A STUDY OF SUBSONIC FLOW LOSSES

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ABSTRACT

Shapiro's pressure loss calculation method, applied to flows in constant-area passages, is compared to Benedict's variable-area method. Both methods are applicable to steady, one-dimensional, adiabatic compressible flows. Although the two methods are different in their respective derivations and mathematical formulations, they prove to be numerically identical for the constant-area case.

Based on the approaches of Shapiro and Benedict, an alternative method is developed for estimating pressure losses in variable-area flow passages. The developed method is a more general and direct approach to flow loss calculation than either of the two reviewed methods.
ACKNOWLEDGEMENTS

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NOMENCLATURE

A  Cross-Sectional Area
A_w  Wetted Surface Area
C_p  Specific Heat at Constant Pressure
D  Hydraulic Diameter
F  Frictional Head
f  Friction Factor
f_F  Fanning Friction Factor
f_f  Mean Fanning Friction Factor
f_n  Denotes Function
g  Acceleration of Gravity
K  Loss Coefficient
L_{max}  Maximum Passage Length for Continuous Flow
M  Mach Number
P  Non-Dimensionalized Pressure Factor
P  Absolute Pressure
Q  External Heat Transfer per LB.
R  Pressure Ratio
Re  Renoylds Number
Rg  Gas Constant
T  Absolute Temperature
u  Specific Internal Energy
v  Specific Volume
\( V \)  Velocity
\( w \)  Mass Rate of Flow
\( \bar{W} \)  External Work per LB
\( x \)  Distance in the Direction of Flow
\( z \)  Potential Head
\( \delta \)  Inexact Differential
\( \alpha \)  Flow Number
\( \gamma \)  Isentropic Exponent
\( \Gamma \)  Generalized Compressible Flow Function
\( \rho \)  Density
\( t_w \)  Wall Shearing Stress

**SUBSCRIPTS**

\( o \)  Denotes Stagnation
\( 1 \)  Denotes Initial Condition
\( 2 \)  Denotes Final Condition
\( s \)  Denotes Static
\( t \)  Denotes Total
\( w \)  Denotes Wall

**SUPERSCRIPCTS**

\( * \)  Denotes Critical State or Maximum Value
INTRODUCTION

One of the most common problems in engineering concerns the determination of pressure losses in flow passages, due mainly to viscous wall friction. The one-dimensional approach to this problem includes estimating a loss coefficient for the section of the passage under consideration. This loss coefficient is generally a function of the friction factor $f$ which is defined in the well known Moody diagram*. The friction factor, in turn, depends on the relative roughness of the flow passages and the equivalent flow Reynolds number. After the loss coefficient is obtained one can follow any of the several available procedures to estimate the total or/and static pressure drops.

Two of the better known pressure loss calculation methods, those of Shapiro [1]** and Benedict [2], are discussed in the main body of this study. The presentation emphasizes the main differences and similarities in the derivation of the two approaches. An attempt is made to evolve an alternative procedure which combines both methods and adapts them to the particular needs of aerodynamics flow loss estimation.

All three above mentioned methods consider compressible subsonic flows with friction. The following are the underlying assumptions common to the three flow analyses.

1) The flow under consideration is steady and one-dimensional, i.e. all fluid properties are uniform over any cross-section of the duct.

* see Appendix A-2
** denotes Reference
(ii) No external work is done by the flow during the flow process.

(iii) Differences in elevation produce negligible changes compared with frictional effects.

(iv) The analysis is restricted to perfect gases, i.e. a perfect gas obeys the following two equations:

\[ pv = RgT \]
\[ \gamma = \text{constant} \]

where

- \( p \) denotes pressure
- \( v \) denotes specific volume
- \( Rg \) denotes gas constant
- \( T \) denotes temperature
- \( \gamma \) denotes specific heat ratio

(v) Within any cross-section of the flow passage the various fluid properties are related isentropically. Flow between any two calculation planes, e.g. inlet and exit of duct, is adiabatic with friction.

Chapter 1 of the present study gives a brief description of Shapiro's flow loss calculation method for constant-area flow passages. Shapiro's loss coefficient and pressure correlations are expressed in terms of the inlet and critical (=1.0) Mach numbers.

The method of Benedict, outlined in Chapter 2, is allegedly valid for both constant and variable-area duct flows. The two principal features of the method are:

a) The loss coefficient, expressed in terms of parameters
$R_1$, $R_2$ and the 'total flow number' $\alpha_{t1}$ at inlet. Parameters $R_1$, $R_2$ and $\alpha_{t1}$ are defined as

$R_1 = \text{ratio of static to total pressure at inlet of flow passage}$

$R_2 = \text{ratio of static pressure at exit to total pressure at inlet of flow passage}$

'total flow number', $\alpha_{t1} = \frac{\dot{W}}{\Delta t} \left( \frac{P_1}{\text{At}} \right)^{1/2} = \frac{w \sqrt{(Rg/g)T_1}}{Ap_{t1}}$

The derivation of the loss coefficient relationship is based on the assumption that the flow passage is of constant area.

b) The Generalized Compressible Flow Function, $\Gamma$, which may be defined as the ratio of the 'total flow number' at any arbitrary state to the critical 'total flow number'. The variable-area aspect is treated as an isentropic change of the $\Gamma$-function.

The combination of the two mentioned concepts forms the workings of Benedict's method.

Chapter 3 presents a discussion of the differences as well as the similarities of the two mentioned methods.

The Alternative pressure loss calculation approach, described in Chapter 4, follows Shapiro's line of analysis up to the point of establishing the final working equations. The resultant working equations are then expressed in terms of the inlet and exit Mach numbers. The variable-area case is handled by utilizing the expressions for total and static flow numbers at two arbitrary duct cross-sections. The 'static flow number' $\alpha_s$ may be defined similarly to the 'total flow number' $\alpha_t$ mentioned earlier as
static flow number, \( \alpha_s \) = \( w \cdot \frac{\sqrt{\gamma R T}}{\gamma P_s} \)

By expressing \( \alpha_s \) and \( \alpha_t \) in terms of Mach number one can write that

\[
\frac{p_{t2}}{p_{t1}} = \frac{A_1 \alpha_{t1}}{A_2 \alpha_{t2}} = fn (M_1, M_2, A_1/A_2)
\]

\[
\frac{p_{s2}}{p_{s1}} = \frac{A_1 \alpha_{s1}}{A_2 \alpha_{s2}} = fn (M_1, M_2, A_1/A_2)
\]

where

subscripts 1 and 2 indicate inlet and exit of flow passage respectively

\( p_{t2}/p_{t1} \) is the total pressure ratio for flow passage

\( p_{s2}/p_{s1} \) is the static pressure ratio for flow passage

\( A_1/A_2 \) is the area ratio for flow passage

Following the Conclusions of this study, Appendixes A, B and C present details and/or auxiliary information pertaining to Shapiro's, Benedict's and the Alternative methods respectively.
Chapter 1  THE METHOD OF SHAPIRO

The constant-area flow loss calculation method of Shapiro is described in Chapter 6 of Reference 1. The entire analysis is carried out in differential form employing the infinitesimal control volume flow model of Fig. 1-1.

![Control Surface Diagram]

**FIG. 1-1 INFINITESIMAL CONTROL VOLUME**

1.1 Basic Equations and Definitions

The following is a summary of the basic equations used by Shapiro in his derivation of the calculation method.

**Perfect Gas Law Equation**

\[
p = \rho R_g T \quad (1-1)
\]

**Definition of Mach number**

\[
M^2 = \frac{V^2}{\gamma R_g T} \quad (1-2)
\]

**Steady Flow Energy Equation**

\[
C_p dT + d \left( \frac{V^2}{2} \right) = 0 \quad (1-3)
\]

**Continuity Equation**

\[
w/A = \rho V \quad (1-4)
\]

**Momentum Equation**

\[
-Adp - t_w dA_w = \omega dV \quad (1-5)
\]
where \( A \) is the cross-sectional area of passage and \( dA/w \) is the wetted wall area over which the shearing stress \( t_w \) acts.

The friction factor* and hydraulic diameter are defined respectively as follows:

\[
f_f = \frac{\text{wall shearing stress}}{\text{dynamic head}} \quad (1-6)
\]

\[
f_f = \frac{t_w}{p \nu^2/2}
\]

\( f_f \) is also called the Fanning friction factor.

\[
D = \frac{4 \times (\text{cross-sectional area})}{\text{wetted perimeter}} \quad (1-7)
\]

\[
= \frac{4A}{dA/w} = 4 \frac{A}{dA/w} dx
\]

1.2 Main Differential Equations

Applying logarithmic differentiation to relations (1-1), (1-2) and (1-4), and by modifying equation (1-3), Shapiro arrives at the following convenient expressions:

\[
\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (1-8)
\]

\[
\frac{dV^2}{V^2} = \frac{dV}{VT} - \frac{dT}{T} \quad (1-9)
\]

\[
\frac{dp}{\rho} + \frac{1}{2} \frac{dV}{V} = 0 \quad (1-10)
\]

\[
\frac{dT}{VT} + \frac{1}{2} \frac{dV}{V} = 0 \quad (1-11)
\]

A subsequent substitution of equations (1-1), (1-2), (1-4), (1-6) and (1-7) into the Momentum expression (1-5) results in

\[
\frac{dp}{p} + \frac{\rho V^2}{2} \frac{4f\rho}{D} \frac{dx}{D} + \frac{\rho V^2}{2} \frac{dV}{V} = 0 \quad (1-12)
\]

* see Appendix A-1
Another useful relationship is obtained by logarithmically differentiating both sides of the isentropic expression \( p_o/p = (1 + \frac{\gamma-1}{2} M^2)^{\gamma-1} \).

This gives

\[
\frac{dp_o}{p_o} = \frac{dp}{p} + \frac{\gamma M^2/2}{1 + \gamma - 1 M^2} \frac{dM^2}{M^2} \tag{1-13}
\]

Relationships (1-8) to (1-13) make up the basic differential equations of the method.

Next, the loss coefficient \( \frac{4f_F dx}{D} \) is selected as the independent variable. Employing the algebraic procedure of elimination, Shapiro manipulates equations (1-8) to (1-13) to obtain the following essential differential equations:

\[
\frac{dM^2}{M^2} = \frac{\gamma - 1}{1 - M^2} \gamma M^2 \frac{4f_F dx}{D} \tag{1-14}
\]

\[
\frac{dp}{p} = -\frac{\gamma M^2 (1 + (\gamma - 1) M^2)}{2(1 - M^2)} \frac{4f_F dx}{D} \tag{1-15}
\]

\[
\frac{dp_o}{p_o} = -\frac{\gamma M^2}{2} \frac{4f_F dx}{D} \tag{1-16}
\]

Equations (1-14) to (1-16) form the backbone of Shapiro's constant-area flow method. They express the effect of friction upon flow Mach number and pressure. A cursory analysis of these expressions reveals that in an adiabatic flow with friction (for \( M < 1 \)) both static and stagnation pressures decrease, whereas the Mach number and, hence, velocity increase in the direction of flow.

1.3 Working Formulas

In order to develop the necessary working expressions for the method, Shapiro chooses \( N^2 \) as the independent variable. Equations (1-14) to (1-16) then take the following forms:

\[
\frac{4f_F dx}{D} = \frac{1 - M^2}{\gamma N^2 (1 + \frac{\gamma - 1}{2} N^2)} \frac{dM^2}{M^2} \tag{1-17}
\]
\[- \frac{\mathrm{d}p}{\rho} = \frac{1 + (\gamma-1) M^2}{2M^2(1 + Y_{-1} N^2) \rho} \mathrm{d}M^2 \tag{1-18} \]
\[- \frac{\mathrm{d}P}{P_o} = \frac{1 - M^2}{2(1 + Y_{-1} N^2) \rho} \mathrm{d}M^2 \tag{1-19} \]

Subsequent integration of these equations between the limits specified below,

<table>
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<th>Variable</th>
<th>Integration limits</th>
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<tr>
<td>(M^2)</td>
<td>(M^2) and 1</td>
</tr>
<tr>
<td>(x)</td>
<td>0 and (L_{\text{max}})</td>
</tr>
<tr>
<td>(p)</td>
<td>(p) and (p^*)</td>
</tr>
<tr>
<td>(P_o)</td>
<td>(P_o) and (P_o^*)</td>
</tr>
</tbody>
</table>

yields

\[
4 \bar{c}_F L_{\text{max}} = \frac{1 - M^2}{Y M^2} + \frac{\gamma + 1}{2} \ln \left[ \frac{(\gamma+1) M^2}{2(1 + Y_{-1} N^2)} \right] \tag{1-20} \]

where

\[
\bar{c}_F = \frac{1}{L_{\text{max}}} \int_0^{L_{\text{max}}} c_F \mathrm{d}x
\]

\(L_{\text{max}}\) is the constant-area duct length at the end of which the flow is choked

\[
\frac{\bar{p}}{p^*} = \frac{1}{M} \left[ \frac{\gamma + 1}{2(1 + Y_{-1} N^2)} \right]^{1/2} \tag{1-21} \]

\[
\frac{\bar{P}_o}{P_o^*} = \frac{1}{M} \left[ \frac{2(1 + Y_{-1} N^2)^2}{\gamma + 1} \right]^{\gamma + 1} \tag{1-22} \]

The parameters marked with an asterisk in the above relationships represent values at the critical state. The above equations are applicable for ducts of any length irrespective whether the flow reaches a critical condition at the exit of the duct or not. In the latter case the problem is treated by considering an imaginary duct extension where the flow is choked at the exit.

Expressions (1-20) to (1-22) are the working formulas of Shapiro's constant-area flow method. They are represented in tabular...
and graphical forms in Table I-1 and Fig. 1-2 respectively.

For practical applications the change in loss coefficient and the pressure ratios between planes 1 and 2 of a flow channel may be obtained thus:

\[
\frac{4f_{L1}}{D} = (4f_{L1\text{max}})_{M1} - (4f_{L1\text{max}})_{M2} \quad (1-23)
\]

\[
\frac{p_2}{p_1} = \frac{(p/p^*)_{M2}}{(p/p^*)_{M1}} \quad (1-24)
\]

\[
\frac{p_{02}}{p_{01}} = \frac{(p_0/p_0^*)_{M2}}{(p_0/p_0^*)_{M1}} \quad (1-25)
\]

Appendix A-2 presents a worked example to illustrate the use of Fig. 1-2 and equations (1-23) to (1-25).
### Table I-1

Data for Shapiro's Method

Perfect Gas, $\gamma = 1.4$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$p/p^*$</th>
<th>$p_0/p^*$</th>
<th>$4f_p L_{\max}/D$</th>
</tr>
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<tbody>
<tr>
<td>0.00</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
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<tr>
<td>0.05</td>
<td>21.903</td>
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<td>1.00000</td>
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<td>0</td>
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FIG. 1-2 SHAPIRO'S METHOD CURVES
Chapter 2  THE METHOD OF BENEDICT

The method of pressure loss calculation as developed by Benedict is outlined in References 2 and 3. The part of the method presented below is limited to compressible fluid flows in the subsonic range.

2.1 Basic Equations

The following is a list of basic equations used to derive the method:

General Energy equation [4]
\[ \delta Q + \delta W = du + pdv + vdp + Vdv + dz \quad (2-1) \]

First Law of Thermodynamics, corollary [4],
\[ \delta Q + \delta F = du + pdv \quad (2-2) \]

where \( \delta F \) is frictional head loss

Darcy - Weisbach Equation for frictional head loss
\[ \delta F = f \frac{dx}{D} \frac{V^2}{2g} \quad (2-3) \]

where \( V \) is the average velocity of flow. This expression is used here as a definition of the friction factor \( f(=4f_p) \)

Continuity Equation
\[ \frac{W}{\Delta t} = AV/v \quad (2-4) \]

where \( W/\Delta t \) is the mass rate of flow

Isentropic Gas Law relationship
\[ pv^\gamma = c_c = \text{constant} \quad (2-5) \]

Perfect Gas Law Equation
\[ \frac{p}{\rho} = RgT \quad (2-6) \]

* see Appendix A-1
2.2 **Loss Coefficient Relationship**

The loss coefficient is defined by Benedict as

\[
K = \int f \frac{dx}{D}
\]

The derivation of \(K\) relation to other flow parameters is carried out by considering a flow passage of uniform cross-section.

First, equations (2-1) and (2-2) are combined to give

\[
\delta F = -vdp - \left( \frac{vdv}{g} \right)
\]  

(2-7)

Subsequently, relationship (2-5) is introduced into (2-7) and the resultant expression integrated between the static and stagnation states at a point \((\delta F = 0)\), producing

\[
v^2 = \frac{2gy(p_vt - pv)}{\gamma - 1}
\]

(2-8)

A further combination of the preceding equation with the continuity relation (2-4) results in

\[
v = \frac{2(g p_v t)^{1/2}}{\left( \frac{R^2}{\alpha^2} + \frac{2(\gamma - 1)}{\gamma} \right)^{1/2} + \frac{R}{\alpha}}
\]

(2-9)

where

\[
R = \frac{p}{p_t 1}, \text{ a variable, a function of static pressure only}
\]

(for a given inlet total pressure \(p_t 1\)).

\[
a_t 1 = \frac{W/\Delta t}{Ap_t 1} (p_v t)^{1/2}, \ 'total \ flow \ number' \ at \ inlet, \ a \ constant
\]

for given area and \(p_t 1\)

\[
p_v t = R g T, \ 'constant \ for \ the \ adiabatic \ case'
\]

Equation (2-9) concludes the first step in the derivation. In the next step, the joint manipulation of expressions (2-3), (2-4) and (2-6) gives

\[
\frac{dK}{V^2/2g} = - \frac{2gAp_t 1}{W/\Delta t} \frac{dp}{V} - \frac{2dV}{V} - 13 -
\]
Substituting expression (2-9) into the above relationship and integrating [7] the resultant equation, one obtains an expression for the loss coefficient in the following form:

\[
K = \frac{1}{2} \frac{R}{\alpha t_1} \left[ \frac{R^2}{\alpha^2} + \frac{2(y-1)}{y} \right]^{1/2}_{1/2} + \frac{1}{2} \frac{R}{\alpha t_1} \left[ \frac{R^2}{\alpha^2} + \frac{2(y-1)}{y} \right]^{1/2}_{1/2}
\]

\[
\frac{2}{\gamma - 1} \ln \left[ \frac{R^2}{\alpha^2} + \frac{2(y-1)}{y} \right]^{1/2}_{1/2}
\]

(2-10)

In the above expression the friction factor \( f \) is assumed to be either independent of \( L \) or to have an average value over the duct length \( L \).

2.3 Generalized Compressible Flow Function [3]

The Generalized Compressible Flow Function is the chief innovation of Benedict's method. To derive its characteristic relationship, equations (2-4), (2-5) and (2-8) are rearranged to the following forms:

\[
\rho V A = \text{constant} \tag{2-11}
\]

\[
\rho = \rho_t \left[ \frac{\rho}{\rho_t} \right]^{1/\gamma} \tag{2-12}
\]

\[
V = \left[ \frac{2\gamma}{\gamma - 1} \left( \frac{\rho_t}{\rho} - \frac{p_t}{p} \right) \right]^{1/2} \tag{2-13}
\]

Furthermore, introducing relations (2-12), (2-13) and (2-6) into (2-11), Benedict obtains a new expression for continuity:

\[
(R \gamma T_t)^{-1/2} \frac{p_t A}{\frac{p_t}{p_t} - \frac{p_t}{p}} \left[ \frac{\rho_t}{\rho} \right]^{1/\gamma \left[ 1 - \left( \frac{p_t}{p} \right) \frac{\gamma - 1}{\gamma} \right]}^{1/2} = \text{constant}
\]

which after subsequent generalization becomes

\[
\left( \frac{T_t}{T_t} \right)^{1/2} \left( \frac{p_t}{p_t} \right)^{\left( A_1/A_2 \right)} \left( p_t \right)^{1/\gamma \left[ 1 - \left( \frac{p_t}{p_t} \right) \frac{\gamma - 1}{\gamma} \right]^{1/2}} = \left( p_2/p_{t2} \right)^{1/\gamma \left[ 1 - \left( \frac{p_2}{p_{t2}} \right) \frac{\gamma - 1}{\gamma} \right]^{1/2}} \tag{2-14}
\]

- 14 -
Considering now expression

\[ P = (p/p_t)^{1/\gamma} \left[ 1 - (p/p_t)^{\gamma-1} \right]^{1/2} \]

one can determine from equation (2-14) that maximum \( P \) occurs when

\[ \frac{P}{P_t} = \left[ \frac{2}{\gamma+1} \right]^{\gamma/(\gamma-1)} \]  \( (\gamma \neq 1) \)

which is the familiar relation of \( p/p_t \) at the critical state, \( p^*/p_t^* \).

At the critical condition

\[ p^* = \left( \frac{2}{\gamma+1} \right)^{1/(\gamma-1)} \left[ \frac{(\gamma-1)/(\gamma+1)}{1/2} \right] \]

Henceforth, Benedict proceeds to define a Generalized Compressible Flow Function as

\[ \Gamma = \frac{P}{p^*} \]

\[ = \frac{(p/p_t)^{1/\gamma} \left[ 1 - (p/p_t)^{\gamma-1} \right]^{1/2}}{\left[ \left( \frac{2}{\gamma+1} \right)^{1/2} \left[ \frac{(\gamma-1)/(\gamma+1)}{1/2} \right] \right]^{1/2}} \]  \( (2-15) \)

where \( \Gamma \) varies between 0 and 1 for any flow process. A transformed continuity equation, (2-14), may hence be stated as

\[ \Gamma_2 = \Gamma_1 \left[ \left( \frac{T_{t2}}{T_{t1}} \right)^{1/2} \left( \frac{p_{t2}}{p_{t1}} \right) \left( \frac{A_1}{A_2} \right) \right] \]  \( (2-16) \)

An alternative definition of \( \Gamma [5] \) may be obtained by defining the total flow number as

\[ \alpha_t = \frac{(H/\Delta t)}{p_t A} \left( \frac{R \Delta T_t}{g} \right)^{1/2} \]

\[ = \left[ \frac{2\gamma/(\gamma-1)}{(p/p_t)^2/\gamma(1 - (p/p_t)^{\gamma-1}/\gamma)} \right]^{1/2} \]  \( (2-17) \)
The critical total flow number is then

$$\alpha_t^* = \left[ \frac{2\gamma}{(\gamma-1)} \left( \frac{2}{(\gamma+1)} \right) \frac{2}{(\gamma-1)/(\gamma+1)} \right]^{1/2}$$

Hence,

$$\Gamma = \frac{\alpha_t}{\alpha_t^*}$$

$$= \frac{(p/p_*)^{1/\gamma} \left( \frac{1-(p/p_*)^{\gamma-1}}{(2/(\gamma+1))^{\gamma-1}} \right)^{1/2}}{\left( \frac{2}{(\gamma+1)} \right)^{(\gamma-1)/(\gamma+1)}}$$

(2-18)

The preceding equation and expression (2-15) are identical.

2.4 Pressure Loss Calculation Procedure

![Flow Model Diagram](image)

FIG. 2-1 FLOW MODEL

Referring to Fig. 2-1, the entire total pressure loss calculation procedure may be divided into four steps:

1) In the first step Benedict [2,3] handles the effect of area variation by representing it as an isentropic change of state. The change is from a given initial state, corresponding to initial area, to a new initial state, corresponding to final area. From equation (2-16) it follows that

$$\Gamma_1' = \Gamma_1 \left( \frac{A_1}{A_2} \right)$$

where \( \Gamma_1' \) corresponds to the new initial state. From the area-modified
initial state calculation then proceeds via a constant-area flow process. The 'modified duct' is shown in broken lines in Fig. 2-1.

ii) Utilizing equation (2-17) to express the total flow number \( \alpha_{t1} \) in terms of \( R_1 \),

\[
\alpha_{t1} = \left( \frac{2\gamma}{(\gamma-1)} \right) \frac{R_1^2}{\gamma (1-R_1^{\frac{\gamma-1}{\gamma}})} \right)^{1/2}
\]  

(2-19)

Equations (2-10) and (2-19) are then solved* [2] simultaneously to yield \( \alpha_{t2} \) as a function of the initial conditions, \( R_2 \) and \( K \).

iii) Having found \( \alpha_{t2} \) one can thus obtain \( \Gamma_2 \) from

\[
\Gamma_2 = \frac{\alpha_{t2}}{\alpha^*_t}
\]

iv) Henceforth, from equation (2-16) it follows that

\[
\Gamma_2 = \Gamma_1 \left( \frac{p_{t1}}{p_{t2}} \right)
\]

or

\[
\Delta p_t = \frac{p_{t1}(\Gamma_2-\Gamma_1^t)}{\Gamma_2}
\]

The tabulated data and graph pertaining to Benedict's method are presented in Table II-1 and Fig. 2-2 respectively. A worked example on the use of the Table and graph is included in Appendix B-2.

*See Appendix B-1 for a possible solution
## TABLE 2-1
DATA FOR BENEDICT'S METHOD

### GENERALIZED FANNO FLOW TABLE

<table>
<thead>
<tr>
<th>Press Ratio No.</th>
<th>Mach No.</th>
<th>Temp Ratio No.</th>
<th>Flow Factor</th>
<th>(t_{\text{2}}/t_{\text{1}}) VERSUS LOSS COEFFICIENT (K or \alpha/\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EF)</td>
<td>(\eta)</td>
<td>(\eta')</td>
<td>(\eta'')</td>
<td>(\eta''')</td>
</tr>
<tr>
<td>(p_{1}/p_{2})</td>
<td>(M_{1})</td>
<td>(T_{1}/T_{2})</td>
<td>(T_{2}/T_{3})</td>
<td>(T_{3}/T_{4})</td>
</tr>
</tbody>
</table>

### EXIT STATIC PRESSURE TO INLET TOTAL PRESSURE (\(p_{1}/p_{2}\)), AND INLET TOTAL PRESSURE TO EXIT TOTAL PRESSURE (\(T_{1}/T_{2}\))

<table>
<thead>
<tr>
<th>(K^{*})</th>
<th>(\alpha^{*})</th>
<th>(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.999)</td>
<td>(0.990)</td>
<td>(0.990)</td>
</tr>
<tr>
<td>(0.99)</td>
<td>(0.989)</td>
<td>(0.989)</td>
</tr>
<tr>
<td>(0.98)</td>
<td>(0.988)</td>
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<tr>
<td>(0.97)</td>
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<td>(0.987)</td>
</tr>
<tr>
<td>(0.96)</td>
<td>(0.986)</td>
<td>(0.986)</td>
</tr>
<tr>
<td>(0.95)</td>
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</tr>
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<td>(0.94)</td>
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<tr>
<td>(0.93)</td>
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<td>(0.92)</td>
<td>(0.982)</td>
<td>(0.982)</td>
</tr>
<tr>
<td>(0.91)</td>
<td>(0.981)</td>
<td>(0.981)</td>
</tr>
<tr>
<td>(0.90)</td>
<td>(0.980)</td>
<td>(0.980)</td>
</tr>
</tbody>
</table>

### CONTINUATION OF IDETROPHIC ENVELOPE

<table>
<thead>
<tr>
<th>Press Ratio No.</th>
<th>Mach No.</th>
<th>Temp Ratio No.</th>
<th>Flow Factor</th>
<th>(t_{\text{2}}/t_{\text{1}}) VERSUS LOSS COEFFICIENT (K or \alpha/\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_{1}/p_{2})</td>
<td>(M_{1})</td>
<td>(T_{1}/T_{2})</td>
<td>(T_{2}/T_{3})</td>
<td>(T_{3}/T_{4})</td>
</tr>
</tbody>
</table>

### NOTE:
1. \(K^{*}\) is the value of \(K\) evaluated between the passage entry plane 1 (see Fig. 2-1) and the critical condition.

2. The values of \(\alpha\) are the same as \(\alpha\) in the text.
NOTE: 1. The envelope (K=0) represents the isentropic flow solutions;

2. The above graph is for a constant-area case;

3. 'Locus of maximum flow' line separates the subsonic and supersonic regimes.

FIG. 2-2 BENEDICT'S METHOD GRAPH
3.1 Correlation of Loss Coefficients

1) Modification of Benedict's expression for \( K \)

Considering equation (2-10)

\[
K = \frac{1}{2} \frac{R}{\alpha_{tl}} \left[ \frac{R^2 + 2(\gamma-1)}{\alpha_{tl}^2} \right]^{1/2} \left[ \frac{1}{2} + \frac{1}{2} \frac{R}{\alpha_{tl}} \right]^{1/2} + \frac{1}{2} \frac{R}{\alpha_{tl}} \left[ 1 + \frac{\gamma^2}{\gamma^2-1} \right]^{1/2} \left[ \frac{1}{2} + \frac{1}{2} \frac{R}{\alpha_{tl}} \right]^{1/2}
\]

one recalls that

\[
\alpha_{tl} = \frac{W/\Delta t}{A_{ptl}} \left( \frac{r+\gamma}{\gamma} \right)^{1/2} = \frac{w\sqrt{RgT_{tl}}}{A_{ptl}} g
\]

\( R = p/p_{tl} \)

Hence, one can write

\[
\frac{R}{\alpha_{tl}} = \left( \frac{1}{\alpha_{tl}} \right) R = \left( \frac{A_{ptl}/w\sqrt{RgT_{tl}}}{g} \right) \frac{p}{p_{tl}}
\]

or

\[
\frac{R}{\alpha_{tl}} = \frac{1}{w\sqrt{RgT_{tl}}/g} \cdot \frac{A_{ps}}{A_{ps}} = 1/\alpha_s
\]

Subsequent substitution of (3-1) into equation (2-10) results in \( K \) as a function of one variable, the static flow number \( \alpha_s \):

\[
K = \frac{1}{2\alpha_s} \left[ \frac{1}{\alpha_s^2} + \frac{2(\gamma-1)}{\gamma} \right]^{1/2} \left[ \frac{1}{2} + \frac{1}{2} \frac{1}{\alpha_s^2} \right]^{1/2} + \frac{1}{2} \frac{1}{\alpha_s^2} \left[ \frac{1}{2} + \frac{1}{2} \frac{1}{\alpha_s^2} \right]^{1/2}
\]

\[
\frac{\gamma+1}{\gamma} \ln \left( \frac{1/\alpha_{s1} + [1/\alpha_{s1}^2 + 2(\gamma-1)/\gamma]^{1/2}}{1/\alpha_{s2} + [1/\alpha_{s2}^2 + 2(\gamma-1)/\gamma]^{1/2}} \right)
\]

(3-2)
ii) Modification of Shapiro's Expression for Loss Coefficient

Considering equation (1-14)

$$\frac{dM^2}{M^2} = \gamma \frac{M^2(1+M^2(\gamma-1)/2)}{1-M^2} \int_0^{fD} dx$$

(1-14)

denote

$$dK = 4f_D \frac{dx}{f}$$

(since $K = \int_0^{fD} dx = \int_0^{fD} dx$)

$$m = M^2$$

Then equation (1-14) becomes

$$\frac{dm}{m} = \gamma \frac{1+M^2(\gamma-1)/2}{1-M^2} dK$$

or after some rearrangement,

$$dK = -\frac{1-M^2}{\gamma M^2 (1+M^2(\gamma-1)/2)} dm$$

Integrating now both sides of the above equation results in

$$K = \left[ -\frac{1}{\gamma M^2} + \frac{\gamma+1}{\gamma} \ln \left( \frac{1+M^2(\gamma-1)/2}{M^2} \right)^{1/2} \right] \left[ \frac{M^2}{2} \right]_1^{M^2}$$

(3-3)

From the isentropic relationship for $\alpha_s$ one can write that

$$\alpha_s^2 = \gamma M^2 (1+M^2(\gamma-1)/2)$$

which after some algebraic transformation gives

$$M^2 = \sqrt{(\gamma-1)^{-2} + 2\alpha_s^2/\gamma(\gamma-1)} - 1/(\gamma-1)$$

(3-4)
A subsequent combination of equations (3-3) and (3-4) produces

\[ K = \left[ \bar{a} + (\gamma + 1)/(2\gamma \ln(b)) \right] \alpha_s^2 \]
\[ \alpha_s^1 \]

where

\[ \bar{a} = \frac{1}{\gamma} \frac{1}{(\gamma - 1)} \left[ \frac{1}{(\gamma - 1)^2 + \frac{2\alpha_s^2}{\gamma(\gamma - 1)}} \right]^{1/2} \]
\[ \bar{b} = \frac{1 + \gamma - 1}{2} \left[ \frac{1}{(\gamma - 1)^2 + \frac{2\alpha_s^2}{\gamma(\gamma - 1)}} \right]^{1/2} - \frac{1}{\gamma - 1} \]
\[ \left[ \frac{1}{(\gamma - 1)^2 + \frac{2\alpha_s^2}{\gamma(\gamma - 1)}} \right]^{1/2} - \frac{1}{\gamma - 1} \]

Part \( \bar{a} \) is then multiplied and divided by the term \( \left[ \frac{1}{\gamma - 1} + \frac{1}{(\gamma - 1)^2 + \frac{2\alpha_s^2}{\gamma(\gamma - 1)}} \right]^{1/2} \).

The transformed \( \bar{a} \) becomes

\[ \bar{a} = -\frac{1}{2\alpha_s} \left[ \frac{1}{\alpha_s^2} + \frac{1}{\frac{2(\gamma - 1)}{\gamma}} \right]^{1/2} + \frac{1}{2\alpha_s^2} \] \hspace{1cm} (3-6)

Similarly, multiplying and dividing \( \bar{b} \) by the term \( \left[ \frac{1}{(\gamma - 1)^2 + \frac{2\alpha_s^2}{\gamma(\gamma - 1)}} \right]^{1/2} + \frac{1}{\gamma - 1} \)

gives, after some algebraic rearrangement and simplification,

\[ \bar{b} = \frac{\gamma}{4} \left[ \frac{1}{\alpha_s^2} + \frac{2(\gamma - 1)}{\gamma} \right]^{1/2} + \frac{1}{\alpha_s^2} \] \hspace{1cm} (3-7)

Since \( K = \frac{\bar{a} + \gamma + 1}{2\gamma} \ln(b) \), a combination of equations (3-6) and (3-7) results in the following new expression for \( K \).

\[ K = \frac{1}{2\alpha_s} \left[ \frac{1}{\alpha_s^2} + \frac{1}{\frac{2(\gamma - 1)}{\gamma}} \right]^{1/2} + \frac{1}{2\alpha_s^2} \] \hspace{1cm} (3-8)

\[ \frac{1}{\gamma} \ln \left\{ \frac{1/\alpha_s + (1/\alpha_s^2 + 2(\gamma - 1)/\gamma)^{1/2}}{1/\alpha_s^2 + (1/\alpha_s^2 + 2(\gamma - 1)/\gamma)^{1/2}} \right\} \]
A comparison of relations (3-2) and (3-8) reveals them to be identical, proving that both Shapiro's and Benedict's loss coefficients are the same.

### 3.2 Other Differences and Similarities

Considering Shapiro's expressions (1-23) and (1-25) for total pressure one can deduce that

\[
\frac{P_{02}}{P_{01}} = \frac{M_1}{M_2}\left[\frac{1 + (\gamma - 1)M_2^2/2}{1 + (\gamma - 1)M_1^2/2}\right]^\frac{\gamma + 1}{2(\gamma - 1)}
\]

As for Benedict's case, a combination of equations (2-16) and (2-18), when \(T_{t2} = T_{t1}\) and \(A_1 = A_2\), gives

\[
p_{t2}/p_{t1} = \alpha_{t1}/\alpha_{t2}
\]

Utilizing the well known isentropic expression for \(\alpha_t\)

\[
\alpha_t = \frac{\sqrt{T_t R \gamma}}{A p_t} = \frac{f_{PM}}{[1 + (\gamma - 1)M^2/2]((\gamma + 1)/(\gamma - 1))/2}
\]

one can express \(p_{t2}/p_{t1}\) as

\[
p_{t2}/p_{t1} = \frac{M_1}{M_2}\left[\frac{1 + (\gamma - 1)M_2^2/2}{1 + (\gamma - 1)M_1^2/2}\right]^\frac{\gamma + 1}{2(\gamma - 1)}
\]

Since equations (3-9) and (3-10) are identical, one concludes that the total pressure ratios for both methods are the same. Similar reasoning can be applied to the static pressures.
The preceding conclusions prove that both Shapiro's and Benedict's pressure loss calculation procedures are quantitatively identical for the constant-area flow case.

Considering now the qualitative merits of the two methods it may be worthwhile to note the following.

a) From overall considerations, Shapiro's approach appears to have more unity, as a whole, and a greater generality. This is mainly due to the use of Mach number as the common parameter. This fact also makes Shapiro's approach simpler to derive and understand, as opposed to Benedict's.

The \( \Gamma \)-function in Benedict's approach does not play the same unifying or other role as does the Mach number in Shapiro's case. Since values of the total flow number \( c_t \) are generally available from aerodynamics and other tables, the \( \Gamma \)-function concept may be considered as redundant for the present type of flow analysis.

b) A comparison of Tables I-1 and II-1 reveals Benedict's tabulated parameters to be more applicable for solving duct pressure loss problems. This is especially true for 'non-choked' passage flows where Benedict's parameters \( p_t/p_t1, \quad p_2/p_t1, \quad p_t1/p_t2 \) and \( K \) are more to the point than Shapiro's \( p/p^*, \quad p_0/p_0^* \) and \( 4f_p L_{\infty}/D \). The worked examples in Appendixes A and B give ample evidence to this effect.

c) If both methods are considered for analysis by computer, it may be conjectured that Shapiro's method would be easier and more straightforward to use for that purpose. Referring to section 2.3 of this write-up, step ii) in Benedict's method would probably involve some
iteration which is an added complication to the method.

d) An added advantage of Benedict's method over Shapiro's is its adaptability for approximate variable-area flow loss calculation. The Alternative method described in Chapter 4 demonstrates how Shapiro's approach can be utilized for that purpose.
4.1 Definition of Loss Coefficient

Based on equations (1-20) and (3-2), two auxiliary expressions for the loss coefficient $K$ are derived as follows.

a) $K$ Relationship in Terms of $M_1$ and $M_2$

Consider equation (3-3), which if evaluated between the indicated limits gives

$$K = \frac{1}{Y} \frac{M_2^2 - M_1^2}{M_1^2 M_2^2} - \frac{\gamma + 1}{2\gamma} \ln \left( \frac{M_2^2 (1 + [\gamma - 1] M_1^2 / 2)}{M_1^2 (1 + [\gamma - 1] M_2^2 / 2)} \right) \tag{4-1}$$

Expression (4-1) is a more general version of Shapiro's loss coefficient. Putting $M_2$ to 1 in equation (4-1) results in Shapiro's form of the loss coefficient, (1-20).

b) Approximated $K$ Coefficient

Evaluating equation (3-2) between the stated limits one obtains

$$K = \frac{X_1}{2a_{s1}} - \frac{X_2}{2a_{s2}} - \frac{(\gamma + 1)}{\gamma} \ln \left( \frac{X_1}{X_2} \right)$$
where

\[ x_1 = \left[ \frac{1}{\alpha_{s1}^2} + 2 \frac{(\gamma-1)}{\gamma} \right]^{1/2} + \frac{1}{\alpha_{s1}} \]

\[ x_2 = \left[ \frac{1}{\alpha_{s2}^2} + 2 \frac{(\gamma-1)}{\gamma} \right]^{1/2} + \frac{1}{\alpha_{s2}} \]

By further algebraic manipulation the ratio \( \frac{x_1}{x_2} \) can be simplified to

\[ \frac{x_1}{x_2} = \frac{1}{\alpha_{s2}} - \left[ \frac{1}{\alpha_{s2}^2} + \frac{2(\gamma-1)}{\gamma} \right]^{1/2} \]

\[ \frac{1}{\alpha_{s1}} - \left[ \frac{1}{\alpha_{s1}^2} + \frac{2(\gamma-1)}{\gamma} \right]^{1/2} \]

Henceforth, consider the series expansion of the term \( \left( \frac{1}{\alpha_{s}^2} + 2 \frac{(\gamma-1)}{\gamma} \right)^{1/2} \).

\[ \left( \frac{1}{\alpha_{s}^2} + 2 \frac{(\gamma-1)}{\gamma} \right)^{1/2} \equiv \alpha_{s}^{-1} + \alpha_{s} \frac{(\gamma-1)}{\gamma} - \alpha_{s}^3 \frac{(\gamma-1)}{(2\gamma)} + \cdots \]

Since for subsonic flows \( \alpha_{s} < 1 \), the preceding series is a rapidly converging one. Hence, substituting the first two terms of the above series expansion into equation (4-2) results in the following approximation of \( K \) coefficient.

\[ K_{\alpha} = \frac{1}{\alpha_{s1}^2} - \frac{1}{\alpha_{s2}^2} - \frac{\gamma+1}{\gamma-1} \ln \left( \frac{\alpha_{s2}}{\alpha_{s1}} \right) \]  \hspace{1cm} (4-3)*

4.2 Derivation of Pressure Correlations

The development of useful pressure correlations presented in this subsection follows mainly along the lines of Shapiro's approach. All pressure correlations are derived as functions of Mach number, at the inlet and exit of a passage.

* See Appendix C-2 for error estimation
a) **Pressure Ratio Correlations**

From equations (1-22) and (1-25), the total pressure ratio can be expressed as

\[
P_{t2}/P_{t1} = \frac{M_1}{M_2} \left( \frac{1 + (\gamma - 1)M_2^2/2}{1 + (\gamma - 1)M_1^2/2} \right)^{\gamma + 1 \over 2(\gamma - 1)}
\]  

(3-10)

Similarly, by combining expressions (2-21) and (2-24) one obtains

\[
P_{s2}/P_{s1} = \frac{M_1}{M_2} \left[ \frac{1 + M_2^2(\gamma - 1)/2}{1 + M_2^2(\gamma - 1)/2} \right]^{1/2}
\]  

(4-4)

Furthermore, the term \(1 + \frac{\gamma - 1}{2}M^2\) can be represented as a series expansion

\[
[1 + \frac{\gamma - 1}{2}M^2]^{1/2} \equiv 1 + (\gamma - 1)\left[\frac{M}{2}\right]^2 - (\gamma - 1)\left[\frac{M}{2}\right]^4 + \cdots
\]

Where, for our case of \(M < 1\), the series converges rapidly. Substituting then the first two terms of the above series expansion into equation (4-4) results in the following approximation:

\[
\left(\frac{P_{s2}}{P_{s1}}\right)_1 = \left( \frac{1 + \frac{\gamma - 1}{4}M_1^2}{1 + \frac{\gamma - 1}{4}M_2^2} \right) \frac{M_1}{M_2}
\]  

(4-5)

b) **Pressure Difference Correlations**

Referring to equations (3-10) and (4-4), one can define the pressure difference relations as
\[ \frac{\Delta p_t}{p_{t1}} = 1 - \frac{p_{t2}}{p_{t1}} \quad (4-6) \]

and

\[ \frac{\Delta p_s}{p_{s1}} = 1 - \frac{p_{s2}}{p_{s1}} \quad (4-7) \]

Dividing both sides of equation (4-6) by the corresponding sides of equations (4-7) results in

\[ \frac{\Delta p_t}{\Delta p_s} \frac{p_{s1}}{p_{t1}} = \frac{p_{t2} / p_{t1} - 1}{p_{s2} / p_{s1} - 1} = \frac{\left( \frac{Y_2}{Y_1} \right)^{(Y+1)/(Y-1)}}{\left( \frac{Y_1}{Y_2} \right)^{1/2} M_1 / M_2 - 1} \]

where

\[ Y_1 = (1 + \frac{M_1^2}{M_1} (Y-1)/2) \]
\[ Y_2 = (1 + \frac{M_2^2}{M_2} (Y-1)/2) \]

If, again, both sides of the preceding equation are divided by the corresponding sides of the generally known isentropic expression

\[ \frac{p_{s1}}{p_{t1}} = (1 + \frac{M_1^2}{M_1} (Y-1)/2)^{-Y/(Y-1)} \]

one obtains

\[ \frac{\Delta p_t}{\Delta p_s} = \frac{Y_2^\phi - Y_1^\phi}{Y_2^{-1/2} M_2 / M_1} \left( \frac{Y_2}{Y_1} \right)^{1/2} \quad (4-8) \]

where

\[ \phi = (Y+1)/(2(Y-1)) \]

Expression (4-8) can be also presented in an approximated simpler form.

This is achieved by substituting for the terms \( Y^\phi \) and \( Y^{1/2} \) the first two terms of their series expansions:

\[ Y^\phi = (1 + \frac{M^2}{M} (Y-1)/2)^\phi \equiv 1 + M^2 (Y+1)/4 + M^4 (Y+1) (Y+3)/32 + \cdots \]
\[ Y^{1/2} = \left[ 1 + \frac{M^2}{M} (Y-1)/2 \right]^{1/2} \equiv 1 + M^2 (Y-1)/4 - M^4 (Y-1)/32 + \cdots \]
The resulting approximation becomes

\[
\frac{[\Delta p_t]}{\Delta p_S} = \left(1 - (\gamma + 1)M_1^2/4\right)\left(1 + (\gamma - 1)\frac{M_1^2}{4}\right)\left(1 + (\gamma - 1)\frac{M_2^2}{4}\right)
\]

\[
\frac{1}{1 + (\gamma - 1)\left(M_1^2 + \frac{M_2^2}{4}\right)/4}
\]

(4-9)

At this point it should be reiterated that all the preceding pressure correlations of this section were derived on the assumption of an adiabatic (\(T_{t2} = T_{t1}\)), constant-area flow passage. An adaptation, following Benedict, of the above correlations for a variable-area case is presented below.

Utilizing the well known expression for the total flow number \(\alpha_t\),

\[
\alpha_t = \frac{\sqrt{\frac{G_t R}{T_t}}}{A_t} = \frac{\sqrt{\gamma M}}{[1 + (\gamma - 1)M^2/2]}(\gamma + 1)/(\gamma - 1)/2
\]

one obtains

\[
\frac{\alpha_{t1}}{\alpha_{t2}} = \frac{p_{t2}M_2}{p_{t1}M_1} \left[ \frac{1 + (\gamma - 1)M_2^2}{1 + (\gamma - 1)M_1^2} \right]^{\gamma + 1}/2(\gamma - 1)
\]

and thus

\[
\frac{p_{t2}}{p_{t1}} = \frac{M_1}{M_2} \left[ \frac{1 + (\gamma - 1)M_2^2}{1 + (\gamma - 1)M_1^2} \right]^{\gamma + 1}/2(\gamma - 1)\frac{A_1}{A_2}
\]

(4-10)

Since

\[
\alpha_s = \frac{\sqrt{\frac{G_t R}{T_t}}}{A_p} \equiv \left[\gamma M^2 \left(1 + (\gamma - 1)M^2/2\right)\right]^{1/2}
\]

one can state that

\[
\frac{\alpha_{s1}}{\alpha_{s2}} = \frac{p_{s2}A_2}{p_{s1}A_1} = \frac{M_1}{M_2} \left[ \frac{1 + (\gamma - 1)M_2^2}{1 + (\gamma - 1)M_1^2} \right]^{1/2}
\]

- 30 -
Hence,

\[
\frac{p_{s2}}{p_{s1}} = \frac{M_1}{M_2} \left[ \frac{1 + \frac{M_1^2}{2}(\gamma-1)/2}{1 + \frac{M_2^2}{2}(\gamma-1)/2} \right]^{1/2} \frac{A_1}{A_2}
\]  \hspace{1cm} (4-11)

For the constant-area case relations (4-10) and (4-11) become respectively identical to expressions (3-10) and (4-4).

Other useful pressure relationships, \(\Delta p_t/p_{t1}, \Delta p_s/p_{s1}\) and \(\Delta p_t/\Delta p_s\), for the variable-area case can be easily derived, if necessary, employing similar techniques as were used for the case of constant-area flow.

4.3 Working Graphs

To complete the presentation of the Alternative pressure loss calculation method, working graphs are presented in Figures 4-1 to 4-4 at the end of this section. These are plots of equations (4-1), (3-10), (4-4), (4-6), (4-7) and (4-8).

If the graphs are to be used for variable-area passages the following must apply:

1. Instead of \(\frac{p_{t2}}{p_{t1}}\) use \(\frac{p_{t2}}{p_{t1}} A_2/A_1\)

2. Instead of \(\frac{p_{s2}}{p_{s1}}\) use \(\frac{p_{s2}}{p_{s1}} A_2/A_1\)

3. Instead of \(\frac{\Delta p_t}{p_{t1}}\) use \(1 - \frac{p_{t2}A_2}{p_{t1}A_1}\)

4. Instead of \(\frac{\Delta p_s}{p_{s1}}\) use \(1 - \frac{p_{s2}A_2}{p_{s1}A_1}\)

5. Instead of \(\frac{\Delta p_t}{\Delta p_s}\) use \(\frac{(1-p_{t2}A_2/p_{t1}A_1) \ p_{t1}}{(1-p_{s2}A_2/p_{s1}A_1) \ p_{s1}}\)
From the preceding table it appears that for a variable-area case the use of Fig. 4-4 is too involved for efficient calculation.

The use of the working graphs is illustrated in Appendix C-1 by means of two examples. In passing, it must be stated that the plots in Figures 4-1 to 4-4 are only a demonstration of the workings of the Alternative method. In order for the method to become of general use, more detailed graphs and tabulated data may be required. These additional data can be obtained by means of the computer program used in providing the data for Figures 4-1 to 4-4. The listing of the program is included in Appendix C-3.

Expressions (4-3), (4-5) and (4-9) comprise the set of simplified approximations to be used for hand calculations if necessary. Of these, \( \left( \frac{p_2}{p_1} \right) \), is correct to within 0.5% of exact value, (4-4), for Mach numbers up to 1.0. The \( \left( \frac{\Delta p_r}{\Delta p_s} \right) \), is correct to within 1.0% of exact value, (4-8), for values of \( M_1 \) and \( M_2 \) less than 0.4 and 0.6 respectively.
ALTERNATIVE METHOD

FIG. 4-1 LOSS COEFFICIENT VARIATION

\( \gamma = 1.4 \)
ALTERNATIVE METHOD

\[ \frac{R_2}{R_1} = 1.0; \frac{\Delta P}{P_1} = 0.0 \]

\[
\begin{array}{ccc}
0.8 & 0.2 \\
0.7 & 0.3 \\
0.6 & 0.4 \\
0.5 & 0.5 \\
0.4 & 0.6 \\
0.3 & 0.7 \\
\end{array}
\]

\[ \gamma = 1.4 \]

FIG. 4–3 STATIC PRESSURE RATIO VARIATION
ALTERNATIVE METHOD

\[ \gamma = 1.4 \]

**FIG. 4-4**
PRESSURE DIFFERENCE RATIO VARIATION
FIG. 4-5 PRESSURE RATIO VS MACH NUMBER
CONCLUSIONS

The foregoing study consisted primarily of reviewing two known pressure loss calculation methods. A third alternative method, utilizing the advantages of the two other methods, was developed and presented as part of this study. The main features of the three methods are as follows.

1. **Numerical accuracy.** Since the expressions for the loss coefficients of the three methods are identical, numerical calculations by any of the three procedures must be the same for constant-area flow. This is further substantiated by the three solutions of the same worked example included separately in Appendixes A, B and C.

2. **Generality of Application.** As in the case of Shapiro's approach, the use of Mach number for the common independent variable gives the Alternative method a greater unity and generality than that of Benedict for constant-area flow. On the other hand, the Alternative approach, by virtue of its added capability of handling variable-area cases, has a wider range of application than Shapiro's method. Further, for the case of variable area flow, the greater availability from various technical sources of tabulated $\alpha_t$ data, as opposed to $\Gamma$-function data, may give the Alternative approach an added advantage over Benedict's method.

3. **Directness of applications.** Comparing the three solutions of the same example one concludes that Shapiro's solution is the most involved one of the three. Indeed, if one takes Shapiro's method in order to calculate the generally needed pressure loss $\Delta p_s$ or $\Delta p_t$, it is
necessary to go first through determining the flow parameters at the
critical state and only then the final conditions can be found. All
this tends to obscure and lengthen the flow loss calculation procedure.
Both Benedict's and the Alternative methods are more direct and explicit
in this respect.

4. **Method versatility.** Shapiro's approach is easier to adapt
for computer analysis than Benedict's method. This holds also for the
Alternative approach. One other feature, peculiar only to the
Alternative method, is the derived set of simplified approximations to
be used for hand calculations, whenever neither proper graphs nor tables
are available. The approximate expressions include the loss coefficient
$K_a$, the pressure ratio $(p_{s2}/p_{s1})_1$ and $(\Delta p_e/\Delta p_s)_1$. 
REFERENCES

1. A.H. Shapiro

2. R.P. Benedict, N.A. Carlucci


4. R.P. Benedict

5. R.P. Benedict

6. V.L. Streeter

7. H.B. Dwight
APPENDIX 'A'

SHAPIRO'S METHOD
Friction Factor $f$

The values for the friction factor may be obtained from the generally known Moody diagram included in this Appendix. The Moody diagram defines $f$ values for three different flow regimes: the laminar flow, fully developed turbulent flow, and flow in the transitional zone.

The friction factor definition for the laminar zone is based on the Hagen-Poiseuille equation for pipe flow:

$$f = \frac{64}{\text{Re}}$$

where Re is the flow Reynolds number referred to the hydraulic diameter of the flow channel. The transition zone friction factor is defined on the basis of Colebrooks' equation [6]:

$$\frac{1}{f} = 0.86 \ln \left( \frac{\epsilon/D + 2.51}{3.7 \text{ Re}^{1/2}} \right)$$

where

$\epsilon/D$ is the relative roughness

Re is the Reynolds number, as above

In general it may be stated that for

- laminar zone $f = fn(\text{Re})$ only
- transition zone $f = fn(\text{Re}, \epsilon/D)$
- turbulent zone $f = fn(\epsilon/D)$ only

The Fanning friction factor, $f_F$, is defined as

$$f_F = \frac{f}{4}$$
FIG. A-1 MOODY DIAGRAM

NOTE: FOR FANNING
FRICITION FACTOR,
DIVIDE f BY FOUR.
\( f_F = \frac{f}{4} \)

REYNOLDS NUMBER
\[ R_e = \frac{VD}{V} \]
\( V \) in \( \text{ft/sec} \), \( D \) in \( \text{ft} \), \( V \) in \( \text{ft}^2/\text{sec} \)

RELATIVE ROUGHNESS \( \frac{\varepsilon}{D} \)
Worked example

In a constant cross-sectional passage through which flows air (\(\gamma=1.4\)) the following is known:

\[
M_1 = 0.347
\]
\[
p_1 = 18.4 \text{ psia}
\]
\[
p_2 = 16.4 \text{ psia}
\]
\[
p_{s1}/p_{c1} = 0.92
\]
Find the pressure loss and the loss coefficient of the passage flow.

Solution:

Knowing \(p_1/p_{o1}, p_{c1} = 20.0 \text{ psia}\).
From Fig. 2-2 one then determines

\[
p_{o1}/p_{o*} = 1.79
\]
\[
p_1/p* = 3.12
\]
\[
(4fF \ L_{max}/D)_1 = 3.53
\]
which enables to calculate \(p_2/p*

\[
p_2/p* = \frac{p_2 \cdot p_1/p*}{p_1}
\]
\[
= \frac{16.4 \times (3.12)}{18.4} = 2.78
\]

Subsequently, from Fig. 2-2, for \(p_2/p* = 2.78\), it follows that

\[
M_2 = 0.39
\]
\[
p_{o2}/p_{o*} = 1.63
\]
\[
(4fF \ L_{max}/D)_2 = 2.5
\]
Hence, one can determine $\frac{P_{O2}}{P_{O1}}$

$$\frac{P_{O2}}{P_{O1}} = \frac{P_{O2}}{P_{O*}} \cdot \frac{P_{O*}}{P_{O1}}$$

$$= \frac{1.63}{1.79}$$

$$= 0.91$$

Thus:

$$\Delta P_O = (1 - \frac{P_{O2}}{P_{O1}}) P_{O1}$$

$$= (.09) .20$$

$$= 1.8 \text{ psi}$$

$$\Delta p = 18.4 - 16.4 = 2.0 \text{ psi}$$

$$\frac{4f_{FL}}{D} = \frac{(4f_{FL_{max}})_1}{D} - \frac{(4f_{FL_{max}})_2}{D}$$

$$= 1.03$$
APPENDIX 'B'

BENEDICT'S METHOD
APPENDIX B-1

Pressure loss calculation, step (ii)

One of the possible ways to express $\alpha_3$ as a function of $R_2$ may be as follows. First, evaluate equation (2-10) so that $K$ expression obtains the form

$$\frac{\gamma+1}{\gamma} \ln \left( \frac{R_2^2 + \frac{2(\gamma-1)}{\gamma} \alpha_t^2}{R_2 + \frac{R_z}{\alpha_t}} \right) \cdot \frac{1}{2} R_z \frac{\alpha_t^2}{R_z} \frac{\alpha_t^2}{\alpha_t} \frac{2(\gamma-1)}{\gamma} \frac{1/2}{K + C_\gamma} = 0$$

where

$$C_\gamma = \frac{1}{2} \frac{R_1}{\alpha_t} \left( \frac{R_2^2 + \frac{2(\gamma-1)}{\gamma} \alpha_t^2}{R_2 + \frac{R_z}{\alpha_t}} \right)^{1/2} + \frac{1}{2} \frac{R_1}{\alpha_t}$$

Considering $\gamma$, $K$, $C_\gamma$ to be known, one can solve for $R_2/\alpha_t$ by using any of the known numerical method procedures.

In order to proceed further, it is useful to redefine $R_2/\alpha_t$ as

$$\frac{R_2}{\alpha_t} = \frac{1}{\alpha_{s2}} = \frac{1}{\sqrt{\frac{\gamma M^2}{\alpha_t^2}}} \frac{1}{R_2}$$

where

$$\alpha_{s2} = \text{'static flow number'}$$

which can be expressed as a function of Mach number as

$$\alpha_{s2} = \sqrt{M} \left( 1 + M_2^2 (\gamma-1)/2 \right)^{1/2}$$

Solving then (B-1) for $M_2$ gives

$$M_2 = \left( \sqrt{(\gamma-1)^2 + 2\alpha_{s2}^2/(\gamma(\gamma-1))} - 1/(\gamma-1) \right)^{1/2}$$

I-5
On the other hand, the total flow number $a_{t2}$ at station 2 may be also defined in terms of $M_2$ as

$$a_{t2} = \frac{\sqrt{\gamma M_2}}{\left[1 + (\gamma-1)M_2^2/2\right]^{(\gamma+1)/(\gamma-1)/2}} \quad (B-3)$$

A further combination of equations (B-2) and (B-3) produces

$$a_{t2} = \frac{[\gamma(\gamma-1)(\lambda-1)]^{1/2}}{\left[(\lambda+1)/2\right]^\phi}$$

where

$$\lambda = [1 + 2a_{t1}^2(\gamma-1)/\gamma R_2^2]^{1/2}$$

$$\phi = (\gamma+1)/[2(\gamma-1)]$$
APPENDIX B-2

Worked example

Consider the adiabatic flow of air ($\gamma=1.4$, $R_g=53.3 \text{ ft}^2/\text{s}^2/\text{lbm}$) from the inlet total pressure of 20 psia and an inlet total temperature of 573°F. Find the static and total pressure drops through a constant-area duct (25 sq. in.) if the flow rate is 5 lb/sec and the loss coefficient is 1.

Solution (see Fig. 2-2):

At

$$a_{t1} = \frac{W/\Delta t}{A_{pt1}} \left( \frac{R_g T}{g} \right)^{1/2}$$

$$= 0.3825$$

obtain from Table II-1

$$\frac{p_1}{p_{t1}} = 0.92 \quad p_1 = 18.4 \text{ psia}$$
$$\frac{p_2}{p_{t1}} = 0.82034 \quad p_2 = 16.4 \text{ psia}$$
$$\frac{p_{t1}}{p_{t2}} = 1.09866 \quad p_{t2} = 18.2 \text{ psia}$$

Thus

$$p_1 - p_2 = 2.0 \text{ psi}$$
$$p_{t1} - p_{t2} = 1.8 \text{ psi}$$

Considering now Fig. 2-2,

At

$$\Gamma = a/a^*$$

$$= 0.5586$$
obtain

\[
P_1/P_{t1} = 0.92 \\
P_2/P_{t1} = 0.82 \\
P_{t1}/P_{t2} = 1.099
\]

These values give substantially the same results as the tabular method.
APPENDIX 'C'

ALTERNATIVE METHOD
APPENDIX C-1

Worked examples

Example 1:

In a constant cross-sectional passage through which flows air (γ=1.4) the following is known:

\[ M_1 = 0.347 \]
\[ P_{s1} = 18.4 \text{ psia} \]
\[ P_{s2} = 16.4 \text{ psia} \]

Find the total pressure loss.

Solution:

From the static pressures:

\[ \frac{P_{s2}}{P_{s1}} = 0.891 \]

Hence, from Fig. 4-3 one obtains

\[ M_2 = 0.39 \]

Going then to Fig. 4-4 gives

\[ \Delta p_t/\Delta p_s = 0.9 \text{ psi} \]

Hence,

\[ \Delta p_t = (0.9) (18.4 - 16.4) \]
\[ = 1.8 \text{ psi} \]

Example 2:

Change the given conditions in Example 1 as follows

\[ M_1 = .347 \]
\[ P_{tl} = 20 \text{ psia} \]
\[ K = 1.0 \]
\[ A_2/A_1 = .96 \]
Find the total and static pressure losses.

Solution:

From Fig. 4-1 one obtains

\[ M_2 = 0.39 \]

then from Fig. 4-2

\[ \frac{P_{t2}}{P_{t1}} \cdot \frac{A_2}{A_1} = 0.91 \]

Hence,

\[ \Delta P_t = P_{t1} - \frac{P_{t2} A_2}{P_{t1} A_1} \cdot \frac{P_{t1} / A_1}{(A_2 / A_1)} \]

\[ = 20 (1 - 0.91 / 0.96) \]

\[ = 1.02 \text{ psi} \]

Going now to Fig. 4-5:

\[ \frac{P_{s1}}{P_{t1}} = 0.92 \]

Hence,

\[ P_{s1} = 20 \cdot (0.92) = 18.4 \text{ psia} \]

Then from Fig. 4-3:

\[ \frac{P_{s2}}{P_{s1}} \cdot \frac{A_2}{A_1} = 0.891 \]

or

\[ \Delta P_s = 18.4 \cdot (1 - 0.891 / 0.96) \]

\[ = 1.325 \text{ psi} \]
ALTERNATIVE METHOD
(CONSTANT $|M_2 - M_1|$ CURVES)

$K_\infty$ ERROR = $|\frac{K - K_\infty}{K}|$

$\gamma = 1.4$

FIG. C-I $K_\infty$ ERROR VS INLET MACH NUMBER
APPENDIX C-3

COMPUTER PROGRAM CODE

YM1 denotes \( M_1 \)
*DM " " \( \Delta M = M_2 - M_1 \)
NYM1 " " No. of values of \( M_1 \)
*NDM " " No. of values of \( \Delta M \) \}

XK denotes \( K \)
*XX2 " " \( K_1 \)
*XXB1 " " \( K \)
XKA " " \( K_\alpha \)

PTR denotes \( P_{t2}/P_{t1} \)
DPTR " " \( P_{t2}/P_{t1} - 1 \)
PSR " " \( P_{s2}/P_{s1} \)
DP SR " " \( P_{s2}/P_{s1} - 1 \)
PSR1 " " \( (P_{s2}/P_{s1})_1 \)
DPR " " \( \Delta P_t/\Delta P_s \)
DPR1 " " \( (\Delta P_t/\Delta P_s)_1 \)

*XXINC " " \( K' \) or \( K'' \)

EKA denotes error of \( K_\alpha \)
*EK2 " " \( K_1 \)
*EK B1 " " \( K \)

EPSR denotes error of \( (P_{s2}/P_{s1})_1 \)
EDPR1 " " \( (\Delta P_t/\Delta P_s)_1 \)

*EKINC denotes error of \( XKINC \)

* denotes parameters not pertaining to this study.
PROGRAM WIFRBA

C
FLOW LOSSES
DIMENSION YM1(10), DM(15), XK(10, 15), XKA(10, 15), XK1(10, 15), XK2(10, 15)
1X, K3(10, 15), XKB(10, 15), DPR3(10, 15), DPR2(10, 15), XKINC(10, 15)

DIMENSION EKA(10, 15), EK1(10, 15), EK2(10, 15), EK3(10, 15), EKB(10, 15)
DIMENSION PTR(10, 15), PSR(10, 15), DPR(10, 15), DPR1(10, 15), DPR2(10, 15)
1, DP3R(10, 15), X31(10, 15), X3K1(10, 15), EK3(10, 15), EK1(10, 15), PTR1(10, 15),
210, 15), PSR1(10, 15), EPR(10, 15), FDSR(10, 15), FDMR(10, 15), FDP2(10, 15)
35, FDP3R(10, 15), FENP4(10, 15), DNP4(10, 15), FKINC(10, 15)

WRITE(61, 99)
99 FORMAT(12H FLOW LOSSES/!!!)
1 READ(60, 100) LAST_RUN
IF (LAST_RUN.EQ.1.BS) 2, 10

2 RETURN
10 READ(60, 21) NYM1, NDM, G
WRITE(41, 21) NYM1, NDM, G
READ(60, 20) (YM1(I), I=1, NYM1)
READ(60, 20) (DM(J), J=1, NDM)
WRITE(61, 20) (YM1(I), I=1, NYM1)
WRITE(61, 20) (DM(J), J=1, NDM)

21 FORMAT(124=F10.3)
20 FORMAT(14=F10.3)
WRITE(41, 22)

22 FORMAT(/)
G1 = (G-1.)* 5
G2 = (G+1.)* 5/G
G3 = (G+1.)* 5/(G-1.)*
G4 = (G+1.)/ 4.
DO 30 I=1, NYM1
DO 31 J=1, NDM
A=P*DM(J)/(G*YM1(I)**2*(YM1(I)+2.*DM(J)))
A1=1.*G1*YM1(I)**2
A=YM1(I)**2.*DM(J)**2
C=YM1(I)**2.*G1*(YM1(I)**2+2.*DM(J))*YM1(I))
XK1(I, J) = A-G2*ALOG (B/C)
XK2(I, J) = A-G2*ALOG ((1.+2.*DM(J))/(YM1(I)+G1*YM1(I)**2))
XK3(I, J) = A-G2*ALOG (1.+2.*DM(J))/YM1(I)
XK3(I, J) = (2.*DM(J)/(G1*YM1(I)+DM(J))))*(1./YM1(I)**2-G4**2)
YM=B=YM1(I)+DM(J)**.5
D=YM+DM(J)+G1*YM**3
E=YM-DM(J)+G1*YM**3
XK(I, J) = P*DM(J)/(G*YM**2)-G2*ALOG (D/F)
XKR1(I,J)=2*D(M(J))*((1/GYM)=2-G4)*((1+DOM)*YMR)/(M(J)*G/YMR)
YM2=YM1(I)+DM(J)
A2=1+G1*YM2
XK(I,J)=((G*YM2)*2-YM1(I))/((G*YM1(I)*2-G2*)*YM22)-G2*ALOG((YM22)*A1
Y1/(YM1(I)*2*A2))
AS1=G*YM1(I)*2*A2
AS2=G*YM2*2*A2
XKA(I,J)=1/AS1*/AS2=G2*ALOG((AS2/AS1)
EK1(I,J)=(XK(I,J)-XK1(I,J))/XK(I,J)
EK2(I,J)=(XK(I,J)-XK2(I,J))/XK(I,J)
EK3(I,J)=(XK(I,J)-XK3(I,J))/XK(I,J)
EK4(I,J)=(XK(I,J)-XK4(I,J))/XK(I,J)
EKA(I,J)=(XK(I,J)-XK1(I,J))/XK(I,J)
EK3(I,J)=(XK(I,J)-XK3(I,J))/XK(I,J)
EK4(I,J)=(XK(I,J)-XK4(I,J))/XK(I,J)
EK1(I,J)=(XK(I,J)-XK1(I,J))/XK(I,J)
PTR(I,J)=(A2/A1)*G3*YM1(I)/YM2
PTR1(I,J)=(1*G4*YM22)*YM1(I)/((1+G4*YM1(I))*YM2)
DPR1(I,J)=PTR(I,J)-1
DPSR(I,J)=(A1/A2)*G5*YM1(I)/YM2
DPSR1(I,J)=(1+G1*G5*YM1(I))*YM1(I)/((1+G1*G5*YM22)*YM2)
DPSR2(I,J)=DPR1(I,J)-1
A2*G5)/A1*G5)
F=1/2.1G1*YM1(I)*2/4
F2=1.1*YM22/4
DPR1(I,J)=(1+G4*YM1(I)*YM2)*F2/((1+G1/2.1*YM1(I)*2*YM2+Y
M2*2)/F)
DPR2(I,J)=(1+G4*YM22)*G/(G1+1)
DPR3(I,J)=1+G8*YM32*5
DPR4(I,J)=1+G8*YM22*5+G8*YM22*8
XKINC(I,J)=DPR1(I,J)*DPSR1(I,J)/G*YM1(I)*2
FKINC(I,J)=(XK(I,J)-XKINC1(I,J))/XK(I,J)
FPTR(I,J)=(PTR1(I,J)-PTR1(I,J))/PTR1(I,J)
FPSR(I,J)=(PSR1(I,J)-PSR1(I,J))/PSR1(I,J)
ENDPR1(I,J)=(DPR1(I,J)-DPR1(I,J))/DPR1(I,J)
ENDPR2(I,J)=(DPR1(I,J)-DPR2(I,J))/DPR1(I,J)
ENDPR3(I,J)=(DPR1(I,J)-DPR3(I,J))/DPR1(I,J)
ENDPR4(I,J)=(DPR1(I,J)-DPR4(I,J))/DPR1(I,J)
31 CONTINUE
30 CONTINUE
WRITE (61,22)
DO 40 I=1,NYM1
DO 40 J=1,NDM
40 WRITE(61,53)YM1(I),DM(J),XK(I,J),XK1(I,J),XK2(I,J),XK3(I,J),XK4(I,J),
XK1(I,J),XK1(I,J),XK2(I,J),XK3(I,J),XK4(I,J),XK1(I,J),XK1(I,J),XK2(I,J),
XK3(I,J),XK4(I,J),XK1(I,J),XK1(I,J),XK2(I,J),XK3(I,J),XK4(I,J),XK1(I,
J),XK1(I,J),XK1(I,J),XK1(I,J),XK2(I,J),XK3(I,J),XK4(I,J),XK1(I,J),
WRITE (61,22)
DO 41 I=1,NYM1
DO 41 J=1,NDM
41 WRITE (61,52) YM1(I), DM(J), FK1(I,J), EK2(I,J), EK3(I,J), FKA(I,J), FKB(I,J), FK1(I,J), FKR1(I,J)
WRITE (61,22)
DO 42 I=1,NYM1
DO 42 J=1,NDM
42 WRITE (61,53) YM1(I), DM(J), PTR(I,J), PDR1(I,J), DPR1(I,J), DPR4(I,J), DPR3(I,J), DPR4(I,J)
WRITE (61,22)
DO 43 I=1,NYM1
DO 43 J=1,NDM
43 WRITE (61,53) YM1(I), DM(J), XK1(I,J), XK21(I,J), XKB1(I,J), DPR1(I,J), DPS1(I,J), PSR1(I,J)
WRITE (61,22)
DO 44 I=1,NYM1
DO 44 J=1,NDM
44 WRITE (61,52) YM1(I), DM(J), FPTR(I,J), EPDR1(I,J), EPDR2(I,J), EPDR3(I,J), EPDR4(I,J), EKINC(I,J)
   GO TO 1
END

FORTRAN DIAGNOSTIC RESULTS FOR WIERRA