INTRODUCING A DEPENDENCE STRUCTURE TO THE OCCURRENCES IN STUDYING PRECISE LARGE DEVIATIONS FOR THE TOTAL CLAIM AMOUNT

Rob Kaas and Qihe Tang
Introducing a Dependence Structure to the Occurrences in Studying Precise Large Deviations for the Total Claim Amount

Rob Kaas\textsuperscript{a}, Qihe Tang\textsuperscript{a,b} *

\textsuperscript{a} Department of Quantitative Economics
University of Amsterdam
Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

\textsuperscript{b} Department of Mathematics and Statistics
Concordia University
7141 Sherbrooke Street West, Montreal, Quebec, H4B 1R6, Canada

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Abstract

In this paper we study precise large deviations for a compound sum of claims, in which the claims arrive in groups and the claim numbers in the groups may follow a certain negative dependence structure. We try to build a platform both for the classical large deviation theory and for the modern stochastic ordering theory.

Keywords: Consistent variation; Matuszewska index; Negative cumulative dependence; Precise large deviations; Random sums; Stop-loss order.

1 Introduction

Inspired by the recent works of Cline and Hsing (1991), Klüppelberg and Mikosch (1997), Tang et al. (2001), and Ng et al. (2004), in the present paper we are interested in precise large deviations for the random sum

\[ S_t = \sum_{k=1}^{N_t} X_k, \quad t \geq 0. \] (1.1)

Here \( \{X_k, k = 1, 2, \ldots\} \) is a sequence of independent, identically distributed (i.i.d.), and nonnegative heavy-tailed random variables, representing the sizes of successive claims, with

\*Corresponding author. E-mails: r.kaas@uva.nl (R. Kaas), qtang@mathstat.concordia.ca (Q. Tang). Tel.: 1-514-848 2424x5219; fax: 1-514-848 2831.
common distribution function $F = 1 - \overline{F}$ and finite mean $\mu > 0$; $N_t$, $t \geq 0$, is a nonnegative, nondecreasing, and integer-valued process, representing the number of claims by time $t$, with a mean function $\lambda_t = E [N_t] < \infty$ for each $t \geq 0$ and $\lambda_t \to \infty$ as $t \to \infty$; and, as usual, $\sum_{k \in \emptyset} (\cdot) = 0$ by convention. The process $\{N_t, t \geq 0\}$ and the sequence $\{X_k, k = 1, 2, \ldots\}$ are assumed to be mutually independent. Our goal is to establish a precise large deviation result that for any fixed $\gamma > 0$, the relation
\[ \Pr (S_t - \mu \lambda_t > x) \sim \lambda_t F(x), \quad t \to \infty, \] (1.2)
holds uniformly for $x \geq \gamma \lambda_t$. Hereafter, all limit relationships are for $t \to \infty$ unless stated otherwise. The uniformity of relation (1.2) means
\[ \lim_{t \to \infty} \sup_{x \geq \gamma \lambda_t} \left| \frac{\Pr (S_t - \mu \lambda_t > x)}{\lambda_t F(x)} - 1 \right| = 0. \]
This is crucial for our purpose.

As an application of (1.2), we consider the calculation of the stop-loss premium of the random sum $S_t$ with large retention $d$, say $d = d(t) \geq (\mu + \gamma) \lambda_t$. Write $x_+ = \max\{x, 0\}$. As $t$ increases, applying the uniformity of the asymptotic relation (1.2) we have
\[ E \left[ (S_t - d)_+ \right] = \int_d^\infty \Pr (S_t > x) \, dx = \int_d^\infty \Pr (S_t - \mu \lambda_t > x - \mu \lambda_t) \, dx \sim \lambda_t \int_d^\infty \overline{F}(x - \mu \lambda_t) \, dx = \lambda_t E \left[ (X_1 + \mu \lambda_t - d)_+ \right]. \]
Clearly, the calculation of the right-hand side of the above is much simpler than the calculation of the stop-loss premium of $S_t$ itself. For further applications of precise large deviations to insurance and finance, we refer the reader to Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998), and Embrechts et al. (1997, Chapter 8), among many others.

In this paper we shall consider the following special case of the random sum (1.1), in which the claim arrivals follow a compound renewal counting process:

1. the arrival times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots$ constitute a renewal counting process
\[ \tau_t = \sum_{k=1}^\infty 1_{(\sigma_k \leq t)}, \quad t \geq 0, \] (1.3)
where $1_A$ denotes the indicator function of a set $A$ and the interarrival times $\sigma_k - \sigma_{k-1}$, $k = 1, 2, \ldots$, are i.i.d. with $E [\sigma_1] < \infty$.
2. at each arrival time $\sigma_k$, a group of $Z_k$ claims arrives, and $Z_k$, $k = 1, 2, \ldots$, constitute a sequence of nonnegative, integer-valued, and identically distributed random variables, that may be independent, but can also follow a certain dependence structure;

3. the number of claims by time $t$ is therefore a process

$$N_t = \sum_{k=1}^{\infty} Z_k 1(\sigma_k \leq t) = \sum_{k=1}^{\tau_1} Z_k, \quad t \geq 0;$$

(1.4)

4. the sequences $\{X_k, k = 1, 2, \ldots\}$ and $\{Z_k, k = 1, 2, \ldots\}$ and the process $\{\tau_t, t \geq 0\}$ are mutually independent.

Clearly, if each $Z_k$ is degenerate at 1, the model above reduces to the ordinary renewal model. Generally speaking, however, it describes a nonstandard risk model since the random sum (1.1) is equal to

$$S_t = \sum_{k=1}^{Z_1} X_k + \sum_{k=Z_1+1}^{Z_1+Z_2} X_k + \cdots + \sum_{k=Z_1+\cdots+Z_{\tau_1}-1+1}^{Z_1+\cdots+Z_{\tau_1}} X_k = A_1 + A_2 + \cdots + A_{\tau_1},$$

where $\{A_n, n = 1, 2, \ldots\}$, though independent of $\{\tau_t, t \geq 0\}$, is no longer a sequence of i.i.d. random variables.

A reference related to the present model is Denuit et al. (2002), who understood the random variable $Z_k$ above as the occurrence of the $k$th claim, hence as a Bernoulli variate $I_k$, $k = 1, 2, \ldots$, and who assumed that the sequence $\{I_k, k = 1, 2, \ldots\}$ follows a certain dependence structure. In this way, the random sum (1.1) is equal to

$$S_t = \sum_{k=1}^{\tau_1} X_k I_k.$$

See also Ng et al. (2004, Section 5.1).

The remaining part of the paper is organized as follows. Section 2 recalls some preliminaries about heavy-tailed distributions; Section 3 establishes the precise large deviations for a standard case where the claim numbers in groups, $Z_k$, $k = 1, 2, \ldots$, are independent; and, after introducing a kind of negative dependence structure, Section 4 further extends the result to a nonstandard case where the claims numbers $Z_k$, $k = 1, 2, \ldots$, follow this dependence structure.

2 Distributions of consistent variation

In this paper, we will assume that the common claim size distribution $F$ is heavy tailed. More precisely, we assume that $F$ has a consistent variation, denoted by $F \in C$. By definition, a
distribution function $F$ belongs to the class $C$ if and only if $F(x) > 0$ for all real numbers $x$ and, moreover,
\[
\lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{F(xy)}{F(x)} = 1. \tag{2.1}
\]
Discussions and applications of this class can be found, for example, in Cline (1994), Schlegel (1998), Jelenković and Lazar (1999, Section 4.3), Ng et al. (2004), and Tang (2004).

Specifically, the class $C$ covers the famous class $\mathcal{R}$, which consists of all distribution functions with regularly varying tails in the sense that there is some $\alpha > 0$ such that the relation
\[
\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha} \tag{2.2}
\]
holds for any $y > 0$. We denote by $F \in \mathcal{R}_{-\alpha}$ the regularity property in (2.2).

A simple example to illustrate that the inclusion $\mathcal{R} \subset C$ is strict is the distribution function of the random variable
\[
X = (1 + U)2^N,
\]
where $U$ and $N$ are independent random variables with $U$ uniformly distributed on $(0, 1)$ and $N$ geometrically distributed satisfying $\Pr(N = n) = (1 - p)p^n$ for $0 < p < 1$ and $n = 0, 1, \cdots$; see Kaas et al. (2004) and Cai and Tang (2004).

For a distribution function $F$ and any $y > 0$, as done recently by Tang and Tsitsiashvili (2003), we set
\[
\bar{F}_*(y) = \liminf_{x \to \infty} \frac{F(xy)}{F(x)}
\]
and then define
\[
J_+^F = -\lim_{y \to \infty} \frac{\log \bar{F}_*(y)}{\log y}. \tag{2.3}
\]
We call $J_+^F$ the upper Matuszewska index of the distribution function $F$. For more details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1), Cline and Samorodnitsky (1994), as well as Tang and Tsitsiashvili (2003). Clearly, if $F \in C$ then $J_+^F < \infty$ and if $F \in \mathcal{R}_{-\alpha}$ then $J_+^F = \alpha$.

Tang and Tsitsiashvili (2003, Lemma 3.5) proved the following result:

**Lemma 2.1.** For a distribution function $F$ with upper Matuszewska index $J_+^F < \infty$, it holds for any $p > J_+^F$ that $x^{-p} = o(F(x))$.

By this lemma it is easy to see that if $F(x)1_{(0 \leq x < \infty)}$ has a finite mean then $J_+^F \geq 1$. 4
3 A standard case with independent occurrences

3.1 The first main result

Ng et al. (2004) investigated the random sum (1.1) and obtained the following general result:

**Proposition 3.1.** Consider the random sum (1.1). If $F \in C$ and $N_t$ satisfies
\[
E \left[ N_t^p 1_{(N_t > \eta \lambda_t)} \right] = O(\lambda_t) \tag{3.1}
\]
for some $p > J_F^+$ and all $\eta > 1$, then for any fixed $\gamma > 0$, the precise large deviation result (1.2) holds uniformly for $x \geq \gamma \lambda_t$.

Now we state the first main result of this paper.

**Theorem 3.1.** Consider the compound model introduced in Section 1. In addition to the assumptions made there, we assume that the claim numbers $Z_k$, $k = 1, 2, \ldots,$ are independent. If the claim size distribution $F \in C$ and
\[
E \left[ Z_1^p \right] < \infty \tag{3.2}
\]
for some $p > J_F^+$, then for any fixed $\gamma > 0$, the precise large deviation result (1.2) holds uniformly for $x \geq \gamma \lambda_t$.

As can easily be seen, Theorem 3.1 above improves Theorems 2.3 and 2.4 of Tang et al. (2001) in several directions.

3.2 Two lemmas

Before giving the proof of Theorem 3.1, we need some preliminaries. First we show, in the spirit of Fuk and Nagaev (1971) (see also Nagaev (1976) for additional erratum and extension), a general inequality for the tail probability of sums of i.i.d. random variables.

**Lemma 3.1.** Let $\{Z_k, k = 1, 2, \ldots\}$ be a sequence of i.i.d. nonnegative random variables with $E[Z_1^p] < \infty$ for some $p \geq 1$. Then for any $\gamma > E[Z_1]$, there is some $C > 0$ irrespective to $x$ and $m$ such that for all $m = 1, 2, \ldots$ and $x \geq \gamma m$,
\[
Pr \left( \sum_{k=1}^{m} Z_k > x \right) \leq C m x^{-p}. \tag{3.3}
\]

**Proof.** For the case $p = 1$, (3.3) is a trivial consequence of Chebyshev’s inequality. Then, we assume $p > 1$ and define $\tilde{p} = \min \{p, 2\}$. With an arbitrarily fixed constant $v > 0$, by Theorem 2 of Fuk and Nagaev (1971) we obtain
\[
Pr \left( \sum_{k=1}^{m} Z_k > x \right) \leq m Pr (Z_1 > x/v) + P_v(x) \tag{3.4}
\]
with
\[ P_v(x) = \exp \left\{ v - \frac{x - mE[Z_1 \mathbb{1}_{Z_1 \leq x/v}]}{x/v} \log \left( \frac{x(x/v)^{\bar{p} - 1}}{mE[Z_1 \mathbb{1}_{Z_1 \leq x/v}]} + 1 \right) \right\}. \]

Since \( x \geq \gamma m \) and \( \gamma > E[Z_1] \), some simple calculation leads to
\[ P_v(x) \leq e^v \left( \frac{\gamma}{v^{p-1}E[Z_1]} \right)^{-v(1-E[Z_1]/\gamma)} x^{-(\bar{p}-1)v(1-E[Z_1]/\gamma)}. \]

It follows that for all large \( v \), say \( v \geq v_0 > 0 \),
\[ P_v(x) = o \left( x^{-p} \right). \]

For the first term on the right-hand side of (3.4), by Chebyshev’s inequality, it holds that
\[ m \Pr(Z_1 > x/v) \leq (x/v)^{-p}mE[Z_1^p]. \]

This proves that inequality (3.3) holds for some constant \( C > 0 \).

For any fixed \( t > 0 \), the random variable \( \tau_t \) defined by (1.3) has certain finite exponential moments, hence has finite moments of all orders; see Stein (1946). We reformulate Lemma 3.5 of Tang et al. (2001) below.

**Lemma 3.2.** Let \( \{\tau_t, t \geq 0\} \) be a renewal counting process defined by (1.3). Then for any \( p > 0 \) and \( \eta > 1 \),
\[ \sum_{m > \eta E[\tau_t]} m^p \Pr(\tau_t \geq m) = o(1). \]

### 3.3 Proof of Theorem 3.1

In view of Proposition 3.1, it suffices to prove that \( N_t \) satisfies assumption (3.1). For this purpose we recall an elementary inequality that for any real numbers \( a_1, a_2, \ldots \), any \( m = 1, 2, \ldots \) and any \( r \geq 0 \),
\[ \left| \sum_{k=1}^{m} a_k \right|^r \leq \max\{m^{r-1}, 1\} \sum_{k=1}^{m} |a_k|^r. \] (3.5)

By this inequality and relation (1.4), one easily checks that for the number \( p \) given in (3.2) and for each \( t > 0 \),
\[ E[N_t^p] \leq E[\tau_t^p] E[Z_1^p] < \infty. \] (3.6)

Denote by \( \Delta \) the forward difference operator and by \( \lfloor a \rfloor \) the largest integer that is not larger than \( a \). In view of (3.6), for any \( \eta > 1 \) and \( p_1 \in (J_F^+, p) \), we can use summation by parts to
obtain
\[ E[N_t^{p_1} 1_{(N_t > \eta \lambda_t)}] = \sum_{n > \eta \lambda_t} n^{p_1} \Pr(N_t = n) \]
\[ = - \sum_{n > \eta \lambda_t} n^{p_1} \Delta \Pr(N_t \geq n) \]
\[ = \sum_{n > \eta \lambda_t} \Pr(N_t \geq n) \Delta n^{p_1} + (\lfloor \eta \lambda_t \rfloor + 1) n^{p_1} \Pr(N_t \geq \lfloor \eta \lambda_t \rfloor + 1) \]
\[ = I_1(t) + I_2(t). \quad (3.7) \]

Recalling relation (1.4), we deal with \( I_1(t) \) as
\[ I_1(t) \sim p_1 \sum_{n > \eta \lambda_t} n^{p_1-1} \Pr(N_t \geq n) \]
\[ = p_1 \sum_{n > \eta \lambda_t} n^{p_1-1} \left( \sum_{0 \leq m \leq \sqrt{E[\tau_t]}} + \sum_{m > \sqrt{E[\tau_t]}} \right) \Pr\left( \sum_{k=1}^{m} Z_k \geq n, \tau_t = m \right) \]
\[ = I_{11}(t) + I_{12}(t). \]

Note that
\[ \lambda_t = E[N_t] = E[Z_1] E[\tau_t]. \quad (3.8) \]

By Lemma 3.1, we know that for some constant \( C_1 > 0, \)
\[ I_{11}(t) \leq p_1 \sum_{n > \eta \lambda_t} n^{p_1-1} \Pr\left( \sum_{1 \leq k \leq \sqrt{E[\tau_t]}} Z_k \geq n \right) \]
\[ \leq C_1 E[\tau_t] \sum_{n > \eta \lambda_t} n^{p_1-1} n^{-p} = o(E[\tau_t]) = o(\lambda_t). \]

Successively applying Chebyshev’s inequality and (3.5), we obtain that
\[ I_{12}(t) = p_1 \sum_{n > \eta \lambda_t} n^{p_1-1} \sum_{m > \sqrt{E[\tau_t]}} \Pr\left( \sum_{k=1}^{m} Z_k \geq n \right) \Pr(\tau_t = m) \]
\[ \leq p_1 \sum_{n > \eta \lambda_t} n^{p_1-1-p} \sum_{m > \sqrt{E[\tau_t]}} E\left[ \sum_{k=1}^{m} Z_k \right]^p \Pr(\tau_t = m) \]
\[ \leq p_1 E[Z_1]^p \sum_{n > \eta \lambda_t} n^{p_1-1-p} \sum_{m > \sqrt{E[\tau_t]}} m^p \Pr(\tau_t = m). \]

Hence by Lemma 3.2, \( I_{12}(t) = o(1). \) This proves that
\[ I_1(t) = o(\lambda_t). \quad (3.9) \]
Now we turn to $I_2(t)$. Analogously, for some $C_2 > 0$,
\[
I_2(t) \leq C_2 \lambda_t^{p_1} \left( \sum_{0 \leq m \leq \sqrt{\eta E[\tau_t]}} + \sum_{m > \sqrt{\eta E[\tau_t]}} \right) \Pr \left( \sum_{k=1}^{m} Z_k \geq \eta \lambda_t \right) \Pr (\tau_t = m) \\
= I_{21}(t) + I_{22}(t).
\]
Again by Lemma 3.1 and relation (3.8), we have for some $C_3 > 0$,
\[
I_{21}(t) \leq C_3 \lambda_t^{p_1-p} \sum_{0 \leq m \leq \sqrt{\eta E[\tau_t]}} m \Pr (\tau_t = m) = o(E[\tau_t]) = o(\lambda_t).
\]
Since $\tau_t$ has finite moments of all orders, by Chebyshev’s inequality and relation (3.8), one easily sees that
\[
I_{22}(t) \leq C_2 \lambda_t^{p_1} \Pr (\tau_t > \sqrt{\eta E[\tau_t]}) = o(1).
\]
This proves that
\[
I_2(t) = o(\lambda_t). \tag{3.10}
\]
Plugging (3.9) and (3.10) into (3.7) we finally obtain that
\[
E \left[ N_t^{p_1} 1_{(N_t \leq \eta \lambda_t)} \right] = o(\lambda_t).
\]
Hence, $N_t$ satisfies assumption (3.1) with $p_1 > J_F^+$ replacing $p$. This ends the proof of Theorem 3.1. \qed

4 A nonstandard case with dependent occurrences

4.1 An equivalent statement of assumption (3.1)

As we have seen in Section 3, the proof of Theorem 3.1 heavily depends on the independence assumptions made. In the following result, we rewrite the left-hand side of (3.1) as the expectation of a nondecreasing and convex function of $N_t$. This enables us to check some nonstandard cases by using the well-developed stochastic ordering theory.

Lemma 4.1. Let \( \{N_t, t \geq 0\} \) be a nonnegative process with mean function $\lambda_t = E[N_t]$, which satisfies $\lambda_t < \infty$ for any $t \geq 0$ and $\lambda_t \to \infty$. Then for any fixed $p > 0$, the following two assertions are equivalent:

A. for any $\eta > 1$,
\[
E \left[ N_t^{p} 1_{(N_t \geq \eta \lambda_t)} \right] = O(\lambda_t); \tag{4.1}
\]

B. for any $\eta > 1$,
\[
E \left[ (N_t - \eta \lambda_t)^p \right] = O(\lambda_t). \tag{4.2}
\]
Proof. The proof of the implication $A \implies B$ is trivial since

$$E \left[\left( N_t - \eta \lambda_t \right)^p \right] \leq E \left[ N_t^p 1_{(N_t > \eta \lambda_t)} \right].$$

To verify the other implication $B \implies A$, let $\eta > 1$ be arbitrarily fixed. We derive that

$$E \left[\left( N_t - \sqrt{\eta} \lambda_t \right)^p \right] = E \left[\left( N_t - \sqrt{\eta} \lambda_t \right)^p 1_{(N_t > \eta \lambda_t)} \right] \geq \left(1 - \sqrt{\eta}/\eta\right)^p E \left[ N_t^p 1_{(N_t > \eta \lambda_t)} \right].$$

Since by condition $B$ the left-hand side of the above is $O(\lambda_t)$, we immediately obtain relation (4.1) with $\eta > 1$ being arbitrarily given. This ends the proof of Lemma 4.1.

4.2 Negative cumulative dependence and convex order

Recently, Denuit et al. (2001) extended the notion of bivariate positive quadrant dependence (PQD) to arbitrary dimension by introducing the notion of positive cumulative dependence (PCD). The analysis there indicates that PCD can well keep the intuitive meaning of PQD.

In a similar fashion, we introduce here the notion of negative cumulative dependence (NCD) as follows. Let $\{Z_1, Z_2, \ldots, Z_m\}$ be a sequence of random variables. For $I \subset \{1, 2, \ldots, m\}$, we denote by $S_I$ the sum of the random variables $Z_k$ whose indices are in the set $I$. We say that the family of random variables $\{Z_1, Z_2, \ldots, Z_m\}$ is NCD if for any $I \subset \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, m\} - I$, the inequality

$$\Pr (S_I > x_1, Z_k > x_2) \leq \Pr (S_I > x_1) \Pr (Z_k > x_2)$$

holds for all real numbers $x_1$ and $x_2$. We say that an infinite family of random variables $\{Z_k, k = 1, 2, \ldots\}$ is NCD if each of its finite subfamilies is NCD.

Given two random variables $Y_1$ and $Y_2$, we say $Y_1$ precedes $Y_2$ in the stop-loss order, written as $Y_1 \leq_{sl} Y_2$, if the inequality

$$E [\phi(Y_1)] \leq E [\phi(Y_2)]$$

holds for all nondecreasing and convex functions $\phi$ for which the expectations exist. It is worth mentioning that $Y_1 \leq_{sl} Y_2$ and $E [Y_1] = E [Y_2]$ if and only if inequality (4.3) holds for all convex functions $\phi$ for which the expectations exist. Reviews on the stochastic ordering can be found in Dhaene et al. (2002a,b).

As usual, we write by $\{Z_k^+, k = 1, 2, \ldots\}$ the independent version of the sequence $\{Z_k, k = 1, 2, \ldots\}$, that is, the random variables $Z_k^+$, $k = 1, 2, \ldots$, are mutually independent and for each $k \in \{1, 2, \ldots\}$ the random variables $Z_k^+$ and $Z_k$ have the same marginal distribution. We have the following result:
Lemma 4.2. Let \( \{Z_k, k = 1, 2, \ldots\} \) be a sequence of NCD random variables and let \( \{Z^\perp_k, k = 1, 2, \ldots\} \) be its independent version. Then the inequality

\[
\sum_{k=1}^{m} Z_k \leq s_d \sum_{k=1}^{m} Z^\perp_k
\]

holds for each \( m = 1, 2, \ldots \).

Proof. The proof for \( m = 2 \) can be given in a similar way as the proof of Theorem 2 of Dhaene and Goovaerts (1996). The remainder of the proof can be given by proceeding along the same lines as in the proof of Theorem 3.1 of Denuit et al. (2001), only changing the directions of some inequalities. \( \square \)

4.3 The second main result and its proof

Now we are ready to state the second main result of this paper.

Theorem 4.1. Consider the compound model introduced in Section 1. In addition to the assumptions made there, we assume that the claim numbers \( Z_k, k = 1, 2, \ldots \), are NCD. If \( F \in \mathcal{C} \) and (3.2) holds for some \( p > J^+_F \), then for any fixed \( \gamma > 0 \), the precise large deviation result (1.2) holds uniformly for \( x \geq \gamma \lambda_t \).

Proof. As done in the proof of Theorem 3.1, it suffices to prove that \( N_t \) satisfies assumption (3.1). By Lemma 4.1, this amounts to proving that (4.2) holds for some \( p > J^+_F \) and all \( \eta > 1 \). Recall (1.4) and Lemma 4.2. We have

\[
E \left[ (N_t - \eta \lambda_t)_+ \right] = \sum_{m=1}^{\infty} E \left[ \left( \sum_{k=1}^{m} Z_k - \eta \lambda_t \right)_+ \right] \Pr (\tau_t = m)
\]

\[
\leq \sum_{m=1}^{\infty} E \left[ \left( \sum_{k=1}^{m} Z^\perp_k - \eta \lambda_t \right)_+ \right] \Pr (\tau_t = m)
\]

\[
= E \left[ (N^\perp_t - \eta \lambda_t)_+ \right] , \quad (4.4)
\]

where \( N^\perp_t = \sum_{k=1}^{\tau_t} Z^\perp_k \) with \( \{Z^\perp_k, k = 1, 2, \ldots\} \) and \( \{\tau_t, t \geq 0\} \) independent. The proof of Theorem 3.1 has shown that the relation

\[
E \left[ (N_t)_+ \right]^p 1_{(N_t > \eta \lambda_t)} = O (\lambda_t)
\]

holds for some \( p > J^+_F \) and all \( \eta > 1 \). Thus, applying Lemma 4.1 once again, the relation

\[
E \left[ (N^\perp_t - \eta \lambda_t)_+ \right] = O (\lambda_t)
\]

holds for each \( m = 1, 2, \ldots \).
holds for some \( p > J^+_F \) and all \( \eta > 1 \). By (4.4) we conclude that, as desired, the relation
\[
E \left[ (N_t - \eta \lambda t)^p \right] = O(\lambda t)
\]
holds for some \( p > J^+_F \) and all \( \eta > 1 \). This ends the proof of Theorem 4.1.

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