STABILITY OF LEAST SQUARES INVERSE POLYNOMIALS AND REALIZATION OF LOW SENSITIVITY 2-D RECURSIVE DIGITAL FILTERS

SRIDHAR PALACHERLA

A Thesis
in
the Department
of
Electrical & Computer Engineering

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Applied Science at
Concordia University
Montréal, Québec, Canada

February 1991

ISBN 0-315-64709-4

ABSTRACT

Sridhar Palacherla

Least square inverse (LSI) polynomials are used to stablize 1-D recursive digital filters. It is not explicitly proved by Robinson that the 1-D LSI of any polynomial having zeros on the unit circle is stable or not. In this thesis we prove that the LSI of a given 1-D polynomial is always devoid of zeros on the unit circle irrespective of whether original polynomial has such zeros or not. Also if LSI polynomial chosen is lacunary then it need not be stable. Proper explanation is given for this which is missing in the proof given by Robinson.

The second part of the thesis is concerned with the BIBO stability of n-D practical digital filters where only one of the independent variables of n-D signal is temporal and other variables are spatial. Double least square inverse method is used to stablize such filters. We prove that the LSI of the denominator polynomial of a practical n-D digital filter will always be stable in the practical and less restrictive sense.

The third part of the thesis deals with the design and implementation of 2-D recursive digital filters. The second order real coefficient 2-D transfer function can be obtained as a sum of two first order 2-D all-pass complex coefficient transfer functions, in which one complex all-pass transfer function coefficients are complex conjugate of other complex transfer function. The coefficients of complex all-pass transfer function are obtained in the design using non-linear optimization. The resulting 2-D real coefficient filter is implemented using complex multipliers. It is shown that the 2-D filter designed and implemented in this manner is less sensitive in the pass band to the coefficient quantization.

ACKNOWLEDGEMENTS

I wish to express my gratitude to my thesis supervisor, Dr. M.N.S. Swamy for his guidance during this research work. I would like to thank Dr. P.S. Reddy for his assistance during the preparation of this thesis. This work was supported, on part, by a Grant from National Sciences and Engineering Research Council of Canada awarded to Dr. M.N.S. Swamy.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Digital signal processing (DSP) has become an essential element of modern technology and has achieved wide spread acceptance in such areas as filtering, data communication, bio-medical engineering, speech processing, radar and sonar.

The origins of DSP can be found in the computational algorithms of the seventeenth and eighteenth century mathematicians Sir Isaac Newton and Karl Friedrich Gauss. In the late 1950s and early 1960s, the digital processing of signals using general-purpose computers began to appear. J. F. Kaiser popularized the several versions of the Z-transform for use as a digital filter design tool. It was soon discovered that filters, which emulated their classic analog counter parts, could be designed in a straight forward manner. Digital filters were found to be particularly effective in processing low frequency signals. Because of their digital nature, results were always repeatable and therefore superior to analog filters. Furthermore, filters of high precision and large dynamic range can be achieved using digital hardware.

Signal processing is concerned with two basic tasks - information rearrangement and information reduction. Computer aided tomography (CAT), image scanning, image enhancement and spectral analysis are examples of information rearrangement. In a CAT scanner the information about the structure of an unknown object is first transferred to a series of electromagnetic waves, which are then sampled to produce an array of numbers, which, in turn, are processed by a computational algorithm and finally displayed on a cathode ray tube screen. Information reduction is concerned with the removal of extraneous information. Noise removal, parameter estimation are examples of this catagory.

Digital signal processing is concerned with the processing of signals which can be represented as sequence of numbers and multi dimensional signal processing is concerned with the processing of signals which can be represented as multi dimensional arrays, such as sampled images and sampled waveforms. During the past decade, there has been a considerable research activity in the area of two-dimensional and multi-dimensional digital filters. For a given response characteristic, recursive digital filters have less hardware requirements and so wherever a linear phase is not required recursive digital filters are preferred to non-recursive filters.

The major problem associated with the design of recursive 2-D digital filter is maintaining stability at the time of design using computer aided optimization. One way of guaranteeing stability is to incorporate stability constraints while doing optimization itself [1] [2]. But this requires a lot of computer time since stability constraints are to be checked for at the end of every iteration of optimization process. Another method [3] which does not require stability test after each iteration need not always produce an optimal solution as pointed out in [4]. The stability problem of two and multi-dimensional digital filters is well presented in the literature [5][6]. In general, it is difficult to take stability constraints into account during the stage of approximation problem. From this point of view it would be useful to develop techniques by which the stability problem could be separated from the approximation. Then the problem is how an unstable 2-D transfer function (obtained at the end of the approximation procedure) can be stablized with out affecting the magnitude characteristics.

Shanks et al. [7] proposed a technique by which this can be achieved. Their technique is based entirely on the validity of their conjecture that planar least squares inverse (PLSI) polynomial of a given arbitrary 2-D polynomial is stable. Making use of the above conjecture, they suggested that the double PLSI of the denominator of the original (unstable) filter gives its minimum phase version with amplitude spectrum roughly equal to that of original filter. By examples they also verified that the larger the dimensions of the intermediate PLSI filter, the better is the resemblance of the amplitude spectra, but they could not prove the conjecture.

Later, Genin and Kamp [8] came up with a counter example for the above conjecture (known as Shanks' conjecture) by proving that a polynomial of third degree in two variables allows a PLSI polynomial of lower degree in two variables, which violates the stability conditions. Keeping the above counter example in view, Jury [9] introduced a modification to Shank's conjecture, which requires that the PLSI polynomial should be of the same degree or higher than the original polynomial. Anderson and Jury [10][11] verified the modified conjecture for special lower order polynomials which are linear in two variables and for some higher order polynomials. Even though no attempt has been made to prove or disprove the conjecture in [12] the result led its authors to make a conjecture that PLSI polynomial is asymptotically stable. Later, Kayran and king [13] have given a counter example to Jury's modified conjecture showing that a third degree two variable polynomial possessed a third degree PLSI polynomial which is unstable.

Shanks' conjecture has been recently modified [14]. The modification suggested in [14] to Shanks' conjecture is that the original 2-D polynomial should be free from zeros on the unit hyper circle. Reddy et al. [15] gave proof for the same. But there is a flaw in that proof as pointed out by two groups of researchers independently [16][17]. In their paper [15] authors used 2-D to 1-D form preserving transformation. In doing that transformation 1-D polynomial obtained from 2-D polynomial became lacunary with the consequences that its stability can not be guaranteed.

The application of multidimensional (n-D) digital signal processing is in areas like seismic, sonar and television image processing. In many situations which occur in practice the independent variables $i_1, i_2, i_3 \ldots i_n$ of an n-D signal x($i_1, i_2, i_3, \ldots, i_n$) are usually spatial variables except perhaps one variable, say i_j is a temporal variable. Practically the temporal variable is unbounded where as the other spatial variables are bounded. In [18], using this concept of only one variable being bounded, a theorem is developed for the practical-BIBO stability of n-D discrete systems.

In [18] it is shown that the conventional-BIBO stability conditions are too restrictive for many applications. Subsequently in [19] a proof is given to show that the least squares inverse (LSI) of the denominator polynomial of a linear shift invariant n-D digital filter satisfies the practical-BIBO stability conditions given in [18]. But unfortunately there is a flaw in their proof as reported in [20]. In their proof [19] the authors make use of the result of Robinson [21] which says that the least squares inverse polynomial of any 1-D polynomial is always stable. But the result of Robinson is not true if the least squares inverse selected is lacunary in the sense that it has some missing terms between the highest power term and the constant term [22].

The choice of the digital filter structure and the direct design of filters with quantized coefficients remain as important research areas. The problem in digital filter implementation is the sensitivity of the magnitude or phase of the filter frequency response to minor variations of the multiplier coefficients. Sometimes the quantized filter may fail to meet specifications even though unquantized filter does. The sensitivity of the filter response to errors in the parameters is dependent on the structure of the filter realization. Thus, for efficient implementation of digital filters, low sensitivity digital filter structures requiring less number of bits per multiplier coefficients are required.

The wave digital filters [23] obtained from the classical doubly terminated LC networks, are low sensitivity digital filters. Digital lattice filters [24] derived from the LC lattice filters are known for low pass-band sensitivity. Swamy and Thygarajan [25] introduced a new type of wave digital filter structure, which is translated from the classical continuous time networks by considering each element in the continuous domain as a two-port rather than as a one-port. They showed that these structures have low sensitivity properties similar to wave digital filters. Vaidynathan et al. [26] developed low sensitivity digital filter structures using bounded real property of the transfer function.

Vaidynathan, Mitra and Neuve [27] have developed a procedure for synthesizing a 1-D digital transfer function as a parallel connection of two all-pass sections without reference to continuous time LC synthesis. They showed that certain digital transfer functions (which can be decomposed in this fashion) are realizable by structurally passive all-pass structures. They implemented each of the two all-pass sections as a cascade of first and second order building blocks.

The concept of complex coefficient 1-D digital filters is not new. Several authors have discussed these complex filters [28-31]. Fettweis [31] derived complex wave digital filters from complex analog reference filters and suggested their use in the processing of complex signals. Recently, Saramaki et al.[32] realized low sensitivity 1-D recursive filters using complex all-pass structures. They ensured structural passivity property by implementing the second order stage using first order complex all-pass structures.

Swamy et al.[33] introduced 2-D wave digital filters using doubly terminated two variable LC-ladder configuration. These filters have low sensitivities associated with finite word length of coefficients. In this thesis, we consider a 2-D digital filter characterized by complex impulse response sequence, that is, a 2-D digital transfer function with complex coefficients. We show that the real coefficient 2-D digital filter derived from the complex coefficient all-pass sections satisfy the low sensitivity in the pass band as is the case of 1-D digital filters [32].

In this thesis we deal with:

(1) A new look at the proof of guaranteed stability of 1-D Least Square Inverse (LSI) polynomials.

(2) The problem of stablizing the unstable n-D practical digital filters using the double least square Inverse method.

(3) A new technique for designing quarter plane second order low sensitivity 2-D recursive filters using complex coefficient first order all-pass 2-D filter.

CHAPTER 2

LEAST SQUARE INVERSE POLYNOMIALS

2.1 Introduction

As is well known, recursive digital filters are superior to non-recursive digital filters from the point of view of memory and cost requirements to satisfy the same design specifications. So invariably one goes in for recursive digital filters when linear phase is not a requirement to be met by the filter. In this chapter, we explain why a lacunary 1-D Least Square Inverse polynomial is not guaranteed to be stable as pointed out in [16], [17]. Also we deal with some preliminaries by giving some definitions on LSI polynomials and also furnish for ready reference a stability theorem on 1-D polynomials.

2.2 Some Preliminaries

In this section we state for the sake of completeness what is meant by least squares inverse polynomials in the case of both 1-D and 2-D polynomials and give a stability theorem pertaining to 1-D polynomials. We also briefly mention how the least squares inverse polynomials are obtained in these two cases.

1-D Polynomials :

Let

$$A(Z) = \sum_{i=0}^{M} a_i Z^i \tag{2.1}$$

be an 1-D polynomial of degree M, where a_i's are known. Then a polynomial denoted by

$$B(Z) = \sum_{i=0}^{N} b_i Z^i \tag{2.2}$$

is called the LSI polynomial of A(Z), if B(Z) is obtained such that

$$B(Z) \approx \frac{1}{A(Z)} \quad on\ |Z|=1 \tag{2.3}$$

and the coefficients of B(Z) namely b_i's, are obtained by minimizing the error energy function E given by

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| 1 - A(e^{-j\omega})B(e^{-j\omega}) \right|^2 d\omega \tag{2.4}$$

with respect to each one of the coefficients b_i. We assume that the LSI polynomial in (2.2) is a full polynomial without any missing terms between the highest power term and the constant. It can be shown that the following expressions are equivalent to (2.4). If C(Z)=A(Z)B(Z) then,

$$E = \left[1 - c_0\right]^2 + \sum_{i>0}^{M+N} c_i^{\,2} \tag{2.5}$$

where c_i's are coefficients of the polynomial C(Z). Also

$$E = \frac{1}{2\pi j} \oint_{|Z|=1} \left[1 - A(Z)B(Z)\right]\left[1 - A(Z^{-1})B(Z^{-1})\right]\frac{dZ}{Z} \tag{2.6}$$

and

$$E = (1 - 2a_0 b_0) + \sum_{s=-k}^{k} \alpha_s \beta_s \tag{2.7}$$

where α_s and β_s are the autocorrelation coefficients of the polynomials A(Z) and B(Z) respectively and k is an integer equal to either M or N, whichever is smaller [21].

It is more convenient to use equation (2.5) or (2.7) to minimize E with respect to each b_i. For this we obtain $\frac{\partial E}{\partial b_i}$, for i=0,1,...,N and make $\frac{\partial E}{\partial b_i}$ =0 ,for i=0,1,...,N. Thus we get a set of linear algebraic equations involving b_i's and can be put in the compact form as below [21].

$$\mathbf{T}\,\mathbf{b} = \mathbf{a} \tag{2.8}$$

where **T** is an $(N+1)\times(N+1)$ square matrix whose entries are the autocorrelation coefficients of the polynomial A(Z), **b** and **a** are column vectors such that

$$\mathbf{b} = \left[b_0, b_1, \ldots, b_N\right]^t$$

and

$$\mathbf{a} = \left[a_0, 0, 0 \ldots, 0\right]^t$$

where t denotes the transpose. It may be noted that the matrix **T** is Toeplitz, centro-symmetric and positive definite. Since **T** is positive definite we have a unique solution for **b** when we solve (2.8) and hence a unique B(Z).

Definition 2.1 : The 1-D polynomial B(Z) is said to be stable if B(Z) $\neq 0, |Z| \leq 1$.

Robinson [21] has proved that the LSI polynomial B(Z) corresponding to a given A(Z) is always stable. We shall discuss this later in section 3.3.

2-D PLSI polynomials :

Let

$$A(Z_1,Z_2) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} a_{ij} Z_1^{\,i} Z_2^{\,j} \tag{2.9}$$

be a given 2-D polynomial (quarter plane and first-quadrant). Then the 2-D polynomial denoted by

$$B(Z_1,Z_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} b_{ij} Z_1^{\,i} Z_2^{\,j} \tag{2.10}$$

is a PLSI polynomial of $A(Z_1,Z_2)$ if

$$B(Z_1,Z_2) \approx \frac{1}{A(Z_1,Z_2)} \quad \text{for } |Z_1|=1, |Z_2|=1 \tag{2.11}$$

and the coefficients b_{ij}'s are obtained by minimizing the error energy function E given by

$$E = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| 1 - A\left(e^{-j\omega_1}, e^{-j\omega_2}\right) B\left(e^{-j\omega_1}, e^{-j\omega_2}\right) \right|^2 d\omega_1 d\omega_2 \tag{2.12}$$

with respect to each coefficient b_{ij} as in 1-D case. We make a mention that there are three other expressions for E, as in 1-D case, which are equivalent to the one given in (2.12). We can formulate the normal equations in the form

$$\mathbf{T}\,\mathbf{b} = \mathbf{a} \tag{2.13}$$

following the simple procedure given in [34] where **T** is a Toeplitz and centro-symmetric matrix of order $(N_1+1)\times(N_2+1)$ and its entries are 2-D autocorrelation coefficients of $A(Z_1,Z_2)$. **b** and **a** are column matrices given by

$$\mathbf{b} = \left[b_{00}, b_{01}, b_{02}, \ldots, b_{0N_2}, b_{10}, b_{11}, \ldots, b_{N_1N_2}\right]^t$$

and

$$\mathbf{a} = \left[a_{00}, 0, 0, \ldots, 0, 0, 0, \ldots, 0\right]^t$$

The solution of (2.13) gives **b** and it is unique since **T** is positive definite [35]. It may be noted that [36]

$$E_{\min} = 1 - b_{00}a_{00}\,. \tag{2.18}$$

Equation (2.18) means that the element b_{00}, namely, the constant term of $B(Z_1,Z_2)$ is incidentally maximized while minimizing E of (2.12).

2.3 Stability of 1-D LSI Polynomials

If

$$A(Z)A(Z^{-1}) = r_0 + r_1(Z+Z^{-1}) + \ldots + r_M(Z^M + Z^{-M}) \tag{2.19}$$

then $r_0, r_1 \ldots, r_M$ are in fact the autocorrelation coefficients of the 1-D polynomial A(Z)[21]. For example if

$$A_1(Z) = a_0 + a_1 Z + a_2 Z^2$$

then the autocorrelation coefficients of $A_1(Z)$ are

$$r_0 = a_0^2 + a_1^2 + a_2^2$$
$$r_1 = a_0 a_1 + a_1 a_2$$

$$r_2 = a_0 a_2$$

With the above definition for autocorrelation coefficients we discuss in this section the result of Robinson [21] with regard to the stability of the LSI 1-D polynomials. We try to explain why the LSI 1-D polynomial need not necessarily be stable if it is chosen to be lacunary in the sense there are some missing terms in it.

It is shown in [21] that corresponding to the polynomial A(Z) there exist, in general, $2^M - 1$ other 1-D polynomials of order M all of whom have the same autocorrelation coefficients as A(Z). All these 2^M polynomials belong to a 'family'. It has been shown [21, p131] that out of these 2^M polynomials belonging to the same 'family' as A(Z) only that polynomial with the highest constant term is stable. It must be noted here that if A(Z) has a zero on the unit circle, no member of the 'family' is stable. Let us take, for example a simple real polynomial $A_1(Z)$,

$$\begin{aligned} A_1(Z) &= a_0 + a_1 Z + a_2 Z^2 \\ &= (p_1 + q_1 Z)(p_2 + q_2 Z) \end{aligned}$$

Let p_i and q_i be real such that $|p_i| > |q_i|$ which makes $A_1(Z)$ a stable polynomial according to Theorem 2.1. The other three polynomials which have the same autocorrelation coefficients as $A_1(Z)$ are

$$\begin{aligned} A_{1a}(Z) &= (p_1 + q_1 Z)(q_2 + p_2 Z) \\ A_{1b}(Z) &= (q_1 + p_1 Z)(p_2 + q_2 Z) \\ A_{1c}(Z) &= (q_1 + p_1 Z)(q_2 + p_2 Z) \end{aligned} \tag{2.20}$$

Of these four polynomials only $A_1(Z)$ is stable since $|p_i| > |q_i|$ making its constant term a_0 being the highest when compared to the constant term of each one of the three polynomials given in (2.20). It may be noted that the members of the ' family ' of $A_1(Z)$ can be constructed either by factorizing $A_1(Z)$ as product of first order factors as done above or by solving the following set of equations :

$$\begin{aligned} a_0'^2 + a_1'^2 + a_2'^2 &= r_0 \\ a_0'a_1' + a_1'a_2' &= r_1 \\ a_0'a_2' &= r_2 \end{aligned}$$

where r_0, r_1, r_2 are the autocorrelation coefficients of the given $A_1(Z)$. It may be noted that the existence of solution to the set of equations (2.21) is guaranteed and in fact (2.21) has four solutions including the one corresponding to the coefficients of the given $A_1(Z)$. In general all the 2^M members of the ' family ' of A(Z) can be obtained by solving a set of (M+1) equations of the type given in (2.21).

In [21] Robinson has proved that the LSI polynomial B(Z) of A(Z) is always stable. This is based on the fact that while obtaining B(Z) coefficients we minimize the error energy function E and obtain [21] $E_{\min}$ as

$$E_{\min} = 1 - a_0 b_0 \tag{2.21}$$

Since $E_{\min}$ of (2.21) is the lowest error energy, b_0 which is the constant term of B(Z) is the greatest in magnitude when compared to the other members of the 'family' of B(Z) which are obtainable by solving the set of (N+1) equations of the following type as discussed in this section

$$\begin{aligned} b_0'^2 + b_1'^2 + \dots + b_N'^2 &= r_0 \\ b_0'b_1' + b_1'b_2' + \dots + b_{N-1}'b_N' &= r_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

$$b_0' b_{N-1}' + b_1' b_N' = r_{N-1}$$
$$b_0' b_N' = r_N \tag{2.22}$$

where r_i's are the autocorrelation coefficients of B(Z). In order that B(Z) is the stable polynomial, b_0 has to be greatest in magnitude when compared to the constant terms of each one of the 2^N-1 polynomials whose coefficients are obtainable by solving equations (2.22). What this means is that if you solve (2.22) maximizing b_0', the only solution we get should be such that $b_0'=b_0$. This will be always guaranteed provided the B(Z) polynomial chosen has no missing coefficients since we will be then using exactly identical left hand side expressions as in (2.22) in finding out r_i 's used in (2.22).

We want to mention here that in order for the proof given by Robinson to be complete, we have to show that the LSI polynomial B(Z) is free of zeros on the unit circle. This is required because even if b_0 is the highest in magnitude the polynomial B(Z), though not lacunary, need not be stable if B(Z)=0 on |Z|=1 as mentioned earlier, since no member of the 'family' of B(Z) is stable in that case. So we state and prove the following theorem.

Theorem 2.1 : The LSI polynomial B(Z) of the given A(Z) is always devoid of zeros on the unit circle irrespective of whether A(Z) has such zeros.

Proof: In order that B(Z)=0 on |Z|=1 irrespective of the polynomial A(Z) for some frequency ω other than the frequencies at which $A(e^{j\omega_a}) = 0$,

$$A(e^{j\omega})B(e^{j\omega}) = 0, \text{ for } \textit{some value of } \omega \tag{2.23}$$

This requires,

$$\mathrm{Re}\left[A(e^{j\omega})B(e^{j\omega})\right]=0, \quad \mathrm{Im}\left[A(e^{j\omega})B(e^{j\omega})\right]=0 \tag{2.24}$$

That is,

$$c_0 + \sum_{i=1}^{M+N} f_i(\omega)c_i = 0$$

$$\sum_{i=1}^{M+N} g_i(\omega)c_i = 0 \tag{2.25}$$

where $f_i(\omega)$ and $g_i(\omega)$ are trigonometric functions and c_i are coefficients of the polynomial C(Z) = A(Z)B(Z). We will now see whether a consistent set of c_i can be obtained by solving (2.25) such that $B(e^{j\omega})=0$ for some ω. In (2.25) there are (M+N+1) number of c_i which, of course, are functions of a_i and b_i. So it looks as though we can select (M+N+1-2) = (M+N-1) number of c_i arbitrarily and solve the two equations (2.25) for the remaining two c_i. But we show now that such a method does not work. When we select values for (M+N-1) number of c_i arbitrarily we have these many equations involving a_i and b_i coefficients which are together (M+N+2) in number. So these (M+N-1) equations can be solved for (M+N+2) unknowns, namely a_i and b_i. When once all a_i and b_i are fixed the remaining two c_i are readily known without having to solve the two equations (2.25). In case we solve (2.25) for the remaining two c_i it will yield different values for the two c_i not consistent with the values already obtained. Thus there exists no consistent set of values for c_i such that equations (2.25) are satisfied. So $B(e^{j\omega})$ is not equal to zero for any ω. Hence the proof of Theorem 2.2.

Having completed the proof of supplementary theorem (Theorem 2.2) we have set right the proof of Robinson [21] on the guaranteed stability of the LSI polynomial B(Z). But one more thing which was not explained or mentioned explicitly in [21] by Robinson is that whether we can choose the LSI polynomial as lacunary (with some missing coefficients) and still guarantee its stability. The answer is no, as mentioned in [16],[12]. We give here proper explanation for the same. Suppose there are some missing terms (coefficients) in B(Z), though we have minimized E and thus maximized b_0 (since $E_{min}=1-a_0b_0$ even in this case) we can not guarantee a solution such that $b_0'=b_0$ when we solve equations (2.22) maximizing b_0'. This is because while obtaining r_i 's of (2.22) we have not used exactly identical expressions for the left hand side of (2.22) since some b_i 's are missing in B(Z). So b_0' may be some times greater than b_0. Hence B(Z) need not be stable if it is lacunary.

In 2-D case, in general, the PLSI polynomial is not stable and so further research has to be done to prove Shank's conjecture at least in some modified form. We have clarified certain points regarding Robinson's Proof of guaranteed stability of 1-D LSI Polynomials.

CHAPTER 3

PRACTICAL-BIBO STABILITY OF n-D DIGITAL FILTERS

3.1 Introduction

In this chapter we deal with a new proof of practical-BIBO stability of n-D digital filters and show that the n-D LSI of the filter denominator polynomial satisfies the practical-BIBO stability requirement. For the sake of completeness, we first present some theorems pertaining to conventional and practical-BIBO stability of n-D digital filters. We then present the basic definition of the LSI of an n-D polynomial. We also discuss a basic fact with respect to the stability of 1-D polynomials. A theorem on practical-BIBO stability of n-D digital filters is given by showing that the least squares inverse of the denominator polynomial of an n-D digital filter always satisfies the practical-BIBO stability requirement. The results of this chapter have appeared as a paper[42].

3.2 Some Preliminaries

This section contains some preliminary concepts and results already existing in the literature. The material presented in this section will be very useful for the easy understanding of the results presented in the subsequent sections.

Consider the transfer function of an n-D linear shift invariant digital filter

$$H(Z_1, Z_2, \ldots, Z_n) = \frac{P(Z_1, Z_2, \ldots Z_n)}{Q(Z_1, Z_2, \ldots, Z_n)} \tag{3.1}$$

We assume that $H(Z_1, Z_2, \ldots Z_n)$ has no non essential singularities of the second kind on the unit polydisc.

An n-D polynomial $B(Z_1, Z_2, \ldots Z_n)$ is the LSI polynomial of $Q(Z_1, Z_2, \ldots, Z_n)$ if

$$B(Z_1, Z_2, \ldots, Z_n) \approx \frac{1}{Q(Z_1, Z_2, \ldots, Z_n)} \quad \text{for } |Z_1| = |Z_2| = \ldots = 1$$

It may be noted that the coefficients b_{ij} of $B(Z_1, Z_2, \ldots Z_n)$ are to be obtained by minimizing the error function

$$E = (1 - c_{0,0,\ldots,0})^2 + \sum_{i_1} \sum_{i_2} \cdots \sum_{\substack{i_n \\ i_1+i_2+\cdots+i_n>0}} c^2_{i_1, i_2, \ldots, i_n} \tag{3.2}$$

where c's are the coefficients of the n-D polynomial $C(Z_1, Z_2, \ldots, Z_n) = B(Z_1, Z_2, \ldots, Z_n)\, Q(Z_1, Z_2, \ldots, Z_n)$. Following the proof given for 2-D polynomials [36] it can be shown that the minimum value of E will be

$$E_{\min} = \left[1 - b_{0,0\ldots0}\, q_{0,0,\ldots0}\right] \tag{3.3}$$

where $b_{0,0,\ldots0}$ and $q_{0,0\ldots0}$ are the constant terms of $B(Z_1, Z_2, \ldots, Z_n)$ and $Q(Z_1, Z_2, \ldots, Z_n)$ respectively. Obviously from (3) since $q_{0,0\ldots,0}$ is a known quantity $b_{0,0,\ldots,0}$ is the maximum corresponding to the set of n-D autocorrelation coefficients obtained while minimizing E and thus arriving at all $b_{ij\ldots}$ values [36].

Consider a 1-D real polynomial B(Z),

$$B(Z) = \sum_{i=0}^{N} b_i z^i$$

B(Z) is said to be stable if all its zeros are outside the unit circle. Let

$$B(Z)B(Z^{-1}) = r_0 + r_1(Z+Z^{-1}) + \ldots + r_N(Z^N + Z^{-N})$$

Where $r_0, r_1, \ldots r_N$ are called autocorrelation coefficients of B(Z). In terms of the coefficients b_i we can write r_i as

$$\begin{aligned}
b_0^2 + b_1^2 + \ldots + b_N^2 &= r_0 \\
b_0 b_1 + b_1 b_2 + \ldots + b_{N-1} b_N &= r_1 \\
b_0 b_2 + b_1 b3 + \ldots + b_{N-2} b_N &= r_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot & \\
\cdot \quad \cdot \quad \cdot \quad \cdot & \\
\cdot \quad \cdot \quad \cdot \quad \cdot & \\
b_0 b_N &= r_N
\end{aligned} \tag{3.4}$$

As mentioned in the previous section, in general there are 2^N polynomials including B(Z) having the same autocorrelation coefficients as B(Z) [21]. This means if we rewrite equations (3.4) by replacing each b_i by b_i' on the left hand side of the equations and call them equations (3.5) and solve them for b_i' we will be able to get 2^N sets of solutions for the coefficients b_i'. Out of all these 2^N polynomials, one corresponding to each set of b_i', only one of the polynomials will be stable. It is proved in [21] that the polynomial having the highest constant term will be the stable one. If we are interested in this stable one only we should solve equations (3.5) maximizing b_0'. So this becomes a constrainted optimization problem and can be handled using Lagrangian multiplier method.

3.3 A Proof of Practical stability n-D LSI Polynomials

In this section, we present the main result of the thesis. That is, if $B(Z_1, Z_2, ..., Z_n)$ is the n-D LSI polynomial of the denominator polynomial $Q(Z_1, Z_2, ..., Z_n)$ of an n-D transfer function then we show that $B(Z_1,Z_2,...,Z_n)$ is stable in the practical sense [18],[19]. That is,

$$B(0,0,\ldots,0,Z_k,0,0,\ldots 0) \neq 0 \quad \text{for} \quad |Z_k| \leq 1 \quad k=1,2...n$$

We first prove some simple result with respect to 2-D PLSI polynomials and extend the discussions to n-D LSI polynomials to prove the main result.

Theorem 3.1: If $B(Z_1, Z_2)$ is a 2-D PLSI polynomial of order one in both the variables, then $B(Z, 0)$ and $B(0, Z)$ are stable.

Proof: Let

$$B(Z_1,Z_2)=b_{00}+b_{01}Z_2+b_{10}Z_1+b_{11}Z_1Z_2$$

then

$$B(Z,0)=b_{00}+b_{10}Z$$

<u>Case (i)</u> $B(Z_1, Z_2) \neq 0$ on T^2

Since $B(Z_1,Z_2)$ is a full 2-D polynomial, B(Z,0) as seen above is a full 1-D polynomial. In order that B(Z, 0) be stable b_{00} should be the highest constant term when compared to the constant terms of all other (in this case one other) 1-D polynomials having the same auto correlation coefficients as B(Z, 0). Since b_{00} is incidentally maximized while obtaining the PLSI $B(Z_1, Z_2)$ it satisfies the above requirement. Hence B(Z, 0) is stable.

<u>Case (ii)</u> $B(Z_1, Z_2) = 0$, for some $Z_1 = Z_1'(=e^{j\omega_1'})$ and $Z_2 = Z_2'(=e^{j\omega_2'})$.

In this case though b_{00} is the highest constant term we just can't straight away conclude that B(Z, 0) is stable. It is because b_{00} is the highest subject to the auto correlation constraints corresponding to B(Z, 0) and an additional constraint. These constraints are,

$$\begin{aligned} b_{00}'^2 + b_{10}'^2 &= r_0 \\ b_{00}'b_{10}' &= r_1 \end{aligned} \tag{3.6(a)}$$

$$B'(Z_1', Z_2') = b_{00}' + b_{01}'Z_2' + b_{10}'Z_1' + b_{11}'Z_1'Z_2' = 0 \tag{3.6(b)}$$

We can only say that if we maximize b_{00}' subject to the constraints (3.6), (3.6(a)) and (3.6(b)) We will get a solution as

$$\begin{aligned} b_{00}' &= b_{00} \\ b_{10}' &= b_{10} \end{aligned} \tag{3.7}$$

But if we want to show that B(Z, 0) is stable, we have to prove that if we maximize b_{00}' subject to the constraints 3.6(a) only, the maximum value of b_{00}' is equal to b_{00}.

We first maximize b_{00}' with the constraints (3.6) using Lagrangian multiplier method as before. Let f_4 be defined as

$$f_4 = b_{00}' + \lambda_1(b_{00}'^2 + b_{10}'^2) + \lambda_2(b_{00}'b_{10}') + \lambda_3[B'(Z_1', Z_2')]$$

Equating the first derivatives of f_4 with respect to each b_{ij}' to zero, we have

$$\begin{aligned} 1 + \lambda_1(2b_{00}') + \lambda_2(b_{10}') + \lambda_3 &= 0 \\ \lambda_1(2b_{10}') + \lambda_2(b_{00}') + \lambda_3 Z_1' &= 0 \\ \lambda_3 Z_2' &= 0 \end{aligned}$$

$$\lambda_3 Z_1' Z_2' = 0 \tag{3.8}$$

Appending equations 3.6(a) and 3.6(b) to (3.8), we have to solve the resulting 7 equations for 7 unknowns, the unknowns being four b_{ij}' and three λ_i. The last two equations of (3.8) requires that $\lambda_3 = 0$. Substituting λ_3=0 in the first two equations of (3.8), we then have only 5 equations as follows:

$$\begin{aligned} 1+\lambda_1(2b_{00}')+\lambda_2(b_{10}')&=0 \\ \lambda_1(2b_{10}')+\lambda_2(b_{00}')&=0 \\ b_{00}'^2+b_{10}'^2&=r_0 \\ b_{00}'b_{10}'&=r_1 \end{aligned} \tag{3.9(a)}$$

$$b_{00}'+b_{10}'Z_2'+b_{10}'Z_1'+b_{11}'Z_1'Z_2'=0 \tag{3.9(b)}$$

As mentioned earlier, the solution of (3.9)is the same as given in (3.7).

On the other hand if we try to maximize b_{00}' with the constraints 3.6(a) only, we get the following equations :

$$\begin{aligned} 1+\lambda_1(2b_{00}')+\lambda_2(b_{10}')&=0 \\ \lambda_1(2b_{10}')+\lambda_2(b_{00}')&=0 \\ b_{00}'^2+b_{10}'^2&=r_0 \\ b_{00}'b_{10}'&=r_1 \end{aligned} \tag{3.10}$$

It can be seen that equations (3.10) are the subset of equations (3.9). So we can show easily the solution of (3.10) will give the same values for b_{00}' and b_{10}' as the solution of (3.9) does. That is $b_{00}' = b_{00}$ and $b_{10}' = b_{10}$. It must be commented here that solving (3.10) or equations of (3.9(a)) first and then substituting the resulting b_{00}', b_{10}' values in (3.9(b)) to obtain b_{01}' and b_{11}' does not present any problem since 3.9(b) gives rise to two equations involving b_{01}' and b_{11}' as unknowns. Thus, since the maximum value of b_{00}' we get subject to the constraints 3.6(a) only is equal to b_{00}, we can say that B(Z, 0) is stable. Similarly we can prove that B(0, Z) is stable. Hence the proof of Theorem 3.1.

Now we consider any 2-D full PLSI polynomial

$$B(Z_1,Z_2)=\sum_{i=0}^{M}\sum_{j=0}^{N} b_{ij} Z_1^{\,i} Z_2^{\,j}$$

and the corresponding 1-D polynomials B(Z, 0) and B(0, Z). The 1-D polynomial B(Z,0) has (M+1) coefficients namely b_{ij}, for i=0,1, . . .,M and j=0. Since $B(Z_1,Z_2)$ is a full 2-D PLSI polynomial, B(Z,0) is also a full 1-D polynomial.

When $B(Z_1,Z_2)\neq 0$ on T^2, since B(Z,0) is a full 1-D polynomial of degree M and since its constant term b_{00} is the highest (having been maximized while obtaining the PLSI $B(Z_1,Z_2)$) it is stable. This follows from the fact that if we obtain the autocorrelation coefficients r_i of B(Z,0) from its known (M+1) coefficients and form autocorrelation equations of the type given in (3.4) we will have (M+1) equations as follows:

$$\begin{aligned} b_{00}'^2+b_{10}'^2+b_{20}'^2+\ldots b_{M0}'^2 &= r_0 \\ b_{00}'b_{10}'+b_{10}'b_{20}'+\ldots b_{(M-1)0}'b_{M0}' &= r_1 \\ \cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot \\ b_{00}'b_{M0}' &= r_M \end{aligned} \tag{3.11}$$

If we now try to maximize b_{00}' with equations(3.11) as constraints we get the maximum b_{00}' as equal to b_{00}. Thus we get the same set of coefficients as those of B(Z,0), as the solution in the maximization process. This is mainly because B(Z,0) is a full 1-D polynomial and r_i on the right hand side of (3.11) have been obtained using the coefficients of B(Z,0).

In the case when $B(Z_1,Z_2)=0$ on T^2 for some $Z_1=Z_1'$ and $Z_2=Z_2'$, in addition to the constraints (25), we have an additional constraint

$$B_1(Z_1',Z_2')=0 \tag{3.12}$$

where $B_1(Z_1',Z_2')$ is what we get from $B(Z_1',Z_2')$ by replacing each b_{ij} by b_{ij}'. We know that if we maximize b_{00}' with the equations (3.11) and (3.12) as constraints, we will get the maximum b_{00}' as equal to b_{00} and the solution for b_{ij}' is

$$b_{ij}'=b_{ij}, \quad \text{for } i=0,1...,M \text{ and } j=0 \tag{3.13}$$

But in order to show that B(Z,0) is stable we have to prove that if we maximize b_{00}' subject to the constraints (3.11) alone we still get the maximum b_{00}' as b_{00}.

When we maximize b_{00}' with (3.11) and (3.12) as constraints, as in case (ii) of Theorem 3 the (M+2)th Lagrangian multiplier λ_{M+2} which multiplies the constraining equation (3.12) will be forced to take zero value since all the derivatives of the Lagrangian function with respect to each b_{ij}', for i=0, j=1,2, ..., N, will give rise to single term equations involving λ_{M+2} as the coefficient. This makes the number of equations to be solved for when maximizing b_{00}' with (3.11) and (3.12) as constraints as just one more than what we have to solve while maximizing b_{00}' with (3.11) only as constraints. This extra equation is (3.12) and all other 2(M+1) equations are identical in both these optimizations, involving b_{ij}', i=0,1 . . .,M and j=0, and (M+1) Lagrangian multipliers as unknowns. Thus the equations we get when maximizing b_{00}' with (3.11) only as constraints is a subset of the equations we have when maximizing b_{00}' with (3.11) and (3.12) as constraints. So the solution of these subset of equations for b_{ij}' is unique and is given by (3.13). Here also as in case (ii) of Theorem 3 solving (3.12) for the other unknown coefficients b_{ij}', for i=0 and j=1,2 . . ., N does not pose any problem since it constitutes only two equations (because of the complex nature of Z_1' and Z_2') involving all these N coefficients. Hence in this case of $B(Z_1,Z_2) = 0$ on T^2 for some $Z_1=Z_1'$ and $Z_2=Z_2'$, since the maximum value of b_{00}' we get subject to the constraints (25) only is equal to b_{00} we can say that B(Z,0) is stable.

The above arguments can be applied to B(0,Z) also in a similar way to conclude that B(0,Z) will also be stable.

Thus we have the generalization of Theorem 3.1 in the form of Theorem 4.

Theorem 3.2: If $B(Z_1,Z_2)$ is a 2-D full PLSI polynomial of some 2-D polynomial, then B(Z,0) and B(0,Z) are stable.

In view of the above discussions the proof of the following main theorem follows:

Theorem 3.3 : If $B(Z_1, Z_2, ..., Z_n)$ is the full n-D LSI of a given n-D polynomial, then

$$B(0,0,\ldots,0,Z_k,0,0,\ldots 0) \neq 0 \text{ for } |Z_k| \leq 1 \quad k=1,2 \ldots n$$

irrespective of whether $B(Z_1, Z_2, ..., Z_n)$ has zeros on the polydisc or not.

Without going into the details of the proof of above theorem, we simply state that all the arguments we have applied earlier to prove B(Z,0) and B(0,Z) are stable in the discussions leading to Theorem 3.2 are valid here also, since each one of the polynomials B(0,0, . . . ,0, Z_k , 0,0, . . .,0) is a full 1-D polynomial and the coefficient $b_{0,0,...,0}$ has been maximized. Hence the proof of Theorem 3.3 follows.

We have presented in this chapter some interesting results with regard to the stability of 2-D PLSI polynomials. A proof for the stability of the LSI polynomials of n-D practical digital filters which will be useful in stabilizing such filters is given.

CHAPTER 4

DESIGN OF LOW SENSITIVITY QUARTER PLANE RECURSIVE 2-D DIGITAL FILTERS

4.1 Introduction

For the last several years, digital filtering has been a major area of research. Attention has been focused on the performance of a digital filter under finite word length implementations. We consider a 2-D digital filter characterized by a complex impulse response sequence, that is, a 2-D digital transfer function with complex coefficients. We show that the real coefficient 2-D digital filter derived from the complex coefficient all-pass sections satisfy the low sensitivity in the pass band as is the case of 1-D digital filters [32]. In this chapter, a new method for designing and implementing low sensitivity 2-D recursive filters is presented. Also some preliminaries regarding structural boundedness, complex coefficient digital filter basic building blocks and realization of complex all-pass filter are discussed. The results of this chapter will be appearing as a paper[43].

4.2 Review of Structural Boundedness

The role of structural boundedness in low sensitivity implementations is given for 1-D digital structures in [26]. In this section the concepts of structural boundedness in the case of 2-D digital filter structures is discussed.

Consider an Nth order 2-D digital real coefficient transfer function:

$$H(Z_1,Z_2) = \frac{\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij} Z_1^{-i} Z_2^{-j}}{1+\sum_{\substack{i=0\,j=0\\ i+j>0}}^{N\;\;N} b_{ij} Z_1^{-i} Z_2^{-j}} \tag{5.1}$$

where coefficients a_{ij} and b_{ij} are real. Our aim is to design a structure with multiplier coefficients $m_1,m_2,\ldots$such that the sensitivity of $|H(e^{j\omega_1},e^{j\omega_2})|$ with respect to each m_k is reduced in the passband. If the quantity $|H(e^{j\omega_1},e^{j\omega_2})|$ is bounded by unity, that is, $|H(e^{j\omega_1},e^{j\omega_2})|\le 1$ for all frequency pairs of (ω_1,ω_2), then $H(Z_1,Z_2)$ is called a bounded real (BR) transfer function and such a function is less sensitive to multiplier coefficients. If $|H(e^{j\omega_1},e^{j\omega_2})|=1$ for all frequency pair (ω_1,ω_2), then $H(Z_1,Z_2)$ is called a lossless bounded real(LBR) transfer function. It is also known as a stable all-pass function.

$|H(e^{j\omega_1},e^{j\omega_2})|$ is bounded by unity, regardless of actual values of the multiplier coefficients m_k. Let $(\omega_1^{(l)},\omega_2^{(m)})$ be a frequency pair in the passband such that $|H(e^{j\omega_1^{(l)}},e^{j\omega_2^{(m)}})| = 1$ for $m_k=m_{k0}$. If the multiplier coefficient m_k is perturbed, then $|H(e^{j\omega_1^{(l)}},e^{j\omega_2^{(m)}})|$ can only decrease regardless of the sign of the perturbation. Thus the slope of $|H(e^{j\omega_1^{(l)}},e^{j\omega_2^{(m)}})|$ with respect to m_k is zero at this frequency pair. The plot of $|H(e^{j\omega_1^{(l)}},e^{j\omega_2^{(m)}})|$ as a function of m_k is as shown in Fig. 4.1.

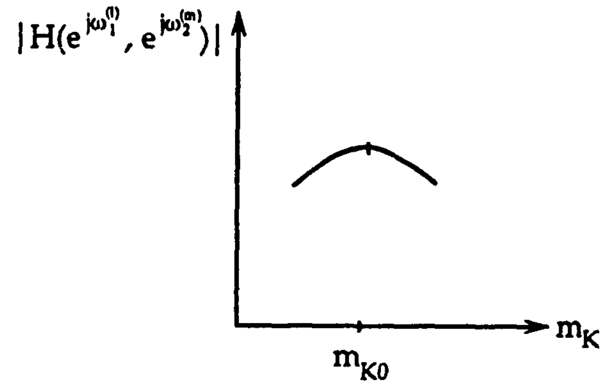

Fig. 4.1 Magnitude as a function of multiplier coefficient.

The zero first order sensitivity at frequency pair (ω_1,ω_2) is given by

$$\frac{\partial |H(e^{j\omega_1^{(l)}}, e^{j\omega_2^{(m)}})|}{\partial m_k} \Big|_{m_k = m_{k0}} = 0 \quad \forall l\ , \forall m, \forall k \tag{4.2}$$

If there are number of closely spaced points in the pass band region such that $|H(e^{j\omega_1}, e^{j\omega_2})| = 1$ at all these points, then we can expect the low pass-band sensitivity. Thus the structure which limits the upper bound on $|H(e^{j\omega_1}, e^{j\omega_2})|$ to unity irrespective of perturbations in the multiplier coefficients is called structurally passive or structurally bounded [26].

In summary, these are the properties required for a structure to have low sensitivity in the pass-band.

1. The boundedness property $|H(e^{j\omega_1}, e^{j\omega_2})| \leq 1$ for all ω_1 and ω_2.
2. $|H(e^{j\omega_1}, e^{j\omega_2})| = 1$ at certain frequencies in the pass-band.
3. Property 1 and 2 holds regardless of the values of the multiplier coefficients, as long as they remain in certain range.

It may be noted that the low sensitivity in the stop-band is obtainable by using cascade realization of lower order 2-D sections.

4.3 Realization of a Real Second Order 2-D Digital Filter

In this section we realize a 2-D real coefficient digital filter using a cascade of second order structures, where each stage is realized in a structurally passive form. We assume that the input u(m,n) to be processed is real and the output y(m,n) required is also real. As mentioned in the previous section, all-pass transfer functions are BR functions. In this section, a second order 2-D digital filter transfer function (with real coefficients) is derived from a first order complex coefficient 2-D all-pass transfer function.

Let us consider the first-order complex all-pass transfer function,

$$G(Z_1,Z_2) = \frac{\alpha_3^* + \alpha_2^* Z_1^{-1} + \alpha_1^* Z_2^{-1} + \alpha_0^* Z_1^{-1} Z_2^{-1}}{\alpha_0 + \alpha_1 Z_1^{-1} + \alpha_2 Z_2^{-1} + \alpha_3 Z_1^{-1} Z_2^{-1}} \tag{4.3}$$

where α's are complex coefficients and α^* by:

$$\begin{aligned} \alpha_0 &= a_0 + jb_0 \\ \alpha_1 &= a_1 + jb_1 \\ \alpha_2 &= a_2 + jb_2 \\ \alpha_3 &= a_3 + jb_3 \end{aligned} \tag{4.4}$$

where a's and b's are real coefficients. In the case of a 2-D real coefficient first order filter the stability conditions are known [37]. It is straightforward to extend the same results to the 2-D first order complex coefficient filter transfer function. The necessary and sufficient conditions for the first-order 2-D complex all-pass transfer function given in (4.3) to be stable are:

$$\sqrt{(a_0-a_1)^2+(b_0-b_1)^2} > \sqrt{(a_2-a_3)^2+(b_2-b_3)^2} \tag{4.5a}$$

$$\sqrt{(a_0+a_1)^2+(b_0+b_1)^2} > \sqrt{(a_2+a_3)^2+(b_2+b_3)^2} \tag{4.5b}$$

$$\sqrt{a_1^2+b_1^2} < \sqrt{a_0^2+b_0^2} \tag{4.5c}$$

Using the transfer function given in (4.2), let

$$H(Z_1,Z_2) = \frac{1}{2}\left[G(Z_1,Z_2) + G^*(Z_1,Z_2)\right] \tag{4.6a}$$

where $G^*(Z_1,Z_2)$ is a first order complex all-pass transfer function with coefficients that are conjugates of those of $G(Z_1,Z_2)$. $H(Z_1,Z_2)$ will be a 2-D real coefficient filter transfer function of order 2 whose coefficients are in terms of coefficients of $G(Z_1,Z_2)$ as follows:

$H(z_1, z_2) =$

$$\frac{[1\ z_1^{-1}\ z_1^{-2}]\begin{bmatrix} (a_0a_3 - b_0b_3) & (a_0a_1+a_2a_3-b_0b_1-b_2b_3) & (a_1a_2-b_1b_2) \\ (a_0a_2+a_1a_3-b_0b_2-b_1b_3) & (a_0^2+a_1^2+a_2^2+a_3^2-b_0^2-b_1^2-b_2^2-b_3^2) & (a_0a_2+a_1a_3-b_0b_2-b_1b_3) \\ (a_1a_2-b_1b_2) & (a_0a_1+a_2a_3-b_0b_1-b_2b_3) & (a_0a_3-b_0b_3) \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \\ z_2^{-2} \end{bmatrix}}{[1\ z_1^{-1}\ z_1^{-2}]\begin{bmatrix} (a_0^2-b_0^2) & 2(a_0a_2+b_0b_2) & (a_2^2+b_2^2) \\ 2(a_0a_1+b_0b_1) & 2(a_1a_2+a_0a_3+b_1b_2+b_0b_3) & 2(a_2a_3+b_2b_3) \\ (a_1^2+b_1^2) & 2(a_1a_3+b_1b_3) & (a_3^2+b_3^2) \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \\ z_2^{-2} \end{bmatrix}} \tag{4.6b}$$

As both $G(Z_1,Z_2)$ and $G^*(Z_1,Z_2)$ are lossless bounded real functions, the addition of these functions gives a bounded real function. Thus the implementation of $H(Z_1,Z_2)$ without affecting the all-pass property of $G(Z_1,Z_2)$ gives rise to a structurally bounded realization.

From (4.6a), it is clear that real coefficient 2-D filter $H(Z_1,Z_2)$can be implemented as a parallel connection of two complex filters characterized by transfer functions $G(Z_1,Z_2)$ and $G^*(Z_1,Z_2)$ respectively, as shown in Fig. 4.2(a). The solid line indicates a real signal and dotted line indicates a complex signal. If input is real, then the output of $G(Z_1,Z_2)$ is the complex conjugate of $G^*(Z_1,Z_2)$. So, the real part of the complex filter output from $G(Z_1,Z_2)$ is the desired real sequence y(m,n). That means only a single complex filter is enough for processing the real input signal u(m,n) as shown in Fig. 4.2(b).

Basic Building Blocks

The basic building blocks required in the case of 2-D complex filters are complex unit delays, complex multipliers and complex adders as shown in Fig. 4.3, where the solid and dotted lines indicate, respectively, the real and imaginary parts of a complex signal.

Let the complex signal v(m,n) with real and imaginary parts, r(m,n) and i(m,n), given by:

$$v(m,n) = r(m,n) + ji(m,n) \tag{4.7}$$

be the input to the complex unit delay (in both Z_1 and Z_2), then its output v_0(m,n) will be

$$v_0(m,n) = v(m-1,n-1) = r(m-1,n-1) + ji(m-1,n-1) \tag{4.8}$$

The operations of the complex adder and the complex multiplier are quite similar to their real counter parts. Thus, if the inputs to a complex adder are

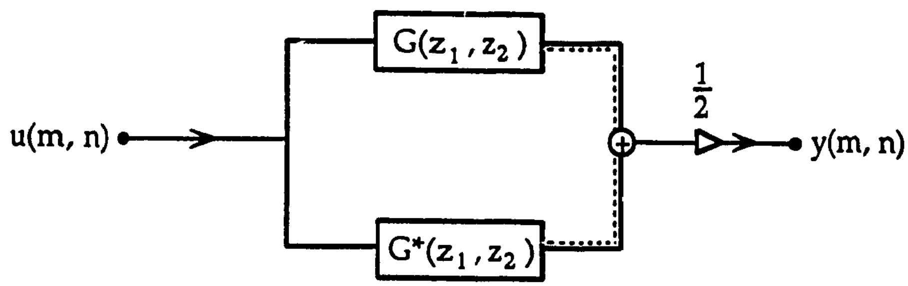

Fig. 4.2(a) Parallel connection of two all-pass sections.

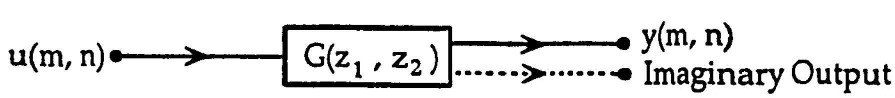

Fig. 4.2(b) Single complex all-pass section for processing u(m,n).

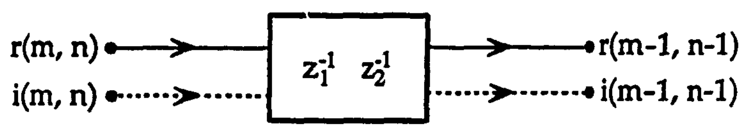

Fig. 4.3(a) Complex delay (both in Z_1 and Z_2).

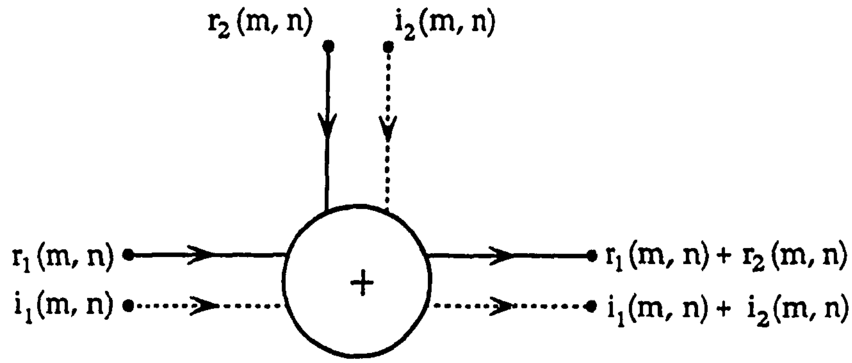

Fig. 4.3(b) Complex adder.

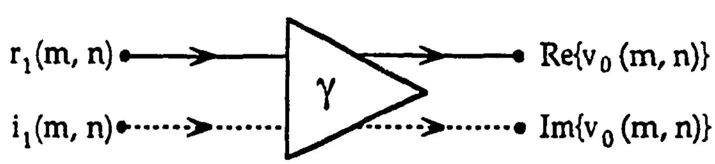

Fig. 4.3(c) Complex multiplier.

$v_1(m,n) = r_1(m,n) + ji_1(m,n)$ and $v_2(m,n) = r_2(m,n) + ji_2(m,n)$, respectively, its output $v_0(m,n)$ is given by:

$$v_0(m,n) = \left[r_1(m,n) + r_2(m,n)\right] + j\left[i_1(m,n) + i_2(m,n)\right] \tag{4.9}$$

In the case of complex multiplier, the output $v_0(m,n)$ with a multiplier coefficient $\gamma=\alpha + j\beta$ for an input v(m,n)=r(m,n)+j i(m,n) is given by:

$$v_0(m,n) = \left[\alpha r(m,n) - \beta i(m,n)\right] + j\left[\alpha i(m,n) - \beta r(m,n)\right] \tag{4.10}$$

Realization

We mentioned earlier that the 2-D second order digital transfer function $H(Z_1,Z_2)$ can be obtained from a sum of two complex coefficient 2-D all-pass digital transfer functions and is given in (4.6b). We have also mentioned that $H(Z_1,Z_2)$ can be realized as in Fig. 4.2(b) using a single complex coefficient all-pass first order section and taking the real output when the input is real. We now obtain a realization for a single complex all-pass 2-D section having $G(Z_1,Z_2)$ as its transfer function. From (4.3),

$$G(Z_1,Z_2) = \frac{Y(Z_1,Z_2)}{U(Z_1,Z_2)} = \frac{Z\left\{y'(m,n)\right\}}{Z\left\{u(m,n)\right\}}$$

$$= \left[A_0+jB_0\right]\frac{\left[(A_3-jB_3)+(A_2-jB_2)Z_1^{-1}+(A_1-jB_1)Z_2^{-1}+Z_1^{-1}Z_2^{-1}\right]}{\left[1+(A_1+jB_1)Z_1^{-1}+(A_2+jB_2)Z_2^{-1}+(A_3+jB_3)Z_1^{-1}Z_2^{-1}\right]}$$

$$= \beta G'(Z_1,Z_2) \tag{4.11}$$

where

$$A_0 = \frac{a_0^2-b_0^2}{a_0^2+b_0^2}$$

$$B_0 = \frac{-2a_0b_0}{a_0^2+b_0^2}$$

$$A_i = \frac{a_0a_i+b_0b_i}{a_0^2+b_0^2} \quad \text{for } i=1,2,3$$

$$B_i = \frac{a_0b_i-a_ib_0}{a_0^2+b_0^2} \quad \text{for } i=1,2,3$$

From (4.11), if we let

$$Z^{-1}\left[G'(Z_1,Z_2)\right] = \frac{y'(m,n)}{u'(m,n)}$$

we have

$$
\begin{aligned}
y'(m,n) &= (A_3-jB_3)u'(m,n) + (A_2-jB_2)u'(m-1,n) \\
&\quad + (A_1-jB_1)u'(m,n-1) + u'(m-1,n) \\
&\quad - (A_1+jB_1)y'(m-1,n) - (A_2+jB_2)y'(m,n-1) \\
&\quad - [(A_3+jB_3)y'(m-1,n-1) \\
&= A_3\left[u'(m,n)-y'(m-1,n-1)\right] + A_2\left[u'(m-1,n)-y'(m,n-1)\right] \\
&\quad + A_1\left[u'(m,n-1)-y'(m-1,n)\right] - jB_3\left[u'(m,n)+y'(m-1,n-1)\right] \\
&\quad - jB_2\left[u'(m-1,n)+y'(m,n-1)\right] - jB_1\left[u'(m,n-1)+y'(m-1,n)\right] \\
&\quad + u'(m-1,n-1)
\end{aligned}
\tag{4.12}
$$

If we realize (4.12) and cascade a multiplier $\beta = A_0+jB_0$ at the input, $G(Z_1,Z_2)$ is realized as shown in Fig. 4.4. The transfer function $H(Z_1,Z_2)$ is realized as shown in Fig. 4.2(b).

The realized complex all-pass configuration is structurally passive. That is, in spite of the multiplier quantization, the mirror image property of the all-pass transfer function is preserved. The proposed structure of Fig. 4.2(b), where $G(Z_1,Z_2)$ is implemented using the structure shown in Fig. 4.4, requires 7 complex multipliers or equivalently 14 real multipliers, where as the direct realization of a second order 2-D filter proposed by Mitra et al. [38] requires 17 multipliers. The other advantage of our structure is the low multiplier coefficient quantization sensitivity, which will be discussed in detail later. With presently available VLSI technology, complex multipliers can be fabricated efficiently on a single chip.

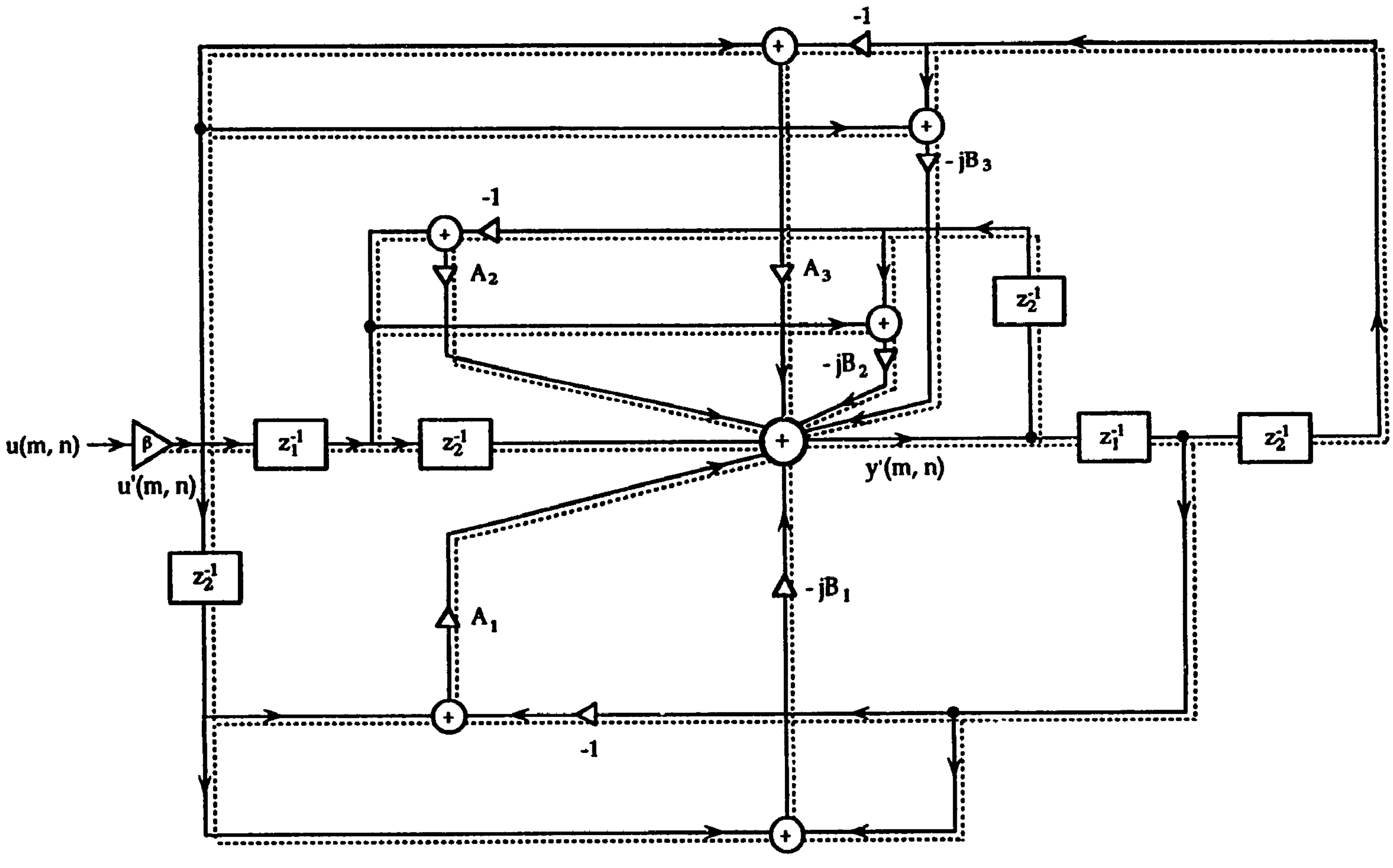

Fig. 4.4 Realization of first order complex coefficient 2-D all-pass section.

4.4 Design Of Quarter Plane 2-D Recursive Digital Filters

Several methods in designing 2-D recursive digital filters are discussed in [39] by Ramamoorthy and Bruton. An l_p design technique for a 2-D digital filter was proposed by Maria and Fahmy [1]. In a recent paper, Wan and Fahmy [41] designed an N-D digital filter with a finite word length and assured stability of the filter by using non-optimal step size. All these design techniques are based on nonlinear optimization algorithms.

As it is not possible, in general, to decompose a given $H(Z_1,Z_2)$, we obtain it from a complex coefficient first order all-pass transfer function as in (4.5b). It may be noted that the coefficients of $H(Z_1,Z_2)$ thus obtained are functions of a_i and b_i. Our aim is to design a 2-D filter with desired magnitude specifications. That is, to find a 2-D transfer function $H(Z_1,Z_2)$ from the given magnitude characteristic $H_{mn}{}^d$ specified as a set of real numbers corresponding to a discrete set of frequency pairs $\left[\omega_m{}^{(1)},\omega_n{}^{(2)}\right]$ for m=1,2,. . ., M and n=1,2,. . ., M. The objective function is defined as

$$J(x) = \sum_{m=1}^{M}\sum_{n=1}^{M}\left[|H(Z_{1m},Z_{2n})| - H_{mn}{}^d\right]^p \tag{4.13}$$

where $Z_{1m}=e^{j\omega_m^{(1)}}$, $Z_{2n}=e^{j\omega_n^{(2)}}$, p is any even positive integer, and **x** is the parameter vector which contains a_i and b_i coefficients of $G(Z_1,Z_2)$.

There are several optimization algorithms available for designing 2-D digital filters. In all these, the Fletcher-Powell algorithm [41] is extensively used in the design of the 2-D digital filter. In [40], Wan and Fahmy used nonoptimal step size in the algorithm to increase the computational efficiency per iteration. We have followed the same optimization procedure for our design. Here briefly, we give the theorems presented in [40].

Theorem 4.1: In the (j+1)th iteration of the Fletcher-Powell algorithm, a mapping matrix $\mathbf{D}_j$ is positive definite (PD) if and only if the step size λ_j satisfies:

$$g'(\lambda_j) + \nabla \mathbf{J}(\mathbf{y}_j)^T \mathbf{D}_j \nabla \mathbf{J}(\mathbf{y}_j) > 0 \tag{4.14}$$

where

$$g'(\lambda_j) = \frac{d\mathbf{J}\left[\mathbf{y}_j + \lambda \mathbf{d}_j\right]}{d\lambda}\Bigg|_{\lambda=\lambda_j} = \nabla \mathbf{J}\left[\mathbf{y}_j + \lambda \mathbf{d}_j\right]^T \mathbf{d}_j$$

Theorem 4.2: If $H^j(Z_1,Z_2)$ is stable with coefficient vector $\mathbf{x} = \mathbf{y}_j$, and the searching direction $\mathbf{d}_j$ is descent, then the smallest value λ_0 satisfying

$$g'(\lambda_0) = \frac{d\mathbf{J}_N\left[\mathbf{y}_j + \lambda \mathbf{d}_j\right]}{d\lambda}\Bigg|_{\lambda=\lambda_0} = 0 \tag{4.15}$$

will ensure the stability of $H^{(j+1)}(Z_1,Z_2)$ with coefficient vector $\mathbf{x}=\mathbf{y}_j\lambda_0\mathbf{d}_j$. Proof of this theorem is given in [40].

<u>Example:</u>

A 2-Dimensional low-pass filter with a cut off frequency equal to one-tenth of Nyquist frequency is considered here. The magnitude specifications are taken to be the same as in [1].

$$H_{mn}{}^d = 1, \qquad \left[(\omega_m{}^{(1)})^2 + (\omega_n{}^{(2)})^2\right]^{1/2} \le 0.08$$

$$= 0.5, \quad 0.08 \le \left[(\omega_m^{(1)})^2 + (\omega_n^{(2)})^2\right]^{1/2} \le 0.12$$

$$= 0, \quad \left[(\omega_m^{(1)})^2 + (\omega_n^{(2)})^2\right]^{1/2} \ge 0.12$$

$\omega_m^{(1)} = 0.00, 0.02, \ldots, 0.18, 0.2, 0.4, \ldots, 1.0$

$\omega_n^{(2)} = 0.00, 0.02, \ldots, 0.18, 0.2, 0.4, \ldots, 1.0$

Once we obtain the design parameters, namely, the coefficients a_i and b_i, we have the transfer function $G(Z_1,Z_2)$ of the complex coefficient first-order 2-D all-pass filter. We have used David Fletcher Powell algorithm with non-linear optimizing algorithm with non optimum step size [40] for minimizing the performance index J(x). This is an unconstrained optimization problem. A new approach given by Wan and Fahmy [40] to ensure stability of the filter in each iteration with out having to perform a stability test used (Theorem 2) in designing the filter. Our parameter vector x is:

$$\mathrm{x} = [a_0\, a_1\, a_2\, a_3\, b_0\, b_1\, b_2\, b_3]^T$$

where T indicates transpose. The initial parameter vector is chosen as:

$$\mathbf{x} = [\, 1.3258\; -0.5412\; -0.7841\; 0.1538\; 1.3449\; 0.4381\; -0.4132\; -0.1257\,]^T$$

The final parameter vector after nonlinear optimization is found to be:

$$\mathbf{x} = [2.0498\; -0.1661\; -0.1667\; -1.1661\; 1.8922\; -0.3210\; -0.3215\; -1.4318]^T$$

The final error function value is given as:

$$\mathrm{J(x)} = 2.3657$$

The multiplier coefficients used in the realization are given as:

$$G(z_1, z_2) = (0.07936 - j\,0.9968)\frac{\begin{bmatrix}1 & z_1^{-1}\end{bmatrix}\begin{bmatrix}-0.6556 - j\,0.0934 & -0.1218 - j\,0.0441 \\ -0.1218 - j\,0.0441 & 1\end{bmatrix}\begin{bmatrix}1 \\ z_2^{-1}\end{bmatrix}}{\begin{bmatrix}1 & z_1^{-1}\end{bmatrix}\begin{bmatrix}1 & -0.1218 - j\,0.0441 \\ -0.1218 - j\,0.0441 & -0.6555 - j\,0.0934\end{bmatrix}\begin{bmatrix}1 \\ z_2^{-1}\end{bmatrix}}$$

We also designed a 2-D digital filter with two first order complex coefficient all-pass filters in cascade which gives in effect two second order real coefficient 2-D filters in cascade. The parameter vector is:

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_1^T \mathbf{x}_2^T\end{bmatrix}$$

where $\mathbf{x}_1$ is the first single order section parameter vector and $\mathbf{x}_2$ is the second first order section parameter vector. The final error function in this case is $J(\mathbf{x}) = 1.8464$. The parameter vector after optimization is:

$$\mathbf{x}_1 = [1.988\ -0.1055\ -0.1059\ -1.0846\ 1.9673\ -0.3110\ -0.3117\ -1.4765]^T$$

$$\mathbf{x}_2 = [2.2939\ -0.1348\ -0.1356\ -1.1352\ 1.9367\ -0.3074\ -0.3095\ -1.4175]^T$$

The multiplier coefficients used in the cascade realization are given as:

$$H_1(z_1, z_2) = (0.0107 - j\,0.9999)\frac{[1\ \ z_1^{-1}]\begin{bmatrix} -0.6469 + j\,0.1025 & -0.1050 + j\,0.0525 \\ -0.1053 + j\,0.0526 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}{[1\ \ z_1^{-1}]\begin{bmatrix} 1 & -0.1053 - j\,0.0526 \\ -0.105 - j\,0.0525 & -0.6469 - j\,0.1025 \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}$$

$$H_2(z_1, z_2) = (0.1677 - j\,0.9858)\frac{[1\ \ z_1^{-1}]\begin{bmatrix} -0.5935 + j\,0.1168 & -0.1004 + j\,0.0493 \\ -0.1010 + j\,0.0496 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}{[1\ \ z_1^{-1}]\begin{bmatrix} 1 & -0.1010 - j\,0.0496 \\ -0.1004 - j\,0.0493 & -0.5935 - j\,0.1168 \end{bmatrix}\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}$$

where $H_1(Z_1,Z_2)$ and $H_2(Z_1,Z_2)$ are the first and second section respectively of the cascade realization. The amplitude response for one section and two sections in cascade are as shown in Figs. 4.5 and 4.6.

It may be observed that the specified magnitude response has circular symmetry. This results in certain symmetry in the coefficients of the transfer function if the 2-D filter is designed directly with real coefficients using, for example, the l_p design of Maria and Fahmy [1]. The 2-D real coefficient second order filter we obtain via first order complex coefficient 2-D all-pass filter using (4.6b) does not possess this symmetry. So naturally the error J(x) in our case is a little more than what it is in [1] for one second order real coefficient 2-D filter. But our 2-D filter is less sensitive to coefficient truncation as will be discussed in the next section.

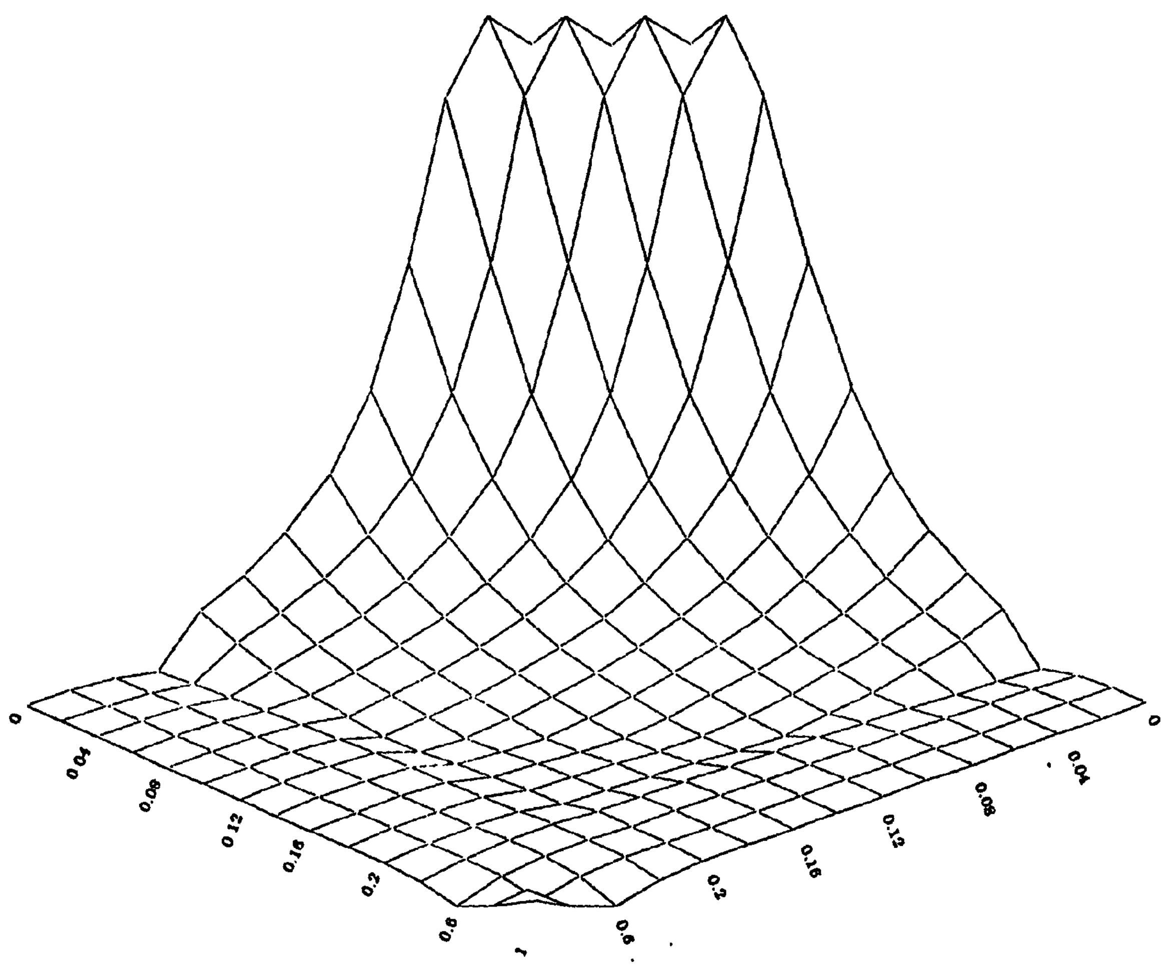

Fig. 4.5 Amplitude response of the one section low-pass filter.

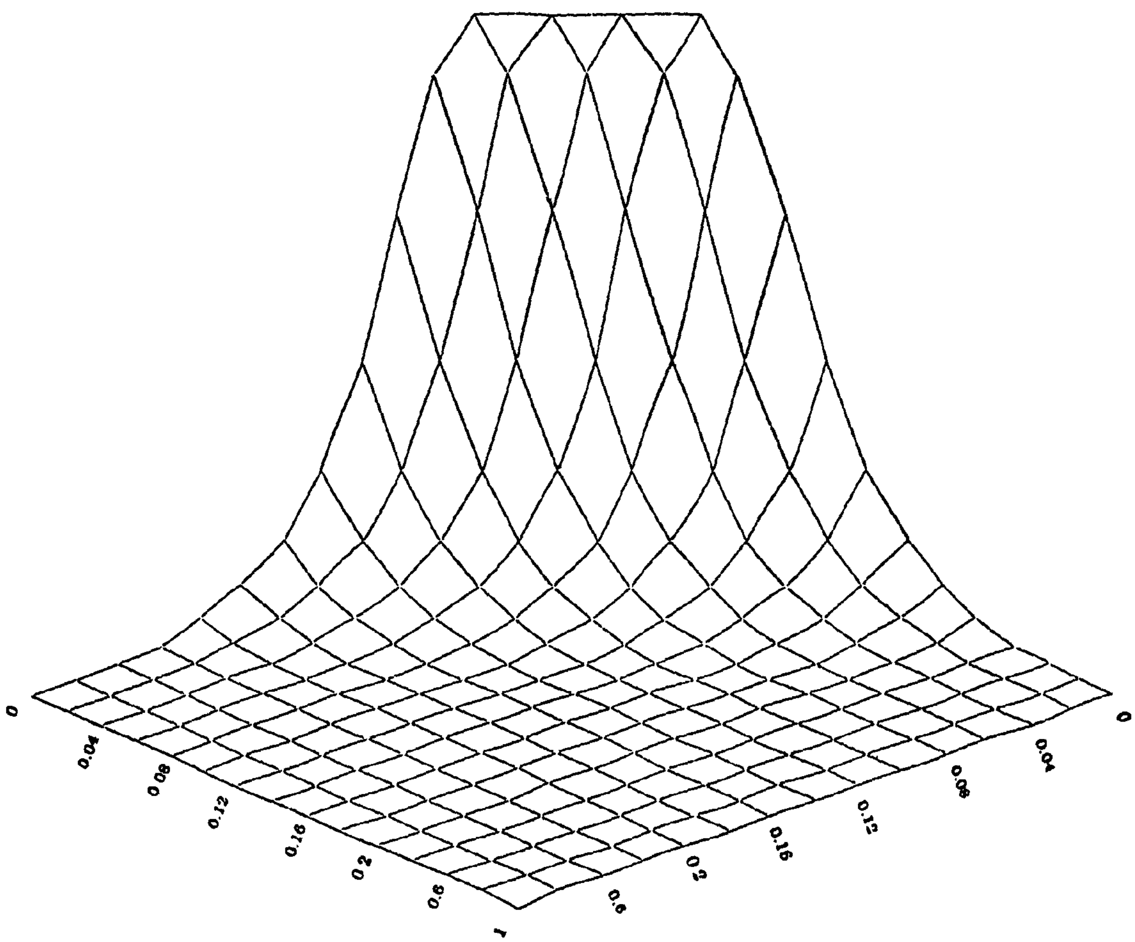

Fig. 4.6 Amplitude response of the cascaded two section low-pass filter.

The number of design parameters, namely, a_i's and b_i's are only 8 in our case whereas they are 11 (with symmetry) in l_p design [1] for designing a single second order 2-D real coefficient digital filter. This results in less computational time in the design. As mentioned earlier, the number of real multipliers required in the implementation of the filter designed by our approach is only 14, whereas it is 17 when real coefficient second order filter designed by l_p design is implemented by direct realization [38]. Also, there is no overshoot in the magnitude characteristic in the pass band of our filter unlike in [1].

4.5 Effects of Finite Word Length

When the multiplier coefficients of a digital filter are quantized, the frequency response is perturbed from the original frequency response. If the structure is sensitive to perturbations of the multiplier coefficients, the resulting system may no longer meet the original design specifications.

Effect of the finite word length is studied by truncating the binary multiplier coefficients to 8 bits, 10 bits and 16 bits for filters obtained both by our method and l_p design (normal) method. These are given in Table 4.1. From Table 4.1, it is clear that truncation of coefficients does not have much effect on the error function in the case of complex coefficient multiplier filter. In the case of normal design, for 8 bit truncation, the filter is becoming unstable. Even for 10 bit truncation, error is increased to more than 6 times. So our designed filter is less sensitive to coefficient quantization.

In this chapter a new design technique for designing quarter plane second order 2-D recursive filters using a complex coefficient first order all-pass 2-D filter is given.

	Error Function $J(\underline{x})$			
	Original (before truncation)	After 8 bits truncation	After 10 bits truncation	After 16 bits truncation
One section complex multiplier filter	2.3657	2.5164	2.3736	2.3657
Two section complex multiplier filter	1.8464	1.9614	1.8565	1.8464
l_p design one second-order section	1.8689	unstable	11.8707	1.8710

Table 4.1 Error function values before and after truncation of binary multiplier coefficients.

CONCLUSIONS

Following the introductory chapter, certain points regarding Robinson's proof [21] of guaranteed stability of 1-D LSI polynomials have been clarified in chapter 2. This is very important in stablizing 1-D recursive digital filters. In chapter 3, a proof for the stability of the LSI polynomials of n-D practical digital filters has been given, and this is useful in stablizing such filters. In the conventional sense, stablizing an unstable filter through the double LSI approach meets with difficulties particularly when the original denominator polynomial has zeros on the unit polydisc. Theorem 3.3 presented is valid even if the original and the corresponding LSI polynomials have zeros on the unit polydisc.

In chapter 4, a new method for designing a quarter plane second order real 2-D recursive filter using a complex coefficient first order all-pass 2-D filter has been discussed. An implementation scheme for these filters using complex multipliers has been given. This implementation effectively requires less number of real multipliers. It is also shown that the sensitivity of these filters to the coefficient is very low quantization when compared to the real filter designed directly.

Suggestions For Further Study:

The problem of proving that the planar least square polynomial of a 2-D polynomial is stable is important in the area of digital filters. So further research may be directed towards proving Shank's conjecture by suitably modifying it. This can solve the problem of stablizing 2-D recursive digital filters. Once this result is available for 2-D case, it can be extended to n-D "first quadrant" and 2-D non symmetric half plane (NSHP) polynomials.

REFERENCES

[1] G.A. Maria and M.N. Fahmy, "An l_p design technique for two-dimensional digital recursive filters," IEEE Trans. Acoustics, Speech, Signal Processing, vol. ASSP-22, pp. 15-21, Feb. 1974.

[2] M.P. Ekstrom, R.E. Twogood and J.W. Woods, " Two Dimensional Recursive filter design - A spectral factorization approach," IEEE Trans.Acoustics, Speech and signal processing, vol. ASSP-28, pp. 16-26, Jan 1980.

[3] K.P. Prasad, and P.S. Reddy, " Computer aided design of stable recursive filters from magnitude specifications," Int. J. Circuits Theory Appl., Vol. 7, pp. 121-126, 1979.

[4] P.K. Rajan, Haranath Reddy and M.N.S. Swamy, " Comments on " Computer aided design of stable recursive filters from magnitude specifications," Int. J. Circuits Theory Appl., Vol. 8, pp. 183, April 1980.

[5] E.I. Jury, "Stability of multidimensional scalar and matrix polynomials," Proc. of IEEE , vol. 66, pp. 1018-1047, Sept. 1978.

[6] B.T.O'Connor and T.S. Huang, " Stability of general two-dimensional recursive digital filters," IEEE Trans. Acoustics speech and signal processing, vol. ASSP-26, pp. 550-560, Dec. 1978.

[7] J.L. Shanks, Sven Treitel and J.H. Justice, " Stability and synthesis of two-dimensional Recursive filters," IEEE Trans. Audio and Electroacoustics, vol. AU-20, pp. 115-128, June 1972.

[8] Y. Genin and Y. Camp, "Counter example in the least square inverse stabilization of 2-D recursive filters," Electron. Lett. vol. 11, pp. 130-131, July 1975.

[9] E.I. Jury, "An overview of Shanks' conjecture and comments on its validity," in Proc. Tenth Asilmor Conf. Circuits Systems and Computers, CA, Nov. 1976.

[10] B.D.O. Anderson and E.I.Jury, " Proof of a special case of Shanks' conjecture," IEEE Trans. Acoustic, Speech and Signal processing, vol. ASSP-24, pp. 574-575, Dec. 1976.

[11] E.I. Jury et al., "Stablization of certain two-dimensional recursive digital filters," Proc. IEEE, Vol. 65, pp. 887-892, June 1977.

[12] P. Delsarte, Y.V. Genin and Y.G. Kamp, "Planar least square inverse polynomials part I - Algebraic properties," IEEE Trans. Circuits and Systems, Vol. CAS-26, pp. 59-66, January 1979.

[13] A.H. Kayran and P.King, "Comments on least-squares inverse polynomials and a counter example for a Jury's conjecture," Electron. Lett., vol. 16, no. 21, pp. 795-796, Oct. 1980.

[14] P.S. Reddy, D.R. Reddy and M.N.S. Swamy, "Modification of Shank's conjecture on stability of 2-D planar Least-square inverse polynomials," Electron. Lett., vol. 18, pp. 456-457, May 1982.

[15] P.S. Reddy, D. Raghurami Reddy and M.N.S. Swamy, " Proof of a modified form of Shank's conjecture on the stability of 2-D planar least square inverse polynomials and its implications", IEEE Trans. Circuits and systems, vol. CAS-31, no. 12, pp. 1009-1015, December 1984.

[16] P. Delsarte, Y. Genin and Y. Kamp , "Comments on proof of a modified form of Shank's conjecture on the stability of 2-D planar least square inverse polynomials and its implications," IEEE Trans. circuits and systems, vol. CAS-32, no. 9, pp. 966, September 1985,

[17] P.K. Rajan and H.C. Reddy, " A note on LSI polynomials and comments on a proof of a modified form of Shank's conjecture," IEEE Trans. circuits and systems, vol. CAS-32, no. 9, pp. 966-968, Sept. 1985.

[18] P. Agathokli and L. Bruton, " Practical-BIBO Stability of n-dimensional discrete systems," IEE Proc. G, Electron. Circuits and Systems, no. 6, pp. 236-242, Dec. 1983.

[19] M.N.S. Swamy, L.M. Roytman and E.I. Plotkin, "Planar least squares inverse polynomials and practical-BIBO stabilization n-dimensional linear shift-invariant filters," IEEE Trans. Circuits and systems, vol. CAS-32, no. 12, pp. 1255-1260, Dec. 1985.

[20] M.N.S. Swamy, L. Roytman and E.I. Plotkin, "Corrections to planar least square Inverse Polynomials and practical BIBO Stabilization of N-Dimensional Linear Shift-Invariant filters," IEEE Trans.on Circuits and systems, vol. CAS-33, pp. 1038-1039, Oct. 1986.

[21] E.A. Robinson, "Statistical Communication and Detection", London, Griffin, 1967.

[22] P. Delsarte, Y. Genin and Y. Kamp , "Comments on proof of a modified form of Shank's conjecture on the stability of 2-D planar least square inverse polynomials and its implications," IEEE Trans. circuits and systems, vol. CAS-32, no. 9, pp. 966, September 1985,

[23] A. Fettweis, "Digital filter structures related to classical filter networks," Arch. Elek. Ubertrangungstechink, Vol. 25, pp. 79-81, Feb. 1971.

[24] A. Fettweis, "Wave digital lattice filters," Int. J. Circuit Theory Appl., Vol. 2, pp. 203-211, June 1974.

[25] M.N.S. Swamy and K.S. Thygarajan, "A new type of wave digital filters,"

J. Franklin Inst., Vol. 300, pp. 41-58, July 1975.

[26] P.P. Vaidyanathan and S.K. Mitra, "Low pass-band sensitivity digital filters: A generalized view point and synthesis procedures," Proc. IEEE, pp. 404-423, Apr. 1984.

[27] P.P. Vidynathan, S.K. Mitra and Y. Neuve, "A new approach to the realization of low sensitivity IIR digital filters," IEEE Trans. on Accoustics, Speech and Signal processing, Vol. ASSP-34, no.2, pp. 350-361, April 1986.

[28] T.H. Crystal and L. Ehrman, "The design and applications of digital filters with complex coefficients," IEEE Trans. Audio Electroacoust., AU-16, pp. 315-320, Sept. 1968.

[29] J.K. Skwirzynski and J.O. Scanlan, Network and Signal Theory, pp. 344-351, Peter Peregrinus, London, 1973.

[30] T. Takebe, K. Nishikawa and M. Yamamoto, "Complex coefficient digital all-pass networks and their applications to variable delay equalizer design," IEEE Int. Symp. Circuits and Systems (Houston, Texas), pp. 605-608, April 1980.

[31] A. Fettweis, "Principles of complex wave digital filters," Int. J. Circuit Theory Appl., Vol. 9, pp. 119-134, 1981.

[32] T. Saramaki, Tian-Hu Yu and S.K. Mitra, "Very low sensitivity realization of IIR digital filters using a cascade of complex all-pass structures," IEEE Trans. Circuits and Systems, Vol. CAS-34, No. 8, pp. 876-886, Aug. 1987.

[33] M.N.S. Swamy, K.S. Thygarajan and V. Ramachandran, " Two-dimensional wave digital filters using doubly terminated two-variable LC-ladder configurations," J. Franklin Inst., Vol. 304, pp. 201-215,

Oct./Nov. 1977.

[34] D. Raghurami Reddy and R. Unbehauen, "Formation of the block Toeplitz matrix for two-dimensional PLSI polynomials,", Circuits, Systems and Signal processing, Vol. 6, no. 4, pp. 450-456, 1987.

[35] E.I. Jury, V.R.Kolavennu, and B.D.O. Anderson, "Stabilization of certain two-dimensional recursive Digital filters," Proc. IEEE, vol. 65, no. 6, pp. 887-892, June 1977.

[36] D.R. Reddy, G. Schmeisser and R. Unbehauen, "PLSI Polynomails : Stabilization of 2-D recursive filters, " IEEE Trans. Circuits and systems, vol. CAS-33, no. 10, pp. 1015-1018, October 1986.

[37] T.S. Huang, "Stability of two-dimensional recursive filters," IEEE Trans. Audio Elecroacoust., AU-20, pp. 158-163, June 1972.

[38] S.K. Mitra, A.D. Sagar and N.A. Pandergrass, "Realization of two- dimensional recursive digital filters," IEEE Trans. Circuits and Systems, CAS-22, pp. 177-184, March 1975.

[39] T.S. Huang (editor), Two-Dimensional Digital Signal Processing I - Linear Filters, pp. 41-67, Springer-Verlag, New York, 1981.

[40] Y. Wan and M. Fahmy, "Design of N-D digital filters with finite word length coefficients," IEEE Trans. Circuits and Systems, CAS-36, pp. 375-382, March 1989.

[41] R. Fletcher and M.J.D. Powell, "A rapidly convergent descent method for minimization," The Computer Journal, Vol. 6, pp. 163-168, 1963.

[42] P.S. Reddy, S.R. Palacherla and M.N.S. Swamy, "Least squares inverse polynomial and a proof of practical-BIBO stability of N-D digital filters," IEEE Trans. Circuits and Systems, CAS-37, pp. 1565-1567, Dec. 1990.

[43] S.R. Palacherla, P.S. Reddy and M.N.S. Swamy,"Design of low sensitivity quarter plane recursive 2-D digital filters," to appear in International Journal of Circuit Theory and Applications, 1991.