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# On Teaching and Learning the Concept of Fractal.

Craig S. Bowers

A Thesis
in
The Department
of
Mathematics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master in the Teaching of Mathematics
Concordia University
Montreal, Quebec, Canada

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# ABSTRACT

On Teaching and Learning the Concept of Fractal.

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This research study describes the acquisition of the concept of fractal by identifying: some of the difficulties encountered in constructing the meaning of a fractal, certain basic acts of understanding necessary in the process of constructing the meaning of fractal, and the didactic conditions of teaching the notion of fractal.

Such identifications, will be considered through a historical review of fractals and dimension, and case studies of three, grade 12 students. The case studies take place in two settings: a clinical interview, and a teaching experiment. The interview and experiment will be founded on a questionnaire and a preliminary epistemological analysis of the concept of fractal. The analysis considers possible acts of understanding and obstacles the students may encounter in learning about fractals. The main pedagogical reference for the experiment is that of the van Hiele model of geometric thinking.

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To both my advisors, Dr. Bill Byers and Dr. Anna Sierpinska, my deepest thanks. Dr. Byers introduced me to the notion of fractal through the study of dynamical systems. His comments were essential in the focussing of my ideas regarding the study of fractals. The analysis used in this thesis is founded on the work of Dr. Sierpinska and her efforts were crucial in enhancing the clarity and presentation of my work.

I wish to thank my parents for their support throughout my academic endeavours. Finally, my appreciation to my partner Margaret who reviewed this work for grammatical errors and our daughter Sarah for her timely arrival.

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## INTRODUCTION

This study will explore the timely topic of fractals. Fractal geometry has been the cause of much excitement. The television shows and volumes of recently published literature are testament to such excitement. Many high school educators have become interested in this dynamic new geometry and have begun to design courses of study on the topic of fractals. The visual appeal and number of applications for the topic make it an inviting topic indeed. However, if the Ministry of Education is to add a topic to an already crowded curriculum, it is justified in asking for legitimate reasons for such action. The beautiful graphics and numerous applications may not justify implementation of fractals into the high school curriculum. The answers to questions such as, what does the student need in his/her mathematics repertoire to study such a topic? and what will the student come to understand through the study of fractals? are essential in determining if the topic is worthy of study and how the topic should be taught. In essence what will the teaching of fractals mean to the individual student studying the topic?

This study undertakes to identify: 1) some of the difficulties encountered in constructing the meaning of a fractal (for the purposes of this study a fractal will refer to an object that is self-similar and has non-integral dimension), 2) certain basic acts of understanding necessary in the process of constructing the meaning of fractal, and 3) the didactic conditions of teaching the notion of fractal: prerequisites, pedagogical exploitation of the conflict generating situations for the introduction of the new concept of dimension, and problems used to develop the notion of fractal.

These aims, will be considered through a historical review of fractals and dimension, and case studies of three, grade 12 students. The case studies take place in two settings: a clinical interview, and a teaching experiment. The interview and experiment will be founded on a questionnaire and a preliminary epistemological analysis of the concept of fractal. The questionnaire was designed to determine the student's existing notion of dimension and similarity. The analysis considers possible acts of understanding and obstacles the students may encounter in learning about fractals. The main pedagogical reference for the experiment is that of the van Hiele model of geometric thinking.

The thesis is developed over five chapters. Chapter one presents the notion of fractal, explains the reasons the study was undertaken, introduces the reader to the literature relevant to this dissertation and a short history of the notion of dimension and fractals. Chapter two contains the pre-experiment analysis. In particular, the two theoretical tools for this analysis, namely - a preliminary epistemological analysis of the concept of a fractal and van Hiele model of geometric thinking. The analysis identifies obstacles to be overcome and acts of understanding needed to be achieved if the notion of fractal is to be developed. The van Hiele model discriminates between three levels of development of a concept: visualization, description, and theoretical. Achievement of each of the consecutive levels is guided by an instructional sequence: inquiry, directed orientation, explication, free orientation, and integration, in that order. The pedagogy of van Hiele model and the preliminary epistemological analysis are then used in determining the didactic conditions necessary for the teaching of fractals.

The third chapter reports the three individual stories regarding the development of the notion of a fractal during the interview and experiment. The clinical interview and three teaching lessons are presented in detail. The teaching experiment involves the teaching of four activities. Eight tasks are developed as a part of the four activities. The experiment focuses on the students development through these tasks. An evaluation, given upon the completion of the teaching undertakes to determine if the aims of the activities (tasks) have been met.

The fourth chapter confronts these stories with the analysis of chapter two. Using the analysis, hypotheses will be made regarding the elements that the three students had in common while progressing through the tasks.

The final chapter will involve four aspects relevant to this thesis. The four aspects include the teaching of fractals, how the teaching undertaken in this study could be extended or modified, in what sense were the objectives of the study met and future areas to be researched.

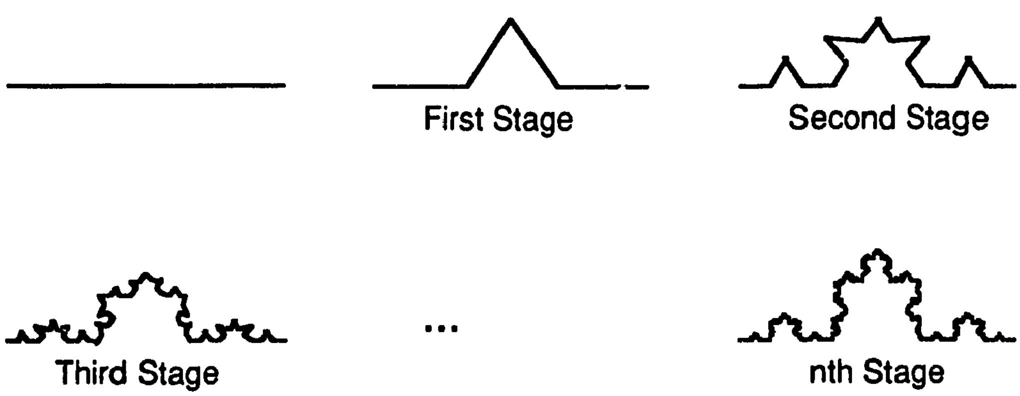
# CHAPTER 1: The notion of fractal, its literature and history

We start with a presentation of the notion of fractal. The reasons for studying fractals form a part of this presentation. This is followed by a review of literature related to fractals. Chapter one closes with an outline of the history related to this subject. This part will focus on the analytic, geometric and theoretical work used in creating the current notions of fractal and dimension. The history sets the stage for the epistemological analysis that follows in chapter two.

## 1. What is a fractal?

Our introduction of fractals comes by way of the von Koch curve. Through the triadic Koch curve, the ideas of self-similarity and fractional dimension will be considered. The relationship between dimension and self-similarity will be highlighted.

The von Koch curve is formed with a recursive technique: start with a line, omit the middle third, substitute the middle third with 2 lines having 1/3 the length of the original line, then do the same thing for each of the four remaining line segments.



The Koch curve is the limit of this process and is deemed self-similar. A set is self-similar if it can be formed by the union of N non-overlapping subsets of S, $S_1$, $S_2$, .., $S_N$. These subsets are all congruent to r(S), $o<r<1$. Congruent meaning, they, the subsets $S_i$, can be formed by r(S) through rotations and translations.

The property of "self-similarity" can be used to discover the object's dimension. A line segment, a square and a cube will be used to show the relationship between self-similarity and dimension. If a line is scaled down by a ratio r then the number of smaller copies, N, is given by N=1/r. For example if a line is scaled down by a factor 1/3 then the number of smaller copies, N, is 3=1/(1/3). If a square is scaled down by a factor r then the number of smaller copies $N=1/r^2$. Similarly, the number of smaller copies of a cube with scaling ratio r is $N=1/r^3$. This may be generalized into the form $N=1/r^D$ or D=logN/log(1/r), where D is the dimension of the figure (Peterson, 1988).

Th von Koch curve can be analyzed in the same manner. The original line is scaled down by a ratio 1/3 and there are 4 smaller copies of the original line, thus, it has dimension D=log4/log3. This is referred to as the object's similarity dimension.

This is the beginning of the relationship between dimensions of non-integral value and self-similarity. These two properties will be common to the fractals dealt with in this paper.

## 2. Why study fractals?

Thorpe (1989), determines three criteria by which curriculum is deemed relevant or not worthwhile for study. The curriculum must be intrinsically exciting, have pedagogic value and/or intrinsic value. The study of fractals is intrinsically exciting. The patterns and pictures created are beautiful. The students can create pictures, and test conjectures regarding the pictures (Thorpe, 1989).

The intrinsic value can be found in many applications which encompass the fields of chemistry, anatomy, physics, and chaotic dynamical systems. High school students are able to characterize fine particle boundaries (eg. carbon black), describe respirable dust found in industry and metal crystals. These descriptions can be used in the context of improving the workplace and improving the quality of manufactured goods.

Mandelbrot states that fractals should be used as an introduction to the concept of derivative, as students should be aware that continuous lines that are nowhere differentiable are the rule and not the exception. Thus, fractals can be used in a pedagogically significant manner and, as grade 12 math is the last course before calculus, this may be an appropriate time for the introduction of fractals (Barcellus, 1984).

A number of high school students have been introduced to the topic of fractals, through the media and after-school activities. The standard presentation begins with the question, "How long is the coastline of Great Britain?". The answer is developed by using the idea that the length depends on your "measuring sticks". Students are then asked to use different measuring sticks and find the length of the coastline. The length versus measuring stick is then plotted on log-log paper. The points can be joined by a straight line. The slope of the line, m, is used to determine the dimension of the coastline through the equation D=1+|m|. The dimension is said to describe the coastlines ruggedness. Often no mathematical explanation is given as to how D=1+|m| is derived. The implication is that yet another rule has been discovered, in the field of mathematics. Students must be given the opportunity to understand that this is not simply another rule, but a new way to perceive the world.

## 3. Literature Review

The word fractal was first coined in 1975 by Benoit B. Mandelbrot. To get an idea of what a fractal means, a comparison of Euclidean and fractal shapes is presented in Pietgen and Saupe's, The Science of Fractal images (p. 26). The comparison in the following table outlines what it means for an object to be a fractal.

**Mathematical language to describe, relate, and manipulate shapes**

| EUCLIDEAN | FRACTAL |
|---|---|
| traditional (>2000 yr) | modern monsters (~10 yr) |
| based on characteristic size | no specific size or scaling |
| suits man made objects | appropriate for natural shapes |
| described by formula | (recursive) algorithm |

The Pietgen's explanation is only accurate in a general sense. If by traditional the author means shapes such as circles and cubes, and monsters is taken to mean sets such as the Cantor set and von Koch snowflake, the reader is given some feeling as to an apparent difference between the two geometries. The second and third points used to describe a fractal should be clarified. The second point, no specific size or scale, would exclude many of the fractals found in nature. Natural objects may exhibit such qualities, but only within a certain range of sizes. With respect to the third point, appropriate for natural shapes, Stewart states "The fractal cow is not of necessity more realistic than the spherical one" (Stewart, 1988, 242). Even this new mathematical representation of the world has several limitations.

Mandelbrot (1982) posed the first mathematical definition of fractal as "a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension" (p.15). This definition is such that it excludes many fractals found in the study of physics. Although this definition is confining it enabled researchers to explore what fractals are in a more mathematically systematic manner. Once this process had begun then refinements to the definition could be based on sound mathematical theory.

In Fractals Everywhere, Barnsley's first definition regarding fractals reads as follows, "If we were to define a deterministic fractal, we might say that it is a fixed point of a contractive transformation.." (p. 80). The context and clarifications given are of interest. The comments "If I were to define" and "we might" demonstrate Barnsley's apprehension in presenting a definition for a fractal. The author clarifies that this is not a definition for fractals, but "deterministic fractals".

As the first definition was proposed in 1982 it isn't surprising that an all encompassing mathematical definition hasn't been developed. Also, mathematicians maybe be reluctant to offer such definitions until the field of study has "stabilized". At present the definition that offers a great deal of flexibility is that a fractal is an entity whose parts are in some way similar to the whole (Feder, 1988, 11).

This paper will be concerned with self-similar fractals. To develop an understanding of what self-similar fractals are, a mathematical description and an example will be given.

Self-similar fractals are those objects whose subsets, when magnified, are all similar to the whole. Such a definition encompasses both exact and statistically self-similar fractals.

An exactly "self-similar object is composed of N copies of itself (with possible translations and rotations) each of which is scaled down by the ratio r ... Consider a set S of points at positions $\mathbf{x}=(x_1,\ldots,x_E)$ in Euclidean space of dimension E. Under a similarity transform with real scaling ratio $0<r<1$, the set S becomes rS with points at $r\mathbf{x}=(rx_1,\ldots,rx_E)$" (Falconer, 1985, 59). A bounded set S is self-similar when S is the union of N distinct (non-overlapping) subsets $S_1,\ldots,S_N$ each of which is congruent to rS. Congruent refers to the fact that a set of points $S_i$ can be made identical to rS under translations and/or rotations . Falconer (1985) gives the Cantor set as an example of such a set. S=[0,1], $S \cap [0,1/3]$ and $S \cap [2/3,1]$ are similar to S with a scaling ratio of 1/3; $S \cap [0,1/9]$, $S \cap [2/9,1/3]$, $S \cap [2/3,7/9]$ and $S \cap [8/9,1]$ are similar to S with a scaling ratio of 1/9.

Feder states that a fractal is statistically self-similar if it is the union of N non-overlapping subsets each of which is scaled down by a ratio r from the whole and if each is identical in all statistical respects to rS. The author doesn't make clear the meaning of "identical in all statistical respects".

Possibility of misinterpreting the term fractal is great. This is exemplified by Peterson's words, "the line, square, and cube are also fractals, although mathematically they count as "trivial" cases. The line contains within itself little line segments, the square contains little squares, and the cube little cubes." (Peterson, 1988, 119). A straight line, square and cube are not fractals as their Hausdorff dimension is the same as their topological dimension. Peterson's misinterpretation results from the development of what he calls the Hausdorff dimension, but which would be more appropriately called the similarity dimension. The ideas of self-similar and dimension are connected first by objects of one, two and three dimensions, this is then extended to fractal objects (Mandelbrot, 1983, Feder, 1988, Peterson, 1988, and Peitgen and Saupe 1988). There is nothing wrong with this in terms of the mathematics, but if a distinction has been made between Euclidean and fractals geometries it must remain clear. If fractal geometry is seen as an extension of Euclidean geometry then lines, squares and cubes can be considered as fractals.

The concept of dimension is another idea th needs clarification. "The problem with dimensionality is dual: there is no constancy at the perceptual level nor is there any universally agreed upon constancy at the theoretical level." (Kaye, 1989, 102). This leaves us with an operational definition at the perceptual level and many overlapping definitions at the theoretical level. Kaye tells the story of the changes in (operational) dimension when observing a ball of string. Perceptually, the string has dimension zero when it is far from the observer as it appears to be a point. When the string is brought closer it can be said to have dimension two as it looks like a circle. The dimension of the string is three when it is brought even closer because the observer can see the contours of the ball of string. Mathematically (theoretically) the problem exists as to what definition should be used to determine an object's dimension. To choose a correct theoretical definition of dimension a relatively detailed idea as to the object's properties is needed. Tricot (1981) presents 12 definitions that can be used to determine the dimension of an object.

Similarity dimension can be used to describe the dimension of those objects that are self-similar. The similarity dimension of an entity may be determined by breaking a line up into N pieces each of ratio $r=1/N$ or $Nr^1=1$ (Mandelbrot, 1983, Feder, 1988 and, Peitgen and Saupe 1988). They proceed to show $Nr^2=1$ for a square, $Nr^3=1$ for a cube and $Nr^D=1$ in general. D is deemed the similarity dimension of the set. If each of the N subsets are scaled down by a different scaling ratio $r_n$ then the dimension can be found using $\Sigma r_n{}^D=1$. Falconer (1985) is far less clear when discussing the idea of similarity dimension. Nowhere does he use the term similarity dimension, although his mathematical definition of similarity is unsurpassed in detail within the scope of my readings.

A mood of change is embodied in Stewart's book Does God Play Dice?. It presents a revolutionary view of the world given through fractals and the modern theory of chaotic dynamical systems. Since the idea of fractal seems to be largely based on the notion of dimension and the definitions this paper uses were available in the early nineteen hundreds the idea that a new way to view the world has been found may seem a bit strange.

For example Hurewicz and Wallman developed Hausdorff dimension in Dimension theory, 1943 and used it to find the dimension of the Cantor set, yet they seem to apply little value to it. In their introduction they talk of the interesting concepts involved in establishing entities of -1, 0, 1, 2, ..., n dimensions, but don't mention that an object can have a dimension of non-integral value. In this writer's unbiased opinion, considering objects of non-integral value is far more interesting. This lack of insight is also found in the article What is dimension?. Why then did the topic of fractals not develop earlier? Certainly computer and the graphics that they produce play a major role in the study of fractals. The questions remain as to what this new perception is and how is it useful?

The new perception can be developed in a broader context, that is the theory of chaotic dynamical systems. This thesis will not undertake to explain this theory. Books such as Chaos and Does God Play Dice? can be used to provide a general understanding of the theory.

Stewart (1988) states that the math of fractal geometry can be used "for describing and analyzing the structured irregularity of the natural world". The term "structured irregularity" provides a great deal of insight into how fractals are used to describe the world. Euclidean geometry encounters difficulty in depicting such irregularity.

"Thus what originated as a concept in pure mathematics has found many applications in the sciences. These in turn are a fruitful source of further problems for the mathematician." (Falconer, ix). The pure and applied aspects of mathematics are very complementary with respect to fractals. This point is pedagogically relevant as it can be used as a motivator when students are studying fractals. Those students who do not intend on pursuing a career in pure mathematics can be motivated by the subject's applicative value. Those who wish to study higher level mathematics have a rich topic at their disposal, one full of complexity and beauty.

## 4. A history of fractals

And so we begin a sojourn into a century of mathematics, the 1870's to the 1970's. Our attention will first be fixed on the period 1870 to 1920, the time in which the foundation for the definition of fractal was built, and the link between continuous curves without tangents and dimension was formed. We take a quantum leap when the works of Fatou and Julia are brought into the computer age with the help of Mandelbrot.

Weierstrass and Riemann were the first to convince the mathematics community that it is wrong to assume that a continuous function always has a finite non-zero derivative, except for a finite number of isolated points. The first counter example was presented by Riemann in 1861.

The example was the summation of the series $\sin(p^2x)/p^2$; unfortunately, his demonstration is unknown to us. In 1872, Weierstrass presented $\sum b^n\cos(a^n\pi x)$ as another example of a function that doesn't possess a derivative (finite or indefinite) for any value of x. (Note: a is an odd, whole number, and $ab>1+3\pi/2$). Hardy (1916), showed that $0<b<1$ and $ab>1$ were sufficient for the function to have no finite derivative, at any point (Chabert, 1990).

This revelation was not to be accepted without apprehension, "Hermite wrote in 1893 in a letter to Stieljes:

> I divert myself with fright and horror of lamentable wound of the function without derivative.

or again, Poincarre (1890) comments:

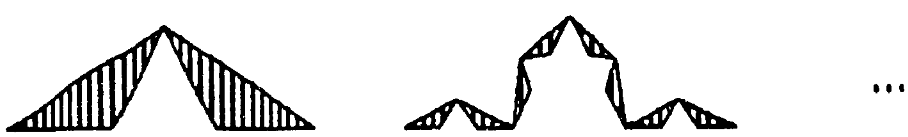
> In the past when we invented a new function it was for practical purpose, but today, we invent purposefully to prove wrong our fathers, and that is all we get out of it."
> (Chabert, 1990)

The work of Bolzano and von Koch brought the analytic works of Riemann and Weierstrass to that of visual geometry.

Bolzano was the first to develop a continuous curve without derivative. The process consists of building a succession of polygonal lines, replacing at each stage all segments of the polygonal lines by polygonal lines, formed by four segments having the same extremity as the initial segments. The extremities of the segments (a,p) and (b,q), where a≠b and p≠q, is replaced by four segments joining the points (a,p); ((5a+3b)/8, (3p+5q)/8); ((a+b/2, (p+q)/2); ((a+7b)/8, (-p+9q)/8); (b,q). Unfortunately, Bolzano's work was not discovered until 1921 (Chabert, 1990). Thus, von Koch was left with the task of bringing Weierstrass' work to a more intuitive (visual) form.

The triadic von Koch curve allows one to perceive the idea of a curve not having a tangent, and is commonly used to bring together the notions of self-similarity and dimension. The curve is obtained through the process shown below in figure 2.

Cesaro (1905), considered the von Kocn curve with respect to area, as seen in figure 3. The area goes to zero.

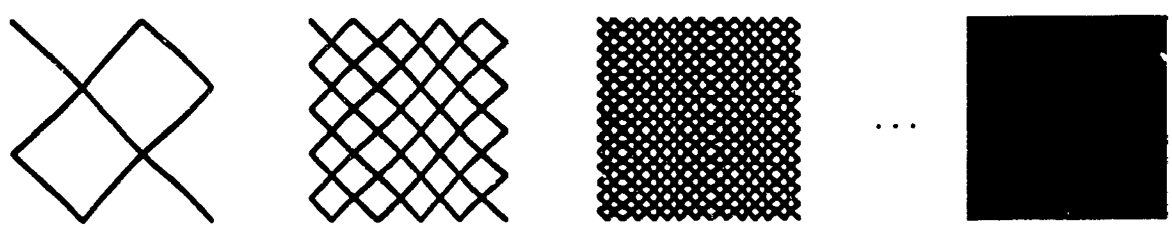


Before finding a link between continuous curves and dimension, I will explore what is meant by dimension.

All the objects we encounter physically are 3-dimensional. However, paper is considered to approach that of a 2-dimensional object (a surface) and string represents our notion of 1-dimensional entities (lines). (Menger, 1943).

The question to be answered is (mathematically) "what is the difference between objects of different dimensions?" (Menger, 1943). Initially, it was thought the difference lay in the number of points. That is, a line has fewer points than a plane, which in turn, has fewer points than a solid. If two sets are to have the same number of elements, then a 1 to 1 correspondence can be established between the elements. Cantor set up such a correspondence between the line, surface and solid, thus proving they had the same number of elements.

The next condition was that of continuity. Maybe a line could be traversed by a continuously moving point, but could shapes of higher dimension? Peano found that a point could traverse a surface. An example is shown in Figure 4.



Menger (1943) defines dimension inductively as follows :

> A set of points of our space is at most n-dimensional if each point in S lies in arbitrarily small neighbourhoods whose boundaries have at most (n -1) - dimensional intersections with S. The set S is n-dimensional if it is at most n-dimensional but not at most (n-1)-dimensional. That S is not at most (n-1)-dimensional means that S contains at least one point at which S is at least n-dimensional , that is to say, a point which does not lie in arbitrarily small neighbourhoods whose boundaries have at most (n-2)-dimensional intersections with S; the boundaries of all sufficiently small neighbourhoods of such a point have at least (n-1)-dimensional intersection with S. The vacuous set, called -1-dimensional,

is the starting point for this recursive definition.
Such a definition is topological in nature and is consistent with the writing of Hurewicz and Wallman, Dimension Theory, 1941.

In 1919, Hausdorff was able to calculate "d-dimensional measures, where $d \geq 0$ was not an integer " (Barcellos, 1984). Besicovitch found that this could be used to find the dimension of various sets, thus the name Hausdorff-Besicovitch dimension.

For certain (self-similar) sets a simplified version of Hausdorff dimension can be used. This is known as similarity dimension. Mandelbrot developed similarity dimension when studying Brownian motion.

The work of Mandelbrot (1980, 1982) and Peitgen et al (1986, 1988) produced many visually appealing images. These images are what have come to be known as fractals. Their research is based largely on the work of Fatou (1906,1919/1920) and Julia (1918).

It is interesting to note that most of these beautiful images recently obtained are found in the theoretical work of Julia. He studied the behaviour of the iterations of the series $z_n = R^n(z_0)$ where $z_0$ is a complex number and R, a rational function of degree > 1. These fractals are concerned with the invariants of non-linear transformations as opposed to the self-similar fractals (invariant by linear transformation) (Chabert, 1990).

Fractals have developed as a result of the analytic work of Weierstrass and Riemann, the geometry of Bolzano, von Koch and Peano, and the theory of Cantor, Julia and Fatou. Mandelbrot was able to unify these works. He grouped the notions of a continuous curve without tangent and dimension to obtain a new means of describing natural and mathematical shapes.

# CHAPTER 2 : The questionnaire and the pre-experiment analysis

This chapter starts with a presentation of the general theoretical framework. The framework comprises a theory of learning, and a theory of instruction.

The framework is then developed for the particular case of the notion of fractal. It consists of a preliminary epistemological analysis of the concept of fractal. The analysis is based on results of a questionnaire administered to a group of grade 12 students, on the one hand, and on the history of fractals, on the other. The latter analysis leads to an identification of a certain number of acts of understanding and obstacles pertaining to the notion of fractals.

Analysis of the results of the questionnaire also helps to outline the prerequisites that, a priori, seem necessary for the learning of fractals (by a grade 12 student). There are a number of prerequisites to such learning whether numerical, algebraic or logical.

## 1. A theory of learning

In developing an appropriate problem to be researched, a considerable amount of thought is evoked. The problem should be relevant to the educational setting, which for the purpose of this paper is high school, and the problem must be consistent with the writer's philosophical ideals. Once this consistency is met, the next step is to develop or adopt a learning theory. This learning theory must be congruent with a theory of instruction or teaching.

As philosophy is developed through activity, so is learning. Kneller (1971) views philosophy as an activity which encompasses speculation, prescription and analysis. To establish theories that encompass as much of life as possible constitutes a speculative activity. The prescriptive activity considers what is worthwhile. To clarify is to be analytic. These three descriptors provide a framework within which to develop a learning theory.

The theory of learning we adopt in our research focusses on the discontinuities rather than on the continuous parts of the process. We admit that learning knows long periods of small steady progress, but we also believe that important cognitive events have the character of "leaps". These leaps result in a complete change of a way of knowing, of focus. Moving from the Euclidean perception of dimension to that of non-integral dimension is not easy; it must be done at a significant cognitive depth and therefore it demands the student go through a number of such discontinuities.

This philosophy of the learning process seems to be shared by quite a few philosophers, psychologists, mathematicians and mathematics educators. Discontinuities are crucial in Bachelard's idea of progress of scientific thought through overcoming of epistemological obstacles (Bachelard, 1938); Kuhn's scientific revolutions (1962) and Piaget's periods of disequilibrium mark exactly the big jumps in the history of human learning. Mathematician Willem Kuyk developed a cusp catastrophy model of mathematical intellectual concentration and discovery (1982): "In mathematical learning jump features are prominent: the sudden recognition of a pattern in problem solving, but also a discovery that certain features fit into a comprehensive framework" .

Byers (FLM 4, 1) applies and develops Piaget's idea of equilibrium of cognitive structures in the context of learning mathematics at the tertiary level. He says:

> Learning can be broken down into two stages. In the first a previously held equilibrium is shown to be inadequate and is broken down. In the second a new state of equilibrium is established. New structures only emerge from awareness of conflict within the cognitive system. One role of the teacher is to provide experiences that promote cognitive conflict - experiences that disequilibrate the system. Without the sustained tension generated by these conflicts there is no hope for students to break out of the equilibrium within which they are trapped (..) The second stage of learning involves the establishment of a higher order equilibrium, thereby resolving the tension generated by a particular dilemrna.

This tension needs to be managed, not eliminated, as Byers states:

> The point which is relevant for us here is that the attempt to avoid the tension which arises naturally in a valid teaching-learning environment often short-circuits the educational process as a whole. There is no painless road to learning. The true educational task consists of managing, not eliminating, tension.

One of the fundamental didactical questions is: what does it mean to understand (a particular notion)? This question has been approached mainly by distinguishing kinds, modes and levels of understanding: apprehension versus comprehension (Dewey, 1910, How we think?); visualization, analysis, informal deduction, deduction and rigour as levels of geometric understanding (van Hiele, 1957, 1959); instrumental, relational, logical and symbolic modes of understanding mathematics on two levels: intuitive and reflective (Skemp, 1978; Herscovics & Bergeron, 1989); the level of intuitive understanding, logico-physical procedural, and logico-physical abstraction, and the level of logico-mathematical procedural, logico-mathematical abstraction and formalization (Bergeron-Herscovics, 1988).

All these approaches focus their attention rather on kinds of ways of knowing rather than on intellectual and emotional acts that make one change one's way of knowing.

The latter point of view is taken by Sierpinska who proposes to define an understanding of a mathematical notion by exactly those intellectual acts which fundamentally change one's ways of knowing, i.e. by the "jumps".

These jumps can be seen in two, complementary, ways. If one looks back, on the old ways of knowing, one sees difficulties and obstacles to the new way of knowing. Looking forward focusses on the new way of knowing.

The first image is called an act of overcoming a difficulty or obstacle. The second - an act of understanding. These images are complementary and neither can be omitted in describing what it means to understand a particular mathematical notion.

Particularly important kind of obstacles are epistemological obstacles, as

> ...they seem to belong to the meaning of the concepts themselves, they are not just results of particular ways of teaching these, they are not idiosyncratic, not something that occurs in a person or two. They are common in the frame of some culture, whether present or past and thus seem to be the

> most <objective> obstacles to a new way of knowing. (Sierpinska, 1991)

Things that function as epistemological obstacles in our ways of knowing very often have the status of deep rooted beliefs, convictions concerning such fundamental categories as space, time, number, cause; or - of philosophical attitudes towards the nature of knowledge (scientific knowledge and mathematical knowledge, in particular); or they are some unconscious schemes of thought which limit our view of a problem and ways in which we might approach its solution.

Acts of understanding are distinguished through four categories: identification, discrimination, generalization and synthesis. Identification (of an object amongst other objects) is related to acts of understanding that pertain to a concept being seen "arising" from the background into the forefront. This is the case when a person first perceives that an object is worthy of study. the act that occurs when significant differences and relevant properties are noticed is referred to as discrimination (between two objects). Generalization occurs when possible extensions to an idea are illuminated. Irrelevant features become visible and new interpretations are perceived under this category. The fourth category, synthesis, is the perception of links between ideas; as a result new organisations of properties and relations are developed.

One important feature of the approach outlined above is the way it views the role of mistakes. It is believed that this role is positive and that by trying to shield students from making mistakes we, as educators, do them a disservice. If mistakes are used appropriately they can provide a deepened awareness and an internalization of the properties relevant to the concept.

**2. A theory of instruction**

A theory of instruction satisfies two functions. The theory is prescriptive in that it sets forth rules for a means of achieving understanding of a mathematical concept. Secondly, a theory of instruction is a normative theory in that "it sets up criteria and states conditions for meeting them" (Bruner, 1966, 40) Thus, the learning theory, which is descriptive should be congruent to the prescriptive nature of the instruction theory. Bruner (1966) developed an instruction theory with four features, the theory should specify:

1. the experiences which most effectively implant in the individual a predisposition toward learning,
2. the ways in which a body of knowledge should be structured,
3. the most effective sequences in which to present the materials to be learned, and
4. the nature and pacing of rewards and punishment in the process of learning and teaching.

As the research described in the thesis will take place in a regular school setting and over a relatively short period of time, the first and fourth functions of the instruction will not be considered. The structure in which the knowledge is presented affects the ability of the learner to understand said knowledge. The way that knowledge is presented, known as the economy of the representation is important. The knowledge may be presented through actions, images or symbols. The economy with which each one of the three representations is presented may be varied. Mathematics lends itself to all three representations. The economy of such representations can be increased as a greater understanding of the concept evolves.

The sequencing of material is important if understanding of a concept is to take place.

Our choices on the level of sequencing of teaching material have been guided by the van Hieles model of instruction, congruent with their model of levels of geometric understanding.

The van Hiele levels of geometric understanding were formulated by Dina van Hiele-Geldof and Pierre M. van Hiele in their dissertations in 1957 and 1959 respectively. The model was first composed of five levels of understanding: visualization, analysis, informal deduction, deduction and rigor.

The first level or level 0 as it is referred has the student at a level of understanding such that he views the geometric constructs as a whole rather than seeing the attributes of the parts. Recognition takes place through physical appearance and not properties.

Level 1 has the student characterizing the figures and conceptualizing classes of shapes. The student is unable to find relations between properties, and between shapes and definitions aren't yet understood.

In level 2 the relations between and within objects is understood. Class inclusion is understood and definitions have meaning. Deductions are based on empirical evidence and the student can't construct a proof from an unfamiliar premise.

The fourth level sees the student constructing proofs and relating theorems, definitions and proofs. The converse of a statement is understood.

The van Hieles state that the final level is rarely reached. At this level different systems can be compared, for example Euclidean and fractal geometry. Geometry is seen in the abstract.

The current version of the model consists of three levels; visual, descriptive, and theoretical. These three level correspond to the original model as follows; visual- level 1, descriptive- levels 2 & 3, and theoretical- levels 4 & 5.

The students proceed from one level to the next and can't skip a level. The intrinsic object at one level is the object of study at the next level. The appropriateness and accuracy of the linguistic symbols used increases as the students proceed through the level.

Within each level of the van Hiele model a sequence of instruction is to be followed: inquiry, directed orientation, explication, free orientation, and integration. These are referred to as phases. An outline of the phases is presented below:

1. students observe and raise questions regarding the particular objects, the purpose is to have the teacher determine the student's existing knowledge and for the students to understand the direction the study will take,

2. the students are given short tasks to obtain specific responses, gradually the structures characteristic of that level are seen,
3. students express and exchange their views about the structures, the teacher assists the students in the use of accurate language,
4. the students work on tasks involving many steps and which are open-ended in nature, and
5. students review and summarize with the help of the teacher.

The learners are now ready for the next level of understanding.

**3. The questionnaire**

Before implementing a clinical interview or teaching experiment a questionnaire was designed. It was designed to serve two purposes. The questionnaire is to be used as an hypothesis builder regarding the student's concept(s) of dimension and similarity, and through such information determine which, if any, of the students questioned can be used in the study. It was expected that the student's conception of dimension would vary greatly as they had not previously studied dimension in an explicit manner.

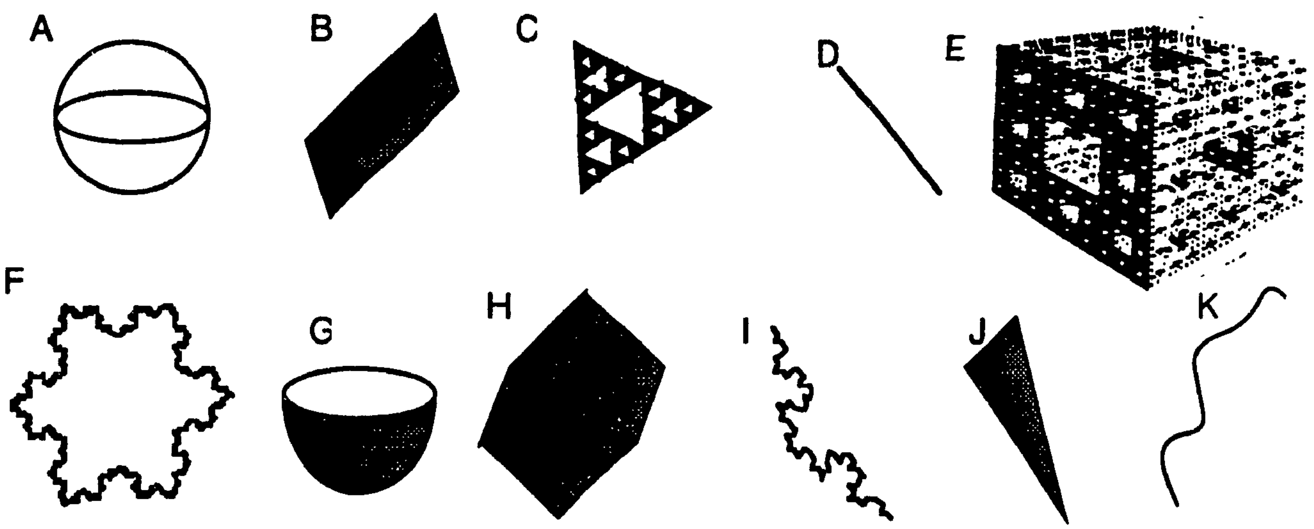
The questionnaire was administered to a class of 27, grade 12 advanced mathematics students ("advanced" classes are for those planning on an university education). The students were from a rural high school in Prescott, Ontario. Their ages ranged from 16 to 18 years.

The presentation to follow will include the questions answered by the students and the aim of such questions. Part A deals with questions relating to the idea of dimension and part B involves the concept of similarity. The figures shown are smaller then those shown to the students.

The Questionnaire:

Part A: Questions concerning dimension

1. Categorize the following objects in terms of their dimension.




Categories (dimension):

2. For each dimension state what the objects have in common.

3. Add an object to each category.

4. a) What is the difference between an object with dimension 1 and an object with dimension 2?

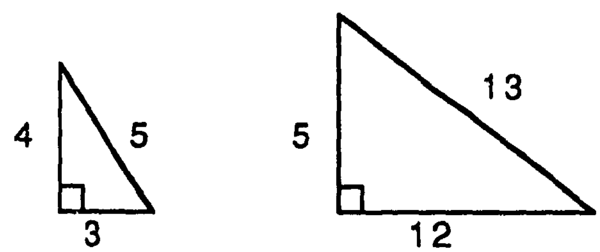
b) What is the difference between an object of dimension 2 and an object of dimension 3?

Part A has four aims: to establish how the student catergorize a variety of shapes with respect to dimension (question 1), to explore the criterion s/he used for categorizing (question 2), to determine if the students could create and draw an object that fits the criterion stated in question2 (question3), and to ascertain if the student can compare characteristics of different classes of objects, in this case the classes are created in terms of dimension (question 4).

Part B on Similarity

5. Are the following pairs of figures similar?

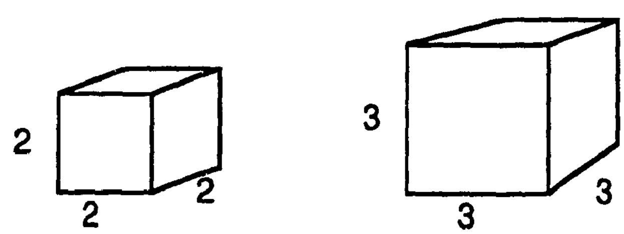
a)




answer:

why?

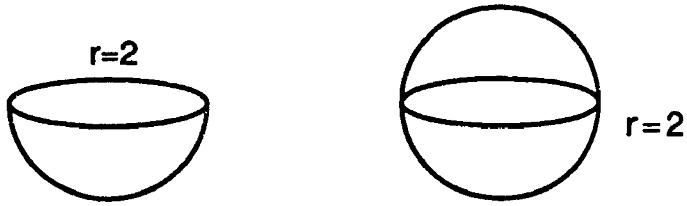
b)




answer:

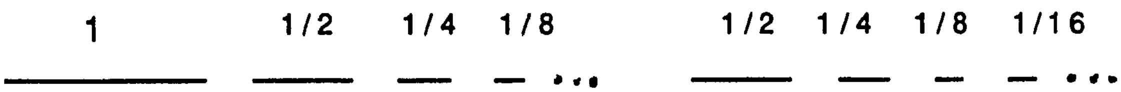
why?

c)




answer:
why?
d)

answer:
why?
e)




answer:
why?

Part B was designed to determine what definition the student attaches to the notion of similarity be it mathematical or otherwise.

**Results**

**Part A**

a) 11 students viewed shapes B, D, F, I, J, and K as one dimensional. Their answers expressed the idea that if the object is flat ("one view"), then it has one dimension,

b) 1 student saw F as one dimensional and G as two dimensional,

c) 10 students viewed D, I and K as one dimensional,

d) 9 students viewed B, C, F and J as two dimensional,

e) 2 students in the class did not express any criterion for determining an objects dimension,

f) 10 students classified shapes on basis of length, width and height,

e.g.: "1 dimensional - length but no width; 2 dimensional: length and width; 3 dimensional - length, width, depth. (...) An object with dimension 1 has only one dimension (a length), an object with dimension 2 has two dimensions (a length, width). An object with 2 dimensions has an area and can be expressed in units squared. An object of dimension 3 has three dimensions (in length, width, depth) and can be expressed in units cubed."

This conception of dimension is very closely related to practical problems of measuring and units. The classification goes along the units, units squared, units cubed.

g) 9 students viewed E, H, and G as three dimensional (A varied as they weren't clear as to the shape).

**Part B**

a) All the students used the idea of proportionality in defining similarity,

b) 1 student determined similarity by "mathematical" and "logical" means. The student in answering 5 a) wrote,

answer: "depends on your point of view".

why? "Mathematically no, because the sides aren't proportional to each other. Logically yes because they are both right < (angled) triangles."

**Comments**

It is of note that many avenues may have been studied upon completion of the questionnaire as seen from the results. For our purposes we will focus on the students who have consider dimension as Euclid presented it. The 17 students who did not view dimension as pertaining to length, width and height were not considered for further study. For example, the 11 students who saw 1-dimensional objects as those that are flat were unsuitable subjects for this study. From the remaining 10, four were selected, the one student who saw F as 1-dimensional and G as 2-dimensional and the student who viewed similarity in terms of mathematics and logic, were two of the four. The remaining two were chosen on the basis of the clarity of their writing. Two females and two males were selected.

## 4. A preliminary epistemological analysis of the concept of fractal.

The analysis presented below will deal with the "Acts of Understanding" (**U.**) necessary for a (grade 12) student to realize the meaning of fractals and the epistemological obstacles (**E.O.**) the student may encounter in constructing such a meaning.

It seems that the very first intellectual act leading to an understanding of the concept of fractal must be:

**U(dim)1**: Identification of dimension as a non-intuitive concept that is in need of study and a precise definition.

This identification can be regarded as overcoming the obstacle:

**E0(dim)1**: (a belief concerning the notion of dimension) There exists one universal concept of dimension.

The above obstacle is rooted in Greek geometry. Our intuitions concerning dimension develop from the seeds of Euclid's classification of geometrical objects as presented in "Elements", Book 3:

"1. A point is that which has no part.

2. A line is breadthless length.

3. The extremities of a line are points .

5. A surface is that which has length and breadth only.

6. The extremities of a surface are lines."

... and Book XI:

"1. A solid is that which has length, breadth and depth.

2. An extremity of a solid is a surface".

This clear idea of classifying geometrical objects into those of dimension one, two and three is already present in Plato's Republic, Book VII: "after plane surfaces ... the right way is next in order after the second dimension to take the third ..., the dimension of cubes and everything that has depth" (Mandelbrot, 1983).

This Euclidean classification has been observed in students' reactions to the questionnaire (see p. 31).

Although the notion of fractional dimension was introduced by Hausdorff in 1919, to this date it is unknown by the general public. Also, I am unaware of many undergraduate mathematics students encountering Hausdorff dimension.

Another feature of the common conception of dimension is that:

**EO(dim)2**: (conception of dimension) Dimension is a unique characteristic of an object.

Some students seemed to believe that the world of geometrical (and maybe also real world) objects can be divided once and for all into three categories: objects of dimension 1, objects of dimension 2 and objects of dimension 3. Some students spoke, in fact of "dimensions" as characteristics that situate an object in space. Those dimensions are: length, width and depth. Each of these characteristics is a "dimension". This reminds one of the notion of basis of a vector space: a non-redundant set, sufficiently large to span the whole space. "One dimension" in students' way of speaking corresponds to "one element basis", "two dimensions" - to "two-element basis" etc.

This idea of an absolute dimension has to be overcome in an act of identification:

**U(dim)2**: identification of dimension as an abstract concept which has to be defined in a way which is specific to the aspects and relationships found in the mathematical domain on which the definition is being applied (figures in Euclidean geometry, topological spaces, physical models such as coastlines and rivers, etc...).

It is hoped that if the student's mathematical knowledge is used to determine a dimension which is non integral s/he will experience a cognitive conflict which will result in the above act of understanding.

"The problem with dimensionality is dual: there is no constancy at the perceptual level nor is there any universally agreed upon constancy at the theoretical level (Kaye, 1989, p. 102). This leaves us with an operational definition at the perceptual level and many overlapping definitions at the theoretical level. Mathematically (theoretically) the problem exists as to what definition should be used to determine an object's dimension. To choose an adequate theoretical definition of dimension a relatively detailed idea as to the object's properties is needed.

A prerequisite for the above act of understanding is, of course, as awareness of the meaning and role of definitions in mathematics. This means overcoming, at least, the obstacle:

**EO(def)3**: (A conception of definition) A definition in mathematics is similar to a description in science; it need not be understood too literally, it is not binding,

The above conception can function as an obstacle in mathematics. Definitions in mathematics are logically binding. In mathematics a mathematical object is nothing more and nothing less than is implied by its definition. If the definition leads to "monsters" which we do not like we can change the definition. But if we do not, we are bound to accept the monsters as belonging to the object defined. However, in physics and other sciences which propose themselves to explain phenomena, it is important to be aware that

"A thing is never what we say it is; it is always something more, or something else; perhaps because what we say is words, and generally what we mean is not words" (Korzybski, quoted by Bohm, 1987) .

**U.(def)3**: discrimination between definitions in mathematics and science; definitions in mathematics are logically binding,

**U.(def)4**: identification of the freedom in mathematics to get rid of tacit assumptions which are traditionally regarded as universal laws and seek the logical implications of these.

For example,

**U(dim)5**: identification of fractional dimension as plausible answer to search for the dimension of a geometrical object.

This act of understanding can be regarded as overcoming the obstacle observed in the students' responses to the questionnaire...

**E0(dim)4**: dimension can only be whole numbers.

(In fact only 3 whole numbers have been taken into account).

The notion of self-similarity has to be built on the notion of similarity but at the same time - against this notion. The discrimination between the two concepts lies not so much in the distinct natures of their definitions as in the application of the definitions. Similarity is applied to two separate objects if the ratios of their corresponding sides are equal. Self-similarity refers to a characteristic that an object may have or not. In this case the object is similar to each of the parts that define the whole. The ratio, $r$ , is such that $0<r<1$ and $r$ does not vary with the part being compared to the object.

**U.(sim)6**: discrimination between similarity and self-similarity.

To work with $r$ one must develop a synthesis:

**U.(sim)7**: synthesis of the practical notions of approximation and scaling in the mathematical concept of self-similarity.

With such synthesis the student can then extend these notions beyond the physical and into the domain of mathematics. These ideas can then be used in developing similarity dimension.

Another common conviction that people seem to share when they are asked to determine the dimension of an object concerns the shape of the object.

**EO(dim)5**: (conception of dimension) Shape is irrelevant in determining dimension.

The shape I was qualified as 2-dimensional not because of the complexity of shape but rather because it seemed thick, having width.

This obstacle is associated with our "Euclidean" intuitions; in this case - with the topological invariance of the dimension of Euclidean spaces. When the definition of dimension is changed to that of "space-filling", shape becomes essential in determining an object's dimension.

The notion of space-filling requires the student to build another discrimination:

**U(fractal)8**: Discrimination between shapes that are smooth and those that are rugged (i.e. have corners everywhere).

...and an identification:

**U(fractal)9**: Identification of "jaggedness" that does not "dissolve" under magnification (non-rectifiable curves).

The concept of fractal is linked to the process upon which it is built. A fractal is constructed using a generator and specific transformations (rotations, dilations and translations). This construction may be, in a student's mind, substituted for the fractal itself:

**E0(fractal)6**: (a conception of fractal) A fractal is not a finished entity, but merely the process of construction or a sequence of objects.

If the iterative process is used as a (tacit) definition of fractal then shapes such as a line segment, the square and the cube are clearly fractals. Fractals would no longer be tied to objects that are nowhere differentiable. The category of fractal would become so broad that its relevance is diminished. If the focus is on the process then the use of r, the scaling ratio, and N, the number of subsets becomes very complicated. Is r found through a comparison of the generator and the nth level (iteration or between the nth and (n-1)th levels? And as N appears to be changing with each iterative process, how do we know which N to select.

Another related obstacle is that which is linked with the difficulty to grasp the meaning of actual infinity, inherent in the notion of fractal:

**E0(fractal)7**: A fractal is the nth approximation (where n is big but not infinite).

Overcoming the above obstacle leads to:

**U(fractal)10**: identification of a class F of geometrical objects which are generated through an infinite iteration of a certain construction.

and ...

**U(fractal)11**: discrimination between an object of class F and the process upon which the object is built.

## 5. Prerequisites for learning fractals and didactic conditions for the teaching of fractals

A priori there seems to be some algebraic, numerical and logical knowledge the absence of which makes the overcoming of the above mentioned obstacles and acts of understanding impossible to occur. Below we list a number of mathematical capabilities that appear to be relevant:

1. Algebraic:
   a) represent relationships symbolically,
   b) interpret formulae, and
   c) manipulate relationships (expression)

2. Numerical:
   a) intuition as to what the solution of an equation like $4=3^d$ will be;
   b) familiarity with logarithms.

3. Logical: capability of understanding and applying a formal definition.

4. Geometrical: Familiarity with at least the elementary geometrical transformations.

In general, a student in grade twelve, advanced mathematics would meet such standards. However, the each student's level of ability in dealing with such prerequisites will vary.

The a priori analysis points to three areas that may be of particular difficulty for the students. First, the students must be made aware that dimension is a non-intuitive concept. This may be accomplished by having the students develop the idea of similarity dimension for the line, square and cube, generalize to get the equation $N=1/r^D$ ($D=\log N/\log(1/r)$), and then apply the equation to the triadic Koch curve.

The next conflict situation may arise when the students have to determine if a shape is self-similar on the basis of a mathematical definition. To do this the student has to use the notions of translation, rotation, and scaling (dilation).

As noted in the a priori analysis the construction of a fractal is not obvious. The distinction between the process of building the object and the object itself, and the relation between the iterative process the definition of fractal are unclear.

These three conflict generating situations will be fundamental in the development of the teaching experiment. Each teaching lesson can be created around one of the conflict generating situations.

# CHAPTER 3: The experiment

The study will consist of two parts: 1) the development and application of a clinical interview, and 2) the development and implementation of a teaching experiment.

The introduction of non-integral dimension will occur in a clinical interview, a one-to-one interview with each of the four preselected students. The interview will function as an intermediary between gaining information on the student's methods of action and the content of their thoughts. This is not purely an assessment type of interview as teaching does take place, its aim is to bring conflict to the surface as opposed to merely hypothesizing as to what may be at the root of a student's problem. This interview is not an end in itself, but lays the foundation for the teaching experiment (it attempts to bring the student to the realization that dimension is a non-intuitive concept in need of study).

## 1. The Clinical Interview

The interview was designed to observe the effect of the students development of the notion of non-integral dimension (through similarity dimension). All four interviews took place in the high school library. The room is sound proof, has three walls made of glass, and is adjacent to another room which is similar in design. The room contained three tables, one video camera (V), one interviewer (I), an observer (O) and the student (S).

The discussions began with an informal talk regarding their answers to the questionnaire. The interview questions were presented in written and graphic form. If the student did not understand the question, the interviewer gave the question orally.

The questions:

1. If we take a straight line of unit length (1) and divide it into N=3 lines, what is the length, r, of each of the lines?

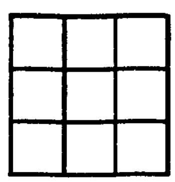
2. How are N and r related? (write N in terms of r, eg. N=...)

Picture:

3. If we take the unit square and divide it into $N=3^2$ squares, what is the length of a side, r, of each of the squares?

4. How are N and r related? (write N in terms of r, eg. N=...)

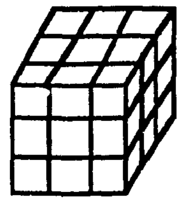
Picture:



5. If we take the unit cube and divide it into $N=3^3$ cubes, What is the length of an edge, r, of each of the cubes?

6. How are N and r related? (write N in terms of r, eg. N=...)

Picture:



These first six questions were used to have the students see the effect or the pertinence the object's dimension has in the context of self-similar objects (they are unaware of the context). All the objects and the objects dimension are known by the students.

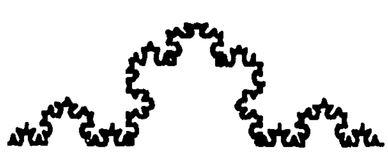
7. Can you see how the object's dimension is related to the equations involving r and N? Explain

The desired answer was $N=1/r^D$ or $r^D=1/N$. If the student did not see this it would been given, however, of the three students who reached this point none had a problem developing the formula.

8. The Koch curve can be subdivided into lengths, r=1/3 of the original size. How many smaller Koch curve, N, do you get?

9. Write N in terms of r.

Picture:



The students were to use the previously establish formula and apply it to this question. Thus, the student would determine that the object's dimension was non-integral. Note that this question's wording is not consistent with that of the other questions and it would have caused less confusion had N been a given as opposed to r. The students answers to the interview questions can be found in Appendix B.

**1.1 Critique of the interview** (as seen by the observer).

The interview was conducted in a library seminar room with walls of clear glass. Any student in the library who wished to glance at what was going on could do so at will.

The presence of a third person and that of the video camera may have affected some of the subjects. To eliminate or at least reduce the potential uneasiness of the students due to these two factors, the observer was seated at a distance from the interview table and the purpose of the camera was explained to the students prior to the interview. The students were told that the observer was there to observe the interview - a statement to which the fourth student responded, "so he is a psychologist." Each session of the interview began with questions regarding the questionnaire so the students would first deal with something familiar.

The subjects were presented with a list of the questions. This may have had two effects: first, the subjects may have thought that they were taking a test and second, it may have contributed to the uneasiness of the subjects. As all the questions were on one page the students may have felt pressured by the amount of work to be done. Having the questions on separate pages and showing the next question only upon completion of the previous question may have been beneficial.

The interviewer's offering of praise may have affected the students. This may have a negative impact if the subject fails to get a question correct as the subject will associate the praises with correct responses.

Use of the word "parts" instead of "lines" in the first question could have made the question clearer. The question were set up in a hierarchical order from simple to complex -- which is positive as a simple question may motivate the subject and prepare him/her for the next question (psychologically). The inductive approach adopted enabled the first three students to obtain the equation involving r, D and N.

### 1.2 Report

Once the reciprocal relation in question 2 was hinted at, Student 1 (Linda) proceeded through the questions with certainty and skill. She developed the expression $4=3^d$ where $1<d<2$. It was only after the interviewer asked "Did you think this ($1<d<2$) was going to be like this?" that a conflict was evident. Linda expressed this conflict in two words, "Oh, No!" Upon seeing the significance of her result, Linda was not uncomfortable with her findings as she said, "It seems possible, because when I looked at it I didn't know if it was 1 or 2, but I didn't know there was any in between dimensions".

Rob had to have the reciprocal relation between N and r given to him for question 2. Setting up similar relationships in questions 4 and 6 came slowly, but he was able to use them in determining the general form $N=1/r^D$ and apply it to the Koch curve. However, Rob could not estimate a value for D, he only knew it was not 1 or 2.

As with the other students Sue had difficulty determining the reciprocal relationship in question 2. Similar to Linda, Sue had little difficulty answering the questions upon finding the relation between r and N in question 2. She determined that $(1/3)^n=(1/4)$ where $1<n<2$. She did not see any significance in her answer until the interviewer brought it to her attention. At this point she said, "How can you have 1.5 dimensions? How can you have length and a little bit of width? Half a width? You can't have a half. So this is really weird. This is wrong somewhere."

Sam had a great deal of trouble focusing on any of the questions tendered and was unable to use any suggestions given by the interviewer.

### 1.3 Analysis

Upon completion of the interview, the two female students, Linda and Sue would appear to have identified dimension as a non-intuitive concept that is in need of further study. They have become aware of and possibly overcome the obstacles related to the existence of a universal concept of dimension, dimension being an invariable characteristic of an object, and that dimension can only be a whole number.

Rob was very slow and cautious when manipulating expressions and seemed nervous when trying to estimate answers (he was unable to estimate the answer to $(1/3)^D=(1/4)$). It will be essential that he is able to work with logarithms if he is to take part in the experiment. Working in the glassed in room seemed to be very distracting to him and he was generally nervous.

Sam seemed the most bothered during the interview. When the conversation began he was quite content to keep talking about what we were doing, but it was difficult to get him to begin the interview. Throughout the interview his thoughts seemed to wander. He tried to find patterns within questions, but not from question to question. This is not surprising as he said the questions reminded him of a test. The interview was ended as the student was becoming too nervous.

Sam was not deemed suitable for the teaching experiment, but the three remaining students will take part.

## 2. Teaching Experiment

This section will state the tasks and questions used in the experiment. The rationale for the outline will be presented during and after the discussion of the tasks and questions (eg. the use of the van Hiele levels and in relation to the epistemological analysis).

In the interview, Rob was able to derive $(1/3)^D=(1/4)$, for the triadic Koch curve, yet was unable to estimate a value for D or even on what interval D may exist. Upon meeting this student during one of his math classes, (to set up a time when we could begin the teaching experiment) he expressed his finding of D through the use of logarithms.

The teaching took place over three lessons. During the course of the three lesson the students were involved in four activities. Each activity contains a specific objective in the development of the concept of fractal (within the van Hiele framework):

Activity 1 (visual) - discernment of characteristics of the object.

Activity 2 (visual) - discernment of shapes which are self-similar, have non-integral dimension and are fractals.

Activity 3 (descriptive) - mathematical definition of self-similarity and use of a new definition of dimension (space-filling ability).

Activity 4 (descriptive) - creation of fractals through an iterative process.

The presentation that follows exhibits the tasks as they appeared in their respective activity. For each task (with the exception of Activity 1) the dialogue between the teacher and one of the students will be presented to enhance the reader's perception of what took place during the teaching experiment.

## Activity 1

**Task 1.** The student is presented with pairs of shapes:

a) the teacher states something that is common to the shapes, and asks the student to state something that is not common to both shapes,

b) the roles are reversed for the next pair of shapes.

The student is encouraged to manipulate the shapes presented, and informal vocabulary isn't corrected. In the discussion regarding the four pairs of figures the teacher names two characteristics of the objects (only if the student has not already done so): dimension & self-similarity (new vocabulary isn't used).

Pair 1 - Figures A & B



Figure A

Figure B

Pair 2 - Figures C & D

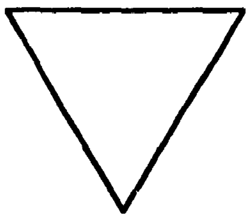


Figure C

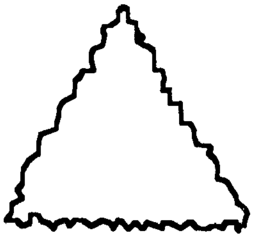


Figure D

Pair 3 - Figures G & H

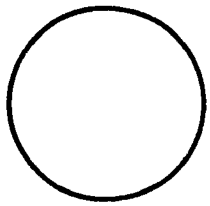


Figure G

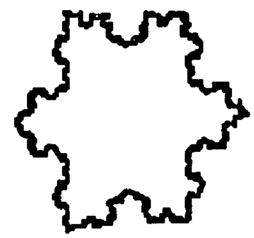


Figure H

Pair 4 - Figure J & K

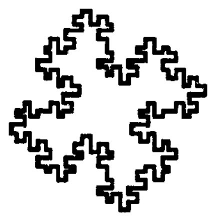


Figure J

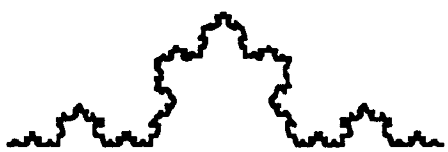


Figure K

As was expected, the students were very vague in their stating of the characteristics of the objects. It would seem that they may never have tried to determine the properties of such shapes without being given a very narrow context of consideration.

Linda didn't seem to see much in the pictures offered. The prominent feature that she referred to was that of closed or open (this maybe as a result of her finding that closed figures (eg. figure J) can be 1-D, a point discussed upon completion of her questionnaire).

Rob was most intrigued by the exploration of the characteristics of the objects. He attempted to compare the shapes with respect to area perimeter, and ruggedness. He attached a physical quality on these "new" shapes, they truly seemed real to him. This relates to an interesting comment he made in relation to figures A & B. He believed that figure A may look like B under magnification.

Sue was the most shocked by the discovery of non-integral dimension and never offered the dimension as a possible characteristic of an object. As was the case with Linda, the shape being closed seemed significant. She saw the area of figure H building toward that of figure G .

## Activity 2.

The student is asked to sort shapes into piles. The teacher starts each pile with one shape and then asks the student to place the remaining shapes into the piles. The first piles are arranged in terms of fractal and non-fractal shapes, the second, in terms of dimension, and the final arrangement was made with respect to self-similarity . The shapes used in activity two are:



Figure A

Figure B

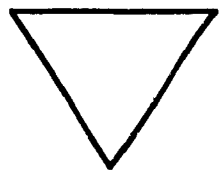


Figure C

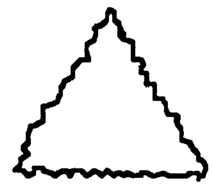


Figure D

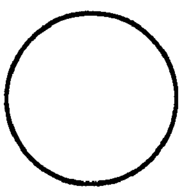


Figure G

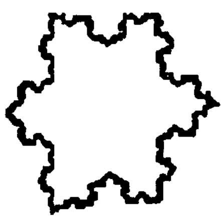


Figure H

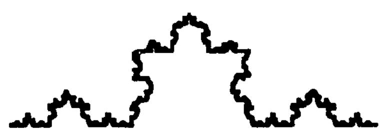


Figure K

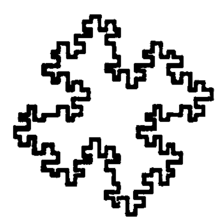


Figure J

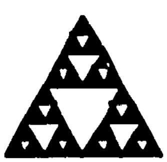


Figure L

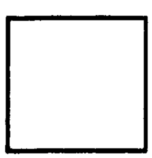


Figure M

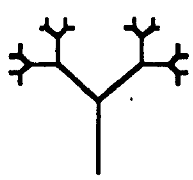


Figure N

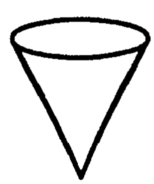


Figure Q

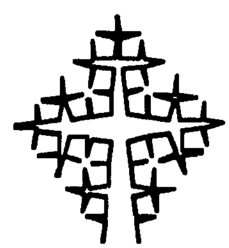


Figure R

Note: each shape was presented on a separate page.

Teacher intervention takes the form of interjections upon the students sorting of a shape into a certain pile.

**Task 2**: fractal and non-fractal shapes

The teacher explains that a fractal is an object that is self-similar and has dimension that is, to quote Sue, "in between" and places figure A in one pile and figure K in the other. The discussion with Rob (S2) went as follows:

S2 O.k. I'll start with the easy ones!

I O.k. that's good enough

S2 I would probably put that one (the square) right there (non-fractal) because it's got the straight lines right here. This is basically a smooth line and you could.. right down there are smooth lines in there too, somewhere (sets aside figure B), but this one is easier to identify (puts figure H in fractal pile). This one is a whole bunch of triangles within triangles (Sierpinski triangle) so you put this one right here since you've got a straight line; you could also put it over here, but because it's triangles and triangles and they keep getting smaller and smaller and smaller right there, inside there (decides to put it in the fractal pile).

I O.k.

S2 That one (another version of the Koch curve) looks basically like this right here (figure K). It looks similar to that at least in the idea that you've got things coming off it. Isosceles triangles, are they?

I That's what they look like....

S2 This right here (fractal tree). This one here does basically the same thing only it has a little line and fork. I take it these are supposed to be the same.

I That's right

S That would probably go on there again. That (figure J) I would probably put there (non-fractal) because its got straight lines and they don't seem to get any smaller from that picture. That one (figure D) I would probably put over there (non-fractal) because these really don't have much conformity to them, its a random thing so its not a pattern like these are , like these tend to be. That one (cone) I would probably put over there because, again, straight lines. This (figure B) would go right here because it looks like (figure D) basically the same thing only smaller. Now this (figure G) is filled in so I would tend to put it in this one right here because since it is filled in it is more like representing an actual object like these represent...

Most of the difficulties the students encountered related to the fact that they could not determine if the process of "bumps on bumps on bumps" stopped. For example it was unclear if the Sierpinski triangle (figure L) shown was a fractal or just a triangle with a few holes in it.

A table will be presented representing each students initial sorting of the figures. This table will follow the comments regarding the students work.

Linda used the definition of fractal presented until faced with figures B & D at which time she sorted on the basis of "old" vs "new". She was uncomfortable with such a decision and looked to me for reassurance.

| Fractal | B, D, H, K, J, N, R |
| --- | --- |
| Non-fractal | A, C, G, L, Q |

Rob had many approaches to classifying such figures, through (self) similarity, straight lines and random lines. He used these aspects quite effectively until figure G, at this point he switched the notion of the shape representing "real" vs "unreal" objects, thus he placed G in the fractal pile.

| Fractal | B, D, H, K, J, N, R, G |
| --- | --- |
| Non-fractal | A, C, L, Q |

Sue classified non-fractal shapes as those with straight or crooked (smooth) lines and fractals as the shapes with "bumps on bumps on bumps..". For Sue shapes with "bumps on bumps..." had dimension that was in between 1 and 2.

| Fractal | B, D, H, K, J, N, R |
| --- | --- |
| Non-fractal | A, C, L, Q, G |

**Task 3**: dimension of 1, 1<D<2, and 2

The teacher explains to the student the new idea of dimension in terms of space filling and uses figures A, K and G as pile generators.

I I want you to look at these shapes and tell me the ones that are in between -- and which will be one or two (dimensions).

S3 What about this one (figure J)? You have to wonder if it has bumps on the bumps.

I That actually looks like it doesn't have bumps on the bumps... it stops. If that is the way it looks, you have to go with what you perceive.

S3 (Asks about figure H)

I I tell you it has bumps on the bumps. (she places it in the in between pile)

S3 This is tricky (the square); that or that (1or 2-dimensional piles), depending on whether there is something in there.

I There is nothing in there because it is not shaded.

S3 So anything that is not shaded...

I Right...

S3 Now, is this the same as that; triangles in the middle of triangles in the middle of triangles?

I No, say you stop with what you have. So, what that actually is... if you are given a triangle, you take out certain parts, but you are still left with certain parts of the triangle..... Why are you putting it there?

S3 Well it appears to have length and width.

I .. And if we took out more of them, we would end up with just lines, and we'll put it here ($1<D<2$).

S3 Ya.

Most of the problems are as a result of the visual interpretation of the picture, as was the case in task 2. The students viewed figures L & J as "stopping" and in turn were not initially placed in pile 1<D<2. As with task 2 a table showing how each of the figures was sorted is presented.

Linda believed that figure J didn't have the characteristic 1<D<2 as "the lines stop", thus it is a line of dimension 1.

| Dim. | Figures |
|---|---|
| 1 | A, B, C, J, M |
| 1<D<2 | H, K, N, R |
| 2 | G, L, Q |

Although Rob was able to sort the figures into the appropriate piles he stated that the shapes where 1<D<2, actually represented 3-D shapes.

| Dim. | Figures |
|---|---|
| 1 | A, B, C, M |
| 1<D<2 | H, K, N, R, J |
| 2 | G, L |
| 3 | Q |

Sue uses the "bumps on bumps.." to categorize shapes where 1<D<2 , and with all other shapes she uses length, width and height as the criterion.

| Dim. | Figures |
|---|---|
| 1 | A, B, C, J, M |
| 1<D<2 | H, K, N, R |
| 2 | G, L, Q |

**Task 4:** based on property of self-similarity

The teacher explains the idea of scaling the figure down and then building it back up with the smaller versions (the parts look like the whole shape). The piles are started with figures A & K in one and figure G in the other.

S2 Because these parts look like self.

I Right.

S2 You can't really do that with this (figure G), you could take little circles out here but you have little spaces in between that are left

I Right

S2 Like factories when you see them they have their little papers and they're punched out.

I That's right.

S2 Now this one (figure J) you could. The lines look smaller to each other even though it is a computer drawing and you could... If it was a drawing you could tell if these lines were similar, but the lines don't look like the whole thing, but this little section here looks like that right there.

I Yes.

S2 So these little things probably could go right there. That's random lines so it really doesn't look like the whole thing....

All the students found this the easiest task of the three. Figure J caused some difficulty as they found it hard to find smaller versions of the whole shape within the figure. They were unable to determine how one might go about breaking up such a figure into smaller, and similar pieces.

The lesson ended with a summary of what had been learned. An explanation of the two notions that comprise the fractals which were presented was given. Self-similarity and the ruggedness of the fractals were described.

**General Comments**

The focus of this lesson was the intuitive comments of the student and the purpose was to introduce a direction in their thinking. Initially, the vocabulary presented was informal and misinterpretations were left uncorrected. The vocabulary became more formalized as the lesson progressed, but this was still difficult as the names of many of the shapes were unknown to the students. Presenting the lesson in forty minutes may have been too lengthy to remain informal mathematically. This time frame may be more appropriate for students of lesser ability.

## Activity 3

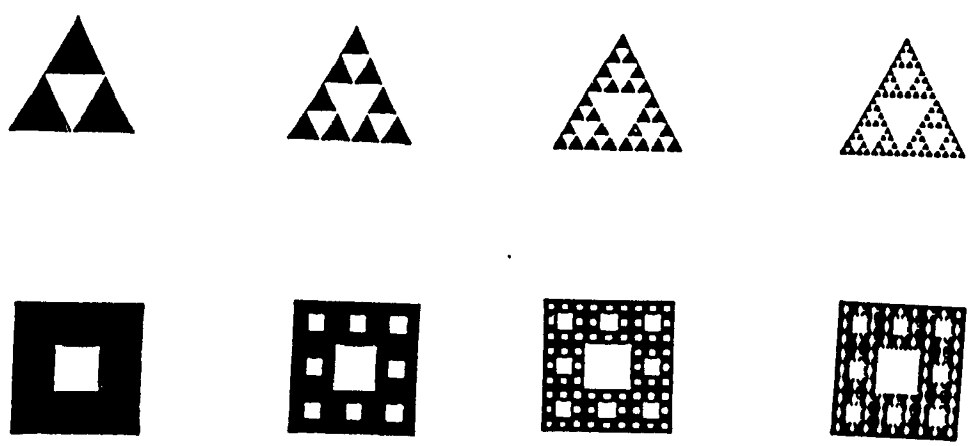
The teacher defines self-similarity and illustrates the definition by means of the triadic Koch curve. The following definition was shown and explained to the student:

Self-Similar

A set is self-similar if it can be formed by the union of N non-overlapping subsets of S, $S_1$, $S_2$, .., $S_N$. These subsets are all congruent to r(S), $o<r<1$. Congruent meaning, they, the subsets $S_i$, can be formed by r(S) through rotations and translations.

**Task 5**

The student is to use the definition of self-similarity in determining if the Sierpinski triangle and Sierpinski carpet are self-similar (see figures below),



a) The student applies the definition to determine whether the Sierpinski triangle is self-similar.

I ... Tell me how you check if this is self-similar...

S1 Starting with this?

I We'll say that that's the one.

S1 Okay, r=1/4.

I Are you sure about that?

S1 No.

I It takes four to make up the whole thing , right? The factor that we're shrinking it down by is actually the length of this side in comparison to the whole side. So by a factor of?

S1 Factor of a half (she then builds the shape through translations along the x and y axis).

b) The construction of the Sierpinski gasket (SG) is explained and the student applies the definition to determine if the Sierpinski gasket is self-similar.

S2 Let me see. One of these little sections right here looks like its similar to the whole thing.

I O.k. and with that can we find the scaling ratio, which is?

S2 Which is 1/3. Because this is one right here.

I So I take the picture.. I scale it down by 1/3 and I'm right here. See if you can form the whole picture again through rotations and translations.

S2 You can translate this one along the x twice here to get the middle one and here to get the far one. You can translate this one along the y here and here. From these here you can translate them, you can translate this over here in this corner up the y to get this over here and you can translate this you across the x to get that. You've got that space up there now.

All the students considered the scaling ratio as a percentage of the original shape. For example the Sierpinski triangle (ST) is made up of three smaller versions each consisting of 1/4 of the area of the original triangle, thus r=1/4. After this error was explained to the student through the act of "shrinking" the problem did not reoccur. Linda was the only student to see that the Slerpinski triangle could be formed through translations. Rob and Sue used rotations about one of the vertices of the triangle.

**Task 6**

a) The student estimates the relative dimension (eg. shape 1 < shape 2) of Sierpinski triangle and Sierpinski carpet on the basis of percentage of their space being filled (density).

All 3 students believed the Sierpinski carpet's dimension was bigger because it was "more black".

b) The formula $((r)^D=1/N$ or $D=\log N/\log(1/r))$ developed in the clinical interview is applied to determine the value of D for the Sierpinski triangle and Sierpinski carpet.

The three students were able to calculate the dimension of the two figures. Sue used the form $r^D=1/N$ as she had yet to study logarithms.

All the students guessed that the Sierpinski carpet (SC) had a greater dimension (with respect to space-filling), but as the triangle had less total area than the square this question may have been biased. Each student was able to determined the dimension of both sets.

A summary of the lesson was then undertaken. The definition of self-similarity was restated and an explanation as to when the formula for similarity dimension can be used was provided. The idea that an object is a fractal if it satisfies the mathematical definition of self-similarity and it's similarity dimension, D, is non-integral.

## Activity 4

The Sierpinski triangle and carpet are shown and their construction is reviewed. As well, the construction of the triadic Koch curve is reiterated.

**Task 7**

The student is then asked to build a fractal using a scaling ratio of r=1/4 which is similar to the triadic Koch curve, but through rotations of $90^0$.

I .. One thing that we will know is that the ratio is 1/4 and our rotations will be $90^0$. Now , the thing about fractals is that you're not going to be able to just draw the whole thing in one step because this is acually five steps ( picture of a triadic Koch curve). First I'll call it zero, not first, is just the straight line. You divide it up in four, that's the first part...

S2 16 cm

I No problem

S2 There.

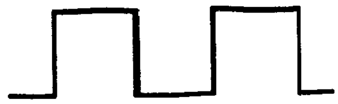
I O.k., that's level zero. Now , the next step is .. and you can figure out with that straight line and divide it into ...

S2 Four.

I Right.

S2 There's 1, 2, 3, 4.

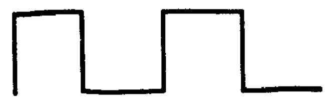
I Great. Now what do you do? Tell me what you are doing. Your dividing it..

S2 Like this! 

I Now are all of these lines equal, 1/4 of the original?

S2 No.

I That's a problem. They have to be. You don't actually have to do much to modify the lengths. Any ideas how we can modify it so they are all the same?

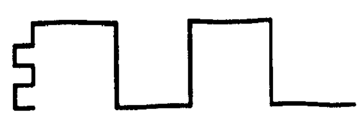
S2 Raise that and put it over there. 

I Great. That's our first step. What would the next level be?.. Tell me what your doing.

S2 I'm doing it wrong!

I Tell me why your saying that.

S2 Because I'm not drawing 1/4 of the bar, I'm taking half of the line.

He then draws 

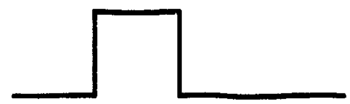
It is pointed out that this could be done in many different ways.

The idea of using one shape to build another seemed foreign to the students and this complicated by the understanding of how one should use the information r=1/4 and rotations of $\pm 90^{0}$. After the generator was created, iterating the procedure was less difficult.

Sue became quite enthusiastic upon realizing the many ways one might create such generators.

**Task 8**

At this point, the student is asked to determine if the new shape is self-similar. Sue created the shape with a generator of:



I Now, next question. Is it self-similar?

S3 We scale it down by a factor of whatever.

I A factor of?

S3 1/4.. And then can we use that to make the shape through rotations and translations?

I Is that the case here?

S3 Ya. (She walks through it with her hands)

I Tell me which ones are strictly translations.

S3 These two (two end sections).

I This one? (first vertical section)

S3 Rotation.. Rotate it that $90^0$ and then translate.

I And this one? (third section (horizontal section))

S3 You can slide it up, then over and over again.....

I This one? (second vertical section)

S3 Translate it that way and then rotate it $-90^0$.

The student then calculates the dimension.

Upon completing stages 0, 1, and 2 Linda no longer knew how to determine N as she saw r as being 1/4 at each stage, yet N varied. This problem was resolved when began to focus on r within a particular level as opposed to between levels. Also, the use of different values of r and N were used to calculate the dimension of the set.

Rob's problem was somewhat different. He realized that he could use more than one value for r, for example r=1/4, r=1/16, .... When he discovered this he became unsure of which one he should use. He appeared to have reconciled this problem when he found that it did not matter what r he chose (of the form $r=(1/4)^n$, n=1,2,3,4...) the dimension would be the same.

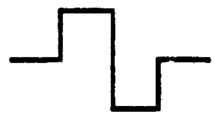
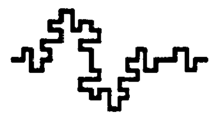
Sue felt the most comfortable of the three with this new approach and confident that she could chose any relevant value for r and determine the set's dimension.

### 3. Evaluation

The evaluation has 4 parts, which consist of questions representing the aim set out during each lesson. For the four questions the dialogue with each of the students is presented.

1. Determine if an object is a fractal...

student was shown 2 objects, the quadratic Koch curve and the generator of the triadic Koch curve, and then was asked if either was a fractal and how she ascertained her answer. The shapes shown were:

 &   ...(three stages)

2. A shape's relative dimension...
student was shown the quadratic and triadic Koch curves and asked if the dimension of the quadratic Koch curve (qKc) had a greater, smaller or the same dimension as compared to the triadic Koch curve (tKc).

3. A shape's dimension...
student was asked to determine the dimension of the quadratic Koch curve.

4. The generation of the triadic Cantor set was shown and explained. The questions: a) what is a possible range for its dimension? and b) calculate its dimension?, were asked of the student.

**Linda**

1. She acturately determined the shape that was a fractal and that which was not a fractal. In verbalizing the properties that a fractal has she only mentioned the characteristic of self-similarity.

I .. I start with this generator. There's stage two and there's three. Do you have any questions. We go on and on with this. Is this a fractal?

S1 Yes.

I Why?

S1 Because you can reduce it.

I And this?

S1 Oh, no. Because if you reduce it you don't build the same thing back up.

2. She believes that the quadratic Koch curve has greater dimension on the basis of the complexity of its structure, "more building on it".

I .. What do you think is going to have the greater dimension, just looking at these two. These are both in the third stage.

S1 This one (qKc).

I O.k., Why?

S1 Oh, I don't know how to say, with the triangle and the square, it has more building on it.

3. The formula for dimension is applied correctly and the dimension calculated.

S1 Ya. Should I say how many lines there are? I could do that and then repeat it?

I Well, if you have to.

S1 Oh, no, because its the same reduced. So that would be.. (tries to figure it out "in her head")

I Why don't you write it down.

S1 1.5?

I If you want to just write it out D= that's okay, just so I know what you are doing.

S1 Okay, D=log8/log4.

4. She is unable to hypothesize a possible range of dimension for the triadic cantor set. As in lesson 3 she became confused with the idea that r was constant and N was changing, but she quickly regrouped her thoughts and determined the dimension of the set.

a) I That's good. Now, I'm going to tell you how to start a certain fractal. You start out with the line segment and we're going to be scaling it by 1/3. Just so you know which to select.

S1 (she draws the cantor set)...

I You can see where its going.

S1 An infinite number of points.

I Ok. What do you think it's dimension is going to be? Give me a ballpark figure.

S1 It would be 2? I don't know. Okay, can I figure it out with the formulae? I don't have an N.

b) I Tell me what you do know about your question.

S1 I know that r=1/3, but N changes between stages.

I Yes, it always does. When you used r=1/3 for the first step...

S1 N is 3. Wait a minute.

I First step, N is what?

S1 Then N is 2, so it would be less than 1.

I Does it make sense?

S1 Yes, because there can be dimensions between 1 and 2, and less than 1.

From the evaluation the teaching would be considered successful in meeting the aims it set out to fulfill.

**Rob**

1. He uses primarily visual cues in establishing whether or not an object is a fractal and in doing so encounters difficulty in determining if the quadratic Koch curve is self-similar (and a fractal).

I First question. I'll show you one object. This was formed by a ratio of 1/4.

S2 That was formed by ratios of 1/3 (looks at the other figure).

I I didn't say. This is the object I'm looking at. Here I'm talking about as far as we could go with a pen or pencil, this is as far as I could go, is this a fractal? That's the first question.

S2 I don't think so. Doesn't each segment have to look like the whole thing? In fractals can't you just take any given section and it would look like the rest of the ...?

I That's good. So let's try ratio's of 1/4. This one started with 1/4.

S2 you could take that (he looks at 1/2 of the figure).

I No, I said fourths.

S2 Oh, fourths. ... I don't think so. I might be able to divide it into halves but not into fourths.

(the picture seemed to confusing to him, he was not able to discern if the object was a fractal and the process upon which the figure was built had to be explained)

2. As with Linda the complexity of the figure is that which determines it's dimension.

I ... with those three steps which looks like its going to have greater dimension?

S2 That one (qKc).

I Why?

S2 Already by three steps these are hanging all over the place and are filling up a bigger part of the page than this.

3. He correctly applies the formula for similarity dimension.

I .. So what's the dimension of this? I want you to calculate it. What's our ratio?

S2 That would be, so if 1 is across there, that would be... 1,2,3,4..

I I'll tell you the ratio we'll us is 1/4.

S2 O.k. The ratio is 1/4. That would be right because that's 1/4 there.

I Right.

S2 N equals, then this here is 1, 2, 3, 4, 5, 6. From here it's 1, 2, 3, 4, 5, 6, 7, 8. 8 right there. And right there is 1, 2, 3, 4, 5, 6, 7, 8, right there (he then calcuates the dimension using D=logN/log(1/r))

4. He does not see that the Cantor set's dimension will lie between 0 and 1, but believes that it will be close to 1 and definitely less than the dimension of other shapes he has seen. He has no difficulty in determining the shapes dimension using the formula.

a) I Start with the unit line segment, our scaling ratio is 1/3. You're taking out the middle 1/3 each time. What do you think its dimension is? (begins to calculate) Don't try to calculate it. Guess it. What's the range that it's going to be in? On its space filling ability

S2 Probably lower than 1.5. Probably like right about 1. That doesn't look like it's going to take up much space there.

b) I Work it out.

S2 That would be 1/3. You've got 1, 2 segments here. That would be 1/2. Lower there. About 1/2... (he writes down the answer, and looks for his calculator).

I Last time.. you use it (the calculator).

S2 So that's (mumbles). Less than 1.

I Does that make sense?

S2 Sort of because this right here is taking up more space around here so this number is getting greater. This is taking up less space, the number would be getting smaller.

I It's going to be smaller, but why should it be less than 1.

S2 Because the whole object isn't there anymore.

Rob has an intuitive sence as to the notion of fractal, but as it appears to be based on visual cues he is unable to deal with sets that are more visually complex. He may still see these shapes as representing physical objects and has not abstracted that which is referred to as a fractal.

**Sue**

1. She was able to determine if the shapes shown were fractals or not, but verbalizes that a fractal is that which is self-similar (as did Linda).

I ... This is our initiater, r's 1/4 and you have eight pieces.

S3 I did something like that.

I What you did was right too. And then you do the next step and get this.

S3 Wow, congradulations for going that far.

I So what kind of a shape would you call this?

S3 A fractal.

I This one (triadic Koch curve generator) is a ?

S3 This is something different.

I ... What properties do fractals have?

S3 Self-similar -- you can scale it down and you can use this scaled down version to build the original shape through rotations and translations.

From this question she questioned the relation between recursion and self-similarity by questioning if  ... was a fractal. She determined that it was not a fractal.

2. She is able to guess which shape will have a greater dimension, but she states that she saw both of these objects as unending. This may be due to the form  used to represent a line segment in many school.

I Which of these two do you think has a greater dimension?.. In terms of space filling? Which would fill more space?

S3 Well there are bumps on these bumps on these bumps, and there's bumps on these bumps on these bumps. I'd say this one (qKc).

I Why?

S3 Cause this one goes on. Oh, is that the end point.

3. D is found for the set using the form $r^D = 1/N$ as she had not been taught logarithms.

S3 I have to write down the dimension. That's 1, 2, 3, 4, 5, 6, 7, 8, .. and we scale it down by 1/4, didn't we?

I Correct. (she then writes out $(1/4)^D = 1/8$ and solves for D, D=1.5)

4. She was the only student to relate that the Cantor set would have dimension between 0 and 1 before calculating it as such. She was the only student who questioned the existence of such a set (lesson 3).

a) I Take a line and remove the middle third. So actually this isn't the end and we keep taking out the middle thirds. What do you think the dimension of this set is going to be?

S3 I think it is between zero and one (seems very pleased that her earlier question had been answered).

I Why?

S3 It's not a line, this was between a line and a plane and this one is in between a point and a line.

b) S3 We've divided it up in three, how much did we scale it down by, we scaled it down by 1/3

I And how many parts?

S3 We have 2, yes 2. (she then writes down $(1/3)^D = 1/2$ and finds D≈.64)

Sue more than met the aims set out by the teaching experiment. It appears that she has a firm grasp on what is meant by fractal.

# CHAPTER 4: Post-experiment analysis

The post-experiment analysis endeavors to identify the acts of understanding and obstacles that were prominent in the clinical interview and teaching experiment. In essence the teaching experiment is confronted by the pre-experiment analysis. Upon identifying prominent acts of understanding and obstacles an examination will be undertaken with regard to: student reactions (eg. language), and the role of teaching and the particular task with respect to the promotion of understanding and of overcoming obstacles.

To facilitate the reader's progress through this analysis, a table is given that outlines the acts that are deemed prominent. The meaning of each of the acts of understanding and obstacle is then stated.

The analysis is developed through the clinical interview, activities 2, 3, and 4, in that order. Although the role of activity 1 was important, in that it gave the students time to consider their initial identification (dimension is a concept worthy of study), it does not have the student deal with any of the new ideas directly and as such is not considered in the analysis.

<u>Table of acts of understanding and epistemological obstacles that appeared in the experiment</u>:

| Clin. Int. | Activity 2 | Activity 3 | Activity 4 |
|---|---|---|---|
| U.1<br>E.O.1<br>E.O.2 | task 2 U.9<br>task 3 E.O.4<br>task 4 U.6 | task 5 U.7<br>task 6 E.O.5 | task 7 U.10<br>task 8 U.11, E.O.6 |

U.1: Identification of dimension as a non-intuitive concept that is in need of study and a precise definition.

E.O.1: There exists one universal concept of dimension.

E.O.2: Dimension is a unique characteristic of an object.

U.9: Identification of "jaggedness" that does not "dissolve" under magnification (non-rectifiable curves).

E.O.4: dimension can only be whole numbers.

U.6: discrimination between similarity and self-similarity.

U.7: synthesis of the practical notions of approximation and scaling in the mathematical concept of self-similarity.

E.O.5: Shape is irrelevant in determining dimension.

U.10: identification of a class F of geometrical objects which are generated through an infinite iteration of a certain construction.

U.11: discrimination between an object of class F and the process upon which the object is built.

E.O.6: A fractal is not a finished entity, but merely the process of construction or a sequence of objects.

1. **Clinical Interview**

It may well be that the clinical interview was the most important component of the experiment. Through the interview the students were able to identify that the concept of dimension was worthy of study. They found that their previous assumptions regarding dimension were not universally applicable, and thus overcame the obstacle that there was only one concept of dimension and the belief that dimension was a unique characteristic of an object.

The students reactions to these new ideas were pronounced. The discovery that dimension could be non-integral was truly "felt" by Linda and Sue. Linda expressed her feelings with the two words, "Oh, No!" and Linda stated emphatically, "How can you have 1.5 dimensions? How can you have length and a little bit of width? Half a width? You can't have a half. So this is really weird. This is wrong somewhere." In my short tenure as a teacher of mathematics, I have never seen such a strong emotion response to a mathematical notion. As previously stated, Rob came to these realizations and identifications only after working with logarithms.

In particular, two features made the interview effective. The mathematics the students dealt with was at their "level", the students satisfied the necessary prerequisites, and they were able to construct the framework necessary to identify that the concept of dimension was worthy of study. This framework would be the source of future dialogue. These two features were promoted through the questions presented and the setting, which was that of a one to one interaction with the student. The students were able to work through the questions at their own pace and as such they were "connected to" and "affected by" their results.

2. **Activity 2**

In Activity 2, task 2, the students sorted with respect to fractal and non-fractal shapes. The students were able to identify a jaggedness that does not dissolve under magnification. This identification was marked by descriptions such as, "it has bumps on bumps on bumps..." (Sue) and "if you looked at it under a microscope, it would look the same (as the original structure)" (Rob).

It was in task 3, sorting on the basis of dimension, that the students were able to apply their new definition of dimension to a variety of shapes. Upon completing this act of categorizing the students no longer used expressions such as "half a width". For this reason the obstacle that dimensions can only be whole numbers is said to be overcome at this point as opposed to the point in the clinical interview when the student first saw D as non-integral.

The last task concerning categorizing was task 4. In this task the students categorized in terms of self-similarity. All three students were able to test if the object "was made up of " parts that were similar to it (the whole object). They appeared to have no difficulty discriminating between similarity and self-similarity, although the idea of self-similarity is still being developed. The "definition" stated above seemed sufficient for Linda and Rob as they broke the object up into parts that were congruent. Sue did not make this assumption and in turn was able to see the circle as a shape that was self-similar. As she said, "you can do that with a circle (parts similar to the whole object), you can imagine a circle (she draws a circle within the larger circle) and then blowing it up in the big circle." and this could be done with as many circles as necessary. At this point the condition that the parts had to be congruent was given.

The role teaching plays is minimal. The teacher's role is to have the students check that they are sorting the shapes of the basis of the definition provided and highlight assumptions within the definition. As with the clinical interview, the tasks are of primary importance.

3. **Activity 3**

In this activity the students were presented with a definition of self-similarity which was based on the notion of sets. The students enjoyed using the definition of self-similarity. This task, task 5, saw the students categorizing shapes with the aforementioned definition, but still using visual cues from the geometric shape. A problem arose as a result of an assumption the students made with respect to the notion of scaling ratio. All three students considered this ratio to be 1 : the number of subsets required to form the set.

In task 6 the student is confronted with the use of the object's shape in determining its dimension relative to that of another object. The two objects were the Sierpinski triangle and Sierpinski carpet. The idea that an object's shape has information pertaining to its dimension was unknown to the students before beginning the experiment. By finding an approximate value of the object's dimension the student has some information to use when considering if the dimension found through, D=logN/log(1/r) is reasonable. The process through which the object was formed was intrinsic to determining the object's relative dimension. The first four stages of development were shown for both the Sierpinski triangle and carpet.

In using the process in this manner the students had to have an intuition as to the object's "space-filling ability" at the limit of the process.

The teacher's role in Activity 3 increased. In task 5 it was important to make the students aware of any assumptions that they may be making with regard to the given definition. The emphasis in task 6 was to enable the students to see the non-integral dimensions as forming a continuum as opposed to just isolated values, be it only on $1<D<2$. The teacher had to explain how the relative dimension could be used in correspondence with D=logN/log(1/r). The combination Sierpinski carpet and triangle were fundamental in unifying these ideas.

4. **Activity 4**

Activity 4 made the process of constructing a fractal the extrinsic object of study. This activity engaged the students in two tasks, the first of which was the construction of a fractal using r=1/4 and rotations of 90°. The students' tendency was to try to draw the object in its entirety, when it was realized that this was not possible, they seemed able to construct the various levels or stages of the fractal. Through this process the students were able to identify fractals as part of a class of objects which are generated through an iterative process.

The second task in this activity caused a considerable amount of tension for Rob and Linda. Once fractals were seen in the form of an iterative process, determining information from the object as a whole seemed unmanageable. Upon completing stages 0, 1, and 2 Linda no longer knew how to determine N as she sa[illegible] being 1/4 at each stage, yet N varied. Rob's problem was [illegible]at different. He realized that he could use more than one value for r, for example r=1/4, r=1/16, .... When he discovered this he became unsure of which one he should use. These problems seemed to be resolved when different values of r (of the form $r=(1/4)^n$, n=1,2,3,4,...) and N were used to calculate the dimension of the set and the dimension was found to be the same for each such selection of r.

The task was important in confronting the problem of discriminating between the process and the object, but it was only through the teacher's intervention that the students were shown how to "step back" from the construction and consider the object as a whole. Being able to process and the object simultaneously is crucial, and cannot be forced upon the student, thus, the importance of the one to one interaction.

## 5. Comments for the Classroom

To adapt the material found in the clinical interview to the classroom, the one to one situation would have to be established in a modified form. In using the questionnaire the teacher could preselect a number of students, do the interview with these students, who in turn could present the material to their peers. This technique would not be without problems, but it may be necessary if the students are to ever truly begin to "understand" the concept of dimension and in turn, the idea of a fractal.

The three categorizing tasks in Activity 2, appeared to give the students the time they needed to question the definitions in use and apply them in a visual manner. This type of exploration may have been enhanced had the students been able to manipulate (eg. dilate, and magnify) the given shapes. This manipulation may come by way of the computer.

Task 5, determining whether an object is self-similar, is another task were the dynamic nature of the computer would have been beneficial. Using a program such as DrawII on the Macintosh, the students could have transformed a variety of figures using different scaling ratios.

Although the notion of fractal may have blossomed as a result of the computer, we must not overlook the benefits of paper and pencil. Using the computer in the construction of the quadratic Koch curve, task 7 and 8, would have made it difficult for a student to distinguish between the process and the object. This is as a result of the ease with which the computer can iterate a given instruction.

To establish the one to one situations set out in the clinical interview is a challenge in a "typical" classroom. Activities1, 2 and

3 are such that they could be used in the classroom with little or no significant changes, and Activity 4 and beyond... presents an ideal environment for group discussions regarding assumptions in mathematics and the nature of a mathematical object vis a vis the distinction and commonalities between process and object.

## Chapter 5: Conclusion

Three areas will be examined in this conclusion: the teaching of fractals, how such teaching could be extended and implemented into the high school curriculum, and in what sense the objectives of the study were met. The thesis will close by suggesting aspects highlighted by this study that are in need in research. Conclusions pertaining to specific acts of undertanding and epistemological obstacles were dealt with in chapter 4.

The teaching undertaken was established through the van Hiele model and an epistemological analysis The analysis was used to determine what ideas were fundamental to understanding fractals and the van Hiele model provided a framework within which the ideas could be taught. As well as how and what was taught, the role of motivation within teaching and what was relatively absent during the teaching, the notion of infinity and limit and the computer, may also be significant.

The epistemological analysis pointed to three areas that may be of particular difficulty for the students. First, the students were made aware that dimension is a concept worthy of study by means of the clinical interview. This was accomplished by having the students develop the idea of similarity dimension for the line, square and cube, generalize to get the equation $N=1/r^D$ ($D=\log N/\log(1/r)$), and then apply the equation to the triadic Koch curve. This yielded the result that dimension could be non-integral.

The analysis predicted that the next conflict situation would arise when the students have to determine if a shape is self-similar on the basis of a mathematical definition. To do this the student has to use the notions of translation, rotation, and scaling (dilation), as in Activity 3, Lesson 2. Assumptions the students made regarding the scaling ratio caused them considerable problems.

As noted in the a priori analysis, the construction of a fractal is not obvious. Given the distinction between the process of building the object and the object itself, and the relation between the iterative process, the definition of fractal would present obstacles to the students understanding. These obstacles were confronted in Activity 4, Lesson 3.

The teaching in this experiment was developed through the van Hiele model of geometric thought. The model distinguishes three levels, visual (level 1), descriptive (level 2), and theoretical (level 3). The teaching experiment involved levels 1 and 2. The model was chosen due to its longstanding success in teaching geometric concepts as discussed in Fuys, Geddes, and Tischler, 1988. As well the model incorporates an instructional sequence: information, directed orientation, explication, free orientation, and integration. A significant feature of this model is the fact that what is intrinsic at one level is extrinsic at the next level. An example of this is the characteristic of self-similarity. In activity 2, the students identified objects with "bumps on bumps...". Activity 3 shifts the focus from the object to the property and a definition of self-similarity is presented and considered. Another example was the intrinsic use of process in Activity 3 and its extrinsic development in Activity 4. This approach is effective in that, upon identifying an idea, the student then confronts such ideas with his/her beliefs. This in turn may enable the teacher to determine obstacles related to these beliefs.

It was apparent in the clinical interview and the final lesson that the students encountered considerable difficulty. It is believed that the tension created by such situations should not be avoided, however, the students need support to overcome such obstacles, which takes both time and individual attention. This attention can be supplied by the teacher and peer tutors. These tutors would have previously received individual instruction from the teacher. As indicated from the begining of the thesis, this approach is designed for high school students in advanced mathematics course.

Though the students were successful in meeting the objectives set out in the lessons some difficulties were noted. Rob was unsuccessful in abstracting the notion of self-similarity, but as stated earlier this may have been due to the teacher not making explicit the relationship between the iterative process and self-similarity. One obstacle to the understanding of similarity dimension that was unforeseen was the association of magnitude to dimension. This may result from the student relating dimension (whole numbers) in terms of basic magnitudes, length (cm), surface ($cm^2$), and volume ($cm^3$). It is difficult to think of a solution to $4=3^D$ if D is conceived of as an amount of something, D must be seen as a number in the abstract. Rob encountered this obstacle during the clinical interview.

Another problem not anticipated in the pre-experiment analysis involved the scaling ratio. All three students associated the scaling ratio to that of a percentage of the whole object. For example, r=1/3 for the triadic Koch curve was seen as 1/4. This may be as a result of the static nature of the paper and pencil environment. The problem may have been resolved more effectively if the computer had been used to show the dilation of an object. This may have enabled the student to identify the dynamic process involved in determining the scaling ratio and in turn the student would no longer see scaling ratio as pertaining to area.

In any interaction between student, teacher and content, the role that motivation plays cannot be underestimated. All but one student in the original grade 12 class was willing to participate in the experiment. With the exception of activity one, the motivation of the three students who participated was high throughout the experiment. This is interesting as each student had a very different perception regarding mathematics. Sue referred to math as a "hobby", Rob saw math as useful in the applied sciences and Linda merely observed math as a subject in which she was successful.

The teacher's level of enthusiasm no doubt affects learning. As the topic of fractals was a topic of choice and interest to this researcher, an enthusiastic regard for the material may have influenced the students. My interest in fractals began upon seeing the "pretty pictures" associated with dynamical systems such as the Mandelbrot set and fractal planets. An introduction to the topic through such pictures may not be appropriate for all students. Most of my colleagues have quickly lost interest in the "pretty pictures" because they were unaware of the mathematics necessary to explore these sets. The sets initially introduced to the three students in this experiment were not as beautiful as some, but were chosen to allow the students to use their mathematical skills in learning about such objects. As the teaching progressed, the students' motivation appeared to increase with each lesson and upon presenting all the beautiful pictures in lesson 3, they seemed disappointed that our discussions had come to a close.

At this point a reader may be questionning what level of understanding any person could achieve in studying fractals without knowledge of the concepts of infinity and limit. It is assumed that the reader is already aware that students have gained significant understanding with regard to fractals. In time the question may become, "how much understanding can one have of infinity if they have not studied fractals?" There is the possiblity that fractals may provide a visual intuition for infinity (through scaling) upon which the teaching of infinity could be developed. In particular, fractals provide information as to the overcoming of obstacles involving infinity that result from the distinction between process and object.

The concept of limit is an important characteristic of what is known as a fractal. Certainly the idea of limit will promote new acts of understanding when the fractal is explicitly considered as the limit of a specific iterative process.

The computer is a third aspect that would appeared to be neglect in the experiment. Three considerations affected this decision: student attitudes toward the computer, classroom resources and the material to be taught.

The two female students stated their dislike of computers (because of their experience with high school programming) and Rob expressed a general disinterest in the computer. Many students have not enjoyed their computer experience in high school and most mathematics classrooms in Ontario do not have direct access to computers. Using the computer in the construction of the quadratic Koch curve, task 7 and 8, would have made it difficult for a student to distinguish between the process and the object. This is as a result of the ease with which the computer can iterate a given instruction. As this distinction was fundamental to the understanding of fractal, as presented in the a priori analysis the computer was not an integral part of the experiment.

In an extended teaching outline, the role of the computer could be increased gradually. The computer could be introduced through short presentations. After the three lesson described in this thesis the computer could be used to "revisited" said lessons. For example, using DrawII on the MacIntosh, dilations (scaling ratio), rotations, and translations of a variety of fractals could be demonstrated. Also, if the students were given a general program format for creating fractals, they could determine the transformations for specific fractals and have the program run on software such as Quickbasic. From this point forward, the students' use of the computer could be expanded to be more creative and independent. Appendix D includes programs written in basic for creating a number of fractals. The fractals in the text of this thesis were formed using DrawII.

This study undertook to identify: 1) some of the difficulties encountered in constructing the meaning of a fractal , 2) certain basic acts of understanding necessary in constructing the meaning of fractal, and 3) the didactic conditions of teaching the notion of fractal: prerequisites, pedagogical exploitation of the conflict generating situations for the introduction of the new concept of dimension, and problems used to develop the notion of fractal.

Point 1) and 2) were determined through the epistemological analysis and the teaching experiment. Seven obstacles were predicted by the a priori analysis and two more were observed during the teaching. Eleven acts of understanding were hypothesized by the epistemological analysis. Chapter 2 saw the stating of prerequisites and the development of three conflict generating situations. The first situation involved the identification that dimension is a concept worthy of study. The second situation pertained to the notions needed to understand self-similarity and the third situation saw the students discriminating between the process of building the object and the object itself. The conflict generating situations were develped as a result of the epistemological analysis.

The first and third situation pertained to fundamental mathematical ideas and since the students have begun to question and understand these ideas, as found in Chapter 4, the experiment has significance. Two other aspects upon which this thesis may be judged are: the questions that are raised as a result of the research and the positive affect that fractals may have on other topics.

Fractals offer the possibility of introducing concepts such as infinity in a visual manner through recursive algorithms. The ideas of process and object are fundamental to such an introduction. How such a visual presentation will affect the understanding of infinity is worthy of further study. Also, how such a topic would be introduced through this new approach is yet unanswered. Another consideration for future study is the assessment of student's understanding of fractal geometry upon the completion of a first course in calculus. The role of the computer in an extended program of study, and both the social and mathematical dynamics (understanding) created in teaching to a small group of students are areas yet to be investigated.

This thesis has focussed on the importance of fractals in and of itself. However, fractal geometry may affect the study of Euclidean geometry. In the past only Euclidean geometry was taught in high schools and it has not been well received by many students.

S3 Oh, Euclidean shapes, we did that in geometry ... but I didn't do well.

I You didn't do well in geometry?

S3 No, that is the one part of mathematics I dislike.

A possible consequence of the introduction of fractal geometry, is that Euclidean geometry may be swept aside. A far more useful approach may be the study of the understanding students gain from a dialogue between these two points of view. Bohm (1987) considers dialogue between different perspectives primary to creative processes.

"The mind is then able to respond to creative new perceptions going beyond the particular points of view that have been suspended." (p.243)

Fractal geometry facilitates a creative dialogue among students and teacher involving dimension, similarity and possibly infinity and limit, and as such has an important place in the world of mathematics education.

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# APPENDIX

A. The answers to the questionnaires given to the four students who participated in the clinical interview.

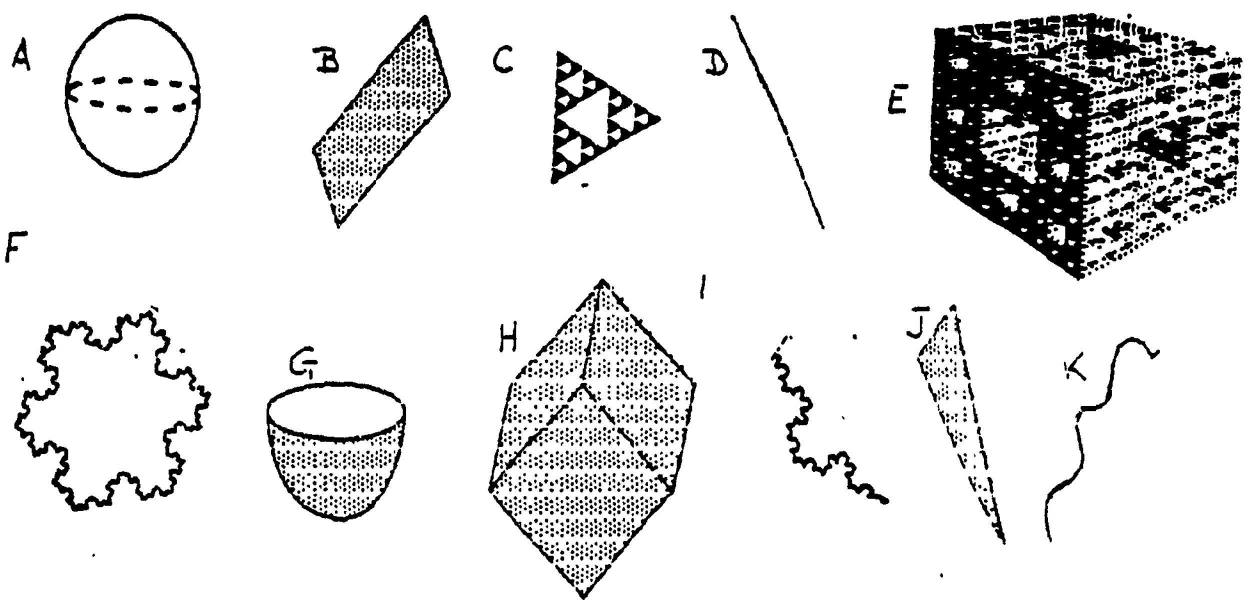
**Name:** Aryn Martin

## Questionnaire

**Instructions:** 1. answer each question as best you can,
2. if you have a question, raise your hand and I will attempt to answer it.

### Questions

1. Categorize the following objects in terms of their dimension.




Categories (dimensions):

1 dimensional - D, K

2 dimensional - B, C, F, J, I

3 dimensional - A, E, H, G

2. For each dimension state what the objects have in common.

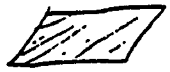
1 dimension - length but no width

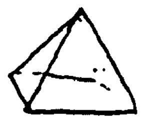
2 dimension - Length & width but no depth

3 dimension - length, width, depth

3. Add an object to each category.

1 dimension 

2 dimensions 

3 dimensions 

4. a) What is the difference between an object with dimension 1 and an object with dimension 2?

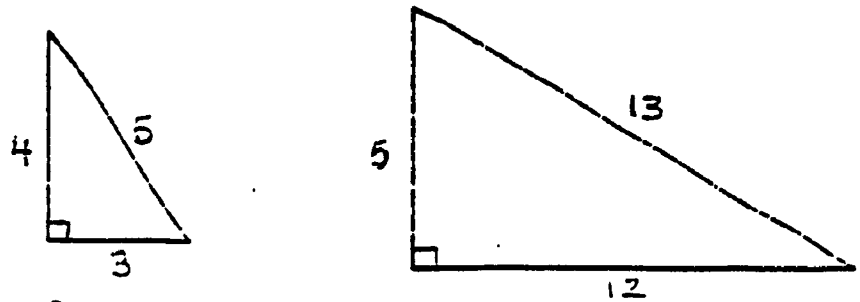
An object with dimension 1 has only 1 dimension (ie length) an object with dimension 2 has 2 dimensions (ie length, width). An object with 2 dimensions has an area and can be expressed in units squared

b) What is the difference between an object of dimension 2 and an object of dimension 3;

An object with dimension 3 has 3 dimensions (ie length, width, depth) and can be expressed in units cubed.

5. Are the following pairs of figures similar ?

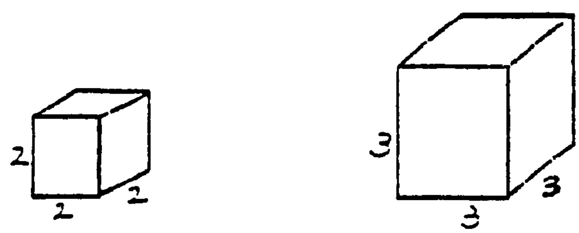
a)




answer: no

why? their sides are not equally proportional

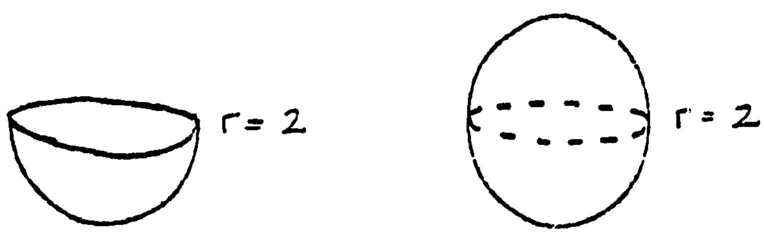
b)




answer: yes

why? their sides are equally proportional (2:3)
they are the same shape

(c)




answer: no

why? They are not the same shape

5. Are the following pairs of figures similar?

d)

answer: no

why? They are not precisely the same shape

e)


answer: yes

why? They are both sequences where the factor is 1/2, they just begin with different numbers.

* Would you be available for a discussion about these questions? yes

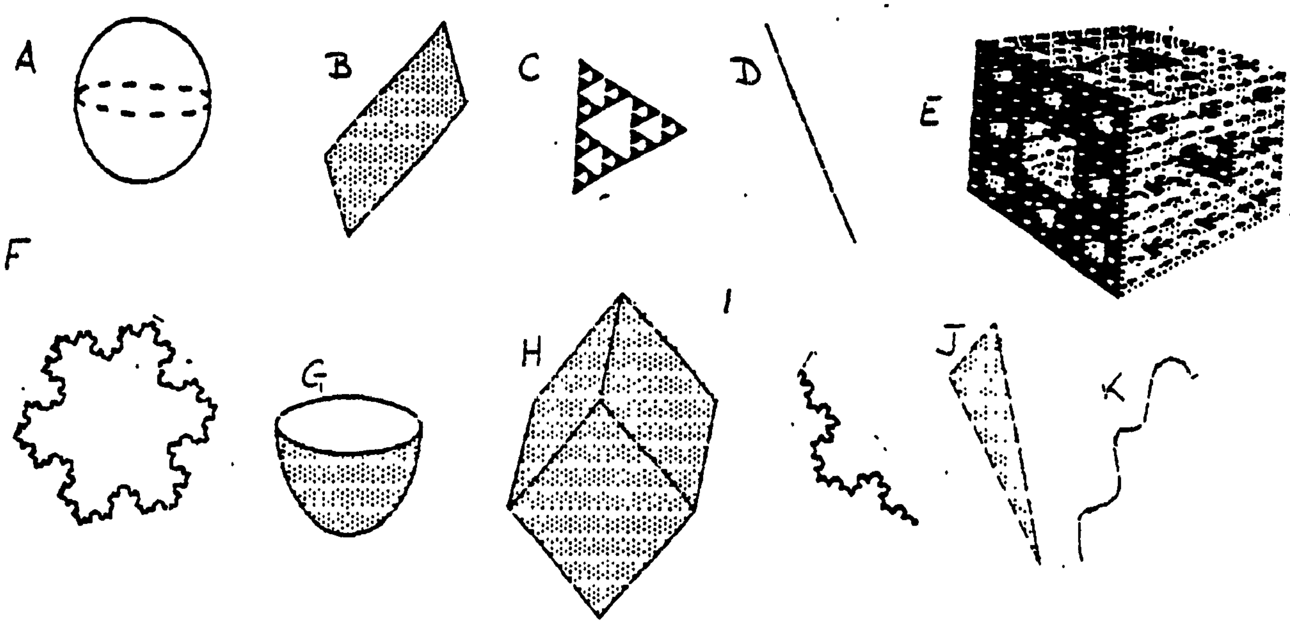
Name: Glenn Wardrop

## Questionnaire

**Instructions:** 1. answer each question as best you can,
2. If you have a question, raise your hand and I will attempt to answer it.

### Questions

1. Categorize the following objects in terms of their dimension.

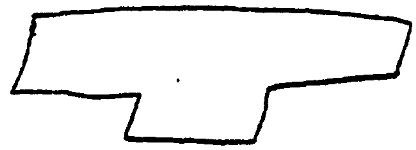



Categories (dimensions): A – "3D" B – "2D" C – "2D" D – "1D"
E - 3D F – "2D" G – "3D" H – "3D" I – "1D" J – "2D"
K – "1D"

2. For each dimension state what the objects have in common.
"1D" – just a line, lacks "substance" (has form but not "na:
"2D" – no depth just images
3D – realistic could exist as a model

3. Add an object to each category.

1D – 

2D – 

3D – 

4. a) What is the difference between an object with dimension 1 and an object with dimension 2?

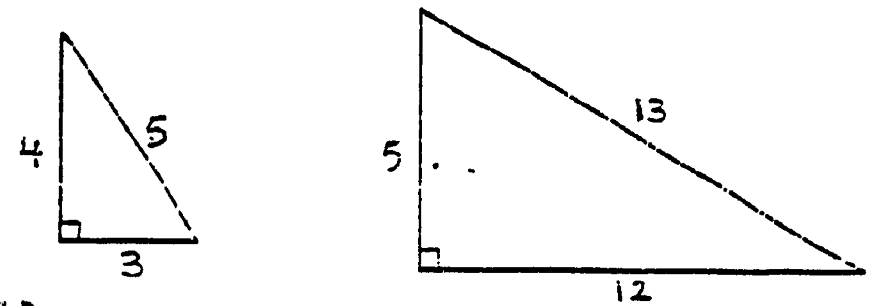
width in dim 2

b) What is the difference between an object of dimension 2 and an object of dimension 3;

depth in dim 3

5. Are the following pairs of figures similar ?

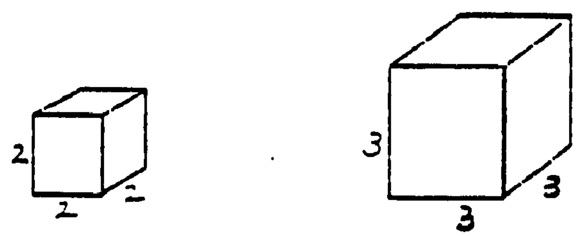
a)




answer: no

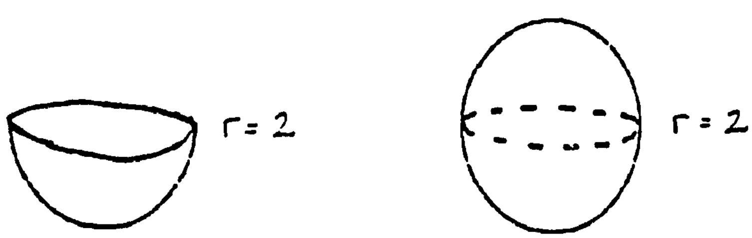
why? ratios of "similar" sides dissimilar even after moving positions

b)




answer: yes

why? ratios of sides similar




answer: yes

why? the object on the left is ½ the object on the right

5. Are the following pairs of figures similar?

d)

answer: no

why? shapes dissimilar

e)


answer: yes

why? pattern is same for both
1/2 of previous figure

* Would you be available for a discussion about these questions?

yes

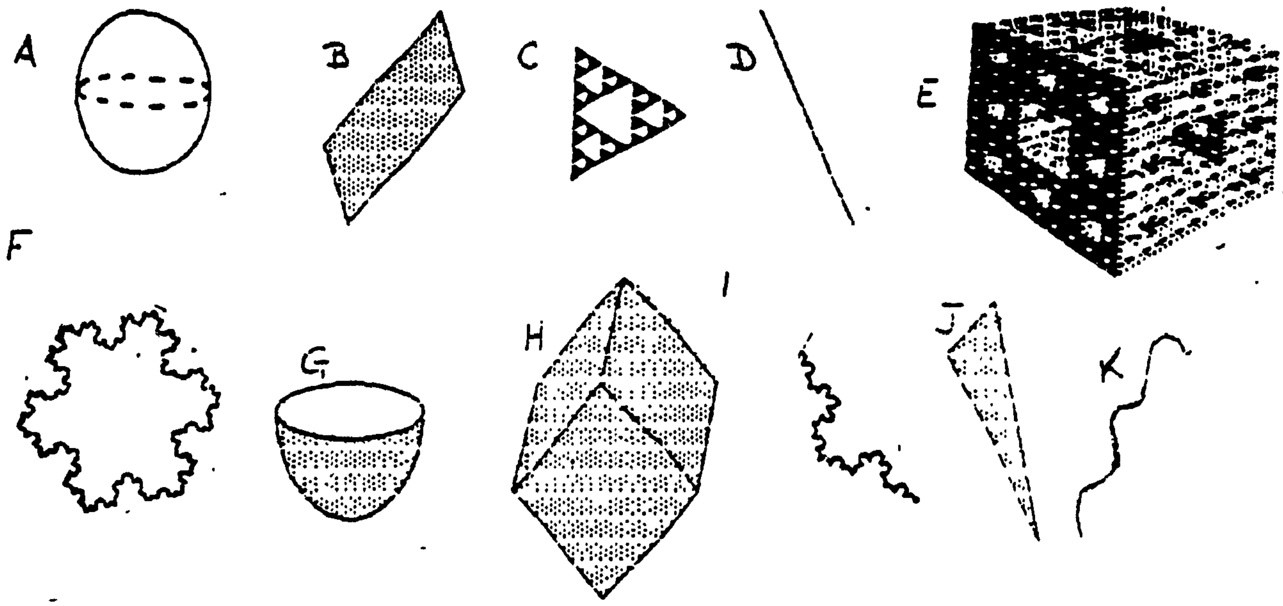
**Name:** LYDIA SCRATCH

## Questionnaire

**Instructions:** 1. answer each question as best you can,
2. If you have a question, raise your hand and I will attempt to answer it.

### Questions

1. Categorize the following objects in terms of their dimension.




Categories (dimensions):

3 dimensions → A, E, G, H

2 dimensions → B, C, D, F, I, J, K

2. For each dimension state what the objects have in common.

3 dimensions - in all of the 3 dimensional shapes I get the sense that they jump out, that I can see them as they would be if they weren't on the page.

2 dimensions - They each have length & width

3. Add an object to each category.

3 →

2 →

4. a) What is the difference between an object with dimension 1 and an object with dimension 2?

→ you can't draw a figure with dimension 1 because when you draw anything on paper it automatically has 2 dimensions so difference is you can draw a 2 dimens shape and not a 1 dimensional shape

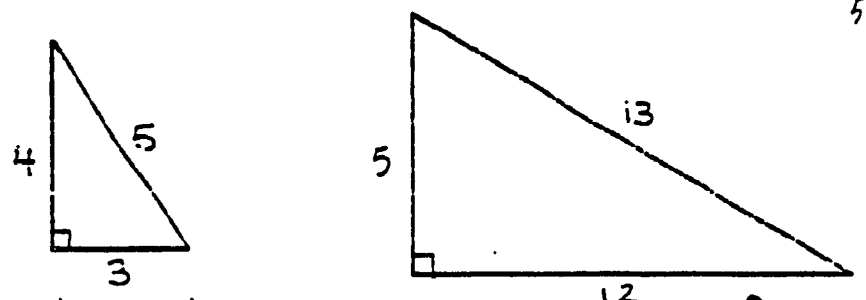
b) What is the difference between an object of dimension 2 and an object of dimension 3;

With Dimension 3 figures you can visual the whole figure. and dimension 2 figures are rather dull and one wouldn't want to visualize the whole figure

5. Are the following pairs of figures similar ?

$\frac{4}{5}$ $\frac{5}{13}$ $\frac{3}{12}$

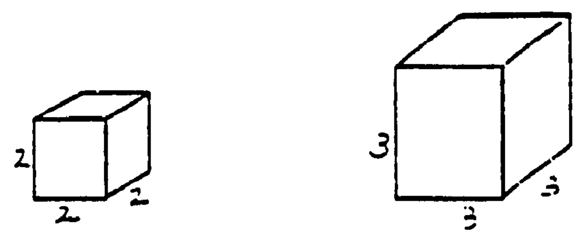
a)




answer: depends on your point of view.

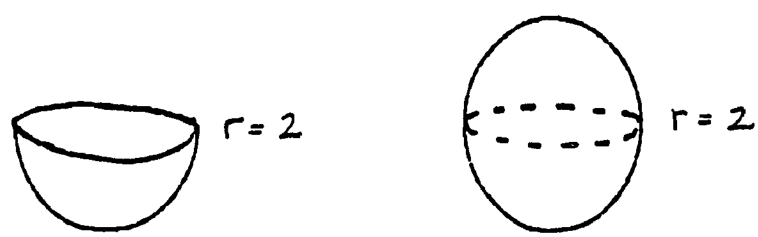
why? Mathematically no because the sides aren't proportional to each other. Logical yes because they are both right ∠ triangle

b)




answer: yes

why? the sides are proportional to each other.




answer: no

why? because they aren't the same figures

5. Are the following pairs of figures similar?

d)

answer: no

why? the 1st one is a triangle, the second one isn't

e)


answer: yes

why? each number in the second set is ½ of the corresponding number in the first set.

* Would you be available for a discussion about these questions?

If you want. My apologies for the answers it's been a really weird day so therefore I'm in a really weird frame of mind.

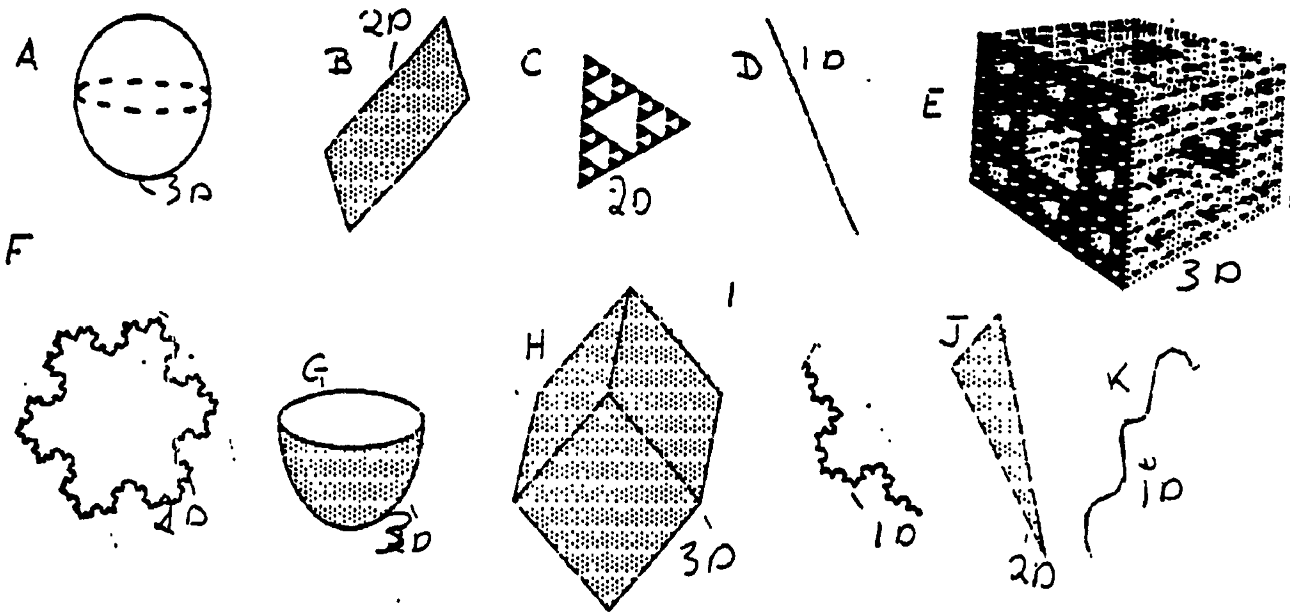
Name: *Scott Carr*

## Questionnaire

**Instructions:** 1. answer each question as best you can,
2. if you have a question, raise your hand and I will attempt to answer it.

## Questions

1. Categorize the following objects in terms of their dimension.




Categories (dimensions):

*A , E,G,and H → 3 demensions*
*B , C,F,~~E~~, J → 2 demensions*
*D , ~~E~~, I , K → 1 demensions*

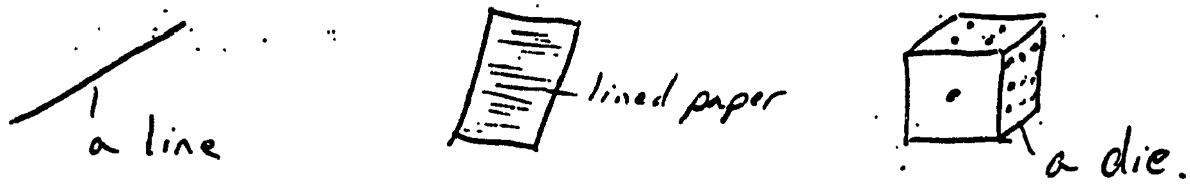
2. For each dimension state what the objects have in common.

*3 demensioned objects have volume.*

*2 demensioned objects have a surface. (can calculate area)*

*1 demensioned objects ~~are~~ are lines or curves*

3. Add an object to each category.




4. a) What is the difference between an object with dimension 1 and an object with dimension 2?

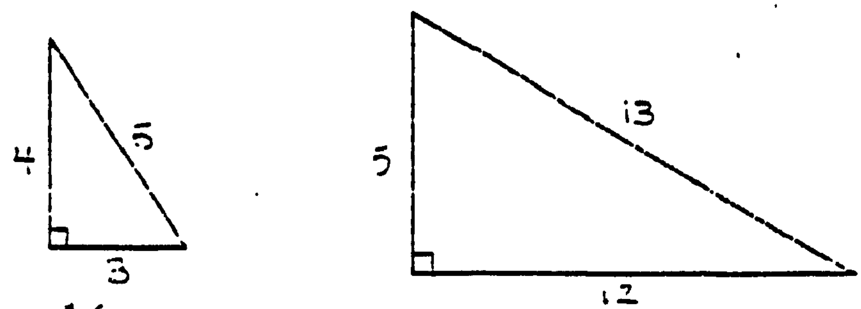
The difference is that the ~~a 2~~ ~~of~~ an object with 2 dimensions has area.

b) What is the difference between an object of dimension 2 and an object of dimension 3;

An object with 2 dimensions has a flat surface. An object with 3 dimensions has more than 1 surface. Therefore it would be able to hold a liquid or some other substance.

5. Are the following pairs of figures similar ?

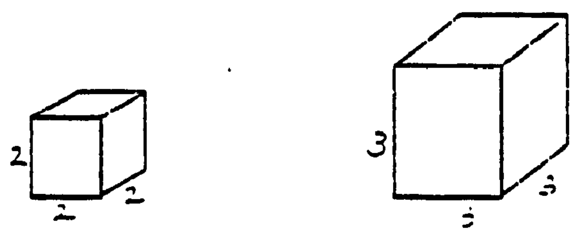
a)




answer: No.

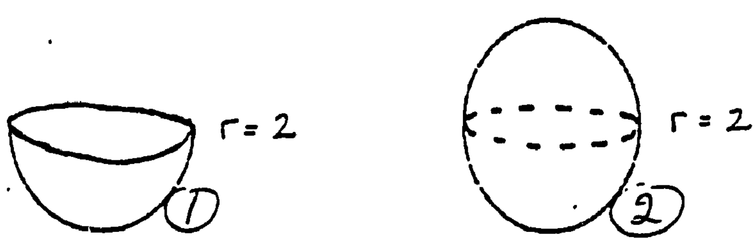
why? The ratio of the lengths of the sides do not correspond with each other.

b)




answer: Yes

why? They are both cubes. The lengths of the sides are increased by one unit. Objects have the same features: 90 angles, 6 sides, square surfaces.




answer: No

why? Obect ① is one half of a sphere, object ② is a whole sphere.

5. Are the following pairs of figures similar?

d)


answer: No

why? Objects do not have the same number of sides.

e)


answer: Yes

why? Lines. All lines are similar no matter what their length is.

* Would you be available for a discussion about these questions?

Yes

B. Responses to questions given during the clinical interview.

*Student 1*

1. If we take a straight line of unit length (1) and divide it into N=3 lines, What is the length, r, of each of the lines?

$r = \frac{1}{3}$

2. How are N and r related? (write N in terms of r, eg. N=...)

$N = 3r$ $\quad N = \frac{1}{r}$

$3 = 9(\frac{1}{3})$

3. If we take the unit square and divide it into $N=3^2$ squares, What is the length of a side, r, of each of the squares?

$\frac{1}{3}$

4. How are N and r related? (write N in terms of r, eg. N=...)

$N = \frac{1}{r^2}$ $\quad N =$

5. If we take the unit cube and divide it into $N=3^3$ cubes, What is the length of an edge, r, of each of the cubes?

$r = \frac{1}{3}$

6. How are N and r related? (write N in terms of r, eg. N=...)

$N = \frac{1}{r^3}$ $\quad N = \frac{1}{(\frac{1}{3})^3} = 27$

7. Can you see how the object's dimension is related to the equations involving r and N? Explain

The exponent of r is the dimension $\quad N = \frac{1}{r^d}$

8. The Koch curve can be subdivided into lengths, r=1/3 of the original size. How many smaller Koch curves, N, do you get?

4

9. Write N in terms of r.

$4 = \frac{1}{(\frac{1}{3})^d}$

$4 = 3^d$

Student 2



1. If we take a straight line of unit length (1) and divide it into N=3 lines, What is the length, r, of each of the lines? $\frac{1}{3}$

2. How are N and r related? (write N in terms of r, eg. N=...)

$r=\frac{1}{N}$ $\quad r=\frac{1}{3}$ $\quad N=3r$

3. If we take the unit square and divide it into $N=3^2$ squares, What is the length of a side, r, of each of the squares?

N= 9 sq $\quad \frac{1}{3}$

4. How are N and r related? (write N in terms of r, eg. N=...)

$r^2=\frac{1}{9}$ $\quad r^2=\left(\frac{1}{3}\right)^2$

$r^2=\frac{1}{n}$

5. If we take the unit cube and divide it into $N=3^3$ cubes, What is the length of an edge, r, of each of the cubes?

$N=3^3$
$=27$ $\quad \frac{1}{3}$

6. How are N and r related? (write N in terms of r, eg. N=...)

$r^3=\frac{1}{n}$

7. Can you see how the object's dimension is related to the equations involving r and N? Explain

r is to the power of the # of the dimension
ie: $r^2 \to 2D$, $r^3 \to 3D$ $r \to 1D$

8. The Koch curve can be subdivided into lengths, r=1/3 of the original size. How many smaller Koch curves, N, do you get?

$r^D=\frac{1}{n}$ $\quad N=4$

9. Write N in terms of r.

$D=\{D\in R / D\neq 0, 1, + D\geq 1\}$
$D\notin \mathbb{Q}$

$\left(\frac{1}{3}\right)^D=\frac{1}{4}$

Student 3



1. If we take a straight line of unit length (1) and divide it into N=3 lines, What is the length, r, of each of the lines?

$1/3$.

2. How are N and r related? (write N in terms of r, eg. N=...)

$N = 3 \quad r = \frac{1}{3} \quad N = 3 \quad r = \frac{1}{N} \quad N = \frac{1}{r}$

3. If we take the unit square and divide it into $N=3^2$ squares, What is the length of a side, r, of each of the squares?

$r = \frac{1}{3}$.

4. How are N and r related? (write N in terms of r, eg. N=...)

$r^2 = \frac{1}{N}$

5. If we take the unit cube and divide it into $N=3^3$ cubes, What is the length of an edge, r, of each of the cubes?

$r = \frac{1}{3}$.

6. How are N and r related? (write N in terms of r, eg. N=...)

$r^3 = \frac{1}{N}$

length
number of
little things
dimension

7. Can you see how the object's dimension is related to the equations involving r and N? Explain

as you increase the number of dimensions you increase the power of r.

$r^n = \frac{1}{N}$

8. The Koch curve can be subdivided into lengths, r=1/3 of the original size. How many smaller Koch curves, N, do you get?

4

9. Write N in terms of r.

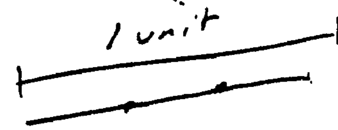
$N=4 \quad r=\frac{1}{3}$

$r \quad \left(\frac{1}{3}\right)^n = \frac{1}{4} \quad 1 < n < 2$

Student 4



1. If we take a straight line of unit length (1) and divide it into N=3 lines, What is the length, r, of each of the lines?




∴ each of the lines is $\frac{1}{3}$ of a unit. $\frac{1}{3}$

2. How are N and r related? (write N in terms of r, eg. N=...)

$\frac{r}{\sqrt{}} = \frac{0.\overline{33}}{3 \text{ lines}}$ $3\sqrt{c}$ $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $r = \frac{1}{3}$

$N = 3r$ $\frac{N}{3} = r$ $r : N = \frac{1}{3} : 1$

3. If we take the unit square and divide it into $N=3^2$ squares, What is the length of a side, r, of each of the squares?

$r = \frac{1}{3}$ $N = 9 \text{ squares } (3^2)$ $N = 27r$

$4r = \frac{4}{3}$

$r = \frac{4}{3} \times \frac{1}{4}$

$r = \frac{4}{12}$

$r = \frac{1}{3}$

4. How are N and r related? (write N in terms of r, eg. N=...)

5. If we take the unit cube and divide it into $N=3^3$ cubes, What is the length of an edge, r, of each of the cubes?

$\frac{1}{3}$

6. How are N and r related? (write N in terms of r, eg. N=...) $3r, 27r, 81r$

$N = 27(3^2) = 27(9) = 243$ $N = \frac{243}{3}$ $N = 81$ $3^3 = xr$ $27 = \frac{x}{3}$ $81 = x$

7. Can you see how the object's dimension is related to the equations involving r and N? Explain

8. The Koch curve can be subdivided into lengths, r=1/3 of the original size. How many smaller Koch curves, N, do you get?

9. Write N in terms of r.

C. <u>The written work completed by the three students involved in the teaching experiment.</u>

Student 1, Lesson 2

$N = \frac{1}{r^d}$

$N = 2^d$

$3 = 2^d$

$\log 3 = d \log 2$

$d = \frac{\log 3}{\log 2}$

$= 1.58$

$N = \frac{1}{r^d}$

$8 = 3^d$

$d = \frac{\log 8}{\log 3}$

$= 1.89$

Student 1, Lesson 3

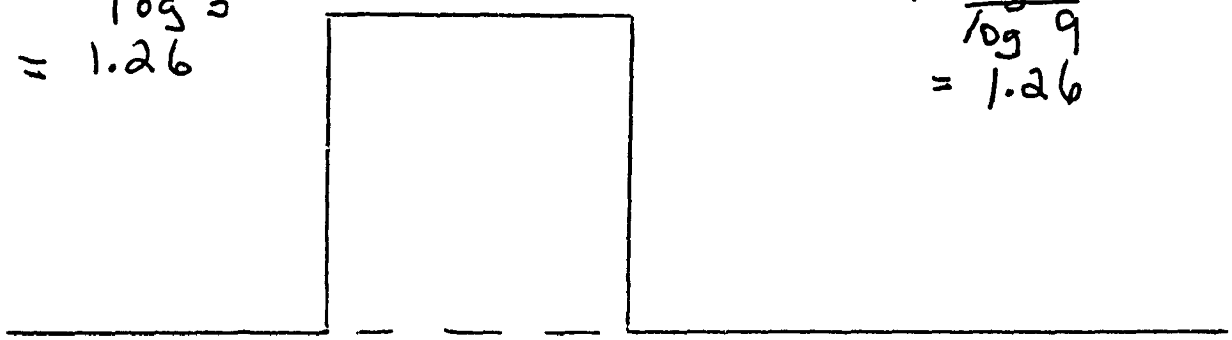
$d = \frac{\log 8}{\log 4} = 1.5$

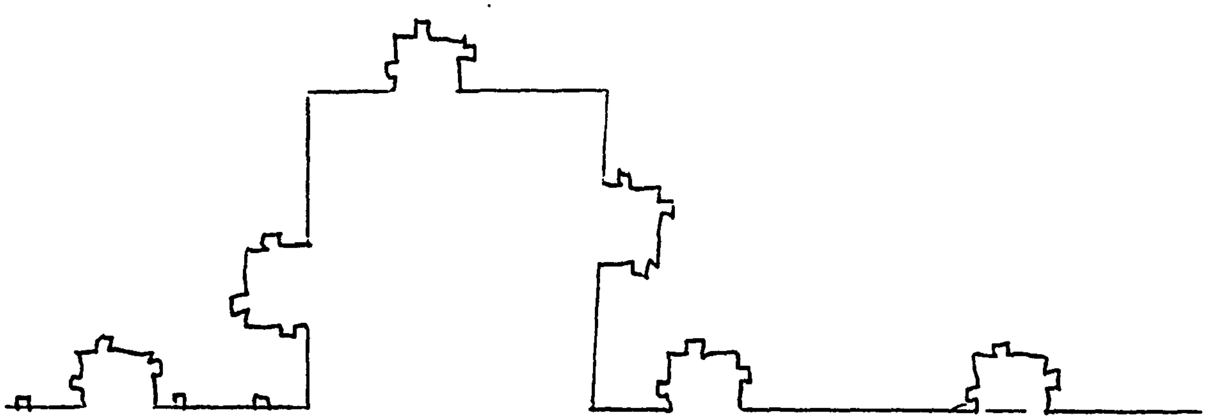
$d = \frac{\log 6}{\log 4} \approx 1.29$

$d = \frac{\log 4}{\log 3} \approx 1.26$

$d = \frac{\log 4}{\log 3} = 1.26$

$d = \frac{\log 16}{\log 9} = 1.26$





Student 2, Lesson 2

N = number of smaller subsets
r = scaling ratio
D = dimension

$$r^D = 1/N$$

$$\tfrac{1}{2}^D = \tfrac{1}{3}$$

$$D \log \tfrac{1}{2} = \log \tfrac{1}{3}$$
$$D(-0.301) = -0.477$$
$$D \doteq 1.584962501$$

$$\tfrac{1}{3}^D = \tfrac{1}{8}$$

$$D \log \tfrac{1}{3} = \log \tfrac{1}{8}$$
$$D(-0.477) = -0.903$$
$$D \doteq 1.892789261$$

---

$$\tfrac{1}{4}^D = \tfrac{1}{8}$$

$$D \log \tfrac{1}{4} = \log \tfrac{1}{8}$$
$$D = \frac{\log \tfrac{1}{8}}{\log \tfrac{1}{4}}$$
$$D = \frac{\log 8}{\log 4}$$
$$D = \frac{0.903}{0.602}$$
$$D = 1.5$$

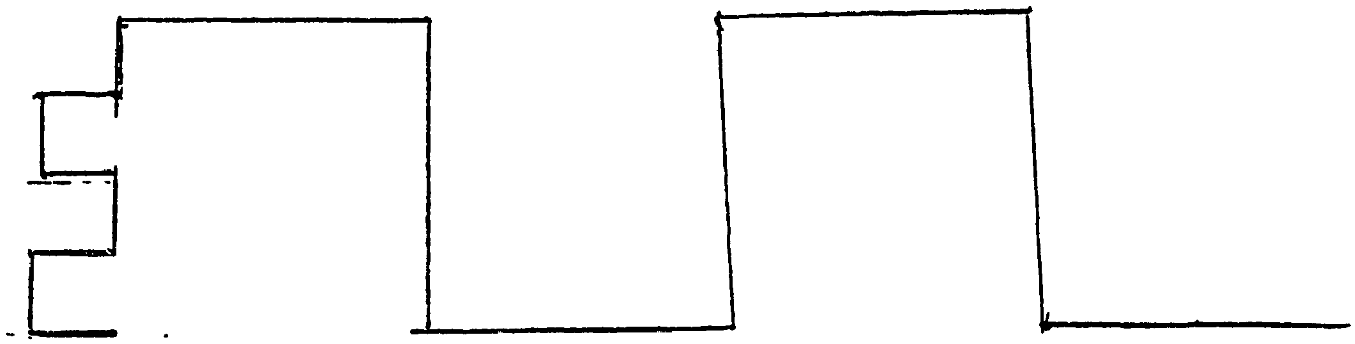
$$\tfrac{1}{3}^D = \tfrac{1}{4}$$

$$D \log \tfrac{1}{3} = \log \tfrac{1}{4}$$
$$D = \frac{\log \tfrac{1}{4}}{\log \tfrac{1}{3}}$$
$$D = \frac{\log 4}{\log 3}$$
$$D = \frac{0.602}{0.477}$$
$$D = 1.26$$

$$\tfrac{1}{4}^D = \tfrac{1}{16}$$

$$D \log \tfrac{1}{9} = \log \tfrac{1}{16}$$
$$D = \frac{\log \tfrac{1}{16}}{\log \tfrac{1}{9}}$$
$$D = \frac{\log 16}{\log 9}$$
$$D = \frac{1.204}{0.954}$$
$$D = 1.26$$

Student 2, Lesson 3



Evaluation

$1/4^D = 1/8$

$1/3^D = 1/2$

$D \log 1/3 = \log 1/2$

$D = \frac{\log 1/2}{\log 1/3}$

$D = \frac{\log 2}{\log 3}$

$D = \frac{0.301}{0.477}$

$D = 0.631$

Student 3, Lessen 2

$r^D \doteq \frac{1}{N}$

$\frac{1}{3}^D = \frac{1}{8}$

$.33^D = 0.125.$

$.33^{1.89} \doteq 0.125.$

$\frac{1}{2}^D = \frac{1}{3}.$

$0.5^D = 0.33$

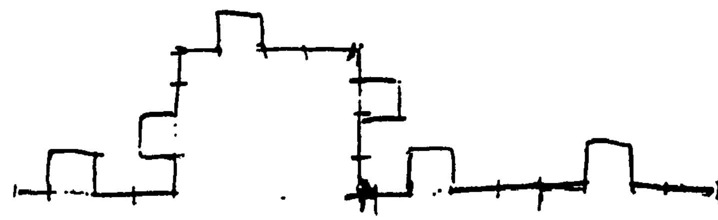
$0.5^{1.59} \doteq 0.33$

$\left(\frac{1}{3}\right)^D = \frac{1}{4}$

$0.33^D = 0.25$

$0.33^{1.25} \doteq 0.25.$

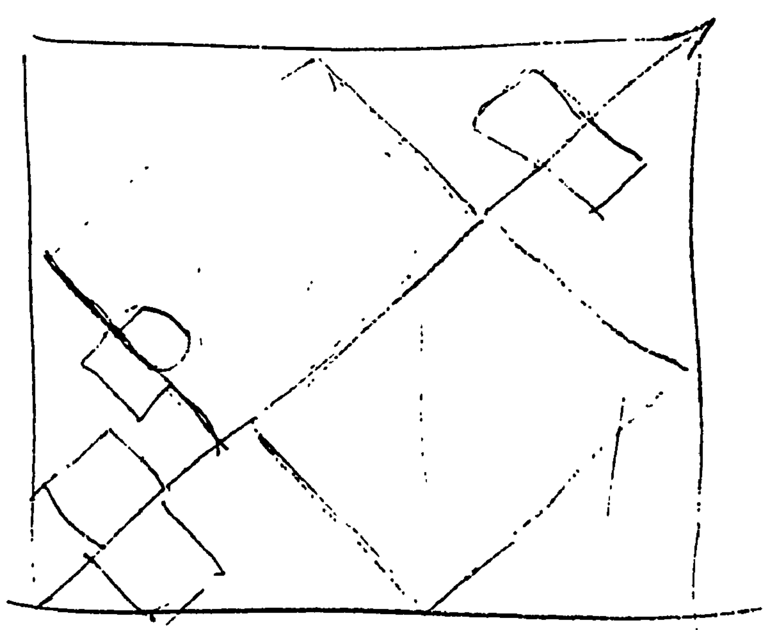
Student 3, Lesson 3



$$\frac{1}{4} = \frac{1}{6}$$

$$.25 = .17$$

$$.25 = .17$$



Evaluation, Student 3

$$\frac{1}{4}^{?} = \frac{1}{8} =$$

$$.25^{?} = 0.125$$

$$.25^{1.5} \doteq 0.125$$

$$\frac{1}{3}^{?} = \frac{1}{2}$$

$$.33^{?} \doteq .5$$

$$0.33^{.64} \doteq 0.5$$

Evaluation, Student 3

$$\frac{1}{4}^{?} = \frac{1}{8} =$$

$$.25^{?} = 0.125$$

$$.25^{1.5} \doteq 0.125$$

$$\frac{1}{3}^{?} = \frac{1}{2}$$

$$.33^{?} \doteq .5$$

$$0.33^{.64} \doteq 0.5$$

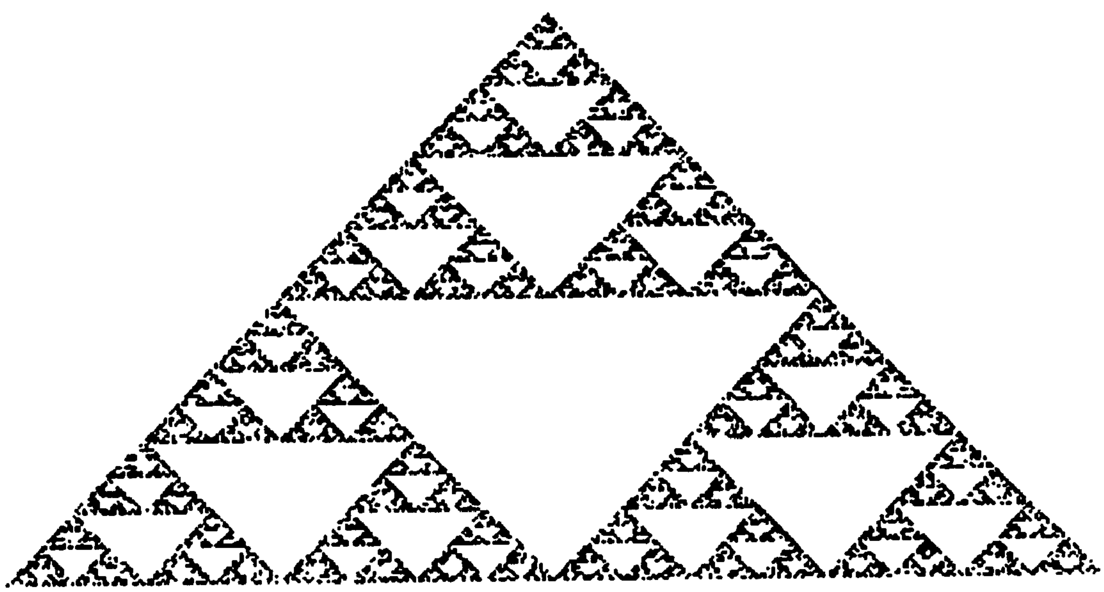
D. The use of Iterated Function Systems in programming fractals.

The fractals found in this appendix are created using specific transformations. The transformations come by way of dilations, translations and rotations. Dilations, translations and rotations can always be expressed as linear equations of the form $x'=ax+by+c$ and $y'=dx+ey+f$ (Bannon, 1991). If we iterate these transformations, choosing one of the transformation at random, the resulting figure can be quite amazing. This is referred to as an iterated function system or IFS.

```
REM KOCH CURVE
REM 4 TRANSFORMATIONS
REM 2000 ITERATIONS
XSC=400
YSC=300
RIGHT=40
DOWN=40
READ N
FOR I=1 TO N
READ  A(I),B(I),C(I),D(I),E(I),F(I),P(I)
P(I)=P(I)+P(I-1)
NEXT
X=0:Y=0
FOR N=1 TO 2000
GOSUB PICKP
XP=A(I)*X+B(I)*Y+C(I)
YP=D(I)*X+E(I)*Y+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
PICKP:
P=RND(1)
I=1
PICKME:
IF P(I)>P THEN RETURN
I=I+1
GOSUB PICKME
POTPOINTS:
XC=XP*XSC+RIGHT
YC=100-YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
PSET (XC,YC)
RETURN
DATA 4
DATA  .33,0,0,0,.33,0,.25
DATA  .17,-.29,.33,.29,.17,0,.25
DATA  .17,.29,.5,-.29,.17,.29,.25
DATA  .33,0,.66,0,.33,0,.25
```

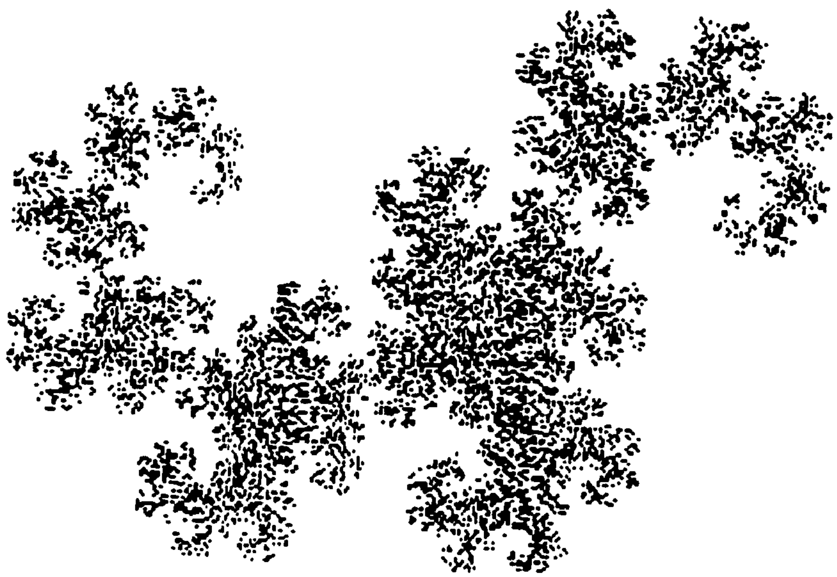
The Koch Curve

```
REM SIERPINSKI TRIANGLE
REM 3 TRANSFORMATIONS
REM 10000 ITERATIONS
XSC=200
YSC=200
RIGHT=50
DOWN=155
READ N
FOR I=1 TO N
READ  A(I),B(I),C(I),D(I),E(I),F(I),P(I)
P(I)=P(I)+P(I-1)
NEXT
X=0:Y=0
FOR N=1 TO 10000
GOSUB PICKP
XP=A(I)*X+B(I)*Y+C(I)
YP=D(I)*X+E(I)*Y+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
PICKP:
P=RND(1)
I=1
PICKME:
IF P(I)>P THEN RETURN
I=I+1
GOSUB PICKME
POTPOINTS:
XC=XP*XSC+RIGHT
YC=100-YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
PSET (XC,YC)
RETURN
DATA 3
DATA    .5,0,0,0,.5,0,.33
DATA   .5,0,1,0,.5,0,.33
DATA   .5,0,.5,0,.5,.5,.34
```



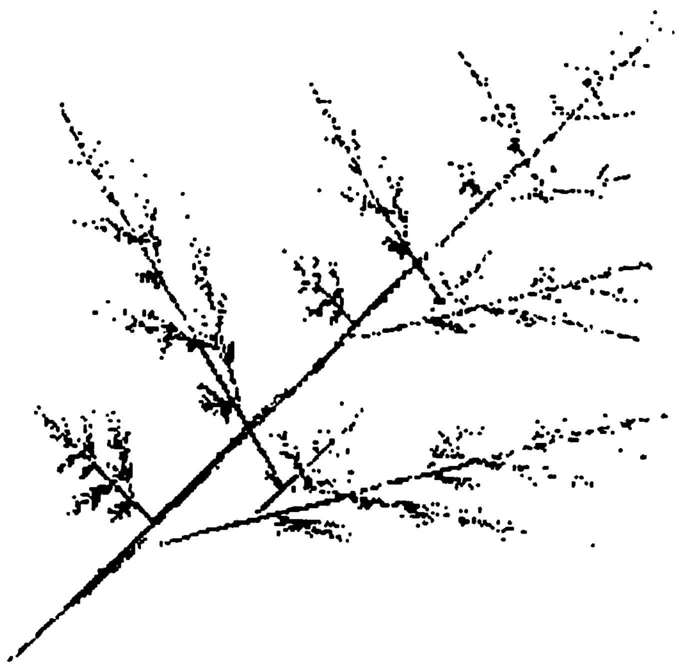
The Sierpinski Triangle

```
REM DRAGON
REM 2 TRANSFORMATIONS
REM 10000 ITERATIONS
XSC=100
YSC=100
RIGHT=140
DOWN=40
READ N
FOR I=1 TO N
READ  A(I),B(I),C(I),D(I),E(I),F(I),P(I)
P(I)=P(I)+P(I-1)
NEXT
X=0:Y=0
FOR N=1 TO 10000
GOSUB PICKP
XP=A(I)*X+B(I)*Y+C(I)
YP=D(I)*X+E(I)*Y+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
PICKP:
P=RND(1)
I=1
PICKME:
IF P(I)>P THEN RETURN
I=I+1
GOSUB PICKME
POTPOINTS:
XC=XP*XSC+RIGHT
YC=100-YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
PSET (XC,YC)
RETURN
DATA 2
DATA  .5,.5,0,-.5,.5,0,.5
DATA  -.5,.5,2,-.5,-.5,0,.5
```



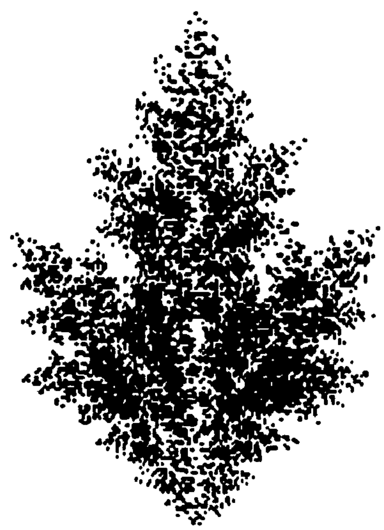
The Dragon

```
REM BRANCH
REM 6 TRANSFORMATIONS
REM 10000 ITERATIONS
XSC=179
YSC=178
RIGHT=-70
DOWN=0
READ N
FOR I=1 TO N
READ  A(I),B(I),D(I),E(I),C(I),F(I)
NEXT
X=0:Y=0
FOR N=1  TO 10000
GOSUB PICKP
XP=A(I)*X+B(I)*Y+C(I)
YP=D(I)*X+E(I)*Y+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
PICKP:
I=INT(6*RND(1))+1
POTPOINTS:
XC=XP*XSC+RIGHT
YC=YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
IF N>10 THEN PSET (XC,YC)
RETURN
DATA 6
DATA 0  ,-.28,0,.29,1.51,.92
DATA  .64,0,0,.64,.82,.06
DATA  -.02,.37,-.31,.29,.85,1.03
DATA  0,-.8,-.22,.01,2.43,1.51
DATA  -.01,.18,-.18,-.01,.88,1.47
DATA  .02,-.48,0,.50,1.6,.8
```



The Branch

```
REM LEAF
REM 4 TRANSFORMATIONS
REM 10000 ITERATIONS
XSC=75
YSC=75
RIGHT=250
DOWN=-50
READ N
FOR I=1 TO N
READ  A(I),B(I),j(I),C(I),F(I)
P(I)=P(I)+P(I-1)
NEXT
X=0:Y=0
FOR N=1 TO 10000
I=INT(4*RND(1))+1
XP=A(I)*X*COS(j(I))-B(I)*Y*SIN(j(I))+C(I)
YP=A(I)*X*SIN(j(I))+B(I)*Y*COS(j(I))+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
POTPOINTS:
XC=XP*XSC+RIGHT
YC=220-YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
PSET (XC,YC)
RETURN
DATA 4
DATA   .6,.6,0,0,-.5
DATA   .6,.6,0,0,.5
DATA   .5,.5,.7854,-.5,-.25
DATA   .5,.5,-.7854,.5,-.25
```



The Leaf

```
REM FERN
REM 4 TRANSFORMATIONS
REM 40000 ITERATIONS
XSC=25
YSC=23
RIGHT=250
DOWN=-0
READ N
FOR I=1 TO N
READ  A(I),B(I),D(I),E(I),C(I),F(I),P(I)
NEXT
X=0:Y=0
FOR N=1 TO 40000&
GOSUB PICKP
XP=A(I)*X+B(I)*Y+C(I)
YP=D(I)*X+E(I)*Y+F(I)
GOSUB POTPOINTS
X=XP:Y=YP
NEXT
LCOPY
END
PICKP:
P=INT(100*RND(1))+1
IF P=100 THEN I=1
IF 0<P AND P<86 THEN I=2
IF 85<P AND P<93 THEN I=3
IF 92<P AND P<100 THEN I=4
POTPOINTS:
XC=XP*XSC+RIGHT
YC=270-YP*YSC+DOWN
IF (XC<0) OR (XC>500) THEN RETURN
IF (YC<0) OR (YC>290) THEN RETURN
PSET (XC,YC)
RETURN
DATA 4
DATA    0,0,0,.16,0,0,.01
DATA   .85,.04,-.04,.85,0,1.6,.85
DATA   .2,-.26,.23,.22,0,1.6,.07
DATA   -.15,.28,.26,.24,0,.44,.07
```

The Fern