SEMI-EMPIRICAL APPROACH TO THE DESCRIPTION
OF
GROUND-STATE ROTATIONAL BANDS
OF
DEFORMED EVEN-EVEN NUCLEI

Norman Robert Lewis

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ABSTRACT

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A semi-empirical model is proposed which permits an excellent fit of level energies of ground-state bands in deformed even-even nuclei. In this model, the rotational energy is expressed as a polynomial in $\omega^2$, $\omega^3$, $\omega^4$ ($\omega$ is the nuclear angular velocity). Each nucleus is thus described by three adjustable parameters, $\alpha$, $B$, and $C$, which are determined by a least-squares fit of all the known levels. The calculated energy levels and parameters are tabulated for 88 even-even nuclei. The range of validity of the model is discussed and compared to that of the VMI model. The role of the parameter $B$ is shown to be that of modifying the nuclear softness, $\sigma$. In contrast to the VMI model, it is shown that negative values of $\sigma$ may exist for $N_A > 2.3$, and may be obtained without requiring a negative ground state moment of inertia.
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ABSTRACT

A semi-empirical model is proposed which permits an excellent fit of level energies of ground-state bands in deformed even-even nuclei. In this model, the rotational energy is expressed as a polynomial in \( \omega^2, \omega^3, \omega^4 \) (\( \omega \) is the nuclear angular velocity). Each nucleus is thus described by three adjustable parameters, \( a, b, \) and \( c \), which are determined by a least-squares fit of all the known levels. The calculated energy levels and parameters are tabulated for 88 even-even nuclei. The range of validity of the model is discussed and compared to that of the VMI model. The role of the parameter \( b \) is shown to be that of modifying the nuclear softness, \( C^* \). In contrast to the VMI model, it is shown that negative values of \( C^* \) may exist for \( R_4 > 2.3 \), and may be obtained without requiring a negative ground-state moment of inertia.
1. INTRODUCTION

In recent years the problem of the prediction of the nuclear rotational energy levels has attracted considerable interest. One of the main reasons is that the development of the high-resolution solid-state $\gamma$ detector has made it possible to follow the rotational bands up to as high as spin 18 (Stephens, Lark, Diamond). The new results showed that the energies of the high-spin rotational states deviate from the well-known rule of a rigid rotor, namely

$$E_{\text{rot}} = \frac{\hbar^2(I+1)}{2J}$$

(1.1)

where $J$ is the effective moment of inertia and is independent of $I$, the nuclear spin. The trend, as revealed by the new measurements, is that $J$ increases with the spin $I$ of the level.

A number of efforts have since been directed at understanding the deviations from the rigid rotor formula. There are two general approaches. One involves a microscopic calculation usually based on the second-order cranking model first suggested by Inglis. Corrections arise from taking into account the centrifugal stretching and the Coriolis-antipairing (CAP) effect first suggested by Mottelson and Valatin (Udagawa and Sheline; Bes, Landowne, and Mariscotti; Krumlinde; and Marshalek). Others have tried to fit the energy levels by two-parameter formulas of various forms. Most of them can be derived either from the fourth-order cranking model (Harris) or from the centrifugal stretching model, namely,
\[ E_{\text{rot}} = \frac{1}{2} C_t (t - t_o)^2 + \frac{I(I+1)}{2J(t)} \]

\[ \frac{dE}{dt} = 0, \]  

(1.2)

with an assumed specific relation between the moment of inertia and the general stretching variable \( t \). Other two- or three-parameter formulations \(^{15,16}\) are based on a Taylor series expansion, however, the convergence of these series is questionable.

In the work presented in this thesis we have followed a semi-empirical approach in which we express \( E_{\text{rot}} \) as

\[ E_{\text{rot}} = \beta \omega^2 + \gamma \omega^3 + \delta \omega^4 \],

(1.3)

where \( \omega \) is the nuclear angular velocity. The expectation value of the angular momentum, \( \langle \psi \mid J \mid \psi \rangle \), is then obtained using Feynman's theorem \(^{17}\) and \( \omega \) is eliminated from the equations for \( E_{\text{rot}} \) and \( \langle \psi \mid J \mid \psi \rangle \), giving a three-parameter expression for the rotational energy. We employed a least-squares fitting procedure to determine the best values of the parameters for each nucleus.

In Chapter 2 the nuclear theory related to the rotation problem is discussed. The various efforts that have been made in understanding the deviations from the rigid rotor formula (1.1) are reviewed in Chapter 3. In Chapter 4 we propose a three-parameter semi-empirical model for the description of ground-state rotational bands and compare it to other models. It is found that our model gives excellent results for the level energies, and is characterized by a parameter which serves to modify the nuclear softness. Thus
negative nuclear softness may be obtained without requiring a
negative ground state moment of inertia.
2. COLLECTIVE MODEL OF THE NUCLEUS

2.1 Introduction

The shell model of the nucleus considers the motions of individual nucleons in an isotropic average nuclear field generated by all the other nucleons. In many nuclei the existing combinations of nuclear orbitals give rise to an overall spherical distribution of nuclear matter, but in others this may not be so. In the latter case the average field will not be given by an isotropic average nuclear potential. In addition, the co-operative motion of many nucleons may result in collective oscillations of the nucleus as a whole about some equilibrium shape, which will play an essential role in the low energy spectra of the nucleus.

The shell model, in its simple form, ignores the important nucleon interactions which are not contained in the average field. If the nucleus had the structure implied by the concept of a liquid drop, it would always have its lowest energy for a spherical shape. But the individual nucleons have the characteristic properties given by the shell model which implies a systematic tendency for distortion of the nuclear shape. The basic mechanism for this lies in the effect first pointed out by Rainwater. A single nucleon moving within the nucleus exerts a centrifugal pressure against the walls of the nucleus in its orbital plane and tends to produce an oblate deformation of the nuclear surface. When the nucleus has a closed shell configuration the deforming effects of many nucleons cancel out because the orbitals are oriented equally in all directions.
Where there are particles not in filled shells the tendency is for the nucleus to adjust its surface to coincide with the density distribution of these particles. If there were no opposing forces, this centrifugal pressure would result in a nucleus with a space distribution equivalent to that of the nucleons in the unfilled shells. There are, however, two effects working in the opposite direction. One is the difficulty in polarizing the closed shell core which strongly prefers spherical symmetry. The other is the pairing forces of the extra-core nucleons. When a nucleus has only a few nucleons beyond a closed shell these effects over-balance the distorting effects of the last odd nucleon and the nuclear equilibrium shape remains spherical. However, the nucleus does become less resistant to shape changes. This softness is evident in the decrease in the energy involved in collective vibrations about the spherical equilibrium shape.

When sufficiently many particles are added outside closed shells the spherical shape becomes unstable and the nucleus assumes a spheroidal equilibrium shape. When this occurs the collective motions of the nucleus will be of two types: rotational and vibrational.

It is possible to approximate these effects by replacing the spherically symmetric binding potential of the simple shell model with an adjustable anisotropic binding potential. There remain, however, some significant residual interactions which arise from the pairing forces between the nucleons outside the closed shells.
These tend to couple two equivalent nucleons to a state of zero angular momentum and thus counteract the tendency of the individual nucleons to deform the nuclear shape.

The most important collective degrees of freedom for the low energy nuclear properties are expected to be those associated with oscillations in shape with approximate preservation of the nuclear volume. The nuclear shape can be expressed in spherical harmonics as follows:

\[ P_\text{c}(\theta, \phi) = R_\infty \left\{ 1 + \sum_{\lambda \mu} \alpha_{\lambda \mu} Y_{\lambda \mu}(\theta, \phi) \right\} \]  

(2.1)

where \( R_\infty \) is the equilibrium radius and \( Y_{\lambda \mu} \) is the normalized spherical harmonic of order \( \lambda \mu \). If we make the assumption that the constants \( \alpha_{\lambda \mu} \) are small and that the frequencies of the single particle excitations are much greater than those involved in collective motions, we can separate the total wavefunction into a part describing the particle motion and a part describing the collective motion. An approximate expression for the Hamiltonian specifying the collective motion is of the form

\[ H_{\text{coll}} = V_n(\alpha) + T_n(\alpha) \]  

(2.2)

Here \( V_n(\alpha) \) refers to the potential energy of the nucleus as a function of the shape defined by the coefficients \( \alpha \). The subscript, \( n \), refers to the group of quantum numbers specifying the motions of all the particles in a nucleus with a shape defined by the \( \alpha \). The second term gives the kinetic energy involved in small oscillations of the nuclear shape. The predominant term is quadratic in the \( \alpha_{\lambda \mu} \) and the normal modes of vibration are of the harmonic oscillator type. In general, the oscillations in shape of the lowest order, \( \lambda = 2 \), are of
primary importance.

The variation of the $V_n(\infty)$ with nuclear shape is determined by the number of particles outside the closed shell, the particular orbits which they fill, and their residual interactions. For a nucleus at a closed shell the interparticle forces of the core nucleons result in a strong preference for spherical symmetry and shape changes are firmly resisted. When a few additional particles are added there is, as we have seen, a competition between nuclear polarization and pairing forces in increasing and decreasing nuclear stiffness towards shape changes. As more and more particles are added beyond the closed-shell configuration the coherent effects of many particles ultimately bring about a stabilized deformation of the nucleus in which a potential energy minimum exists for a non-spherical shape. A stabilized non-spherical shape can be considered to be achieved when the shape changes associated with zero-point vibrations are small compared to the equilibrium deformation.

For non-spherical nuclei, the collective excitations include not only vibrational oscillations but changes in orientation without change in shape — that is, rotational excitation.

2.2 $\beta$- and $\gamma$- Vibrations

In the idealized case of a nucleus with constant density and a sharp surface, the nuclear surface would be defined by the $\alpha_\mu$ in the equation
\[ R(\theta, \phi) = R_0 \left\{ 1 + \sum_{\lambda \mu} \alpha_{\lambda \mu} Y_{\lambda \mu}(\theta, \phi) \right\} \] (2.3)

Here \( \mu \) is the projection of \( \lambda \) on a space fixed axis. For small amplitudes of oscillations the energy may be expanded in powers of \( \alpha_{\lambda \mu} \) and \( \beta_{\lambda \mu} \) and one obtains, to a first approximation,

\[ H_{\text{coll}} = \sum_{\lambda \mu} \left( \frac{1}{2} B_{\lambda \mu} |\alpha_{\lambda \mu}|^2 + \frac{1}{2} C_{\lambda \mu} |\beta_{\lambda \mu}|^2 \right) \] (2.4)

corresponding to a set of independent harmonic oscillators with energy quanta

\[ \hbar \omega_{\lambda \mu} = \hbar \sqrt{\frac{C_{\lambda \mu}}{B_{\lambda \mu}}} \] (2.5)

The \( B_{\lambda \mu} \) represents the mass transport associated with the vibration. A theoretical estimate based on the surface oscillations of an irrotational and incompressible liquid drop would give

\[ (B)_{\text{irrot}} = \frac{1}{\lambda} \frac{2}{\hbar} \frac{1}{A} R^2 \] (2.6)

where \( \frac{1}{A} \) is the mass of the nucleus. The parameter \( C_{\lambda \mu} \) represents an effective surface tension.

The shape oscillations may be classified according to their multipole order \( \lambda \). The lowest frequencies are expected to be of the quadrupole type (\( \lambda = 2 \)) since a surface deformation with \( \lambda = 1 \) corresponds to a simple translational movement.

Let us now consider in more detail the possible types of surface vibration (of the quadrupole type). Choosing the axes of our co-ordinate system to coincide with the principal axes of an ellipsoidal nucleus simplifies the coefficients \( \alpha_{\lambda \mu} \) as follows:

\[ \alpha_{21} = \alpha_{2 -1} = 0 \]
\[ \alpha_{22} = \alpha_{2 -2} \]
Also, 
\[ \alpha_{20} = \beta \cos \gamma \]
and 
\[ \alpha_{22} = \frac{1}{\sqrt{2}} \beta \sin \gamma \]
where \( \beta \) is a measure of the total deformation from a sphere and describes the deviation from rotational symmetry about the principal axis of the ellipsoid. It can be shown that the expansion of (2.3) under these conditions gives 
\[ R = R_0 \left\{ 1 + \beta \cos \gamma \sqrt{\frac{2}{J_{68}}} \left( 3 \cos^2 \theta - 1 \right) + \frac{1}{\sqrt{2}} \beta \sin \gamma \sqrt{\frac{15}{32J_{88}}} \sin^2 \theta \cos 2\psi \right\} \]  
(2.7)

If \( \gamma = 0^\circ \) the last term drops out. In this case, if \( \beta > 0 \), the nuclear shape is a prolate spheroid; if \( \beta < 0 \), it is an oblate spheroid. The term \( \beta \) - vibration refers to an oscillation in the shape parameter \( \beta \). If \( \gamma \neq 0^\circ \) the circular cross section of the nucleus perpendicular to the main axis is changed into an ellipse. The term \( \gamma \) - vibration may refer to oscillations around a spherical equilibrium shape, about a spheroidal equilibrium shape, or about an ellipsoidal equilibrium shape. The common feature of these three types is that the nuclear shape is ellipsoidal at the extremes of the oscillatory motion.

2.3 Rotational Energies and the Moment of Inertia

At a considerable distance from closed shells the nucleus becomes stabilized in a non-spherical shape under the influence of the coherent effects of many particles in unfilled shells. It should be noted that it is necessary to be far removed from closed shells
Figure 1. Shape of the nucleus given by Equation (2.7) for deformation with $\beta$ positive and $\gamma = 0$. 
for both neutrons and protons if conditions are to be favourable for stabilization of the nucleus in a spheroidal shape. Such a location is found among the heavy elements above Pb$^{208}$ and among the heavy rare earths between the mass numbers 150 to 195. In these regions the nucleus acquires a prolate spheroidal shape.

For the strongly deformed nucleus the frequencies of motion which have to do with vibrational, and particularly with rotational, excitation are lower in general than those related to intrinsic particle excitation. In this case one can obtain an approximate separation of the motion of the individual particles in the potential field defined by the equilibrium shape of the core and the relatively slow collective rotation and vibration of the entire system. In other words, the complete wavefunction of the nucleus may be put as

$$\psi = \chi_{\text{part}} \psi_{\text{vib}} D_{\text{rot}}$$ \hspace{1cm} (2.8)

Here $\chi_{\text{part}}$ represents the intrinsic motion of the nucleons which can be expressed in terms of the independent motion of the individual particles in the deformed field, $\psi_{\text{vib}}$ describes the vibrations of the nucleus around the equilibrium shape, and $D_{\text{rot}}$ describes the collective rotational motion of the system as a whole.

The coupling scheme for deformed nuclei is illustrated in Figure 2. The three important constants of motion are $\bar{I}$, $K$, and $M$ where $\bar{I} (\equiv \hbar I(I + 1))$ is the total angular momentum of the nucleus with component along the space axis $\hbar$ and component along the symmetry axis $K \hbar$. Each individual particle of angular momentum $\bar{J}$ has a
Figure 2. Coupling scheme appropriate for deformed nuclei. $\bar{I}$ is the total angular momentum of the nucleus. $K$ and $M$ are the projections of $\bar{I}$ on the axis of symmetry $Z'$ and a space fixed axis $Z$, respectively. $\bar{R}$ is the collective rotational angular momentum. For the ground state and for many low-lying states $K$ is equal to $\bar{L}$ where $\bar{L}$ is defined as the projection of the total angular momentum of the intrinsic particle motion on the symmetry axis.
projection on the symmetry axis \( \mathcal{N}_z \). Since \( \sum_i \mathcal{N}_z = \mathcal{J} \) then
\[
\sum_i \mathcal{N}_z = \mathcal{N}.
\]
The angular momentum of the collective motion is \( \mathcal{R} \). We thus obtain
\[
|\mathcal{R}|^2 = \left[ I(I + 1) - K^2 \right] \hbar^2.
\] (2.9)

For the ground state and for low-lying excited states for which there is no collective rotation about \( Z' \), \( K = \mathcal{N} \). The total angular momentum \( \sum_i \mathcal{J}_z \) of the particle system is not, in general, a constant of motion. The value of \( \mathcal{N} \) for a single particle, \( \mathcal{N}_p \), takes on half-integral values, positive or negative. States differing only in the sign of \( \mathcal{N}_p \) are degenerate since they are identical except for the opposite sense of rotation. The particles fill pairwise into states of opposite \( \mathcal{N}_p \) with no net contribution to \( K \) from the pairs. Thus for an even-even nucleus in its lowest state, \( K = 0 \). \( K \) will be different from \( \mathcal{N} \) for \( \gamma \) - vibrations in which collective angular momentum is contributed along the nuclear axis.

The deformed nucleus may rotate with preservation of shape and internal structure. Since \( \mathcal{R} \) is always perpendicular to the symmetry axis, all members of a rotational band have the same value of \( K \).

Provided we can separate the Hamiltonian as above we can express the nuclear rotational energy in a form similar to that of a free rotor, \( \text{i.e.} \)
\[
E_{\text{rot}} = \frac{1}{2} \mathcal{I} \omega^2.
\] (2.10)

where \( \mathcal{I} \) is the effective moment of inertia about the axis of rotation perpendicular to the nuclear symmetry axis and \( \omega \) is
the nuclear angular velocity. Also, since $\overline{\mathbf{H}} = \mathbf{J} \omega$, we have, using (2.9),

$$E_{\text{rot}} = \frac{\hbar^2}{2J} \left[ I(I + 1) - k^2 \right]. \quad (2.11)$$

Using this formula we see that the spacings should be

$$R_4 = E_{4+}/E_{2+} = 10/3 \quad R_6 = E_{6+}/E_{2+} = 21/3, \text{ etc.}$$

($\hbar^2 k^2/2J$ does not affect the level spacings.) The possible quantum states of the nucleus are restricted by the reflection symmetry of the deformation, which implies that the states labeled by $(k, \Omega)$ must be combined in a definite way with those labeled by $(-k, -\Omega)$. Also, the reflection symmetry about the plane passing through the center of the nucleus and perpendicular to the axis of symmetry implies that the collective motion has even parity and that the parity of a nuclear state is thus determined by that of the particle structure. This symmetry condition limits the acceptable states in the ground state rotational band of even-even nuclei to $I = 0, 2, 4, \ldots$ even parity.

The collective motion which gives rise to the nuclear rotation is essentially different from that of a rigid body, and the effective moment of inertia, $J$, is somewhat less than the rigid body moment, $J_{\text{rigid}}$, given by

$$J_{\text{rigid}} = \frac{2}{5} \frac{N \Lambda R_0^2}{I} \left( I + 0.31 + 0.44/3 + \ldots \right). \quad (212)$$

In the early development of the collective model, estimates of were based on a hydrodynamical model of nuclear motion (to be discussed later). For the irrotational flow of an incompressible fluid, this model estimates the moment of inertia by the following equation:
\[ J_{\text{irrot}} = \frac{2}{5} A N R_0 \beta^2 \]  (2.13)

where \( \beta \) is a deformation parameter of a spheroid given by

\[ \beta^2 = \frac{4}{3} \frac{\Delta R}{R_0} \left( \frac{\Delta R}{R_0} \right)^{1.06} \]  (2.14)

Here, \( R_0 \) is the mean nuclear radius and \( \Delta R \) is the difference between the major and minor semi-axis of the spheroid. However, the values of \( J \) found experimentally are usually two or three times higher than \( J_{\text{irrot}} \) and much closer than predicted to \( J_{\text{rigid}} \). Hence, experimentally one observes that \( 3 J_{\text{irrot}} < J < \frac{J_{\text{rigid}}}{2} \).
3. PHENOMENOLOGICAL MODELS OF NUCLEAR ROTATION

The rotational formulas given in Chapter 2, based on the concept of a free rotor, can give precisely correct values for the spacing of rotational levels only if there is complete separation of rotational from vibrational and intrinsic motions. That this is not always the case is evident from the observed deviations in the spacings of the rotational levels from the $I(I+1)$ law, which will predict a value of $10/3$ for $R_4$ (the ratio of $E_4$ to $E_2$). It is observed that in many cases $R_4$ is less than $10/3$. This leads to the following classification of $R_4$ regions:

I. Rotation Region: $3 < R_4 < 3.33$
II. Transition Region: $2.4 < R_4 < 3$
III. Vibration Region: $2 < R_4 < 2.4$.

It should be noted that $R_4 = 3.33$ corresponds to the ideal rigid rotor, while $R_4 = 2$ corresponds to the ideal quadrupole vibrator.

3.1 The Hydrodynamical Model

Many models have been introduced to describe this departure from the $I(I+1)$ dependence. One of the first was the hydrodynamical model of Bohr and Mottelson. They added an extra term proportional to $I^2(I+1)^2$ to the expression (2.11) in order to take into account rotation-vibration coupling. Thus we have (K being zero for the ground state of even-even nuclei)

$$E_{\text{rot}} = A I(I+1) + B I^2(I+1)^2$$  \hspace{1cm} (3.1)

where $A = \hbar^2/2 J$ and $B$ is the rotation-vibration coupling constant.
similar to that found in molecular spectroscopy. The parameters, A and B, are determined from the experimental positions of the first two excited levels 2+ and 4+ in the ground state band. It has been generally believed that the experimental energy levels can be accurately reproduced by equation (3.1), although the second term represents only a first-order correction and higher-order terms should also be considered.

In the hydrodynamical model the parameter B may be related to the angular velocities of the $\beta$- and $\gamma$- vibrations through the relation

$$B_{\text{theor}} = \frac{k A^2}{(h \omega_\beta)^2} + \frac{12 A^3}{(h \omega_\beta)^2}$$  \hspace{1cm} (3.2)

where $\omega_\beta$ and $\omega_\gamma$ are the angular velocities of the $\gamma$- and $\beta$- vibrations, respectively. However, it was found that the empirical values of B obtained from curve fitting were different by an order of magnitude from those calculated from (3.2). It was soon found necessary to add the next order correction term, proportional to $I^3(I + 1)^3$, to obtain a more satisfactory description of relatively high spin states. Nathan and Nilsson have included even the terms of higher orders thus giving an infinite series of the form

$$E_{\text{rot}} = A I(I + 1) - B I^2(I + 1)^2 + C I^3(I + 1)^3 - \ldots$$  \hspace{1cm} (3.3)

Discussions of possible approaches to this are given by Gupta and Sood.
3.2 The Asymmetric Rotor Model

Since it was found that the power series expansion (3.3) in $I(I + 1)$ was inadequate, several approaches have been made based on a more precise treatment of rotation-vibration coupling. Up to this point it was assumed that the shape of the nucleus in its ground state possesses axial symmetry. Davydov and his coworkers 26-28 have considered the consequences of dropping this assumption. First, Davydov and Filippov 26 considered the rotational states of a nucleus with an ellipsoidal shape under the adiabatic approximation that there was no interchange of energy with the intrinsic or vibrational states during rotation. The Hamiltonian for such a rotation is

$$H = \sum_{\lambda=1}^{3} \frac{\Lambda}{2} \left( \frac{\mathbf{J}^2}{\mathbf{J}_0^2} \right)$$

(3.4)

where $\Lambda = \frac{h^2}{4\mu J^2}$ and $B$ is a mass parameter. The $\mathbf{J}_\lambda$ are operators of the projection of the nuclear angular momentum $\mathbf{J}$ on the axes of the coordinate system which are chosen to correspond to the principal axes of the ellipsoid. For $\nu = 0$ or $\nu/3$, (3.4) represents the rotation of an asymmetric top. At $\nu = 0$ we have a $2^+, 4^+, 6^+$ series of states which is identical in all respects to the ground state band discussed in Chapter 2 with the energy spacings following the $I(I + 1)$ rule. As the deviations from axial symmetry increase, the level energies also increase slightly. (For example, the axial symmetry increases from $\gamma = 16^0$ to $\gamma = 25^0$ between $^{186}O$ and $^{192}O$. ) The minimum value for $\nu$ turns out to be $8/3$. It is found that the parameter $\gamma$ is simply related to $E_{2^+}/E_{2^-}$ (Here $E_{2^-}$ is defined as follows). There are found nuclei between the strongly deformed and "magic number" nuclei with a
near-harmonic pattern which is characterized by a second excited state with an energy approximately twice the energy of the first excited state and \( I = 0, 2, \) or 4. The second excited \( 2^+ \) state is usually denoted by \( 2^+ \) and its energy by \( E_2 \). A study of the branching ratios between the \( \gamma \) - vibrational (\( K = 2 \)) and the ground state bands in the framework of the same model yields values of \( \gamma \) ranging from \( 12^0 \) to \( 23^0 \) for the same nuclei, suggesting a small inconsistency in the axially asymmetric model which is not easily removed.

The fact that most of the quantities predicted by the asymmetric rotor model are closely related to \( \gamma \) which is in turn related to \( E_2/E_2 \) has encouraged a number of comparisons of experimental data as a function of this energy ratio. The most significant results are those of Hallmann who showed that for even-even nuclei with widely differing \( N, Z \), and \( E_2 \) values the energy ratios \( E_6/E_2 \) and \( E_8/E_2 \), plotted against \( E_4/E_2 \), lie on two "universal" curves. This finding suggests that the ground-state bands may indicate features which are common to nuclei lying both inside and outside the deformed region.

The asymmetric rotor model has been extended to include the effects of interaction rotational and vibrational modes by Davydov and Chaban. They assume that the nucleus has an equilibrium value of \( \beta \) and of \( \gamma \) about which it can execute vibrations. The problem is to determine the coupling of these vibrations to the rotation as provided by the centrifugal forces. A parameter \( \mu \) is introduced whose value serves as a guide to the extent of this coupling. With the aid of this parameter \( \mu \) and the older parameter \( \gamma \), Davydov and Chaban were
able to make many significant correlations of experimental spectra for a wide range of nuclei.

3.3 Models Based on Centrifugal Stretching

As was pointed out in Chapter 1, new methods for measuring energies showed that the level spacings at higher \( I \) are smaller than those given by the \( I(I + 1) \) rule. This decrease in the energy spacing may be attributed to an increase in the moment of inertia \( I \). At higher \( I \) the moment of inertia appears to approach a "rigid" value. Morinaga\(^{31}\) introduced the parameter "softness", defined as the percentage increase of the moment of inertia per unit change of angular momentum, and discussed the form of dependence on this quantity on \( I \) as a function of \( N \) and \( Z \).

3.3.1 The Semiclassical Model

One explanation for the increase of the moment of inertia as \( I \) increases is that at higher angular momenta the deformation (\( \beta \)) increases\(^{32}\) (\( \beta \) stretching or centrifugal stretching). The semiclassical model of Diamond, Stephens, and Swaitecki\(^{32}\), based on this assumption, leads to an expression for the energy as the sum of a potential energy term and a kinetic (rotational) energy term:

\[
E_I(\beta) = \frac{1}{2} C (\beta_i^2 - \beta_i) + \frac{I(I + 1)}{2} I (\beta_i) \tag{3.5}
\]

where \( I \) is the moment of inertia in terms of \( \hbar^2 \). In addition, the equilibrium condition \( \frac{\partial E_I}{\partial \beta_i} = 0 \) is applied to obtain the value of \( \beta_i \). With this model a good fit may be obtained for bands of
strongly deformed nuclei, assuming the relation given by the hydrodynamical model \( J \sim \beta^2 \). However, bands outside the deformed region cannot be fitted by this method with reasonable accuracy.

In view of this, the deformation parameter \( \beta \) was replaced by a general stretching variable \( t \) in (3.5) and it was assumed that the moment of inertia could be expressed as \( J \approx t^n \); different values for \( n \) were chosen by different authors. Draper et al.\(^\text{13}\) have taken \( n \) as a non-integer parameter to be determined separately for each nucleus. Their results indicate that the best \( n \) values range from 0.7 to 2.8. Diamond et al.\(^\text{32}\), Moszkowski\(^\text{10}\), and Sood\(^\text{11}\) have used a quadratic dependence, \( J \approx t^2 \), the same as that given by the irrotational flow model.

3.3.2 The Variable Moment of Inertia Model

Mariscotti et al.\(^\text{12}\) have assumed a linear dependence \( J \approx t \) as an empirical fact based on the study\(^\text{33}\) of the general relation \( J \approx t^n \). This is equivalent to taking the moment of inertia, \( J \), itself as the general variable \( t \). They then arrive at the variable moment of inertia model (VMI).

The level energy is thus given by

\[
E(I) = \frac{1}{2} C (J - J_o)^2 + \frac{1}{2} \left[I(I + 1)/J_I\right] (3.6)
\]

subject to the equilibrium condition

\[
\frac{\partial E(J)}{\partial J} = 0. \quad (3.7)
\]

The relation (3.7) determines the moment of inertia \( J_I \) (given in
units of $\hbar^2$) for each state with spin I. $\tilde{J}$ is a parameter defined as the "ground state moment of inertia" and C ($>0$) is the "restoring force constant". The model is successful in justifying Halmann's curves, in going beyond the range of validity of the asymmetric rotor model, in predicting the levels of ground-state bands (they present data showing the predictions of the VMI model for the energy levels of 83 even-even nuclei), and in fitting the rotational bands built on $\tilde{Y}$ - vibrational states in even-even nuclei.

For each spin I there exists an equilibrium value of the variable $\tilde{J}$ determined from (3.7). Using both (3.6) and (3.7), a cubic in $\tilde{J}$ is obtained which has one real root for any finite value of $\tilde{J}$ and C.

$$\tilde{J}^3 - \tilde{J}^2 \tilde{J}_0 - \left[ I(I+1) / 2 C \right] = 0.$$  

(3.8)

Equation (3.8) combined with (3.6) yields the following expression for the energy of the state with spin I:

$$E_I = \frac{I(I+1)}{2 \tilde{J}^2} \left( 1 + \frac{I(I+1)}{2 C \tilde{J}^2} \right)$$  

(3.9)

From (3.8) the "softness" of the nucleus turns out to be

$$\sigma = \left[ \frac{1}{J} \frac{d^2 J}{dI^2} \right]_{I=0} = \frac{1}{2C \tilde{J}^2}.$$

(3.10)

The quantity $\sigma$, which may be taken to be a parameter, provides a convenient mode of discussing the range of validity of the VMI model. Defining $r_I = J / \tilde{J}$, we have from (3.8)

$$r_I^3 - r_I^2 = \sigma I(I+1)$$

(3.11)

In the adiabatic limit (rigid rotor), $\sigma = 0$ and hence $r_I = 1$. 

Thus,

$$E_I(\sigma = 0) = I(I + 1)/2\lambda_0$$  \hspace{1cm} (3.12)

In this limit, $R_I = E_{1/2}/E_2$ is

$$R_I(\sigma = 0) = \frac{I(I + 1)}{6}$$  \hspace{1cm} (3.13)

In the opposite extreme in the limit of very soft nuclei $\sigma \to \infty$, and from (3.8) we have

$$r_I = \left\{ \frac{\sigma}{I(I + 1)} \right\}^{1/3}$$  \hspace{1cm} (3.14)

We then have

$$E_I(\sigma \to \infty) = 3/4 \left\{ \frac{I(I + 1)}{\sigma I} \right\}$$  \hspace{1cm} (3.15)

which leads to

$$R_I(\sigma \to \infty) = \left\{ \frac{I(I + 1)}{6} \right\}^{2/3}$$  \hspace{1cm} (3.16)

Hence the range of validity of the VMI model is

$$\left\{ \frac{I(I + 1)}{6} \right\}^{2/3} \leq R_I \leq \frac{I(I + 1)}{6}$$  \hspace{1cm} (3.17)

In the case of $I = 4$ this gives

$$2.23 \leq R_4 \leq 3.33$$  \hspace{1cm} (3.18)

which is greater than that of the Davydov-Filippov model where

$$2.67 < R_4 < 3.33$$  \hspace{1cm} (see Sec 3.2).

G. Scharrf-Goldhaber and A.S. Goldhaber further extended the VMI model toward the magic nuclei by permitting negative values of the parameter $\lambda$. For large negative values of $\lambda$ they obtain $R_4 \to \sqrt{20/6} = 1.82$, which leads them to the following classification.

$$2.23 < R_4 < 3.33 : \text{Deformed Region}$$

$$1.82 < R_4 < 2.23 : \text{Spherical Region}$$

$$1.00 < R_4 < 1.82 : \text{Magic Region}$$

In the deformed region the ground state moment of inertia $\lambda$ is
positive, while in the spherical region it is negative. They introduce the notion of "internal stress" or "rigidity" to interpret the significance of \( \mathcal{J}_0 < 0 \). The larger the negative value of \( \mathcal{J}_0 \), the more firmly the shell structure resists departure from spherical symmetry. The fact that \( \mathcal{J}_0 \) changes sign at \( R_4 = 2.23 \) while \( C \) remains positive, means that \( C \) becomes negative. This is not mentioned at all by Schraff-Goldhaber at all.

Mariscotti, who also discusses the extended VMI model, develops three solutions for the rotational energy. For two of these solutions \( \sigma \) goes from a value of zero at \( R_4 = 3.33 \) to \( +\infty \) at \( R_4 = 2.23 \). Then there is a discontinuity in \( \sigma \) as it jumps to \( -\infty \) and then back to zero at \( R_4 = 1.825 \). The third solution gives the same \( \sigma \) down to \( R_4 = 2.23 \) but for \( R_4 < 2.23 \) it remains small and negative (\( \sim 10^{-4} \)). Mariscotti simply states that for \( R_4 > 2.23 \), \( C/\mathcal{J}_0^3 \) represents the softness, thus avoiding an interpretation of negative \( \sigma \).

3.4 The Cranking Model

The second main approach to explain the increase in angular momentum with spin is the cranking model as developed by Inglis. If the rotational motion is slow compared to the intrinsic motion (adiabatic condition), we can consider the energy required to rotate the system with its internal motion unchanged. Formally we transform the intrinsic Hamiltonian to a rotating co-ordinate system, which introduces a new term, \( \hat{J}_\omega \), if the angular velocity of the system
is \( \omega \) about the x axis and \( \hbar \mathbf{J} \) is the total angular momentum. The change in energy due to this term is the rotational energy and the moment of inertia, \( \mathbf{J} \), is twice the coefficient of \( \omega^2 \). If second-order adiabatic perturbation theory is used one obtains

\[
\mathbf{J} = 2\hbar^2 \sum_{i \neq 0} \frac{\left| \langle \phi | \mathbf{J}_z | \phi \rangle \right|^2}{\varepsilon_i - \varepsilon_0}
\]

(3.19)

where the label \( i \) runs over all the intrinsic states of the system, and \( \varepsilon_i \) is the total energy of the state \( i \).

If, for the intrinsic states, one substitutes into (3.19) shell-model states of individual particles moving in orbits appropriate to a deformed well, one obtains the rigid body moment of inertia.

This led Harris \(^9\) (HPA) to extend the cranking model by including higher order terms in \( \omega \). He begins by considering a rotating, deformed, self-consistent potential well which, in the laboratory system, has a time-dependent Hamiltonian \( \mathbf{H} \) and state function \( \psi \) which is a solution of the equation

\[
\mathbf{H} \psi = i \frac{\partial \psi}{\partial t}
\]

(3.20)

Assuming that the nucleus is rotating about the x axis, (3.20) is transformed to the intrinsic nuclear reference frame in which the wave function is \( \tilde{\psi} \) where

\[
\psi = U(t) \tilde{\psi}
\]

(3.21)

Substituting (3.21) into (3.20) gives a new Schrödinger equation

\[
\tilde{\mathbf{H}} \tilde{\psi} = i \left( \frac{\partial \psi}{\partial t} \right)
\]

(3.22)

where \( \tilde{\mathbf{H}} \) is given by

\[
\tilde{\mathbf{H}} = U^{-1} \left( \mathbf{H} U - i \frac{\partial U}{\partial t} \right)
\]

(3.23)

A stationary solution for (3.21) for which one can write
\[ \hat{\mathcal{H}} \varphi = \tilde{\mathcal{E}} \varphi \]  

(3.24)

Since \( \mathcal{U} \) is given by

\[ \mathcal{U} = \exp \left( -i \hat{J}_x \omega t \right) \]  

(3.25)

equation (3.23) may be written as

\[ \hat{\mathcal{H}} = \exp \left( -i \hat{J}_x \omega t \right) \mathcal{H} \exp \left( -i \hat{J}_x \omega t \right) - \omega \hat{J}_x \]

(3.26)

The energy eigenvalues in the two systems are also easily related;

\[ E = \langle \psi | \hat{\mathcal{H}} | \psi \rangle \]

\[ = \tilde{E} + \omega \langle \psi | \hat{J}_x | \psi \rangle \]  

(3.27)

Equation (3.24) is then solved using (3.26) with basis states taken as stationary states of \( \mathcal{H}_o \), and with \( \mathcal{H} \) treated as a perturbation. The usual cranking model (Inglis) results from second-order perturbation theory. Harris includes terms up to fourth order in \( \mathcal{H} \). The result is

\[ E = E_o - \omega^2 \sum_{k \neq 0} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{E_o - E_k} 
- 3 \omega^4 \sum_{k \neq 0} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} 
+ 3 \omega^4 \sum_{k \neq 0} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} \]  

(3.28)

All terms containing \( \omega \) contribute to the rotational energy. Equation (3.28) may be written as

\[ E = E_o + \frac{1}{2} J(\omega) \omega^2 \]  

(3.29)

where

\[ J(\omega) = J_o + 3 \omega^2 \]  

(3.30)

and

\[ J_o = 2 \sum_{k \neq 0} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{E_o - E_k} \]  

(3.31)

with

\[ C = 2 \sum_{k \neq 0} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} \frac{\langle \psi | \hat{J}_k | \psi \rangle}{(E_o - E_k)(E_o - E_k)} \]  

(3.34)
Also one obtains

\[ \langle \Psi | J_x | \Psi \rangle = \omega \left( J_o + 2 C \omega^2 \right) \] (3.33)

It is evident that a different effective moment of inertia enters into
the calculation of energy and angular momentum when higher-order terms
are retained in \( J(\omega) \). The usual cranking model treatment leads to
(3.30), but here one has a result for \( J \) which depends on the degree of
rotation. This is similar to the theory of Dewydo and Chaban although
no interaction between rotational and vibrational modes is explicitly
introduced.

Harris proceeds to obtain the main results of the above
without resorting to perturbation theory, by employing a "self-
consistency" argument via Feynman's theorem \(^{17}\) (see Appendix A). The
results are

\[ E_{\text{rot}} = \frac{1}{2} \omega^2 \left( J_o + 3 C \omega^2 + 5 D \omega^4 + \ldots \right) \] (3.34)

\[ \left\{ I(I + 1) \right\}^{1/2} = \omega \left( J_o + 2 C \omega^2 + 3 D \omega^4 + \ldots \right) \] (3.35)

where \( J_o, C, D, \ldots \) are adjustable parameters. In principle, \( \omega \) may
be eliminated from those two equations leaving one equation for \( E_{\text{rot}} \)
as a function of nuclear spin I. Harris presents energies for even-even
nuclei obtained from empirical curve fittings to the experimentally
observed levels in which he has retained two and three parameters,
respectively, in the equations (3.34, 3.35).

It may be shown that the Harris model and the VNI model are
equivalent. The two-parameter Harris model reduces to

\[ \left\{ I(I + 1) \right\}^{1/2} = \omega \left( J_o' + 2 C' \omega^2 \right) \] (3.36)
\[ E'_{I} = \frac{1}{2} \omega^2 \left( J'_o + 2 c' \omega^2 \right). \] (3.37)

If the moment of inertia \( J_I \) is defined as
\[ J_I = \left( \frac{I(I+1)}{\omega} \right)^{1/2} \] (3.38)
one obtains from (3.36)
\[ J_I = J'_o + 2 c' \omega^2 \]
\[ = J'_o + 2 c' \frac{I(I+1)}{J_I^2} \] (3.39)

or, equivalently,
\[ J_I^3 - J_I J'_o - 2 c' I(I+1) = 0 \] (3.40)

which is identical to (3.8) if
\[ c' = \frac{1}{4} c \quad \text{and} \quad J'_o = J_o. \] (3.41)

Using (3.39, 3.41), one may write (3.37) as
\[ E'_{I} = \frac{1}{2} \omega^2 \left( J_I + c' \omega^2 \right). \]

Substituting from (3.38) one obtains
\[ E'_{I} = \frac{I(I+1)}{2 J_I} \left\{ 1 + c' \frac{I(I+1)}{J_I^3} \right\} \] (3.42)

Using (3.41) one finds that (3.42) is identical to (3.9).

3.4 Models Employing Taylor Series Expansions

Gupta\(^{15}\) has developed a nuclear-softness (NS) model in which he treats the moment of inertia \( J_I \) as a function of spin \( I \) and expands \( J_I \) about its ground state value \( J_o \) for \( I = 0 \). This leads to
\[ E_I = \frac{I^2}{2 \partial I} \text{I}(I+1) \]
\[ = \frac{I^2}{2} \left[ \frac{1}{\partial E} \frac{\partial \text{I}}{\partial E} \right] \text{I} + \left( \frac{2}{\partial I} \frac{\partial \text{I}}{\partial I} \right) \text{I}^2 - \frac{4}{\partial I} \frac{\partial \text{I}}{\partial I} \text{I}^2 \left( \text{I} + \text{I} \right) \]
\[ \times \text{I}(I+1) \]

Collecting the various terms in this expansion in terms of Morinaga's softness parameter \( 31 \), he obtains
\[ E_I = \frac{I^2}{2 \partial I} \text{I}(I+1) \left( 1 - \frac{\partial \text{I}}{\partial \text{I}} \right) \left( \text{I} + \frac{\partial \text{I}}{\partial \text{I}} \right) \left( \text{I} + \frac{\partial \text{I}}{\partial \text{I}} \right) \left( \text{I} + \frac{\partial \text{I}}{\partial \text{I}} \right) \cdots \] (3.44)

where
\[ \sigma_1 = \frac{1}{\partial E} \frac{\partial \text{E}}{\partial \text{E}} \sigma_2 = \frac{1}{\partial I} \frac{\partial \text{I}}{\partial \text{I}} \sigma_3 = \frac{1}{\partial I} \frac{\partial \text{I}}{\partial \text{I}} \sigma_4 = \frac{1}{\partial I} \frac{\partial \text{I}}{\partial \text{I}} \cdots \] (3.45)

Treating \( \sigma_1, \sigma_2, \sigma_3, \cdots \) as adjustable parameters allows him to obtain a least-squares fit to the experimental data. Results for two- and three-parameter fits are given for 102 even-even nuclei. His results for two parameters are comparable to those obtained by VMK.

Gupta did not discuss the convergence of his series nor the relative magnitudes of his terms.

Satpathy and Satpathy \( 16 \) have formulated a shape fluctuation (SF) model in which they write
\[ E_I = \text{B}(\psi(I)) + \text{B}(\psi(I) \text{I}(I+1)) \quad (3.46) \]

They then expand \( \psi(I) \) in a Taylor series about \( I = 0 \) and further expand \( E \) and \( B \) about \( \psi(I) = 0 \) to obtain
\[ E_I = \text{B}(\psi(I = 0)) + \text{B}(\psi(I = 0) \text{I}(I+1)) \quad \text{...} \]
\[ + \left\{ \text{B}(\psi(I = 0)) + \text{B}(\psi(I = 0) \text{I}(I+1)) \right\} \text{I}(I+1) \]
\[ \approx E_0 + I \psi_0 + (B_0 + I \psi_0) \text{I}(I+1) \]
\[
E = E_0 + B I(I+1) + I V'E' + I V'B' I(I+1)
= aI + bI^2 + cI^3
\]

(3.47)

where the three adjustable parameters \(a\), \(b\), and \(c\) are given by

\[
\begin{align*}
    a &= B_0 + V'E' \\
    b &= B_0 + V'B' \\
    c &= V'B'
\end{align*}
\]

(3.48)

The three-parameter fit they obtain is comparable to Gupta's three-parameter NS model. Satpathy et al. also do not discuss the convergence of their series or the relative magnitude of various terms.

### 3.5 Expansions in Powers of Nuclear Spin

As was indicated in Sec. 3.1, for the region of strongly deformed nuclei there is sanction for writing the rotational energy as a power series in \(I(I+1)\),

\[
E_I = I(I+1)/2I - B I^2(I+1)^2 + \ldots
\]

(3.49)

For the most strongly deformed nuclei we have \(2IB \sim 10^{-3}\). Therefore for \(I \ll 10\), the second term is only 10% of the first. By contrast, we find that at the end of the deformed region \(2IB \sim 10^{-2}\). Thus mathematical convergence fails at \(I \sim 10\). To deal with this problem Das, Dreizler, and Klein have proposed an anharmonic vibration model (AVM) in which they expand the excitation energy as a polynomial in \(I\) rather than in \(I(I+1)\). Their energy can be written as

\[
E_I = aI + kI(I+1)
\]

(3.50)

where \(a\) and \(k\) are adjustable parameters. Their results are better than the VMI model in the vibration region but not as good in the rotation region.
In the vibrational region and in the beginning of the transition region, in the model of Das et al.\textsuperscript{37}, the first term dominates, while the second and third are in competition. In the rotation region and the end of the transition region their first two terms are in competition but their third term is definitely smaller. They claim that their analysis of the HPA and the VMI model yields the facts that the power series in $\omega$ is dominated by one term in the rotation region but that the first two terms compete in the vibration region. They conclude that to obtain a precision fit in the vibration region requires a formulation with more than two parameters. It is interesting to note that Ejiri\textsuperscript{39} arrived at (3,50) on purely empirical grounds.

Marshalek\textsuperscript{40} has developed a microscopic cranking model for vibrational nuclei. He claims, as a result of a microscopic calculation, that, for spherical nuclei (vibrational region), an expansion in $I$ rather than in $I(I + 1)$ is more likely to converge. Truncation of his expression to accommodate two parameters also yields (3,50).
4. THREE PARAMETER SEMI-EMPIRICAL APPROACH

4.1 General Solution and Parameters

We choose to write $E$ in the laboratory frame as a polynomial in $\omega$ up to $\omega^4$:

$$E = E_0 + \alpha \omega + \beta \omega^2 + \gamma \omega^3 + \delta \omega^4 \quad (\omega > 0) \quad (4.1)$$

where

$$\alpha = \left( \frac{\partial E}{\partial \omega} \right)_{\omega = 0^+}$$
$$\beta = \frac{1}{2} \left( \frac{\partial^2 E}{\partial \omega^2} \right)_{\omega = 0^+}$$
$$\gamma = \frac{1}{3!} \left( \frac{\partial^3 E}{\partial \omega^3} \right)_{\omega = 0^+}$$
$$\delta = \frac{1}{4!} \left( \frac{\partial^4 E}{\partial \omega^4} \right)_{\omega = 0^+} \quad (4.2)$$

For time-reversal invariance, we impose the condition on $E$ that

$$E(\omega) = E(-\omega) \quad (4.3)$$

which implies that

$$\left( \frac{\partial^n E}{\partial \omega^n} \right)_{\omega = 0^+} = (-1)^n \left( \frac{\partial^n E}{\partial \omega^n} \right)_{\omega = 0^-} \quad (4.4)$$

Also, analyticity of $E$ at $\omega = 0$ requires that

$$\left( \frac{\partial E}{\partial \omega} \right)_{\omega = 0^+} = \left( \frac{\partial E}{\partial \omega} \right)_{\omega = 0^-} \quad (4.5)$$

Relation (4.5) and (4.4) require that $\alpha = 0$ in (4.1).

For a description of the nucleus, consider a rotating, deformed, self-consistent potential well. In the laboratory system one has a time-dependent Hamiltonian $H$ and a state-function $\psi$ such that

$$H \psi = i \frac{\partial \psi}{\partial t} \quad (4.6)$$
We can transform (4.6) to the intrinsic nuclear reference frame. The wave function in the latter system is \( \psi \) where
\[
\psi = U(t) \phi.
\] (4.7)

Substituting (4.7) into (4.6), one obtains a new Schrödinger equation
\[
\tilde{H} \psi = i \frac{\partial \psi}{\partial t}.
\] (4.8)

where
\[
\tilde{H} = U^{-1} \left( H U - i \frac{\partial U}{\partial t} \right).
\] (4.9)

We now take a stationary solution of (4.8), i.e., one for which
\[
\tilde{H} \psi = \tilde{E} \psi.
\] (4.10)

Noting that \( U \) is given by
\[
U = \exp(-i J_x \omega t)
\] (4.11)

(assuming that the nucleus is rotating about the x axis) the equation (4.9) may be written as
\[
\tilde{H} = \exp(i J_x \omega t) H \exp(-i J_x \omega t) - \omega J_x
\] = \( H_0 - \omega J_x \)
\[
= H_0 - H' \] (4.12)

The energy eigenvalues are related by
\[
E = \langle \psi | H | \psi \rangle
\] = \( \tilde{E} + \omega \langle \psi | J_x | \psi \rangle \). (4.13)

We now express the expectation value of the angular momentum as the polynomial
\[
\langle \psi | J_x | \psi \rangle = \beta' \omega + \gamma' \omega^2 + \delta' \omega^3.
\] (4.14)

Thus (4.13) and (4.14) give
\[
E = \tilde{E} + \omega^2 (\beta' + \gamma' \omega + \delta' \omega^2)
\] (4.15)

Therefore
\[
\frac{\partial E}{\partial \omega} = \frac{\partial E}{\partial \omega} + 2 \beta \omega + 3 \gamma \omega^2 + 4 \delta \omega^3. \tag{4.16}
\]

Applying now Feynman's theorem (see proof in Appendix A) which states that for a stationary solution of
\[
\tilde{H}(\omega) \psi(\omega) = \tilde{E}(\omega) \psi(\omega) \tag{4.17}
\]

one has
\[
\frac{\partial \tilde{E}}{\partial \omega} = \left\langle \psi \left| \frac{\partial}{\partial \omega} \frac{\partial \tilde{H}}{\partial \omega} \right| \psi \right\rangle, \tag{4.18}
\]

we obtain
\[
\frac{\partial \tilde{E}}{\partial \omega} = - \left\langle \psi \left| J_x \right| \psi \right\rangle. \tag{4.19}
\]

From (4.14), (4.16), and (4.19) we have
\[
\frac{\partial \tilde{E}}{\partial \omega} = \beta \omega + 2 \gamma \omega^2 + 3 \delta \omega^3. \tag{4.20}
\]

But (4.1) gives
\[
\frac{\partial E}{\partial \omega} = \alpha + 2 \beta \omega + 3 \gamma \omega^2 + 4 \delta \omega^3. \tag{4.21}
\]

Equating coefficients of like terms in (4.20) and (4.21) gives
\[
\alpha = 0 \quad \text{(as required)}
\]
\[
\beta = \beta' / 2 \tag{4.22}
\]
\[
\gamma = 2 \delta' / 3
\]
\[
\delta = 3 \delta' / 4
\]

Putting
\[
\beta' = \beta_0, \quad \frac{\gamma'}{3} = B, \quad \frac{\delta'}{4} = C
\]

in (4.1) and (4.14) gives, respectively
\[
E = E_0 + \frac{\omega^2}{2} (\beta_0 + 4 B \omega + 3 C \omega^2) \tag{4.24a}
\]
\[
\left\langle \psi \left| J_x \right| \psi \right\rangle = \omega (\beta_0 + 3 B \omega + 2 C \omega^2) \tag{4.25a}
\]

Since \(\left\langle \psi \left| J_x \right| \psi \right\rangle = \sqrt{I(I + 1)}\), where I is the spin of the band head, we may begin with
\[ E_{rot} = \frac{\omega^2}{2} \left( J_o + 4 B \omega + 3 C \omega^2 \right) \quad (4.24B) \]
\[ \sqrt{1(I+1)} = \omega \left( J_o + 3 B \omega + 2 C \omega^2 \right) \quad (4.25B) \]

The nuclear angular velocity, \( \omega \), may be eliminated from these equations, leaving one equation for \( E_{rot} \) as a function of nuclear spin I, dependent on the three parameters \( J_o, B, \) and \( C \). Solving equation (4.25B), a cubic in \( \omega \), gives three roots \( \omega_1, \omega_1', \) and \( \omega_1'' \)

where
\[
\omega = \frac{-b}{2c} + \left\{ \frac{-r}{2} + \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} + \left\{ \frac{-r}{2} - \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} \quad (4.26A)
\]
\[
\omega_1 = \frac{-b}{2c} + e^{2\pi i/3} \left\{ \frac{-r}{2} + \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} + e^{4\pi i/3} \left\{ \frac{-r}{2} - \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} \quad (4.26B)
\]
\[
\omega_1' = \frac{-b}{2c} + e^{2\pi i/3} \left\{ \frac{-r}{2} + \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} + e^{4\pi i/3} \left\{ \frac{-r}{2} - \sqrt{\frac{2}{4} + \frac{3}{27}} \right\}^{1/3} \quad (4.26C)
\]

where
\[
r = \frac{B}{3} - \frac{J_o B}{4 C^2} - \frac{\sqrt{1(I+1)}}{2C} \quad (4.27)
\]
\[
q = \frac{J_o}{2C} - \frac{3B^2}{4C^2} \quad (4.28A)
\]

From (4.25B) for \( I = 0 \) we have either
\[
\omega = 0 \quad (4.28A)
\]
or
\[
\omega = -3B \pm \sqrt{9B^2 - 8J_o} \quad (4.28B)
\]

That \( \omega = 0 \) for \( I = 0 \) corresponds to the root (4.26A) can be
seen as follows. We note that the parameters $J_o$, $B$, and $C$ calculated by the computer in each case are such that one always has

$$9B^2 < 8J_o C$$  \hspace{1cm} (4.29)

The condition (4.29) implies that the roots (4.26B) are complex. Moreover, the condition, $9B^2 < 8J_o C$, implies that

$$q > \frac{-J_o}{6C}$$  \hspace{1cm} (4.30)

which gives

$$\frac{x^2}{4} + \frac{q^3}{27} > 0.$$  \hspace{1cm} (4.31)

From equation (4.31) we see that the root (4.26A) is real ($I > 0$) (the roots (4.26B) and (4.26C) are then complex). Therefore, for $I = 0$, the root (4.26A) gives $\omega = 0$.

We now consider some alternate expressions for $E_{\text{rot}}$ which are of interest. First, we define the effective moment of inertia, $\bar{J}$, through the semi-classical relation

$$\sqrt{I(I+1)} = \omega \bar{J} = \omega (J_o + 3B\omega + 2C\omega^2)$$  \hspace{1cm} (4.32)

With this, we may write (4.24B) as

$$E_{\text{rot}} = \frac{\omega^2}{2} (J_o + 4B\omega + 3C\omega^2)$$

$$= \frac{\omega^2}{2} (J + B\omega + C\omega^2)$$

\hspace{1cm} \text{i.e.}  \hspace{1cm} \frac{E_{\text{rot}}}{I(I+1)} = \frac{1}{2} B\omega^3 + \frac{1}{2} C\omega^4$$  \hspace{1cm} (4.33)

Now consider

$$\left(\bar{J} - J_o\right)^2 = \frac{9B^2\omega^2 + B\omega^3 + \left(\frac{1}{2}B\omega^3 + \frac{1}{2}C\omega^4\right)^2}{8C}$$  \hspace{1cm} (4.34)

Comparing (4.33) and (4.34) gives
\[ E_{\text{rot}} = \left( \frac{\mathcal{J} - \mathcal{J}_0}{8C} \right)^2 + \frac{I(I+1)}{2\mathcal{J}} - \frac{2E^2\omega^2}{8C} - \frac{3}{2} \omega^3 \]

or, using (4.32)
\[ E_{\text{rot}} = \left( \frac{\mathcal{J} - \mathcal{J}_0}{8C} \right)^2 + \frac{I(I+1)}{2\mathcal{J}} - \frac{2E^2I(I+1)}{8C} - \frac{B^2I(I+1)}{\mathcal{J}^2}. \]

This form of the expression for \( E_{\text{rot}} \) will prove useful in our discussion of nuclear softness. If \( B = 0 \), it is noted that (4.35) reduces to the VMI expression for \( E_{\text{rot}} \).

Using (4.32), the expression for \( E_{\text{rot}} \) may also be written as
\[
E_{\text{rot}} = \frac{1}{2} \omega^2 \left( \mathcal{J}_0 + 4B\omega + 3C\omega^2 \right)
= \frac{1}{2} \omega^2 \left( \mathcal{J} + B\omega + C\omega^2 \right)
= \frac{I(I+1)}{2\mathcal{J}} \left( \mathcal{J} + \frac{B\sqrt{I(I+1)}}{\mathcal{J}^2} + \frac{C(I(I+1))}{\mathcal{J}^3} \right).
\]
i.e.
\[ E_{\text{rot}} = \frac{I(I+1)}{2\mathcal{J}} \left( 1 + \frac{B\sqrt{I(I+1)}}{\mathcal{J}^2} + \frac{C}{\mathcal{J}^3} \right). \quad (4.36)
\]
This demonstrates that \( E_{\text{rot}} \) may be expressed as a polynomial in \( \sqrt{I(I+1)} \). It is seen that (4.36) reduces to an alternate VMI expression for \( E_{\text{rot}} \) if \( B = 0 \).

Equation (4.36) could be used to provide a description in terms of the three parameters \( \mathcal{J}_0, B, \) and \( C \) with \( \mathcal{J} = \mathcal{J}(\mathcal{J}_0, B, C) \), independent of \( \omega \). This form for \( E_{\text{rot}} \) suggests that one may write
\[
E_{\text{rot}} = \sum_{n=0}^{\infty} A_n \frac{1}{n!} \left[ I(I+1) \right]^{n/2}.
\]
This is quite plausible since, as has been noted, an expansion in \( I(I+1) \) gives good results in the rotation region while an expansion in \( I \) gives good results in the vibration region. As will be shown,
(4.36), or, analogously, (4.37), gives an expansion in a quantity which is "in between" I and I(I + 1). Such an expansion will naturally give good fits in both regions.

We now investigate the possible physical significance of the three parameters appearing in the expression for \( E_{rot} \). From (4.29) and (4.32) it is evident that \( \vec{J}(I = 0) = \vec{J}_0 \). Thus \( \vec{J}_0 \) is the "ground state moment of inertia". Noting that the expectation value of the nuclear angular momentum is given by \( \langle \psi | \vec{J}_x | \psi \rangle = \sqrt{I(I + 1)} \) and that \( \vec{J} \) is the rotational analogue of a mass we see that the term \( \frac{I(I + 1)}{2} \) \( \vec{J} \) in (4.35) represents the rotational kinetic energy. Similarly we may interpret the term (in (4.35)) \( (\vec{J} - \vec{J}_0)^2 \) as representing the potential energy of the nucleus due to rotational stretching. The constant \( \frac{1}{\sqrt{c}} \) appears as a "restoring force constant", in effect measuring the slope of the potential energy curve.

It remains to interpret the terms containing the parameter \( B \).

First, we demonstrate that the condition

\[
\sqrt{I(I + 1)} = \omega J = \omega (\vec{J}_0 + 3 B \omega + 2 C \omega^2)
\]

(4.32)

implies the equilibrium condition \( \frac{\partial E}{\partial \omega} = 0 \). From equation (4.33) we get

\[
\frac{\partial E}{\partial \omega} = -\frac{I(I + 1)}{2 \phi^2} \frac{\partial \phi}{\partial \omega} + \frac{3 B \omega^2}{2} + 2 C \omega^2.
\]

(4.38)

But from (4.32)

\[
\frac{\partial \phi}{\partial \omega} = 3 B + 4 C \omega.
\]

(4.39)

Using now (4.39), we have for (4.38)
\frac{\Delta E}{\Delta \omega} = \frac{-1}{2} \sum_{I} \left( 3 B + 4 C \omega \right) + \frac{3 B \omega^2}{2} + 2 C \omega^3
\\seteq \frac{-1}{2} \omega^2 \left( 3 B + 4 C \omega \right) + \frac{3 B \omega^2}{2} + 2 C \omega^3
\\seteq 0 \quad \text{(4.40)}

Hence
\frac{\partial E}{\partial j} = 0 \quad \text{for all } I \quad \text{(4.41)}

since
\frac{\partial E}{\partial j} = \frac{3 B}{\partial \omega \partial \omega} \frac{3 B}{\omega} \quad \text{(4.42)}

This demonstrates that \sqrt{I(I+1)} = \omega \omega' \text{ is equivalent to the}
equilibrium condition \frac{\partial E}{\partial \omega} = 0. \text{ From (4.35) we have}
\frac{\partial E}{\partial \omega} = \frac{3 B}{4 C} \omega + \frac{9 B^2}{4 C} \frac{I(I+1)}{j^2} + \frac{3 B}{2} \frac{\sqrt{I(I+1)}}{j^4} - \frac{I(I+1)}{2 j^2} \quad \text{(4.43)}

Using the equilibrium condition, (4.41) and rearranging we obtain
\sqrt{\omega} \omega' \sqrt{\omega} + 9 B \omega \omega' \sqrt{\omega} + 12 B C \left\{ \frac{I(I+1)}{2 j^2} \right\} - 2 C \frac{I(I+1)}{2 j^2} = 0 \quad \text{(4.44)}

From this equation \omega' can be expressed as a function of the parameters \omega, B, and C and of the nuclear spin I. We may now derive from (4.44)
an expression for the nuclear softness \sigma_3, defined as the relative
increase of the moment of inertia with the angular momentum I.

Explicitly
\sigma_3 = \left[ \int_{\omega}^{\omega'} \frac{\partial E}{\partial \omega} \right]_{\omega'} = 2 \omega \omega' C - 9 B^2 \frac{\omega}{j^2} \quad \text{(4.45)}

Noting that the VMI value for \sigma_3 is \sigma_{VMI} = 2 C \omega\omega' \text{,}
(4.45) may be put as
\sigma_3 = \sigma_{VMI} - \frac{9 B^2}{j^2} \quad \text{(4.46)}
It is thus evident that the parameter B modifies the nuclear softness by reducing it somewhat below that given by the WMI model. A decrease in "softness" would lower the rotational energy levels since the nucleus is more resistant to deformation and thus would not acquire a large moment of inertia. This is also related to the fact that the terms dependent on B in (4.35) are negative, thus reducing the energy. This is obvious for $B > 0$, but for $B < 0$, the last term, $-B \frac{(I(I + 1))^{3/2}}{\mathcal{J}^{3}}$, would be positive but its effect would be much smaller, compared to that of the term $-\frac{9B^{2}}{8C} \frac{I(I + 1)}{\mathcal{J}^{2}}$ which is negative regardless of the sign of B because of the larger power of $\mathcal{J}$ occurring in the denominator. (Note that the computer calculations indicate that $\mathcal{J} > B$.)

4.2 Range of Validity

We now determine the limits of validity of our semi-empirical approach. Defining $r = \mathcal{J} / \mathcal{J}_{o}$, we obtain from (4.44)

$$r^{5} - r^{4} + \frac{3B^{2}}{\mathcal{J}_{o}^{4}} r (I(I + 1))^{3/2} - \frac{2C}{\mathcal{J}_{o}^{5}} r^{2} I(I + 1) = 0$$

(4.47)

This may be put in terms of $\sigma$ as

$$r^{5} - r^{4} + \frac{3B^{2}}{\mathcal{J}_{o}^{4}} (I(I + 1)r(1 - r) + \frac{12BC}{\mathcal{J}_{o}^{5}} (I(I + 1))^{3/2} = \sigma r^{2} I(I + 1)$$

(4.48)

For the case of hard, well-deformed nuclei, $\sigma \rightarrow 0$ implying that $B \rightarrow 0$ and $C \rightarrow 0$ from equation (4.45). Hence (4.48) reduces to

$$r^{5} - r^{4} = 0 \text{ or } r = 1.$$ That is $\mathcal{J} = \mathcal{J}_{o}$. Thus from (4.36)

$$E_{\text{rot}}(\sigma \rightarrow 0) = \frac{I(I + 1)}{2\mathcal{J}_{o}}$$

(4.49)
which corresponds to the energy of a rigid rotor, and

\[ R_4(\sigma \to 0) = \frac{10}{3} = 3.33 \tag{4.50} \]

which is the rigid rotor limit.

For the case of \(|\sigma| \to \infty\), we have \(|J| \to 0\) and hence \(|r| \to \infty\). Equation (4.48) can be put as

\[ r^3 - r^2 + \frac{2}{\beta} (1 - r) I(I + 1) + \frac{12}{\beta} \frac{E}{r^2} \frac{I(I + 1)}{r} = C I(I + 1) \tag{4.51} \]

Since \(\beta \ll 1\) and \(|r| \to \infty\), this reduces to

\[ r^3 \approx C I(I + 1) \]

or

\[ r \approx \left( \frac{C I(I + 1)}{r^3} \right)^{\frac{1}{3}} \]

Thus

\[ J \approx J_0 \left( \frac{C I(I + 1)}{r^3} \right)^{\frac{1}{3}} \tag{4.52} \]

Substituting this value of \(J\) into (4.36) gives

\[ E_{\text{rot}}(\sigma \to \infty) \approx \left( \frac{I(I + 1)}{r^3} \right)^{\frac{1}{3}} + \frac{B}{2} \frac{I(I + 1)}{r^3} + \frac{C}{2} \frac{I(I + 1)}{r^3} \tag{4.53} \]

But since \(r \approx \left( \frac{C I(I + 1)}{r^3} \right)^{\frac{1}{3}}\) and \(|r| \to \infty\) the first term will dominate because it has the lowest power of \(\sigma\) in its denominator.

Thus

\[ E_{\text{rot}}(\sigma \to \infty) \approx \left( \frac{I(I + 1)}{r^3} \right)^{\frac{1}{3}} \tag{4.54} \]

Hence

\[ R_4(\sigma \to \infty) = \left( \frac{20}{6} \right)^{\frac{1}{3}} = 2.23 \tag{4.55} \]

Thus the range of validity of our model is the same as that of the original VMI model, namely \(2.23 \leq R_4 \leq 10/3\).
4.3 Calculations and Results

We have evaluated the energy levels, $E_{\text{rot}}^*$ (using equations (4.243) and (4.256)) for the same 83 nuclei, covering the rotation, transition, and vibration regions, as considered in Reference 12. The results are presented in Table I. The parameters $J_0$, $B$, and $C$, for each nucleus considered, were determined by a least-squares fitting procedure (see Appendix C) involving the energies of all the known spin states. In Table I, the first row gives our calculated energies, expressed in keV, for levels up to five states beyond those experimentally known. The second row gives the experimental energies, which were taken from Reference 12, and the third row gives the energies as calculated using the VMI model\textsuperscript{12}. The $R_4$ value ($R_4 = E(I = 4+)/E(I = 2+)$) for each nucleus is also included.

As can be seen, the results are excellent for all regions (rotation, transition, and vibration), being, in the vast majority of cases, better than those predicted by the VMI model. This is due primarily to the improved fitting of the high spin states. The above conclusion is born out by a comparison of the weighted sum of squares (WSS) for our values with those given by the VMI model. (WSS = \[ \sum I \left[ \frac{E_{\text{exp}}(I) - E_{\text{theor}}(I)}{E_{\text{exp}}(I)} \right]^2 / E_{\text{exp}}(I) \]). This is presented in Table II where the first column gives our WSS value for each nucleus and the second column gives the WSS values for the VMI model. For almost all cases our WSS values are at least an order of magnitude smaller than those obtained on the basis of the VMI model.
### Table I: Comparison of Rotational Energy Levels

For each nucleus, the first row contains the energies obtained with our model. The second row gives the experimental energies and the third row gives energies predicted by the VMI model. All energies are in keV. (The values in the second and third row are taken from reference 16.) \( R(\alpha) = E(\alpha) / E(\alpha = 2) \)

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In many cases the results of the VMI model lie outside the limits of experimental error while ours lie within. (A listing of the experimental errors for the nuclei considered is presented in Reference 12.) Thus if we take, as the experimental energy values for a given set of levels, the outside experimental limit to calculate the WSS (still using the same parameters), the WSS value obtained is not appreciably different, whereas an appreciable difference would be found for the VMI model for those cases where the calculated energies lie outside the limits of experimental error.

It may be argued that the improved fitting of the energy levels obtained in our case is due to the appearance of an extra parameter compared to that of the VMI model. However, the addition of any arbitrary term to the VMI model would bring only an insignificant improvement. The fact that our choice of the new additive term, depending on the parameter \( B \), does improve the WSS values, in many cases by four or five orders of magnitude, suggests that our special choice of this term extends the VMI model in the most consistent manner.

It should be stressed at this point that our model is quite different from the extended cranking model due to Harris. It does not give the same results as would be obtained simply by taking the next two terms of the Harris series as an additional correction. This is evident from the great difference between our values of \( J_0 \) and \( C \) as compared to the values of those parameters which enter formally in the same way in the Harris model.
The values of the parameters $J_0$, $B$, and $C$ obtained for the various cases are presented in Columns 3-5 of Table III. It is seen that generally $J_0 \sim 10^{-2}$ keV$^{-1}$, $B \sim 10^{-5}$ keV$^{-2}$, and $C \sim 10^{-7}$ keV$^{-3}$. For most nuclei, the values of $J_0$ and $C$ are comparable to those predicted by the VMI model. Two notable exceptions are Cd$^{152}$ ($R_4 = 2.20$) and Er$^{156}$ ($R_4 = 2.32$) for which our theory gives $J_0 < 0$. This is in accord with the discussion in Section 4.2 where it was pointed out that for nuclei with $R_4 \sim 2.2$ we have $C^{-} \sim 0$ which, from $(4.52)$, implies $r < 0$. But since $J_0 > 0$, this means that for this case, $J_0 < 0$. With respect to the results for the new parameter $B$, there appears to be no preference in the sign (+ or -), although, as previously mentioned, this is of little consequence since $B$ appears as $B^2$ in most places.

Column 6 of Table III gives the values of the softness parameter, $\sigma^-$, as predicted by our model. With a few exceptions, most of the nuclei in the rotation region ($3.0 < R_4 < 3.33$, far from closed-shell nuclei) have approximately the same value for $\sigma^-$ as predicted by the VMI model, since, for these nuclei, $B$ is quite small ($\sim 10^{-6}$ keV$^{-2}$) and the correction to the nuclear softness is negligible. For most nuclei in the transition and vibration regions (nuclei near and approaching closed shells) our model predicts $\sigma^- < 0$. This is seen more clearly from figure 3 in which we have plotted $\sigma^-$ (on a logarithmic scale) versus the mass number $A$. The numbers in brackets refer to the $R_4$ values of the given nuclei. We note that in the region $A = 92$ to $A = 112$, $\sigma \approx C_{VMI}$, and that the curves show the discontinuities at $N = 98, 104,$ and $108$ as discussed by Mariscotti et al.\textsuperscript{12}. 


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Figure 3. The nuclear softness, $\sigma$ (on a logarithmic scale), plotted against mass number ($A$) for values of constant atomic number ($Z$). The numbers in brackets refer to $R_H$ values.
The anomalous behavior at \( N = 98 \) had been previously observed by Stephens et al.,\(^1\) who speculated, as a possible explanation for this effect, that the pairing correlations are reduced because of the large energy gap in the Nilsson diagram between the levels of
\[
\frac{5}{2}^- (98 \text{ neutrons}) \text{ and } \frac{7}{2}^+ .
\]
More recently, Duckworth\(^2\) showed that breaks are also seen at \( N = 90 \) and \( N = 108 \) in the plot of double neutron separation energies as a function of neutron number. However, outside this region where \( R_4 < 3 \) and where \( N \) and/or \( Z \) begin to approach magic number values, the value of \( \sigma \) becomes negative. This transition in the value of \( \sigma \) occurs most noticeably in the region \( N = 88 \) to 90 where the nuclei go from an almost spherical shape to a well-deformed one.

The significance of "negative softness", which occurs in our model, as well as in the VMI model, is not yet understood. From the graphs presented in Figure 3 one can possibly conclude that the "degree of softness" is related to nuclear shell structure and thus to pairing forces in the nucleus. It would appear that a microscopic calculation might provide a better physical picture.

In Figure 4 we have plotted graphs of the effective moment of inertia, \( \mathcal{J} \), versus nuclear spin, \( I \), for a few representative nuclei from the rotation, transition, and vibration regions. The numbers in brackets give the \( R_4 \) values in each case. From these graphs it is seen that:

1. The \( \mathcal{J}_0 \) values increase in a fairly regular fashion with increasing \( R_4 \).
(ii) for well-deformed, stable nuclei, e.g., $^{180}$Hf, $\mathcal{J}$ is almost constant as $I$ increases,

(iii) the nuclei with the most dramatic relative increase of $\mathcal{J}$ with respect to $I$ are those with small $\mathcal{J}_0$ and $R_4 \lesssim 2.23$,

(iv) despite the fact that many nuclei have a negative value of $\sigma$ their $\mathcal{J}$ versus $I$ curves resemble those having positive values of $\sigma$ in as much as the slope is concerned,

(v) those nuclei which appear to have $d\mathcal{J}/dI < 0$ from the graph, have $\sigma > 0$.

A situation similar to that in (iv) above has been reported by Schraff-Goldhaber et al., in which they predict $\sigma_{VMI} < 0$ for $^{120}$Te while their $\mathcal{J}$ versus $I$ graph for this isotope resembles that of $^{120}$Xe for which $\sigma_{VMI} > 0$. It is hoped that once the concept of negative softness is understood that these paradoxes appearing both in our approach and in the VMI model may be resolved. A complete listing of the effective moments of inertia for each spin state for all the cases considered is given in the first column of Table IV. The values are given in order of increasing spin, $I = 2^+, 4^+, \ldots$. (The value for spin $I = 0$ is not given as this is equal to $\mathcal{J}_0$, the values of which were presented in Table III.)

A measure of the discrepancy between $\mathcal{J}_0$ and $\mathcal{J}$ is $\mathcal{S}$, defined by $\mathcal{S} = \mathcal{J} - \mathcal{J}_0$. Values of $\mathcal{S}$ for each spin state of each nucleus are presented in Column 2 of Table IV. As a general rule it is seen that high values of $\mathcal{S}$ correspond to small $R_4$ values and vice versa.
Figure 4. The effective moment of inertia, $J$, plotted against nuclear spin $I$ for representative nuclei from the rotation, transition, and vibration regions. The numbers in brackets refer to $R_4$ values.
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We have presented in Appendix B a list of the successive terms of \((4.243)\) for each spin state for all the nuclei considered. In the rotation region, the first term is dominant and the second two are either successively smaller or are in competition. (One exception is Er\(^{162}\) \((R_4 = 3.2)\) for which the second term is the dominant term of the three.) This suggests that, at least for this region, the last two terms in our expression may represent corrections to the rigid rotor formula, \((1.1)\), which is to be expected, as for \(3.0 < R_4 < 3.3\) we are near the rigid rotor limit.

In the transition region, one finds that, for many cases considered, the second term is the dominant term of the three. The two exceptions are Pd\(^{108}\) \((R_4 = 2.4)\) and Er\(^{158}\) \((R_4 = 2.7)\) where the third term is dominant. In the vibration region, the third term is either dominant, or is comparable to the second term, the first term being much smaller. Thus the rigid rotor description is not at all valid in these regions and the second and third terms in our expression for \(E_{\text{rot}}\) may not be considered to be simply corrections to the rigid rotor formula for \(2.23 < R_4 < 3.0\). A similar analysis of the relative magnitude of the various terms is also given by Das et al.\(^{36}\) (as discussed in Section 3.5), but there is no one-to-one correspondence between his terms and our terms.

4.4 Conclusions

We have provided a semi-empirical description of the ground-state bands of 88 even-even deformed nuclei. The energies predicted by our approach provide better fits than other models as evidenced by
our WSS values. Although three parameters are employed, this is more than compensated for by the much improved results, especially for the high spin states. The treatment is completely general in that it is independent of the manner in which the variation in $J$ takes place (e.g. $\beta$-stretching, decrease in pairing energy, etc.). Also, since our expression for $E_{\text{rot}}$ is in the form of a polynomial, we do not have to worry about the convergence problem inherent in the power series (or perturbation series) expansions as employed by Harris, Satpathy, and others (as previously discussed).

In addition to the greatly improved fits to the experimental data we have introduced a parameter, $B$, whose significance is unique to our approach in that it modifies the nuclear softness, $\Sigma$. Through it, the role of $\Sigma$, as discussed in the VMI model, is extended. It is hoped that a microscopic investigation will provide further insight into the concept of nuclear softness.

As a further extension of this work, it would also be of interest to see how the parameter $B$ affects the theoretical predictions of the values of nuclear quadrupole moments and the levels of those nuclear excited states which do not belong to the ground-state band, e.g., $\beta$- and $\gamma$-vibrational bands.
Appendix A

Proof of Feynman’s Theorem

Feynman’s Theorem states that if $H(\lambda)$ depends on a parameter $\lambda$, and $\psi_\lambda(\lambda)$ is an eigenfunction which is normalized to unity then

\[
\frac{\partial E_m(\lambda)}{\partial \lambda} = \left \langle \frac{\partial \psi_m(\lambda)}{\partial \lambda} \left| \frac{\partial H(\lambda)}{\partial \lambda} \right| \psi_m(\lambda) \right \rangle.
\]

Proof:

\[
\frac{\partial E_m(\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left \langle \psi_m(\lambda) \left| H(\lambda) \right| \psi_m(\lambda) \right \rangle
\]

\[
= \left \langle \psi_m(\lambda) \left| \frac{\partial H(\lambda)}{\partial \lambda} \right| \psi_m(\lambda) \right \rangle
\]

\[
+ \left \langle \psi_m(\lambda) \left| H(\lambda) \right| \frac{\partial \psi_m(\lambda)}{\partial \lambda} \right \rangle
\]

\[
+ \left \langle \frac{\partial \psi_m(\lambda)}{\partial \lambda} \left| H(\lambda) \right| \psi_m(\lambda) \right \rangle
\]

\[
= \left \langle \psi_m(\lambda) \left| \frac{\partial H(\lambda)}{\partial \lambda} \right| \psi_m(\lambda) \right \rangle
\]

\[
+ E_m \left \langle \frac{\partial \psi_m(\lambda)}{\partial \lambda} \left| \frac{\partial \psi_m(\lambda)}{\partial \lambda} \right| \psi_m(\lambda) \right \rangle
\]

\[
+ E_m \frac{\partial}{\partial \lambda} \left \langle \psi_m(\lambda) \left| \psi_m(\lambda) \right \rangle \right \rangle
\]

\[
= \left \langle \psi_m(\lambda) \left| H(\lambda) \right| \psi_m(\lambda) \right \rangle + E_m \frac{\partial}{\partial \lambda} \left \langle \psi_m(\lambda) \left| \psi_m(\lambda) \right \rangle \right \rangle
\]

\[
= \left \langle \psi_m(\lambda) \left| H(\lambda) \right| \psi_m(\lambda) \right \rangle
\]

since $\left \langle \psi_m(\lambda) \left| \psi_m(\lambda) \right \rangle \right \rangle = 1$. 

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<td>346.344400</td>
<td>-1276.233887</td>
<td>10009.629665</td>
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<tr>
<td></td>
<td>453.427345</td>
<td>399.160043</td>
<td>-1616.844566</td>
<td>13707.926003</td>
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<td>479.041159</td>
<td>4462.623535</td>
<td>-19060.756515</td>
<td>17077.817687</td>
</tr>
<tr>
<td>XE 120 R(4) = 2.47</td>
<td>292.402119</td>
<td>63.591841</td>
<td>216.264263</td>
<td>63.151374</td>
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<tr>
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<td>271.666123</td>
<td>78.643457</td>
<td>523.046601</td>
<td>205.018289</td>
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<td>323.486315</td>
<td>111.349155</td>
<td>852.091954</td>
<td>442.066432</td>
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<td>365.113584</td>
<td>142.628888</td>
<td>1279.935009</td>
<td>676.063370</td>
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<tr>
<td>XE 122 R(4) = 2.50</td>
<td>219.822965</td>
<td>64.942421</td>
<td>209.465252</td>
<td>54.252541</td>
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<td>207.151109</td>
<td>120.513721</td>
<td>529.469155</td>
<td>186.833559</td>
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<td>813.268614</td>
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<td>391.602358</td>
<td>224.569883</td>
<td>1342.393961</td>
<td>649.444113</td>
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<tr>
<td>XE 124 R(4) = 2.48</td>
<td>223.846533</td>
<td>61.703169</td>
<td>228.363834</td>
<td>61.775849</td>
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<tr>
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<td>303.314618</td>
<td>113.296252</td>
<td>556.126603</td>
<td>206.260070</td>
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<td>365.422344</td>
<td>162.107323</td>
<td>972.427657</td>
<td>426.391195</td>
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<td>411.26915</td>
<td>204.454950</td>
<td>1422.043924</td>
<td>707.761919</td>
</tr>
<tr>
<td>XE 126 R(4) = 2.44</td>
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<td>29.793112</td>
<td>270.616142</td>
<td>88.107653</td>
</tr>
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<td>52.20335</td>
<td>629.107181</td>
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<td>92.969947</td>
<td>1691.739748</td>
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<td>XE 128 R(4) = 2.34</td>
<td>259.751779</td>
<td>66.916996</td>
<td>-5.113211</td>
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<td>105.346403</td>
<td>-6.174267</td>
<td>939.626461</td>
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<td>372.059004</td>
<td>136.026389</td>
<td>-6.259916</td>
<td>1613.237202</td>
</tr>
<tr>
<td>XE 130 R(4) = 2.25</td>
<td>208.411674</td>
<td>227.799150</td>
<td>-800.925064</td>
<td>1107.559223</td>
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<tr>
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<td>325.301581</td>
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<td>2230.491329</td>
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<td>400.436126</td>
<td>-1866.693066</td>
<td>3422.389137</td>
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<td>467.665592</td>
<td>-2357.427997</td>
<td>4671.923166</td>
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<tr>
<td>BA 124 R(4) = 2.83</td>
<td>165.699757</td>
<td>124.272710</td>
<td>90.236432</td>
<td>10.269777</td>
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<tr>
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<td>249.443942</td>
<td>291.12674</td>
<td>304.391611</td>
<td>52.98095</td>
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<td>315.167305</td>
<td>464.714945</td>
<td>622.243052</td>
<td>134.79232</td>
</tr>
<tr>
<td>BA 126 R(4) = 2.76</td>
<td>182.204226</td>
<td>139.344149</td>
<td>101.231099</td>
<td>11.421365</td>
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<tr>
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<td>273.727333</td>
<td>314.501564</td>
<td>343.234653</td>
<td>60.216944</td>
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<td>500.477024</td>
<td>649.027350</td>
<td>152.645927</td>
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<td>408.495285</td>
<td>690.174721</td>
<td>1115.824902</td>
<td>289.98032</td>
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<tr>
<td>CE 128 R(4) = 2.93</td>
<td>154.527274</td>
<td>140.856854</td>
<td>51.570069</td>
<td>14.617936</td>
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<td>335.431556</td>
<td>149.737075</td>
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<td>308.91032</td>
<td>544.013684</td>
<td>392.144630</td>
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<td>796.477724</td>
<td>641.960073</td>
<td>482.666683</td>
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<tr>
<td>CE 130 R(4) = 2.83</td>
<td>181.227808</td>
<td>142.501130</td>
<td>63.437424</td>
<td>24.856246</td>
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<tr>
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<td>269.404170</td>
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<td>275.440237</td>
<td>121.381767</td>
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<td>336.422840</td>
<td>491.065795</td>
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<tr>
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<td>391.044104</td>
<td>665.907697</td>
<td>846.239082</td>
<td>542.136022</td>
</tr>
<tr>
<td>CE 132 R(4) = 2.64</td>
<td>218.979447</td>
<td>128.064149</td>
<td>137.356046</td>
<td>67.691442</td>
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<td>305.219079</td>
<td>233.253638</td>
<td>371.885646</td>
<td>205.644916</td>
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</tbody>
</table>
Appendix C

Listing of Curve-Fitting Program Employed to Obtain

Three-Parameter Fit
PROGRAM ENERGY

INPUT, OUTPUT, TAPE4, TAPE5, TAPE6, TAPE7, TAPE9, TAPE10, TAPE11
M = NUMBER OF PARAMETERS
N = NUMBER OF ITERATIONS ALLOWED
N0 = INITIAL CHANNEL OF THE GROUP
N = NUMBER OF CHANNELS IN THE GROUP

DIMENSION Z(256), FM(256), FC(256), DF(256), ERR(256), R13, R1(3), R2(1, 3), RC(41251), S1(6), DAT(256), S1(6), P6(6), P6(6), PH(6), PH(6), 2PH(6), PH(6), PHE(6), A1(6), AE(6), ABC2(2), NN
DATA(256)
DIMENSION WH(25)
COMMON/DATA/ABC, M, COMMON DC
EQUATION (2, DC), (FM, DC(257)), (FC, DC(513)), (DF, DC(769)), 1ERR, DC(10251), (R1, DC(1261)), (R2, DC(13011)), (N, DC(7811)), 2L4, DC(1791), (N, DC(1764)), (R2, DC(17691)), (M, DC(1783)), 318, DC(17191), (1, DC(2180)), (I, DC(1770)), (4L4, DC(1582)), 4TERM, DC(1770), ITERM2, DC(1769), ITERM3, DC(1759), OMEGA, DC(1710)
51, (M, DC(1750)), (N, DC(1716))
EQUATION2(A, DC(1711)), (SIGMA, DC(1721)), (RAT, DC(1713))
EQUATION2CTOR, DC(1741), (PAGE, DC(1719)), (R, PAGE, DC(1716)), (NOM, 10DC(1717), (FM, DC(1718))
DATA(ABC2=2NH, JHYES, 1H=1H, 1HC, 1NM, 1HN)

Q1 AND Q2 ARE THE PRECISIONS OF THE CURVE FITTING

C

PAGE=88
WRITE(7, 449)
WRITE(7, 500)
500 FORMAT(1X, 49TABLE I C iTunes OF ROTATIONAL ENERGY LEVELS///
1)
WRITE(7, 575)
575 FORMAT(1X, 12HFOR EACH NUCLEUS THE FIRST ROW CONTAINS THE ENERGIES
1 OBTAINED WITH OUR MODEL. THE SECOND ROW GIVES THE EXPERIMENTAL ENERGIES AND THE THIRD ROW GIVES THE VALUES PREDICTED BY THE VM 3I MODEL. ALL ENERGIES ARE IN KEV. THE VALUES IN THE SECOND AND TH
3RD ROW ARE FROM REFERENCE 16. THE VALUES IN THE SECOND AND TH
3RD ROW ARE FROM REFERENCE 16.)
WRITE(7, 501)
501 FORMAT(3X, WHSPIN, 6X, 1N2, 6X, 1N4, 6X, 1NH, 7X, 2NH10, 7X, 2NH12, 7X, 2H
11, 7X, 2H10, 7X, 2H12, 7X, 2HD9, 7X, 2HD7, 7X, 2HD6, 7X)
WRITE(7, 550)
550 FORMAT(12X, 63TABLE II LIST OF PARAMETERS EMPLOYED FOR THREE-PA
1RAMETER FIT///
1)
WRITE(8, 576)
576 FORMAT(2S3X, HU, KE IN UNITS OF KEV, B IN 1/KEV**2, AND C IN 1/KE
4**3). (2**2) <<HUGU>> = (2**2) <<HUGU>>
WRITE(7, 502)
502 FORMAT(23X, 3T, 7HNUCLEUS, 7X, 4HRH, 13X, 4H, 15X, 4HRH, 13X, 4H
1, 3TABLE II COMPARISON OF WSS VALUES///
1)
WRITE(9, 540)
PROGRAM ENERGY

541 FORMAT(2X,7HMOM,10X,21HOUR 4SS)
WRITE(6,541)
593 FORMAT(3X,5HMOMENTUM B. LIST OF OMEGA AND TERMS OF SERIES FOR E)
1CH SPIN/17X,4HSPIN,7X,SHOMEga,17X,6HTERM 1,15X,6HTERM 2,15X,6HTERM
2M 3/1
WRITE(10,571)
571 FORMAT(1X,4MHTABLE IV. LIST OF EFFECTIVE MOMENTS OF INERTIA/1X,3)
10AND DELTA. (DELTA = ( = 1/1))
1000 CONTINUE
READ(5,390)CASE,NAME
399 FORMAT(2X,13)
WRITE(6,100)
184 FORMAT(1H1)
WRITE(6,340)CASE,NAME
70 FORMAT(1X,4,13)
READ(5,71)Q1,Q2
7 FORMAT(2F5.3)
WRITE(6,8)Q1,Q2
8 FORMAT (1X,4H01 = ,F5.3,5X,4H02 = ,F5.3)
READ (6,990) NO,N ,L4
READ(5,910) (DAT(I),I=1,N)
RAT = DAT(2)/DAT(1)
READ(5,910) (X4(I),I=1,N)
L4 =100
WRITE(6,950)CASE,NAME,RAT
WRITE(10,573)CASE,NAME,RAT
573 FORMAT(1X,4,13,4X,7HR(4) =,F5.2,1H))
580 FORMAT(1X,4,13,4X,7HR(4) =,F5.2)
C VM(II) REFER TO VALUES CALCULATED BY OTHERS FOR THE SAME NUCLEUS
85 910 FORMAT (12X,10F6.1)
WRITE(6,3)
3 FORMAT(2X,13H INITIAL DATA//)
WRITE(6,331) (DAT(I),I=1,N)
931 FORMAT(13X,10F6.1)
939 FORMAT (12)
WRITE(6,108)NO,N
10 FORMAT12DH INITIAL CHANNEL= I3,25H *NUMBER OF CHANNELS= I3)
M=3
Og 115 I=1,N
KG=I+NO
Z(I)=KG+1
KNO=I+NO=I+1
P4(I)=DAT(KNO)
115 ERR(I)=SQRT(P4(I))
WRITE(6,931)ERR(I),I=1,N)
READ(5,898)
89 FORMAT(2X,3E12.5)
WRITE(6,939)
C B45 = O REFERS TO GAMMA VIBRATIONAL STATES
C B45 = 1 REFERS TO GROUND STATE BANDS
READS,771B45
77 FORMAT(10F6.1)
CALL CURPIL
WRITE(6,100)
105 WRITE (6,135)
PROGRAM ENERGY

135 FORMAT (1X,11H PARAMETERS//3X,1HJ,10X,4HB(J),27X,6HERORS//)
140 WRITE (6,140) (KJ,B(J),B1(J),J=1,3)
141 FORMAT (3X,12.5X,E16.6,15X,E16.6 //)
CALL PLOT(NG,NG,F,NF,FM)
115 WRITE (6,939) (FCI(KJ),KJ=1,N)
946 FORMAT (5H#N=1,5H#N=15,2H//)
116 WRITE (6,936) (FM(KJ),KJ=1,N)
117 NL=NL+1
120 WRITE (6,937) (FCI(KJ),KJ=1,N)
121 WRITE (6,938) (VM(KJ),KJ=1,N)
122 WSS=0.0
123 DO 835 IN=1,N
835 WS5=WSS+(1VMII)-DAT(I)*0.21/0DAT(I)
836 IND=2.*b11**0.3-9.*02**0.4/(b11**0.4)
PAGE=PAGE+1;
DO 710 NOME=1,9
710 FNO=FLOAT(NOME)
SPAX=(PAGE+1.)/10.
IF (SPACE-0NO)710,705,710
130 705 WRITE (7,549)
131 CONTINUE
WRITE (7,522) CASE,NME,RAT
WRITE (7,521) CASE,NME,RAT,01,02,03,
135 SIGMA
WRITE (7,520) CASE,NME,RAT,01,02,03,
502 FORMAT (1X,4X,13X,4X,7X,R(4),=F5.2,1H1)
504 FORMAT (7,503) (FCI(I)=IN)
505 FORMAT (4X,3HOUR,13F9.1)
506 WRITE (7,504) (FCI(I),I=1,N)
507 FORMAT (4X,3HEXP,8F9.1)
508 WRITE (7,505) (VMII(I),I=1,N)
509 FORMAT (4X,3HVM1,8F9.1)
510 WRITE (6,N6) WSS
515 FORMAT (15H WSS = ,F15.6)
935 FORMAT (15H FCI(I)=I=3,4,5)
936 FORMAT (15H FC1(I)=I=3,4,5)
937 FORMAT (15H FC1(I)=I=3,4,5)
938 FORMAT (15H VMII(I)=I=3,4,5)
150 READ (9,10) J0
160 FORMAT (11)
160 IF (J0-2) 1000,1001,1001
1002 STOP
1003 END
SUBROUTINE CURFIT

C SUBROUTINE CURFIT
F O R T R A N 4

DIMENSION Z(256), F(256), FC(256), DF(256), ERR(256), XI(256)

1 1, B(1), B(1), B(1), B(1), B(1), B(1), B(1), B(1), B(1), B(1)


3, N(4), M(4), L(4), K(4), J(4), I(4), H(4), G(4), F(4)

COMMON DATAPAC,F

COMMON DC

EQUIVALENCE (Z , DC(257)), (F, DC(251)), (XI , DC(256)),

1 (II, DC(182)), (B , DC(181)), (E , DC(1204)), (H , DC(1205)),

2 (L , DC(1203)), (J , DC(1204)), (M , DC(1799)),

3 (DC1799), (DC1799), (DC1799), (DC1799), (DC1799)

5 (1 , DC1799), (L , DC1799), (B , DC1799),

EQUIVALENCE (SR, DC1799)

DATA (ABC=2, N=3, YES=1, NO=0, 3, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

B55=0.0

L1 = 0

SA = 0.0

DO 1000 J=1, M

B1(J)=0.0

DO 1000 K=1, M

1000 DO 100 I = 1, N

X1(I) = ERR(I)**2

L=1

CALL FUNC2(I)

DF1 = F(I) - FC1(I)

30 DO 101 J=1, M

B1(J)=0.1*DF1*DF1/J

DO 101 K=1, M

101 B2(J, K)=B2(J, K)-(2.0*DF1*DF1*DF1*DF1*DF1*

100 SA=SA+DF1**2

GMD=0.0

DO 102 J=1, M

102 GMD=GMD+G1(J)**2

WRITE(6,1751)

243 FORMAT(1X,25H INITIAL VALUE SUM OF SQ.=

1L0=13.5)

WRITE(6,1751)

1751 FORMAT(1HDERIVATIVES=)

WRITE(6,1751)

240 FORMAT(15X,5E13.5,4X/)

IF (SA > 11) 110, 110, 200

110 LC=1

DO TO 600

200 S=0.0

GMD=0.0

BND=0.0

PR=0.0

A2=AB0(I)

DO 210 J = 1, M

B1(J)=0.0

DO 210 K = 1, M
SUBROUTINE CURFIT

210 B2(J,K) = 0.0
DO 220 I = 1, N
L1=1
CALL FUNC2
DF(I) = FM(I) - FC(I)
WRITE(6,1114)DF(I)
11140 FORMAT(4h DF(I)=,E17.7)
DO 220 J = 1, M
B1(J) = B1(J) - ((2.0*DF(I)*DL(J))/XI)
DO 220 K = 1, M
220 B2(J,K) = B2(J,K) - ((2.0*DF(I)*DL(J,K)))/X1)
WRITE(6,1111)0
DO 230 J = 1, M
230 GRAD(J) = B1(J)
L1 = L1 + 1
CALL EXAM (02,01,01,01)
WRITE(6,1111)
75 IF (L1) 250, 250, 305
250 DO 231 II=1,1
DO 231 JI=1,1
231 B3(I,J) = B2(I,J)
DO 1120 I=1,3
1120 WRITE(6,1112)B3(I,J),J=1,3
11121 FORMAT(4i12)
DO 1133 I=1, M
1133 WRITE(6,1113)B3(I,J),J=1,3
DO 235 I = 1, M
235 B1(I) = B3(I,I)
A2 = A2 + C2(I)
DO 250 J = 1, M
250 B3(J) = 0.0
DO 270 J = 1, M
270 B3(J) = B1(J)
DO 270 K = 1, M
270 B1(K) = B3(K) + B2(J,K) * GRAD(J)
DO 270 J = 1, M
WRITE(6,1114)B1(J)
1114 FORMAT(4h B1(J)=,E17.7)
IF (B1(J)) 240, 240, 265
240 B1(J) = B1(J)
265 B1(J) = B1(J) + B1(J)
WRITE(6,1111)
DO TO 275
275 CONTINUE
DO 275 J = 1, M
275 CONTINUE
DO 290 J = 1, M
290 B1(J) = 0.0
275 CONTINUE
DO 290 J = 1, M
300 B1(J) = B1(J) + B2(J,K) * D1(K)
305 DO 310 J=1, M
GMD = GMD + GRAD(J)**2
310 CONTINUE
320 CONTINUE
320 CONTINUE
SUBROUTINE CURFIT

310  BMOD = BMOD + B(I,J)**2

1110  FORMATION ACHECK, /E17.7/

1115  FORMATION GMOD,GMOD

120  IF (BMOD - Q2) 315, 315, 320

315  LE = 2

125  CALL FUNC (1)

410  GO TO 590

130  N = 1

440  GO TO 640

135  L3 = L3 + 1

140  IF (S - SI 435, 500, 500

145  B(I,J) = B(I,J)/2.0

150  L3 = L3 + 1

155  IF (L3 = 4) 450,460,460

160  LE = 5

165  WRITE (6,11111) LD,LO

C FOLLOWING CARDS ARE FOR ROTATIONAL LEVELS

WRITE (6,11111) LD,LO

GDC 6600 FORTRAN V3.0-P296 OPT=1 72/09/18. 19:42:07. PAGE 3
SUBROUTINE CURFIT

IF(N55) 22, 22, 22
23 I6=0
DO 20 I1=1, N
IF(N44) 14, 14, 19
170 CONTINUE
F1=FLOAT(I1)*2.
F1=SORT(F1)*(F1+1.1)-SORT(6.1)
GO TO 30
F1=FLOAT(I1)
F1=(2.*F1+1.)
F1=SORT(F1)
30 ETA=FIA
RS = (B(2)**3)/(4.*B(3)**3)-B(2)**2/(4.*B(3)**2)-ETA/(2.*B(3))
1
Q55 = (B(1)**2)*3/(4.*B(2)**2)-B(2)**2/(4.*B(3)**2)
Q9 = (Q55**2)/(4.)+Q55**3/(27.)
IF(FO) 10, 20, 20
10 DO 11 J=1, N
GRAD(J) = B1(J) + B1(J)
11 B1(J) = GRAD(J) - B1(J)
I6 = 1
GO TO 30
20 CONTINUE
IF(I6=1) 122, 12, 23
22 CONTINUE
C UPPER CARDS ARE FOR ROTATIONAL LEVELS
S4 = 0
WRITE(6, 11121) S4
11121 FORMAT(4H S4 = E17.7)
IF (S4 < 01) 507, 507, 530
507 LE = 1
WRITE(6, 11122) L4
11122 FORMAT(4H L4 = E14)
GO TO 600
530 IF (L4) 200, 200, 908
900 WRITE(6, 901) L4, I4, 5, GMOD, (B(J)) J=1, N
920 FORMAT(4H,15H, ITERATION NO = 15), 15, 43
TRANSFORMATION MADE TO PR
1NICIAL AXES = 4, 10, 10.
10K BINARY CHOP USED=13, 6H TIMES/14, 27H NO
2IGHTED SUM OF SQUARES = E14.7, 25H, 32H
3T = E14.7/20H PARAMETERS B1(J) = (EEE7.6)/
IF (L4) 200, 910, 910
910 LE = 6
GO TO 600
210 DO 710 J=1, M
B1(J) = B1(J)
710 B1(J) = B1(J)
L1 = 1
720 DO 720 I = 1, N
720 CALL FUNO(I)
WRITE(4, 575) WIEF, DELTA
575 FORMAT(4H, 15, 6, 5X, 15, 6)
I1 = I1
WRITE(4, 940) I1, OMEGA, TERN1, TERN2, TERN3
SUBROUTINE CURFIT

940 FORMAT (1X,K,12,7X,F15.6,7X,F15.6,7X,F15.6,7X,F15.6,7X,F15.6)
945 D(I) = FM(I) - FC(I)
950 DO 720 J = 1, M
955 D(I) = D(I) - (DF(I)*D(I)*DI(J))/DI(I)
960 DO 720 K = 1, M
965 B2(J,K) = B2(J,K) - (DF(I)*D2(J,K) - DI(J)*DI(K))/DI(I)
970 CALL MATINV (B2, M, B1, 1, DETERM)
975 DO 730 J = 1, M
985 2001 B1(J) = SQRT (-B2(J,J))
990 2002 B1(J) = SORT (B2(J,J))
995 730 CONTINUE

740 GO TO 730

745 B2(J,K) = B2(J,K) / (B1(J) * B1(K))
750 WRITE (6, 551) I, E, SA
551 FORMAT (1AH, 12H EXIT NUMBER=I3, 20H, 25H WEIGHTED SUM OF SQUARES=F15.8

17/)

240 N1 = N+1
245 N5 = N+5
250 GO 39 1=N1, N5
255 CALL FUNC
260 39 CONTINUE
265 RETURN
END
SURROUN D) F U NC 5
SURROUN D) F U NC
DIMENSION DC(243),Q(3),O(I3),DZ(3),Z(256),E(6)
DIMENSION R(4,4),Q(4,4),P(4,4),FAD(4,4),W(4,4),EE(3)
1EEN(3),N(4,4),EMDE(3),RAD(4,4),E(3)
COMMON DC
EQUIVALENCE (2,DC),(4,DC(255)),(6,DC(279)),(10,DC(1799)),
1002,DC(1799)),(4,DC(16)),(1,DC(1249)),
20(4,4,DC(262)),
30(TERM1,DC(1707)),(TERM2,DC(1708)),(TERM3,DC(1709)),(OMEGA),DC(1710)
41,(NIEF,DC(1719)),(DELTA),DC(1710)
5070...11=10
60I(I)=0.0
70...12=1.0
80IF(U04<78,79
90CONTINUE
100IF(X<0.0,91,92)
110IF(X>0.0,93,94)
120CONTINUE
130ETA=1
140R(L,1)=-(B(2)**3)/(4.*B(3)**3)-B(1)*E(2)/(4.*B(3)**2)-ETA/(2.*P)
150R(L,1)=-(B(2)**3)/(4.*B(3)**3)
160R(L,2)=(3.-E(2)**2)/(4.+(E(3)**2))-(3.-E(3)**2)/(4.+(E(2)**2))+B(1)*E(2)/(4.*B(3)**2)+ETA
170R(L,2)=(B(2)**3)/(4.*B(3)**3)
180R(L,3)=0.0
190R(L,1)=1.0-(4.*B(3)**2)
200R(L,2)=1.0-(4.*B(3)**2)
210R(L,3)=1.0-(4.*B(3)**2)
220R(L,2)=1.0-(4.*B(3)**2)
230R(L,3)=1.0-(4.*B(3)**2)
240R(L,2)=1.0-(4.*B(3)**2)
250R(L,3)=1.0-(4.*B(3)**2)
260R(L,2)=1.0-(4.*B(3)**2)
270R(L,3)=1.0-(4.*B(3)**2)
280R(L,2)=1.0-(4.*B(3)**2)
290R(L,3)=1.0-(4.*B(3)**2)
300R(L,2)=1.0-(4.*B(3)**2)
310R(L,3)=1.0-(4.*B(3)**2)
320R(L,2)=1.0-(4.*B(3)**2)
330R(L,3)=1.0-(4.*B(3)**2)
340R(L,2)=1.0-(4.*B(3)**2)
350R(L,3)=1.0-(4.*B(3)**2)
360R(L,2)=1.0-(4.*B(3)**2)
370R(L,3)=1.0-(4.*B(3)**2)
380R(L,2)=1.0-(4.*B(3)**2)
390R(L,3)=1.0-(4.*B(3)**2)
400R(L,2)=1.0-(4.*B(3)**2)
410R(L,3)=1.0-(4.*B(3)**2)
420R(L,2)=1.0-(4.*B(3)**2)
430R(L,3)=1.0-(4.*B(3)**2)
440R(L,2)=1.0-(4.*B(3)**2)
450R(L,3)=1.0-(4.*B(3)**2)
460R(L,2)=1.0-(4.*B(3)**2)
470R(L,3)=1.0-(4.*B(3)**2)
480R(L,2)=1.0-(4.*B(3)**2)
490R(L,3)=1.0-(4.*B(3)**2)
500R(L,2)=1.0-(4.*B(3)**2)
510R(L,3)=1.0-(4.*B(3)**2)
520R(L,2)=1.0-(4.*B(3)**2)
530R(L,3)=1.0-(4.*B(3)**2)
540R(L,2)=1.0-(4.*B(3)**2)
550R(L,3)=1.0-(4.*B(3)**2)
560R(L,2)=1.0-(4.*B(3)**2)
570R(L,3)=1.0-(4.*B(3)**2)
580R(L,2)=1.0-(4.*B(3)**2)
590R(L,3)=1.0-(4.*B(3)**2)
600R(L,2)=1.0-(4.*B(3)**2)
610R(L,3)=1.0-(4.*B(3)**2)
620R(L,2)=1.0-(4.*B(3)**2)
630R(L,3)=1.0-(4.*B(3)**2)
640R(L,2)=1.0-(4.*B(3)**2)
650R(L,3)=1.0-(4.*B(3)**2)
660R(L,2)=1.0-(4.*B(3)**2)
670R(L,3)=1.0-(4.*B(3)**2)
680R(L,2)=1.0-(4.*B(3)**2)
690R(L,3)=1.0-(4.*B(3)**2)
700R(L,2)=1.0-(4.*B(3)**2)
710R(L,3)=1.0-(4.*B(3)**2)
720R(L,2)=1.0-(4.*B(3)**2)
730R(L,3)=1.0-(4.*B(3)**2)
740R(L,2)=1.0-(4.*B(3)**2)
750R(L,3)=1.0-(4.*B(3)**2)
760R(L,2)=1.0-(4.*B(3)**2)
770R(L,3)=1.0-(4.*B(3)**2)
780R(L,2)=1.0-(4.*B(3)**2)
790R(L,3)=1.0-(4.*B(3)**2)
800R(L,2)=1.0-(4.*B(3)**2)
810R(L,3)=1.0-(4.*B(3)**2)
820R(L,2)=1.0-(4.*B(3)**2)
830R(L,3)=1.0-(4.*B(3)**2)
840R(L,2)=1.0-(4.*B(3)**2)
850R(L,3)=1.0-(4.*B(3)**2)
860R(L,2)=1.0-(4.*B(3)**2)
SUBROUTINE FUNC

RAD(4,4) = (R(4,4)**2) / 4.0 + Q(4,4)**3) / 27.0
DO 130 J = 1, M
  130 RAD(4, J) = R(4, J) / 2.0 + Q(4, J)**2) * Q(4, J) / 9.0
DO 131 K = 1, M
  131 RAD(K, J) = R(K, J) / 2.0 + Q(K, J)**2) * Q(K, J) / 9.0
DO 132 K = 1, M
  132 RAD(K, J) = RAD(4, J) * RAD(K, J)
DO 133 J = 1, M
  133 RAD(K, J) = RAD(4, J)**2
END

SUBROUTINE FAD

PAD(4,4) = (R(4,4)**2) / 2.0 + RAD(4,4)
DO 1134 J = 1, M
  1134 PAD(4, J) = RAD(4, J)**2 + RAD(4, J)
DO 1134 J = 1, M
  1134 FAD(4, J) = RAD(4, J)**2 + RAD(4, J)
DO 1134 J = 1, M
  1134 FAD(4, J) = RAD(4, J)**2 + RAD(4, J)

FAD(4,4) = (R(4,4)**2) / 2.0 + RAD(4,4)
A = 1.0 / 3.0
A = A / 3.0
A = A / 2.0
IF (FAD(4,4) <= 1.0) THEN
  1 PS = 1.0
  GO TO 4
  2 PS = 1.0
  4 CONTINUE
  IF (FAD(4,4) > 5.0) THEN
  5 FS = 1.0
  GO TO 7
  6 FS = 1.0
  7 CONTINUE
W(4,4) = -2.0 * (2.0 % 3.0) * PS * (PS * PAD(4,4))**2) - FS * (FS * FAD(4,4))**2
FC(1) = B(1) * (W(4,4)**2) / 2.0 + B(2) * (W(4,4)**3) + B(3) * (W(4,4)**4)
1/2.
OMEGA = W(4,4)
TERM1 = B(1) * (W(4,4)**2) / 2.0
TERM2 = B(2) * (W(4,4)**3)
TERM3 = (B(3) * (W(4,4)**4)) / 2.0
MIEF = R(1) + R(2) + (W(4,4)**2) * (W(4,4)**3) + (W(4,4)**4)
DELTA = MIEF / R(1)
GO TO (105, 110, 120)
105 CONTINUE
EE11 = (W(4,4)**2) / 2.0
EE12 = (W(4,4)**3)
EE33 = (W(4,4)**4)
EM = (R(1) * (W(4,4)**2) + 0.0 * (R(2) * (W(4,4)**3) + 0.0 * (R(3) * (W(4,4)**4)))
EM = (R(3) * (W(4,4)**2) + 0.0 * (R(2) * (W(4,4)**3) + 0.0 * (R(3) * (W(4,4)**4)))
110 CONTINUE
SUBROUTINE FUCN

END
SUBROUTINE EXAM(A,B,M,LF)
SUBROUTINE EXAM
      F O R T R A N 4
      D I M E N S I O N A ( 3 , 3 ) , B ( 3 ) , C ( 3 )
      D O  40  J = 1 , M
        C ( J , J ) = A ( J , J )
        I F ( A ( 1 , 1 ) ) 60 , 200 , 70
        60  A ( 1 , 1 ) = S O R T ( - A ( 1 , 1 ) )
      G O T O  300
    70  A ( 1 , 1 ) = S O R T ( A ( 1 , 1 ) )
    G O T O  100
    100  I F ( M ) . E Q . 400 , 400 , 110
    110  D O  115  K = 2 , M
      115  A ( 1 , K ) = A ( 1 , K ) / ( A ( 1 , 1 ) )
    D O  120  J = 2 , M
      J = J + 1
      S = A ( J , J )
    D O  125  L = 1 , J
      125  S = S - A ( L , J ) * * 2
      I F ( S ) . G T . 50.200 . 40
      90  A ( J , J ) = S O R T ( - S )
    G O T O  300
    40  A ( J , J ) = S O R T ( S )
    G O T O  130
    130  I F ( J ) . E Q . 35 , 400 , 400
      135  J = J + 1
    D O  126  K = J , M
      145  S = S - A ( L , K ) * * 2
      I F ( S ) . G T . 50.200 . 40
      120  A ( J , K ) = S O R T ( - S )
    400  B ( 1 ) = B ( 1 ) / A ( 1 , 1 )
      I F ( H ) . G T . 328 , 420 , 405
    325  D O  410  J = 2 , M
      410  S = B ( J )
      J = J - 1
    D O  415  L = 1 , J + 1
      415  S = S - A ( L , J ) * * 2
      I F ( S ) . G T . 50.200 . 40
      410  B ( J ) = S O R T ( - S )
    420  H ( N ) = H ( M ) / A ( M , M )
      J = J - 1
    425  I F ( J ) . L T . 450 , 450 , 425
    425  S = B ( J )
    45  J = J + 1
    D O  430  L = J + 2 , M
      430  S = S - A ( J , L ) * * 2
      I F ( S ) . G T . 50.200 . 40
      410  H ( J ) = S O R T ( - S )
      J = J - 1
    G O T O  435
    435  L F = 1
    G O T O  460
    200  L F = 0
    G O T O  460
    300  L F = - 1
SUBROUTINE EXAM

460 DO 465 J=1,M
465 A(J,J)=C(J)
470 IF(J=M)470,475,475
470 J=J+1
465 DO 465 K=J,M
465 A(J,K)=A(K,J)
475 RETURN
END
SUBROUTINE MATINV(A,N,B,M,DETERM)
SUBROUTINE MATINV

MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR EQUATIONS
DIMENSION IPIVOT(3),AR(3,3),B(3,1),INDEX(3,3),PIVOT(3)
EQUIVALENCE (IRON,JRON),(ICOLUM,JOCLUM),(IAMA,T,SHAP)

DETERM=1.0
DO 20 JT=1,N
   DO 10 J=1,N
      AMXX=0.0
      DO 10 K=1,N
         IF(PIVOT(J,J).NE.K) GO TO 120
         IF(PIVOT(J,J).GT.1.0E10) GO TO 130
         IF(PIVOT(J,J).LT.-1.0E10) GO TO 130
         AMXX=AMXX+ABS(A(J,J))
      10     CONTINUE
   20    CONTINUE

IF (IRON=ICOLUM) IPIVOT(ICOLUM)=JT
IF (IRON.IE.ICOLUM) DO 140 L=1,N
      DETERM=DETERM/A(IRON,L)
      A(IRON,L)=A(IRON,L)/A(ICOLUM,L)
   140 CONTINUE

IF (IRON.LE.ICOLUM) DO 200 L=1,N
      IF(A(L,L).EQ.0.0) GO TO 220
      SWAP=A(L,IRON)
      A(L,IRON)=A(IRON,L)
      A(IRON,L)=SWAP
   200 CONTINUE

IF (IRON.GE.ICOLUM) DO 210 L=1,N
      IF(A(L,L).EQ.0.0) GO TO 220
      SWAP=A(L,IRON)
      A(L,IRON)=A(IRON,L)
      A(IRON,L)=SWAP
   210 CONTINUE

DO 250 L=1,M
      IF (INDEX(I,L).EQ.0.0) GO TO 250
      INDEX(I,L)=INDEX(I,L)/A(ICOLUM,L)
      PIVOT(I,J)= INDEX(I,J)/INDEX(I,L)
   250 CONTINUE

IF (IRON.LE.ICOLUM) DO 350 L=1,N
      IF(A(L,L).EQ.0.0) GO TO 350
      INDEX(I,J)=INDEX(I,J)/A(L,L)
      IF (INDEX(I,J).EQ.0.0) GO TO 350
   350 CONTINUE

IF (IRON.GE.ICOLUM) DO 370 L=1,N
      IF(A(L,L).EQ.0.0) GO TO 370
      INDEX(I,J)=INDEX(I,J)/A(L,L)
   370 CONTINUE

DO 400 L=1,N
   IF (L.E.ICOLUM) GO TO 400
   DETERM=DETERM/A(L,L)
   ALL(I,ICOLUM)=0.0
   DO 450 L=1,N
      IF (L.LT.ICOLUM) GO TO 450
      INDEX(I,L)=INDEX(I,L)-A(I,ICOLUM)*A(L,L)
      IF (INDEX(I,L).EQ.0.0) GO TO 450
   450 CONTINUE
   400 CONTINUE

DO 500 L=1,N
   DETERM=DETERM*A(L,L)
   ALL(I,ICOLUM)=ALL(I,ICOLUM)-A(I,ICOLUM)*A(L,L)
   IF (ALL(I,ICOLUM).GT.0.0) GO TO 500
   IF (ALL(I,ICOLUM).LT.0.0) GO TO 500
   INDEX(I,L)=INDEX(I,L)/A(L,L)
   ALL(I,ICOLUM)=0.0
   500 CONTINUE

DO 55 I=1,N
   DETERM=DETERM/A(I,I)
   ALL(I,ICOLUM)=ALL(I,ICOLUM)-A(I,ICOLUM)*A(I,L)
   IF (ALL(I,ICOLUM).GT.0.0) GO TO 55
   IF (ALL(I,ICOLUM).LT.0.0) GO TO 55
   INDEX(I,I)=INDEX(I,I)/A(I,I)
   ALL(I,ICOLUM)=0.0
55 CONTINUE
SUBROUTINE MATINV

L=N+1-I
IF(INDEX(L,1)-INDEX(L,2)) L=30,710,630

DO 710 K=1,N
SWAP=A(K,JROM)
A(K,JCOLUM)=A(K,JROM)
A(K,JROM)=SWAP

CONTINUE

RETURN
END
SUBROUTINE PLOTB (NO, N, AA, BB)

C SUBROUTINE PLOT B
C A = LARGEST OF FC AND FH, B = SMALLEST
5 DIMENSION X(116), AA(12), BB(12), W(4), AB(2)
COMMON/DATA/ABC, N
DATA(AAC=+100.0, M=10, HH, 1HH, 1HH)
A=AA(1)
B=AA(2)
10 DO 990 I=1,N
   IF (AA(I)-A) 905, 905, 910
910 A=AA(I)
905 IF (BB(I)-A) 915, 915, 920
920 B=BB(I)
15 IF (AA(I)-A) 930, 925, 925
930 B=AA(I)
925 IF (AB(I)-B) 935, 900, 900
935 BB(I)
900 CONTINUE
20 FACTOR = 1.0
520 IF ((A=B)-1000.0) 500, 510, 510
500 A = 2.0*A
   B = 2.0*B
   FACTOR = 2.0*FACTOR
   GO TO 520
25 K0 = (A+N/112.0 + 1.0
   KS = IFIX(K0) = 2*K0
   WRITE (6,1)
   1 FORMAT(119H0...1...2...3...4...5...6...7...8...9...10...11....)
   15 DO 100 I=1,N
   DO 110 K=1,116
   110 X(K)=W(I)
      K = AA(I)*FACTOR
      L = BB(I)*FACTOR
   35 X(K) = (K-KS)/K0
   40 IF (L-K) 120, 130, 120
   120 X(L)=W(I)
      K = AA(I)*FACTOR
   100 WRITE (6,90) INO, X
   90 FORMAT(119H0...1...2...3...4...5...6...7...8...9...10...11....)
   WRITE (6,1)
   RETURN
END
SUBROUTINE JACOBI
C
SUBPROGRAM FOR DIAGONALIZATION OF MATRIX Q BY SUCCESSIVE ROTATIONS
DIMENSION Q(3,3),X(3,3),IM(3)
C
NEXT 6 STATEMENTS FOR SETTING INITIAL VALUES OF MATRIX V
10 IF(J(J))<10,15,10
10 I=1,N
I=J+1,N
I=J,11,12
11 X(I,J)=0.
12 CONTINUE
14 CONTINUE
15
NEXT 8 STATEMENTS SCAN FOR LARGEST OFF DIAG. ELEM. IN EACH ROW
X(I) CONTAINS LARGEST ELEMENT IN ITH ROW
IM(I) HOLDS SECOND SUBSCRIPT DEFINING POSITION OF ELEMENT
C
M=N-1
DO 30 I=1,N
M=I+1
DO 30 J=M,N
IF (X(I, J)=ABS (Q(I, J))) 20,20,30
20 X(I, J)=ABS (Q(I, J))
IM(I)=J
30 CONTINUE
C
NEXT 7 STATEMENTS FIND FOR MAXIMUM OF X(I)S FOR PIVOT ELEMENT
40 IF (X(MAX)=X(I)) 60,60,45
45 IF (X(MAX)=X(I)) 60,70,70
50 X(MAX)=X(I)
IM(I)=I
JP=IM(I)
70 CONTINUE
C
NEXT 2 STATEMENTS TEST FOR XMAX, IF LESS THAN 10**-3, GO TO 1000
EPSI=1.E-8
IF (X(MAX)<EPSI) 1000,1000,145
C
145 M=M+1
C
NEXT 11 STATEMENTS FOR COMPUTING TANG,SINE,COSINE,Q(I,I),Q(J,J)
C
IF (Q(I,I)=Q(J,J)) 150,151,151
150 TANG =-2.*Q(I,J)/(ABS(Q(I,I)-Q(J,J))+SORT((Q(I,I)-Q(J,J))*Q(I,J,J))**2))
131***(Q(I,J,J)**2))
GO TO 160
151 TANG =2.*Q(I,J)/(ABS(Q(I,I)-Q(J,J))+SORT((Q(I,I)-Q(J,J))*Q(I,J,J))**2))
131***(Q(I,J,J)**2))
GO TO 160
C
160 CONTINUE
SUBROUTINE JACOBI

SINE = TANGL*COSN
Q11 = GIIP(IP)
Q1P(IP) = COSN*Q11*TANG*(2.*GII(IP,JP)+TANG*(Q1IP,JP))
Q1P(IP,JP) = COSN*Q11*TANG*(2.*GII(IP,JP)+TANG*(Q1I))

60 C Q1P(IP,JP) = 0.

CC NEXT 4 STATEMENTS FOR PSEUDO RANK OF THE EIGENVALUES
IF (Q1P(IP,JP) = Q(IP,JP)) 152, 153, 159
152 TEMP = GII(IP,JP)
Q1P(IP,JP) = GII(IP,JP)
Q(IP,JP) = TEMP

70 CC NEXT 6 STATEMENTS ADJUST SINE,COS FOR COMPUTATION OF Q(I,K),Q(J,K)
IF (SINE) 154, 155, 156
154 TEMP = COSN
GO TO 170
155 TEMP = -COSN
156 SINE = TEMP

76 CC NEXT 10 STATEMENTS FOR INSPECTING THE I'S BETWEEN I+1 AND N+1 TO
Determine whether a new maximum value should be computed since
THE PRESENT MAXIMUM IS IN THE I OR J ROW

80 DO 350 M = 1, N
IF (I(IP) = 210, 350, 200)
200 IF (I(IP) = 210, 350, 210)
210 IF (I(IP) = 230, 240, 230)
230 IF (I(IP) = 230, 240, 350)
240 XII = TANG*(M)
TEMP = GII(I,K)
Q1(K) = 0.
M = I(IP)
XII = 0.

90 CC NEXT 5 STATEMENTS SEARCH IN DEPLETED ROW FOR NEW MAXIMUM

95 DO 320 J = M + 1, N
IF (XII(J) = TANG*(M)) 300, 300, 320.
300 XII = TANG*(J)
320 CONTINUE

350 CONTINUE

C X1P(IP) = 0.
X(IP,JP) = 0.

105 CC NEXT 30 STATEMENTS FOR CHANGING THE OTHER ELEMENTS OF Q

106 DO 530 M = 1, N
110 IF (I(IP) = 370, 530, 420)
SUBROUTINE JACOBI

370 TEMP = QI(IP)
375 QI(IP) = COSN*TEMP + SINE*Q(I,JP)
380 IF ( I(IP) = ABS(QI(IP))) 380, 390, 390
390 XI(I) = ABS(QI(IP))
400 IM(I) = JP
405 GO TO 530

420 IF ( I(IP) = JP) 430, 530, 480
430 TEMP = Q(IP, I)
435 Q(IP, I) = COSN*TEMP + SINE*Q(I,JP)
440 IF ( I(IP) = ABS(QI(IP, I))) 440, 450, 490
450 XI(IP) = I
455 Q(IP, I) = SINE*TEMP + COSN*Q(I,JP)
460 IF ( I(IP) = ABS(QI(IP, I))) 460, 530, 530

480 TEMP = Q(IP, I)
485 Q(IP, I) = COSN*TEMP + SINE*Q(IP, I)
490 IF ( I(IP) = ABS(QI(IP, I))) 490, 500, 500
495 IM(IP) = I
500 Q(IP, I) = SINE*TEMP + COSN*Q(IP, I)
510 IF ( I(IP) = ABS(QI(IP, I))) 510, 530, 530
515 XI(IP) = ABS(QI(IP, I))
520 IM(IP) = I
530 CONTINUE

C C C
C NEXT 6 STATEMENTS TEST FOR COMPUTATION OF EIGENVECTORS
C
540 IF ( IUCEC ) 540, 40, 540
545 DO 550 I = 1, N
550 IF ( I(IP) = 1, N)
555 XI(IP) = COSN*TEMP + SINE*V(I, JP)
560 XI(IP) = SINE*TEMP + COSN*V(I, JP)
565 GO TO 40
570 RETURN
580 END
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