SENSITIVITY ANALYSIS AS APPLIED TO OPTIMAL CONTROL SYSTEMS

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ABSTRACT

This thesis is concerned with Sensitivity Analysis as applied to Optimal Control Systems. Sensitivity techniques are presented as important design tools to ensure good system performance. It is concluded that by applying these techniques to analysis and design problem, valuable information about system behaviour can be obtained. These techniques are readily applicable to computer-aided system design. They can be used to find the optimal system configuration, subject to variations in system parameters such as system order or the values of some system components.

The optimal control problem is defined for deterministic systems. The different techniques for finding the control law that optimizes a given system performance index are described. The case of linear models is stressed because many practical systems behave linearly within the range of interest or can be closely approximated by a linear model. The implementation of the control law can be either open-loop or closed-loop using a controller.

In the design of optimal control systems, nominal values are usually assigned to the different parameters such as system components, initial states and system order. Due to physical factors such as aging or environmental conditions, many parameters tend to deviate from their nominal values. Errors induced due to these deviations are obtained for infinitismal and large variations. Trajectory and/or Performance Index sensitivities are used to compare

different system configurations. If a system is susceptible to variations in parameters, a configuration which is less sensitive to these variations is considered superior. In particular, it is shown that open-loop and closed-loop implementations have equivalent performance index sensitivities. However, examples show the superiority of closed-loop configuration from a practical point of view.

Design techniques used to reduce system sensitivity are presented. They are classified into three main categories, namely: adaptive, choice of performance index and use of dynamic compensators. In the adaptive technique, the control system is modified through the inclusion of a corrective term to compensate for the output (State) error due to parameter variations. One form of the second technique is to augment the performance index by adding terms weighing the sensitivity coefficients to variations in some parameter. Alternatively, multiplying the integrand, in a quadratic cost functional by an exponential term leads to a less sensitive closed-loop system. Dynamical compensators are being used increasingly in control systems to serve different purposes such as system stabilization or decoupling. Their use in sensitivity reduction seems natural and it forms the third category.

LIST OF IMPORTANT ABBREVIATIONS AND SYMBOLS

A	System matrix
B	Input distribution matrix
C	Output distribution matrix
D	Input-Output coupling matrix
e_c	State error vector for closed-loop systems
e_0	State error vector for open-loop systems
E	State error vector
f,F	Arbitrary function of its arguments
g	Arbitrary function of its arguments
G(s)	System transfer matrix
h	Scalar function of its arguments
H	Hamiltonian of the dynamical system
I	Unit matrix
J	Cost functional or performance index
J*	Optimal value of the cost functional
J_c	Cost functional of a closed-loop system
J_0	Cost functional of an open-loop system
δJ_c	Variation in performance index of a closed-loop system
δJ_0	Variation in performance index of an open-loop system
K	Sampling instant
K(t)	Gain matrix
L	Scalar analytic function of its arguments
n,N	Order of the control system
P,P(t)	A symmetric matrix

$Q(t)$	Semi-definite symmetric state weighting matrix
$r(t)$	Input vector
$R(t)$	Positive definite control weighting matrix
R^n	Real space of dimension n
s	Laplace variable
S	Target set of dimension n-1
$S(t)$	Positive semi-definite symmetric matrix
S_μ^T	Scalar sensitivity function
t	Time
t_0	Initial time
t_f, T	Final time
$\underline{u}, u, u(t)$	Input or control vector
$\underline{u}^*, u^*, u^*(t)$	Optimal control vector
$\delta u(t), \delta u$	Variation in the control vector
v	Constant parameter vector
w	State vector of augmented system
$\underline{x}, x, x(t)$	State vector
$\underline{x}_0, x_0, x_0(t)$	Initial state vector
$\delta x, \delta x(t)$	Variation in the state vector
$\delta x/\partial\mu$	Sensitivity function matrix
$\underline{y}, y, y(t)$	Output vector
$z, z(t)$	Non-negative symmetric state weighting matrix
Γ	Upper limit of the control vector
γ_i	Upper limit on the magnitude of the i^{th} component of the control vector

ϕ	Vector function of its arguments
$\phi(t,t_0)$	State transition matrix
$\phi(t,\mu)$	Vector function of the parameter vector μ
ψ	Vector function of its arguments
ε	Infinitismally small real number
λ	Costate vector
λ^i	Sensitivity function vector with respect to parameter μ_i
ρ	A real positive number ≥ 1
μ	Parameter vector
$\delta\mu$	Variation in the parameter vector
$\eta^i(t,\mu)$	Output sensitivity vector with respect to the i^{th} component of the parameter vector
θ	A continuously differentiable scalar function of its arguments

CHAPTER 1

INTRODUCTION

1.1 General

In engineering, we are usually faced with two main types of problems. The first is to analyze a given system in order to improve some of its desired characteristics, suppress others which are undesirable, or have this system execute a given task. The second is to design and build a system to do a specific function. In both cases, we try to find a suitable mathematical model[1] that represents the system as closely as possible or as needed for our intended use. No matter how complicated this mathematical model is, there exist discrepancies between its response and that of the physical system to a given input.

In engineering analysis and design, it is a common practice to look for the simplest model that represents adequately the system for a particular purpose.[2,3] A simple model results in a simple mathematical problem that necessitates use of uncomplicated mathematical tools. In practice, a more accurate model leads to mathematical problems that might be more complex, e.g. higher order systems, or nonlinear systems. However one usually finds out that this extra complication is not justifiable, because the results obtained from the latter model could be within permissible limits of tolerances. On the other hand, any physical phenomena can rarely be described with very high accuracy by a finite dimensional model.

Normally systems are designed to serve a specific function and consequently many parameters are known to vary within a priori known ranges, e.g. a motor for a home appliance will operate at an input of 110V ± 10 Volts.

In many cases, the nature of the expected input to the system is known. Typical types of inputs are impulse, step, ramp and sinusoidal. A model that represents the system "closely" under normal operating conditions can be considered adequate for design as well as for analysis.

However a system may be described by more than one mathematical model. To select the best model, a measure should be available to judge the quality of the possible models under operative conditions. One possible measure is "Sensitivity".[4] Sensitivity could be simply defined as the rate of change of response with respect to change in some parameter. Inputs as well as initial conditions could be dealt with as parameters.

Sensitivity analysis deals with the calculation of the expected differences between the physical system and its mathematical model; when operating conditions are slightly altered from their nominal values. By defining a cost functional, one can measure how accurate the model represents the system. The choice of a particular cost functional depends upon the particular application for which the system is used, and our understanding of the process. It is nevertheless an arbitrary choice, and this point should be kept in mind when evaluating the system. Using sensitivity analysis, the engineer can interpret, with profit, the results obtained using an approximate model. He can predict the extent of the differences that might exist between the response of the model and that of the system. If these differences are within the limits of tolerance, the model can be considered adequate, otherwise a different model or a different cost functional should be considered. Thus, sensitivity analysis enables the engineer to assess the validity of a model and to apply the results obtained from model analysis to the physical system with far greater confidence. In this report, sensitivity analysis as applied to control systems is discussed.

1.2 Control Systems

A control system consists of two major parts, a plant and a controller. A plant is an object assembled to perform a given task, while operating within limitations, usually imposed by physical constraints, such as acceleration, voltage, etc.(5) In many instances the performance of the plant could be improved by the employment of another system connected to it in a particular fashion. This system is called the "Controller".

The specific choice of the controller for a given plant depends upon the desired objectives to be achieved, such as:

(a) Optimization with respect to some "Performance Index".(6)

(b) Minimization of the effect of undesirable disturbances.(7,8)

(c) Reduction of the sensitivity of the system to parameter variations.

(d) To meet design specifications.

A controller can be roughly classified as being either open-loop or closed-loop. Open-loop controllers are feeding the plant mainly with a priori determined information. They are in general simple and cheap from the practical point of view. In the absence of appreciable disturbances and errors or when the natural stable operational mode of the system coincides with the desired operating mode, open-loop control can be applied successfully. A closed loop controller is processing measured information from the plant to modify its action upon the plant. This could be easily achieved by comparing the desired operating point (input) of the system to the actual operating point (output) and feeding back the difference (error) to the input to drive the actual operating point towards the desired one.

1.2.1 Mathematical Formulation of the Control Problem

In control problems, usually we are given the plant whose characteristics are fixed. For the moment we will assume that the plant can be represented by a continuous model of the form[9]

$$\dot{\underline{x}} = \phi\,(\underline{x},\ \underline{u},\ t),\quad \underline{x}(t_0) = \underline{x}_0 \qquad \ldots (1.1)$$

where $\underline{x}$: state vector of dimension n

$\underline{u}$: input vector of dimension m

t : time, the system operates from time t_0 to t_f

$\cdot$: $\frac{d}{dt}$

ϕ : an n vector function of its arguments

In 1.2 some functions of the controller were stated. The determination of the function of the controller is equivalent to finding the control $\underline{u}$ (t) over a given period of time (t_0, t_f).[6] If $\underline{u}$ is generated as a function of time, the controller is of the open-loop type. This obviously does not necessitate the knowledge of the state vector $\underline{x}(t)$. On the other hand, $\underline{u}$ might be expressed as a function of the state vector $\underline{x}(t)$, i.e.

$$\underline{u} = \underline{u}\ (\underline{x}, t) \qquad \ldots (1.2)$$

In this case the controller is of the feed back type. A continuous estimate of $\underline{x}$ (t) is necessary to generate $\underline{u}$. If some of the components of the state vector $\underline{x}$ are not directly accessible (measureable), an observer may be useful if the system is observable.[10-12]

1.2.2 Plant Identification[1-3]

In case the model of the plant is not supplied by its manufacturer or if some modifications were introduced to the original plant, a mathematical model will be sought. The process of deriving a model is called "plant identification". The basic form of the model is first assumed, based on previous experience with similar plants. The nature of the model is a function of the way the plant operates and the type of controller to be used. The plant mathematical model can be of the form of either:

(a) An ordinary differential equation (O.D.E.)

(b) A difference equation (D.E.)

(c) A partial differential equation (P.D.E.) or

(d) A combination of some of the above types, with possible time delays.

If a digital computer is used as a controller or if the plant is intrinsically of discrete nature, a difference equation representation is most suitable, and is derived either directly, or by discretizing a continuous model.

Linear models form an important class of plant representation. They are the most extensively used form of model. This is due to the fact that many systems can be adequately represented by such models and many of the non-linear systems can be approximated by linear ones within a given range of input variations. This type of model is simpler to work with for analysis or synthesis purposes.

1.2.3 Linear Models

A continuous linear model is usually represented by

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t), \quad \underline{x}(t_0) = \underline{x}_0 \qquad \ldots (1.3)$$

$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t) \qquad \ldots (1.4)$$

where $\underline{x}$: n dimensional state vector

$\underline{u}$: m dimensional input vector

$\underline{y}$: r dimensional output vector

t : time, the system operates from t_0 to t_f

A, B, C and D are matrices of dimensions (n x n), (n x m), (r x n) and (r x m) respectively.

If all the elements of the matrices A, B, C and D are constant, the model is called fixed or time invariant.

A discrete linear time-invariant model has the form:

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \qquad \ldots (1.5)$$

$$\underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \qquad \ldots (1.6)$$

where k is the sampling instant and all other variables have similar definitions as above.

Two important properties of linear systems are "controllability" and "observability". They play a major rôle in establishing the existence and uniqueness of the optimal control. They may be defined as follows.

1.2.4 Controllability[9,13]

A system is said to be controllable if for each initial state $\underline{X}_0$ and a final state $\underline{X}_1$ there exists a piecewise continuous control $\underline{u}(t)$ on $[t_o,T]$ such that the application of this control to the system will drive it from $\underline{X}_o$ at $\underline{t}_o$ to $\underline{X}_1$ at T.

It has been shown that[13], for linear time-invariant continuous systems, the following condition is necessary and sufficient for a system to be controllable

$$\text{Rank } [B \,\vdots\, AB \,\vdots\, \ldots\ldots\ldots\ldots \,\vdots\, A^{n-1} B] = n \qquad \ldots (1.7)$$

1.2.5 Observability

A system is said to be observable if for any initial state $\underline{x}_0$ at t_0 there exists a time $t_1 \geq t_0$ such that a knowledge of $\underline{u}[t_0,t_1]$ is sufficient to determine $\underline{x}_0$. Mathematically, for linear time-invariant continous systems, this is equivalent to

$$\text{Rank } [C' \,\vdots\, A' C' \,\vdots\, \ldots\ldots \,\vdots\, A'^{n-1} C'] = n \qquad \ldots (1.8)$$

1.2.6 The State Transition Matrix[14]

A continuous linear system is represented by the equations:

$$\dot{\underline{X}}(t) = A(t)\,\underline{X}(t) + B(t)\,\underline{u}(t),\ \underline{x}(t_0) = \underline{x}_0$$

$$\underline{Y}(t) = C(t)\,\underline{X}(t) + D(t)\,\underline{u}(t) \qquad \ldots (1.9)$$

To determine the output (or the state) at any time $t > t_0$, it is sufficient to know the control $\underline{u}(t)$ over the interval $[t_0,t]$. The autonomous system corresponding to (1.9) is obtained by putting $\underline{u}=0$ to get:

$$\dot{\underline{X}}(t) = A(t)\,\underline{X}(t),\ \underline{X}(t_0) = \underline{x}_0 \qquad \ldots (1.10)$$

The solution of (1.10) is of the form

$$\underline{X}(t) = \phi(t, t_0)\,\underline{x}(t_0) \qquad \ldots (1.11)$$

The matrix $\phi(t, t_0)$ satisfies the following conditions

$$\frac{\partial}{\partial t}\phi(t, t_0) = A(t)\phi(t,t_0) \qquad \ldots (1.12)$$

$$\phi(t_0,t_0) = I \qquad \ldots (1.13)$$

$\phi(t,t_0)$ is known as the state transition matrix.

The complete solution of (1.9) is now given by

$$\underline{X}(t) = \phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t}\phi(t,\tau)\,B(\tau)\,\underline{u}(\tau)\,d\tau \qquad (1.14)$$

In case of time-invariant systems

$$\phi(t, t_0) = e^{A(t-t_0)} \qquad \ldots (1.15)$$

and the system transfer matrix G(S) can be obtained from (1.9) and (1.10) by taking the Laplace transform of both sides of each equation, assuming zero initial conditions

$$G(S) = [C(SI-A)^{-1}\,B + D] \qquad \ldots (1.16)$$

and $\quad \underline{Y}(S) = G(S)\,\underline{u}(S)$.

1.3 Optimal Control Systems (6)

A control system is called "optimal" if it optimizes (maximizes or minimizes) a prespecified "index of performance". This index of performance could be a function of the controller, system parameters and time, and its choice is arbitrary and based upon the nature of the purpose that the system serves. Therefore the word "optimal" has to be interpreted as being a "relative optimal", since it is optimal for a specific index of performance. In practice, the model of the system is used to arrive at the control that optimizes the "index of performance". However, due to differences between the system and the model, the control obtained using the model might not be the optimal one for the system. These differences are due to parameter variations, intial condition variations and control variations from the nominal values used while finding the response of the model.

1.3.1 Mathematical Formulation of the Optimal Control Problem

Mathematically, the optimal control problem can be stated as follows:

Given a continuous plant represented by

$$\dot{\underline{X}} = \phi(\underline{x},\underline{u};t),\ \underline{x}(t_0) = \underline{x}_0 \qquad \ldots (1.18)$$

Find the piecewise continuous control u(t) over the interval $[t_0,T]$ which minimizes the scalar cost functional given by

$$J = F\,[\underline{x}(t),\ T] + \int_{t_0}^{T} N(\underline{x},\underline{u},t)\,dt \qquad \ldots (1.19)$$

Under special conditions on ϕ, N and F, a unique optimal control $\underline{u}^*(t)$ that minimizes J, exists. It is very difficult to show that a unique solution exists. Further, finding a closed form expression for the solution is a very large problem. Often it is necessary to search for an acceptable suboptimal solution. Numerical techniques are then used to find $\underline{u}^*(t)$ or its subóptimum. Gradient techniques, dynamic programming and techniques based on necessary conditions of optimality are commonly used to find $\underline{u}^*(t)$.

1.3.2 The Maximum Principle

The control problem as defined above has some similarities with the calculus of variations, which has been extended by Pontryagin(16) in his well known "Maximum Principle". By defining a Hamiltonian H as:

$$H = N(\underline{x},\underline{u},t) + \lambda^T\phi(\underline{x},\underline{u},t) \qquad \ldots (1.20)$$

where λ is a time varying vector called the costate, Pontryagin showed that the optimum u*(t) would satisfy the following set of necessary conditions:

$$\text{(a)} \quad \left.\frac{\partial H}{\partial u}\right|_* = 0 \qquad \ldots (1.21)$$

This is the case only if u is unconstrained. This normally gives an expression for $\underline{u}^*(t)$ in terms of the other variables.

$$\text{(b)} \quad \frac{\partial H}{\partial \lambda} = \dot{x} \;,\; x(t_0) = x_0 \qquad \ldots (1.22)$$

This gives the system dynamics (1.18).

$$\text{(c)} \quad \frac{\partial H}{\partial x} = -\dot{\lambda} \;,\; \lambda(t_f) = \frac{\partial F[x(T),T]}{\partial x(T)} \qquad \ldots (1.23)$$

This gives a differential equation for $\lambda(t)$ with a boundary

(Transversality) condition at t_f.

The solution of (1.22) and (1.23) constitutes a two point boundary value problem (TPBVP) since half the boundary conditions are given at $t = t_0$ and the other half is given at $t = t_f$. This problem is normally solved by assuming an initial control $u_1(t)$ over $[t_0, T]$. $X(t)$ is integrated over $[t_0, T]$ using $u_1(t)$. The value of $\lambda_1(t_f)$ corresponding to $u_1(t)$ is compared to $\lambda(t_f)$ given by (1.23). If $\lambda_1(t_f) \neq \lambda(t)$, $u_1(t)$ is modified to get $u_2(t)$. This process is repeated iteratively till satisfactory results are obtained.

1.3.3 Linear Systems with Quadratic Cost Functionals[6]

It has been shown that finding the optimal control $\underline{u}^*(t)$ might be tedious process. One special case in which a simple expression for $\underline{u}^*(t)$ is guaranteed, is that of a linear system with quadratic cost functional. In this case, one can prove that the optimum control $\underline{u}^*(t)$ is unique and can be obtained by direct feedback from the state vector $\underline{x}$ (t) to the input, with an appropriate time-varying transformation. Since this closed-loop control does not depend upon the initial conditions, this leads to substantial simplification in implementing the control law. Simplicity is even greater in the case when all states are accessible.

The particular choice of the cost functional defines the nature of the desired control system response. In the case of a terminal controller, we choose our functional as a summation of a term measuring the state deviation from its desired value and an integral weighing the state deviation and control energy expenditure over the control interval. The particular choice of the weighting matrices S,R and Q is reached as a compromise between cost of control energy and the state deviation from zero. A large R indicates that energy is expensive and results in low energy expenditure. However, the corresponding deviations will be relatively large.

An important special case of the terminal controllers is that of a regulator. In this case the final time is large and there is no clear motivation for choosing a particular value of T, consequently, T is chosen as ∞.

An important advantage of linear regulators is that the resulting feedback gain is time invariant. In this case, the plant is represented by:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad \ldots (1.24)$$

and the cost functional J is

$$J = 1/2\ \underline{X}^T(T)\ S(T)\ \underline{X}(T) + 1/2 \int_{t_0}^{T} [\underline{x}^T(t)\ Q(t)\ \underline{x}(t) + \underline{u}^T(t)\ R(t)\ \underline{u}(t)]\ dt \qquad \ldots (1.25)$$

where S(T) and Q(t) are positive semidefinite symmetric real matrices.

R(t) : positive definite symmetric matrix.

Kalman[14] showed that a unique control $\underline{u}(t)$ exists which minimizes the cost (1.25) if:

(a) The pair (A,B) is controllable.

(b) Writing $Q = CC^T$, the pair (A,C) is observable.

Wonham[17] showed that a control $\underline{u}^*(t)$ might exist if (A,B) is stabilizable.

To use the maximum principle, we define the Hamiltonian H as:

$$H = 1/2\ (\underline{x}^T Q \underline{x} + u^T R u) + \lambda^T\ (A\underline{x} + B\underline{u}) \qquad \ldots (1.26)$$

The necessary conditions for a maximum are

$$\text{(a)} \quad 0 = \frac{\partial H}{\partial \underline{u}} = R\underline{u} + B^T \underline{\lambda} \qquad \ldots (1.27)$$

This gives

$$\underline{u}^* = -R^{-1}\ B^T\ \underline{\lambda} \qquad \ldots (1.28)$$

R^{-1} exists since R is positive definite.

$$\text{(b)} \quad \frac{\partial H}{\partial \lambda} = \dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad \ldots (1.29)$$

$$\text{(c)} \quad \frac{\partial H}{\partial x} = -\dot{\underline{\lambda}} = Q\underline{x} + A^T \lambda \qquad \ldots (1.30)$$

Putting $\underline{\lambda} = P\underline{x}$, (1.28), (1.30) and (1.24) become

$$\underline{u}^* = -R^{-1}\ B^T\ P\ \underline{X} \qquad \ldots (1.31)$$

$$-\dot{p}\ x - p\ \dot{x} = Q\ x - A^T\ p\ x \qquad \ldots (1.32)$$

$$\dot{x} = (A - B\ R^{-1}\ B^T\ P)\ X \qquad \ldots (1.33)$$

Consequently the differential equation for the matrix P (t) becomes

$$\dot{P}\ (t) = -A^T P - PA + PBR^{-1}B^T P - Q \qquad \ldots (1.34)$$

This is the well known matrix Riccati equation. The transversality condition gives

$$P(T) = S \qquad \ldots (1.35)$$

so equation (1.34) can be integrated subject to the boundary conditions.

An important case is that of infinite final time. This is called Regulator problem. In this case equation (1.34) reduces to the following algebraic matrix equation:

$$0 = A^T P + P A - P B R^{-1} B^T P + Q \qquad \ldots (1.36)$$

In general this equation has more than one solution. The solution we are interested in is the one that stabilizes the closed loop system given by (1.33). This is the unique positive definite solution of (1.36). In this case P is a constant and the control u given by (1.31) is obtained by constant feedback from the state $\underline{x}(t)$.

1.3.4 Inequality Constraints on the Control[6,15]

For many systems the control vector $\underline{u}$ is bounded. Physical limitations normally impose constraints of the form:

$$|| u || \le \Gamma \qquad \ldots (1.37)$$

$$| u_i | \le \gamma_i, \; i = 1, 2, \ldots, m \qquad \ldots (1.38)$$

Constraints of the form (1.38) are frequently encountered in practice. The solution of the optimal control problem subject to such constraints can be obtained using the maximum principle. Equation (1.21), however, is not applicable since $\underline{u}$ is constrained. In this case, it can be shown that for nonsingular problems, the control u(t) will be such that

$$|u_i(t)| = \gamma_i, \; i = 1, 2, \ldots, m \qquad \ldots (1.39)$$

A control of the form (1.39) is called "bang-bang" control. In bang-bang control of type (1.39), control components $u_i(t)$ change from $+\gamma_i$ to $-\gamma_i$ (or vice versa) at times t_{ij} called the switching times. The number of switching times depend upon the order of the system and the initial state x_0.

In bang-bang control systems the state space is divided by switching curves, switching surfaces or hypersurfaces. If the initial state x_0 does not lie on the switching surface, the initial control $u(t_0)$ must be chosen to move the system toward the switching curve. When the state $x(t)$ hits the switching curve at time t_s, some control components switch signs. The state space trajectory then lies in the switching surface till another switching surface is met, where some controls switch sign and the trajectory follows the new switching surface. The process continues till the final state is reached.

1.3.5 Suboptimal Control[18-21]

If the form of the optimal control $\underline{u}^*(t)$ is difficult to implement or generate, suboptimal control might be sought. The form of the suboptimal control should be chosen so as to simplify the synthesis of the controller. For instance, if the control $\underline{u}^*(t)$ is obtained in the form of a time varying feedback $\underline{u}^*(x,t)$, a time invariant feedback $u_s(x)$ can be considered.

It is obvious that the suboptimal control is not unique. However, one may limit the choice of u_s to $u_s \epsilon U$, where U is the class of all admissible controls that satisfy some additional conditions. For example, U may be the class of all controls of the form

$$\underline{u}(t) = K(t)\,\underline{X}(t) \qquad \ldots (1.40)$$

where K(t) is constrained to the set of all matrices whose K_{ij} elements are piecewise constant.

Suboptimal control might be considered in case of inaccessible states. One would like to generate the control as a function of the available output only. In practice, high order systems are often approximated by lower order models. The optimal control is obtained using the model. This control however, cannot be considered optimal for the original system. This is definitely a suboptimal control. Inaccuracies in the model parameters lead to a control which is suboptimal with respect to the actual system.

The quality of suboptimal control resulting from system or controller simplifications should be assessed. It can be considered satisfactory if the deterioration in system performance due to departing from the optimal control lies within permissible limits of tolerance for the particular application.

1.4 Scope

The term sensitivity arises in many mathematical and engineering applications. In numerical analysis, for example, one studies the sensitivity of the roots of a polynomial to changes in its coefficients or sensitivity of eigenvalues of a matrix when its elements are perturbed. In circuit theory, sensitivity of circuit reponse to variations in the circuit components is studied extensively. In fact some mathematical representations are preferred to others because they lead to lower sensitivity coefficients.

In industrial applications, system parameters change due to aging, temperature variation and change in the quality of the raw material used. This applies almost to any industry but it is particularly obvious in the petrochemical industries, cement plants, pulp and paper mills and steel mills.

In this report we shall confine ourselves to the application of sensitivity analysis to find the errors encountered due to the above mentioned variations. Methods and techniques used to render the system optimal subject to these variations are also reported and analyzed. Numerical examples illustrating the different techniques are given. The structure of this report is as follows:

In Chapter II the sensitivity to parameter variations is considered. This includes sensitivity to errors in the initial conditions which can be treated as parameters. A comparison between open-loop and closed-loop implementations is given.

Chapter III treats the problem of sensitivity to control variations. Limits of tolerance on the control signal are found so that the target set will be reached using any control within tolerance. The degradation of performance due to control errors is assessed.

Chapter IV deals with the design of low sensitivity systems. Several techniques are reported and classified according to the design philosophy. The different approaches are compared on the basis of design and operating costs.

Chapter V gives a summary of the problems treated in this report and possibilities of extension to other related topics are indicated.

CHAPTER II

SENSITIVITY TO PARAMETER VARIATIONS

2.1 Introduction

To analyze a given system or to design a new one, the control engineers deal with the mathematical model of the system rather than the physical system. Many of the parameters of the mathematical model are assigned nominal values. For example, a nonlinear process may be linearized about an operating point and the parameters of the linearized model are calculated for this specific operating point. In practice, however, the actual operating point might be different from the designed one. In this chapter we will present some of the sensitivity analysis techniques that can be applied to estimate the errors due to parameter variations.

Optimal control problems represent another domain of interest for sensitivity analysis. They are often formulated and solved as open-loop; that is, the control is a preprogrammed function of time, depending or not on particular initial conditions. Such a control is, of course, not responsive to actual errors in the initial state and to subsequent disturbances of the system's trajectory. To account for these automatically a closed-loop configuration becomes desirable whereby the optimal control is expressed and implemented as a function of the current state. The two configurations are equivalent in the sense that if both are started at the same intial state, then under deterministic conditions and for parameters at nominal values the state trajectories of both systems will be identical.

It is well known from the theory of linear control systems that feedback may reduce the sensitivity to parameter variations of the controlled plant.

Therefore a system that is insensitive to parameter variations is considered desirable. A measure of sensitivity should be established. Trajectory and performance sensitivities are the most practical measures of system response to parameter changes. If parameter variations are of infinitismal nature, i.e., very small compared to their nominal values, the concept of differential trajectory sensitivity and differential performance index sensitivity should be considered.

These sensitivies are respectively, the first-order variation $\delta x(t)$ of the state trajectory $x(t)$ and the first-order variation δJ of the performance index J, due to a first-order variation $\partial\mu(t)$ of a parameter $\mu(t)$. A treatment of these concepts is given in Sections 2.2 and 2.4.

It is worth mentioning here that trajectory and performance sensitivities are by no means the only concepts of sensitivity available. There can be no universal definition or measure of sensitivity since each class of systems suggests its own suitable definition. In an oscillator, for example, the sensitivity of the frequency and amplitude of the oscillations is a more suitable concept by far than trajectory or performance sensitivity.

If the changes in parameters are large with respect to their nominal values, the concept of ρ-sensitivity (global sensitivity) introduced by McClamroch et al[46] should be used. This technique is discussed in Section 2.5.

Applications of these concepts to optimal control problems are given in section 2.5.

2.2 Trajectory Sensitivity Functions and Sensitivity Equations

Differential equations for the trajectory sensitivity functions may be obtained readily from the system equations. Suppose a system is described by

$$\dot{x} = f(x,\mu,t) \quad \ldots (2.1)$$

where the n-dimensional vector x is the state vector, and the p-dimensional vector μ represents a set of p parameters. We assume that unique solutions exist for all initial conditions and for all values of μ. Furthermore, we assume f is continuously twice differentiable with respect to x and μ.[22,23]

Any solution of (2.1) may be considered a function of the parameter vector :

$$x = \phi(t,\mu) \quad \ldots (2.2)$$

This solution satisfies the initial condition

$$x_0 = \phi(t_0,\mu) \quad \ldots (2.3)$$

For a particular value of μ, the "nominal" value μ_n, we obtain the nominal solution

$$x_n = \phi(t, \mu_n) \quad \ldots (2.4)$$

We wish to study perturbations in this solution due to perturbation in μ. The "variation δx" is defined by

$$\delta x \triangleq \phi(t,\mu_n) - \phi(t,\mu) \quad \ldots (2.5)$$

and is due to a parameter variation

$$\delta\mu = \mu_n - \mu \qquad \ldots (2.6)$$

By using Taylor's theorem (2.5) may be written

$$\delta x = \left(\frac{\partial x}{\partial \mu} \right)_n \delta\mu + 0(||\delta\mu||)^2 \qquad \ldots (2.7)$$

where

$\left(\frac{\delta x}{\partial \mu} \right)_n$ is an nxp matrix of sensitivity functions evaluated for nominal x, the ij component of which is

$$\left(\frac{\partial x}{\partial \mu_j} \right)_n = \frac{\partial \phi_i(t,\mu)}{\partial \mu_j} \Big|_{\mu=\mu_n} \qquad \ldots (2.8)$$

The existence of these sensitivity functions is guaranteed by our assumptions on f given above.(22) The columns of the sensitivity matrix, $\left(\frac{\partial x}{\partial \mu} \right)_n$ are the vector sensitivity functions

$$\lambda^j \triangleq \frac{\partial x}{\partial \mu j},\ j = 1, \ldots p \qquad \ldots (2.9)$$

The differential equations for the vector sensitivity functions may be found either by Taylor's series expansion or by direct differentiation of (2.1). In either case, the necessary mathematical operations are justified by our assumptions on f. The result is

$$\frac{d}{dt} \lambda^j = \dot{\lambda}^j = \left(\frac{\partial f}{\partial x} \right)_n \lambda^j + \left(\frac{\partial f}{\partial \mu} \right)_n,\ j = 1, 2, \ldots p \qquad \ldots (2.10)$$

These equations are called the "sensitivity equations". The nxp matrix $\left(\frac{\partial f}{\partial x} \right)_n$ with ij component given by $\left(\frac{\partial f_i}{\partial x_j} \right)_n$

is the jacobian matrix evaluated on the nominal solution. It is noted that (2.10) is a linear differential equation, with time-varying coefficients, in general. Thus the effects of small perturbations $d\mu$ on a linear system may be evaluated, to first order, by solving a linear equation.

The initial conditions for (2.10) may be obtained from (2.3)

$$\lambda_0^j = \frac{\partial \phi(t_0,u)}{\partial \mu_j} = \frac{\partial x_0}{\partial \mu_j} \quad \ldots (2.11)$$

Sensitivity functions for variables other than state variables (such as outputs) may be obtained readily from the equations relating these variables to the state:

$$y = g(x,\mu,t) \quad \ldots (2.12)$$

Then

$$\frac{\partial y}{\partial \mu_j} = \dot{\eta}^j = \left(\frac{\partial g}{\partial x} \right)_n \lambda^j + \left(\frac{\partial g}{\partial \mu_j} \right)_n \quad \ldots (2.13)$$

If an input or control variable u is present, depending on μ,

$$\dot{x} = f(x,u,\mu,t) \quad \ldots (2.14)$$

Then

$$\dot{\lambda}^j = \left(\frac{\partial f}{\partial x} \right)_n \lambda^j + \left(\frac{\partial f}{\partial u} \right)_n \frac{\partial u}{\partial \mu_j} + \left(\frac{\partial f}{\partial \mu_j} \right)_n \quad \ldots (2.15)$$

and, if

$$y = g(x,u,\mu,t) \quad \ldots (2.16)$$

then

$$\dot{\eta}^j = \left(\frac{\partial g}{\partial x} \right)_n \lambda^j + \left(\frac{\partial g}{\partial u} \right)_n \frac{\partial u}{\partial \mu_j} + \left(\frac{\partial g}{\partial \mu_j} \right)_n \quad \ldots (2.17)$$

2.3 Comparison Sensitivity

For single-input single-output linear time-invariant systems, Bode[24] proposed the logarithmic derivative of the system transfer function T with respect to a parameter μ as a sensitivity function

$$S_{\mu}^{T}(s) = \frac{\partial \ln T(s, \mu)}{\partial \ln \mu} \qquad \ldots (2.18)$$

The sensitivity of one transfer function T with respect to some other transfer function P, such as the plant transfer function, was then defined as

$$S_{P}^{T}(s) = \frac{\partial \ln T(S, P(s))}{\partial \ln P(s)} \qquad \ldots (2.19)$$

If S_P^T is a real number, it can be interpreted as the ratio of the percentage change in T to the percentage change in P for differentially small changes.

If S_P^T is not real or the system is multivariable or time-varying, the above definitions can not be applied. Cruz and Perkins [25] proposed "Comparison Sensitivity" as a means of measuring sensitivity for multivariable, time-varying and even nonlinear systems.

Let the system be described by:

$$\dot{X}(t) = A(t)\,X(t) + B(t)\,U(t),\ X(t_0) = X_0 \qquad \ldots (2.20)$$

As mentioned in Chapter I, a control law $u_n(t)$ might be selected to achieve some given objectives. This control law is normally calculated using the nominal values $A_n(t)$ and $B_n(t)$ of the system parameters. If $u_n(t)$ is applied in an open-loop fashion to the nominal system, the corresponding trajectory will be

$$X_n(t) = \phi_n(t, t_0)\ X(t_0) + \int_{t_0}^{t} \phi_n(t,\tau)\ B_n(\tau)\ u_0(\tau)\ d\tau \qquad \ldots (2.21)$$

where $\phi_n(t,t_0)$ is the state transition matrix corresponding to $A_n(t)$. If $u_n(t)$ can be expressed as a linear function of the current state $x(t)$, i.e.

$$u_n(t) = C(t)\ x(t)$$

where $C(t)$ is an mxn matrix, the corresponding closed-loop system is

$$\dot{X}(t) = [A_n(t) + B_n(t)\ C(t)]\ X(t), \quad X(t_0) = X_0$$

and the closed-loop trajectory will be given by

$$X_n(t) = \phi_n(t,t_0)\ X(t_0) \qquad \ldots (2.22)$$

where ϕ_n is the state transition matrix corresponding to

$$[A_n(t) + B_n(t)\ C(t)]$$

Under the assumption that the initial states of both the open-loop and closed-loop systems are identical and system parameters are at nominal values, the open-loop and closed-loop trajectories given by (2.21) and (2.22) are identical. The two systems are called "nominally equivalent".[25]

Let the actual system parameters be $A(t)$ and $B(t)$. The open loop trajectory becomes

$$X_0(t) = \phi(t,t_0)\ X(t_0) + \int_{t_0}^{t} \phi(t,\tau)\ B(\tau)\ u(\tau)\ d\tau \qquad \ldots (2.23)$$

where $\phi(t,t_0)$ is the state transition matrix corresponding to $A(t)$.

Let $e_0(t)$ be the error in the open-loop trajectory due to changes in parameters, i.e.

$$e_0(t) = x_n(t) - x_0(t) \qquad \ldots (2.24)$$

Similarly we define the closed loop trajectory error as

$$e_c(t) = x_n(t) - x_c(t) \qquad \ldots (2.25)$$

Cruz and Perkins[25] suggest system sensitivity comparison in terms of a functional involving a quadratic form in parameter-induced errors

$$\text{Performance Index} = \int_{t_0}^{t_1} e^T(t)\ q(t)\ e(t)\ dt \qquad \ldots (2.26)$$

The feedback structure of Fig. 2.1(a) is said to be less sensitive than that of the nominally equivalent open-loop structure of Fig.2.1(b), if, for a fixed positive semidefinite matrix Q

$$\int_{t_0}^{t_1} e_c^T(t)\ Q\ e_c(t)\ dt \leq \int_{t_0}^{t_1} e_0^T(t)\ Q\ e_0(t)\ dt \qquad \ldots (2.27)$$

for all inputs r(t) such that

$$\int_{t_0}^{t_1} r(t)\ Q\ r(t)\ dt \leq \infty$$

and for all $t_1 \in [0,\infty]$. The inequality (2.27) can be expressed in terms of a frequency domain criterion as follows

Theorem [25]

A sufficient condition for

$$\int_{t_0}^{t_1} e_c^T(t)\ Q\ e_c(t)\ dt \leq \int_{t_0}^{t_1} e_0^T(t)\ Q\ e_0(t)\ dt \qquad \ldots (2.28)$$

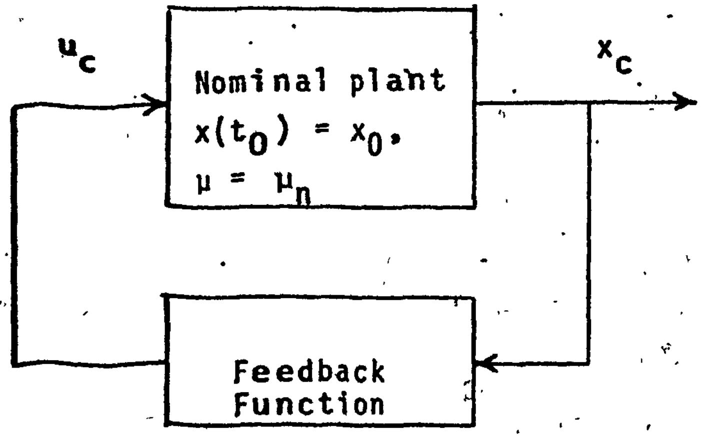

Fig. 2.1(a) Closed-loop Control System

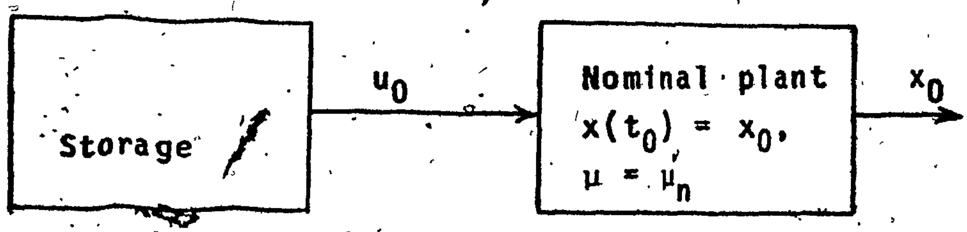

Fig. 2.1(b) Open-loop Control System

for all r such that

$$\int_{t_0}^{t_1} r\, Q\, r\, dt \leq \infty$$

and all $t_1 \in [0,\infty)$ is

$$Q - S^T(-j\omega)\, Q\, S\,(-j\omega) \geq 0 \qquad \ldots (2.29)$$

for all real ω.

2.4 Performance Index Sensitivity

Another approach to the problem of comparing the performance of closed-loop to open-loop configurations is the sensitivity of the performance index to parameter variations. It should be noted however that the value of the perturbed performance index $J^* + \delta J^*(\delta\mu, \mu^*)$ will not be in general optimal and may be either higher or lower than $J^*(\mu^*)$.

Pagurek(26) considered the case of a linear system and a quadratic cost functional with free end point $x(t_f)$. He proved that for this formuation the variation in performance index is the same for both open-loop and closed-loop, i.e.

$$\delta J_0(\delta\mu, \mu^*) = \delta J_c(\delta\mu, \mu^*) \qquad \ldots (2.30)$$

For all continuous parameter variations $\delta\mu(t)$. Witsenhauser(27) proved that the same result holds for non-linear systems and a more general class of cost functionals and constraints. Kokotovic et al(28) and Kokotovic and Heller (29) and Youla and Dorato(30) considered comparative performance index sensitivity for the nonconstrained case. Kreindler (31) extended the same result to the case of controls bounded by both the instantaneous inequality constraints and the isoperimetric inequality constraints. The Pagurek Witsenhauser result (2.30) also emerges as a special case of expression of the performance index for optimal adaptive control in Werner and Cruz.[32]

Equation (2.30) implies that feedback does not improve system sensitivity to parameter variations, a result which appears to contradict most of what we have proved earlier. This contradiction can be explained by the fact that all the previous results are based on trajectory sensitivity rather than performance sensitivity. Actually (2.30) has a simple intuitive explanation. For a cost functional of the form

$$J = g(t_f, x(t_f)) + \int_{t_0}^{t_f} h(t,x,u)\, dt \qquad \ldots (2,31)$$

Kreindler[33] showed that the variation δJ of J due to variations $\delta u(t)$ and $\delta x(t)$ is given by

$$\delta J = g_x^T\, \delta x(t_f) + \int_{t_0}^{t_f} (h_x^T\, \delta x + h_u^T\, \delta u)\, dt \qquad \ldots (2.32)$$

For open-loop systems, $\delta u(t)$=zero and consequently the second term in the integrand (2.32) vanishes. In the closed-loop system δu is given by

$$\delta u(t) = -k_x[t, x(t), u^*]\, \delta x(t) \qquad \ldots (2.33)$$

In the optimal system J* is stationary with respect to all admissible control variations, including the one given by (2.33). Therefore the part of δJ_c^* due to δu vanishes and (2.30) follows.

For the case of performance index sensitivity to variations in the initial state x_0 equation (2.30) does not hold. The reason is that the nominally equivalent systems are derived on the basis that their respective initial states are identical. It is expected that if x_0 varies from its nominal value x_0^*, the subsequent trajectories will be different even if other parameters are kept identical. By optimality of the closed-loop system, it follows that

$$\Delta J_0^* \left(\Delta x_0, \mu^*\right) \geq \Delta J_c^* \left(\Delta x_0, \mu^*\right) \qquad \text{. . . (2.34)}$$

The relation (2.30) is quite general and includes discontinuous and bang-bang optimal controls. It is valid provided δJ given by (2.32) exists. In particular (2.30) is violated when the plant is unstable and the upper limit of the integral in the performance index is $t_f = \infty$. Then $\delta J_0^* = \infty$, while the closed-loop system is stable δJ_c^* is finite.

The use of performance sensitivity for optimal systems is quite logical in principle; in practice, however, it is for several reasons of somewhat limited value. Quite often the performance index is a weighted sum reflecting several design objectives, and the optimization is merely a design device. The optimal values and sensitivities of the components of J, rather than of J itself, are then of interest. Performance sensitivity is inapplicable for the case where J is independent of x(t), that is,

$$J = \int_{t_0}^{t_f} h(t,u)\, dt \qquad \text{. . . (2.35)}$$

as in the minimum time (h=1) and minimum effort (h=1) and minimum effort ($h=|u|^P$, $P \geq 1$) problems, which are prime examples for the case where the optimal value of J is of interest. This is so because in these cases, $\delta J = 0$ for the open-loop system, and the only effect of parameter variation is a terminal error. In general, δJ is not (by itself) a valid sensitivity concept, in cases where $x(t_f)$ is not free. In such systems, meeting the terminal condition on $x(t_f)$ is part of the control objective; and, unless special methods are used, one cannot expect $\delta x(t_f)$ to be zero (or equal for both systems compared) for all $\delta\mu$. Therefore, in systems where $x(t_f)$ is not free, some combination of δJ and $\delta x(t_f)$ is an appropriate sensitivity concept. Often a terminal condition on $x(t_f)$ is handled by appending it to the performance index as a heavily weighted additional term. In this case of the use of δJ is applicable and result holds; however, the presence of an arbitrary weighting coefficient in J renders the values of J^* and δJ^* uninteresting in practice.

2.5 Estimation of Errors Due to Large Parameter Variations

Parameter variations are not always of infinitesimal size. Large variations do occur in many optimally controlled plants.

Uncertainty may occur in the plant due to environmental and aging effects. Also, there are always inherent uncertainties in the choice of a mathematical model both for the controlled system and the controller. Thus, for several reasons there will be discrepancies between any physical process and the mathematical model which is chosen as its representation.

There have been various techniques in the literature to develop theoretical methods to study this problem. The so-called "performance index sensitivity vector", as discussed in the previous section, was proposed by Dorato[34], has been used by Pagurek[26], Witsenhausen[27] and Dunn[36] to investigate the change in a performance index due to sufficiently small changes in the system parameters. In particular, for a certain class of problems it has been shown that various implmentations of the optimal control, e.g. feedback or open loop, lead to the same performance index sensitivity vector. However, this approach has certain inherent disadvantages due to the fact that the sensitivity vector is defined as the gradient of the performance index with respect to a parameter vector. Thus, this approach can yield information only of a local nature. Several other approaches based on the consideration of essentially finite changes in the system characteristics have been considered. Howard and Rekasius[37] have considered the worst possible parameter variations within some class in the sense that the performance index is maximized. In addition, Rissanen[38] and McClamroch[39] have considered the problem of specifying an upper bound on the change in the performance index and determining admissible variations in the parameters. Similar results have appeared in papers by Rissanen and Durbeck[40] and Sarma and Deekshatulu[41].

The related problem of determining the admissible parameter variations so that the value of the performance index does not change has also been studied. Various results have been reported in papers by Barnet and Storey[42], by McClamroch and Aggarwal[43,44], and by McClamroch, Aggarwal, and Clark[45].

McClamroch et al[46] defined the concept of ρ sensitivity, for linear systems with quadratic type cost functional, to large parameter variations. For some real number ρ there exists a class of parameter variation ε for which the system is ρ sensitive.

ρ- Sensitivity occurs if the value of the cost functional does not increase by more than a factor of ρ for any change in the class ε in comparison with a nominal or errorless system. For fixed ρ several methods are developed which allow determination of certain error classes ε for which ρ-sensitivity occurs. Analogous results are obtained for both the finite time problem and the infinite time problem.

2.5.1 Development of the Technique[46]

The system is treated as a linear regulator problem. The plant is represented by the linear differential system:

$$x = F(t)x + B(t)u, \; x(t_0) = x_0 \qquad \ldots (2.36)$$

We wish to choose the control u from the set of all bounded piecewise continuous functions defined on the time interval $[t_0, t_1]$, so that the value of the performance index given by

$$J = \frac{1}{2} x'(t_1)\Lambda x(t_1) + \frac{1}{2} \int_{t_0}^{t_1} \{x'(t)\ H(t)\ x(t)$$

$$+ u'(t)\ R(t)\ u(t)\}\ dt \qquad \ldots (2.37)$$

is a minimum with respect to all admissible controllers.

The usual assumptions on vector and matrix dimensions as well as controllability are assumed. It is well known that the optimal control, in feedback form, is given by

$$u^*(t) = -R^{-1}(t)\ B^T(t)\ P(t)\ x(t) \qquad \ldots (2.38)$$

and the optimal value of the performance index is given by

$$J^* = \frac{1}{2} x_0^T\ P(t_0)\ x_0 \qquad \ldots (2.39)$$

where $P(t)$ is a symmetric nxn matrix which is the unique solution to

$$\dot{P} + PF + F^T P - PBR^{-1}B^1 P + H = 0$$

$$P(t_1) = \Lambda \qquad \ldots (2.40)$$

Due to variations in the parameters, it is assumed that the differential system (2.36) changes as

$$\dot{x} = F(t)x + B(t)u^* + E(t)x \qquad \ldots (2.41)$$

where the nxn matrix $E(t)$ represents the change or error in the system (2.36). The matrix $E(t)$ could correspond to a variation in either the matrix $F(t)$ or $B(t)$. For physical reasons there is no need to assume that the matrices A, $H(t)$, or $R(t)$ change. Thus the value of the performance index (2.37) changes because the trajectory changes. Note also that since the optimal control u^* is given in feedback form by (2.38) it obviously changes, since the trajectory changes.

Now consider a slightly more general problem in which the control does not appear explicitly. The linear differential system is given by

$$\dot{x} = A(t)x + E(t)x, \quad x(t_0) = x_0 \qquad (2.42)$$

and the performance index is

$$J = x^T(t_1)\, Ax(t_1) + \int_{t_0}^{t_1} x^T(t)\, Q(t)\, x(t)\, dt \qquad (2.43)$$

where A is non-negative definite and Q(t) is positive definite. The matrix E(t) is again considered to represent the system error, the errorless or nominal system being given by (2.41) with $E(t) = 0$. If the optimal feedback control (2.37) is substituted into (2.37) and (2.41) then we obtain (2.43) with $A = F - PBR^{-1}B^TP$ and $Q = H + PBR^{-1}B^TP$. In fact (2.37) and (2.40) can be obtained by using any linear feedback control for (2.37) and (2.38) optimal or not. Thus it suffices to consider only the system (2.41) with the performance measure (2.38).

Since the matrix E(t) represents an error its value is not known exactly; it is assumed that the error is a member of some appropriate class of errors, i.e. $E\epsilon\varepsilon$. In order to include the possibility of an errorless system it is also assumed that $0\epsilon\varepsilon$.

With these preliminaries the following definition makes clear the concept of sensitivity.

<u>Definition</u>: For some real numbers ρ and some class of errors the system (2.41) and the performance measure (2.43) are said to be ρ-sensitive if for each $x_0 \epsilon R^m$

$$J_E \leq \rho J_0 \quad \text{for all } E \epsilon \varepsilon. \qquad \ldots (2.44)$$

Here J_E denotes the value of the performance index (2.44) evaluated along the solution of (2.41), assuming an error matrix $E\epsilon\varepsilon$. J_0 denotes the value of the performance measure for no system error.

Since $0\epsilon\varepsilon$ by assumption, it is sufficient to consider only $\rho \geq 1$. It should be noticed that if $0\epsilon\varepsilon_1$, $0\epsilon\varepsilon_2$ and ρ sensitivity holds for the class ε_1, then $\varepsilon_2 \subset \varepsilon_1$ implies that ρ-sensitivity holds for the class of errors ε_2. Thus these various classes can be ordered by inclusion. It would be desirable to know the maximum class under the above ordering; however for practical reasons the definition is in terms of an arbitrary error class ε.

2.6 Applications

2.6.1 Trajectory Sensitivity of an Open-Loop Linear Time-Invariant System(47,48)

As mentioned above, many control systems are controlled in an open-loop fashion. The main advantages of this type of control are its low cost and ease of application. However such a control is inherently irresponsive to variations in initial conditions or system parameters. One may still apply an open-loop control to a system if the errors resulting from parameter variations could be tolerated. The limits of tolerance are dependent upon the specific application.

For any given system, there are certain outputs (or states) whose variations from nominal values should be kept within pre-assigned limits. Variations in other outputs (or states) may be allowed without any serious degredation of performance. Therefore the control engineer usually computes the maximum deviations for only a few important outputs. These deviations are functions of the expected variations in system components and of the sensitivity functions which are dependent upon plant configuration. It is a challenging engineering problem to optimize a plant layout so as to minimize the sensitivity functions of some particular variables.

Consider a linear time-invariant system with parameter vector μ described in state form by

$$\dot{x} = \frac{d}{dt} x(t,\mu) = A(\mu)\, x(t,\mu) + B(\mu)\, u(t) \quad \ldots (2.45)$$

with the initial condition

$$x(0,\mu) = x_0(\mu) \quad \ldots (2.46)$$

The output is related to the state and the input by

$$y(t,\mu) = C(\mu)\, x(t,\mu) + D(\mu)\, u(t) \quad \ldots (2.47)$$

The functional dependence of the state and the output on both t and μ has been shown explicitly. The input $u(t)$ is considered to be independent of μ although μ-dependent inputs may be considered by a simple modification of the following derivation. Differentiating (2.45) with respect to μ (assuming such derivatives exist), and interchaning the μ_i and t partials on the left-hand side, we obtain

$$\dot{\lambda}^i = \frac{d}{dt} \lambda^i (t,\mu) = A(\mu)\lambda^i(t,\mu) + \frac{\partial A(\mu)}{\partial \mu_i} x(t,\mu) + \frac{\partial B(\mu)}{\partial \mu_i} u(t) \quad \ldots (2.48)$$

where

$$\lambda^i \triangleq \frac{\partial x}{\partial \mu_i} \quad \ldots (2.49)$$

The coefficient matrices A, $\partial A/\partial \mu_i$ and $\partial B/\partial \mu_i$ are to be evaluated at nominal μ, $\mu=\mu_n$ yield the trajectory sensitivity functions for small changes about $\mu=\mu_n$.

Similarly, differentiating (2.47) we obtain

$$\eta^i(t,\mu) = \frac{\partial C(\mu)}{\partial \mu_i} x(t,\mu) + C(\mu)\lambda^i(t,\mu) + \frac{\partial D(\mu)}{\partial \mu_i} u(t) \quad \ldots (2.50)$$

The initial conditions for (2.48) may be obtained as follows

$$\lambda_0^i \triangleq \lambda^i(t,\mu)\Big|_{t=0} = \frac{\partial x(t,\mu)}{\partial \mu_i}\Big|_{t=0}.$$

Assuming appropriate continuity conditions, we may interchange the order of the partial with respect to μ and the substitution of t=0. Thus

$$\lambda_0^i = \frac{\partial}{\partial \mu_i}\left[x(t,\mu)\Big|_{t=0}\right] = \frac{\partial x_0}{\partial \mu_i} \quad \ldots (2.51)$$

Equations (2.48) and (2.50) define the sensitivity model. The equations have a very interesting form, as is displayed pictorially in the matrix block diagram of Fig. 2.2 showing both the system model, described by (2.45) and (2.47) and the sensitivity model (for a single-parameter μ_i), described by (2.48) and (2.50).

The system model and the sensitivity model both have the same A matrix, and thus the same state transition

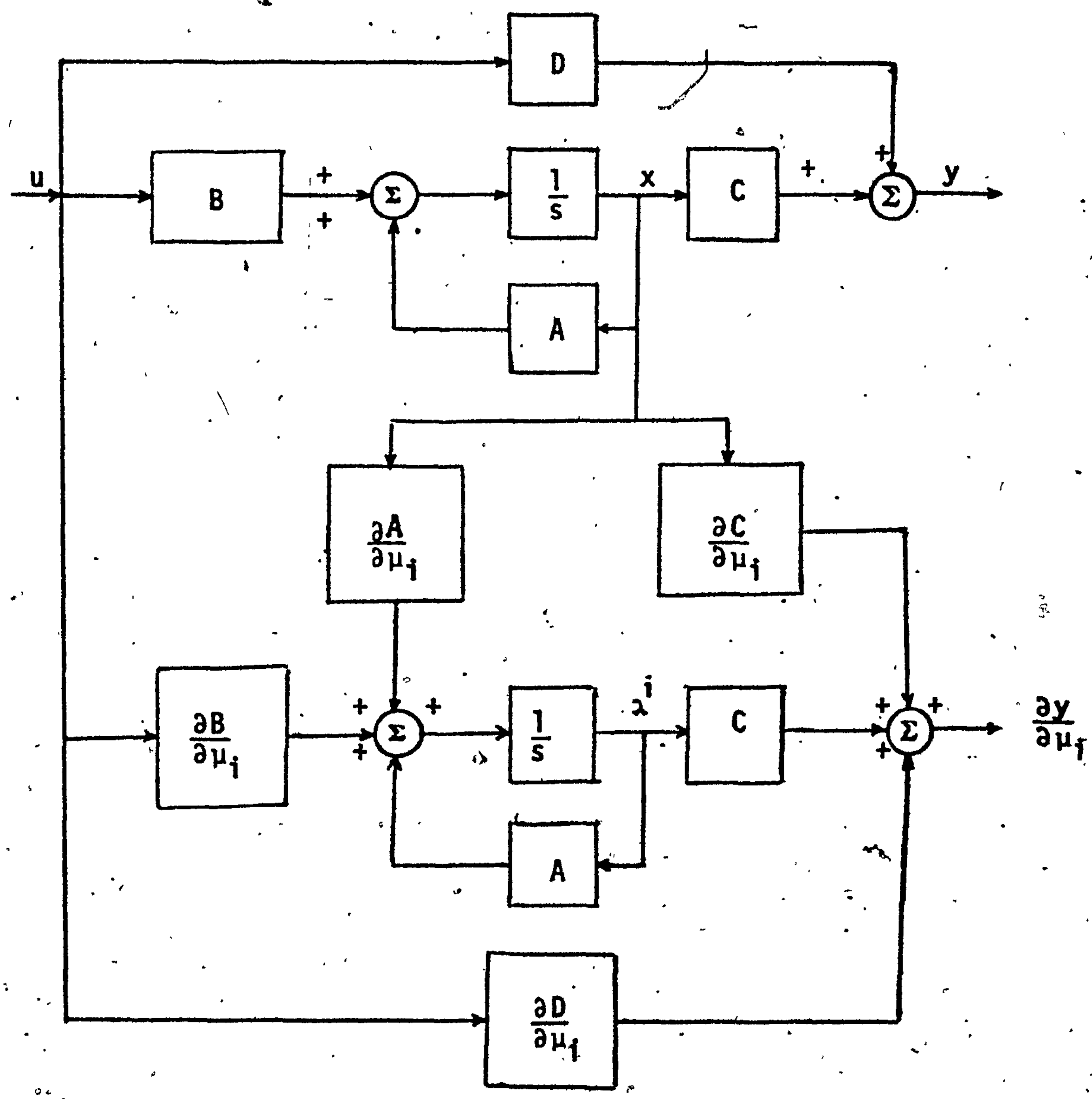

Fig. 2.2 Simulation of System and Associated Sensitivity Model

matrix. Figure 2.2 could represent an analog simulation of the system and the sensitivity model. The partial derivatives are obtained as analog outputs with no arithmetical calculations of differences being required. If K parameters are varying, equations of the form of (2.48) and (2.50) describe the situation; thus sensitivity models may be used to obtain all the sensitivity functions.

Wilkie & Perkins[48] proposed a technique by which only one sensitivity model in addition to that of the original system are needed to obtain all the sensitivity functions. The overall dimensions of such configuration is 2n which leads to a big saving in computational time or simulation equipment.

We can, therefore, using the above technique, find the differential equations and the initial conditions for the sensitivity functions for each component of the state vector with respect to each component of the parameter vector. We showed that the sensitivity differential equation is always linear even if the original system is nonlinear which makes the solution quite straightforward. The solutions of these differential equations give us an indication of the degree of sensitivity of an x_i with respect to an μ_j.

The expected variations of the parameters should now be considered. It is well known that some states (e.g. the outputs) are more important than others and therefore we have to ensure that their expected variations should be within reasonable limits. If a component of the sensitivity matrix

λ_j^i is found to be high and if x_j is an important state and μ_i is expected to vary widely, it is reasonable to assume that x_j will vary excessively. In this case we can conclude that the design is unsatisfactory and alternatives should be sought. One such alternative is implementing the control law in closed-loop form. If this turns out to be unsatisfactory too, the design techniques of Chapter IV should be considered.

2.6.2 Nonlinear Systems with Constrained Control[31,33]

Comparison sensitivity can be applied to linear as well as nonlinear systems. Even the case of constraints on the control can be handled without adding much to the complexity of the problem. In this case we assume that the controls of both the nominal and the perturbed systems satisfy the mathematical equations representing the constraints.

Let the plant be described by vector differential equation

$$\dot{X} = F(t, x, \mu, u), \quad X(t_0) = X_0 \quad \ldots (2.52)$$

where the scalar t is time, x is the n-dimensional state, u the r-dimensional control function and μ a p-dimensional continuously time-varying parameter. The function F is assumed to be twice continuously differentiable in t, x, u and μ. The nominal values of all variables are denoted by a subscript n.

The control u(t) will be considered piecewise continuous, and the values of u(t) may be considered to be in a possibly time-varying region of r-space given by

$$\phi(t,u) \leq 0 \quad \phi = (\phi^1, \phi^2, \phi^3, \ldots \phi^q) \quad \ldots (2.53)$$

where ϕ is twice continuously differentiable. $|u| \leq 1$ is a special case of (2.53) with

$$\phi^1 = u - 1 \quad \text{and} \quad \phi^2 = u + 1$$

The objective of the control is to transfer the state x from an initial point $x(t_0) = x_0$ to some point $x(t_1) = x_1$ in a smooth m-dimensional terminal manifold in (t,x) space $m=n-1$, given by

$$\psi(t,x) = 0 \quad , \quad \psi = (\psi^1, \psi^2, \psi^3, \ldots \psi^m) \qquad (2.54)$$

while minimizing the performance index

$$I = g[t_1, x(t_1)] + \int_{t_0}^{t_1} h(t,x,u)\, dt \qquad \ldots (2.55)$$

where g and h are scalar functions twice continuously differentiable in t, x and u, and where t_0 is a fixed initial time and t_1 is either fixed or free terminal time.

The system is subject to another constraint of the form

$$\int_{t_0}^{t_1} \theta(t,u)\, dt \leq E \qquad \ldots (2.56)$$

where θ is a continuously differentiable scalar function.

The optimal control $u^*(t)$ can be found as an explicit function of time for a specific t_0 and x_0; that is, it is an open-loop control. If the dependence on t_0 and x_0 is eliminated and $u^*(t)$ is expressed as a function of the current t and x, that is

$$u^*(t) = -k\,(t,x) \qquad \ldots (2.57)$$

then we have a closed-loop or feedback control.

Consider the effect of differential parameter variations from their nominal values μ_n. For small, time-varying, multivariable parameter variations, $\Delta\mu(t)$ we can write

$$\mu(t) - \mu_n(t) = \Delta\mu(t)$$

From (2.52) we can write

$$\delta\dot{x} = f_x \partial x + f_\mu \partial\mu + f_u \partial u \quad \text{. . . (2.58)}$$

for the open loop system, u is independent of μ, and thus $\delta u = 0$ and from (2.58) it follows

$$\dot{e}_0 = f_x e_0 + f_\mu \delta\mu, \quad e_0(t_0) = 0 \quad \text{. . . (2.59)}$$

In the closed loop system, u(t) is given by (2.57) and therefore

$$\dot{e}_c = f_x e_c + f_u \delta u - f_u k_x e_c, \quad e_c(t_0) = 0 \quad \text{. . (2.60)}$$

Let $\phi(t,t_0)$ be the state transition matrix corresponding to f_x, then the solution of (2.59) is

$$e_0(t) = \int_{t_0}^{t} \phi(t,\tau) f_\mu \delta\mu(\tau) d\tau \quad \text{. . . (2.61)}$$

and hence from (2.60)

$$e_c(t) = \int_{t_0}^{t} \phi(t,\tau) f_\mu \mu(\tau) d\tau - \int_{t_0}^{t} \phi(t,\tau) f_u k_x e_c(\tau) d\tau \quad \text{. . . (2.62)}$$

Kreindler[31] gives the following lemma:

For the pair of nominally equivalent open-loop and closed-loop systems given respectively by

$$\dot{X} = f[t, x, u(t)], \quad x(t_0) = x_0 \quad \text{. . . (2.63)}$$

and

$$\dot{X} = f[t, x, -k(t,x)], \quad x(t_0) = x_0 \quad \text{. . . (2.64)}$$

where the feedback function $k(t,x)$ is assumed to be continuously differentiable in t and x. The open-loop and closed-loop trajectory sensitivities given by (2.61) and (2.62) respectively are related by

$$e_0(t) = e_c(t) + \int_{t_0}^{t} \phi(t,\tau)\, f_u\, k_x\, e_c(\tau)\, d\tau \quad . . (2.65)$$

Furthermore, the closed-loop system is less sensitive than the open-loop one according to the inequality

$$\int_{t_0}^{t'} e_c^T(t)\, Z\, e_c(t)\, dt \le \int_{t_0}^{t'} e_0^T(t)\, Z\, e_0(t)\, dt \quad (2.66)$$

where t' is an arbitrary time $t' > t_0$, if and only if

$$\int_{t_0}^{t'} [2\, e_0^T(t)\, Z\, v(t) + v^T(t)\, Z(t)\, v(t)]\, dt \ge 0 \quad (2.67)$$

where $Z(t)$ is some continuously differentiable, non-negative definite symmetric matrix and

$$v(t) = \int_{t_0}^{t} \phi(t,\tau)\, f_u\, k_x\, e_c(\tau)\, d\tau$$

The inequality in (2.66) holds if and only if (2.67) is satisfied with an equality.

A result similar to the above lemma, for time-invariant linear systems was first given by Cruz and Perkins(49,50) and has been extended and generalized by them and by others in several directions.(51-58)

A special case of the above is when the system dynamics can be written in the form

$$\dot{X} = a\,[t,\ x,\ \mu] + b\,[t,\ u,\ \mu] \qquad \ldots (2.68)$$

and

$$x(t_0) = x_0\ ,\quad \psi[t_1,\ x(t_1)] = 0 \qquad \ldots (2.69)$$

and the performance index to be optimized is

$$J = g\,[t_1,\ x(t)] + \int_{t_0}^{t_1} [q(t,x) + r(t,u)]\ dt \qquad . (2.70)$$

if all the functions a, b, q and r are to satisfy the smoothness condition (Legendre-Clebsch sufficiency conditions), the nominally equivalent closed loop system is less sensitive than the open loop system to continuous first order parameter variations according to the inequality

$$\int_{t_0}^{t'} e_c^T\ Z\ e_c\ dt \leq \int_{t_0}^{t'} e_0^T\ Z\ e_0\ dt \qquad \ldots (2.71)$$

for all $t_0 \leq t' \leq t_1$

where Z is a continuously differentiable non-negative symmetric matrix given by

$$Z = k_x^T\ H_{uu}\ k_x \qquad \ldots (2.72)$$

where H is the Hamiltonian of the optimization problem.

2.6.3 Linear Regulator Problem

The linear regulator problem is again a special case of the nonlinear problem treated above. In this case the system dynamics are

$$\dot{X}(t) = A(t,u)\ X(t) + B(t,u)\ U(t),\quad x(t_0) = x_0 \qquad \ldots (2.73)$$

and the performance index to be minimized is given by

$$J = \frac{1}{2} X^T(t_1)\ G\ X(t_1) + \frac{1}{2}\int_{t_0}^{t_1} [X^T\ Q(t)X + U^T\ R(t)\ U]\ dt \quad (2.74)$$

The matrices A, B, Q and R are continuously differentiable in t and u. R(t) is a positive definite symmetric matrix, G and Q(t) are non-negative definite and symmetric matrices, t_1 is some fixed terminal time. In this case

$$H_{uu} = -K(T)\ X \quad \ldots (2.75)$$

where

$$K(t) = R^{-1}(t)\ B^T(t)\ P(t) \quad \ldots (2.76)$$

where P is the solution of the matrix ricatti equation.

Kreindler[31] proved that for the linear optimal regulator problem, the nominally equivalent closed-loop system is less sensitive to continuous first order parameter variations than the open-loop system, according to the inequality

$$\int_{t_0}^{t'} e_c^T\ Z\ e_c dt \leq \int_{t_0}^{t'} e_0^T\ Z\ e_0\ dt \quad \text{for all } t',\ t_0<t'\leq t_1 \quad . (2.77)$$

where Z is a non-negative definite symmetric matrix given by

$$Z(t) = K^T(t)\ R(t)\ K(t)$$

The inequality sign in (2.77) occurs if and only if

$$v^T(t')\ P(t')\ v(t') + \int_{t_0}^{t'} v^T\ Q\ v\ dt = 0 \quad \ldots (2.78)$$

where

$$v(t) = \int_{t_0}^{t} \phi(t,\tau)\ B(\tau)\ K(\tau)\ e_c(\tau)\ d\tau \quad \ldots (2.79)$$

If Q(t) is positive definite, the inequality sign in (2.77) can occur if and only if

$$e_c(t) \equiv e_0(t), \quad t_0 \leq t \leq t' \qquad \ldots (2.80)$$

This applies, of course, to the time-invariant case, where all the matrices in (2.73) and (2.74) are time-invariant and in the performance index G = 0, t = 0 and $t_1 = \infty$, the plant (2.73) is then assumed to be completely controllable and Q to be such that $X^T(t)\, Q\, X(t) \not\equiv 0$ for all the solutions of $\dot{X} = AX$. Then the feed back matrix K and consequently Z are time-invariant. For this case, the nominal values of the parameters must be, of course, time-invariant, but the differential parameter variations can be time-varying. Inequality (2.77) is now

$$\int_0^{t'} e_c^T\, Z\, e_c\, dt \leq \int_0^{t'} e_0^T\, Z\, e_0\, dt \qquad \ldots (2.81)$$

For the further special case of a single-input plant (i.e., where u is a scalar) in phase-variable form we have the following result.

Kreindler[59] also showed that for a single-input, time-invariant, linear regulator where the plant equations are in the form

$$\dot{x}^i = x^{i+1}, \quad i = 1, 2, \ldots, n-1 \qquad \ldots (2.82)$$

$$\dot{x}^n = a^1x^1 + a^2x^2 + \ldots + a^nx^n + u \qquad \ldots (2.83)$$

and the performance index is

$$J = 1/2 \int_0^{\infty} [X^T\, QX + (u)^2]\, dt \qquad \ldots (2.84)$$

The closed-loop system is less sensitive than the nominally equivalent open-loop system to continuous first order parameter variations according to

$$\int_0^{t'} (e_c^i)^2 \, dt \le \int_0^{t'} (e_0^i)^2 \, dt \qquad \ldots (2.85)$$

for all $t' < 0$, $i = 1, 2, \ldots, n$

2.6.4 Sensitivity of Time Varying Linear Systems to Large Parameter Variations

Consider the general time-varying problem for the state equation (2.41) and performance measure (2.43). The generic matrix $E(.)\epsilon\varepsilon$ denotes the unknown system error in (2.41). The following theorems are based on the ρ-sensitivity technique developed in section 2.4.

Theorem 1. The system (2.41) and (2.43) is ρ-sensitive with respect to ε if and only if for each $E(.)\epsilon\varepsilon$,

$$(\rho-1)\, P_0\,(t_0) - P_1(t_0) \geq 0 \qquad \ldots (2.86)$$

where $P_0(t)$ and $P_1(t)$ satisfy

$$\dot{P}_0 + P_0A + A^TP_0 + Q = 0\ , \quad P_0(t_1) = \Omega \qquad \ldots (2.87)$$

$$\dot{P}_1 + P_1(A+E) + (A^T+E^T)P_1 + P_0E + E^TP_0 = 0$$

$$P_1(t_1) = 0 \qquad \ldots (2.88)$$

The conditions of Theorem I define the maximum class of ρ-sensitive errors. However, from a practical viewpoint Theorem 1 is very difficult to apply. The following theorem is easier to apply, but the conclusion is weaker than that of Theorem 1.

Theorem 2. The system (2.41) and (2.43) is ρ-sensitive with respect to ε if for each $E(.)\epsilon\varepsilon$,

$$(\rho-1)Q(t) - \rho P_0(t)E(t) - \rho E^T(t)\, P_0(t) \geq 0 \quad . \quad . \quad (2.89)$$

for all $t_0 \leq t \leq t_1$

where $P_0(t)$ satisfies (2.87).

If a class of errors ε consists only of matrices which satisfy condition (2.89), then we are guaranteed that the system (2.41) and (2.43) is ρ-sensitive with respect to ε. It is straightforward to apply Theorem 2 since, as a result of Sylvester's criterion, (2.89) represents n inequalities to be satisfied for all $t_0 \leq t \leq t_1$. Further the matrix $P_0(t)$ is determined solely from knowledge of the nominal system. Condition (2.89) in Theorem 2 is only one condition which guarantees ρ-sensitivity. It is possible to determine other conditions; however, conditions (2.89) represents a simple as well as a relatively strong condition.

The above results can be directly applied to the situation where the nominal system is optimal. Using the optimal feedback control (2.38) in (2.41) the state equation is

$$\dot{x} = [F(t) - B(t)\, R^{-1}(t)\, B^T(t)\, P(t) + E(t)]\, x,$$
$$x(t_0) = x_0 \quad . \quad . \quad . \quad (2.90)$$

and the performance measure is given by

$$J = \frac{1}{2}\, x^T(t_1)\, \Omega\, x(t_1) + \frac{1}{2} \int_{t_0}^{t_1} x^T(\sigma)[H(\sigma)$$
$$+ P(\sigma)\, B(\sigma)\, R^{-1}(\sigma)\, B^T(\sigma)\, P(\sigma)\, x(\sigma)]d\sigma \quad . \quad . \quad . \quad (2.91)$$

Theorem 2 applied to (2.38) and (2.41) yields

__Theorem 3__. The optimal system (2.90) and (2.91) is

ρ-sensitive with respect to ε if for each $E(.)\epsilon\varepsilon$

$$(\rho-1)(H + PBR^{-1} B^TP) - \rho PE - \rho E^TP \geq 0 \quad . . . (2.92)$$

for all $t_0 \leq t \leq t_1$

where P(t) satisfies (2.40).

Example[46]

For example consider the linear time-varying system

$$\dot{x}_1 = x_2 \quad . . . (2.93)$$

$$\dot{x}_2 = -\alpha_1(t)x_1 - \alpha_2(t)x_2 \quad . . . (2.94)$$

with the nominal system given by

$$\alpha_1(t) = \alpha_2(t) = 0 \quad . . . (2.95)$$

The Performance measure is given by

$$J = \int_0^{1.0} (x_1^2 + x_2^2)\, dt \quad . . . (2.96)$$

In order to apply Theorem 2, $P_0(t)$ is first obtained from (2.87) as

$$P_0(t) = \begin{bmatrix} 1 - t & 1/2(1-t)^2 \\ 1/2(1-t)^2 & (1-t)^2 + 1/3(1-t)^3 \end{bmatrix} \quad (2.97)$$

Applying Sylvester's conditions for (2.89) obtain

$$\alpha_1 \geq \frac{-(\rho-1)}{2\rho P_{12}} \quad . . . (2.98)$$

$$(\alpha_2 P_{12} - \alpha_1 P_{22})^2 \leq \frac{(\rho-1)^2}{\rho^2} + \frac{2(\rho-1)}{\rho}\alpha_2 P_{22}$$

$$+ \frac{2(\rho-1)}{\rho}\alpha_1 P_{12} \quad . . . (2.99)$$

which if satisfied for all $0 \leq t \leq 1.0$ guarantees that the system is ρ-sensitive, where

$$P_{12}(t) = \frac{1}{2}(1 - t)^2 \qquad \ldots (2.100)$$

$$P_{22}(t) = (1 - t)^2 + \frac{1}{3}(1 - t)^3 \qquad \ldots (2.101)$$

The two inequalities can be thought of as representing two geometrical constraints in a three-dimensional $\alpha_1 - \alpha_2 - t$ space. In Figs. 2.3(a) and (b) the two cases $\alpha_1 \equiv 0$, and $\alpha_2 \equiv 0$ are considered. The indicated curves are the lower bounds for the corresponding errors. Note that an error for which the conditions of Theorem 2 are not satisfied but for which the system is ρ-sensitive is also indicated.

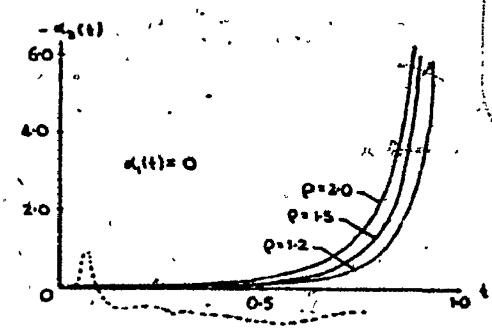

Fig. 2.3(a) Region for ρ-sensitivity, $\alpha_1=0$

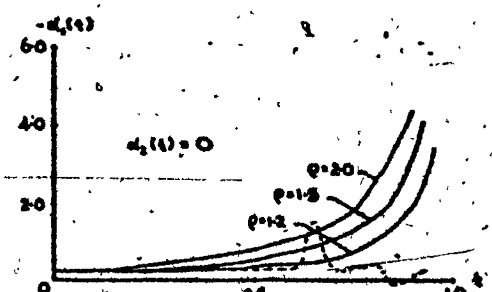

Fig. 2.3(b) Region for ρ-sensitivity, $\alpha_2=0$

CHAPTER 3

SENSITIVITY TO CONTROL VARIATIONS

3.1 Introduction

In optimal control problems, a model and an index of performance are defined for the system and a theoretical value for the optimal control $\underline{u}^*(t)$ over the interval $[t_0, \tau]$ is computed using a suitable numerical technique.

For linear systems, the optimal control can be implemented either in an open-loop or a closed-loop form. If the control is to be implemented in the form of closed-loop, the feed back gains are found and generated by amplifiers or compensators. Amplifiers are used for static gains while compensators are needed for dynamic gains. In case of state feed back, all components of the state vector $\underline{x}(t)$ should be observable.

In case of closed-loop control, errors may be introduced either in the process of observation or by the feedback loop implementation due to inaccuracies in the values of the components used. The latter could be considered as a variation in system parameters which was studied in Chapter 2. The errors due to observers are beyond the scope of this report.

Therefore, we are left in this chapter with the case of open-loop control. In practice, the optimal control $\underline{u}^*(t)$ cannot be precisely generated. Instead a different (suboptimal) control $u(t)$ is obtained. The error $\underline{u}^*(t) - u(t)$ will depend upon the tightness of the tolerances imposed.

The effects of control variations are twofold:

(a) If the object of the control task is to transfer the state of the system to some target set, variations in the control may cause the target to be missed.

(b) If a cost function is defined, variations in the control will produce cost variations.

Garvilovic, Petrovic and Siljak[60] treated the problem of sensitivity to control variation by considering the relation between control variations and variations on the initial conditions of the adjoint system. In this way the problem becomes one of initial conditions variation treated in Chapter 2.

A direct approach is due to Belanger[61]. He considered the case of continuous and bang-bang types of control. Our treatment will be based mainly on the results of this work.

3.2 Types of Control Variations and Tolerances

3.2.1 Continual Variations

Suppose the desired nominal control function $\underline{u}(t)$ is continuous and let the generated $\underline{u}(t)$ vary from the optimum $\underline{u}^*(t)$ by an infinitesimal amount. More precisely

$$\underline{u}(t) = \underline{u}^*(t) + \varepsilon\underline{n}(t) \qquad \ldots (3.1)$$

where ε is infinitesmally small over the whole control interval. This type of variation is called "continuous" and is illustrated in Fig. 3.1.

In the classical calculus of variations, this type

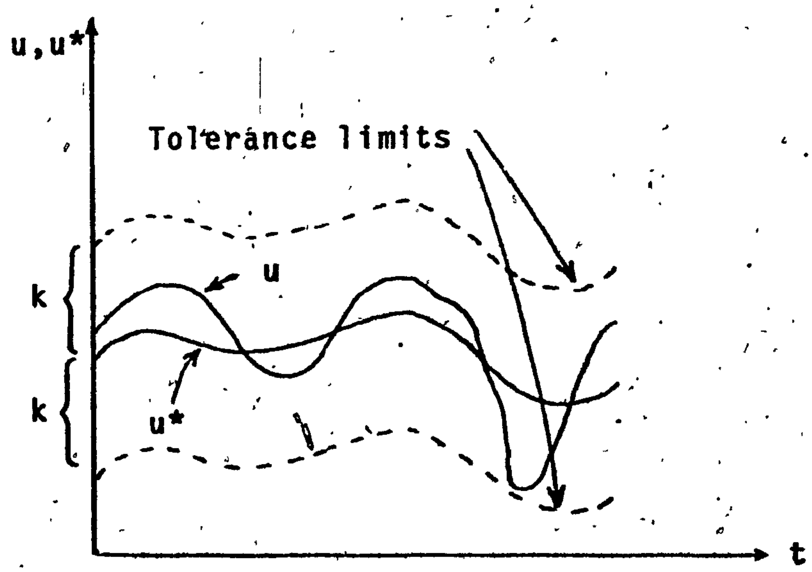

Fig. 3.1 The Continual Variation

of variation is called "weak" variation.

The tolerance placed on the control function is defined as a limit of the magnitude of the variations. This is the only physical constraint that can be imposed on the control variations.

To impose a tolerance on a continual variation is to limit $\eta(t)$ in some sense; one way for instance, requires that $||\underline{\eta}(t)|| \leq k$, or that

$$|\eta_j| \leq k_j \quad , j = 1, 2, 3, \ldots, r \qquad \ldots (3.2)$$

where $||.||$ is the Euclidean norm.

3.2.2 Intermittent Variations

On the other hand, when the desired nominal control function $u(t)$ is of the "bang-bang" type, $\underline{u}(t)$ takes on a discrete set of values. If the i^{th} component of $\underline{u}(t)$, $\underline{u}_i(t)$ has switching times at $t_{i1}, t_{i2}, \ldots t_{iN_i}$.

The i^{th} component of $u(t)$ might have its switching times at $t_{i1} + \varepsilon\delta t_{i1}, t_{i2} + \varepsilon\delta t_{i2}, \ldots t_{iN_i} + \varepsilon\delta t_{iN_i}$

where ε is a suitably small positive number. The variation in the i^{th} component is then:

$$u_j(t) - u_j^*(t) = \begin{cases} u_j^*(t_{ji}+) - u_j^*(t_{ji}-), & \text{if } \delta t_{ji} < 0 \\ \text{and if } t_{ji} + \varepsilon\delta t_{ji} < t < t_{ji} & \\ u_j^*(t_{ji}-) - u_j^*(t_{ji}+), & \text{if } \delta t_{ji} > 0 \\ \text{and if } t_{ji} \leq t < t_{ji} + \varepsilon\delta t_{ji} & \\ 0 \text{ otherwise} & \end{cases} \qquad (3.3)$$

This variation is illustrated in Fig. 3.2. It is clear that $\underline{u}^*(t)$ and $\underline{u}(t)$ differ by large amounts but during infinitismal intervals of time. This is called "intermittent" variation and it is a special type of strong variation, in the language of the classical calculus of variation. To impose a tolerance on an intermittent variation is to limit the size of the δt_{ji}'s.

3.3 Control Variations and the Control Problem

Let the system be represented by the vector differential equation

$$\dot{\underline{x}} = \underline{f}\ (\underline{x},\underline{u}) \qquad \ldots (3.4)$$

where $\underline{x}$ and $\underline{f}$ are n vectors and u is an r vector. The plant is assumed to be completely known. The control problem is to transfer the system state from $\underline{x}_0$ at time t_0 to some target set S at time t_f, described by

$$\underline{g}(\underline{x}(t_f)) = \underline{0} \qquad \ldots (3.5)$$

where g is an m vector. It is assumed that g fulfills the condition that makes S a smooth manifold.

It is also assumed that a nominal control $\underline{u}^*(t)$ exists which solves the control problem. A tolerance is then placed on the size of the variation between $\underline{u}^*$ and its implementation $\underline{u}$. It is necessary to ensure that the target set S be reached using any control within tolerance. To achieve that the effect of control variation on the state space trajectory be evaluated.

Let $\underline{x}^*(t)$ be the state space trajectory corresponding

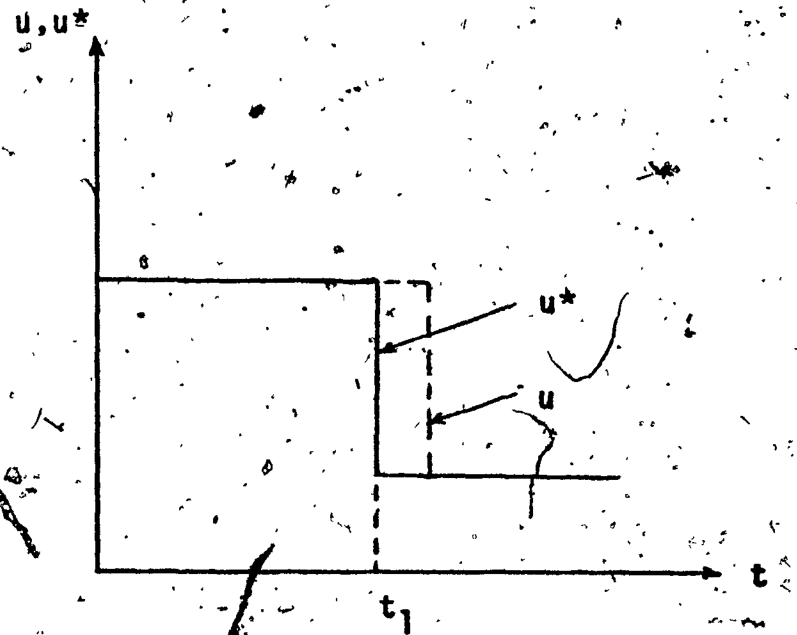

Fig. 3.2 The Intermittent Variation

to $\underline{u}^*(t)$ with $\underline{x}(t_0) = x_0$. Let $\underline{x}(t)$ be the system response to $\underline{u}(t)$ with $\underline{x}(t_0) = x_0$. For a continual variation to first order

$$\underline{x}(t) - \underline{x}^*(t) = \varepsilon\delta\underline{x}(t) \quad \ldots (3.6)$$

where x(t) is the solution of

$$\delta\dot{\underline{x}}(t) = \frac{\partial \underline{f}}{\partial \underline{x}} [\underline{x}^*, \underline{u}^*]\, \delta\underline{x} + \frac{\partial \underline{f}}{\partial \underline{u}} (\underline{x}^*, \underline{u}^*)\, \underline{n}(t) \quad \ldots (3.7)$$

with $\underline{x}(t_0) = 0$.

The matrix $\partial\underline{f}/\partial\underline{x}$ is the jacobian matrix with ij[th] component as $\partial fi/\partial xj$. The matrix $\partial f/\partial u$ is similarly defined. It is assumed that the various derivatives exist. In the case of an intermittent variation

$$\delta\underline{x}(t) = \sum_{j=1}^{\gamma} \sum_{i=1}^{N_j} \phi\,(t_f, t_{ji})\, [\underline{f}(\underline{x}^*(t_{ji}), \underline{V}_j(t_{ji}))$$

$$- \underline{f}\,(\underline{x}^*(t_{ji}), \underline{w}(t_{ji}))]\,|\delta t_{ji}| \quad \ldots (3.8)$$

where $V_j(t_{ij})$ and $w(t_{ji})$ are γ-vectors defined as follows

$$\delta t_i < 0\,, \quad v_k^j = \begin{cases} u_x^*(t_{ji}-) & k \neq j \\ u_k^*(t_{ji}+) & k = j \end{cases}$$

$$w(t_{ji}) = u^*(t_{ji}-)$$

$$\delta t_i > 0\,, \quad v_k^j(t_{ji}) = \begin{cases} u_k^*(t_{ji}+) & k \neq j \\ u_x^*(t_{ji}-) & k = j \end{cases}$$

$$w(t_{ji}) = u^*(t_{ji}+) \quad \ldots (3.9)$$

If the matrix $\phi(t, t')$ is the state transition matrix of the system

$$\delta\dot{x}(t) = \frac{\partial f}{\partial x}(x^*(t), u^*(t))\ \delta x(t) \qquad \text{. . . (3.10)}$$

From (3.7) or from (3.8), $\delta x(t_f)$ can be found. Since t_f is free, it is required that S be reached at some time $t_f+\varepsilon\delta t_f$. Assume $u(t_f)$ be constant at t_f, then to first order

$$\delta x(t_f+\delta t_f) = \delta x(t_f) + f(x^*(t_f), u^*(t_f))\delta t_f \qquad \text{. (3.11)}$$

If $x(t_f+\delta t_f)$ is to lie on S one have

$$< \frac{\delta gi}{\delta x}(x^*(t_f), \delta x(t_f+\delta t_f)> = 0 \qquad \text{. . . (3.12)}$$

If there exist an α, such that $<\partial g_\alpha/\partial x, f> \neq 0$, then δt_f can be obtained in terms of $\delta x(t_f)$ and $\partial g_\alpha/\partial x$. This yields

$$<\{< \frac{\partial g_\alpha}{\partial x}, f > \frac{\partial g_i}{\partial x} - < \frac{\partial g_i}{\partial x}, f > \frac{\partial g_\alpha}{\partial x}\}, \delta x(t_f)> = 0$$

$$\text{. . . (3.13)}$$

This equation implies that $\delta x(t_f)$ must lie in the intersection of m-1 hyperplanes. It may then be calculated that, in general, only a target set described by one equation may be reached using all controls within tolerance.

3.4 The Ideal and Actual Target Sets

If the control task is to take the system from an initial state to some target set, it can be shown that no matter how tight the control tolerances, there is always, in general, some control within tolerances which will cause the target set to be missed.

An exception is the case where the target is of dimensionality one less than that of the state - time space. Target sets not falling in this latter class are used in design as idealization of actual target sets, e.g. a point is used instead of a small sphere.

If the target set S is of dimensionality less than n-1, a target set T of dimensionality n-1 is introduced whose points are close to, i.e. points of T are one order of ϵ away from S. It is required that if u* takes the system from $x_0(t_0)$ to $x^*(t_f)\epsilon S$, then $u(t)$ takes $x_0(t_0)$ to $x(t_f+\delta t_f)\epsilon T$ and u lies within the specified tolerance. Conversely that max. tolerance that ensures the target set T to be reached can be found.

If $x - x^*(t_f) = \delta x$, T may be described by

$$h(\delta x) = 0 , \quad \ldots (3.14)$$

Assuming small control variations, (3.14) need only hold in a small neighborhood of $x^*(t_f)$. If $x(t_f+\delta t_f)$ is to lie in T, then, to first order;

$$\begin{aligned} h(x(t_f+\delta t_f) - x^*(t_f)) &= h(\delta x(t_f+\delta t_f)), \quad \ldots (3.15) \\ &= h(\delta x(t_f)+f(x^*(t_f))\delta t_f) = 0, \end{aligned}$$

from this, δt_f can be found in terms of $\delta x(t_f)$. Since δt_f must be real, a condition on $\delta x(t_f)$ of the form

$$\alpha_1 \leq g(\delta x(t_f)) \leq \alpha_2 \quad \ldots (3.16)$$

will result.

3.5 Examples[61]

Example 1.

Let

$$\dot{x}_1 = x_2 \qquad \ldots (3.17)$$

$$\dot{x}_2 = -2x_1 - 3x_2 + u \qquad \ldots (3.18)$$

with $x_1(0) = 10$, $x_2(0) = 0$.

The task is to drive the state to the origin in one second. A control function u which performs this task is

$$u(t) = 88e^{1.90\ t} + 4.89e^{-1.90\ t} - 154e^{1.18\ t} - 11.8e^{-1.18\ t} \qquad \ldots (3.19)$$

Let T be the circle $x_1^2 + x_2^2 = R^2$, and let the final time be "free", i.e., let $t_f = 1+\delta t_f$, where δt_f is small.

Since

$$\dot{x}(x^*(t_f),\ u^*(t_f)) = \begin{bmatrix} 0 \\ 87 \end{bmatrix} \qquad \ldots (3.20)$$

it is seen that $\delta x(t_f)$ must lie in the set W described by $|\delta x_1(1)| \leq R$ (see Fig. 3.3). The linearized system is

$$\delta\dot{x}_1 = \delta x_2 \qquad \ldots (3.21)$$

$$\delta\dot{x}_2 = -2\delta x_1 - 3\delta x_2 + \eta \qquad \ldots (3.22)$$

with $\delta x_1(0) = \delta x_2(0) = 0$.

If $|\eta| \leq k$, the maximum principle can be used to maximize

$$|\delta x_1(1)| \leq 0.206K \qquad \ldots (3.23)$$

Therefore, the tolerance in this case would be

$$|\eta| \leq 4.86\ R \qquad \ldots (3.24)$$

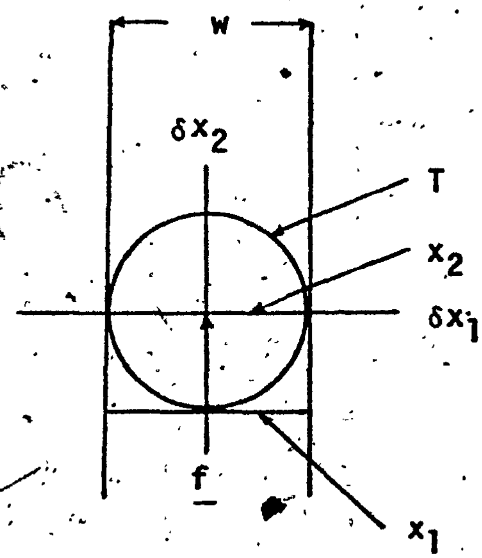

Fig. 3.3 Illustrating the Sets T, W, X_1 and X_2 for Example 1

Example 2:

Let

$$\dot{x}_1 = x_2 \qquad \ldots (3.25)$$

$$\dot{x}_2 = x_3 \qquad \ldots (3.26)$$

$$\dot{x}_3 = -3x_3 - 2x_2 + u \qquad \ldots (3.27)$$

with $x_1(0) = -39$, $x_2(0) = 84.5$, $x_3(0) = -174$, it is required to drive the state to the origin, the final time being free. A piecewise-constant control sequence [1, -1, 1] with switchings at t = 1.1, 2.2 is used to accomplish the task. The final time is 2.89. If the actual target T is a small sphere of radius R centered at the origin, what deviations in the switching points can be tolerated if T is still to be reached?

Since the velocity vector at the final time and state has a component in the 3 directions only, the set W is the cylinder described by the equation $\delta x_1^2 + \delta x_2^2 = R^2$ which is shown in Fig. 3.4.

Applying (3.8)

$$\delta x(t_f) = \phi(t_f, t_1)\, b\, (V(t_1) - W(t_1))|\delta t_1| + \phi(t_f, t_2)\, b\, (V(t_f) - w(t_2))|\delta t_2| \qquad \ldots (3.28)$$

where b is the vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and where ϕ is the state transition matrix of system (3.25-27) using (3.9)

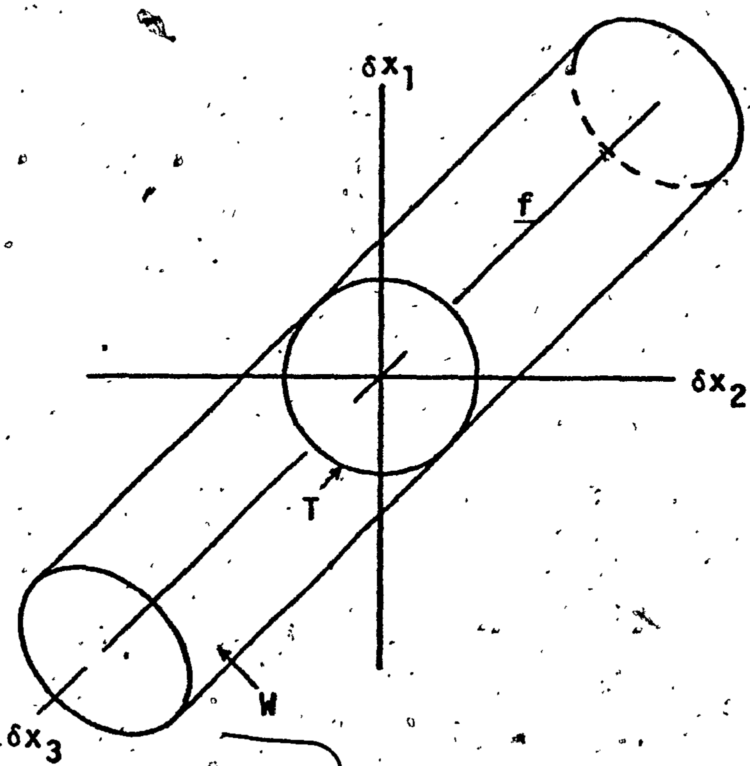

Fig. 3.4 Illustrating T and W for Example 2

$$\delta x(t_f) = 2\phi\ (t_f,t_1)\ b\delta t_1 - 2\phi\ (t_f,t_2)\ b\delta t_2 \quad \ldots (3.29)$$

$$\delta x_1(t_f) = \frac{25}{36}\,\delta t_1 - \frac{1}{4}\,\delta t_2 \quad \ldots (3.30)$$

$$\delta x_2(t_f) = \frac{5}{18}\,t_1 - \frac{1}{2}\,\delta t_2 \quad \ldots (3.31)$$

$$\delta x_3(t_f) = -\frac{2}{9}\,\delta t_1 \quad \ldots (3.32)$$

and

$$\delta x_1^2\ (t_f) + \delta x_2^2\ (t_f) = 0.558\delta t_1^2 - 0.624\delta t_1\delta t_2$$

$$+ 0.31\delta t_2^2 \leq R^2 \quad \ldots (3.33)$$

If tolerances of the form $|\delta t_1| < \Delta_1$ and $|\delta t_2| < \Delta_2$ are required then

$$0.558\ \Delta_1^2 - 0.624\ \Delta_1\Delta_2 + 0.31\ \Delta_2^2 \leq R^2 \quad \ldots (3.34)$$

Any tolerances satisfying (3.34) are acceptable.

CHAPTER 4

DESIGN OF LOW SENSITIVITY SYSTEMS

4.1 Introduction

In Chapter 2 we studied the analysis of system sensitivity to parameter variations. It was shown that when the plant parameters deviate from their nominal values, the system characteristics are expected to change. These characteristics are obtained to achieve specific objectives. Any deviation from the design characteristics will degrade the system performance. In some cases the parameters of the plant are known to vary from their nominal values or only approximate values of some parameters are available. In designing controllers for such plants, it is desirable that plant parameter variations will have minimum effect on the resulting system performance. A design that leads to low performance sensitivity is considered superior.

Several techniques for the design of minimum sensitivity systems have been reported in the literature. There are basically three main techniques, namely:

(a) Adaptive

(b) Choice of the Performance Index

(c) Use of Dynamic Compensators.

Many researchers, however, have investigated the possibility of combining more than one of the above techniques to obtain better results.

In this chapter the above mentioned techniques will be discussed.

4.2 The Adaptive Technique

This approach to the problem was introduced by Kelley[62,63], Bryson et al.[64], Werner and Cruz[32] and Taylor[65].

The main first steps of this technique is to solve the optimization problem. The resulting optimal control and state trajectories are stored (sometimes the trajectory of the costates might also be needed). In the actual operation of the system, the magnitudes of the different states are continuously measured (or sampled at reasonable intervals and compared with their nominal values).

Parameter-induced errors are used to generate a corrective signal, which is applied to the input. It can be shown that the resulting control signals render the system close to optimum for parameter variations that lie within given regions.

The state $x(t)$, the control $u(t)$ and the costate $\lambda(t)$ are expanded in a Taylor series around their nominal trajectories. Truncating the series to a finite number of terms, linear, quadratic, cubic, etc., feedback signals can be computed.

4.2.1 The Method of Kelley[63]

Kelley formulated his approach to solve the optimal guidance problem. The parameters that are succeptible to variations in this problem are the magnitudes of the initial states. He has also shown that parameters that are not initial states can still be handled by treating them as extra states whose initial values are subject to errors. In this technique trajectory comparison is used to identify the actual values of the parameters by estimating their differences from the assumed values.

Kelley introduced the concept of transverse comparison which is very useful for problems with free end time. Transverse comparison means that points on the actual trajectory are not compared to points on the nominal trajectory for the same value of time t. Rather the corresponding points on the nominal trajectory is the one with the same increment of the function to be minimized. For the minimum time problem, where t_f is to be minimized, a point A at time t has an incremant of $\Delta = t_f - t$ which is the time-to-go. The nominal problem has a minimum time t_f. The point on the nominal trajectory that corresponds to A should be at time t_1 such that $t_f - t_1 = \Delta$.

Kelley considered both linear and quadratic feedback. The results of his work show that quadratic feedback does not necessarily improve the system performance. However, when used with transverse comparison, excellent results were obtained.

4.2.2 Mathematical Formulation

The system to be controlled is represented by

$$\dot{x} = f(x,u,t,v) \qquad \ldots (4.1)$$

$$x(t_0,v) = x_0(v) \qquad \ldots (4.2)$$

and

$$y = g(x,u,t,y) \qquad \ldots (4.3)$$

where x is an n-state vector, y is the n-output vector, u is an n-control vector, t is time and v is an n-vector of unknown constant parameters. The n-vector function f and the n-vector function g are known analytic functions in all the arguments. The initial condition $x(t_0,v)$ is a known analytic function $x_0(v)$ of v. The terminal state may or may not be specified. All dimensions n_x, n_y, n_u and n_v are assumed to be known. However, there is a system uncertainty because v is not known. The only information about v is that it lies in a given domain. Some components of v may be unknown initial conditions. It is desired to minimize the scalar performance index.

$$J(u,v) = G(x,t,v)\big|_{t=t_f} + \int_{t_0}^{t_f} L(x,u,t,v)dt \quad . \quad . \quad (4.4)$$

with respect to u for all v where G and L are scalar analytic functions of their arguments. The optimal control will be called "optimally adaptive" control and it will be denoted by $u_a(t,v)$. For simplicity of discussion will be assumed to be unique, and analytic in v.

The basic idea for the determination of $u_a(t,v)$ will be developed for n=1. The generalization for a vector may be easily seen after the basic procedure for $n_v=1$ is established.

Let the n_x state differential equation be appended in (4.4) with Lagrange multipliers $\lambda_i(t,v)=1$, i=1, . . n_x to obtain

$$J^*(u,v) = G_{t=t_f} + \int_{t_0}^{t_f} \{-L+\lambda^T[f-\dot{x}]\}dt \quad \ldots (4.5)$$

$$= G(x,t_f,v)\Big|_{t=t_f} + \int_{t_0}^{t_f} L(z,\dot{z},t,v)\,dt$$

where

$$z(t,v) = [u^T(t,v) \vdots x^T(t,v) \vdots \lambda^T(t,v)]^T \quad \ldots (4.6)$$

is a (2n+1)-dimensional vector.

Let $z_a(t,v)$, $t_0 \leq t \leq t_f$ be the solution that minimizes J for every $v \in V$ and let $y_a(t,v)$ be the corresponding output. Then, $z_a(t,v)$ must satisfy the Euler-Lagrange equation

$$\left[\frac{\delta L}{\partial z} - \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} \right] = 0^T \quad \ldots (4.7)$$

for $t_0 \leq t \leq t_f$ and $x_a(t,v) = x_0(v)$ for every $v \in V$ is specified for some, i, then z_a must satisfy

$$x_{a_i}(t_f,v) = x_i(t_f) \quad \ldots (4.8)$$

If $x_j(t_f)$ is free (unspecified), then z_a must satisfy

$$\left[\frac{\partial G}{\partial x_j} + \frac{\partial L}{\partial \dot{x}_j} \right] = 0 \quad \ldots (4.9)$$

at $z = z_a$ and $t = t_f$ for every $v \in V$.

In the case when $n_v = 1$ so that v is scalar, the Taylor series for $z_a(t,v)$ about an arbitrary v_0 is

$$z_a(t,v) = z_{a0}^0 + z_{a1}^0(v-v_0) + \frac{1}{2} z_{a2}^0(v-v_0)^2$$

$$+ \ldots + \frac{1}{n!} z_{an}^0(v-v_0)^n + \ldots \quad \ldots (4.10)$$

where

$$z_{ak}^0 = \frac{\partial^k z_a}{\partial v^k}(t,v)\Big|_{v=v_0} \quad \ldots (4.11)$$

The vectors $z_{ak}^0 = z_{zk}^0(t)$, k=0,1,. . ., are time-varying Taylor series coefficients. The optimal output vector $y_a(t,v) = g(x_a(t,v), u_a(t,v),(t,v))$ can similarly be expressed as a Taylor series in v about v_0.

The method of finding the optimal Taylor series coefficients successively for k=0,1,2,. . ., is briefly summarized in the following steps.

(a) Compute the optimal nominal solution $z_a(t,v_0) = z_{a0}^0$ by setting $v=v_0$ in the Euler-Lagrange equation (4.7) and the boundary equations (4.2), (4.8) and (4.9). This is the usual optimal control problem.

(b) Take the total derivative with respect to v of (4.2), (4.7), (4.8) and (4.10) and set $v=v_0$. The result is a differential equation in z_{a0}^0 and z_{a1}^0. Substitute z_{a1}^0 from step (1). The result is a linear differential equation in z_{a1}^0 with appropriate boundary conditions at t_0 and t_f.

(c) Assume that $z_{a0}^0, z_{a1}^0, \ldots, z_{a(n-1)}^0$ are now available. Take the nth total derivative with respect to v of (4.2), (4.7), (4.8) and (4.9) and set $v=v_0$. Substitute the previously obtained $z_{a0}^0, z_{a1}^0, \ldots, z_{a(n-1)}^0$. The result is a linear differential equation in z_{an}^0 with appropriate boundary conditions. Computer z_{an}^0. By induction $z_{a0}^0, z_{a1}^0, \ldots, z_{an}^0$ are computed successively.

Observe that by using the preceding three steps it is not necessary to solve for all of the Taylor series coefficients simultaneously. Each z_{ak}^0 depends only on z_{a0}^0,

$z^0_{a1}, \ldots,$ up to $z^0_{a(k-1)}$

4.2.3 Example[63]

Consider a system defined by

$$\dot{x} = -\frac{1+v}{v}\,x+u,\quad x(0) = 1.0,\quad x(\infty) = \text{free} \qquad \ldots (4.12)$$

$$y=x$$

$$v < 0 \qquad \ldots (4.13)$$

The performance index is

$$J(u,v) = \int_0^\infty (x^2+u^2)\,dt \qquad \ldots (4.14)$$

Then

$$J(u,v) = \int_0^\infty \left[x^2+u^2+\lambda\left(u - \frac{1+v}{v}\,x-\dot{x}\right)\right] dt \qquad \ldots (4.15)$$

denoting $[u:x:\lambda]$ by z, the Euler-Lagrange equation becomes

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0^T \qquad \ldots (4.16)$$

$$\left[2u+\lambda,\ 2x - \frac{1+v}{v}\,\lambda+\dot{\lambda},\ u - \frac{1+v}{v}\,x-\dot{x}\right] = 0^T \qquad \ldots (4.17)$$

The nominal value v_0 of -2 is chosen, from step (a),

$$u^0_{a0} = -\frac{1}{2}\,\lambda^0_{a0} \qquad \ldots (4.18)$$

$$\dot{x}^0_{a0} = -\frac{1}{2}\,x^0_{a0} - \frac{1}{2}\,\lambda^0_{a0} \qquad \ldots (4.19)$$

$$\dot{\lambda}_{a0} = -2\,x^0_{a0} + \frac{1}{2}\,\lambda^0_{a0} \qquad \ldots (4.20)$$

The boundary conditions

$$x^0_{a0}(0) = 1.0$$

and $$\lambda^0_{a0}(\infty) = 0 \qquad \ldots (4.21)$$

The solutions for (4.18) and (4.19) are

$$x^0_{a0}(t) = \exp(-\sqrt{5}t/2)$$

$$\lambda^0_{a0}(t) = (\sqrt{5}-1)\exp(-\sqrt{5}t/2) \qquad \ldots (4.22)$$

$$u^0_{a0}(t) = (1-\sqrt{5}t/2)\exp(-\sqrt{5}t/2)$$

For n=1 step (b) leads to

$$u^0_{a1} = -\frac{1}{2}\lambda^0_{a1} \qquad \ldots (4.23)$$

$$\dot{x}^0_{a1} = -\frac{1}{2}x^0_{a1} - \frac{1}{2}\lambda^0_{a1} + \frac{1}{4}x^0_{a0} \qquad \ldots (4.24)$$

$$\dot{\lambda}^0_{a1} = -2x^0_{a1} + \frac{1}{2}\lambda^0_{a1} - \frac{1}{4}\lambda^0_{a0} \qquad \ldots (4.25)$$

With the boundary conditions

$$x^0_{a1}(0) = 0 \text{ and } \lambda^0_{a1}(\infty) = 0 \qquad \ldots (4.26)$$

The solutions for x^0_{a0} and λ^0_{a0} from (4.18) and (4.19) are substituted in (4.24). Then the differential equations may be integrated to yield the solutions for

$$x^0_{a1}(t),\ \lambda^0_{a1}(t) \text{ and } u^0_{a1}(t).$$

These are

$$x^0_{a1}(t) = (t/4\sqrt{5})\exp(-\sqrt{5}t/2) \qquad \ldots (4.27)$$

$$\lambda^0_{a1}(t) = ((\sqrt{5}-1)/2\sqrt{5})(1+t/2)\exp(-\sqrt{5}t/2) \qquad \ldots (4.28)$$

and $$u^0_{a1}(t) = ((1-\sqrt{5}/4\sqrt{5})(1+t/2)\exp(-\sqrt{5}t/2) \qquad \ldots (4.29)$$

Step (c) with n=2 yields a set of differential equations in x^0_{a2}, λ^0_{a2} and u^0_{a2}. Thus z^0_{a0}, z^0_{a1}, z^0_{a2}, . . . can be obtained (using steps (a) through (c) for n=1, 2, 3, . . .)

Having found these time-varying coefficients, $z_a(t,v)$ and $y_a(t,v) = x_a(t,v)$ can be expressed as Taylor series in (4.10).

The procedure for multidimensional v is similar except that a Taylor series in several variables must be used. It is given by

$$z_a(t,v) = z_a(t,v_0) + \sum_{i=1}^{n_v} \left.\frac{\partial z_a}{\partial v_i}\right|_{v_0} (v_i - v_i^0) + \cdots$$

$$+ \sum_{i,j}^{n_v} \left.\frac{\partial^2 z_a}{\partial v_i \partial v_j}\right|_{v_0} (v - v_i^0)(v_j - v_j^0)/2$$

$$+ \sum_{i,j,k}^{n_v} \left.\frac{\partial^3 z_a}{\partial v_i \partial v_j \partial v_k}\right|_{v_0} (v_i - v_i^0)(v_j - v_j^0)(v_k - v_k^0)/3!$$

$$+ \cdots \qquad \cdots (4.30)$$

Steps analogous to steps (a) through (c) are used to compute the Taylor series coefficients. The mixed partial derivatives in (4.30) may be computed in successive order starting from z_{a0}^0.

4.3 The Method of Werner[32]

Kelley's technique requires an explicit identification of v which is assumed to be unknown. This might be an expensive process in terms of computer time and storage. To circumvent this, Werner and Cruz[32] suggested a technique that generates the feedback control as a function of the output only and does not explicitly depend on v. This control will be denoted by $u_c(t,y)$ and is obtained by representing u_c as a Taylor series in y about y_a^0 which corresponds to the optimal control for v_0. The undetermined coefficients in the series are obtained by equating the series to Taylor series of $u_a(t,v)$ obtained by Kelley.

For a scalar parameter v only one output y_1 is needed for feedback and therefore $u_a(t,y)$ need only be expressed in terms of y_1. Thus

$$u_c(t,y_1) = u_c(t,y_{a1}^0) + \left.\frac{\partial u_c}{\partial y_1}\right|_{y_{a1}^0} (y_1-y_{a1}^0) + \ldots$$

$$+ \frac{1}{n!} \left.\frac{\partial^n u_c}{\partial y_1{}^n}\right|_{y_{a1}^0} (y_1-y_{a1}^0)^n + \ldots \quad (4.31)$$

Denote the k^{th}-order sensitivity function coefficients in (4.31) by u_{ck}^0 so that (4.31) becomes

$$u_c(t,y_1) = u_{c0}^0 + u_{c1}^0(y_1-y_{a1}^0) + \frac{1}{2} u_{c2}^0(y_1-y_{a1}^0)^2 +$$

$$\ldots + \frac{1}{n!} u_{cn} (y_1-y_{a1}^0)^n + \ldots \quad (4.32)$$

The object is to compute the coefficients u_{ck}^0 for k=0,1,2,.. . Once these time-varying coefficients are determined, the feedback control function is obtained from the series in (4.32).

The control function $u_c(t,y_1)$ is not an explicit function of v, but since y_1 depends on v, u_c depends on v through y_1. If $u_c(t,y_{a1})$ is to equal $u_a(t,v)$ for all t and all v, all the total derivatives of $u_a(t,v)$ with respect to v must match those of $u_a(t,v)$ for $v=v_0$. Starting with the zeroth derivative,

$$u_c(t,y_{a1}^0) = u_{c0}^0(t) = u_a(t,v_0) \doteq u_{a0}^0(t) \quad \ldots (4.33)$$

Hence,

$$u_{c0}^0(t) = u_{a0}^0(t) \quad \ldots (4.34)$$

The first derivative with respect to v of $u_c(t,y_{a1})$ is

$$\left.\frac{du_c}{dv}\right|_{v_0} = \left[\frac{\partial u_c}{\partial y_{a1}} \frac{\partial y_{a1}}{\partial v} \right]_{v_0} = u_{c1}^0 \left(\frac{\partial y_{a1}}{\partial v} \right)_{v_0} \quad \ldots (4.35)$$

Note that for $v=v_0$, $y_{a1}=y_{a1}^0$ which is also the first component of y_a^0. The first derivative of u_a with respect to v is

$$\left.\frac{du_a}{dv}\right|_{v_0} = u_{a1}^0 \quad \ldots (4.36)$$

Since

$$\left.\frac{du_c}{dv}\right|_{v_0} = \left.\frac{du_a}{dv}\right|_{v_0} \quad \ldots (4.37)$$

must hold, (4.35) and (4.36) yield

$$u_{c1}^0 = \frac{u_{a1}^0}{\left(\frac{\partial y_{a1}}{\partial v} \right)_{v_0}} \quad \ldots (4.38)$$

provided $(\partial y_{a1}/\partial v)_{v_0} \neq 0$. For the second derivative of $u_c(t,y_{a1})$ with respect to v

$$\frac{\partial^2 u_c}{\partial v^2} = \left(\frac{\partial^2 u_c}{\partial y_1^2}\right)\left(\frac{\partial y_{a1}}{\partial v}\right)^2 + \left(\frac{\partial u_c}{\partial y_1}\right)\left(\frac{\partial^2 y_{a1}}{\partial v^2}\right) \quad \ldots (4.39)$$

Evaluating (4.39) at $v=v_0$ and setting it equal to u_{a2}^0

$$u_{c2}^{0}\ (y_{a1}^{0})_{1}^{2} + u_{c1}^{0}\ (y_{a2}^{0})_{1} = u_{a2}^{0} \qquad \ldots (4.40)$$

where $(y_{ak}^{0})_{1}$ denotes

$$\left(\frac{\partial^{k} y_{a1}}{\partial v^{k}}\right)_{v_0} = (y_{ak}^{0})_{1} \qquad \ldots (4.41)$$

Hence, solving (4.40)

$$u_{c2}^{0} = \frac{u_{a2}^{0} - u_{c1}^{0}(y_{a2}^{0})_{1}}{(y_{a1}^{0})_{1}^{2}} \qquad \ldots (4.42)$$

provided $(y_{a1}^{0})_{1} \neq 0$.

The sensitivity functions u_{a2}^{0}, $(y_{a2}^{0})_{1}$, $(y_{a1}^{0})_{1}$ are available from the expansion of $z_a(t,v)$ and $y_a(t,v)$ and u_{c1}^{0} has been previously calculated. Thus, the coefficients may be calculated successively from

$$\left(\frac{d^{k}u_{c}}{dv^{k}}\right)_{v_0} = \left(\frac{\partial^{k}u_{a}}{\partial v^{k}}\right)_{v_0} = u_{ak}^{0} \qquad \ldots (4.43)$$

at $y_1 = y_{a1}$ for $k=0,1,2, \ldots$ For example, $k=3$

$$u_{a3}^{0} = u_{a3}^{0}(y_{a1}^{0})_{1}^{3} + 3u_{c2}^{0}(y_{a1}^{0})(y_{a2}^{0})_{1} + u_{c1}^{0}(y_{c3}^{0})_{1} \qquad \ldots (4.44)$$

Hence,

$$u_{c3}^{0} = \frac{u_{a3}^{0} - 3u_{c2}^{0}(y_{a1}^{0})_{1}(y_{a2}^{0})_{1} - u_{c1}^{0}(y_{a3}^{0})_{1}}{(y_{a1}^{0})_{1}^{3}} \qquad \ldots (4.45)$$

where all the factors in the right-hand side of (4.45) are known from previous calculation for $k=0,1,$ and 2 in (4.43).

The procedure for multiple feedback and multi-dimensional is similar, but a Taylor series for several variables must be employed. Instead of a single feedback signal y, several or even all components of y must be fed-back.

If the dimension of the parameter vector v is greater than that of the output vector, y, the output vector has to be augmented.

In this case, instead of a control $u_c(t,y)$, a control $u(t,w)$ is sought where w is a vector consisting of y augmented by signals based on y. For example, may be taken as

$$w = [y \vdots \int_{t_0}^{t} y \; dt]^T \qquad \ldots (4.46)$$

Higher order integrals may be used for augmentation to increase the dimension of w further.

4.3.1 <u>Truncation of $u_a(t,v)$ and $u_c(t,w)$</u>

For practical considerations in the implmentation of the controller, it is desirable to truncate the series u_a and u_c. The effect of such a truncation on the performance index $J(u,v)$ is examined.

Let $J(u,v)$ as given by (4.4) be expanded in the Taylor series in v:

$$J(u,v) = J(u,v_0) + \left(\frac{dJ}{dv}\right)_{v_0} (v-v_0) + \frac{1}{2}\left(\frac{d^2J}{dv^2}\right)_{v_0} (v-v_0)^2 + \ldots + \frac{1}{k!}\left(\frac{d^kJ}{dv^k}\right)_{v_0} (v-v_0)^k + \ldots \qquad \ldots (4.47)$$

where v is assumed to be scalar for simplicity in notation. Denote the n^{th} order approximation of u_a by u_a^n

$$u_a^n = u_{a0}^0 + u_{a1}^0(v-v_0) + \ldots + \frac{1}{n!} u_{an}^0(v-v_0)^n \quad \ldots (4.48)$$

and the n^{th}-order approximation for $u_c(t,w)$ by u_c^n

$$u_c^n(t,w) = u_{c0}^0 + u_{c1}^0(w-w_a^0) + \ldots$$

$$+ \frac{1}{n!} u_{cn}^0 (w-w_a^0)^n \quad \ldots (4.49)$$

Let the system equations be as given in (4.1), (4.2) and (4.3) with the final state free. We have

$$\frac{d^kJ}{dv^k}(u_a,v)\Big|_{v=v_0} = \frac{d^kJ}{dv^k}(u_a^n,v)\Big|_{v=v_0} = \frac{d^kJ}{dv^k}(u_c^n,v)\Big|_{v=v_0} \quad \ldots (4.50)$$

for k=0,1,2, . . . , (2n+1).

This implies that an n^{th}-order truncation of the control results in a $(2n+1)^{th}$-order approximation of the optimally adaptive performance index.

4.4 Choice of Performance Index

Cassidy(66) and Higginbotham(67) proposed a technique by which system sensitivity can be reduced. It represents an extension of the optimal control approach to the design problem. They considered the linear regulator problem reformulated so that the sensitivity is controlled by the resulting control system configuration.

The design steps are:

(a) Augment the state vector by a set of sensitivity vectors.

(b) Augment the control vector by vectors which are functions of sensitivity variation.

(c) Minimize a quadratic cost functional in the augmented space.

The resulting optimal control is obtained by linear feedback from the state and the set of sensitivity vectors.

4.4.1 Higginbotham's Technique[67]

Let the system dynamics be

$$\dot{x}(t) = A(t,\mu)\; x(t) + B(t,\mu)\; u(t), \quad x(t_0) = x_0 \qquad (4.51)$$

where μ is a constant vector of p plant parameters.

State sensitivity dynamics are established by first defining a set of sensitivity vectors.

$$v_j(t) = \frac{\partial x(t)}{\partial \mu_j}, \quad (j=1,2, \ldots p) \qquad \ldots (4.52)$$

Then from equation (4.51), the sensitivity equations become

$$\dot{v}_j(t) = A(t,\mu_0)\; v_j(t) + \frac{\partial A(t,\mu)}{\partial \mu_j}\, x(t)$$

$$+ B(t,\mu_0)\, \frac{\partial u}{\partial \mu_j} + \frac{\partial B(t,\mu)}{\partial \mu_j}\, u(t) \qquad \ldots (4.53)$$

where $v_j(t_0) = 0$ and μ_0 is the vector μ evaluated using nominal values. Since the optimal state control law $u(t)$ will be a function of state and sensitivity expansion of $\partial u/\partial \mu_j$ yields

$$\frac{\partial u}{\partial \mu_j} = [J_x u]\; v_j(t) + \sum_{i=1}^{p} [J_{v_i} u]\; \frac{\partial v_i}{\partial \mu_j} \qquad \ldots (4.54)$$

where $J_x u$ and J_{v_i} are Jacobian matrices substituting equation (4.54) and (4.53) yields

$$\dot{v}_j(t) = [A(t,\mu_0) + B(t,\mu_0)\ J_x u]\ v_j(t) + \frac{\partial A}{\partial \mu_j}\ x(t)$$

$$+ \frac{\partial B}{\partial \mu_j}\ u(t) + B(t,\mu_0) \sum_{i=1}^{p} [J_{v_i} u] \frac{\partial v_i}{\partial \mu_j} \quad . . . (4.55)$$

Let

$$m_j(t) = \sum_{i=1}^{p} [J_{v_i} u] \frac{\partial v_i}{\partial \mu_j} \quad . . . (4.56)$$

where the dimension of each m_j is y. Also let z(t) be defined as the (p+1) n-dimensional composite vector of the state and sensitivity vectors; that is, $z(t) = [x'(t), v_1'(t), . . . v_p'(t)]$ where the prime (') denotes the transpose.

Equations (4.51) and (4.55) can be written as

$$\dot{z}(t) = A_1(t,\mu)\ z(t) + B_1(t,\mu)\ u_1(t) \quad . . . (4.57)$$

where

$$u_1(t) = [u'(t), m_1'(t), . . . , m_p'(t)]'$$

A quadratic cost functional is assumed to correspond to the performance requirements. Thus

$$J = \frac{1}{2} z'(t_f)\ F\ z(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [z'(t)Qz(t)$$

$$+ u_1'(t)\ R(t)\ u_1(t)]\ dt \quad . . . (4.58)$$

where the terminal time t_f is fixed and specified, F and Q(t) are symmetric positive semidefinite (p+1) n x (p+1)n matrices, and R(t) is a symmetric positive definite (p+1) r x (p+1)r matrix.

In summary, the problem is to find the optimal state control law u(t) and the set of optimal sensitivity control laws that minimize the quadratic cost of functional (4.58).

Figure 4.1 shows the structure of the optimal sensitivity and state controlled regulating system.

It is clear that this technique requires the generation of the sensitivity dynamics which adds to the memory requirements. Another drawback is that the optimization process is done in an augmented space with the dimensionality greatly increased if several parameters are expected to vary.

4.4.2 Cassidy's Technique(66)

Cassidy treated the specific optimal control problem (s.o.c.). This is the case of systems with inaccessible states where feedback is available only from the outputs. To achieve the control goals the elements of the matrices of the cost functional have to satisfy certain conditions. These conditions guarantee that gains corresponding to unavailable states will be zero.

The cost functional is chosen as

$$J = \frac{1}{2} \int_0^{\infty} [x^T sx + x^T \dot{s}x + x^T wu + u^T Qu] \; dt \quad \ldots (4.59)$$

Subject to

$$y = Cx \quad \ldots (4.60)$$

$$\dot{x} = Ax + Bu \; , \; x(t_0) = c \quad \ldots (4.61)$$

where x is an NS element state vector and u is a NC element control vector. The necessary conditions defining the optimum are

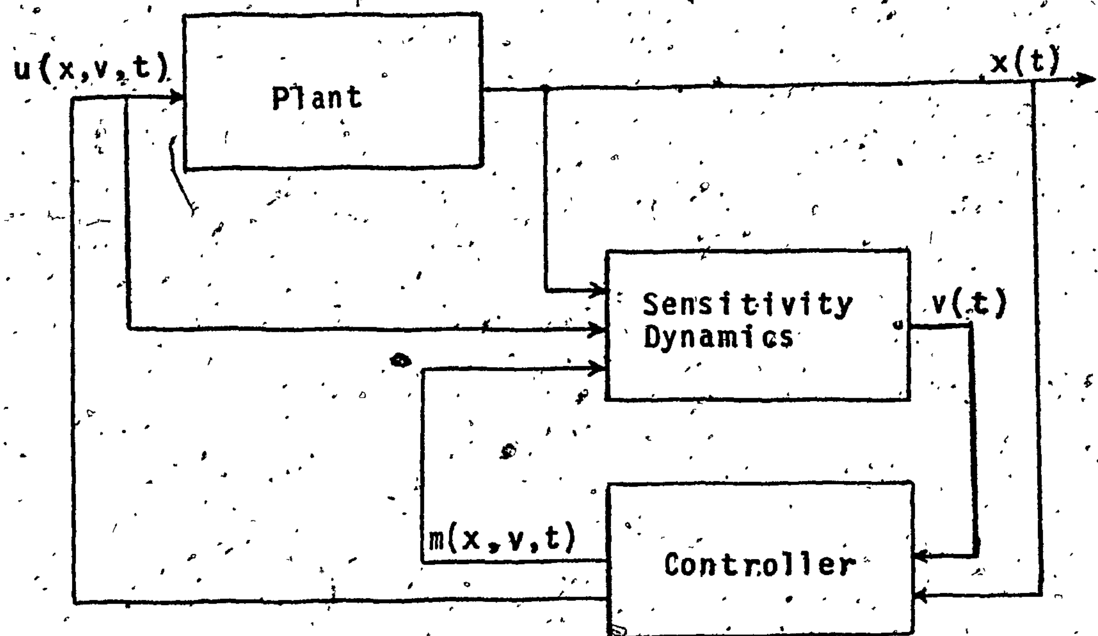

Fig. 4.1 Structure of the Optimal Sensitivity and Stated Controlled Regulating System

$$A^TP + PA + s + \hat{s} - (PB + \frac{w+\hat{w}}{2})\ Q^{-1}\ (B^TP + \frac{w^T+\hat{w}^T}{2}) = 0 \quad \ldots (4.62)$$

$$u = -Q^{-1}\ (B^TP + \frac{w^T+\hat{w}^T}{2})\ x \quad \ldots (4.63)$$

$$\dot{x} = Ax + Bu\ ,\ x(0) = c \quad \ldots (4.64)$$

Assume that the last L states on the state vector are unavailable, $G = [I_{NS-L}, 0]$ and consider the following definitions of $\hat{s}$ and $\hat{w}$

$$\hat{w} = -2\ I_2(PB + \frac{w}{2}) \quad \ldots (4.65)$$

$$\hat{s} = \frac{1}{2}\ ((w+\hat{w})k^TC + C^Tk(w^T+\hat{w}^T)) \quad \ldots (4.66)$$

where

$I_2 = \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I_L \end{bmatrix}$ is a NS by NS matrix and I_L and I_{NS-L} are the L by L and NS-L identity matrices respectively. In addition define

$$I_1 = I - I_2 \quad \ldots (4.67)$$

These definitions insure that the desired control structure is obtained and simplify the reduced necessary conditions which result from eliminating $\hat{w}$ and $\hat{s}$ from equations (4.62) to (4.64).

This technique is superior to that of Higginbotham in that dimensionality of the problem can be reduced.

The reduction of dimensionality is a result of the fact that the resulting Ricatti equation can be partitioned into blocks with dimensions equal to that of the original

2 OF/DE 2

system and if the Newton-Raphson approach is used, the computational effort increases only linearly with the number of parameters corresponding to a dramatic reduction in computational time.

To obtain starting values for the matrices in the cost functional, an initial gain that stabilizes the system is assumed. The reverse SOC problem is then solved. This is defined as:

Given a linear system and feedback corresponding to a stable closed-loop system, find a SOC index for which the given gains are optimum.

After the reverse problem has been solved, a new SOC problem may be obtained by perturbing the reverse problem weightings S, Q and W. For any values of S, Q, and W in some neighborhood of the reverse problem weightings, the new problem has a unique solution.

The design procedure is summarized in the following steps:

(a) For the given system determine an appropriate feedback structure and calculate a stable set of feedback gains.

(b) Solve the reverse SOC problem to obtain a benchmark set of weighting matrices.

(c) Perturb the weighting matrices and solve the corresponding SOC problems.

(d) Evaluate system performance via simulation.

pole positions, or cost index calculation.

(e) Repeat steps (c) and (d) if necessary.

4.4.3 The Exponentially Weighted Quadratic Cost Functional

Anderson and Moore[68] introduced a cost functional which is basically the same as the usual quadratic cost functional for linear system. The only difference is that the intergrand is multiplied by an exponential term.

This leads to results similar to those obtainable for a usual quadratic cost with the only exception that A is replaced by $(A+\alpha I)$. The control is still generated by linear feedback from the states. One of the advantages of such a choice of cost is the reduction of trajectory sensitivity due to parameter variations. In fact it can be shown that sensitivity is inversely proportional to α.

4.5 Use of Dynamic Compensators

In control systems, if all the states of the plant are available for measurement or are re-constructible through the use of an observer[10-12], Wonham[17] showed that arbitrary pole allocation can be achieved through constant feedback from all states. In particular a system can be stabilized by placing all poles in the left hand side of the complex plane excluding the imaginary axis. It is obvious that there exists more than one gain that can achieve this objective. Specific feedback gain can be chosen to stabilize the system and at the same time reduce the system sensitivity.

The existence of a dynamic compensator that can stabilize a plant by acting on the available outputs only was established by Potter and VanderVelde[69]. The necessary and sufficient conditions for the existence of such a compensator are that the unstable modes of the plant should be observable and controllable. There will exist a class of compensators that can stabilize a given plant. The particular choice can be made so as to reduce system sensitivity.

The problem of sensitivity reduction using dynamic controllers was studied by Sims & Melsa[70] for the case of linear and nonlinear time-invariant systems. Perkins, Cruz and Gonzales[71] approached the problem from a minimax point of view.

4.5.1 Sims-Melsa Technique[70]

The specific optimal control problem is considered here. This is the fixed structure controller problem in which the configuration of the controller are pre-specified and the values of the parameters are to be found to optimize a given functional.

Problem Statement

The problem is to obtain a closed-loop control for the dynamical system

$$\dot{x} = f(x,u,t) \qquad \ldots (4.68)$$

based on the available output

$$y = g(x,t) \qquad \ldots (4.69)$$

such that the integral performance index:

$$J = \int_{t_0}^{t_f} L(x,u,t)dt \qquad \ldots (4.70)$$

for fixed values of t_f and t_0 is minimized. In the usual specific optimal control formulation, u is chosen as:

$$u = h(y,a) \qquad \ldots (4.71)$$

where a is a set of constant parameters which are selected to minimize J. Figure 4.2 illustrates the usual structure of the specific optimal control problem, while Figure 4.3 presents the suggested structure for the dynamical approach to specific optimal control.

In the dynamical controller form of specific optimal control, involves an intermediate dynamical system:

$$\dot{z} = k(z,y,u,t,c) \; ; \quad z(t_0) = d \qquad \ldots (4.72)$$

and is written as

$$u = m(y,z,b) \qquad \ldots (4.73)$$

The elements of c may be thought of as internal gains, those of b as control coefficients, and the elements of d are the initial conditions of the intermediate dynamical structure of (4.72). The constant parameters b, c and d are chosen to minimize the performance index, J. The motivation for the dynamical controller structure is derived from a knowledge of the nature of the solutions of stochastic control problems.

It is shown that use of dynamical controller can

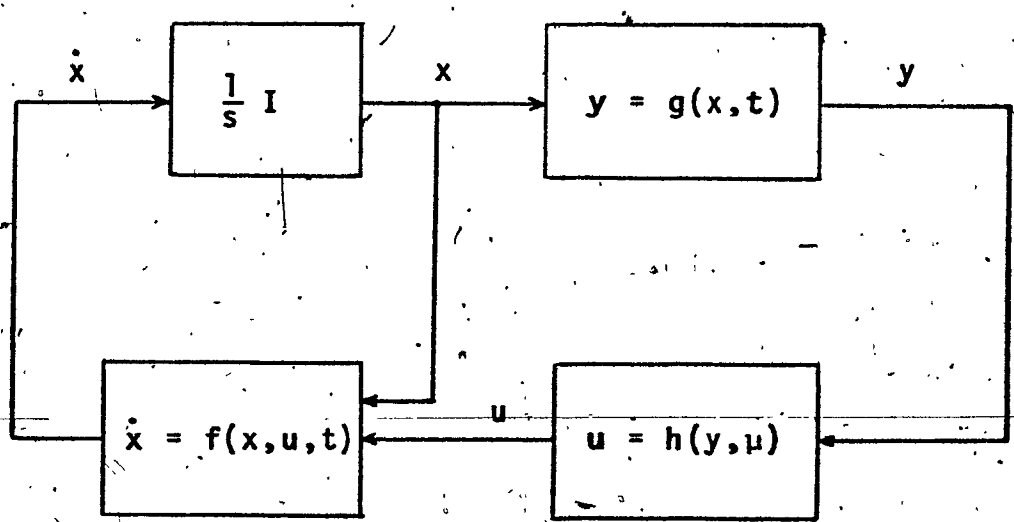

Fig. 4.2 Usual Structure of the Specific Optimal Control Problem

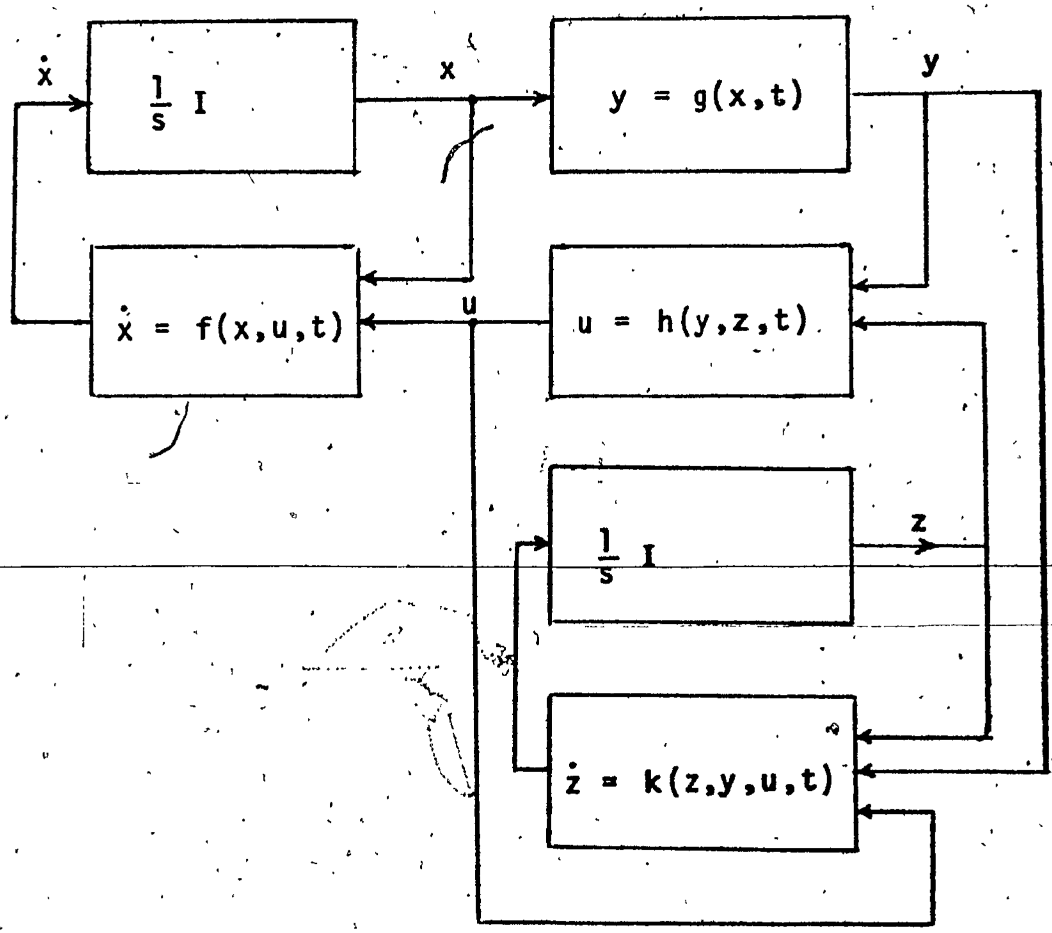

Fig. 4.3 Dynamical Structure of the Specific Optimal Control Problem

achieve lower sensitivity and a lower value for the cost functional. However, no systematic procedure is given for the choice of the dynamics of the controller. In the examples worked out in the paper, the controller was chosen to act as an observer of the given plant.

It is felt that if a fixed order controller is chosen with parameters to be selected so as to minimize sensitivity, better results would have been obtained. However, this will increase the dimensionality of the minimization process.

4.5.2 Method of Gonzales[71]

In this technique the desired overall transfer function of the system is first found so as to satisfy a specific objective. In the example given, the transfer function is chosen in a diagonal form so that the system is decoupled. The diagonal elements are chosen so that each channel will be of second order with a damping ratio of 0.707.

A two degree of freedom structure is proposed where by two controllers G and H are to be selected as shown in Figure 4.4. The constraints G and H are that:

(a) G and H must represent parameter-independent (fixed) stable compensating systems.

(b) It may be desirable to avoid differentiation in either G or H.

(c) The final value of the parameter-induced error must be zero.

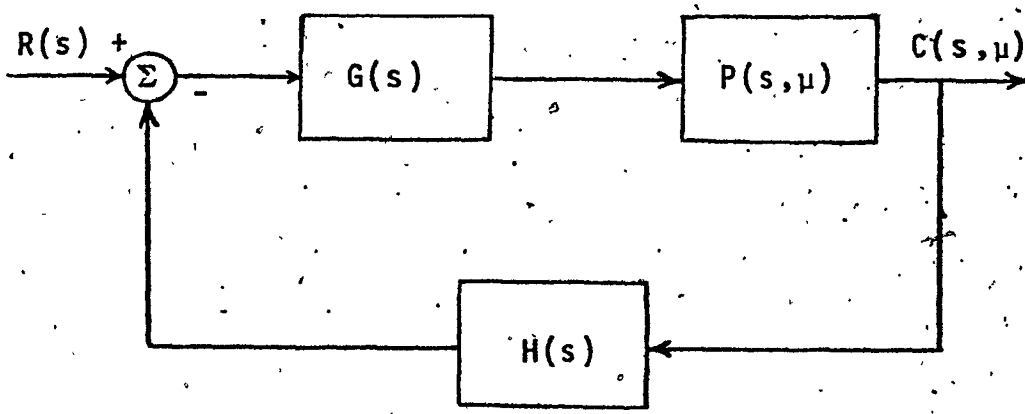

Fig. 4.4 Multivariable Feedback Control System

(d) The number of poles and zeros of G and H may be further restricted for simplicity of design, desired asymptotic frequency behaviour, avoidance of infinite gain, etc.

The problem can be stated as follows:

Consider the linear time-invariant multivariable control system shown in Fig. 4.4. The plant is characterized by the transfer function matrix $P(s,\mu)$ which is rational in s, where s is the complex Laplace transform variable, and μ is a plant parameter whose components are unknown, but time-invariant. The compensating networks, represented by the transfer function matrices G(s) and H(s) are parameter-invariant and rational in s. It is supposed that the system transfer function matrix T is specified to be T_n when the plant parameters are at their nominal values $\mu=\mu_n$.

$$T = T(s,\mu_0)$$
$$= [I + P(s,\mu_0)\ G(s,\mu_0)\ H(s)]^{-1}\ P(s,\mu_0)\ G(s) \qquad \ldots (4.74)$$

The matrix T_n could be specified to obtain some desired time response, or to optimize some performance index, for example.

In any physical realization of this system, μ will differ from μ_0 and, thus, the output will differ from the desired output. The output error induced by the parameter variation is

$$E(s,\mu) = C(s,\mu_0) - C(s,\mu) \qquad \ldots (4.75)$$

where

$$\mu = \mu_0 + \Delta\mu \qquad \ldots (4.76)$$

A measure of the effects of this parameter variation is the scalar sensitivity index,

$$J = \int_0^\infty e'(t,\mu)\, Q\, e(t,\mu)\, dt \qquad \ldots (4.77)$$

where $e(t,\mu)$ is the inverse Laplace transform of $E(s,\mu)$, Q is a positive definite weighting matrix, and the prime denotes of the matrix transpose. Using Parseval's theorem, (4.77) becomes:

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E'(-s,\mu)\, Q\, E(s,\mu)\, ds \qquad \ldots (4.78)$$

In (4.78) the parameter induced error is evaluated for a specific system input R. Notice that J is a functional of G and H. G and H are to be selected to minimize (4.78) subject to certain restrictions. These restrictions depend on the details of the specified system being designed.

4.5.3 Design Procedure

Consider an open-loop system, Fig. (4.5) having the same nominal plant input and the same nominal output as the closed loop system of Fig. 4.4. Such systems are called nominally equivalent. It has been shown that the parameter variation errors are related by

$$E(s,\mu) = SE_0(s,\mu) \qquad \ldots (4.79)$$

where

$$E_0(s,\mu) = C_0(s,\mu) - C_0(s,\mu_0) \qquad \ldots (4.80)$$

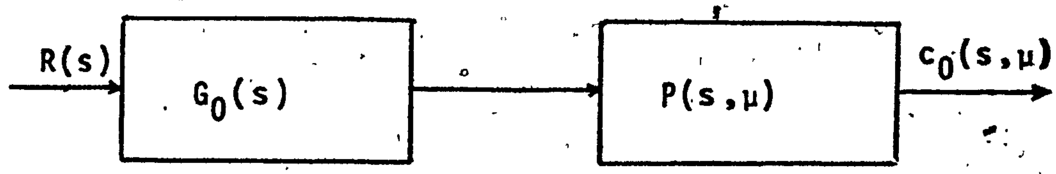

Fig. 4.5 Multivariable Open-loop Control System

is the open-loop parameter-induced error, $E(s,\mu)$ is as defined in (4.75) and where the sensitivity matrix S is given by

$$S = [I + P(s,\mu)\ G(s,\mu)\ H(s)]^{-1} \quad \ldots (4.81)$$

In the following only differentially small parameter variations $\Delta\mu = d\mu$ will be considered. For this case E_0 and E are differentially small. Thus, to first order, (4.79) becomes:

$$E(s,\mu) = [I + P(s,\mu_0)\ G(s)\ H(s)]^{-1}\ E_0(s,\mu) \quad \ldots (4.82)$$

Therefore, the sensitivity matrix depends only on the nominal plant:

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E_0'(-s,\mu)\ S'(-s)\ Q\ S(s)\ E_0(s,\mu)\ ds \quad \ldots (4.83)$$

The sensitivity index J now may be regarded as a functional of the sensitivity matrix, S, to be minimized by choice of S. Recall that the input R is fixed. Once S is obtained, G and H can be found from (4.81) and (4.74)

$$PG = S^{-1}T \quad \ldots (4.84)$$

and

$$TH = I - S \quad \ldots (4.85)$$

In the event (4.84) and (4.85) have nonunique solutions for G and H the designer has additional freedom in the choice of compensation networks. Equations (4.84) and (4.85) also allow physical realizability conditions on G and H to be incorporated easily into S.

However, S is not completely at the designer's disposal. An examination of (4.81) and (4.74) reveals that the denominator polynominal of all entries in the matrix S is the same as the denominator polynominals of all entries in the specified system transfer function matrix T_n namely, det(I+PGH). Thus, only the numerator polynominals in the matrix S are free. The minimum sensitivity problem, then, has been expressed as one of parameter optimization, the parameters being the coefficients in the numerator polynominals in S, and the performance index optimized being the scalar sensitivity index (4.83).

If the plant parameter variations are represented by $d\mu\epsilon a$ and if the S matrix numerator coefficients are represented by the vector β with $\beta\epsilon B$, then the sensitivity index J may be regarded as a function $J(\beta, d\mu)$ of the plant parameter deviations and the sensitivity numerator coefficients. Consequently the optimization of (4.83) proposed here may be indicated by

$$J^{\circ} = \min_{\beta\epsilon B} \{ \max_{d\mu\epsilon a} [J(\beta,d,\mu)] \} \qquad \ldots \quad (4.86)$$

For a single-input single-output system case, the sensitivity matrix (4.81) becomes the familiar Bode transfer function sensitivity. If the plant contains only one parameter μ, the problem becomes that considered by Mazer[72]. This case simplifies considerably because the scalar sensitivity index (4.83) is homogeneous in $(d\mu)^2$. Thus, the optimum parameters β are independent of $(d\mu)$, and the minimum of J with respect to β may be found without first maximizing with respect to $d\mu$. The most complicated situation of several variable plant parameters, but still single-input single-output, have been studied by Gonzales[73].

The key to simplicity of the procedure is (4.79) which expresses the closed-loop error as the output of a system whose transfer function matrix is S and whose input is the open-loop error of the structure of Fig. 4.5. The matrix S is independent of $d\mu$, while the input E_0 is independent of β. Sensitivity models are employed to generate E_0. The numerator coefficients in S are then adjusted iteratively until (4.83) is minimaxed, i.e., maximized over $d\mu$ and minimized over the numberator coefficients of S. The procedure is relatively simple to implement on a digital computer.

CHAPTER V

SUMMARY AND CONCLUSIONS

We have presented the Sensitivity Analysis Techniques as applied to Optimal Control Systems. We have shown that these techniques are readily applicable to design as well as analysis problems, to ensure good system performance.

It is concluded that by applying these techniques to analysis and design, the control engineer can obtain extra valuable information about system behaviour. Many apparently simple design problems, if overlooked, might lead to serious deficiencies in system performance under operating conditions. Optimal control system design should, therefore, be backed up by sensitivity analysis to guarantee a near-optimal performance under the expected varying conditions.

In analysis and design, the control engineer deals with a mathematical model of the plant rather than the plant itself. Usually, the model form is assumed and the values of the model parameters are computed using the relevant identification techniques or imperical formulas. The optimal control law for the plant is then obtained based on the model. In practice, however, the values of the plant parameters will differ from those of the model. Consequently, the control law obtained for the model will not be optimal when applied to the plant. To assess the quality of the system resulting from applying the computed control to the plant, sensitivity analysis techniques are used. The errors induced in system trajectory and/or performance index, due to deviations in plant parameters or initial states from their nominal values are obtained.

A comparison between open-loop and closed-loop implementations of the control law is presented from the point of view of sensitivity. It is shown that the sensitivity measure is an essential part of the analysis and use of different measures might lead to contradictory results. This contradiction can be easily resolved if the measure is stated explicitly.

Since large parameter variations are to be expected in some plants, the ρ-sensitivity technique is presented to analyze this important case. This technique compares the magnitude of the performance index at extreme conditions to that obtained at nominal parameter values. In design, the value of ρ, which can be tolerated, is assumed and the corresponding parameter variations range is computed. This serves as a quality control criterion in choosing the components to be used to build the plant. On the other hand, if the parameter variations are known, the corresponding value of ρ could be evaluated.

Errors due to erroneous implmentation of the control law are treated. It is shown that in case of closed-loop implementation, the problem reduces to that of parameter variations. Open-loop configuration is then analyzed and performance degradation is estimated.

If the performance degradation due to expected parameter variations is unacceptable, some measures should be taken to improve the system. This could be achieved by using higher quality components or change of system configurations. In Chapter IV some design techniques used to reduce system sensitivity are discussed. They are classified into three main categories namely, adaptive, choice of performance index and use of dynamic compensators. Techniques using a combination of these categories can be rewarding, which is worth investigating.

In this report, only deterministic continuous systems have been considered. Many excellent papers have appeared in the literature treating the stochastic and discrete cases. These are, however, beyond the scope of this report.

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