, 2, ..., n, \\
0 & \text{if } N = 0,
\end{cases}
\]

where \( X = 2N\tilde{a}_1 \sum_{i=1}^{D} \tilde{a}_{K_i+1} \) and \( Y = \sum_{i=1}^{D} \sum_{j=1}^{D} \tilde{a}_{K_i+1} \tilde{a}_{K_j+1} \). Then

\[
E[S_2^2] = E[E(S_2^2|N)] = \sum_{n=1}^{N} E[S_2^2|N = m] P_{nm}(t p_x),
\]

\[
= (1 + i)^{2t} \sum_{n=1}^{N} \left( \frac{m^2 \bar{a}_1^2 + E[X|N = m] + E[Y|N = m]}{m^2} \right) P_{nm}(t p_x). \quad (2.46)
\]

Using the fact that all lives are mutually independent we have

\[
E[X|N = m] = 2m\tilde{a}_1(n - m) a_1, \quad (2.47)
\]

\[
E[Y|N = m] = \sum_{i=1}^{n-m} \sum_{j=1}^{n-i} E[\tilde{a}_{K_i+1} \tilde{a}_{K_j+1}|K_i < t, K_j < t].
\]

Since

\[
E[\tilde{a}_{K_i+1} \tilde{a}_{K_j+1}|K_i < t, K_j < t] = \delta_{ij} v_1 + a_1^2,
\]

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then
\[ E[Y | N = m] = (n - m) v_1 + (n - m)^2 a_1^2. \] (2.48)

Replacing (2.47) and (2.48) in (2.46) we get

\[ v^2 E[S_2^2] - \bar{a}_n^2 (1 - \bar{q}_n^m) = \]
\[ \sum_{m=1}^n \left[ \frac{2m \bar{a} q(n - m) a_1 + (n - m) v_1 + (n - m)^2 a_1^2}{m^2} \right] P_{nm}(t_{\bar{p}x}) = \]
\[ 2 \bar{a} q a_1 \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] + \frac{v_1}{n} R_n(t_{\bar{p}x}) + a_1^2 \left[ R_n(t_{\bar{p}x}) - B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] = \]
\[ 2 \bar{a} q a_1 \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] - a_1^2 \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] + R_n(t_{\bar{p}x}) \left( \frac{v_1}{n} + a_1^2 \right) = \]
\[ (2 \bar{a} q a_1 - a_1^2) \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] + R_n(t_{\bar{p}x}) \left( \frac{v_1}{n} + a_1^2 \right) = \]
\[ a_1^2 \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] - (\bar{a} q - a_1)^2 \left[ B_n(t_{\bar{p}x}) - (1 - \bar{q}_n^m) \right] + R_n(t_{\bar{p}x}) \left( \frac{v_1}{n} + a_1^2 \right), \]
from where the result follows. \hfill \Box

As we see in the proof of Proposition 2.13, the notation used by Ramsay is not helpful to find expectations since we are dealing with various random variables at the same time. This complication is due to the use of \( D = n - N \) random terms in the sum that defines \( S_2 \). The following proposition and its proof present a new derivation for the variance of \( S_2 \) based on conditional expectations.

**Proposition 2.14** The variances of \( S_1 \) and \( S_2 \) are given by

(i) \( V[S_1] = (1 + i)^2 t \left( Q_n(t_{\bar{p}x}) - B_n^2(t_{\bar{p}x}) \right) \).

(ii) \( v^2 V[S_2] = a_1^2 \left[ Q_n(t_{\bar{p}x}) - B_n^2(t_{\bar{p}x}) \right] + 2 i q_n^m a_1 \left( \bar{a} q - a_1 \right) B_n(t_{\bar{p}x}) + i q_n^m (1 - i q_n^m) \left( \bar{a} q - a_1 \right)^2 + \left[ Q_n(t_{\bar{p}x}) - B_n(t_{\bar{p}x}) \right] v_1/n. \)

**Proof.** (i) We can write
\[ S_1 = (1 + i)^t X, \] (2.49)
where \( X \) is given by (1.13) and apply (1.14).
(ii) We may write

\[ S_2 = (1 + i)^t U, \quad (2.50) \]

\[ U = \begin{cases} 
\frac{Y}{N} & \text{if } N = 1, \ldots, n, \\
0 & \text{if } N = 0,
\end{cases} \]

\[ Y = \sum_{i=1}^{n} Y_i, \quad (2.51) \]

\[ Y_i = \begin{cases} 
\tilde{a}_{K_i+1} & \text{if } K_i < t, \\
\tilde{a}_l & \text{if } K_i \geq t,
\end{cases} \]

\[ = a_{K_i+1} I_{[K_i < t]} + \tilde{a}_l I_{[K_i \geq t]}. \]

Therefore,

\[ E[Y] = n \tilde{a}_{x;\tilde{a}}, \quad \text{by } (1.2), \quad (2.52) \]

\[ E[Y|N] = (n - N) a_1 + N \tilde{a}_l, \quad (2.53) \]

\[ V[Y|N] = (n - N) v_1, \quad (2.54) \]

\[ E[U|N] = a_1 X + (\tilde{a}_l - a_1) I_{[N > 0]}, \quad \text{by } (2.53) \text{ and } (1.13). \quad (2.55) \]

Using (2.55), (2.1) and (1.14) we get

\[ V[E(U|N)] = \]

\[ a_1^2 \left[ Q_n(tp_x) - B_n^2(tp_x) \right] + 2 q^n_a a_1 \left( \tilde{a}_l - a_1 \right) B_n(tp_x) + q^n_a \left( 1 - q^n_a \right) \left( \tilde{a}_l - a_1 \right)^2. \]

Since (2.54) and (1.13) imply \( V[U|N] = (X^2 - X) v_1/n \), then (1.14) yields

\[ E[V(U|N)] = [Q_n(tp_x) - B_n(tp_x)] v_1/n. \]

Finally from (2.50) we have \( v^{2t} V[S_2] = V[U] = V[E(U|N)] + E[V(U|N)]. \)

\[ \Diamond \]

**Corollary 2.7** For \( S_1 \) and \( S_2 \) we have

(i) \( \lim_{n \to \infty} S_1 = t s_x. \) \quad (ii) \( \lim_{n \to \infty} S_2 = \tilde{s}_{x;\tilde{a}}. \)

**Proof.** (i) Using (2.7) and (2.14) in Proposition 2.13 - (i) we get \( \lim_{n \to \infty} V[S_1] = 0 \) and from Proposition 2.11 - (iii) we see that \( \lim_{n \to \infty} E[S_1] = t s_x. \)

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(ii) Use of (2.7) and (2.14) in Proposition 2.13 - (ii) yields

$$\lim_{n \to \infty} E[S^2_2] = (1 + \tau)^2 \left[ \tilde{d}^2 \frac{1}{t \tilde{p}_x} - (\tilde{a} - a_1)^2 \left( \frac{1}{t \tilde{p}_x} - 1 \right) + \left( \frac{1}{t \tilde{p}_x^2} - \frac{1}{t \tilde{p}_x} \right) a_1^2 \right],$$

$$= t s_x^2 \left[ \tilde{a} \tilde{p}_x - \tilde{a} a_1 \right]^2 t \tilde{p}_x^{-1} t q_x + a_1^2 t q_x^2,$$

$$= t s_x^2 \left[ \tilde{a} \tilde{p}_x + a_1 t q_x \right]^2 = t s_x^2 \tilde{a}_{x;\tilde{q}} = \tilde{s}_{2;\tilde{q}}.$$  

Then Propositions 2.11 - (iv) and 2.13 - (ii) imply \( \lim_{n \to \infty} V[S_2] = 0. \)

Another advantage of Proposition 2.14 is the fact that the proof of Corollary 2.7 is straightforward since from there, \( \lim_{n \to \infty} V[S_2] = 0 \) is trivial.
Chapter 3

Other Approaches to Future Shares

The first part of this chapter is dedicated to random loss variables, their means and variances. In the second part a reformulation of future random shares is proposed under the principle of equivalence (c.f. Bowers et. al. [1], p. 169).

3.1 Random loss variables

In this section we define random losses for Problems 1 and 2, defined in Chapter 1, and find their expectations and variances. We also prove that under the equivalence principle Ramsay’s stochastic definitions of future share are not unbiased.

Definition 3.1 Random losses. Let $X_1$ and $X_2$ be any random variables proposed as solution to Problems 1 and 2, respectively. Denote by $\pi_1 = \pi_1(X_1) = \nu^tE[X_1]$ and $\pi_2 = \pi_2(X_2) = \nu^tE[X_2]$. Therefore, if $Y$ is defined as in (2.51), the random losses at time 0 are respectively given by

\begin{align}
L_1(X_1) &= N\pi_1 - n, \quad (3.1) \\
L_2(X_2) &= N\pi_2 - Y. \quad (3.2)
\end{align}
Proposition 3.1 The expectations and variances of $L_1$ and $L_2$ are found to be

$$E[L_1(X_1)] = n (\mu_x \pi_1 - 1),$$
$$E[L_2(X_2)] = n (\mu_x \pi_2 - \bar{a}_{x;\eta}),$$
$$V[L_1(X_1)] = n \mu_x \eta q_x \pi_1^2,$$
$$V[L_2(X_2)] = n \mu_x \eta q_x \left[ \pi_2 (\pi_2 + a_1 - \bar{a}_\eta)^2 + v_1 \right],$$

where $a_1$ and $v_1$ are given by (2.2) and (2.3).

Proof. The first formula is true since $E[N] = n \mu_x$. The second equality is obtained from (2.52). The third comes from $V[N] = n \mu_x \eta q_x$. Finally,

$$V[L_2(X_2)] = V\{E[L_2(X_2)|N]\} + E\{V[L_2(X_2)|N]\},$$
$$= V[N \pi_2 - E(Y|N)] + E[V(Y|N)], \text{ from (3.2)},$$
$$= V\{N \pi_2 - [(n - N)a_1 + N\bar{a}_\eta]\} + E[(n - N)v_1], \text{ by (2.53) and (2.54)},$$
$$= V[(\pi_2 + a_1 - \bar{a}_\eta)N] + E[(n - N)v_1],$$
$$= n \mu_x \eta q_x (\pi_2 + a_1 - \bar{a}_\eta)^2 + n \mu_x \eta q_x v_1,$$
$$= n \mu_x \eta q_x \left[ \pi_2 (\pi_2 + a_1 - \bar{a}_\eta)^2 + v_1 \right]. \quad \diamond$$

Under the principle of equivalence we want unbiased random losses, i.e.,

$$E[L_1(X_1)] = E[L_2(X_2)] = 0.$$

From Proposition 3.1 this is equivalent to

$$E[X_1] = \mu_x \eta,$$
$$E[X_2] = \bar{\eta}. \quad (3.3)$$

Then solving Problems 1 and 2 under the principle of equivalence leads to find $X_1$ and $X_2$ such that (3.3) and (3.4) hold, respectively. The deterministic actuarial solutions $\mu_x \eta$ and $\bar{\eta}$ trivially satisfy (3.3) and (3.4), respectively. In the meantime, Ramsay's stochastic solutions $S_1$ and $S_2$, in general do not satisfy conditions (3.3)
and (3.4). In fact, from Proposition 2.12, the random losses \( L_1(S_1) \) and \( L_2(S_2) \) never are unbiased when \( n = 1 \). For \( n \geq 2 \), \( L_1(S_1) \) is unbiased only if \( \iota p_x = \beta_n \), while \( L_2(S_2) \) is unbiased only if \( \iota p_x = r \gamma_n \). We keep in mind that these values tend to 0 as \( n \) increases.

Equations (3.3) and (3.4) imply that the actuarial present value of the future benefit equals 1 in Problem 1 and \( \bar{a}_{x:n} \) in Problem 2. Under Ramsay’s definitions these actuarial present values are given respectively by \( \rho_1 \) and \( \bar{a}_{x:n} \rho_2 \).

### 3.2 New definitions of stochastic future shares

We need to modify \( S_1 \) and \( S_2 \) so that the new random variables satisfy (3.3) and (3.4), respectively. Since there are many different ways to do this we present here two of them.

**Definition 3.2** Define the random future share for Problem 1 as any of the following

\[
S_{11} = S_1 + \left( \frac{\iota s_x - E[S_1]}{1 - \iota q_x^n} \right) I_{[N > 0]}, \tag{3.5}
\]

\[
S_{12} = S_1 + \iota s_x - \left( \frac{E[S_1]}{1 - \iota q_x^n} \right) I_{[N > 0]}.
\] \tag{3.6}

For Problem 2 it is defined as any of the following

\[
S_{21} = S_2 + \left( \frac{\iota \bar{s}_x:n - E[S_2]}{1 - \iota q_x^n} \right) I_{[N > 0]}, \tag{3.7}
\]

\[
S_{22} = S_2 + \iota \bar{s}_x:n - \left( \frac{E[S_2]}{1 - \iota q_x^n} \right) I_{[N > 0]}.
\] \tag{3.8}

Observe that in (3.5) and (3.7) we added a correction term only when \( N > 0 \), while in (3.6) and (3.8) we added two terms, one when \( N > 0 \) and another when \( N = 0 \). The terms which provide minimal variance of the future random share are given in Lemma 2.2. We note that if \( n = 1 \) then \( S_{12} = \iota s_x \) and \( S_{22} = \iota \bar{s}_x:n \), i.e., they are deterministic.
Proposition 3.2 Consider the random variables $S_{11}$, $S_{12}$, $S_{21}$ and $S_{22}$ given by (3.5), (3.6), (3.7) and (3.8), respectively. Then

(i) They have unbiased random losses, defined by (3.1) and (3.2).

(ii) Their variances are found to be

\[
V[S_{11}] = V[S_1] + \left( \frac{\epsilon q_x^n}{1 - \epsilon q_x^2} \right) (s_x^2 - E[S_1]^2),
\]

\[
V[S_{12}] = V[S_1] - \left( \frac{\epsilon q_x^n}{1 - \epsilon q_x^2} \right) E[S_1]^2,
\]

\[
V[S_{21}] = V[S_2] + \left( \frac{\epsilon q_x^n}{1 - \epsilon q_x^2} \right) (s_x^n - E[S_2]^2),
\]

\[
V[S_{22}] = V[S_2] - \left( \frac{\epsilon q_x^n}{1 - \epsilon q_x^2} \right) E[S_2]^2,
\]

where $V[S_1]$ and $V[S_2]$ are given by Proposition 2.13 or 2.14, while $E[S_1]$ and $E[S_2]$ are as in (1.23) and (1.24).

(iii) The variances of the corresponding losses are given by

\[
V[L_1(S_{11})] = V[L_1(S_{12})] = V[L_1(s_x)] = \frac{n \epsilon q_x}{\epsilon p_x},
\]

\[
V[L_2(S_{21})] = V[L_2(S_{22})] = V[L_2(s_x^n)] = \frac{n \epsilon q_x}{\epsilon p_x} [a_1^2 + \epsilon p_x v_1],
\]

where $a_1$ and $v_1$ are defined in (2.2) and (2.3).

(iv) We also have:

\[
V[S_1|N > 0] = V[S_{11}|N > 0] = V[S_{12}|N > 0] = \frac{E[S_1^2]}{1 - \epsilon q_x^2} - \left( \frac{E[S_1]}{1 - \epsilon q_x^2} \right)^2,
\]

\[
V[S_2|N > 0] = V[S_{21}|N > 0] - V[S_{22}|N > 0] = \frac{E[S_2^2]}{1 - \epsilon q_x^2} - \left( \frac{E[S_2]}{1 - \epsilon q_x^2} \right)^2,
\]

\[
E[S_1 S_2] = (1 + \epsilon)^{2n} \left( a_1 Q_n(\epsilon p_x) + (\hat{a}_1 - a_1) B_n(\epsilon p_x) \right),
\]

\[
Cov[S_1, S_2|N > 0] = Cov[S_{11}, S_{21}|N > 0] = Cov[S_{12}, S_{22}|N > 0],
\]

\[
= \frac{E[S_1 S_2]}{1 - \epsilon q_x^2} - \frac{E[S_1] E[S_2]}{(1 - \epsilon q_x^2)^2}.
\]
Proof. (i) Clearly (3.3) holds for \( S_{11} \) and \( S_{12} \), while (3.4) holds for \( S_{21} \) and \( S_{22} \).

(ii) It follows from (2.1).

(iii) Here \( \pi_1 = 1/\kappa_p \), \( \pi_2 = \bar{a}_x / \kappa_p \), and the result follows from Proposition 3.1.

(iv) We need to prove only the formula for \( E[S_1S_2] \). To this

\[
E[S_1S_2] = E\{E[S_1S_2|N]\},
\]

\[
= (1 + i)^2 E\{XE[U|N]\}, \quad \text{by (2.49) and (2.50)},
\]

\[
= (1 + i)^2 E\{X [a_1X + (\bar{a}_t - a_1) I_{[N>0]}]\}, \quad \text{from (2.55)},
\]

\[
= (1 + i)^2 E\left[a_1X^2 + (\bar{a}_t - a_1) X\right],
\]

\[
= (1 + i)^2 \left[a_1 Q_n(\kappa_p) + (\bar{a}_t - a_1) B_n(\kappa_p)\right], \quad \text{by (1.14)}. \quad \diamond
\]

Variances of \( L_1(S_1) \) and \( L_2(S_2) \) can be computed from Proposition 3.1 with \( \pi_1 = v^t E[S_1] \) and \( \pi_2 = v^t E[S_2] \). We note that for \( \kappa_p \) large enough, say \( \kappa_p > \beta_n \) (see Proposition 2.7 and Corollary 2.6) for \( L_1(S_1) \) and \( \kappa_p > 2.25/n \) for \( L_2(S_2) \), these variances are larger than those given in Proposition 3.2 - (iii).
Chapter 4

An Application and Numerical Results

This chapter presents the deterministic approach of retrospective reserves in a particular case and also suggests a stochastic definition. The chapter ends with some numerical results and conclusions.

4.1 Reserves

In the modern theory of life contingencies there is no stochastic definition for retrospective net premium reserves. They are defined only as the mean of the stochastic prospective loss, then by definition both retrospective and prospective net reserves are the same.

In this section the results from Chapter 3 are extended to retrospective net reserves for a t-year pure endowment insurance. Other cases can be considered in a similar way. We start presenting the deterministic approach.

4.1.1 The actuarial deterministic approach

Let \( P_{x\tilde{\eta}} = P(A_{x\tilde{\eta}}) \) be the net level annual premium for a unit t-year endowment insurance for \((x)\). The loss for this insurance is

\[
L = Z - P_{x\tilde{\eta}} Y,
\]

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where \( Y \) and \( Z \) are given by (1.1) and (1.5), respectively. Taking expectations and using the equivalence principle we get
\[
P_{x: \overline{t}} = \frac{A_{x: \overline{t}}}{\overline{a}_{x: \overline{t}}} = \frac{1 - d \overline{a}_{x: \overline{t}}}{\overline{a}_{x: \overline{t}}}.
\]
Since \( Y = (1 - Z)/d \) then
\[
L = \left(1 + \frac{P_{x: \overline{t}}}{d}\right)Z - \frac{P_{x: \overline{t}}}{d}.
\]
Taking variance and using (1.7)
\[
V[L] = \left(1 + \frac{P_{x: \overline{t}}}{d}\right)^2 \left(2A_{x: \overline{t}} - A_{x: \overline{t}}^2\right).
\]
Let \( 0 \leq h \leq t \) be an integer and \( J \) the curtate future life time of \( (x + h) \). The p.f. of \( J \) is given by \( j p_{x+h} q_{x+h+j} \), for \( j = 0, 1, \ldots, w - x - h - 1 \). At time \( h \) the present value of future benefits for this insurance is
\[
Z_1 = \begin{cases} 
  u^{J+1} & \text{if } J < t - h, \\
  u^{t-h} & \text{if } J \geq t - h.
\end{cases}
\]
Denote by \( \overline{a}_{\overline{t}} = 0 \). Then the present value of future contributions is \( P_{x: \overline{t}} Y_1 \), where
\[
Y_1 = \begin{cases} 
  \overline{a}_{J+1} & \text{if } J < t - h, \\
  \overline{a}_{t-h} & \text{if } J \geq t - h.
\end{cases}
\]
The difference
\[
hL = Z_1 - P_{x: \overline{t}} Y_1, \quad (4.1)
\]
is called the prospective loss at time \( h \) for this insurance. Its mean, denoted by \( _h V_{x: \overline{t}} \) and given by
\[
_h V_{x: \overline{t}} = \begin{cases} 
  A_{x+h; \overline{t-h}} - P_{x: \overline{t}} \overline{a}_{x+h; \overline{t-h}} & \text{if } h < t, \\
  1 & \text{if } h = t,
\end{cases} \quad (4.2)
\]
is called the prospective net premium reserve at time \( h \). At the beginning the reserve has been defined using the principle of equivalence, then \( 0 V_{x: \overline{t}} = 0 \).

Since \( Y_1 = (1 - Z_1)/d \) then
\[
_h L = \left(1 + \frac{P_{x: \overline{t}}}{d}\right)Z_1 - \frac{P_{x: \overline{t}}}{d}.
\]
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From this and (1.7) we get

\[ V[h,L] = \begin{cases} 
(1 + \frac{P_{x:1}}{d})^2 \left( 2A_{x+h:1} - A_{x+h,1} \right) & \text{if } h < t, \\
0 & \text{if } h = t.
\end{cases} \] (4.3)

Formula in (4.2) can be written as

\[ hV_{x:1} = P_{x:1} \bar{s}_{x:1} - A_{1:1} h s_x. \] (4.4)

The expression \( A_{1:1} h s_x \) is usually denoted by \( hK_x \) and is called the accumulated cost of insurance. Formula in (4.4) is called the retrospective net premium reserve of the \( t \)-year endowment insurance. We observe that by definition the retrospective and the prospective net premium reserves are the same.

According to the deterministic definition of future values, expression \( P_{x:1} \bar{s}_{x:1} \) is the actuarial accumulated value of an annuity of \( P_{x:1} \) per year at the end of \( h \) years, while \( hK_x \) is the assessment against each of the \( l_{x+h} \) survivors to provide for the accumulated value of the death claims in the survivorship group between ages \( x \) and \( x + h \). Thus, the reserve given by (4.4) can be interpreted as the difference between the net premiums, accumulated with interest and shared among the survivors at age \( x + h \), and the accumulated cost of insurance (see Bowers et. al. [1], pp. 216-217).

### 4.1.2 A stochastic approach

Let \( \pi \) be the individual net level annual premium for a unit \( t \)-year endowment insurance for each member of \( n \) mutually independent lives age \( x \). At time 0 the individual loss for this insurance can be defined to be

\[ L(n) = \frac{Z - \pi Y}{n}, \]

where \( Z = \sum_{i=1}^{n} Z_i \), \( Z_i \) is defined as in (1.5) with \( K = K_i \) and \( Y \) is given by (2.51). Since all lives are mutually independent then from (1.6), solving \( E[L(n)] = 0 \) we get \( \pi = P_{x:1} \).

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Now suppose that each member of the group is participating in an $h$-year term life insurance of 1 payable at the end of the year of death. Let $\pi_h$ be the individual net single premium for it. The individual loss for this insurance can be defined as

$$L_1(n) = \frac{Z - n\pi_h}{n},$$

where $Z = \sum_{i=1}^{n} Z_i$ and $Z_i$ is given by (1.3) with $K = K_i$ and $t = h$. Solving $E[L_1(n)] = 0$ we get $\pi_h = A_{1,0}^{1,0}.$

For a $t$-year endowment insurance we want to define a possible retrospective random loss at time $h$ given that there is at least one survivor, that is, $N > 0$. As one of many possible definitions we propose the following. The individual retrospective loss at time $0 \leq h \leq t$ is denoted by $\mathbb{L}(n)$ and is defined as the accumulated value of past premiums minus the accumulated value of past death benefits. For $h = 0$ it gives 0. For $1 \leq h \leq t$, from (3.7) we may interpret the expression $P_{x:0}S_{21}$ (here $S_{21}$ is computed using $h$ instead of $t$) as the future value at time $h$ of the individual annuity $P_{x:0}$, with accumulation under interest and survivorship. Similarly, from (3.5), expression $A_{1,0:0}^{1,0}S_{11}$ ($S_{11}$ computed using $h$) can be seen as the future value at time $h$ of individual past death benefits up to time $h$ (or the accumulated random cost of insurance). Therefore,

$$\mathbb{L}(n) = \begin{cases} 
P_{x:0}S_{21} - A_{1,0:0}^{1,0}S_{11} & \text{if } 0 < h \leq t, \\
0 & \text{if } h = 0.
\end{cases}$$

The retrospective net premium reserve at time $h$, denoted by $\mathbb{V}_{x:0}^h(n)$, is given by

$$\mathbb{V}_{x:0}^h(n) = E[\mathbb{L}(n)|N > 0] = \frac{\mathbb{V}_{x:0}^h(n)}{1 - hq_x^n}.$$ 

We see that $\mathbb{V}_{x:0}^h(n) > \mathbb{V}_{x:0}^h(n)$ for $h \geq 1$. Also $\mathbb{V}_{x:0}^h(n) \approx \mathbb{V}_{x:0}^h$ for large values of $n$. It is also clear that $\mathbb{V}_{x:0}^h(1)$ is deterministic. Besides, we observe that when $h$ is close to $t$ our definition gives $\mathbb{V}_{x:0}^h(n) > 1$. In particular when $h = t$ we get $\mathbb{V}_{x:0}^h(n) = \frac{1}{1 - tq_x^n} > 1$. This property is not desirable since we would like that $\mathbb{V}_{x:0}^h(n) \leq 1$. To fix this problem we can use in $\mathbb{L}(n)$ the accumulation factors $S_{12}$ and $S_{22}$ instead of $S_{11}$ and
\( S_{21}, \text{i.e.,} \)

\[
\begin{align*}
\mathcal{h}L(n) &= \begin{cases} 
P_{x;\tilde{y}} S_{22} - A_{\frac{1}{2};\tilde{y}} S_{12} & \text{if } 0 < h \leq t, \\
0 & \text{if } h = 0. 
\end{cases} 
\tag{4.5}
\end{align*}
\]

Now since for \( 1 \leq h \leq t \) we have \( E[S_{12}|N > 0] = \mathcal{h}S_x \) and \( E[S_{22}|N > 0] = \mathcal{s}_{x;\tilde{y}} \), then

\[
\mathcal{h}V_{x;\tilde{y}}(n) = E[\mathcal{h}L(n)|N > 0] = \mathcal{h}V_{x;\tilde{y}}.
\]

Therefore, \( \mathcal{h}V_{x;\tilde{y}}(n) = \mathcal{h}V_{x;\tilde{y}} \) for all \( 0 \leq h \leq t \).

**Proposition 4.1** For \( \mathcal{h}L(n) \) given in (4.5) we have \( V[\mathcal{h}L(n)|N > 0] = 0 \). If \( 1 \leq h \leq t \) then

\[
V[\mathcal{h}L(n)|N > 0] = P^2_{x;\tilde{y}} \left[ \frac{E[S^2_{2}]}{1 - hq^2_x} - \left( \frac{E[S_{2}]}{1 - hq^2_x} \right)^2 \right] + A^2_{\frac{1}{2};\tilde{y}} \left[ \frac{E[S^2_{1}]}{1 - hq^2_x} - \left( \frac{E[S_{1}]}{1 - hq^2_x} \right)^2 \right]
\]

\[
- 2P_{x;\tilde{y}} A_{\frac{1}{2};\tilde{y}} \left[ \frac{E[S_{1}S_{2}]}{1 - hq^2_x} - \frac{E[S_{1}]E[S_{2}]}{(1 - hq^2_x)^2} \right].
\]

**Proof.** It follows from (4.5) and Proposition 3.2 - (iv). \( \diamond \)

There is a way similar to that given in (4.1) to define a prospective random loss such that the retrospective and the prospective net premium reserves are the same. In fact, for a \( t \)-year endowment insurance we denote by \( \mathcal{h}PL(n) \) the average of the individual prospective random losses at time \( 0 \leq h \leq t \). Given that \( N > 0 \) we denote by \( J_i, i = 1, 2, \ldots, N \), the curtate future life time of the \( i^{th} \) life and by \( \mathcal{h}L_i \) the prospective loss given by (4.1) with \( J = J_i \). Therefore, the prospective random loss is given by

\[
\mathcal{h}PL(n) = \begin{cases} 
\frac{\sum_{i=1}^{N} \mathcal{h}L_i}{N} & \text{if } N = 1, \ldots, n, \\
0 & \text{if } N = 0.
\end{cases}
\]

The prospective net single premium is then

\[
E[\mathcal{h}PL(n)|N > 0] = E\{E[\mathcal{h}PL(n)|N, N > 0]\} = E[\mathcal{h}V_{x;\tilde{y}}] = \mathcal{h}V_{x;\tilde{y}}.
\]

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The variance of the prospective loss is

\[ V[hPL(n)|N > 0] = V\{E[hPL(n)|N, N > 0]\} + E\{V[hPL(n)|N, N > 0]\}, \]
\[ = V[V_{x\bar{n}}] + E\{V[hL]/N|N > 0\}, \text{ by (4.1) and (4.3)}, \]
\[ = V[hL]b_1(n, hp_x). \tag{4.6} \]

We see that

\[ \lim_{n \to \infty} V[hPL(n)|N > 0] = 0, \]
\[ V[hPL(n)|N > 0] \leq V[hL], \]
\[ V[0PL(n)|N > 0] = V[0L]/n, \]
\[ V[hPL(1)|N > 0] = V[hL]. \]

### 4.2 Numerical results

Here we present some numerical results using the formulas for variances of future shares and reserves found in previous sections and chapters. Ramsay's assumptions in page 12 are used in our computations when no other assumptions are specified.

#### 4.2.1 Future shares

Tables 4.1 and 4.2 show the variances of the different future random shares defined in pages 10 and 51. The formulas are given in Propositions 2.14 and 3.2 - (ii). We can see that differences occur only for small values of \( n p_x \), precisely when Ramsay's formulas differ from the deterministic formulas. The cases \( n = 500 \) and \( n = 5000 \) are presented to see how fast they decrease to 0 as \( n \) increases. There is a substantial difference between both tables. Variances in Table 4.1 are small and decrease quickly, but variances in Table 4.2 are large and decrease slowly. This is natural since in Problem 1 [see page 4] there is only a single payment, while payments in Problem 2 depend on the random variables \( K_i \), for \( i = 1, \ldots, n \), and hence involve more uncertainty, especially when \( t \) is large as seen in Table 4.2. When \( n = 1 \), the variances of \( S_{12} \) and \( S_{22} \) are null since in this case these expressions are deterministic.
Table 4.1: Variances of $S_1$, $S_{11}$ and $S_{12}$ when $(x)$ survives to 65

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<tr>
<th>$n$</th>
<th>$x = 20$</th>
<th>$x = 30$</th>
<th>$x = 40$</th>
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Table 4.2: Variances of $S_2$, $S_{21}$ and $S_{22}$ when $(x)$ survives to 65

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</table>
In Tables 4.3 and 4.4 we present the variances of the random losses \( L_1(S_1) \), \( L_1(t s_x) \), \( L_2(S_2) \) and \( L_2(\bar{s}_{x:n}) \), defined in (3.1) and (3.2). The formulas are given in Proposition 3.1. As expected, variances increase as the size of the group increases. Also, when \( ts_x \) and \( \bar{s}_{x:n} \) (or any of \( S_{11}, S_{12}, S_{21}, S_{22} \)) are used, the variances are smaller, except for very small values of \( n \). For large values of \( n \) they are asymptotically the same.

Table 4.3: Variances of \( L_1(S_1) \) and \( L_1(t s_x) \) when \( (x) \) survives to 65

<table>
<thead>
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<th>( n )</th>
<th>( x = 20 )</th>
<th>( x = 30 )</th>
<th>( x = 40 )</th>
<th>( x = 50 )</th>
<th>( x = 60 )</th>
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Table 4.4: Variances of \( L_2(S_2) \) and \( L_2(\bar{s}_{x:n}) \) when \( (x) \) survives to 65

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<th>( x = 30 )</th>
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61
4.2.2 Reserves

The variance of \( hL(n) \) for a 45-year endowment insurance of 1 for (20), given in Proposition 4.1, is presented in Table 4.5. If \( n = 1 \) we see that \( V[hL(1)|N > 0] = 0 \), showing that the variable is deterministic, as well as when \( h = 0 \). \( V[hL(n)|N > 0] \) increases with \( h \) and tends to 0 as \( n \) increases. Hence, for large \( n \), \( hL(n) \) degenerates to the deterministic \( hV_{20}\).

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The variance of \( hPL(n) \) for a 45-year endowment insurance of 1 for (20), given by (4.6), is presented in Table 4.6. It equals \( V[hL] \) for \( n = 1 \) and tends to 0 as \( n \) increases. Hence, for large \( n \), \( hPL(n) \) and \( hV_{20} \) are asymptotically the same. \( V[hPL(n)] \), for a fixed \( n \), first increases and then decreases to 0. This is seen in Figure 4.1.
Table 4.6: $V[hPL(n)]$ for a 45-year endowment insurance of 1 for (20)

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<th>$n = 2$</th>
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</table>

Figure 4.1: $V[hPL(n)]$ for a 45-year endowment insurance of 1 for (20)
4.3 Conclusions

The results on the polynomials $B_n$ are important to study the ratios $\rho_1$ and $\rho_2$. They also can be used to investigate when some approximations for the first inverse moment of a positive binomial variate, found in [13], [10], [5] and [12], are appropriate.

Two different ways of fixing the bias problem of Ramsay’s stochastics future shares under the principle of equivalence are proposed. $S_{12}$ and $S_{22}$ seem to be more convenient definitions. With these new definitions it is proved that the use of the deterministic formulas is adequate. These new random future shares can be used to estimate the inaccuracy involving the use of $t_s^2$ and $\bar{s}_{t}^2$.

We suggest a random retrospective loss variable that enables to estimate the inaccuracy of $\lambda V_{t}$]. It seems that the use of the deterministic $\lambda V_{t}$ is quite adequate. Under the proposed stochastic approach the retrospective and the prospective net single premium reserves are the same.

The stochastic life contingencies problems considered here focus on the definition of future shares with constant interest but random survivorship. Clearly, this research needs to be extended to the stochastic interest case, as proposed in the earlier works of Martin-Löf [8] and Frees [2].
Bibliography


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