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VARIATIONAL BOUNDS FOR MULTIPHASE TRANSVERSELY ISOTROPIC COMPOSITES

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Abstract
A general algorithm for the calculation of variational bounds for any type of anisotropic linear elastic composite is presented. Analytical expressions for the bounds are derived using this general approach. Bounds for multiphase, linear, transversely isotropic, elastic composites are reported. Bounds for the two-dimensional case are calculated. Some comparisons with other models are shown.

1. Introduction

Due to the anisotropic character of a composite, its geometric and physical nonlinear behavior, the geometric shape of the reinforcements and their distributions in the composite, among other factors, exact analytic expressions of overall properties can not be provided. Sometimes, however, the range of validity of the overall properties of a composite using variational principle can be estimated.

The study of the variational bounds for linear and non-linear composite materials has been considered by many authors (see, for instance, Pobedrya (1984), Ponte Castañeda (1992) and Willis (1991)). In Rodriguez-Ramos et al. (2004) a generalization of the Hashin-Shtrikman variational principles is reported.

In this work, starting from the general expression of the energy derived in Rodriguez-Ramos et al. (2004) for non-linear composites, an application to a particular case of linear, elastic, transversely isotropic composites is studied and the analytical expressions of the variational bounds for each material constant are derived. In particular, the bounds for two and three phase transversely isotropic composites are reported.

2. Statement of the problem

The solution of the non homogeneous problem follows from:

$$Div\tilde{\sigma}\left(\vec{u},\vec{X}\right)=0,\ \text{or}\ \sigma_{ij,j}=0,\quad \sigma_{ij}=F\left(\tilde{\varepsilon},\vec{X}\right)=\frac{\partial \mathrm{W}(\tilde{\varepsilon})}{\partial\tilde{\varepsilon}}=C_{ijlk}\varepsilon_{kl}, \tag{2.1}$$

$$\vec{u}\big|_{\Sigma}=\vec{u}_o=\tilde{\varepsilon}^{\,0}, \tag{2.2}$$

where $\tilde{\sigma}$, $\tilde{\varepsilon}$ denote the second order stress and strain tensors, respectively. $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ where u_i denotes the component of mechanical displacement and the comma denotes partial differentiation. $\tilde{C}$ is the stiffness tensor, and it is a fourth range tensor. $\tilde{\varepsilon}^0$ and $\vec{u}^0$ are respectively the prescribed second order constant tensor and the mechanical displacement vector.

The solution of the above problem (heterogeneous problem) can be written as the sum of the solution of the comparison problem (homogeneous medium), denoted by Problem A, and the solution of the auxiliar problem related to the second order stress polarization tensor $\tilde{p}$, denoted by Problem B.

Therefore,

$$\vec{u} = \vec{u}^c + \vec{u}^p \,, \qquad \tilde{\varepsilon} = \tilde{\varepsilon}^c + \tilde{\varepsilon}^p \,, \qquad \tilde{\sigma} = \tilde{\sigma}^c + \tilde{p} \,, \tag{2.3}$$

where $\tilde{\sigma}^c$, $\tilde{\varepsilon}^c$ and $\vec{u}^c$ are the stress, strain tensors and mechanic displacement of the comparison body, respectively. The magnitudes $\tilde{\varepsilon}^p$, $\tilde{p}$ and $\vec{u}^p$ denote the strain, stress polarization tensors and the mechanical displacement of the polarization contribution, respectively.

The solution of the Problem A is the vector $\vec{u}^c$ and it satisfies the following boundary value problem

$$Div\, \tilde{\sigma}^c\left(\vec{u}\right) = 0 \,, \qquad \tilde{\sigma}^c = \frac{\partial W^c(\tilde{\varepsilon}^c)}{\partial \tilde{\varepsilon}^c} = \tilde{C}^c : \tilde{\varepsilon}^c \,, \tag{2.4}$$

$$\vec{u}^c\Big|_{\Sigma} = \vec{u}_0 \,, \tag{2.5}$$

where the quadratic potential energy of the comparison body W^c can be introduced as $W^c(\tilde{\varepsilon}^c) = \frac{1}{2} C^c_{ijkl}\varepsilon^c_{ij}\varepsilon^c_{kl}$ and the stiffness constant C^c_{ijkl} does not depend on the spatial coordinate $\vec{x}$.

The solution $\vec{u}^p$ of the auxiliary problem related to the stress polarization tensor $\tilde{p}$ (Problem B) can be found from

$$Div\left(\tilde{p} + \tilde{C}^c \tilde{\varepsilon}^p\right) = 0, \quad \text{or} \quad p_{ij,j} + C^c_{ijkl}\varepsilon^p_{kl,j} = 0 \,, \tag{2.6}$$

$$\vec{u}^p\Big|_{\Sigma} = 0 \,, \tag{2.7}$$

where the potential energy of the polarized contribution is denoted by $W^p(\tilde{\varepsilon})$, and it is defined by $W^p(\tilde{\varepsilon}) = W(\tilde{\varepsilon}) - W^c(\tilde{\varepsilon})$. Then we can obtain the expression for the stress polarization tensor in components as

$$p_{ij} = F^p(\varepsilon_{ij}) = \frac{\partial W^p(\varepsilon_{ij})}{\partial \varepsilon_{ij}} = \sigma_{ij} - C^c_{ijkl}\varepsilon_{kl} = C_{ijkl}\varepsilon_{kl} - C^c_{ijkl}\varepsilon_{kl} = (C_{ijkl} - C^c_{ijkl})\varepsilon_{kl} \,. \tag{2.8}$$

The associate constitutive relation (strain as a function of the polarization tensor) from (2.8) can be written in the form,

$$\varepsilon_{ij} = \frac{\partial \omega^p(p_{ij})}{\partial\, p_{ij}} = H_{ijkl}\, p_{kl} \,, \quad H_{ijkl} = (C_{ijkl} - C^c_{ijkl})^{-1}, \tag{2.9}$$

where

$$\omega^p(\tilde{p}) = \frac{1}{2} H_{ijkl}\, p_{ij}\, p_{kl} \,, \tag{2.10}$$

is the inverse potential energy related to the potential energy $W^{p}(\tilde{\varepsilon})$ of the polarization contribution defined in (2.8).

The boundary value problem for the composite can be summarized as follows:

Composite		*Comparison body*		*Polarization contribution*	
$\sigma_{ij,j}=0$		$\sigma^{c}_{ij,j}=0$		$\sigma^{p}_{ij,j}=0$	
$\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$	$=$	$\sigma_{ij}=C^{c}_{ijkl}\varepsilon^{c}_{kl}$	$+$	$\sigma^{p}_{ij}=p_{ij}+C^{c}_{ijkl}\varepsilon^{p}_{kl}=0$	(2.11)
$u_i\big\vert_{\Sigma}=\varepsilon^{o}_{ij}x_j$		$u_i\big\vert_{\Sigma}=\varepsilon^{o}_{ij}x_j$		$u^{p}_i\big\vert_{\Sigma}=0$	

3. Some considerations related to elastic energy. General expression of the energy

The analytical expressions for the variational bounds of a linear transversely isotropic elastic composite are obtained using the energy functional given by the formula (53) reported in Rodriguez-Ramos et al. (2003):

$$\Phi=\frac{1}{2}\int(\sigma^{c}_{ij}\varepsilon^{c}_{ij}+2p_{ij}\varepsilon^{c}_{ij}+p_{ij}\varepsilon'_{ij}-2\omega^{p}(\tilde{p}))dV\,. \tag{3.1}$$

Averaging the functional (3.1) and using the average operator $<\bullet>$ on the volume V, the expression (3.1) can be written in the following form:

$$A=\frac{2\Phi}{V}=<\sigma^{c}_{ij}\varepsilon^{c}_{ij}>+2<p_{ij}\varepsilon^{c}_{ij}>-2<\omega^{p}(\tilde{p})>+<p_{ij}\varepsilon^{p}_{ij}>. \tag{3.2}$$

Each term of (3.2) can be denoted, for simplicity, in the following form:

$$A_1=<\sigma^{c}_{ij}\varepsilon^{c}_{ij}>,\quad A_2=2<p_{ij}\varepsilon^{c}_{ij}>,\quad A_3=-2<\omega^{p}(\tilde{p})>\ \text{and}\ \ A_4=<p_{ij}\varepsilon^{p}_{ij}>. \tag{3.3}$$

Therefore, the equation (3.2) can be given in an equivalent form as $A=A_1+A_2+A_3+A_4$.

Now, consider a composite made of q components (q-phases). The stress polarization tensor $\tilde{p}$ depends on the space variable $\vec{x}$, and the stress polarization tensor for each phase α of the composite is denoted by $\tilde{p}_{\alpha}$. Therefore, we have:

$$\langle\tilde{p}\rangle=\sum_{\alpha=1}^{q}v_{\alpha}\tilde{p}_{\alpha}\,, \tag{3.4}$$

where v_{α} is the volume fraction of the component α in the composite, and

$$\sum_{\alpha=1}^{q}v_{\alpha}=1. \tag{3.5}$$

The magnitudes A_1, A_2 and A_3 can be written as functions of the spherical and deviator contributions of tensors $\tilde{\varepsilon}^{0}$ and $\tilde{p}$. The decomposition is $p_{ij}=\frac{1}{d}p\delta_{ij}+\bar{p}_{ij}$ and

$\varepsilon_{ij}^{o} = \frac{1}{d} e \cdot \delta_{ij} + \bar{e}_{ij}$, where p, e are the spherical and $\bar{p}_{ij}$, $\bar{e}_{ij}$ are the deviator parts of the two tensors $\tilde{\varepsilon}^{0}$ and $\tilde{p}$, respectively, and d is the dimension of the Euclidean space, i.e. $d = 2$ (two-dimensional space) or $d = 3$ (three-dimensional space).

After some algebraic manipulations, the magnitudes A_1, A_2 and A_3 have the following compact and general form in terms of spherical and deviator parts,

$$A_1 = \frac{1}{d^2} e^2 C_{iijj}^{c} + \frac{2}{d} e \cdot \bar{e}_{ij} C_{ijkk}^{c} + \bar{e}_{ij} C_{ijkl}^{c} \bar{e}_{kl},$$

$$A_2 = \sum_{\alpha=1}^{q} v_\alpha \left(\frac{2}{d} p_\alpha e + 2\tilde{\bar{p}}_\alpha : \tilde{\bar{e}} \right), \tag{3.6}$$

$$A_3 = -\sum_{\alpha=1}^{q} v_\alpha \tilde{p}_\alpha : \tilde{H}_\alpha : \tilde{p}_\alpha = -\sum_{\alpha=1}^{q} v_\alpha \left[\frac{1}{d^2} p_\alpha^2 H_{iijj_\alpha} + \frac{2}{d} p_\alpha \bar{p}_{ij_\alpha} H_{ijkk_\alpha} + \bar{p}_{ij_\alpha} H_{ijkl_\alpha} \bar{p}_{kl_\alpha} \right],$$

with $\omega^{p}(\tilde{p}_\alpha) = \frac{1}{2} \tilde{p}_\alpha : \tilde{H}_\alpha : \tilde{p}_\alpha$ and $\tilde{H} = \left(\tilde{C} - \tilde{C}^{c} \right)^{-1}$. For the sake of brevity, the proof of the expression (3.6) is omitted.

The expression A_4 is related to polarization contribution (Problem B) and it depends on the unknown polarization displacement $\vec{u}^{\,p}$. Therefore, we need to solve the auxiliary problem (2.6)-(2.7) related to stress polarization tensor $\tilde{p}$ where the vector $\vec{u}^{\,p}$ is the solution of the problem. The problem is solved using the Fourier transform.

Applying the Fourier transform to the equation (2.6), we obtain

$$C_{ijkl}^{c} \omega_j \omega_l U_k = \mathrm{i} \omega_j P_{ij}, \tag{3.7}$$

where U_k and P_{ij} denote the Fourier transforms of the functions u_k^{p} and p_{ij}, respectively. The magnitude A_4 can be written in the Parsevall form,

$$A_4 = \frac{1}{V} \int_V p_{ij} \varepsilon_{ij}^{p} dV_x = \frac{1}{8\pi^3 V} \int_V P_{ij}(\vec{\omega}) E_{ij}(\vec{\omega})\, dV_\omega = \frac{1}{8\pi^3 V} \int_V P_{ij}(\vec{\omega}) G_{ij}(\vec{\omega}) dV_\omega, \tag{3.8}$$

where $\tilde{P}(\vec{\omega})$ and $\tilde{E}(\vec{\omega})$ are the Fourier images of the stress polarization tensor $\tilde{p}$ and strain tensor $\tilde{\varepsilon}$, respectively, and $E_{ij} = \frac{1}{2}(G_{ij} + G_{ji})$ with $G_{ij} = \mathrm{i} U_i \omega_j$.

Now, our aim is to derive the integrand function given in (3.8) as a function only of the polarization tensor $\tilde{p}$. From the vector $\vec{\omega}$ we can introduce a unit vector, i.e. $k_i = \frac{\omega_i}{\| \vec{\omega} \|}$ and then equation (3.7) is written in the following form, $C_{ijkl}^{c} k_j k_l U_k = \mathrm{i} \frac{k_j}{\| \vec{\omega} \|} P_{ij}$. We therefore have,

$$U_k = \mathrm{i} (C_{ijkl}^{c} k_j k_l)^{-1} \frac{k_j}{\| \vec{\omega} \|} P_{ij}. \tag{3.9}$$

We can derive from formula (3.9) the expression,

$$C^c_{ijkl}\, k_j\, k_l\, G_{km} = -\, k_j\, k_m\, P^*_{ij}\,, \qquad (3.10)$$

where P^*_{ij} is the Fourier image of the component $p^*_{ij} = p_{ij} - < p_{ij} >$.

The tensor $\tilde{C}^c$ can be decomposed as sum of two tensors; one isotropic tensor denoted by $\tilde{C}^c_{iso}$ (the components are denoted as $c^c_{ijkl_{iso}}$) and the other tensor, the so called remainder tensor, denoted by $\tilde{C}^c_{res}$ (the components are denoted by $c^c_{ijkl_{res}}$). Thus, $c^c_{ijkl} = c^c_{ijkl_{iso}} + c^c_{ijkl_{res}}$. Therefore, the equation (310) can be rewritten in the form:

$$(c^c_{ijkl_{iso}}\, k_j\, k_l + c^c_{ijkl_{res}}\, k_j\, k_l)\, G_{km} = -\, k_j\, k_m\, P^*_{ij}\,. \qquad (3.11)$$

The tensor $\tilde{T} = t_{ik}\, \vec{e}_i \otimes \vec{e}_k$, with components $t_{ik} = c^c_{ijkl_{iso}}\, k_j\, k_l = \dfrac{c^c_{11} - c^c_{12}}{2}\delta_{ik} + \dfrac{c^c_{11} + c^c_{12}}{2} k_i\, k_k$ has an inverse tensor of the form $\tilde{Z} = z_{ik}\, \vec{e}_i \otimes \vec{e}_k$, with $z_{ik} = \dfrac{2}{c^c_{11} - c^c_{12}}\delta_{ik} - \dfrac{c^c_{11} + c^c_{12}}{c^c_{11}\,(c^c_{11} - c^c_{12})} k_i\, k_k$. The equation (3.11) is multiplied by the inverse tensor $\tilde{Z}$ and we have

$$(1 + \frac{2}{c^c_{11} - c^c_{12}} c^c_{ijil_{res}}\, k_j\, k_l - \frac{c^c_{11} + c^c_{12}}{c^c_{11}\,(c^c_{11} - c^c_{12})} c^c_{ijkl_{res}}\, k_i\, k_j\, k_k\, k_l)\, G_{km} = -\, k_j\, k_m\, P^*_{ij}\, z_{ik}\,. \qquad (3.12)$$

Let us introduce a unit vector $\vec{N} = n_i \vec{e}_i$ with the following properties,

$$< n_i n_j >= \frac{1}{V}\int_V n_i n_j\, dV = \begin{cases} \dfrac{1}{2}\delta_{ij} & if \quad d = 2 \\ \dfrac{1}{3}\delta_{ij} & if \quad d = 3 \end{cases},$$

$$< n_i n_j n_k n_l >= \frac{1}{V}\int_V n_i n_j n_k n_l\, dV = \begin{cases} \dfrac{1}{8}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) & if \quad d = 2 \\ \dfrac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) & if \quad d = 3 \end{cases}.$$

The equation (3.12) is multiplied by the tensor $\tilde{P}^* = P^*_{ij}\, \vec{e}_i \otimes \vec{e}_j$, and using the above properties, the following expression can be derived for two dimensional case ($d = 2$)

$$\tau_2 < G_{km}\, P_{km} >= \frac{c^c_{11} + c^c_{12}}{8\, c^c_{11}\,(c^c_{11} - c^c_{12})} < P^*_{ii}\, P_{jj} > + \frac{c^c_{12} - 3\, c^c_{11}}{4\, c^c_{11}\,(c^c_{11} - c^c_{12})} < P^*_{ij}\, P_{ij} >, \qquad (3.13)$$

where $\tau_2 = 1 + \dfrac{c^c_{ijij_{res}}}{c^c_{11} - c^c_{12}} - \dfrac{(c^c_{11} + c^c_{12})}{8\, c^c_{11}\,(c^c_{11} - c^c_{12})}(c^c_{iijj_{res}} + 2\, c^c_{ijij_{res}})$. Notice that for the isotropic case $\tau_2 = 1$, since the reminder tensor is the null tensor, i.e. $c^c_{ijkl_{res}} = 0$.

The expression A_4 in (3.8) using (3.13) can be written in the following form,

$$A_4 = < p_{ij}\, \varepsilon_{ij} > = \frac{c_{11}^c + c_{12}^c}{8\tau\, c_{11}^c (c_{11}^c - c_{12}^c)} < p_{ii}^*\, p_{jj} > + \frac{c_{12}^c - 3c_{11}^c}{4\tau\, c_{11}^c (c_{11}^c - c_{12}^c)} < p_{ij}^*\, p_{ij} > . \qquad (3.14)$$

Finally, the components p_{ij}^* and p_{ij} of the expression $p_{ij}^* = p_{ij} - < p_{ij} >$ can be split into deviators and spherical parts, and therefore we obtain

$$A_4 = a_c \sum_{\alpha=1}^{q} v_\alpha (p_\alpha^2 - < p >^2) + b_c \sum_{\alpha=1}^{q} v_\alpha (\bar{p}_{ij_\alpha}^2 - < \bar{p}_{ij} >^2)\ , \qquad (3.15)$$

where $a_c = -\dfrac{1}{4\tau_2\, c_{11}^c}$ and $b_c = -\dfrac{3c_{11}^c - c_{12}^c}{4\tau_2\, c_{11}^c (c_{11}^c - c_{12}^c)}$.

Now, we can derive the general expression of A_4 for the three-dimensional case ($d = 3$). The equation (3.12) can be written in the form,

$$\tau_3 < G_{km}\, P_{km} > = \frac{c_{11}^c + c_{12}^c}{15 c_{11}^c (c_{11}^c - c_{12}^c)} < P_{ii}^*\, P_{jj} > + \frac{2c_{12}^c - 8c_{11}^c}{15 c_{11}^c (c_{11}^c - c_{12}^c)} < P_{ij}^*\, P_{ij} > , \qquad (3.16)$$

where $\tau_3 = 1 + \dfrac{2c_{ijij_{res}}^c}{3(c_{11}^c - c_{12}^c)} - \dfrac{(c_{11}^c + c_{12}^c)}{15 c_{11}^c (c_{11}^c - c_{12}^c)} (c_{iijj_{res}}^c + 2c_{ijij_{res}}^c)$.

Analogous to the above procedure for $d = 2$, we obtain the general expression A_4 for three-dimensional case ($d = 3$) in the form,

$$A_4 = a_{c_3} \sum_{\alpha=1}^{q} v_\alpha (p_\alpha^2 - < p >^2) + b_{c_3} \sum_{\alpha=1}^{q} v_\alpha (\bar{p}_{ij_\alpha}^2 - < \bar{p}_{ij} >^2)\ , \qquad (3.17)$$

where $a_c = -\dfrac{1}{9\tau_3\, c_{11}^c}$ and $b_c = -\dfrac{8c_{11}^c - 2c_{12}^c}{15\tau_3\, c_{11}^c (c_{11}^c - c_{12}^c)}$.

Finally, the expression for the energy can be calculated for two dimensional problems ($d = 2$) from the equations (3.6) and (3.15) and for three-dimensional problem ($d = 3$) using the expressions (3.6) and (3.17).

4. General approach for variational bounds of multiphase anisotropic elastic composite

It is well known that the solution of the auxiliary problem (2.6)-(2.7) (Problem B) is a stationary point of the functional (see, Rodríguez-Ramos et. at. (2004))

$$\Phi = \frac{1}{2} \int_V (\sigma_{ij}^c\, \varepsilon_{ij}^c + 2p_{ij}\, \varepsilon_{ij}^c + p_{ij}\, \varepsilon_{ij}^p - 2\omega^p(\tilde{p}))\, dV \quad \text{i.e.} \quad \delta\, \Phi(\tilde{p}, \delta\, \tilde{p}) = 0\, . \qquad (4.1)$$

From the variational principle of Hashin-Shtrikman we can state that the stationary point is maximum or minimum according to the following statements:

The functional (4.1) has maximum if the tangential modulus $\dfrac{\partial F^p}{\partial \varepsilon_{ij}}$ is positive, i.e. from (2.8), we can see that $C_{ijkl} > C^c_{ijkl}$.

The functional (4.1) has minimum if the tangential modulus $\dfrac{\partial F^p}{\partial \varepsilon_{ij}}$ is negative, i.e. from (2.8), we can see that $C_{ijkl} < C^c_{ijkl}$.

The maximum or minimum of the magnitude A can be obtained with respect to the stress polarization tensor $\tilde{p}$ from the condition of the existence of extrema applied to the spherical (p) and deviator ($\bar{p}_{ij}$) components of the stress polarization tensor. In that case, a system of equations is written in the following form,

$$[\frac{1}{d^2} H_{iijj_\alpha} - a_c]\, p_\alpha + \frac{1}{d} \bar{p}_{ij_\alpha} H_{ijkk_\alpha} = \frac{1}{d} e - 2 a_c < p >, \tag{4.2}$$

$$\frac{1}{d} p_\alpha H_{ijkk_\alpha} + \bar{p}_{kl_\alpha} H_{ijkl_\alpha} - b_c \bar{p}_{ij_\alpha} = \bar{e}_{ij} - b_c < \bar{p}_{ij} >. \tag{4.3}$$

The two unknown magnitudes $< p >$ and $< \bar{p}_{ij} >$ are derived from equations (4.2) and (4.3). Then, the unknown functions $< p >$ and $< \bar{p}_{ij} >$ are substituted into the expression for the energy (3.2), but they are now written in a compact form for any anisotropic media as

$$A = \frac{1}{d^2} e^2 C^c_{iijj} + \frac{2}{d} e \cdot \bar{e}_{ij} C^c_{ijkk} + \bar{e}_{ij} C^c_{ijkl} \bar{e}_{kl} + \frac{1}{d} < p > + < \bar{p}_{ij} > \bar{e}_{ij}, \tag{4.4}$$

and applying the sufficient condition of extrema $C_{ijkl_\alpha} > C^c_{ijkl}$ (maximum) and $C_{ijkl_\alpha} < C^c_{ijkl}$ (minimum) an analytical expression of the bounds can be derived. The analysis of the bounds is done for the composite with well ordered q-phases, i.e. $C_{ijkl_1} > C_{ijkl_2} > \cdots > C_{ijkl_q}$. The condition $C_{ijkl_\alpha} > C^c_{ijkl}$ is satisfied if we take the properties of the comparison body as the properties of the q-phase elastic moduli, i.e., C_{ijkl_q}; on the other hand, the condition $C_{ijkl_\alpha} < C^c_{ijkl}$ is satisfied if the comparison body is chosen to be one of highest phase elastic moduli C_{ijkl}, i.e. the phase with properties C_{ijkl_1}.

5. Example of bounds for transversely isotropic elastic composite

In this section, an example of bounds using the above procedure is presented. For a better explanation, bounds for two dimensional case (d = 2) are studied. In particular, we consider a composite with transversely isotropic properties of their constituents. The Hooke law for the transversely isotropic elastic materials is given in the following matrix form,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{23} \\ \sqrt{2}\,\sigma_{13} \\ \sqrt{2}\,\sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\,\varepsilon_{23} \\ \sqrt{2}\,\varepsilon_{13} \\ \sqrt{2}\,\varepsilon_{12} \end{bmatrix}. \tag{5.1}$$

The reduced Hooke law for the plane deformation in the plane x_1x_2 where $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ can be written as,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\,\sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{11} & 0 \\ 0 & 0 & c_{11} - c_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\,\varepsilon_{12} \end{bmatrix}. \tag{5.2}$$

In this case, the expressions given in (3.6) and (3.14) can be reduced for transversely isotropic case to

$$A_1 = \frac{1}{4}(2c_{11}^c + 2c_{12}^c)\,e^2 + (c_{11}^c - c_{12}^c)\bar{e}_{ij}^2\,, \quad \bar{e}_{ij}^2 = \bar{e}_{11}^2 + \bar{e}_{22}^2 + 2\bar{e}_{12}^2\,, \tag{5.3}$$

$$A_2 = \sum_{\alpha=1}^{q} v_\alpha \left(p_\alpha e + 2\bar{p}_{ij\,\alpha}\bar{e}_{ij} \right), \tag{5.4}$$

$$A_3 = -\sum_{\alpha=1}^{q} v_\alpha \left[\frac{1}{4} p_\alpha^2 \xi_1 + \xi_2 \bar{p}_{ij_\alpha}^2 \right], \quad \bar{p}_{ij}^2 = \bar{p}_{11}^2 + \bar{p}_{22}^2 + 2\bar{p}_{12}^2\,, \tag{5.5}$$

where the constants ξ_1 and ξ_2 are listed as follows,

$$\xi_1 = H_{iijj_\alpha} = \frac{2}{c_{11_\alpha} - c_{11}^c + c_{12_\alpha} - c_{12}^c}\,, \quad \xi_2 = H_{1111_\alpha} - H_{1122_\alpha} = \frac{1}{c_{11_\alpha} - c_{11}^c - c_{12_\alpha} + c_{12}^c}\,.$$

The magnitude A_4 is given by the formula,

$$A_4 = a_c \sum_{\alpha=1}^{q} v_\alpha (p_\alpha^2 - \langle p \rangle^2) + b_c \sum_{\alpha=1}^{q} v_\alpha (\bar{p}_{ij_\alpha}^2 - \langle \bar{p}_{ij} \rangle^2)\,, \tag{5.6}$$

where the constants a_c, b_c are written in the form,

$$a_c = -\frac{1}{4(m^c + k^c)}\,, \quad b_c = -\frac{k^c + m^c}{4m^c\,(k^c + m^c)}\,, \text{ where } k^c = \frac{c_{11}^c + c_{12}^c}{2} \text{ and } m^c = \frac{c_{11}^c - c_{12}^c}{2}\,.$$

From the condition of extremum with respect to spherical and deviator parts, i.e. $\dfrac{\partial A}{\partial p_\alpha} = 0$ and $\dfrac{\partial A}{\partial \bar{p}_{ij_\alpha}}$, we obtain the following system of equations,

$$\frac{\partial A}{\partial p_\alpha} = v_\alpha\, e - \frac{1}{2} v_\alpha\, p_\alpha\, \xi_1 + 2\,a_c v_\alpha\,(p_\alpha - \langle p \rangle) = 0\,, \tag{5.7}$$

$$\frac{\partial A}{\partial \overline{p}_{ij_\alpha}} = 2 v_\alpha \, \overline{e}_{ij} - 2 v_\alpha \, \overline{p}_{ij_\alpha} \, \xi_2 + 2 b_c \, v_\alpha \, (\overline{p}_{ij_\alpha} - < \overline{p}_{ij} >) = 0 \,. \tag{5.8}$$

The unknown functions <p> and $< \overline{p}_{ij} >$ are calculated directly from (5.7) and (5.8). Therefore, we can write,

$$< p > = \frac{1}{2} \frac{\gamma_1}{(1 + a_c \, \gamma_1)} e \,, \quad \text{with} \quad \gamma_1 = \sum_{\alpha=1}^{q} v_\alpha \, (\frac{1}{4} \xi_1 - a_c)^{-1} \,, \tag{5.9}$$

$$< \overline{p}_{ij} > = \frac{\gamma_2}{(1 + b_c \gamma_2)} \overline{e}_{ij} \,, \text{ with} \quad \gamma_2 = \sum_{\alpha=1}^{q} v_\alpha \, (\frac{1}{4} \xi_2 - b_c)^{-1} \,. \tag{5.10}$$

The functional A given in the formula (4.4) can be written in the compact form $A = A_1 + \frac{A_2}{2}$. Thus replacing (5.9) and (5.10) in the compact form, we obtain,

$$A = \frac{1}{2} (c_{11}^c + c_{12}^c) e^2 + (c_{11}^c - c_{12}^c) \overline{e}_{ij}^2 + \frac{1}{2} < p > e + < \overline{p}_{ij} > \overline{e}_{ij} \,,$$

$$A = k^c \, e^2 + m^c \, \overline{e}_{ij}^2 + \frac{1}{4} \frac{\gamma_1}{(1 + a_c \, \gamma_1)} e^2 + \frac{\gamma_2}{1 + b_c \, \gamma_2} \overline{e}_{ij}^2 = (k^c + \frac{1}{4} \frac{\gamma_1}{(1 + a_c \, \gamma_1)}) e^2 + (2 m^c + \frac{\gamma_2}{1 + b_c \, \gamma_2}) \overline{e}_{ij}^2 \,.$$

Let us introduce the following notation in the above quadratic form,

$$k = k^c + \frac{\gamma_1}{4 (1 + a_c \, \gamma_2)} \quad \text{and} \quad m = m^c + \frac{\gamma_2}{2 (1 + b_c \, \gamma_2)} \,.$$

The expressions of the lower and upper bounds related to multiphase composites can be obtained from the above formula, taking into consideration the relevant statement from the variational principle of Hashin-Shtrikman. They are applied in the following way,

$$k_L = k_1 + \frac{\gamma_1^{(1)}}{4 [1 + a_c^{(1)} \, \gamma_1^{(1)}]} \quad \text{and} \quad k_U = k_q + \frac{\gamma_1^{(q)}}{4 [1 + a_c^{(q)} \, \gamma_1^{(q)}]} \,, \; (k_1 < k_2 < \cdots < k_{q-1} < k_q), \tag{5.11}$$

$$m_L = m_1 + \frac{\gamma_2^{(1)}}{2 [1 + b_c^{(1)} \, \gamma_2^{(1)}]} \quad \text{and} \quad m_U = m_q + \frac{\gamma_2^{(q)}}{2 [1 + b_c^{(q)} \, \gamma_2^{(q)}]} \,, \; (m_1 < m_2 < \cdots < m_{q-1} < m_q). \tag{5.12}$$

The superscripts 1 and q between brackets mean the replacement of the comparison body by the inclusion or matrix respectively.

The bounds (5.11)-(5.12) for two phases composite can be written in the form,

$$k_L = k_1 + \frac{v_2}{\dfrac{1}{k_2 - k_1} + \dfrac{v_1}{m_1 + k_1}} \quad \text{and} \quad k_U = k_2 + \frac{v_1}{\dfrac{1}{k_1 - k_2} + \dfrac{v_1}{m_2 + k_2}} \,, \tag{5.13}$$

$$m_L = m_1 + \frac{v_2}{\dfrac{1}{m_2 - m_1} + \dfrac{(k_1 + 2 m_1) v_1}{2 m_1 (m_1 + k_1)}} \quad \text{and} \quad m_U = m_2 + \frac{v_1}{\dfrac{1}{m_1 - m_2} + \dfrac{(k_2 + 2 m_2) v_2}{2 m_2 (m_2 + k_2)}} \,. \tag{5.14}$$

The expressions (5.13)-(5.14) are the same bounds to (4.25)-(4.28) reported by Hashin (1965).

Now, the bounds for the shear module μ (G) are studied. We consider the antiplane deformation problem, where the Hooke law in that case has the form $\sigma_{12} = \mu\,\varepsilon_{12}$, $\sigma_{13} = \mu\,\varepsilon_{13}$. In matrix form we have,

$$\begin{bmatrix} \sqrt{2}\,\sigma_{12} \\ \sqrt{2}\,\sigma_{13} \end{bmatrix} = \begin{bmatrix} 2\mu & 0 \\ 0 & 2\mu \end{bmatrix} \begin{bmatrix} \sqrt{2}\,\varepsilon_{12} \\ \sqrt{2}\,\varepsilon_{13} \end{bmatrix}.$$

Therefore, the only deviator contributions different from zero are $\overline{e}_{12}$ and $\overline{e}_{13}$. The magnitudes A_1, A_2, A_3 and A_4 are obtained as follows,

$$A_1 = \mu^c\,\overline{e}_{ij}^2 , \quad \overline{e}_{ij}^2 = 2\,\overline{e}_{12}^2 + 2\overline{e}_{13}^2 ,$$

$$A_2 = 2\sum_{\alpha=1}^{q} v_\alpha\,\overline{p}_{ij_\alpha}\overline{e}_{ij} , \quad A_3 = -\sum_{\alpha=1}^{q} \frac{v_\alpha}{\mu_\alpha - \mu^c}\,\overline{p}_{ij_\alpha}^2 , \quad \overline{p}_{ij}^2 = 2\,\overline{p}_{12}^2 + 2\,\overline{p}_{13}^2 ,$$

$$A_4 = b_c \sum_{\alpha=1}^{q} v_\alpha\,(\overline{p}_{ij_\alpha}^2 - <\overline{p}_{ij}>^2) \text{ , with } b_c = -\frac{1}{2\,\mu^c} .$$

The necessary condition $\dfrac{\partial A}{\partial\,\overline{p}_{ij_\alpha}} = 0$ for the extrema of the functional A is given in the following form:

$$\frac{\partial A}{\partial\,\overline{p}_{ij_\alpha}} = 2\,v_\alpha\,\overline{e}_{ij} - 2\frac{v_\alpha}{\mu_\alpha - \mu^c}\,\overline{p}_{ij_\alpha} + 2\,b_c\,v_\alpha\,(\overline{p}_{ij_\alpha} - <\overline{p}_{ij}>) = 0 . \quad (5.15)$$

From the equation (5.15) we can get $<\overline{p}_{ij}> = \dfrac{\gamma}{1 + b_c\,\gamma}\overline{e}_{ij}$, with $\gamma = \sum_{\alpha=1}^{q} v_\alpha \left(\dfrac{1}{\mu_\alpha - \mu^c} - b_c \right)^{-1}$.

Thus, the functional $A = A_1 + \dfrac{1}{2}A_2$ has the particular form, $A = \mu^c\,\overline{e}_{ij}^2 + \dfrac{\gamma}{1 + b_c\,\gamma}\overline{e}_{ij}^2 = (\mu^c + \dfrac{\gamma}{1 + b_c\,\gamma})\overline{e}_{ij}^2$.

Let us introduce the notation $\mu = \mu^c + \dfrac{\gamma}{1 + b_c\,\gamma}$. Therefore the bounds for the shear module in the composite with q-phase has the form

$$\mu_L = \mu_1 + \frac{\gamma^{(1)}}{1 + b_c^{(1)}\,\gamma^{(1)}} \quad \text{and} \quad \mu_U = \mu_q + \frac{\gamma^{(q)}}{1 + b_c^{(q)}\,\gamma^{(q)}} \text{ , } (\mu_1 < \mu_2 < \cdots < \mu_{q-1} < \mu_q).$$

In particular, for a two-phase composite we obtain

$$\mu_L = \mu_1 + \frac{v_2}{\dfrac{1}{\mu_2 - \mu_1} + \dfrac{v_1}{2\,\mu_1}} \quad \text{and} \quad \mu_U = \mu_2 + \frac{v_1}{\dfrac{1}{\mu_1 - \mu_2} + \dfrac{v_2}{2\,\mu_2}} . \quad (5.16)$$

The bounds (5.16) are the same bounds to (5.12) – (5.13) given by Hashin (1965).

Finally, the bounds for the parameters c_{13}, c_{33} and c_{44} are derived. Our analysis is reduced to the plane $x_1 x_3$. The Hook law for the deformation plane $x_1 x_3$ where $\varepsilon_{12} = \varepsilon_{22} = \varepsilon_{23} = 0$ can be written in the matrix form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{13} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & 2c_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{33} \\ \sqrt{2}\,\varepsilon_{13} \end{bmatrix}.$$

Analogous to the above procedure for the other bounds we can calculate the magnitudes A_1, A_2, A_3 and A_4 in the following way,

$$A_1 = \frac{c_{11}^c + 2c_{13}^c + c_{33}^c}{4} e^2 + (c_{33}^c - c_{11}^c)\, e\, \bar{e}_{33} + (c_{33}^c + c_{11}^c - 2c_{13}^c)\, \bar{e}_{33}^2 + 4c_{44}^c\, \bar{e}_{13}^2,$$

$$A_2 = \sum_{\alpha=1}^{q} v_\alpha \,(p_\alpha\, e + 4\,\bar{p}_{33_\alpha}^2\; \bar{e}_{33} + 4\,\bar{p}_{13_\alpha}^2\; \bar{e}_{13})$$

$$A_3 = -\sum_{\alpha=1}^{q} v_\alpha [\frac{1}{4}\xi_1\, p_\alpha^2 - \xi_2\, p_\alpha\, \bar{p}_{33_\alpha} + \xi_3\, \bar{p}_{33_\alpha}^2 + 2\xi_4\, \bar{p}_{13_\alpha}^2\,],$$

$$A_4 = a_c \sum_{\alpha=1}^{q} v_\alpha \,(p_\alpha^2 - < p >^2) + b_c \sum_{\alpha=1}^{q} v_\alpha \,(2\,\bar{p}_{33_\alpha}^2 - 2 < \bar{p}_{33} >^2 + 2\,\bar{p}_{13_\alpha}^2 - 2 < \bar{p}_{13} >^2),$$

where

$$\xi_1 = \frac{c_{11_\alpha} - c_{11}^c + c_{33_\alpha} - c_{33}^c - 2(c_{13_\alpha} - c_{13}^c)}{(c_{11_\alpha} - c_{11}^c)(c_{33_\alpha} - c_{33}^c) - (c_{13_\alpha} - c_{13}^c)^2}, \quad \xi_2 = \frac{c_{11_\alpha} - c_{11}^c - c_{33_\alpha} + c_{33}^c)}{(c_{11_\alpha} - c_{11}^c)(c_{33_\alpha} - c_{33}^c) - (c_{13_\alpha} - c_{13}^c)^2},$$

$$\xi_3 = \frac{c_{11_\alpha} - c_{11}^c + c_{33_\alpha} - c_{33}^c + 2(c_{13_\alpha} - c_{13}^c)}{(c_{11_\alpha} - c_{11}^c)(c_{33_\alpha} - c_{33}^c) - (c_{13_\alpha} - c_{13}^c)^2}, \quad \xi_4 = \frac{1}{2(c_{44_\alpha} - c_{44}^c)}, \quad \text{with} \quad a_c = -\frac{1}{4\tau c_{11}^c},$$

$$b_c = -\frac{3c_{11}^c - c_{12}^c}{4\tau c_{11}^c (c_{11}^c - c_{12}^c)}, \text{ and}$$

$$\tau = \frac{(c_{11}^c - c_{12}^c)(4c_{11}^c - 4c_{12}^c + 16c_{13}^c + 24c_{44}^c) + (11c_{11}^c - 3c_{12}^c)(2c_{44}^c - c_{11}^c + c_{12}^c)}{16c_{11}^c (c_{11}^c - c_{12}^c)}.$$

From the necessary condition $\frac{\partial A}{\partial p_\alpha} = 0$ and $\frac{\partial A}{\partial \bar{p}_{ij_\alpha}} = 0$ of the extrema of the functional A we can write,

$$(\frac{1}{4}\xi_1 - a_c)\, p_\alpha - \frac{1}{2}\xi_2\, \bar{p}_{33_\alpha} = \frac{e}{2} - a_c < p >, \tag{5.17}$$

$$\frac{\xi_2}{4}\, p_\alpha + (\xi_3 - b_c)\,\bar{p}_{33_\alpha} = \bar{e}_{33} - b_c < \bar{p}_{33} >, \tag{5.18}$$

$$(\xi_4 - b_c)\,\bar{p}_{13_\alpha} + b_c < \bar{p}_{13} > = \bar{e}_{13}\,. \tag{5.19}$$

An expression for $< \bar{p}_{13} >$ can be calculated directly from the equation (5.19), i.e.,

$$< \bar{p}_{13} > = \frac{n_{13}}{1 + b_c\, n_{13}}\,\bar{e}_{13} \quad \text{, where} \quad n_{13} = \sum_{\alpha=1}^{q} \frac{v_\alpha}{\xi_4 - b_c}\,.$$

The system (5.17) – (5.18) can be solved with respect to p_α and $\bar{p}_{33_\alpha}$ and a new system is obtained,

$$(1 + a_c\,\vartheta_1) < p > + \frac{\vartheta_2\, b_c}{2} < \bar{p}_{33} > = \frac{\vartheta_1}{2}\, e + \frac{\vartheta_2}{2}\,\bar{e}_{33}\,, \tag{5.20}$$

$$-\frac{\vartheta_2\, a_c}{4} < p > + (1 + b_c\,\vartheta_3) < \bar{p}_{33} > = \vartheta_3\,\bar{e}_{33} - \frac{\vartheta_2}{8}\, e\,, \tag{5.21}$$

where

$$\vartheta_1 = \sum_{\alpha=1}^{q} \frac{v_\alpha\,(\xi_3 - b_c)}{(\frac{1}{4}\xi_1 - a_c)(\xi_3 - b_c) + \frac{\xi_2^2}{8}} \;,\quad \vartheta_2 = \sum_{\alpha=1}^{q} \frac{v_\alpha\,\xi_2}{(\frac{1}{4}\xi_1 - a_c)(\xi_3 - b_c) + \frac{\xi_2^2}{8}} \quad \text{and}$$

$$\vartheta_3 = \sum_{\alpha=1}^{q} \frac{v_\alpha\,(\frac{1}{4}\xi_1 - a_c)}{(\frac{1}{4}\xi_1 - a_c)(\xi_3 - b_c) + \frac{\xi_2^2}{8}}\,.$$

The system (5.20) – (5.21) is solved with respect to unknown functions $< p >$ and $< \bar{p}_{33} >$ and we get, $< p > = \varphi_1\, e + \varphi_2\,\bar{e}_{33}$ and $< \bar{p}_{33} > = \varphi_3\,\bar{e}_{33} - 4\varphi_2\, e$, with

$$\varphi_1 = \frac{8\,\vartheta_1\,(1 + b_c\,\vartheta_3) + \vartheta_2^2\, b_c}{2[8(1 + a_c\,\vartheta_1)(1 + b_c\,\vartheta_3) + \vartheta_2^2\, a_c\, b_c]} \;,\quad \varphi_2 = \frac{4\,\vartheta_2}{8(1 + a_c\,\vartheta_1)(1 + b_c\,\vartheta_3) + \vartheta_2^2\, a_c\, b_c}$$

and $\varphi_3 = \dfrac{8\,\vartheta_3\,(1 + a_c\,\vartheta_1) + \vartheta_2^2\, a_c}{8(1 + a_c\,\vartheta_1)(1 + b_c\,\vartheta_3) + \vartheta_2^2\, a_c\, b_c}$.

Thus, the functional $A = A_1 + \frac{1}{2} A_2$ can be written in the form,

$$\begin{aligned} A = {} & \frac{1}{4}[2c_{11}^c + 2c_{13}^c + 2\varphi_1 + \frac{15}{2}\varphi_2]e^2 + [c_{33}^c - c_{11}^c - \frac{15}{2}\varphi_2](\frac{1}{2}e + \bar{e}_{33})^2 + \\ & + [2c_{11}^c - 2c_{13}^c + 2\varphi_3 + \frac{15}{2}\varphi_2]\bar{e}_{33}^2 + 2[2c_{44}^c + \frac{n_{13}}{1 + b_c\, n_{13}}]\bar{e}_{13}^2 \end{aligned}\,.$$

The following notation can be introduced,

$$2c_{11} + 2c_{13} = 2c_{11}^c + 2c_{13}^c + 2\varphi_1 + \frac{15}{2}\varphi_2\,, \tag{5.22}$$

$$c_{33} - c_{11} = c_{33}^c - c_{11}^c - \frac{15}{2}\varphi_2 , \tag{5.23}$$

$$2c_{11} - 2c_{13} = 2c_{11}^c - 2c_{13}^c + 2\varphi_3 + \frac{15}{2}\varphi_2 , \tag{5.24}$$

$$2c_{44} = 2c_{44}^c + \frac{n_{13}}{1 + b_c\, n_{13}} . \tag{5.25}$$

The system (5.22)–(5.25) is solved with respect to the unknown functions c_{11} , c_{13}, c_{33} and c_{44} and we obtain,

$$c_{11} = c_{11}^c + \frac{\varphi_1}{2} + \frac{15\varphi_2}{2} + \frac{\varphi_3}{2} , \qquad c_{13} = c_{13}^c + \frac{\varphi_1}{2} - \frac{\varphi_3}{2} ,$$

$$c_{33} = c_{33}^c + \frac{\varphi_1}{2} - \frac{15\varphi_2}{2} + \frac{\varphi_3}{2} \quad \text{and} \quad c_{44} = c_{44}^c + \frac{n_{13}}{2(1 + b_c\, n_{13})} .$$

The expressions for the bounds of the materials constants c_{11}, c_{13}, c_{33} and c_{44} can be listed as,

$$c_{11_L} = c_{11}^{(1)} + \frac{\varphi_1^{(1)}}{2} + \frac{15\varphi_2^{(1)}}{2} + \frac{\varphi_3^{(1)}}{2} , \qquad c_{11_U} = c_{11}^{(q)} + \frac{\varphi_1^{(q)}}{2} + \frac{15\varphi_2^{(q)}}{2} + \frac{\varphi_3^{(q)}}{2} ,$$

$$c_{13L} = c_{13}^{(1)} + \frac{\varphi_1^{(1)}}{2} - \frac{\varphi_3^{(1)}}{2} , \qquad c_{13U} = c_{13}^{(q)} + \frac{\varphi_1^{(q)}}{2} - \frac{\varphi_3^{(q)}}{2} ,$$

$$c_{33_L} = c_{33}^{(1)} + \frac{\varphi_1^{(1)}}{2} - \frac{15\varphi_2^{(1)}}{2} + \frac{\varphi_3^{(1)}}{2} , \qquad c_{33_U} = c_{33}^{(q)} + \frac{\varphi_1^{(q)}}{2} - \frac{15\varphi_2^{(q)}}{2} + \frac{\varphi_3^{(q)}}{2} ,$$

$$c_{44L} = c_{44}^{(1)} + \frac{n_{13}^{(1)}}{2(1 + b_c^{(1)}\, n_{13}^{(1)})} , \qquad c_{44U} = c_{44}^{(q)} + \frac{n_{13}^{(q)}}{2(1 + b_c^{(q)}\, n_{13}^{(q)})} .$$

6. Conclusions

A general procedure for deriving the variational bounds for any type of anisotropic linear elastic composite is studied. Analytical expressions for the bounds of linear, transversely isotropic elastic composite are given. In particular, bounds for two-dimensional case are then calculated. Non limitation about the quantities of inclusions and the geometrical shape of the inclusions are considered for the estimation of the bounds. The method allows us the calculation of all bounds for the material parameters involved in different type of anisotropies and good estimation of the effective properties are obtained.

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