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The $\lambda\phi^n$ Theory: Feynman Rules, Renormalizability, Regularization and Renormalization

Mohamed Amjad Husain

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ABSTRACT

The $\lambda\phi^n$ Theory: Feynman Rules, Renormalizability, Regularization and Renormalization

Mohamed Amjad Husain

Functional techniques are used to establish the Feynman Rules for the $\lambda\phi^n$ theory. Dimensional analysis of the coupling constant (λ) along with Dyson's power counting method is utilized to examine the renormalizability (non-renormalizability) of the theory in a variable space time dimension. A prescription for achieving dimensionally regularized integrals from divergent Feynman integrals is given. The method is applied to the one-loop graphs from $\lambda\phi^3$, $\lambda\phi^4$ and $\lambda\phi^n$ theories. The results are found to be consistent with the already existing ones.

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INTRODUCTION

The structure of quantum field theory simplifies when one considers a theory involving the boson field only. These simplifications are important in that they improve our ability to deal with the more complicated theories. The purpose of this thesis is to investigate certain aspects of the $\lambda\phi^n$ theory such as Feynman Rules, Ultraviolet (UV) divergences, regularization and renormalization.

The Feynman Rules, developed in Chapter I for $\lambda\phi^4$ -theory and in Chapter II for $\lambda\phi^n$ theory, would generally lead to Feynman amplitudes containing loop integrations which are infinite for both large and small loop momenta. For large loop momenta the divergences are known as UV divergences. These divergences are discussed in Chapter IV for the $\lambda\phi^n$ theory. The divergences that arise for small loop momenta are called Infra-red (IR) divergences and are present only in massless theories.

Divergences are removed by renormalization. Renormalization is essential, for otherwise most field theories do not exist. Pertubative renormalization is dealt with in Chapter III in a general sense while Chapter VI is devoted to regularization and renormalization of some simple examples. The fundamental result of renormalization theory states that to all order in pertubation theory the UV divergences of a quantum field theory may be formally absorbed into the parameters defining the theory while locality, unitarity and Lorentz invariance are maintained. Theories with considerable predictive powers are specified by only a finite number of parameters. Such theories are called renormalizable theories. Thus there are renormalizable and nonrenormalizable theories.

The word "nonrenormalizable" may be misleading. It does not mean that such theories cannot be made finite but rather that the multiplication of their divergences, and hence of counterterms, make then unrealistic in the framework of the pertubation expansion. After renormalization they will depend on an infinite number of parameters.

A necessary precusor of renormalization is the regulatization of Feynman integrals. There are many techniques available to achieve regularized integrals. In Chapter V a few of the standard techniques used are reviewed with special emphasis on the latest technique, dimensional regulatization.

Regularization is the introduction of a cut-off parameter into a Feynman integral in such a way as to make the divergences appear only as the cut-off parameters tends to some limiting value. By this cunning trick, it becomes possible to make mathematically respectable what would otherwise be purely formal manipulation of divergent quantities.

A good regularization method maintains as many of the desirable features of the theory as it can. The early proofs in renormalization used Pauli-Villars[2] regulators which maintains manifest Poincaré symmetry. The most recent regulatization technique is dimensional regularization invented by t'Hooft and Veltman[3]. A very important and attractive feature of this method is that is can also regulate infra-red divergences.

The originality of this thesis is contained mainly in Chapters II and IV. To my knowledge, renormalization of the one-loop diagram from ϕ^n theory (Section 6.5) is also original work.

CHAPTER I

FEYNMAN RULES FOR THE $\lambda\phi^4$ THEORY

In this chapter the equation of motion for the $\lambda\phi^4$ theory is solved by expanding the generating functional, to be defined later, in powers of the coupling constant λ. The two and four-points functions are then used to obtain the Feynman Rules for this theory.

1.1 Equation of Motion for the $\lambda\phi^4$ Theory

The quantum field theory that is considered in this thesis may be described by the Lagrangian given by Equation (1.1).

$$L = \tfrac{1}{2}\, \partial_\mu\phi(x)\, \partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) + L_{int} \qquad (1.1)$$

L_{int}, the interaction Lagrangian, can take many forms; the most general form consists of the fermion field (ψ), the electromagnetic interaction (A_μ), a polynomial in the boson field (ϕ) and their derivatives. In particular, when L_{int} is set equal to zero, the resulting Equation (1.2) describes a free particle of mass m.

$$L_0 = \tfrac{1}{2}\partial_\mu\phi(x)\, \partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) \qquad (1.2)$$

A much more interesting Lagrangian is that obtained by setting L_{int} equal to $\frac{-\lambda\phi^4(x)}{4!}$. Equation (1.1) now reads

$$L = \tfrac{1}{2}\partial_\mu\phi(x)\, \partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi^2(x) - \frac{\lambda\phi^4(x)}{4!} \qquad (1.3)$$

The quantum field theory described by Equation (1.3) is known in the literature as the $\lambda\phi^4$ theory or the ϕ^4 theory. The minus sign in the interaction term of (1.3) is the one used in a classical field theory to ensure stability of the solution $\phi(x) = 0$ of the equation of motion (Itzykson and Zuber)[4]. The 4! in the denominator will take care of combinatorial factors.

The equation of motion for the $\lambda\phi^4$ theory is obtained by inserting Equation (1.3) into the variational equation given by

$$\frac{\partial}{\partial x_\mu} \frac{\delta L}{\delta \partial_\mu \phi(x)} - \frac{\delta L}{\delta \phi(x)} = 0 \tag{1.4}$$

The resulting equation of motion is

$$(\Box + m^2) \phi(x) = \frac{-\lambda\phi^3}{3!} \tag{1.5}$$

1.2 Vacuum Expectation Value of the Time-Ordered Product

The nonlinear character of Equation (1.5) makes it difficult to obtain solutions by some general method. The usual technique is to obtain a pertubation expansion of Feynman diagrams for the Lagrangian (1.3). This is exactly what was done by Nash[5]. His work will be followed in the development of the Feynman Rules for the $\lambda\phi^4$ theory.

The solutions of Equation (1.5) are the Green's functions. There exists a simple relation between the Green's functions and the vacuum expectation value (VEV) of the time-ordered product. Equation (1.6) defines the vacuum expectation value of the time-ordered product for two scalar fields

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \theta(x_o-y_o)\,\langle 0|\phi(x)\phi(y))|0\rangle + \theta(y_o-x_o)\,\langle 0|\phi(y)\phi(x)|0\rangle \tag{1.6}$$

where θ is the step function defined as

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

If $G_N(x_1 \ldots x_N)$ represents the N-point Green's function and $\langle 0|T(\phi(x_1)\phi(x_2) \ldots \phi(x_N))|0\rangle$ is the vacuum expectation value of the time-ordered product for N scalar fields, the simple relation connecting them is

$$G_N\,(x_1 \ldots x_N) = \langle 0|T(\phi(x_1)\phi(x_2) \ldots \phi(x_N))|0\rangle \tag{1.6a}$$

There are two equivalent formulae expressing the vacuum expectation value of the time-ordered product of N scalar fields in terms of the interaction Lagrangian. The first is due to Gell-Mann and Low[6]; their formula takes the form

$$\langle 0|T(\phi(x_1)\phi(x_2) \ldots \phi(x_N))|0\rangle = \langle 0|T\{\exp[i\int L_{int}(y)d^4y]\phi_{in}(x_1)\ldots \ldots \phi_{in}(x_N)\}|0\rangle \,/\, \langle 0|T\{\exp[i\int L_{int}(y)d^4y]\}|0\rangle \tag{1.7}$$

where $\phi_{in}(x)$ is the Heisenberg picture field operator which is related to the free field operator $\phi(x)$ by the formula

$$\phi_{in}(x) = U^{-1}(t)\phi(x)\,U(t) \tag{1.8}$$

with $U(t)$ being a unitary operator (Itzykson and Zuber)[4]. A more familar form of (1.7) is obtained by expanding the exponentials of the numerator and denominator in series.

The second form of $\langle 0|T(\phi(x_1)\phi(x_2)\ \ldots\ \phi(x_N))|0\rangle$ appears when dealing with the non-pertubative form of the theory and was first written down in print by Mathews and Salam[7]. The expectation value takes the form

$$\langle 0|T(\phi(x_1)\phi(x_2)\ \ldots\ \phi(x_N))|0\rangle = \frac{\int D[\phi]\exp[i\int d^4x L(x)]\ \prod_{j=1}^{N}\phi(x_j)}{\int D[\phi]\exp[i\int d^4x L(x)]} \tag{1.9}$$

where $D[\phi] = \prod_x d\phi(x)$

Derivation of Equation (1.9) requires the modification of the action (Equation (1.10)) by adding the term $\int d^4xJ(x)\phi(x)$ to it. This technique is due to Schwinger[8]. $J(x)$ is an ordinary scalar function known as the source function.

$$S = \int d^4x\ (\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!}) \tag{1.10}$$

A functional integral $Z[J]$ is now defined as

$$Z[J] = \frac{1}{A}\int D[\phi]\exp[i\int d^4x(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!} + J(x)\phi)] \tag{1.11}$$

where

$$A = \int D[\phi]\exp\{i \int d^4x\, [\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!}]\} \qquad (1.12)$$

is a normalization factor. The functional integral $Z[J]$ is very useful because the Green's functions can be obtained from it by performing simple functional differentation with respect to $J(x)$. (For some information on functional differentation and functional Taylor Series see Appendix A). For example

$$\frac{\delta Z[J]}{\delta J(x_1)} = \frac{i}{A} \int D[\phi]\phi(x_1)\exp[i \int d^4x\, (\tfrac{1}{2}\, \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!} + J(x)\phi)] \qquad (1.13)$$

and in general

$$\frac{(-i)^N\, \delta^N Z[J]}{\delta J(x_1)\delta J(x_2)\, \ldots\, \delta J(x_N)} = \frac{1}{A}\int D[\phi]\exp[i \int d^4x\, (\tfrac{1}{2}\, \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!} + J(x)\phi)] \prod_{j=1}^{N} \phi(x_j) \qquad (1.14)$$

Setting $J = 0$ gives

$$\frac{(-i)^N\, \delta^N Z[0]}{\delta J(x_1)\delta J(x_2)\, \ldots\, \delta J(x_N)} =$$

$$= \frac{\int D[\phi]\exp[i \int d^4x\, (\tfrac{1}{2}\, \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!})] \prod_{j=1}^{N} \phi(x_j)}{\int D[\phi]\exp[i \int d^4x\, (\tfrac{1}{2}\, \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4!})]} \qquad (1.15)$$

The right hand sides of (1.9) and (1.15) are identical. Hence

$$\langle 0|T(\phi(x_1)\phi(x_2)\ \ldots\ \phi(x_N))|0\rangle = \left.\frac{(-i)^N\ \delta^N\ Z[J]}{\delta J(x_1)\delta J(x_2)\ \ldots\ \delta J(x_N)}\right|_{J=0} \tag{1.16}$$

The right hand side of (1.16) is the N^{th} order moment of the functional integral $Z[J]$.

1.3 The Free Particle Green's Functions

Direct evaluation of Equation (1.11) is possible only for exponents of quadratic form. For $\lambda = 0$.

$$Z[J]\Big|_{\lambda=0} = F[J] = \int D[\phi]\exp[i\int d^4x\ (\tfrac{1}{2}\ \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2}$$

$$+ J(x)\phi)]/\int D[\phi]\exp[i\int d^4x\ (\tfrac{1}{2}\ \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2})] \tag{1.17}$$

defines a new functional $F[J]$ whose exponent is of quadratic form. Of course, the Green's functions obtained from (1.17) will describe a free particle. A lengthy but comprehensive evaluation of $F[J]$ is given by Nash[5]. The result is

$$F[J] = \exp\ [-\frac{i}{2}\int dx\ dy\ J(x)\ \Delta_F(x-y)J(y)] \tag{1.18}$$

where $\Delta_F(x-y)$ is the Feynman propagator satisfying the equation

$$(\Box + m^2)\ \Delta_F(x-y) = \delta(x-y) \tag{1.19}$$

The solution of (1.19) is

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\,\varepsilon} \ . \tag{1.20}$$

In the above equation, p denotes momentum. Note also that $F[0] = 1$ from Equation (1.18). The functional $F[J]$ has a functional Taylor series obtained by expanding the exponential (see Appendix A). The terms in this series turn out to be the free particle Green's function. In particular, the two point Green's function is

$$G_2(x_1 x_2) = \left. \frac{\delta^2 F[J]}{\delta J(x_1)\delta J(x_2)} \right|_{J=0} = -i\ \Delta_F(x_1-x_2) \tag{1.21}$$

But $\Delta_F(x_1-x_2)$ is a solution of Equation (1.19), therefore $\frac{i\delta^2 F[0]}{\delta J(x_1)\delta J(x_2)}$ is the propagator of a scalar particle of mass m. In diagramatic notation $\Delta_F(x_1-x_2)$ is represented by

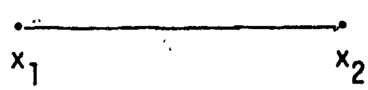
$\Delta_F(x_1-x_2) \equiv$ 

FIGURE 1.1

Similar computations show that the 4-point Green's function is

$$\frac{\delta^4 F[0]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = - \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) - \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) - \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3) \tag{1.22}$$

and its diagramatic representation is

$$\frac{\delta^4 F[0]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} =$$

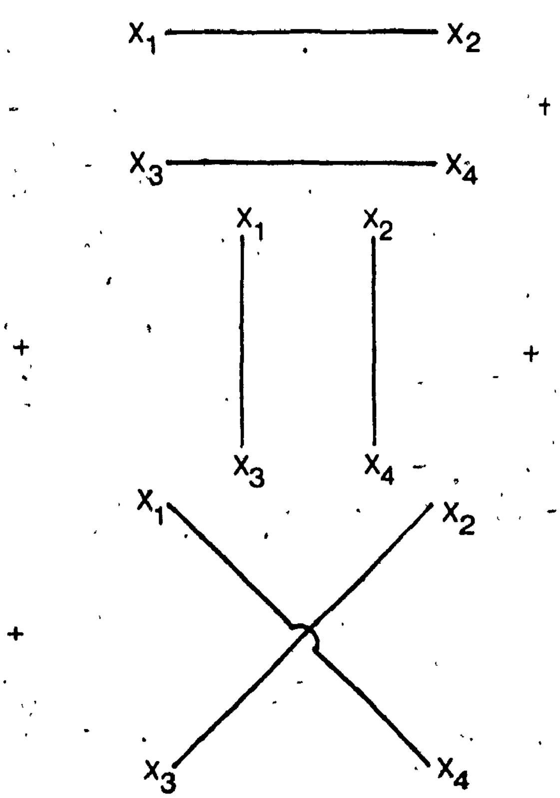


FIGURE 1.2

It is obvious from Equation (1.18) that the higher derivative $\delta^N F[0]/\delta J(x_1) \ldots \delta J(x_N)$ vanishes for odd N; and for even N are given by sums of products of the function $\Delta_F(x-y)$. This means that $\delta^N F[0]/\delta J(x_1) \ldots \delta J(x_N)$ are simply the free particle Green's functions, i.e., the quantities $<0|T(\phi(x_1) \ldots \phi(x_N))|0>$ where $\phi(x)$ is a scalar field.

1.4 Green's Functions for $\lambda\phi^4$ Theory

It was already mentioned that the Green's functions for a free particle are simply given by the moments of the functional F[J]. The great importance of this property is that it remains true for interacting fields. When considering the $\lambda\phi^4$ interaction the functional F[J] is replaced by Z[J]. The presence of the quartic term in the action now makes it impossible to perform the functional integral directly. Instead the numerator and denominator of Equation (1.11) are expanded in powers of λ and the functional integral becomes sums of moments of the free action.

The following definitions will be of use in the future:

(a) Connected diagrams are those which are made up of only one piece. Figure 1.3 illustrates the meaning of this statement. Figure 1.3 is the 4-point function of $\lambda\phi^4$ theory. Using the representation of Figure 1.1 for the propagator together with Equation (1.6a) Figure 1.3 transforms to

$$
\begin{aligned}
<0|T(\phi(x_1) \ldots \phi(x_4))|0> = \; & <0|T(\phi(x_1)\phi(x_2))|0><0|T(\phi(x_3)\phi(x_4))|0> \\
& + <0|T(\phi(x_1)\phi(x_3))|0><0|T(\phi(x_2)\phi(x_4))|0> \\
& + <0|T(\phi(x_1)\phi(x_4))|0><0|T(\phi(x_2)\phi(x_3))|0> +
\end{aligned}
$$

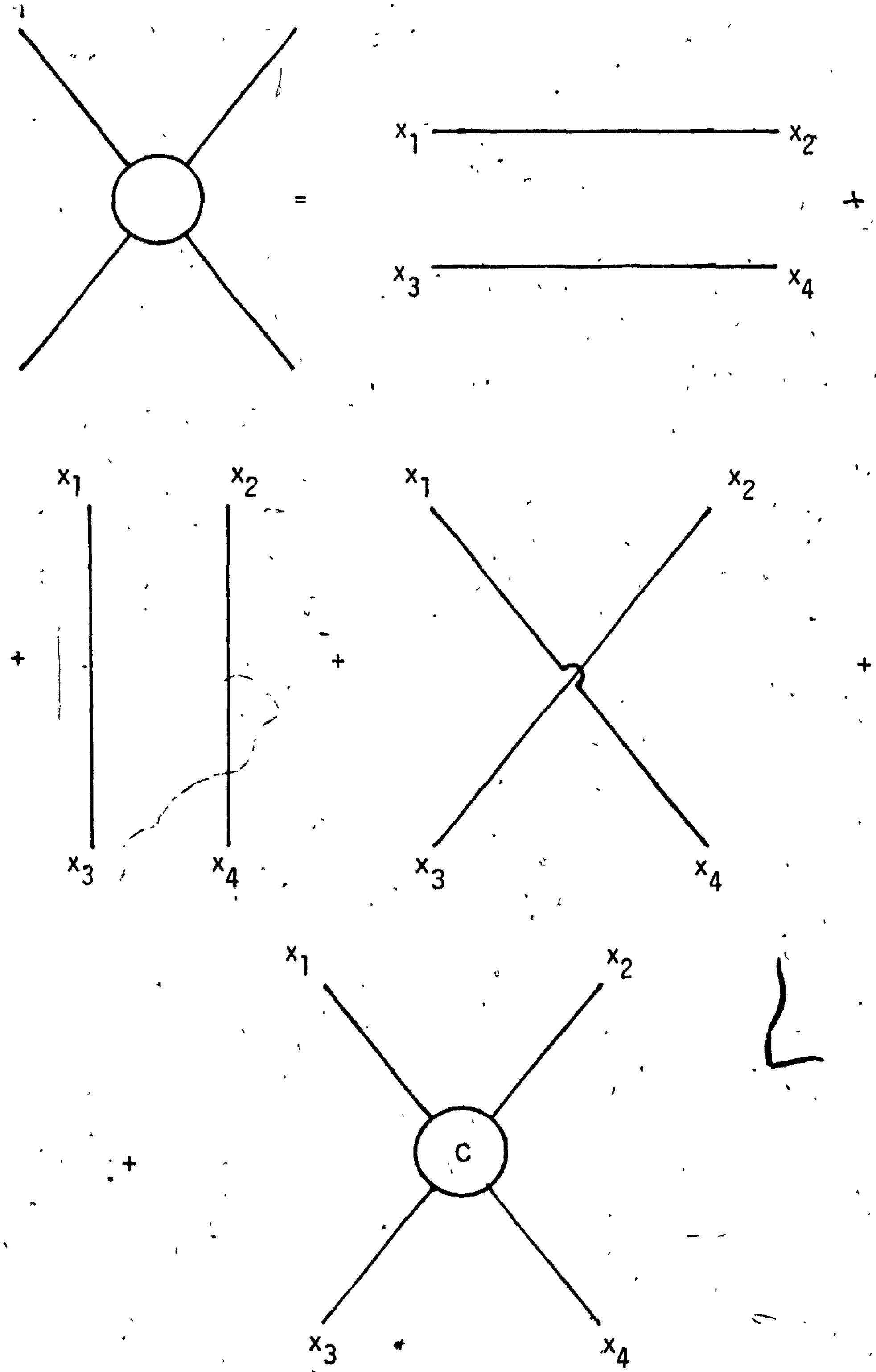


FIGURE 1.3

$$+ <0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0>_C \qquad (1.23)$$

The subscript C on the last term of Equation (1.23) indicates that the corresponding diagram is connected.

(b) It is also common practice to work with "amputated connected" diagrams. Such diagrams are obtained by removing the external propagators from the connected diagrams. Equation (1:24) relates the amputated connected 4-point Green's function to the connected 4-point Green's function.

$$<0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0>_{\overline{C}} = \int [\prod_{j=1}^{4} d\xi_j \Delta_F^{-1}(x_j-\xi_j)] \times <0|T(\phi(\xi_1)\phi(\xi_2)\phi(\xi_3)\phi(\xi_4))|0>_C \qquad (1.24)$$

The inverse propagator satisfies

$$\int d\xi \; \Delta_F^{-1}(x-\xi)\Delta_F(\xi-y) = \delta(x-y) \qquad (1.25)$$

in coordinate space and

$$\hat{\Delta}_F^{-1}(p)\;\hat{\Delta}_F(p) = 1 \qquad (1.26)$$

in momentum space. In passing to momentum space the amputated connected Green's functions depend, because of translation invariance, only on the differences of the x's. Therefore there is always an energy-momentum conserving delta function appearing in the Fourier transform. If the contribution to the series of Feynman diagrams in momentum space is

$T(p_1 \ldots p_4)$ for $\langle 0|T(\phi(x_1) \ldots \phi(x_4))|0\rangle$ then

$$\int \prod_{j=1}^{4} dx_j e^{-ip_j x_j} \langle 0|T(\phi(x_1) \ldots \phi(x_4))|0\rangle =$$

$$T(p_1 \ldots p_4)(-i)(2\pi)^4 \delta(p_1+p_2+p_3+p_4) \qquad (1.27)$$

In momentum space the only diagram contributing to the amplitude for $\langle 0|T(\phi(x_1) \ldots \phi(x_4))|0\rangle$ is shown in Figure 1.4b; its coordinate representation is shown in Figure 1.4a.

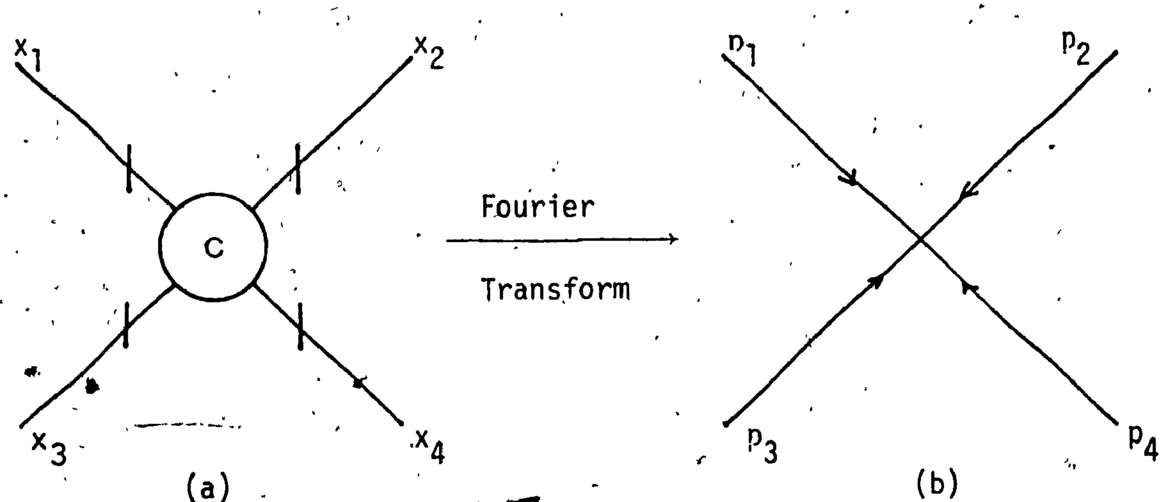


FIGURE 1.4

The slash appearing on a propagator indicates that the propagator is amputated.

Returning to Equation (1.11) for $Z[J]$ and expanding numerator and denominator in powers of λ, it is found that

$$Z^{(1)}[J] = \left\{ i\lambda \int D[\phi] \exp\{i \int d^4x \, (L_0+J\phi)\} \cdot \int D[\phi] \int d^4x\right.$$

$$\frac{\phi^4}{4!} \exp\{i \int d^4x(L_0)\}\Big\} \cdot \Big\{ D[\phi] \exp\{i \int d^4x L_0\}\Big\}^{-2} -$$

$$i\lambda \int D[\phi] \int d^4x \frac{\phi^4}{4!} \exp[i \int d^4x\ (L_0 + J\phi)] \cdot \Big\{\int D[\phi] \exp\{i \int$$

$$d^4x(L_0)\}\Big\}^{-1} \qquad (1.28)$$

L_0 is the free Lagrangian defined in Equation (1.2). The superscript on Z indicates that (1.28) is the first order term in the expansion. To obtain the 4-point Green's function from (1.28) one must evaluate

$$\left.\frac{(-i)^4\ \delta^4 Z^{(1)}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}$$

Inspection of (1.28) shows that

$$\frac{(-i)^4 \delta^4 Z^{(1)}[0]}{\delta J(x_1)\ \dots\ \delta J(x_4)} = \frac{\lambda}{4!} \frac{\delta^4\ F[0]}{\delta J(x_1)\ \dots\ \delta J(x_4)} \cdot \int dx \frac{\delta^4\ F[0]}{\delta J(y_1)\ \dots \delta J(y_4)}$$

$$- \frac{\lambda}{4!} \int dx \frac{\delta^8\ F[0]}{\delta J(x_1)\ \dots\ \delta J(x_4)\ \delta J(y_1)\ \dots\ \delta J(y_4)} \qquad (1.29)$$

where $y_1 = y_2 = y_3 = y_4 = x$. $J = 0$.

Since $\frac{\delta^N\ F[0]}{\delta J(x_1)\ \dots\ \delta J(x_N)}$ is sums of products of $\Delta_F(x-y)$ it is clear that the right hand side of (1.29) is also sums of products of the free propagator. All diagrams, except one, arising from (1.29) contain at least one "bubble". A bubble can be taught of as a line leaving and returning at the same point. For example

$$\Delta_F(x-x) = \Delta_F(0) =$$ 

or

$$\int \frac{d^4x\ \delta^4\ F[0]}{\delta J(x_1)\delta J(x_1)\delta J(x_1)\delta J(x_1)} =$$ 

where $\lambda\int d^4x$ signifies the presence of a vertex.

1.5 The Feynman Rules for $\lambda\phi^4$ Theory

The only bubble free diagram of Equation (1.29) is also the only connected diagram and is shown in Figure 1.5

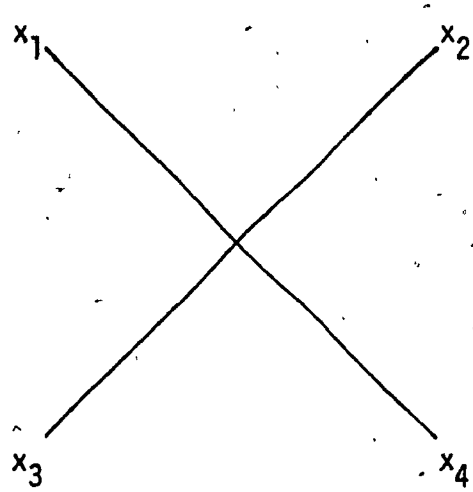


FIGURE 1.5

In terms of the free propagator, Figure 1.5 transforms to

$$\langle 0|T(\phi(x_1)\ \ldots\ \phi(x_4))|0\rangle_C = -\lambda \int dx \Delta_F(x_1-x)\Delta_F(x_2-x) \Delta_F(x_3-x)\Delta_F(x_4-x) \tag{1.30}$$

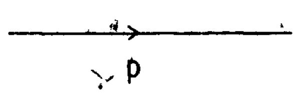
The contribution from the amputated connected diagram is obtained from (1.24) by making use of (1.25) and (1.30). The result is

$$\langle 0|T(\phi(x_1)\ \ldots\ \phi(x_4))|0\rangle_{\tilde{C}} = -\lambda\ \delta(x_2-x_1)\delta(x_3-x_1)\delta(x_4-x_1) \tag{1.31}$$

The momentum contribution $T(p_1,p_2,p_3,p_4)$, arising from the vertex (Figure 1.4b) is obtained by substituting (1.31) into (1.27). The result is simply $-i\lambda$. Similarly, the momentum representation of the propagator is found to be

$$\tilde{G}_2(p_1-p) = \frac{1}{p^2-m^2+i\varepsilon} \tag{1.32}$$

The diagramatic representation of (1.32) is

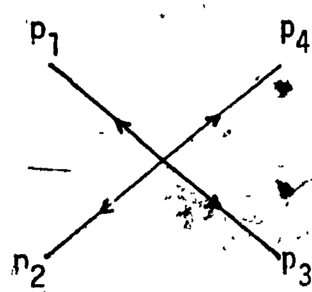


$$= \frac{i}{p^2-m^2+i\varepsilon} \tag{1.33}$$

Table 1.1 contains a summary of the Feynman Rules for $\lambda\phi^4$ Theory.

TABLE 1.1

FEYNMAN RULES FOR $\lambda\phi^4$ THEORY

	REPRESENTATION	CONTRIBUTION
j^{th} external line	$\xrightarrow{\hspace{1em}}$ p_j	$\frac{i}{p_j^2-m^2+i\varepsilon}$
ℓ^{th} internal line	$\xrightarrow{\hspace{1em}}$ k_ℓ	$\frac{i}{k_\ell^2-m^2+i\varepsilon}$
ℓ^{th} loop integration		$\int \frac{d^4k_\ell}{(2\pi)^4}$
vertex	 $\sum_{i=1}^{4} p_i = 0$	$-i\lambda$
symmetry factor		S (varies according to the diagram)

The symmetry factor appearing in Table 1.1 results from the number of possible contractions leading to the same diagram. In this context two diagrams are different if they are topologially distinct. The phrase "topologically distinct" may best be understood by an example. For instance, Figure 1.6(a) and (b) are topologically distinct but Figure 1.6(c) are (d) are not.

The symmetry factor of a diagram may be easily forgotten. However once one knows how to calculate the symmetry factor of a diagram there is no need to memorize it. Given below are a few simple steps involved in the calculation of the symmetry factor of a diagram. Figure 1.7(a) is used as an example. The first step is to disconnect the diagram as in Figure 1.7(b). Then try to find out the number of different ways in which the disconnected vertices of Figure 1.7(b) can form Figure 1.7(a). This is done by first finding out the number of ways in which one can choose the external line at any one of the vertices. Consider the vertex x of Figure 1.7(b). The following diagrams of Figure 1.8 shows the possible choices. The arrows specifiy the directions of the momenta. Therefore there are eight choices of the external line at x. Once a choice is made the number of choices for the external line at vertex y is reduced to four. The reason for this becomes clear by looking at what happens if Figure 1.8(a) is choosen for vertex x. The line AB through the vertices x and y of Figure 1.7(a) is a single line. Once a choice is made at vertex x, the direction of the external line at vertex y must be consistent with this choice. The external line of Figure 1.8(a) is into the vertex therefore the external line at vertex y must be choosen to point away from the vertex. The possible choices are shown in Figure 1.9. Therefore there are four choices for the external line at vertex y after the choice at vertex x is made. The remaining six lines may be joined in six possible ways. (No two lines from the same vertex are connected to each other). See Figure 1.10 below. Therefore, the total number of ways to form the diagram of Figure 1.7(a) is 8x4x6 and the symmetry factor is

$$S = \frac{8x4x6}{2!(4!)^2} = \frac{1}{6}$$

where $\frac{1}{4!}$ is the initial vertex factor for each vertex. The $2!$ in the denominator appears because there are two identical vertices in Figure 1.7(a).

A graph may be thaught of as a collection of vertices and lines. In the scalar theory, the number of lines assigned to a vertex is determined by inspection of the interaction Lagrangian. For instance, the $\lambda\phi^4$ theory has vertices with four lines only. A theory like $\lambda\phi^3 + g\phi^6$ would contain diagrams with two types of vertices, some with three lines and some with six lines. In future the words graphs and diagrams will be used in the same context unless otherwise stated.

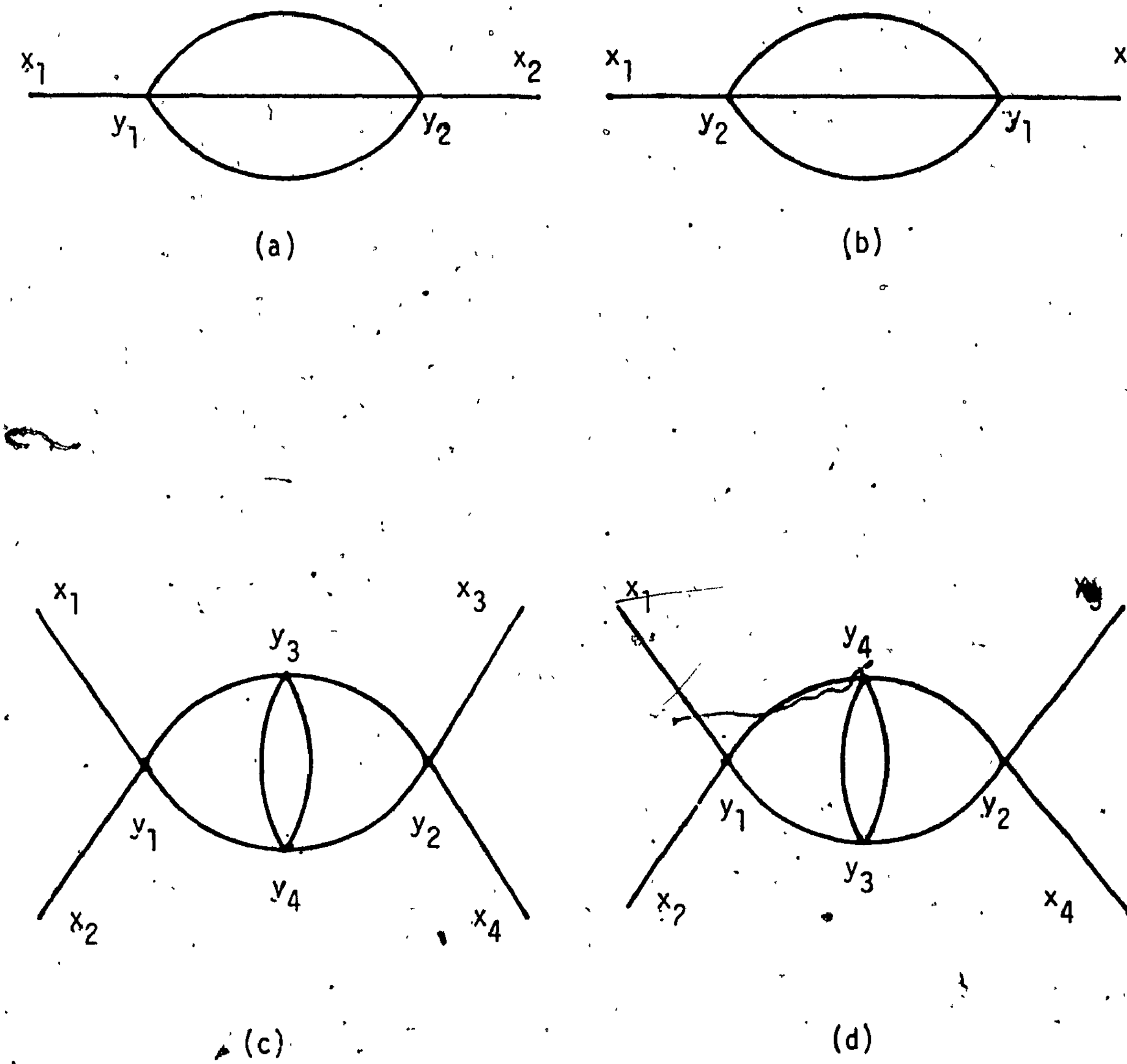


FIGURE 1.6 (a) and (b) are topologically distinct but (b) and (c) are not

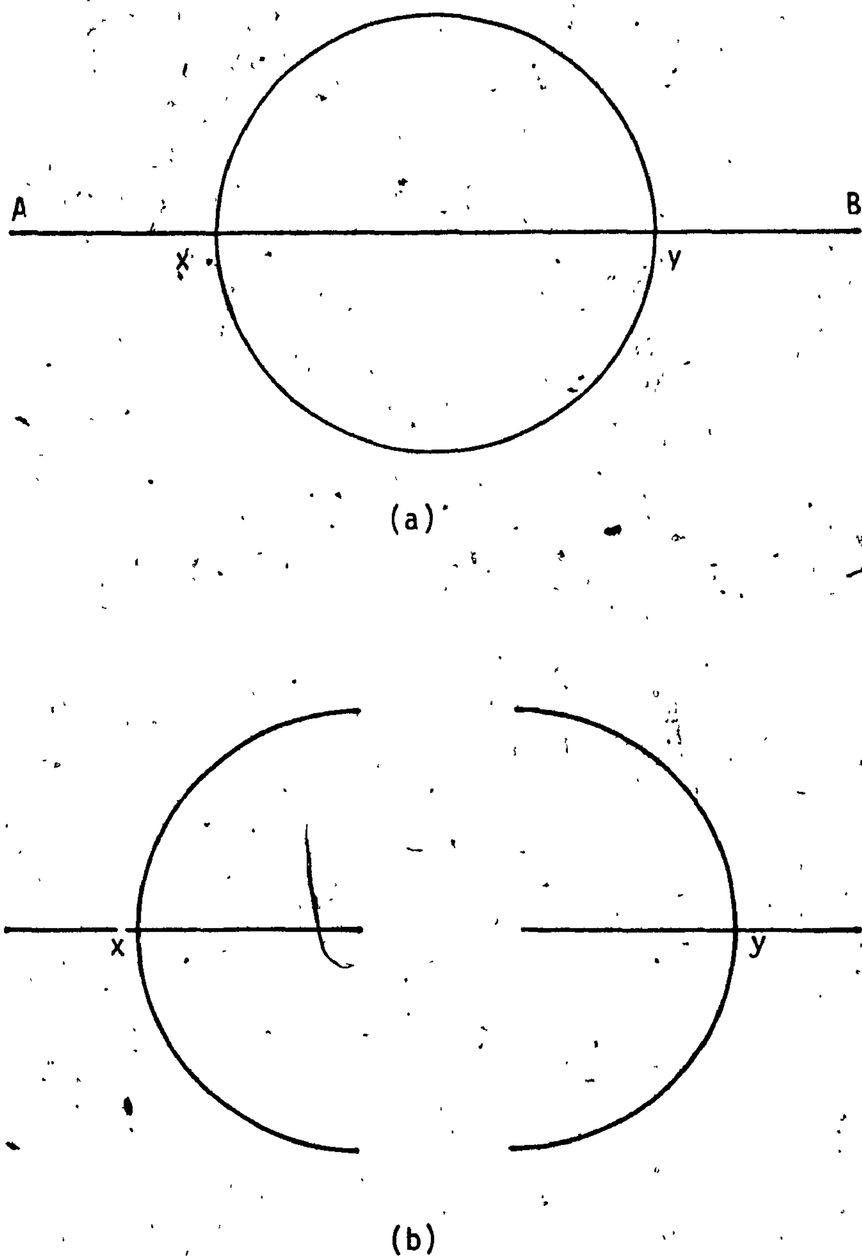


FIGURE 1.7

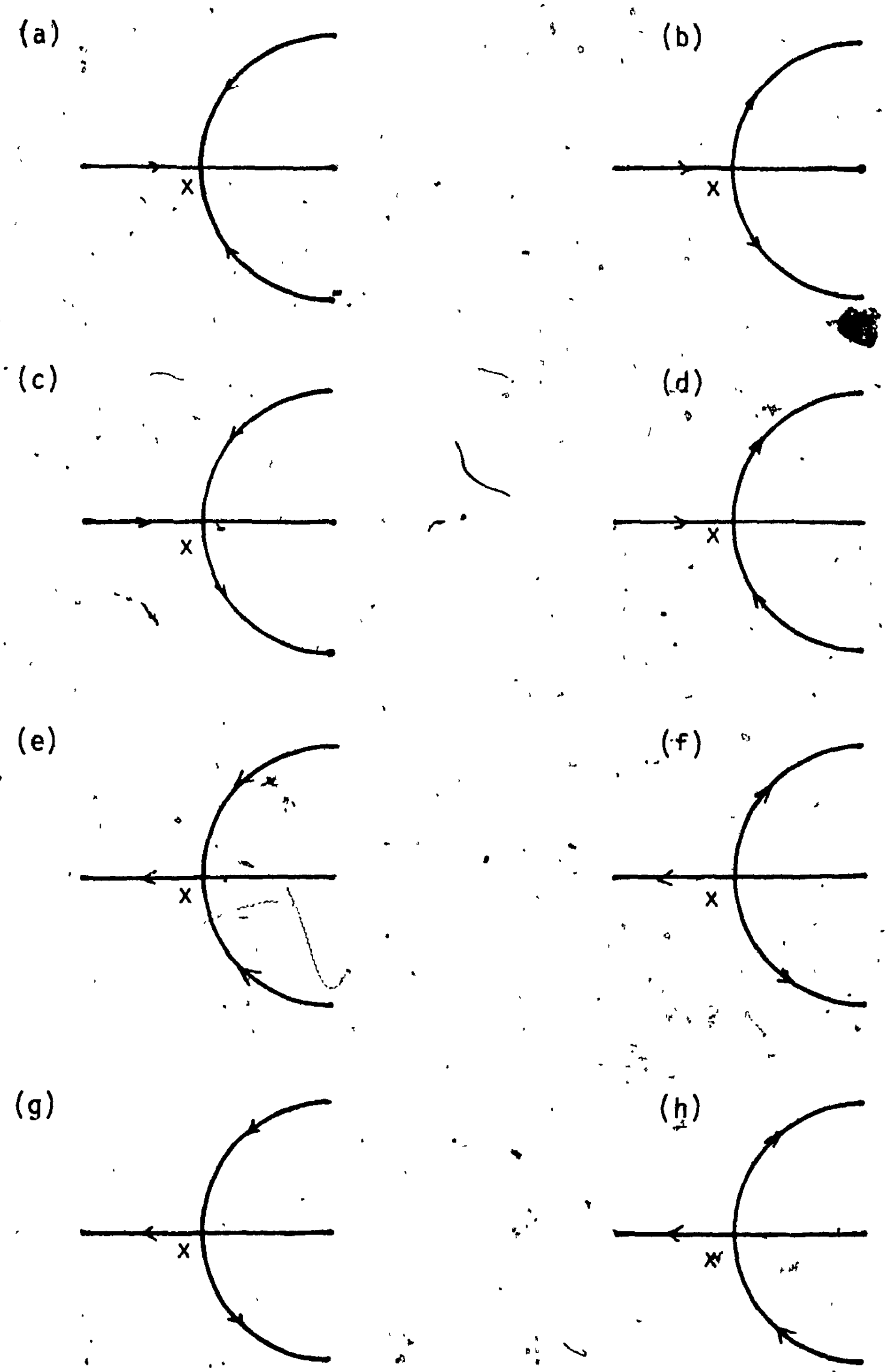


FIGURE 1.8 The eight possible choices of vertex x of Figure 1.7(b)

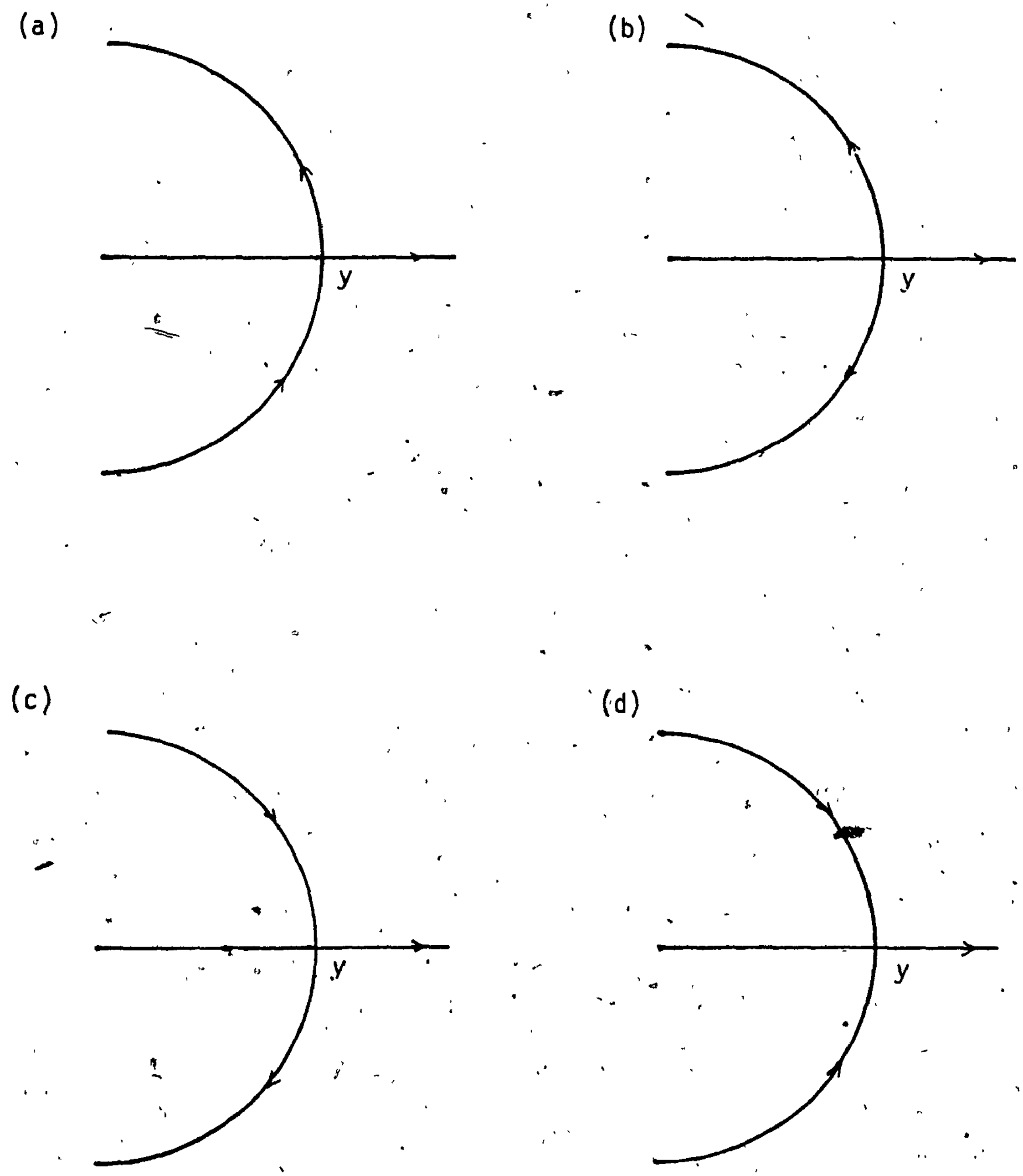


FIGURE 1.9 The four possible choices of vertex y of Figure 1.7(b) after vertex x was choosen

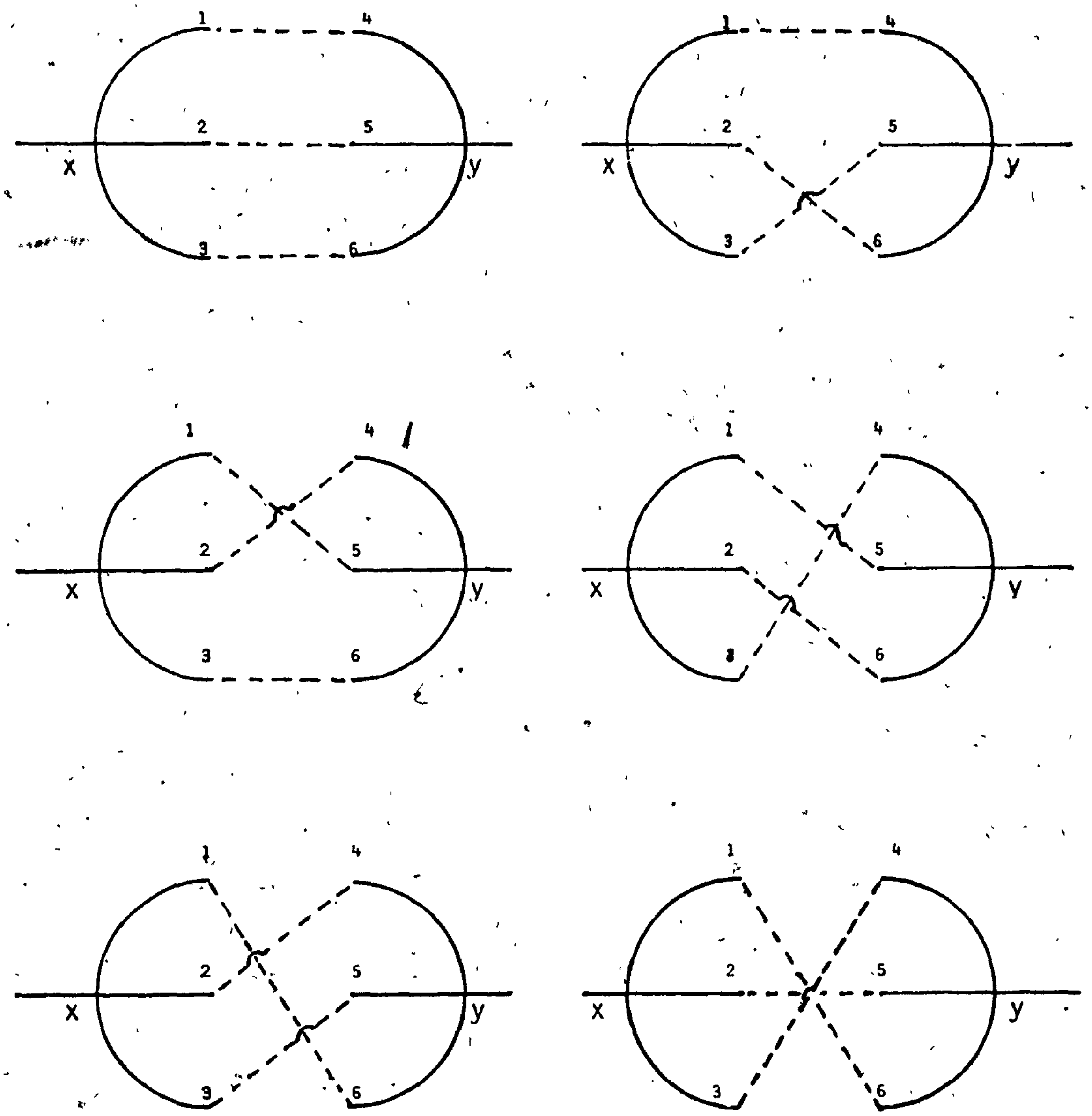


FIGURE 1.10 The six possible ways of joining the internal lines of Figure 1.7(b)

CHAPTER II

THE $\lambda\phi^n$ THEORY

The material in this chapter is a generalization of that found in the preceeding chapter. A general polynomial interaction of the boson field takes the form

$$\sum_{i=3}^{\infty} \lambda_i[\phi(x)]^i/i!$$

The absence of an $i=1$ term ensures that the current $j(x)$ derived from the interaction has zero vacuum expectation value (Jaffe)[9] while the $i=2$ term is omitted because L_0 (see Equation (1.2)) already contains a term of this order. It will suffice to consider a single term from the above interaction which will be written in the form $\lambda\phi^n/n!$.

2.1 The Equation of Motion for $\lambda\phi^n$ Theory

The interaction term of the Lagrangian (1.1) is now $-\lambda\phi^n/n!$. This, together with Equation (1.4) gives the equation of motion for the $\lambda\phi^n$ theory as

$$(\Box + m^2)\,\phi(x) = \frac{-\lambda\phi^{n-1}}{(n-1)!} \tag{2.1}$$

when $n=4$, Equation (1.5) is recovered.

2.2 The Generating Functional $Z[J]$

After replacing $\frac{\lambda\phi^4}{4!}$ by $\frac{\lambda\phi^n}{n!}$ in Equations (1.11) and (1.12) it is found that the generating functional is

$$Z[J] = \frac{1}{A'} \int D[\phi] \exp\left\{i \int d^4x \left(\tfrac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^n}{n!} + J\phi\right)\right\} \quad (2.2)$$

where

$$A' = \int D[\phi] \exp\{i \int d^4x \; (\tfrac{1}{2} \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^n}{n!})\} \quad (2.3)$$

Therefore, in analogy with Equation (1.14) the N-point Green's function for $\lambda\phi^n$ theory is

$$G_N(x_1 \ldots x_N) = \frac{(-i)^N \; \delta^N \; Z[J]}{\delta J(x_1) \; \ldots \; \delta J(x_N)} =$$

$$= \frac{1}{A'} \int D[\phi] \exp \left\{i \int d^4x \; (\tfrac{1}{2} \partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^n}{n!} + J\phi\right\} \times$$

$$\prod_{j=1}^{N} \phi(x_j) \quad (2.4)$$

2.3 Green's Functions for $\lambda\phi^n$ Theory

Now expand the right hand side of Equation (2.2) as in Chapter I and retain terms of order λ. Equation (2.5) shows the result.

$$Z^{(1)}[J] = \left\{i\lambda \int D[\phi] \exp \{i \int d^4x \; (L_0 + J\phi)\} \cdot \int D[\phi] \int d^4x\right.$$

$$\left.\frac{\phi^n}{n!} \exp \{i \int d^4x \; L_0\}\right\} \left\{ \int D[\phi] \exp \{i \int d^4x \; L_0\}\right\}^{-2}$$

$$-\left\{i\lambda \int D[\phi] \int d^4x \frac{\phi^n}{n!} \exp \{i \int d^4x \, (L_o+J\phi)\}\right\} \cdot \left\{\int D[\phi] \exp [i \int d^4x \, L_o]\right\}^{-1} \tag{2.5}$$

When n=4, Equation (2.5) reduces to Equation (1.28). In terms of the generating functional F[J] for the free particle, Equation (2.5) reads

$$Z^{(1)}[J] = -\frac{i\lambda}{n!} F[J] \int \frac{(-i)^n \, \delta^n \, F[0] \, d^4x}{\delta J(y_1) \, \ldots \, \delta J(y_n)} + \\ -\frac{i\lambda}{n!} \int \frac{d^4x \, \delta^n \, F[J]}{\delta J(y_1) \, \ldots \, \delta J(y_n)} \tag{2.6}$$

It follows that the N-point Green's function for the $\lambda\phi^n$ theory is

$$\frac{(-i)^N \, \delta^N \, Z^{(1)}[0]}{\delta J(x_1) \, \ldots \, \delta J(x_N)} = \frac{-i\lambda}{n!} \frac{(-i)^N \, \delta^N \, F[0]}{\delta J(x_1) \, \ldots \, \delta J(x_N)} \int \frac{(-i)^n \, \delta^n \, F[0] \, d^4x}{\delta J(x_1) \, \ldots \, \delta J(x_n)} \\ -\frac{i\lambda}{n!}(-i)^n \int d^4x \, \frac{\delta^N \, F[0] \, \delta^n \, F[0]}{\delta J(x_1) \, \ldots \, \delta J(x_N) \delta J(y_1) \ldots \delta J(y_n)} \tag{2.7}$$

In particular, when N = n then

$$\langle 0|T(\phi(x_1) \, \ldots \, \phi(x_n))|0\rangle = \frac{-i\lambda}{n!} \frac{(-i)^n \, \delta^n \, F[0]}{\delta J(x_1) \, \ldots \, \delta J(x_n)} \int \frac{(-i)^n \delta^n \, F[0] \, d^4x}{\delta J(y_1) \, \ldots \, \delta J(y_n)} \\ -\frac{i\lambda}{n!}(-i)^n \int d^4x \, \frac{\delta^{2n} \, F[0]}{\delta J(x_1) \, \ldots \, \delta J(x_n) \delta J(y_1) \, \ldots \, \delta J(y_n)} \tag{2.8}$$

2.4 Feynman Rules for $\lambda\phi^n$ Theory

The only connected diagram arising from Equation (2.8) is the first order vertex diagram shown in Figure 2.1. In terms of the free propagator Figure 2.1 transforms to

$$<0|T(\phi(x_1) \ldots \phi(x_n))|0> = -\lambda \int \Delta_F(x_1-x)\, \Delta_F(x_2-x) \ldots \Delta_F(x_{n-1}-x)$$
$$\Delta_F(x_n-x)\, d^4x \qquad (2.9)$$

After generalizing Equations (1.24) and (1.27) to give

$$<0|T(\phi(x_1) \ldots \phi(x_n))|0>_{\overline{C}} = \int \prod_{j=1}^{n} d\xi_j \; \Delta_F^{-1}(x_j-\xi_j)] \times$$
$$<0|T(\phi(\xi_1) \ldots \phi(\xi_n))|0>_C \qquad (2.10)$$

and

$$T(p_1,\ldots p_n)(-i)(2\pi)^4\, \delta(\sum_{j=1}^{n} p_j) = \int \prod_{j=1}^{n} dx_j \exp(-ip_jx_j) \times$$
$$<0|T(\phi(x_1) \ldots \phi(x_n))|0> \qquad (2.11)$$

they are used along with (2.9) to obtain

$$T(p_1,\ldots p_n)(-i)(2\pi)^4\, \delta(\sum_{j=1}^{n} p_j) = -\lambda\left\{\int \prod_{j=1}^{n} d^4x_j \exp(-ip_jx_j)\right.$$
$$\left.\delta(x_2-x_1)\, \delta(x_3-x_1) \ldots \delta(x_n-x_1)\right\}$$
$$= -\lambda \int d^4x_1 \exp[-ix_1 \sum_{j=1}^{n} p_j]$$

$$= -\lambda(2\pi)^4\delta(\sum_{j=1}^{n} p_j) \tag{2.12}$$

Therefore $T(p_1,\ldots p_n)$, the momentum contribution of Figure 2.2, is simply $-i\lambda$.

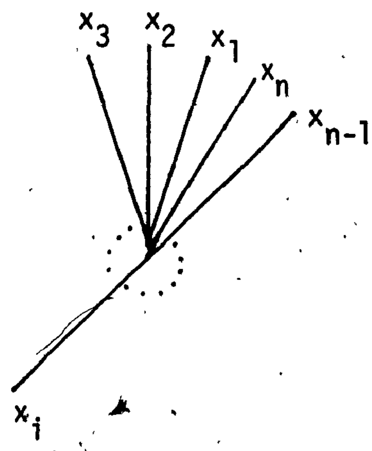


FIGURE 2.1

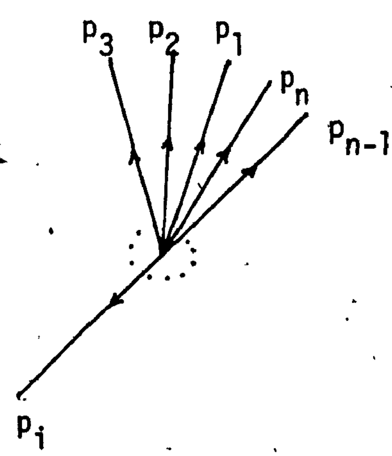


FIGURE 2.2

Below are the rules for obtaining the Feynman amplitude from an arbitrary diagram. Although actual calculations would not involve the contributions arising from the external lines of the diagram, the rule for obtaining their contribution is nevertheless stated below.

(1) Assign the factor $\frac{i}{(p_j^2-m^2+i\varepsilon)}$ to the j^{th} external line.

(2) To the ℓ^{th} internal line whose internal momentum is k_ℓ, assign the factor $\frac{i}{k_\ell^2-m^2+i\varepsilon}$.

(3) For the ℓ^{th} loop is assigned the factor $\int \frac{d^4k_\ell}{(2\pi)^4}$

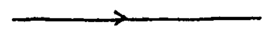
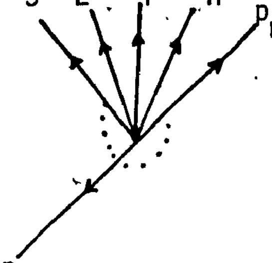
(4) The factor $(-i\lambda)(2\pi)^4\,\delta^4(q_j)$ is assigned to the j^{th} vertex. q_j is the sum of the incoming momenta at the j^{th} vertex.

(5) The product of the above factors along with the appropriate symmetry factor is the Feynman amplitude for the diagram in question.

For a more compact form of the above rules refer to Table 2.1. The only difference between Table 1.1 and Table 2.1 is clearly the vertex. It is obvious that there are a total of n lines at any vertex of a diagram belonging to $\lambda\phi^n$ theory. But whatever the value of n, the contribution from a vertex is always the same, namely $-i\lambda$.

TABLE 2.1

FEYNMAN RULES FOR $\lambda\phi^n$ THEORY

	REPRESENTATION	CONTRIBUTION
j^{th} external line	 p_j	$\frac{i}{p_j^2-m^2+i\epsilon}$
ℓ^{th} internal line	 k_ℓ	$\frac{i}{k_\ell^2-m^2+i\epsilon}$
ℓ^{th} loop integration		$\int \frac{d^4k_\ell}{(2\pi)^4}$
Vertex	 p_3 p_2 p_1 p_n p_{n-1} p_i $[\sum_{j=1}^{n} p_j=0]$	$(-i\lambda)(2\pi)^4\delta^4(\sum_{j=1}^{n} p_j)$ $= -i\lambda$
Symmetry factor		S

CHAPTER III

PERTUBATIVE RENORMALIZATION

As used here, the word renormalization means the removal of infinities from Feynman amplitudes in pertubation theory, for Lagrangian field theory with polynomial interaction. In particular non-pertubative renormalization (Bui Duy)[10] is outside the scope of this thesis, as are the properties of non-polynomial interactions. Field theories involving only the boson field will be concerned with in what follows.

3.1 Divergent Integrals

Attempts to calculate scattering amplitudes with Equation (3.1), following the conventional Feynman rules, soon leads to divergent Feynman diagrams. As an example, consider the four-point function (Figure 3.1) for $\lambda\phi^4$ theory. Using the Feynman Rules appearing in Table 1.1 the Feynman amplitude of Fig. 3.1 is

$$I = \frac{1}{(2\pi)^4} \frac{\lambda^2}{2} \int d^4k \frac{1}{[k^2-m^2 + i\epsilon][(k-p_1-p_2)^2 - m^2 + i\epsilon]} \tag{3.1}$$

By introducing Feynman parameters (see Appendix B) Equation 3.1 can be written as

$$I = \frac{\lambda^2}{2(2\pi)^4} \int \frac{d^4k \, d\alpha}{[\{k + (1-\alpha)(p_1+p_2)\}^2 - m^2 + (1-\alpha)s - (1-\alpha)^2 s]^2} \tag{3.2}$$

where s is the Mandlestam variable defined as

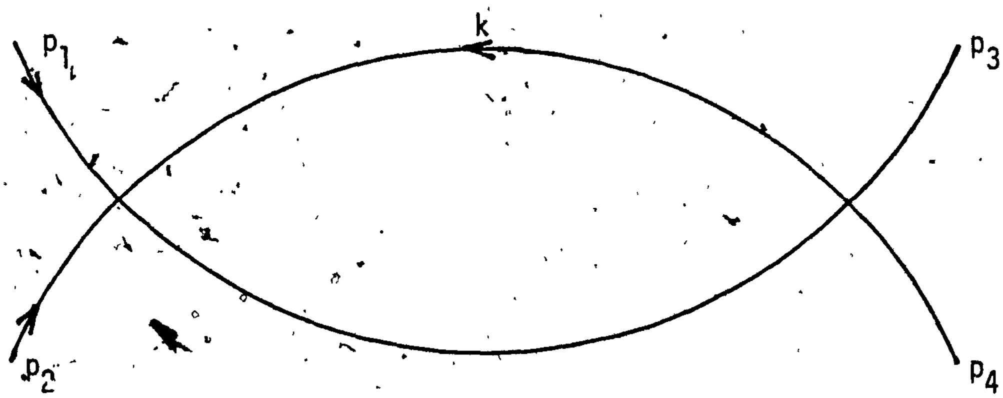


FIGURE 3.1 A second order diagram from $\lambda\phi^4$ theory.

$$s = (p_1+p_2)^2 \tag{3.3}$$

After a shift of the origin, Equation (3.2) reads

$$I = \frac{\lambda^2}{2(2\pi)^4} \int \frac{d^4k\ d\alpha}{(k^2+b^2)^2} \tag{3.4}$$

where

$$b^2 = -m^2 + \alpha(1-\alpha)s \tag{3.5}$$

But

$$\int \frac{d^4k}{(k^2+b^2)^n} = i\pi^2 \frac{\Gamma(n-2)}{\Gamma(n)} \frac{1}{(b^2)^{n-2}} \tag{3.6}$$

where Γ is the Euler gramma function. When $n=2$ Equation (3.6) involves $\Gamma(0)$, which is infinite because $\Gamma(n)$ has poles at $n=0,-1,-2$ .... To avoid working with these infinites the propagators are usually regularized (see Chapter V) by introducing a cut-off parameter. The renormalization procedure of Bogoliubov[11] consists of adding to the Lagrangian, extra terms, the so-called renormalization counterterm whose function is to cancel the cutoff-dependence of the amplitude.

Understanding the construction of the counterterms requires the following three definitions.

1) One particle-irreducible diagrams (1PI).

A Feynman diagram is said to be one particle-irreducible if it is connected and cannot be disconnected by cutting one internal line.

Figure 3.2 shows three Feynman diagrams in $\lambda\phi^4$ theory. Figures 3.2(a) and (b) are 1PI but (c) is not.

2) Superficial degree of divergence.

A Feynman integral is, in general a multiple integral. The superficial degree of divergence of such an integral is the difference between the momentum power in the numerator of the integral (arising from loop integration variables and from explicit momenta at vertices due to derivative interactions if any) and the momentum power in the denominator (Dyson)[1] arising from the propagator. Figure 3.3 shows three Feynman diagrams in $\lambda\phi^4$ theory, with their superficial degree of divergence denoted by $\delta\Gamma$. The integrals are taken to be in 4-space-time dimension.

Figures 3.3(a) and (b) are respectively logarithmically and quadratically divergent while Figure 3.3(c) is superfically quadratically convergent. Although Figure 3.3(c) is superficially convergent, it is in fact divergent and it is for this reason the word "superficial" is used; the integration along the lower loop is divergent no matter what happens in the rest of the diagram. A large portion of Chapter IV is devoted to superficial degree of divergence so there is no reason to dwell on the subject any longer.

3) The third definition is that of the Taylor expansion of a Feynman amplitude about the point zero. A Feynman amplitude with n external lines is a function of $n-1$ independent four momenta. Furthermore if there are no massless particles in the theory it is an analytic function of these momenta in some neighbourhood of the point zero, the point where all external momenta vanish. Thus, it may be expanded in a Taylor series of these variables.

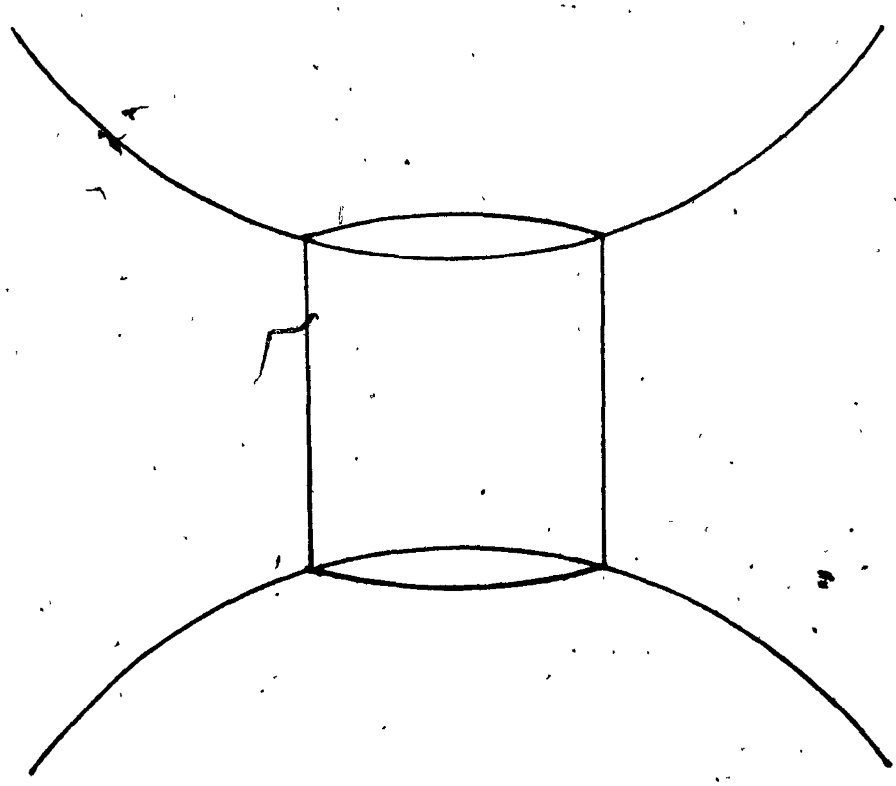


FIGURE 3.2(a) A fourth order 1PI diagram from $\lambda\phi^4$ theory

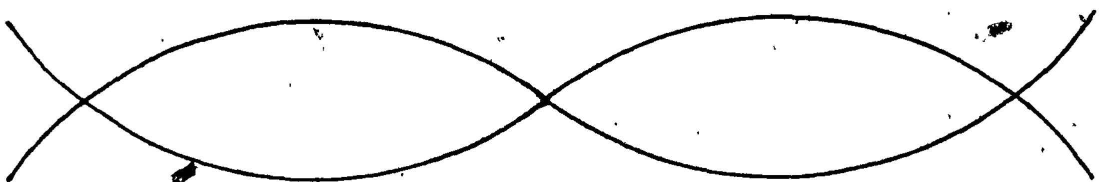


FIGURE 3.2(b) A third order 1PI diagram from $\lambda\phi^4$ theory

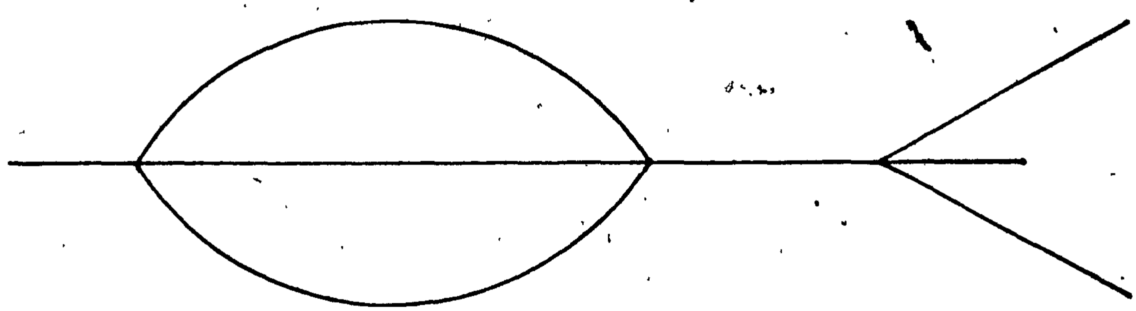


FIGURE 3.2(a) A 1PR diagram from $\lambda\phi^4$ theory

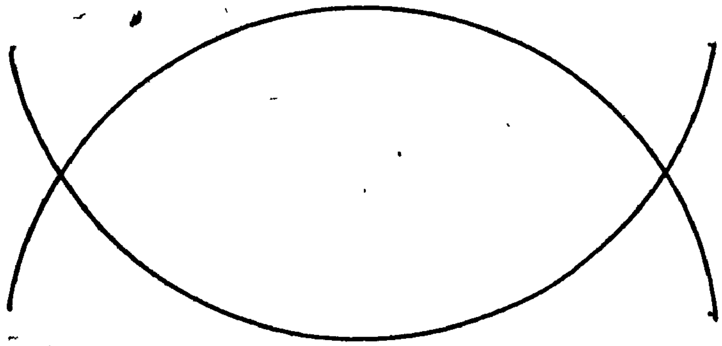


FIGURE 3.3(a) $\delta\Gamma = 0$

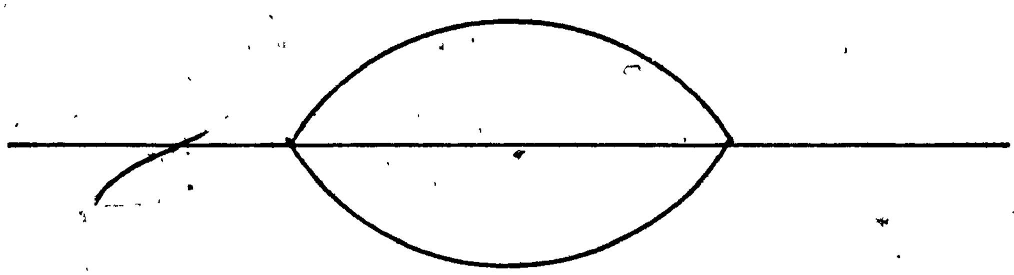


FIGURE 3.3(b) $\delta\Gamma = 2$

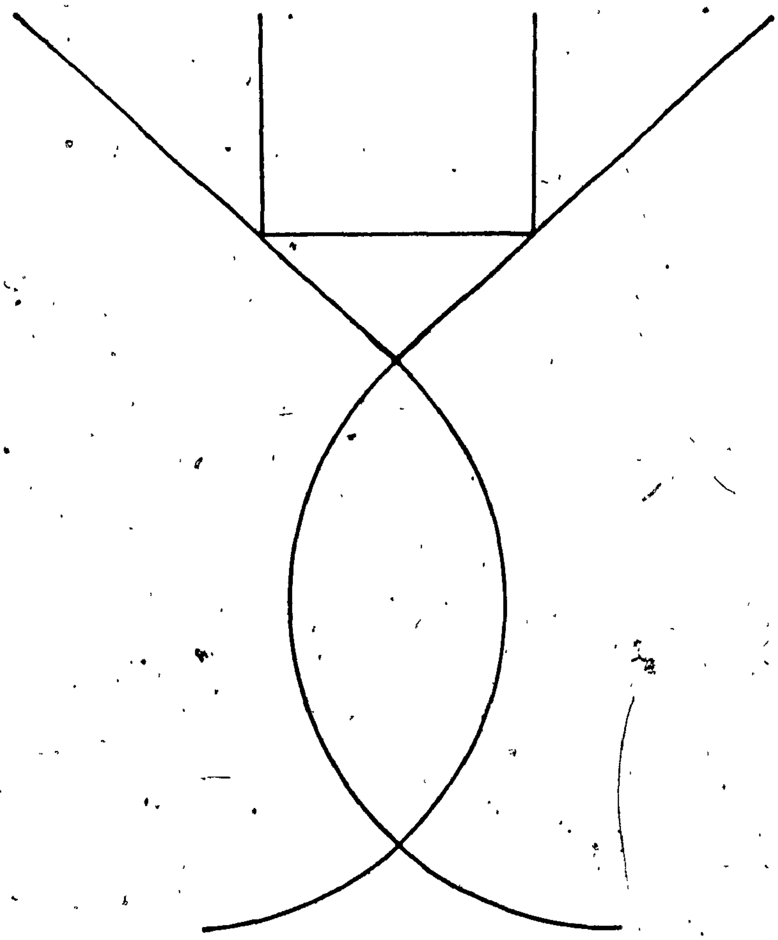


FIGURE 3.3(c) $\delta\Gamma = -2$

3.2 Renormalization Prescription

The renormalization prescription of Bogoliubov and Parasiuk[11] is as follows

(1) Calculate in pertubation theory until a 1PI diagram whose superficial degree of divergence $(\delta\Gamma)$ is greater than or equal to zero is encountered.

(2) Add to the Lagrangian extra terms (the counterterms) choosen to precisely cancel, to this order, all terms in the Taylor expansion of this diagram of order $\delta\Gamma$ or less. As an example of this procedure, consider $\lambda\phi^4$ theory, for which the Lagrangian is

$$L = \tfrac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\,m^2\phi^2 - \frac{\lambda\phi^4}{4!} \tag{3.7}$$

The two second order diagrams encountered in $\lambda\phi^4$ theory are shown in Figure 3.3(a) and (b). According to step 2 above, L must be altered by adding extra terms so that

$$L \to L - \frac{A_2\phi^4}{4!} + \tfrac{1}{2}\,B_2\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}C_2\phi^2 \tag{3.8}$$

The subscripts on A,B,C, are reminders that these are second order terms. The A_2 term is choosen to cancel the zeroth-order term in the Taylor expansion of Figure 3.3a; the B_2 and C_2 terms will cancel the zeroth and second order terms in the Taylor expansion of Figure 3.3b. There is no need for a first order counterterm because Lorentz invariance forbids a first order term in the Taylor expansion.

(3) Continue computing, now using the corrected Lagrangian. This

procedure eliminates all divergences in the second order.

This procedure can be continued for all order diagrams, with the original Lagrangian and new counterterms. Each time the counterterms are choosen to cancel the divergences in that order. The remarkable thing about this procedure when applied to $\lambda\phi^4$ and other renormalizable theories, is that the counterterms needed to remove the divergences all generate new interactions of the same type as those present in the original Lagrangian. As a result, the modified Lagrangian is

$$L = [\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2} m^2\phi^2 - \frac{\lambda\phi^4}{4!}] - \frac{A}{4!}\phi^4 + \frac{B}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2} C\phi^2 \tag{3.9}$$

where A,B and C are power series in λ, with coefficients that are, in general, cutoff dependent. Each term in these power series takes care of the divergence of a diagram from $\lambda\phi^4$ theory. Equation (3.9) usually appears in the literature as

$$L = \tfrac{1}{2}(\partial_\mu\phi_o)^2 - \tfrac{1}{2} m_o^2\phi_o^2 - \frac{\lambda_o}{4!}\phi_o^4 \tag{3.10}$$

with

$$\phi_o = (1+B)^{\frac{1}{2}}\phi \tag{3.11}$$

$$m_o = (m^2+C)(1+B)^{-1} \tag{3.12}$$

and

$$\lambda_o = (\lambda+A)(1+B)^{-2} \tag{3.13}$$

Equation (3.10) takes the exact form of the original Lagrangian, except that the coefficients have been changed. ϕ_0 is known as the unrenormalized field. The quantities m_0 and λ_0 are called the "bare" mass and "bare" coupling constant respectively. Thus, when the bare mass and coupling constant are choosen in a cut-off-dependent fashion, all the divergences of $\lambda\phi^4$ theory disappear, order by order, in pertubation theory. A Lagrangian that has this property is said to be renormalizable.

In the next chapter the subject of renormalizability and non-renormalizability of $\lambda\phi^n$ theory is considered by examination of the superficial degree of divergence of a general Feynman amplitude in an arbitrary space-time dimension.

CHAPTER IV

RENORMALIZABILITY AND NONRENORMALIZABILITY OF $\lambda\phi^n$ THEORY

The concept of renormalization and renormalizability are quite distinct and there is no reason why the divergences of even nonrenormalizable theories cannot be absorbed into an infinite family of local counterterms. What is important is, for all renormalizable theories, not only is the number of primitively divergent graphs limited, but further, perhaps more important, the degree of divergence of the integrals corresponding to these graphs do not depend on the order of the graphs.

4.1 Superficial Degree of Divergence of a Feynman Graph

Almost every book on quantum field theory contains some material on superficial degree of divergence and its relation to renormalizable and nonrenormalizable theories. Itzykson and Zuber[4] gave a formula for the degree of divergence of a graph in 4-space-time dimension for a theory involving spin 0 or spin 1 boson fields and spin ½ fermion fields. Similar work was done by Pohlmeyer[12] but in d-space-time dimension. It is a trivial task to derive from these results, the degree of divergence of a graph originating from a theory with boson fields only (Ramond)[13].

Only ultraviolet divergences, those divergences associated with large loop momenta, will be looked at in this chapter. Another type of divergence is the infra-red (IR) divergence associated with small loop momenta. It occurs only in massless theories.

Consider a 1PI graph, $\Gamma^{(E)}$, from $\lambda\phi^n$ theory. The list below shows some symbols used in the derivation of the degree of divergence ($\delta\Gamma^{(E)}$) of $\Gamma^{(E)}$.

V	:	Number of Vertices
n	:	as in $\lambda\phi^n$
E	:	Number of external lines
I	:	Number of internal lines
d	:	Number of space-time dimension
L	:	Number of loops
$\delta\Gamma^{(E)}$	:	Degree of divergence of $\Gamma^{(E)}$

The number of internal momenta is equal to the number of loops in the diagram. The I internal momenta satisfy $(V-1)$ relations among themselves. The -1 appears because of overall momentum conservation. Therefore

$$L = I - (V-1) \tag{4.1}$$

The general form of $\Gamma^{(E)}$ (apart from symmetry factors) is shown in Equation (4.2)

$$\Gamma^{(E)} \propto (-i\lambda)^V \int \prod_{i=1}^{L} \frac{d^d k_i}{(2\pi)^d} \prod_{j=i}^{I} \frac{1}{(\ell_j^2 - m^2 + i\varepsilon)} \tag{4.2}$$

There are V vertices each contibuting $(-i\lambda)$, hence the factor $(-i\lambda)^V$. The quantity ℓ_j is a linear combination of external and loop momenta.

$\delta\Gamma^{(E)}$ is obtained by counting the powers of loop momenta in the integral of (4.2). The power counting method is described in Dyson's[1] paper and an elementary proof is also given by Hahn and Zimmerman[14]. Each loop integration of (4.2) provides d powers of momenta. The factor

$$\prod_{j=1}^{I} \frac{1}{(\ell_j^2 - m^2 + i\varepsilon)}$$

provides $-2I$ powers of momenta. The minus sign appears because the momenta is in the denominator. Together, the degree of divergence is

$$\delta\Gamma^{(E)} = dL - 2I \tag{4.3}$$

Remembering that each vertex of the $\lambda\phi^n$ theory has n lines attached to it, the total number of lines in the diagram is nV. But each internal line must be counted twice because they are attached to two vertices. It follows from the conservation of the number of lines in the diagram that

$$nV = E + 2I \tag{4.4}$$

Substituting (4.4) and (4.1) into (4.3) yields

$$\delta\Gamma^{(E)} = d - \frac{E}{2}(d-2) + V[\frac{d}{2}(n-2) - n] \tag{4.5}$$

4.2 Dimensional Analysis of the Coupling for $\lambda\phi^n$ Theory

Consider the action in d-space-time dimension

$$S = \int L \, d^dx \tag{4.6}$$

where L is given by

$$L = \tfrac{1}{2} (\partial^\mu \phi)^2 - \frac{m^2 \phi^2}{2} - \frac{\lambda \phi^n}{n!} \tag{4.7}$$

With the knowledge that the Feynman path integral involves the expression (Feynman and Hibbs)[15].

$$\exp \{\tfrac{i}{\hbar} S(x,t)\}$$

It is clear that the action is dimensionless. (The natural units $\hbar = c = 1$ were used). Denoting the dimension of a quantity R by [R] and recalling that

$$[L] = \frac{[\text{ENERGY}]}{[\text{VOLUME}]}$$

it follows from (4.6) that

$$[S] = 1 = \frac{[\text{ENERGY}]}{L^{d-1}} \quad L^d \tag{4.8}$$

Therefore $[\text{ENERGY}] = L^{-1}$ where L is the dimension of length. Equation (4.6) and (4.7) gives

$$\left\{\iint \frac{m^2 \phi^2}{2} \; d^d x\right\} = \left\{\iint \frac{\lambda \phi^n}{n!} \, d^d x\right\} = \left\{\iint \frac{(\partial_\mu \phi)^2}{2} \; d^d x\right\} = 1 \tag{4.9}$$

So in units of mass dimension

$$[\lambda] = -\left\{\frac{(n-2)}{2} d - n\right\} \tag{4.10}$$

and Equation (4.5) may now be written as

$$\delta\Gamma^{(E)} = d - \frac{E}{2}(d-2) - V[\lambda] \qquad (4.11)$$

Therefore there is an explicit relation between the degree of divergence and the dimension of the coupling.

4.3 Degree of Divergence of $\lambda\phi^4$ Theory

The interpretation of (4.5) may be achieved by considering some specific examples. Take for example the case $d=4$. Equation (4.5) reduces to

$$\delta\Gamma^{(E)} = 4 - E + (n-4)V \qquad (4.12)$$

Evidently the case $n=4$, the $\lambda\phi^4$ theory, is a very interesting one for then $\delta\Gamma^{(E)}$ is independent of the number of vertices [see Equation (4.13)] and there are only a finite number of graphs for which $\delta\Gamma^{(E)}$ is non-negative. These graphs are the primitively divergent graphs of the $\lambda\phi^4$ theory.

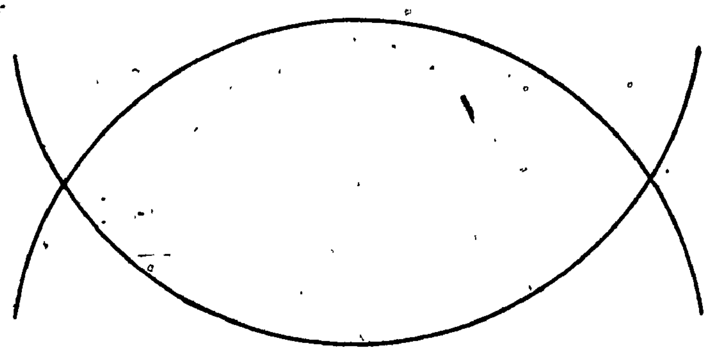
$$\delta\Gamma^{(E)} = 4 - E \qquad (4.13)$$

The two primitively divergent graphs of $\lambda\phi^4$ theory are

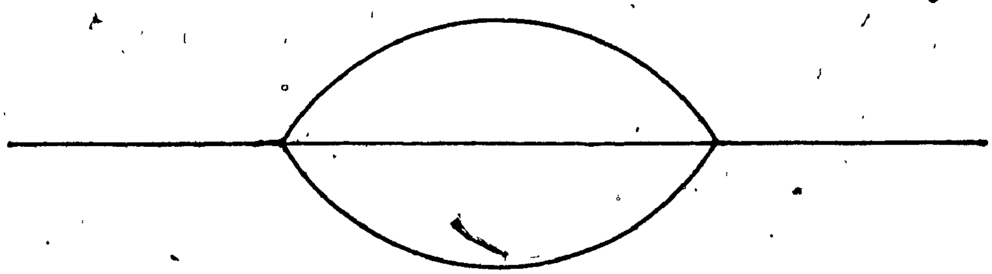
$\Gamma^{(2)}$ with $\delta\Gamma^{(2)} = 2$: superficial quadratic divergence

and

$\Gamma^{(4)}$ with $\delta\Gamma^{(4)} = 0$: superficial logarithmic divergence



(a)



(b)

FIGURE 4.1 The primitively divergent diagrams of $\lambda\phi^4$ theory in 4 dimensions

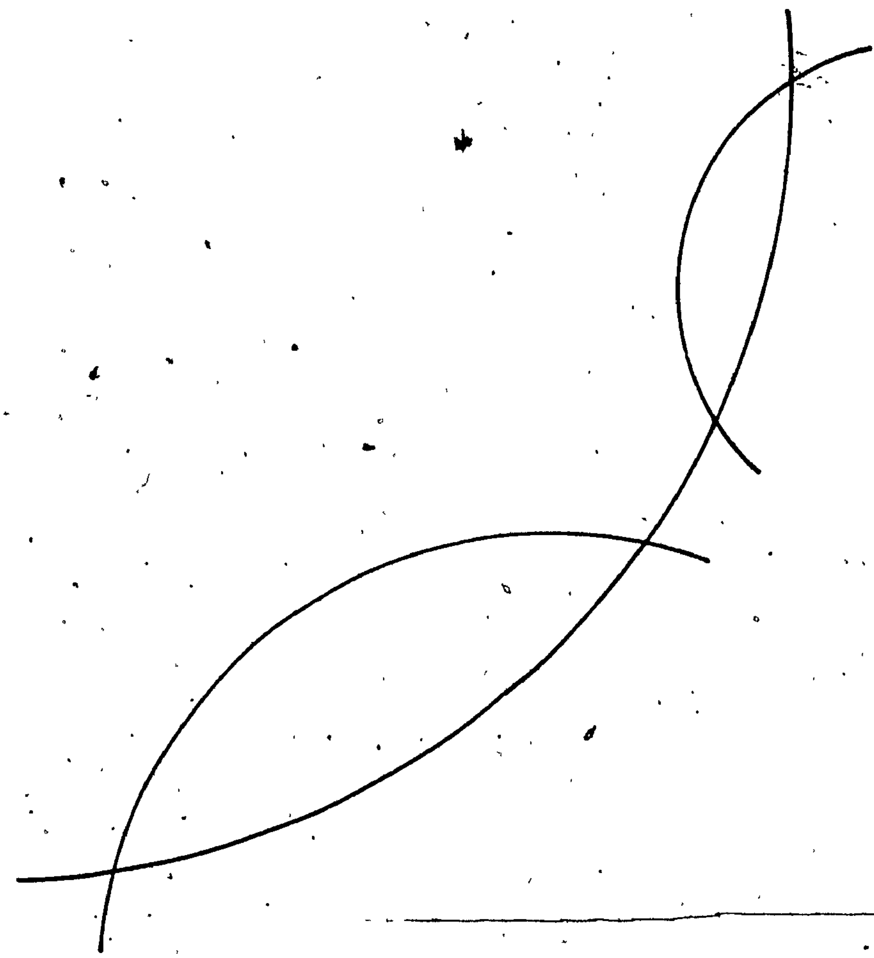


FIGURE 4.2 A two loop 1PR diagram from $\lambda\phi^4$ theory

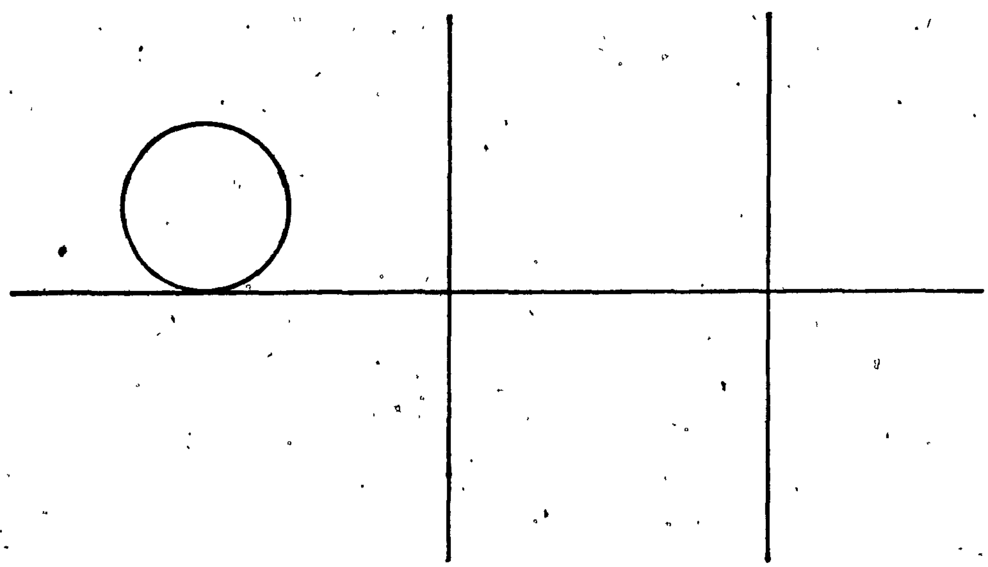


FIGURE 4.3 A 1PR diagram from $\lambda\phi^4$ theory with a bubble on one leg

Figures 4.1(a) and (b) show the primitively divergent graphs from $\lambda\phi^4$ theory.

The above analysis does not mean that graphs with negative $\delta\Gamma^{(E)}$ will converge. To understand this statement consider the diagrams of Figures 4.2 and 4.3. Although the degree of divergence of Figure 4.2 is -2 the graph still diverges. The reason for this is the presence of the divergent loops. The graph of Figure 4.3 is also divergent but its degree of divergences is -2, apparently convergent. Now the source of divergences is the "bubble" shown on one of its legs. Figures 4.2 and 4.3 are one particle reducible (1PR) graphs. There are also 2,3, ... N particle reducible graphs containing divergent subgraphs but yet appear to be superficially convergent.

In general, if in a Feynman graph, there are subgraph(s) whose degree of divergence is (are) greater than or equal to zero, the graph is divergent. This result is due to Weinberg[16] and is stated more elegantly in his theorem which says

> The general Feynman graph converges if its degree of divergence together with the degree of divergence of all its subgraphs are negative..

Although all scattering amplitudes of $\lambda\phi^4$ theory may have infinite parts there are only two real sources of divergence in the $d=4$ dimension. These are the two and four point functions of Figure 4.1.

When $d=4$ and $n>4$ ($\lambda\phi^5$, $\lambda\phi^6$ etc) the dimension of the coupling is negative and

$$\delta\Gamma^{(E)} = 4 - E + (n-4)V \tag{4.14}$$

But $V(n-4) > 0$. Therefore $\delta\Gamma^{(E)}$ can be made as large as possible by choosing the appropriate number of vertices. For this reason $\lambda\phi^n$ theory for $n>4$, is said to be nonrenormalizable in 4-space-time dimensions. On the other hand, when $d=2$

$$\delta\Gamma^{(E)} = 2(1-V) \tag{4.15}$$

and the degree of divergence is seen to be independent of the theory. The only primitively divergent diagrams are those for which $V=0,1$. Since divergences occur because of loop integrations, this means that the divergence occurs only when a leg from one vertex is connected to the same vertex, and not from the interaction between two or more vertices. Such self-inflicted divergences are called "normal ordering" and are usually removed by working with a Lagrangian whose interaction term is normal ordered. For instance, for the $\lambda\phi^n$ theory the replacement

$$\frac{\lambda\phi^n}{n!} \longrightarrow :\frac{\lambda\phi^n}{n!}: = \frac{\lambda}{n!} :\phi^n:$$

is made. The operators enclosed within the symbol $:\quad:$ are said to be normal ordered. For two operations the definition is

$$:\phi(x)\phi(y): = T(\phi(x)\phi(y)) - \langle 0|T(\phi(x)\phi(y))|0\rangle \tag{4.16}$$

In the limit as $y \to x$, (4.16) give

$$:\phi(x)\phi(x): = \phi(x)\phi(x) - \langle 0|\phi(x)\phi(x)|0\rangle \tag{4.17}$$

The time ordered symbols are omitted because the arguments of the operators are the same. It was seen in Chapter I that the last term in (4.17) was

$$\langle 0|\phi(x)\phi(x)|0\rangle = \Delta_F(x-x) = \Delta_F(0) \tag{4.18}$$

Therefore

$$:\phi(x)\phi(x): = \phi(x)\phi(x) - \Delta_F(0) \tag{4.19}$$

shows that the effect of normal ordering is to subtract out the diagrams containing $\Delta_F(0)$. It can be shown in a similar way that

$$:\phi^4(x): = \phi^4(x) - \text{diagrams containing } \Delta_F(0) \tag{4.20}$$

The definition for the normal product of N operators is given by Itzukson and Zuber[4].

The $\lambda\phi^3$ theory, considered by Collins[17] in detail and mentioned by Bogoliubov and Shirkov,[18] is the simplest model of a real scalar field one can consider. Its superficial degree of divergence in d-space-time dimensions is

$$\delta\Gamma^{(E)} = d - \frac{E}{2}(d-2) + V\left(\frac{d}{2} - 3\right) \tag{4.21}$$

This theory is renormalizable only for those values of d which makes $(\frac{d}{2} - 3) \leq 0$ i.e. for $d \leq 6$. Again it is evident that the non-negative dimension of the coupling is responsible for the theory being renormalizable. The following classification in terms of $[\lambda]$ are usually made

$[\lambda] = 0$: Renormalizable

$[\lambda] > 0$: Super-renormalizable

$[\lambda] < 0$: Non-renormalizable

Finally $[\lambda]$ is less than zero for all values of $d \geq 7$ regardless of the value of n. For $n > 2$ the $\lambda\phi^n$ theory is non-renormalizable for $d \geq 7$. Table 4.1 shows the summary of the results.

TABLE 4.1

Dimensions in which $\lambda\phi^n$ Theory is Renormalizable, Super-Renormalizable and Non-Renormalizable

n	Dimensions in which theory is		
	Super-renormalizable	Renormalizable	Non-Renormalizable
3	2, 3, 4, 5	6	$d > 6$
4	2, 3	4	$d > 4$
>>4	2		$d > 2$

CHAPTER V

REGULARIZATION TECHNIQUES

The purpose of this chapter is to introduce a few regularization techniques used in quantum field theory. Special attention will be paid to dimensional regularization and a prescription for regularizing any Feynman amplitude will be given (Leibbrandt)[19].

5.1 Pauli-Villars Regularization

A regularization technique is any mathematical prescription which renders a divergent amplitude finite by means of a special cut-off procedure. There are numerous regularization techniques available. However there are a few standard techniques used, the first of which is due to Pauli and Villars[2]. Their technique introduced massive auxillary fields called regulators in order to eliminate singularities from propagators and other ill defined functions. In its simplest version the Pauli-Villars method consists of replacing the free progator

$$\frac{i}{p^2-m^2}$$

in a scalar theory by

$$S_F(p,m;M) = \frac{i}{p^2-m^2} - \frac{i}{p^2-M^2} = \frac{i}{(p^2-m^2)}\,\frac{(m^2-M^2)}{(p^2-M^2)} \quad (5.1)$$

Notice that as $M \to \infty$ the original propagator is recovered. The behaviour for large p has clearly been improved. Thus the degree of divergence of the Feynman graph in the theory has been reduced.

5.2 Analytic Regularization

The method of analytic regularization differs completely from that of Pauli and Villars in that it makes use of the concept of analytic continuation in some complex parameter α.

To obtain an insight into the workings of this method consider the Feynman propagator

$$(p^2 - m^2 + i\varepsilon)^{-1}$$

for a scalar particle of mass m. The crucial step in the prescription is to replace the above propagator by

$$(p^2 - m^2 + i\varepsilon)^{-\alpha}$$

where the regulating parameter α may be complex. The result of such a replacement is to transform originally divergent integrals into well behaved analytic functions of α. The integrals are now convergent and can be evaluated unambiguously by performing the usual operations of integration by parts, shift of integration variables etc. After these manipulations one can continue the resulting expressions analytically to $\alpha=1$. The original UV divergences now show up as poles at $\alpha=1$. Subtraction of these poles at the end of the calculation yields the desired finite portion of the integral.

5.3 Dimensional Regularization

The technique of dimensional regularization, first introduced by t'Hooft and Veltman[3], is the latest and probably the best regularization technique available. Because the notion of analytic continuation of the space time dimension is utilized, the method is also commonly referred to as "the continuous dimensional method". To understand the basic motivation behind this technique consider the four-dimensional integral

$$I(4) = \int \frac{d^4k}{(2\pi)^4 \, k^2[(k-p)^2 + m^2]} \tag{5.2}$$

defined over Euclidean momentum space. To transfer from Minkowski space to Euclidean space it is necessary to perform a rotation through $\pi/2$ in the complex energy plane. Such a rotation is called "Wick-rotation" and it has the effect of changing the time component k_o of a Minkowski vector to $-ik_o$. If

$$k = (k_o, k_1, k_2, \ldots k_{d-1}) \tag{5.3}$$

with

$$k^2 = k_o^2 - \sum_{i=1}^{d-1} k_i^2 \tag{5.4}$$

represents a d-dimensional vector in Minkowski space then

$$k = (-ik_o, k_1, k_2, \ldots k_{d-1}) \tag{5.5}$$

with

$$k^2 = - \sum_{i=0}^{d-1} k_i^2 \tag{5.6}$$

represents the same vector but in Euclidean space. Evidently the square of the Euclidean vector is always negative. This procedure is regarded as analytic continuation to the region where k^2 is negative. Results of expressions for positive k^2 are obtained by analytic continuation at the end of the calculations.

The integral of Equation (5.2) is logarithmically divergent for large momentum $(k^2 \to \infty)$. However the corresponding integral in 3-dimension is convergent. So a reduction in the number of space-time dimension rendered the divergent integral convergent. The idea, therefore, is to write all Feynman integrals in an arbitrary space-time dimension (d) and use d as a regularising parameter. Once all manipulations involving integrals have been made, the principle of analytic continuation can be utilised to return to four-dimensional space. The concept of analytic continuation in the number of space-time dimensions is the most important single feature in the technique of dimensional regularization.

The famous theorem of Knopp[20], stated below, will clarify the meaning of analytic continuation.

THEOREM: Let an analytic function $g_1(z)$ be defined in a region D_1 and let D_2 be another region which has a certain subregion R, but only this one, in common with D_1. Then if a function $g_2(z)$ exists which is analytic in D_2 and coincides with $g_1(z)$ in R there can only be one such function. $g_1(z)$ and $g_2(z)$ are said to be analytic continuations of each other. See Figure 5.1.

A nice example which is the basis for the method of analytic continuation, is the difference between the Euler and Weierstrass representations of the gamma functions. For $\mathrm{Re}z>0$, the Euler representation is

$$\Gamma_E(z) = \int_0^\infty dt\, t^{z-1}e^{-t} \tag{5.7}$$

The domain of analyticity is shown in Figure 5.2. For $\mathrm{Re}z<0$ and $t\to 0$ $\Gamma_E(z)$ behaves as

$$\frac{dt}{t^{(1+|\mathrm{Re}z|)}}$$

which leads to an infinity. In order to discuss points outside the analytic region it is necessary to find first an analytic continuation of $\Gamma_E(z)$. Such a continuation is obtained by splitting the integration limit in (5.7).

$$\Gamma_E(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\alpha dt\, t^{n+z-1} + \int_\alpha^\infty dt\, e^{-t}\, t^{z-1} \tag{5.8}$$

where α is totally arbitrary. The second integral is well defined even when $\mathrm{Re}z<0$ as long as $\alpha>0$. The first integral has simple poles whenever z is a negative integer or zero. The Weierstrass representation of the gamma function is obtained by setting $\alpha=1$ so that

$$\Gamma_W(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^\infty dt\, e^{-t}\, t^{z-1} \tag{5.9}$$

which is analytic everywhere except at $z=0, -1, -2, \ldots$ See Figure 5.3.

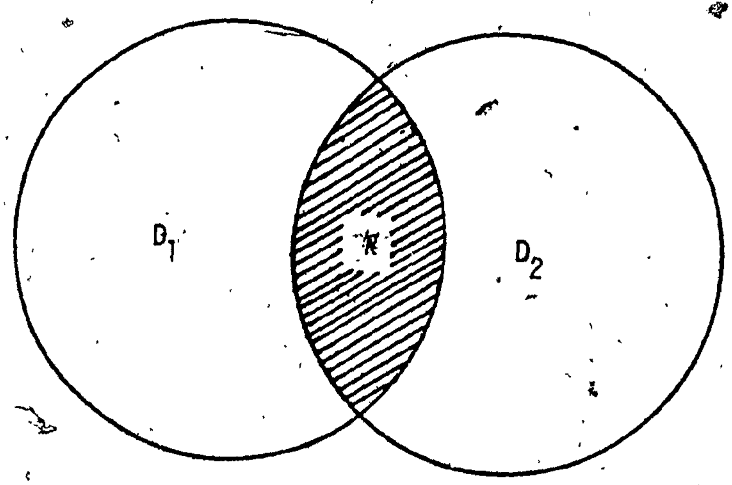


FIGURE 5.1 The common region R is the region of analytic continuation.

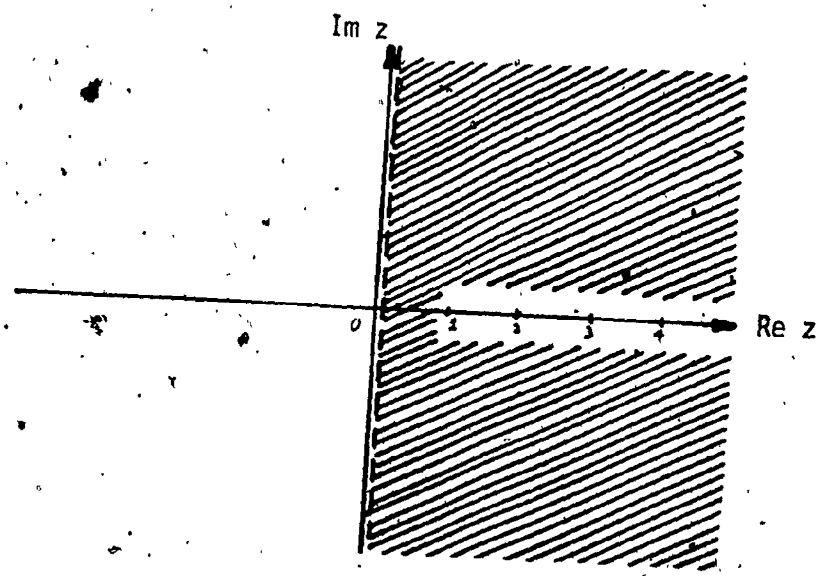


FIGURE 5.2 Domain of analyticity of $\Gamma_E(z)$

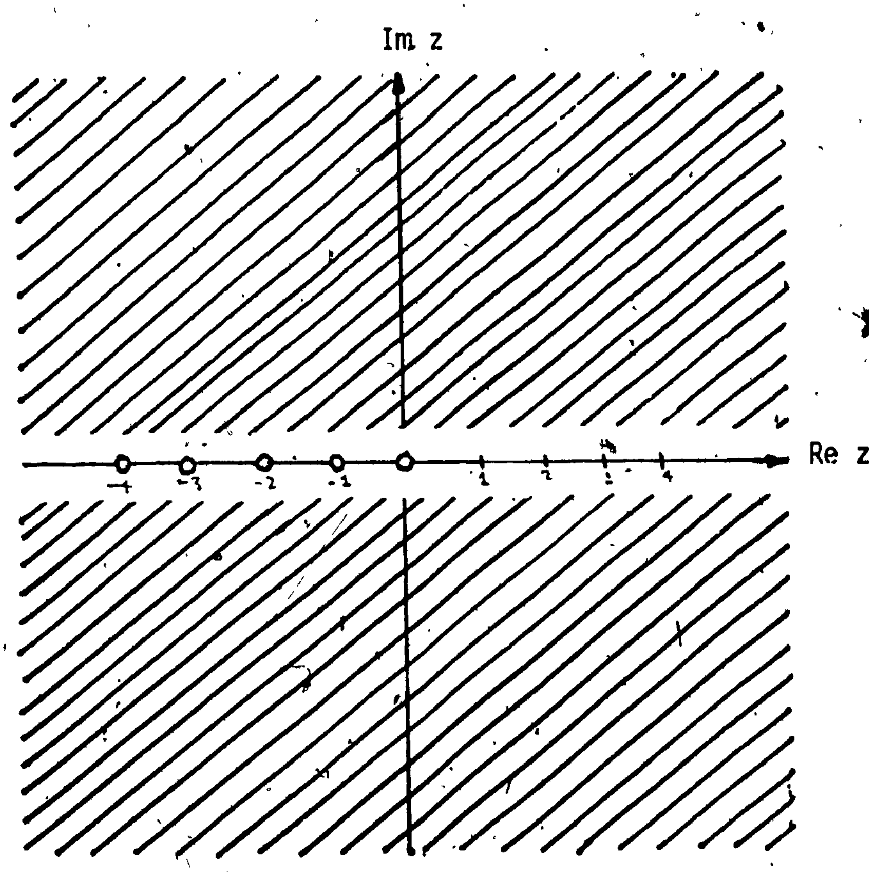


FIGURE 5.3 Domain of analyticity of $\Gamma_W(z)$

$\Gamma_W(z)$ is a unique analytic continuation of $\Gamma_E(z)$, since its domain of definition clearly overlaps that of $\Gamma_E(z)$.

5.4 Dimensional Regularization Prescription

This section contains a multistep prescription which should be helpful in achieving regularized integral by the continuous dimensional method. Suppose that the four-dimensional integral

$$I(p) = \int \frac{d^4k}{(2\pi)^4} J(k^2, k.p) \tag{5.10}$$

is ultraviolet divergent. For massive fields, the basic steps in the method of dimensional regularization as given by Leibbrandt[19] are

(i) Define all vector products over a complex d-dimensional space

(ii) Parametrize all momentum-space propagator according to

$$\frac{1}{k_i^2 + m^2} = \int_0^\infty d\alpha_i \exp[-\alpha_i(k_i^2 + m^2)] \tag{5.11}$$

Notice that in Euclidean space the $i\varepsilon$ term in the propagator is not required. To understand this consider the Minkowski space propagator without the $i\varepsilon$ term:

$$S_M = \frac{i}{k^2-m^2} \tag{5.11a}$$

S_M has real poles at $k=\pm m$. So integration of S_M immediately diverges. To avoid this divergence the small complex quantity $i\varepsilon$ is added to S_M. On the other hand the Euclidean space propagator (5.11) has imaginary poles at $k_i = \pm im$. Therefore integration of this propagator does not produce divergence so there is no need for the $i\varepsilon$ term.

(iii) Use the generalized gaussian integral (5.12) to integrate over momentum space

$$\int \frac{d^4k}{(2\pi)^d} \exp[-xk^2 + 2k.b] = \frac{(\pi/x)^{\frac{d}{2}}}{(2\pi)^d} \exp \frac{b^2}{x} \quad (x>0) \tag{5.12}$$

(IV) The resulting amplitude is now well defined as a function in a finite domain of the complex d-plane. To obtain an amplitude defined outside this domain analytic continuation must be performed.

(V) Integration over Feynman parameters lead, in regions where the integral exists, to gamma (Γ) functions. Analytic continuation is achieved by using the Weierstrass representation of this Γ function.

(VI) All d dependent quantities are expanded in Laurent series about the point d=4, so that the integral I(p) of Equation (5.10) becomes

$$I(p) = \frac{G(p^2)}{d-4} + F(p^2) + O(d-4) \tag{5.13}$$

The original UV infinites now manifest themselves as poles at the "physical" value d=4.

(VII) Cancel the pole term $G(p^2)/(d-4)$ by adding appropriate counterterms to the original Lagrangian in which case the regularized integral is shown in (5.14). $F(p^2)$ is the renormalized integral and the counterterm is $-G(p^2)/(d-4)$.

$$I_{reg}(p) = F(p^2) + O(d-4) \tag{5.14}$$

(VIII) Analytically continue Equation (5.14) to four space-time dimension by taking the limit $d \to 4$ so the value of the integral is given by the finite portion $F(p^2)$ of the Laurent expansion, properly continued to Minkowski space.

The above prescription is sufficient to regularize UV divergent integrals associated with massive scalar fields.

In the preceeding chapter the Feynman integral for a 1PI graphs with L loops, I internal lines and V vertices was written as

$$I = S(-i\lambda)^V \int \prod_{i=1}^{L} \frac{d^d k_i}{(2\pi)^d} \prod_{j=1}^{I} \frac{1}{(\ell_j^2 - m^2 + i\varepsilon)} \tag{5.15}$$

where ℓ_j is a linear combination of external and loop momenta and S is the symmetry factor. To parametrize (5.15) the explicit dependence of the ℓ's on the k's must be known.

Regularization of the general Feynman graph of a scalar theory was done by Itzykson and Zuber[4]. To fully understand their work it is necessary to study the analytic properties of Feynman graphs (Todorov)[21]. Such a study is beyond the scope of this thesis and will not be considered any further. Instead, a simple graph from $\lambda\phi^n$ theory will be regularized in the next chapter.

CHAPTER VI

DIMENSIONAL REGULARIZATION AND RENORMALIZATION OF ONE LOOP GRAPHS

In this chapter the techniques of dimensional regularization and minimal subtraction (MS) (Collins)[17] are utilized so as to achieve renormalization of some simple graphs.

6.1 The d-Dimensional Integral

It was already mentioned that the method of dimensional regularization treated the space-time dimension as a continuous variable. Since vector spaces of non-integer dimension do not exist as such, it is not obvious that the concept has any consistency, let alone validity, even in the purely formal sense. So the concept of integration on a space of finite non-integer dimension, d, cannot be taken completely literally. Either it is a set of purely formal rules for obtaining answers or it is an operation that is not literally integration in d-dimensions, but only behaves in many respects as if it were integration in d dimensions. Nevertheless uniqueness and existence of the d-dimensional integrals will be assumed and standard manipulations will be applied to them. For an explicit definition of the d-dimension integration and some axioms necessary in application to Feynman integrals see Collins[17].

6.2 The Basic Graphs of $\lambda\phi^3$ Theory

The basic graphs of $\lambda\phi^3$ theory in 4 space-time dimensions are shown in Figure 6.1. Figures 6.1(a) and (c) are called tadpole graphs.

(a)

(b)

(c)

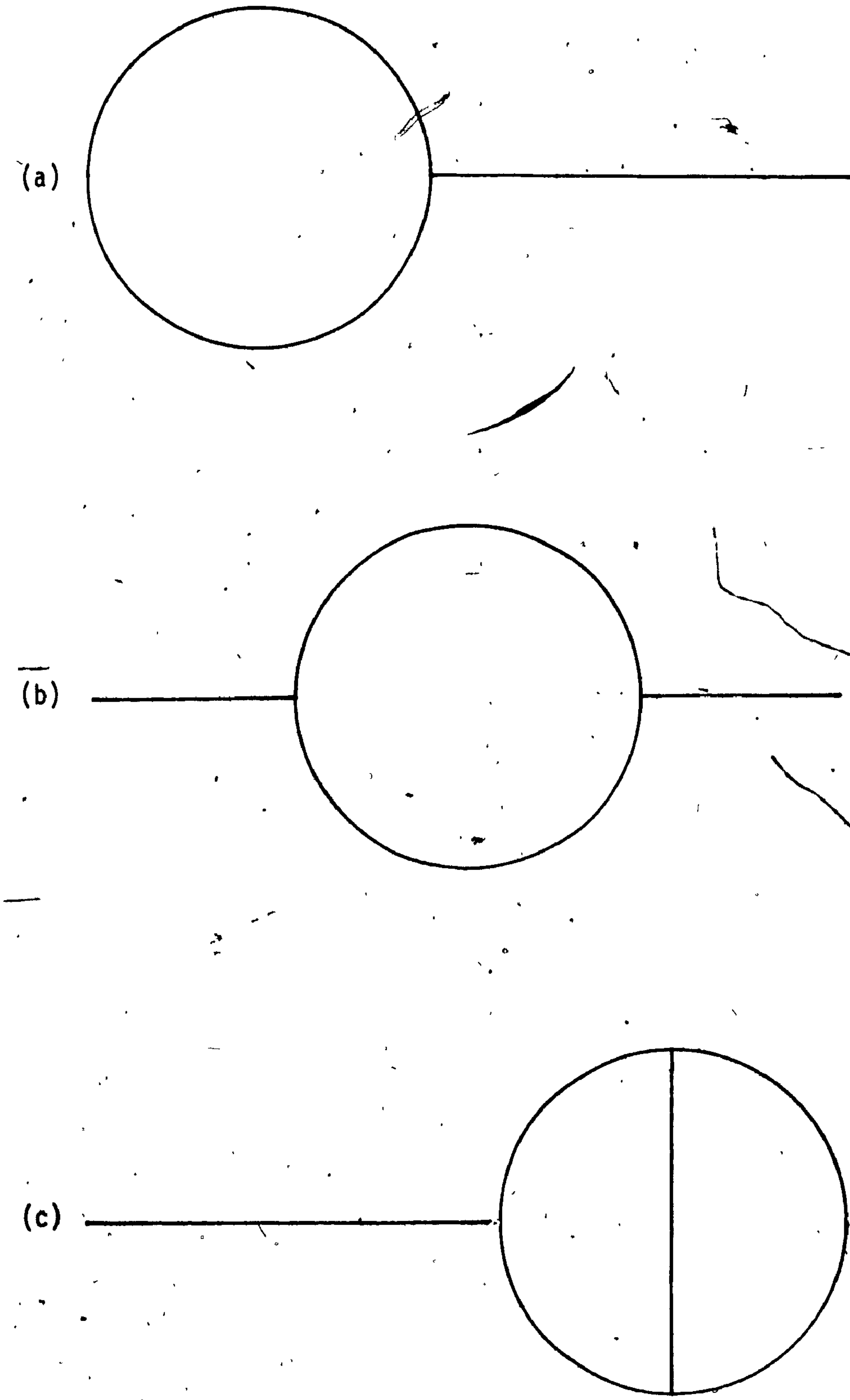
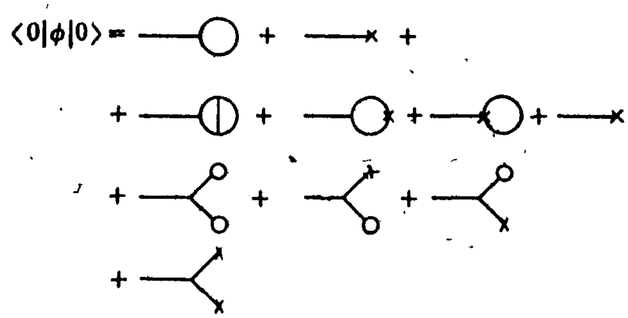


FIGURE 6.1 The basic graphs from $\lambda\phi^3$ theory in 4-dimensions

The divergences caused by these, and all other tadpole graphs, are removed by imposing the renormalization condition

$$<0|\phi|0> = 0 \tag{6.1}$$

To third order, the left hand side of (6.1) is as given in Equation (6.2)



(6.2)

Figures 6.1(a) and (c) and their counterterms are contained in (6.2) as a result, divergences caused by them are removed by imposing condition (6.1). Hence, the only basic graph that need be renormalized in 4 dimension is the one-loop self-energy graph shown in Figure 6.1(b).

6:3 Regularization and Renormalization of the One-Loop Self-Energy Graph from $\lambda\phi^3$ Theory

The contribution of Figure 6.2 to the self-energy is defined as i times the Feynman integral for the amputated graph of Figure 6.2. In d space time dimensions this is given by Equation 6.3.

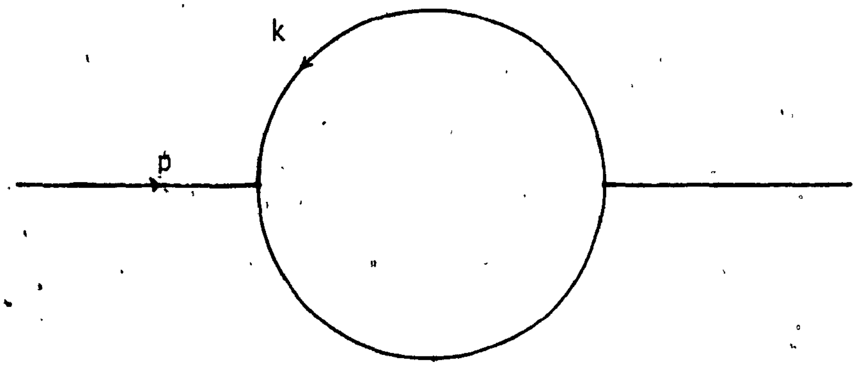


FIGURE 6.2 The one-loop self-energy diagram from $\lambda\phi^3$ theory

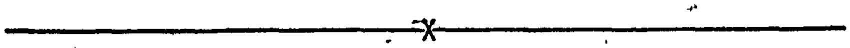


FIGURE 6.3 The counterterm of Figure 6.2

$$\Sigma_1(p^2) = \frac{i}{2}\frac{\lambda^2}{(2\pi)^d}\int d^dk \frac{1}{[k^2-m^2+i\varepsilon][(k+p)^2-m^2+i\varepsilon]} \tag{6.3}$$

Using Schwinger's parametric representation for each Euclidean propagator (Appendix B)

$$\frac{1}{m^2 - k^2 - i\varepsilon} = \int_0^\infty d\alpha \exp[-\alpha(m^2 - k^2 - i\varepsilon)] \tag{6.4}$$

Equation (6.3) reads

$$\Sigma_1 = = \frac{i\lambda^2}{(2\pi)^d}\int^\infty d\alpha \int^\infty d\beta \int d^dk \exp[-(\alpha+\beta)m^2$$
$$\beta p^2 + 2\beta p.k + (\alpha+\beta)k^2] \tag{6.5}$$

Now replace k^μ by $k^\mu - \frac{\beta p^\mu}{(\alpha+\beta)}$ and change variables to

$$z = \alpha + \beta \quad , \quad x = \frac{\alpha}{z} \tag{6.6}$$

The Jacobian of the transformation, defined by

$$d\alpha\, d\beta \longrightarrow \begin{vmatrix} \frac{\partial\alpha}{\partial z} & \frac{\partial\beta}{\partial z} \\ \frac{\partial\alpha}{\partial x} & \frac{\partial\beta}{\partial x} \end{vmatrix} dx\, dz$$

together with the new k gives

$$\sum_1 = \frac{-i\lambda^2}{2(2\pi)^d}\int_0^1 dx \int_0^\infty dzz \int d^dk \exp \quad -z[m^2 - p^2x(1-x)] + zk^2 \tag{6.7}$$

After scaling k by a factor $z^{\frac{1}{2}}$ Equation (6.7) becomes

$$\sum_1 = \frac{i\lambda^2}{2(2\pi)^d} \int dx \int^\infty dz\, z^{1-d/2} \exp\left\{-z[m^2 - p^2x(1-x)]\right\} \times$$

$$\int d^dk \exp(k^2) \tag{6.8}$$

The integral over k may be replaced by $i\pi^{d/2}$ which is correct for integer d. With a change of variable in Equation (6.8),

$$z[m^2 - p^2x(1-x)] \rightarrow z'$$

the integration reads

$$\int \frac{dz'\, z'^{(1-d/2)}}{\left\{m^2 - p^2x(1-x)\right\}^{2-d/2}} e^{-z'}$$

$$= \frac{1}{\left\{m^2 - p^2x(1-x)\right\}^{2-d/2}} \int dz'\, z'^{\{(2-d/2)-1\}} e^{-z'}$$

Comparing the integral in the above expression with Equation (5.8) reveals that the integration over z in Equation (6.8) is nothing more than

$$\Gamma(2 - d/2) \left\{ m^2 - p^2 x(1-x) \right\}^{d/2-2}$$

so that

$$\Sigma_1 = \frac{-\lambda^2}{2(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 dx \left\{ m^2 - p^2 x(1-x) \right\}^{d/2-2} \tag{6.9}$$

The divergence of the original integral now resides in the Euler gamma function of (6.9) which has simple poles at $d = 4, 6, 8, \ldots$. The residue of each pole is a polynomial in p of degree equal to the degree of divergence.

Consider renormalization in 4 space-time dimensions. A rather obvious way of renormalizing (6.9) is to define a term so that when it is added to Σ_1 the singularities are exactly cancelled. The term added is called the counterterm (see Chapter III). The counterterm of Figure 6.2 is shown in Figure 6.3.

Now $\Gamma(y) = \frac{1}{y} - \gamma_E + O(y)$ (6.10)

where the Euler's constant γ_E is given by

$$\gamma_E = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log_e n\right) = 0.577\ldots \tag{6.11}$$

Also $\frac{1}{(b)^m} = 1 - m \ln b$ (6.12)

Substitute (6.10) and (6.12) into (6.9) to find

$$\Sigma_1 = \frac{\lambda^2}{2(4\pi)^{d/2}} \left\{\frac{1}{(2-d/2)} - \gamma_E + O(2-d/2)\right\} \times$$

$$\left\{\int_0^1 dx[1 - (2-d/2)\ln\{m^2 - p^2x(1-x)\} + ..]\right\} \tag{6.13}$$

The infinite part of (6.13) is now a pole at $d = 4$. The residue of this pole is

$$- \frac{\lambda^2}{32\pi^2(2-d/2)}$$

Therefore the counterterm needed is

$$\frac{\lambda^2}{32\pi^2(2-d/2)}$$

The renormalized self-energy is thus

$$\Sigma_{1R} = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left\{\ln\left[\frac{m^2 - p^2x(1-x)}{4\pi}\right] + \gamma_E\right\} \tag{6.14}$$

There is only one fault with (6.14). The problem is that the expansion in powers of $(d - 4)$ did not allow for the fact that λ has a mass dimension dependent on d. Therefore the implicit introduction of a mass scale is inevitable. For this reason (6.14) contains the logarithm of a dimensional quantity. The scale can be made explicit by redefining the coupling constant (Ramond[13], Collins[17]) so that

$$\lambda \to \mu^{2-d/2} \lambda$$

where μ is a parameter with dimensions of mass. Now making use of the relation

$$f(x) = \exp[\ln f(x)] \tag{6.15}$$

gives

$$\begin{aligned} \mu^{2-d/2} &= \exp[(2-d/2)\,\ln\mu] \\ &= 1 + (2-d/2)\ln\mu + \tfrac{1}{2}(2-d/2)^2\ln^2\mu + \ldots \end{aligned} \tag{6.16}$$

The new renormalized self-energy is therefore

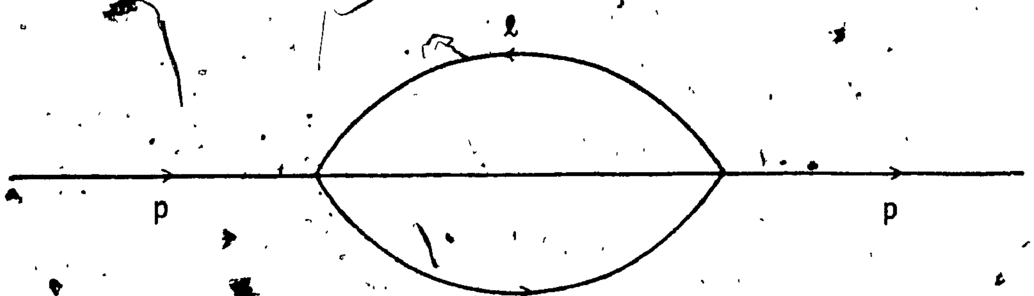
$$\Sigma_{1R} = \frac{\lambda^2}{32\pi^2}\int dx\left\{\ln\left[\frac{m^2 - p^2x(1-x)}{4\pi\mu^2}\right] + \gamma_E\right\} \tag{6.17}$$

The above prescription is called Minimal Subtraction (MS) whenever the counterterms are pure poles at the physical value of d. The same type of calculations can be performed for $d = 6$.

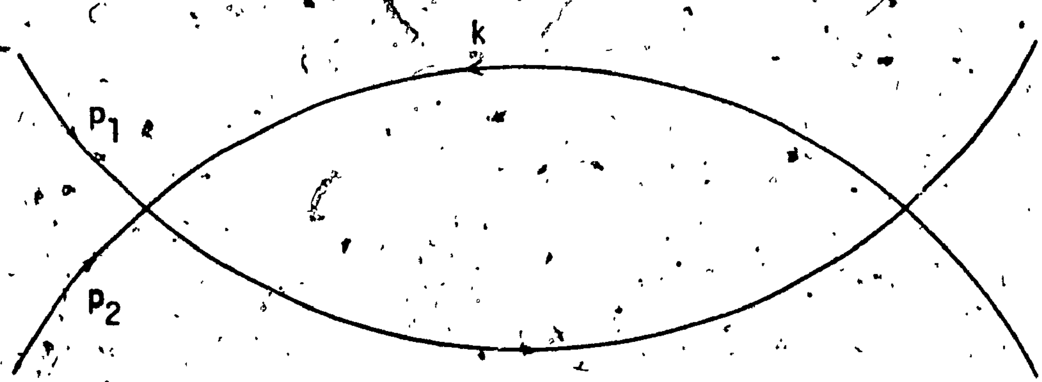
6.4 Overlapping Divergence

The basic graphs of $\lambda\phi^4$ theory were seen to be those of Figure 6.4 below. The symmetry factor of Figure 6.4(a) was calculated in Chapter I and was found to be 1/6. Thus, the Feynman amplitude of Figure 6.4(a) is

$$I = \frac{i}{6}\frac{\lambda^2}{(2\pi)^8}\int \frac{d^4k\ d^4\ell}{[k^2-m^2+i\varepsilon][\ell^2-m^2+i\varepsilon][(p+\ell-k)^2-m^2+i\varepsilon]} \tag{6.18}$$



(a)



(b)

FIGURE 6.4 The two basic graphs from $\lambda\phi^4$ theory (d=4)

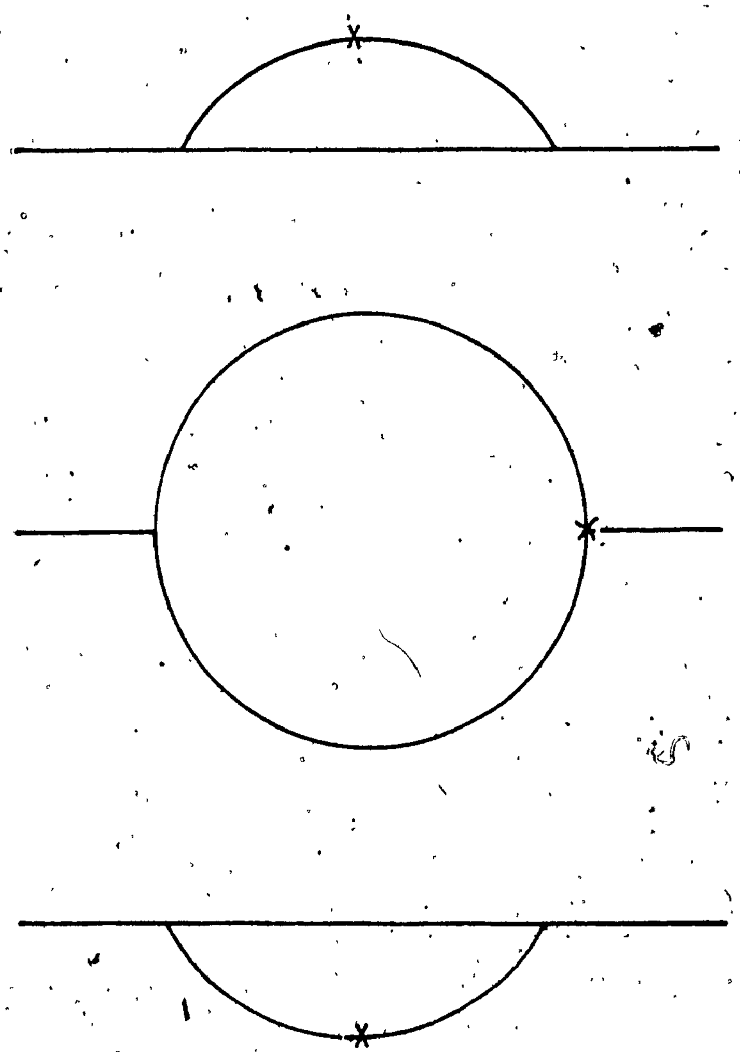


FIGURE 6.5 The counterterms of Figure 6.4(a)

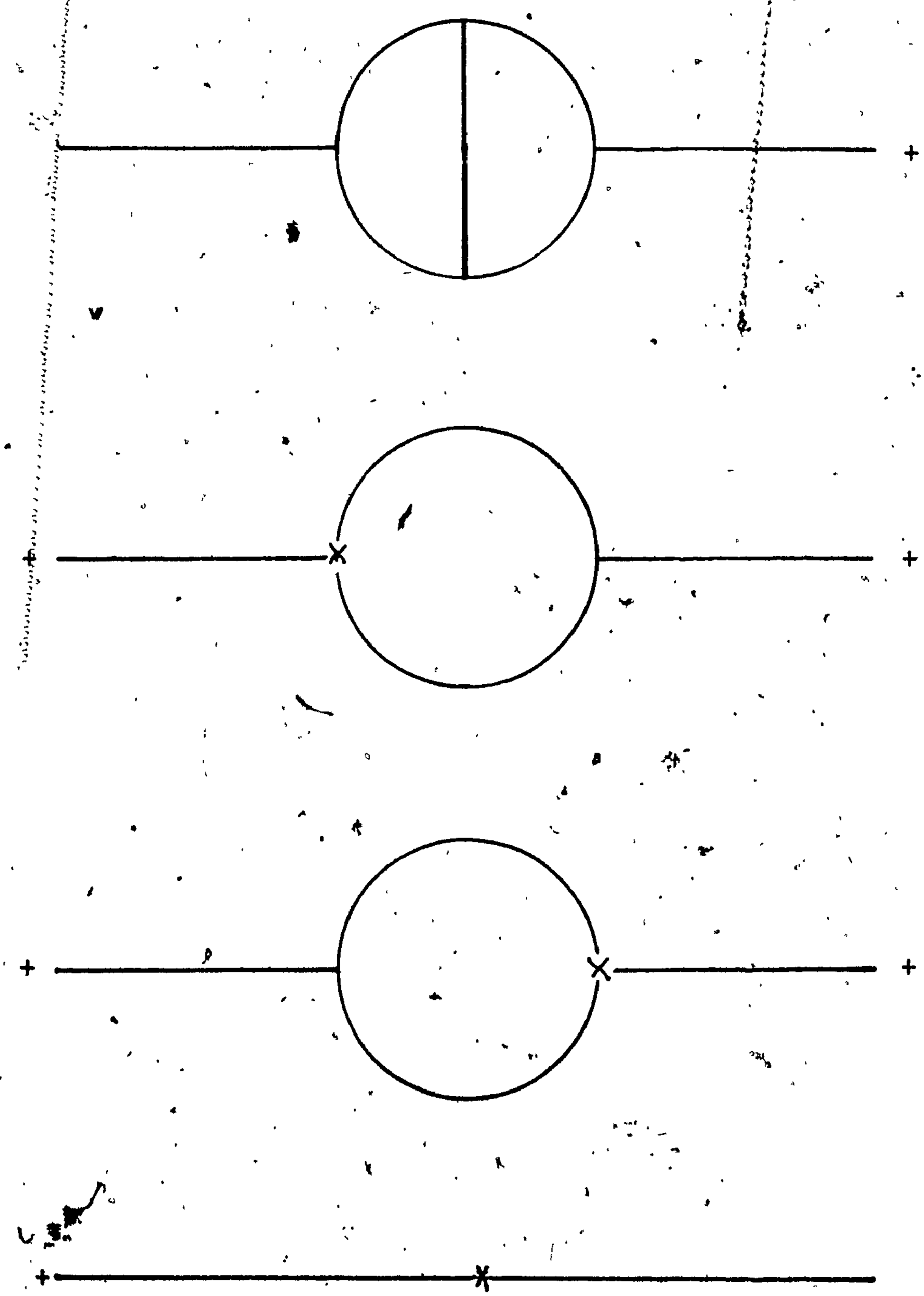


FIGURE 6.6 A two loop diagram from $\lambda\phi^3$ theory with overlapping divergence, and its counterterms

Equation (6.18) diverges logarithmically if k is held fixed and $\ell \to \infty$ or if ℓ is held fixed and $k \to \infty$. However, if both ℓ, $k \to \infty$ the integral diverges quadratically. The latter is called an overlapping divergence. Whenever there are diagrams with loops, at least two of which has an internal line in common, there will be overlapping divergence(s). The counterterms of Figure 6.4(a) are shown in Figure 6.5. Another example of a diagram with overlapping divergence in Figure 6.6. Overlapping divergences are much more difficult to deal with than the simpler kind (i.e. those associated with one loop of a diagram). They have the property that if one tries to evaluate the integral after parametrization, the divergence moves in part from the loop-momenta integrations to the integrations over the Feynman parameters. This makes the dimensional method much more difficult to use and caution must be exercised when dealing with multiloop calculations. For simplicity only one loop diagrams will be treated here.

6.5 The One Loop Diagram from $\lambda\phi^n$ Theory

Figure 6.7 shows the one loop diagram from $\lambda\phi^n$ theory. There are a total of n lines at each vertex. With S as the symmetry factor, the Feynman integral of Figure 6.7 is (in d dimensions)

$$I = \frac{S(-i\lambda)^2}{(2\pi)^d} \int \frac{d^dk}{[k^2-m^2+i\epsilon][(k+\sum_j p_j)^2 - m^2 + i\epsilon]} \tag{6.19}$$

The index j of p_j runs from 1 to $(n-2)$. Using the Schwinger's parametric representation for each propagator (Equation (6.4)) it is found that

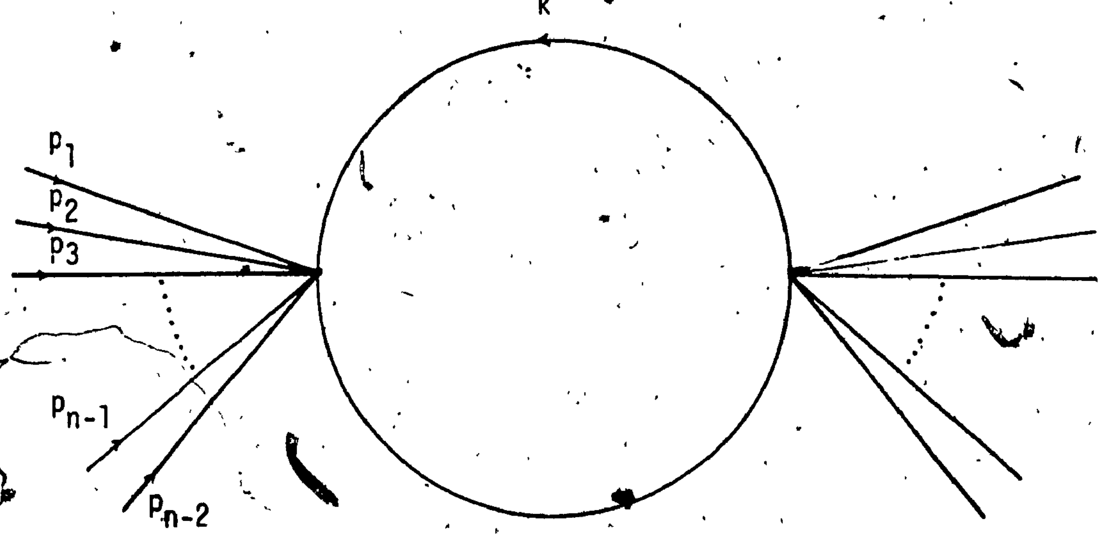


FIGURE 6.7 The one-loop diagram from $\lambda\phi^n$ theory

$$I = C \int d\alpha \int d\beta \int d^d k \exp \left\{ - \alpha(m^2 - k^2 - i\varepsilon) - \beta[m^2 - (k+P)^2 - i\varepsilon] \right\} \tag{6.20}$$

where

$$C = \frac{S(-i\lambda)^2}{(2\pi)^d} \tag{6.21}$$

and

$$P = \sum_{j=1}^{n-2} p_j \tag{6.22}$$

The argument of the exponential may be written as

$$A = -\alpha(m^2 - k^2 - i\varepsilon) - \beta[m^2 - (k+p_1+p_2 + \ldots + p_{n-2})^2 - i\varepsilon]$$

After dropping the $i\varepsilon$ term, the above expression becomes

$$A = -\alpha(m^2 - k^2) - \beta[m^2 - k^2 - 2k \sum_j p_j - \sum p_j^2 - \sum_{i \neq j} p_i p_j] \tag{6.23}$$

$$A = k^2(\alpha+\beta) - m^2(\alpha+\beta) + 2\beta k \sum_j p_j + \beta(\sum p_j^2 + \sum_{i \neq j} p_i p_j) \tag{6.24}$$

Now shift k so that

$$k \to k - \left(\frac{\beta}{\alpha+\beta}\right) \sum_j p_j$$

This procedure will eliminate the terms linear in k.

$$A = (\alpha+\beta)\left\{k - \left(\frac{\beta}{\alpha+\beta}\right)\sum p_j\right\}^2 - (\alpha+\beta)m^2 +$$

$$+ 2\beta\left\{k - \left(\frac{\beta}{\alpha+\beta}\right)\sum p_j\right\}\left(\sum p_j\right) +$$

$$+ \beta\left\{\sum p_j^2 + \sum_{i\neq j} p_i p_j\right\} \qquad (6.25)$$

or

$$A = (\alpha+\beta)k^2 - \frac{\beta^2}{\alpha+\beta}\left(\sum p_j\right)^2 - (\alpha+\beta)\, m^2 +$$

$$+ \beta\left(\sum p_j^2 + \sum_{i\neq j} p_i p_j\right) \qquad (6.26)$$

The change of variables defined by (6.6) lead to

$$A = zk^2 - z\left[m^2 - x(1-x)\left\{\sum p_j^2 + \sum_{i\neq j} p_i p_j\right\}\right] \qquad (6.27)$$

Substitute in (6.20) to find

$$I = C\int_0^1 dx \int_0^\infty dz\, z \int d^dk\, \exp\left\{zk^2 - z\left\{m^2 - x(1-x)\left(\sum p_j^2 + \sum_{i\neq j} p_j p_i\right)\right\}\right\} \qquad (6.28)$$

After scaling k by $z^{\frac{1}{2}}$ and performing the z and k integrations as in section 6.3 the result is

$$I = \frac{i\lambda^2 S}{(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 dx[m^2 - x(1-x)(\sum p_j^2 + \sum_{i \neq j} p_i p_j)]^{d/2-2} \tag{6.29}$$

When the only external momentum is $p_1 = p$, Figure 6.7 becomes Figure 6.2, the one-loop self energy diagram from $\lambda\phi^3$ theory. The symmetry factor of Figure 6.2 is ½ and the term $\sum_{i \neq j} p_i p_j$ vanishes. Hence, from (6.29)

$$I = -\frac{i\lambda^2}{2(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 dx[m^2 - p^2 x(1-x)]^{d/2-2} \tag{6.30}$$

This is exactly what was obtained in Section 6.3 (see Equation (6.9)) except for a factor i, but in Section 6.3 the Feynman integral was multiplied by i.

It was found in Chapter IV that in 2 dimensions the divergent graph from $\lambda\phi^n$ theory are those with 0 or 1 vertex. Since Figure 6.7 does not contain either of these graphs as subgraphs it is expected that Equation (6.29) is finite in 2-dimensions. The gamma function $\Gamma(2-d/2)$ may be written as

$$\Gamma(2 - d/2) = \Gamma(1 - d/2 + 1) \tag{6.31}$$

But $\Gamma(z + 1) = z\Gamma(z)$ (6.32)

$$\therefore \Gamma(1 - d/2 + 1) = (1 - d/2)\Gamma(1 - d/2) \tag{6.33}$$

Also from Equation (6.10)

$$\Gamma(z) = \frac{1}{z} - \gamma_E + O(z)$$

So $\Gamma(1 - d/2 + 1) = (1 - d/2) \left\{ \frac{1}{1 - d/2} - \gamma_E + O(1 - d/2) + \ldots \right\}$

$$ \tag{6.34} $$

or

$$\Gamma(1 - d/2 + 1) = 1 - (1 - d/2)\,\gamma_E + O(1 - d/2)^2) + \ldots \tag{6.35}$$

Using Equations (6.12) and (6.35) in (6.29) it is found that

$$I = \frac{-i\lambda^2 s}{(4\pi)^{d/2}} \{1 - (1-d/2)\gamma_E + O((1-d/2)^2) + \ldots\} \times$$

$$\int_0^1 dx[1 - (2-d/2)\ln\{m^2 - x(1-x)(\Sigma p_j^2 + \Sigma p_i p_j)\} + ..] \tag{6.36}$$

As expected, when $d = 2$ Equation (6.36) is finite and there is no reason to add a counterterm.

6.6 The One Loop Diagram from $\lambda\phi^4$ Theory

To complete the discussion on one-loop diagrams consider Figure 6.4b from $\lambda\phi^4$ theory. In Chapter III parametrization of the Feynman integral shown in Equation (3.1) lead to the integral shown in Equation (3.4). In d-dimensions this integral may be written as

$$I = \frac{\lambda^2}{2(2\pi)^d} \int \frac{d^d k \, d\alpha}{[k^2 + b^2]^2} \tag{6.37}$$

where b^2 is defined by Equation (3.5).

Equation (6.37) was obtained by using the Feynman parametrization defined in Appendix B. Using the identity

$$\int \frac{d^d k}{[k^2 + b^2]^n} = \frac{i\pi^{d/2}\Gamma(n-d/2)}{\Gamma(n)} \quad \frac{1}{(b^2)^{n-d/2}} \tag{6.38}$$

to perform the k integration in (6.37), one finds that

$$I = -\frac{i\lambda^2}{2(4\pi)^{d/2}} \int d\alpha \, [m^2 - \alpha(1-\alpha)(p_1+p_2)^2]^{d/2-2} \tag{6.39}$$

Equations (3.3) and (3.5) were used to replace b^2 and s.

When Equation (6.29) is used to evaluate the same diagram [Figure 6.4b] then

$$\sum p_j^2 = p_1^2 + p_2^2 \tag{6.40}$$

and

$$\sum_{i \neq j} p_i p_j = p_1 p_2 + p_2 p_1 = 2p_1 p_2 \tag{6.41}$$

$$\therefore \; \sum p_j^2 + \sum_{i \neq j} p_i p_j = (p_1+p_2)^2 \tag{6.42}$$

Equation (6.29) now reads, with $S = \frac{1}{2}$,

$$I = \frac{i\lambda^2}{2(4\pi)^{d/2}} \int dx[m^2 - x(1-x)(p_1+p_2)^2]^{d/2-2} \tag{6.43}$$

In obtaining (6.39) the Feynman integral was parameterized using the Feynman parameteric representation of the propagators, but Equation (6.43) was obtained using the Schwinger representation for the propagators. Nevertheless Equation (6.39) and (6.43) are identical. The finite portion of (6.43) is obtained by using Equations (6.10) and

(6.12). The result is

$$\frac{-i\lambda^2}{32\pi^2} \int_0^1 dx\{\ell n[m^2 - x(1-x)(p_1+p_2)^2] + \gamma_E\}$$

This is the same result obtained by Nash[5]. This concludes the discussion of regularization and renormalization of the one loop graph (Figure 6.7) for $\lambda\phi^n$ theory.

CONCLUSION

The results of Chapters I and II showed that the Green's functions of $\lambda\phi^4$ and $\lambda\phi^n$ theories can be expressed in terms of the generating function arising from the free Lagrangian. The Feynman Rules for $\lambda\phi^4$ and $\lambda\phi^n$ theories differ only in the structure of the vertices. A vertex from $\lambda\phi^4$ theory contains 4 lines while a vertex from the $\lambda\phi^n$ theory contains n lines. Hence, diagrams from a theory like $\lambda\phi^p + g\phi^q$ will contain vertices with p and q lines. Although the structure of the vertices are different, the contribution to the Feynman amplitude of any vertex, whether from $\lambda\phi^3$ or $\lambda\phi^4$ theory, is always $-i\lambda$.

Whether a theory is renormalizable or not depends on the mass dimension of the coupling (λ) which in turn depends on d, the space-time dimension. In particular, the $\lambda\phi^4$ and $\lambda\phi^3$ theories were found to be superrenormalizable for $d < 4$ and $d < 6$, renormalizable for $d = 4, 6$ and non-renormalizable for $d > 4$, $d > 6$ respectively. These results are consistent with those found by Nash[5], Ramond[13] and Collins[17]. Furthermore, when $d = 4$ and $n > 4$ the coupling constant has a negative mass dimension and $\delta\Gamma$ can be made as large as required by choosing the appropriate number of vertices. Hence, the $\lambda\phi^n$ theory is non-renormalizable in 4-dimensions for all $n > 4$. When $d = 2$ the situation is reversed, for then the degree of divergence is independent of n. There are only two primitively divergent diagrams and all theories are super-renormalizable in $d = 2$ dimensions. When $d \geq 7$ there are no values of $n > 2$ for which $\lambda\phi^n$ theory is renormalizable.

Equation (6.29) shows the dimensionally regularized Feynman amplitude of the one-loop diagram from $\lambda\phi^n$ theory. For $n = 3, 4$ Equation

(6.29) reduces to (6.9) and (6.43) respectively. These are consistent with the expressions given by Nash[5] and Collins[17].

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APPENDIX A

FUNCTIONAL DIFFERENTIATION AND FUNCTIONAL TAYLOR SERIES

Let $E(f(x))$ be a functional of $f(x)$. The functional derivative of $E(F(x))$ with respect to $f(y)$ is defined as

$$\frac{\delta E(f(x))}{\delta f(y)} = \lim_{\varepsilon\to 0} \frac{E[f(x)+\varepsilon\delta(x-y)] - E(f(x))}{\varepsilon} \tag{A1}$$

There are some reasonable similarities to ordinary differentiation and the well known chain rule for functions of a function has a functional counterpart. Let $E = E(F)$ and $F = F(f(x))$, then the chain rule is

$$\frac{\delta E}{\delta f(y)} = \int dx' \frac{\delta E}{\delta F(x')} \frac{\delta F(x')}{\delta f(y)} \tag{A2}$$

Let F be an arbitrary functional of f, and λ a real variable so that

$$g(\lambda) = F(f + \lambda f') \tag{A3}$$

The functional F has a Functional Taylor series expansion defined by

$$F(f) = \sum_{0} \frac{1}{n!} \int dx_1 \ldots dx_n f(x_1) \ldots f(x_n) \frac{\delta^n F(0)}{\delta f(x_1) \ldots \delta f(x_n)} \tag{A4}$$

providing that the ordinary Taylor Series of $g(\lambda)$ defined by A5

$$g(\lambda) = \sum_{0}^{\infty} \frac{g^n(0)\lambda^n}{n!} \tag{A5}$$

converges at the point $\lambda = 1$.

APPENDIX B.

PARAMETRIC FORMS OF THE PROPAGATOR

THE FEYNMAN PARAMETER

The general Feynman amplitude arising from a graph with V vertices, I internal lines and L loops, was seen to take the form

$$I = (-i\lambda)^V S \int \prod_{j=1}^{L} \frac{d^d k_j}{(2\pi)^d} \prod_{i=1}^{I} \frac{1}{(\ell_j^2 - m^2 + i\varepsilon)} \tag{B1}$$

One very important step necessary to regularize a graph is the parameterization of its propagators. There are two equivalent parametric forms of the propagator usually used. The first is due to Feynman. Its most general form is

$$\frac{1}{A^\alpha B^\beta \ldots E^\gamma} = \frac{\Gamma(\alpha+\beta+\ldots+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\ldots\Gamma(\gamma)} \times \int dx\, dy \ldots dz\, \delta(1-x-y-\ldots z) \frac{(x^{\alpha-1} y^{\beta-1} \ldots z^{\gamma-1})}{[Ax+by+..+Ez]^{\alpha+\beta+..\gamma}} \tag{B2}$$

where A, B etc. are propagators.

THE SCHWINGER PARAMETER

For the propagator $(m^2-k^2)^{-\alpha}$ the Schwinger parameteric representation is

$$\frac{1}{(m^2-k^2)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty dx\, x^{\alpha-1} \exp[-x(m^2 - k^2)] \tag{B3}$$

The generalization of (B3) is straight forward.