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EXTENDING PRICING RULES WITH GENERAL RISK FUNCTIONS

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Extending pricing rules with general risk functions

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Abstract. The paper addresses pricing issues in imperfect and/or incomplete markets if the risk level of the hedging strategy is measured by a general risk function. Convex Optimization Theory is used in order to extend pricing rules for a wide family of risk functions, including Deviation Measures, Expectation Bounded Risk Measures and Coherent Measures of Risk. For imperfect markets the extended pricing rules reduce the bid-ask spread. The paper ends by particularizing the findings so as to study with more detail some concrete examples, including the Conditional Value at Risk and some properties of the Standard Deviation.

KEY WORDS. Incomplete and imperfect market, Risk measure and deviation measure, Pricing rule, Convex optimization.

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1. INTRODUCTION

General risk functions are becoming more and more important in finance. Since the paper of Artzner *et al.* (1999) introduced the axioms and properties of their "Coherent Measures of Risk", many authors have extended the discussion. Hence, it is not surprising that the recent literature presents many interesting contributions focusing on new methods for measuring risk levels. Among others, Goovaerts *et al.* (2004) have introduced the Consistent Risk Measures, Frittelli and Scandolo (2005) have analyzed Risk Measures for Stochastic Processes, and Rockafellar *et al.* (2006) have defined the Deviations and the Expectation Bounded Risk Measures.

Many classical financial problems have been revisited by using new risk functions. So, Mansini *et al.* (2007) deal with Portfolio Choice Problems with complex risk measures, Alexander *et al.* (2006) compare the minimization of the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) for a portfolio of derivatives, Calafiore (2007) studies "robust" efficient portfolios if risk levels are given by standard deviations and absolute deviations, and Schied (2007) deals with Optimal Investment with Convex Risk Measures.

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The extension of pricing rules to the whole space in incomplete markets is a major topic in finance. Several papers have used coherent measures of risk to price and hedge under incompleteness, though the article by Nakano (2004) seems to be an interesting approach that also incorporates previous and significant contributions of other authors. Another line of research is related to the concept of "good deal", introduced in the seminal paper by Cochrane and Saa-Requejo (2000). A good deal is not an arbitrage but is close to an arbitrage, so the absence of good deal may be an adequate assumption if it is useful for pricing.

In recent papers Jaschke and Küchler (2001) and Staum (2004) extended the notion of good deal so as to involve coherent measures of risk, and they introduced the "coherent prices" as upper and lower bounds that every extension of the pricing rule to the whole space must respect. They allowed for imperfections in the initial market and also studied existence properties and other classical issues. Later Cherny (2006) also dealt with pricing issues with risk measures in incomplete markets, though it is not the major focus of the article.

The present paper considers an initial incomplete and maybe imperfect market and deals with the Expectation Bounded Risk Measures and the Deviation Measures of Rockafellar *et al.* (2006) in order to extend the pricing rule to the whole space. As we will see, the Representation Theorems of Risk Measures provided by the authors above are very appropriate to simplify the Mathematical Programming Problems leading to Optimal Hedging Strategies and prices, which permits us to introduce new pricing rules satisfying adequate properties and easy to compute in practice.

The paper's outline is as follows. Section 2 will present notations and the basic conditions and properties of the initial pricing rule π to be extended and the risk function ρ to be used. Since the risk function is not differentiable in general, the optimization problem giving the optimal hedging strategy is not differentiable either, and Section 3 will be devoted to overcome this caveat. We will use Representation Theorems of Risk Measures so as to transform the initial optimal hedging problem in a minimax problem. Later, following an idea developed in Balbás *et al.* (2008*b*), the minimax problem is equivalent to a new convex optimization problem in Banach spaces. In particular, the dual variable belongs to the set of probabilities on the Borel $\sigma-$algebra of the sub-gradient of ρ. Since this fact would provoke high degree of complexity when dealing with the optimality conditions of the hedging problem, Theorem 2 is one of the most important results in this section, because it guarantees that the optimal dual solution will be a Dirac Delta, and thus we can leave the use of general probability measures in order to characterize the optimal solutions. The section ends by proving its second important result. Theorem 4 yields simple necessary and sufficient optimality conditions as well as guarantees the existence of Stochastic Discount Factors of π in the sub-gradient of ρ, a property that will imply many consequences throughout the paper.

Section 4 starts by introducing the extension π_ρ of π. Theorem 7 shows the interesting properties of π_ρ, that is convex, continuous and bounded by π and ρ. Theorem 8 states that π_ρ is a genuine extension of π if the initial market is free of frictions, and reduces the transaction costs caused by π otherwise. Theorem 10 states that the Stochastic Discount Factors of π and π_ρ that belong to the sub-gradient of ρ coincide, which enables us to prevent the existence of arbitrage opportunities for π_ρ in Corollary 11. The section ends by proving that π_ρ outperforms the classical extension of pricing rules in incomplete (and maybe imperfect) markets if ρ is coherent.

Section 5 considers a General Deviation Measure and focuses on this particular case. Special attention is paid to the Standard Deviation, since it is often used in finance to extend pricing rules (see Schweizer, 1995, or Luenberger, 2001, among others). Some relationships between the proposed extension and another classical ones are analyzed. Section 6 deals with the $CVaR$, since it is becoming a very popular Coherent and Expectation Bounded Risk Measure that respects the second order Stochastic Dominance (Ogryczak and Ruszczynski, 2002) and has been used for several authors in different types of Portfolio Choice Problems. Theorem 14 characterizes the proposed extension in this special case and its Corollary 15 analyses some particular situations.

The last section of the paper points out the most important conclusions.

2. Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure μ. Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the Banach space of $\mathbb{R}-$valued measurable functions y on Ω such that $E\left(|y|^p\right) < \infty$, $E\left(\right)$ representing the mathematical expectation ($E\left(|y|^q\right) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem, we have that L^q is the dual space of L^p.

Consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In general, $(S_t)_{t\in\mathcal{T}}$ will denote an adapted stochastic price process.

Let us assume that $Y \subset L^p$ is a convex cone composed of super-replicable payoffs, *i.e.*, for every $y \in Y$ there exists at least one self-financing portfolio whose final pay-off is $S_T \geq y$ (see Jouini and Kallal, 1995, and De Wagenaere and Wakker, 2001, among many others, for further details about self-financing portfolios and pricing issues in markets with or without transaction costs). Denote by $\mathcal{S}\left(y\right)$ the family of such self-financing portfolios, and suppose that there exists

$$\pi\left(y\right) = Inf\ \left\{S_0;\ \ (S_t)_{t\in\mathcal{T}} \in \mathcal{S}\left(y\right)\right\} \tag{1}$$

for every $y \in Y$. We will say that $\pi(y)$ is the price of y. The market will be said to be perfect if Y is a subspace and $\pi : Y \longrightarrow \mathbb{R}$ is linear, and imperfect otherwise. In general, we will impose the natural conditions, sub-additivity

$$\pi(y_1 + y_2) \leq \pi(y_1) + \pi(y_2) \tag{2}$$

for every $y_1, y_2 \in Y$, and positive homogeneity

$$\pi(\alpha y) = \alpha \pi(y) \tag{3}$$

for every $y \in Y$ and $\alpha \geq 0$. Consequently, π is a convex function. Finally, we will assume the existence of a riskless asset that does not generate any friction, *i.e.*, almost surely constant random variables $y = k$ belong to Y for every $k \in \mathbb{R}$, and there exists a risk-free rate $r_f \geq 0$ such that

$$\pi(k) = ke^{-r_f T} \tag{4}$$

holds. It is easy to see that (4) leads to

$$\pi(y + k) = \pi(y) + ke^{-r_f T} \tag{5}$$

for every $y \in Y$ and $k \in \mathbb{R}$. Indeed $\pi(y + k) \leq \pi(y) + ke^{-r_f T}$ is clear, and

$$\pi(y) = \pi(y + k - k) \leq \pi(y + k) + \pi(-k) = \pi(y + k) - ke^{-r_f T}$$

implies the opposite inequality.

Let

$$\rho : L^p \longrightarrow \mathbb{R}$$

be the general risk function that a trader uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; -E(yz) \leq \rho(y), \ \forall y \in L^p\}. \tag{6}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma(L^q, L^p)$ $-$compact and

$$\rho(y) = Max \ \{-E(yz) : z \in \Delta_\rho\} \tag{7}$$

holds for every $y \in L^p$. Furthermore, we will also impose

$$\Delta_\rho \subset \{z \in L^q; E(z) = 1\}. \tag{8}$$

These are quite natural assumptions closely related to the Representation Theorems of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, and bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for convex functions, it is easy to prove that the $\sigma(L^q, L^p)$ −compactness of Δ_ρ and the fulfillment of (7) and (8) hold if ρ is continuous and satisfies:

a)

$$\rho(y+k) = \rho(y) - k \tag{9}$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

b)

$$\rho(\alpha y) = \alpha \rho(y) \tag{10}$$

for every $y \in L^p$ and $\alpha > 0$.

c)

$$\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2) \tag{11}$$

for every $y_1, y_2 \in L^p$.

d)

$$\rho(y) \geq -E(y) \tag{12}$$

for every $y \in L^p$.[1], [2]

It is easy to see that if ρ is continuous and satisfies Properties *a*), *b*), *c*) and *d*) above then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L^q_+ = \{z \in L^q; \mu(z \geq 0) = 1\}. \tag{13}$$

Particular interesting examples are the Conditional Value at Risk ($CVaR$) of Rockafellar *et al.* (2006), the Dual Power Transform (DPT) of Wang (2000) and the Wang Measure (Wang, 2000), among many others.[3] Furthermore, following the original idea of Rockafellar *et al.* (2006) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho(y) = \sigma(y) - E(y) \tag{14}$$

is continuous and satisfies *a*), *b*), *c*) and *d*) if $\sigma : L^p \longrightarrow \mathbb{R}$ is a continuous (or lower semi-continuous) deviation, that is, if σ satisfies *b*), *c*),

e)

$$\sigma(y+k) = \sigma(y) \tag{15}$$

[1] Actually, the properties above are almost similar to those used by Rockafellar *et al.* (2006) in order to introduce their Expectation Bounded Risk Measures. These authors also impose *a*), *b*), *c*) and *d*), work with $p = 2$, allow for $\rho(y) = \infty$, and impose $\rho(y) > -E(y)$ if y is not constant.

[2] According to Theorem 2.2.20 in Zalinescu (2002), if ρ satisfies *a*), *b*), *c*) and *d*) then ρ is continuous if and only if ρ is lower semi-continuous.

[3] If ρ equals the Wang measure or the DPT (or other risk measures given by distortion functions) then see Cherney (2006) for further details about Δ_ρ.

for every $y \in L^p$ and $k \in \mathbb{R}$, and

$f)$

$$\sigma(y) \geq 0 \tag{16}$$

for every $y \in L^p$.

Particular examples are the $p-$deviation given by $\rho(y) = [E(|E(y) - y|^p)]^{1/p}$, or the downside $p-$semi-deviation given by $\rho(y) = [E(|Max\ \{E(y) - y, 0\}|^p)]^{1/p}$, among many others.

Denote by $g \in L^p$ a new pay-off that we are interested in pricing and hedging. If the trader sells g for P dollars and buys $y \in Y$ in order to hedge the global position, then he will choose y so as to solve

$$\begin{cases} Min\ \rho(y-g) \\ \pi(y) \leq P \\ y \in Y \end{cases} . \tag{17}$$

3. OPTIMAL HEDGING: PRIMAL AND DUAL PROBLEMS AND OPTIMALITY CONDITIONS

In general ρ will be non-differentiable and therefore so will be Problem (17). To overcome this caveat we follow the method proposed in Balbás *et al.* (2008*b*). So, bearing in mind (7), Problem (17) is equivalent to

$$\begin{cases} Min\ \theta \\ \theta + E(yz) - E(gz) \geq 0, \quad \forall z \in \Delta_\rho \\ \pi(y) \leq P \\ \theta \in \mathbb{R},\ y \in Y \end{cases} \tag{18}$$

in the sense that y solves (17) if and only if there exists $\theta \in \mathbb{R}$ such that (θ, y) solves (18), in which case

$$\theta = \rho(y-g)$$

holds. Notice that the objective of (18) is differentiable and even linear. The first constraint is valued on the Banach space $\mathcal{C}(\Delta_\rho)$ of real-valued and continuous functions on the (*weak**) compact space Δ_ρ. Since its dual space is $\mathcal{M}(\Delta_\rho)$, the space of inner regular real valued $\sigma-$additive measures on the Borel $\sigma-$algebra of Δ_ρ (endowed with the *weak** topology), the Lagrangian function

$$\mathcal{L} : \mathbb{R} \times Y \times \mathcal{M}(\Delta_\rho) \times \mathbb{R} \longrightarrow \mathbb{R}$$

becomes

$$\mathcal{L}(\theta, y, \nu, \lambda) =$$
$$\theta \left(1 - \int_{\Delta_\rho} d\nu(z)\right) - \int_{\Delta_\rho} E(yz)\, d\nu(z) + \int_{\Delta_\rho} E(zg)\, d\nu(z) + \lambda\pi(y) - \lambda P.$$

Following Luenberger (1969) the element $(\nu, \lambda) \in \mathcal{M}(\Delta_\rho) \times \mathbb{R}$ is dual feasible if and only if it belongs to the non-negative cone $\mathcal{M}_+(\Delta_\rho) \times \mathbb{R}_+$ and

$$Inf\ \{\mathcal{L}(\theta, y, \nu, \lambda) : \theta \in \mathbb{R},\ y \in Y\} > -\infty,$$

in which case the infimum above equals the dual objective on (ν, λ). Hence, bearing in mind (2) and (3), the dual problem of (18) becomes

$$\begin{cases} Max\ \int_{\Delta_\rho} E(gz)\, d\nu(z) - P\lambda \\ \lambda\pi(y) - \int_{\Delta_\rho} E(yz)\, d\nu(z) \geq 0, \quad \forall y \in Y \\ \nu \in \mathcal{P}(\Delta_\rho),\ \lambda \in \mathbb{R}_+ \end{cases} \tag{19}$$

$\mathcal{P}(\Delta_\rho)$ denoting the set composed of those elements in $\mathcal{M}(\Delta_\rho)$ that are probabilities.

$\mathcal{P}(\Delta_\rho)$ is convex, and the theorem of Alaoglu easily leads to the compactness of $\mathcal{P}(\Delta_\rho)$ when endowed with the $\sigma(\mathcal{M}(\Delta_\rho), \mathcal{C}(\Delta_\rho))$ −topology (Horvàth, 1966). Besides, given $z \in \Delta_\rho$ we will denote by $\delta_z \in \mathcal{P}(\Delta_\rho)$ the usual Dirac delta that concentrates the mass on $\{z\}$, *i.e.*, $\delta_z(\{z\}) = 1$ and $\delta_z(\Delta_\rho \setminus \{z\}) = 0$. It is known that the set of extreme points of $\mathcal{P}(\Delta_\rho)$ is given by

$$ext(\mathcal{P}(\Delta_\rho)) = \{\delta_z; z \in \Delta_\rho\}, \tag{20}$$

though we will not have to draw on this result. The objective function of dual problems in the finite-dimensional case is attained in a extreme feasible solution, which, along with (20), suggest that the solution of (19) could be achieved in $\{\delta_z; z \in \Delta_\rho\}$. Let us show that this conjecture is correct. First we provide an instrumental lemma whose statement and complete proof may be found in Balbás *et al.* (2008*a*).

Lemma 1. *(Mean Value Theorem). Let $\nu \in \mathcal{P}(\Delta_\rho)$. Then there exists $z_\nu \in \Delta_\rho$ such that*

$$\int_{\Delta_\rho} E(yz)\, d\nu(z) = E(yz_\nu) \tag{21}$$

holds for every $y \in L^p$. □

Theorem 2. *If $(\nu, \lambda) \in \mathcal{P}(\Delta_\rho) \times \mathbb{R}_+$ solves (19) then there exists $z \in \Delta_\rho$ such that (δ_z, λ) solves (19).*

Proof. Consider (ν, λ) solving (19) and take $z_\nu \in \Delta_\rho$ satisfying (21). Then, for every $y \in Y$ we have that

$$\begin{aligned} &0 \leq \\ &\lambda\pi(y) - \int_{\Delta_\rho} E(yz)\, d\nu(z) \\ &= \lambda\pi(y) - E(yz_\nu) \\ &= \lambda\pi(y) - \int_{\Delta_\rho} E(yz)\, d\delta_{z_\nu}(z). \end{aligned}$$

and

$$\lambda P-\int_{\Delta_\rho}E\left(gz\right)d\nu\left(z\right)=\lambda P-E\left(gz_\nu\right)=\lambda P-\int_{\Delta_\rho}E\left(gz\right)d\delta_{z_\nu}\left(z\right)$$

which proves that (δ_{z_ν},λ) is (19)-feasible and the objective values of (19) in (ν,λ) and (δ_{z_ν},λ) are identical. □

Remark 1. *The latter theorem leads to significant consequences. In particular, we can consider the alternative and far simpler dual problem*

$$\begin{cases} Max\ E\left(gz\right)-\lambda P \\ \lambda\pi\left(y\right)-E\left(yz\right)\geq 0, \quad \forall y\in Y \\ z\in\Delta_\rho,\ \lambda\in\mathbb{R}_+ \end{cases} \tag{22}$$

where $z\in\Delta_\rho$ is playing the role of $\nu\in\mathcal{P}\left(\Delta_\rho\right)$. □

Proposition 3. *Let be $z\in\Delta_\rho$. The inequality $\lambda\pi\left(y\right)-E\left(yz\right)\geq 0$ for every $y\in Y$ can only hold for $\lambda=e^{r_fT}$.*

Proof. Indeed, the inequality leads to $\lambda e^{-r_fT}-E\left(z\right)\geq 0$ if $y=1$, and $\lambda e^{-r_fT}-E\left(z\right)\leq 0$ if $y=-1$, so the conclusion is obvious because $E\left(z\right)=1$ for every $z\in\Delta_\rho$ (see (8)). □

Remark 2. *The previous proposition enables us to simplify (22) once again. The equivalent problem will be*

$$\begin{cases} Max\ E\left(gz\right)-Pe^{r_fT} \\ \pi\left(y\right)e^{r_fT}-E\left(yz\right)\geq 0, \quad \forall y\in Y \\ z\in\Delta_\rho \end{cases} \tag{23}$$

where the λ variable has been removed. □

Notice that (4) implies that (17) is feasible, and therefore so is (18). Since we are dealing with infinite-dimensional Banach spaces the so called "duality gap" between (18) and (23) might arise. To prevent this pathological situation we will give the next theorem and impose a very weak assumption with clear economic interpretation. We will also connect the statement $b)$ of the theorem below with classical key notions in Asset Pricing Theory.

Theorem 4. *The three following conditions are equivalent:*

$a)$ There exist $P_0\in\mathbb{R}$ and $g_0\in L^p$ such that (18) is not unbounded, i.e., there are no sequences $(y_n)\subset Y$ of feasible solutions such that $\rho\left(y_n-g_0\right)\longrightarrow-\infty$.

b) The (23)-feasible set

$$D_f = \left\{ z \in \Delta_\rho;\ \pi\left(y\right) e^{r_f T} - E\left(yz\right) \geq 0, \forall y \in Y \right\} \tag{24}$$

is non void.

c) Problem (18) is not unbounded for every $P \in \mathbb{R}$ *and* $g \in L^p$.

Furthermore, in the affirmative case (18) and (23) are feasible and bounded, (23) attains its optimal value, the dual maximum equals the primal infimum, and the following Karush-Kuhn-Tucker conditions

$$\begin{cases} \theta + E\left(y^* z^*\right) - E\left(g z^*\right) = 0 \\ \theta + E\left(y^* z\right) - E\left(g z\right) \geq 0, & \forall z \in \Delta_\rho \\ \pi\left(y^*\right) - P = 0 \\ \pi\left(y^*\right) e^{r_f T} - E\left(y^* z^*\right) = 0 \\ \pi\left(y\right) e^{r_f T} - E\left(y z^*\right) \geq 0, & \forall y \in Y \\ \theta \in \mathbb{R},\ y^* \in Y,\ z^* \in \Delta_\rho \end{cases} \tag{25}$$

are necessary and sufficient so as to guarantee that (θ, y^*) *and* $\left(z^*, \lambda = e^{r_f T}\right)$ *solve (18) and (23) respectively.*

Proof. $a) \Rightarrow b)$ Suppose that we prove the fulfillment of the Slater Qualification for (18) (Luenberger, 1969), *i.e.*, the existence of $(\theta_0, y_0) \in \mathbb{R} \times Y$ such that

$$\begin{cases} \theta_0 + E\left(y_0 z\right) - E\left(g_0 z\right) > 0, & \forall z \in \Delta_\rho \\ \pi\left(y_0\right) < P_0 \end{cases}$$

holds. Then Condition $a)$ implies that (19) (and therefore (23)) must be feasible because its feasible set does not depend on P_0 or g_0, and (18) is bounded for at least a couple (P_0, g_0) (Luenberger, 1969).

In order to show the fulfillment of the Slater Qualification notice that (4) implies that (17) is always feasible, and therefore so is (18). Moreover, given a (18)-feasible solution (θ, y), and bearing in mind (8), the element $(\theta_0, y_0) = (\theta + 2, y - 1)$ satisfies the primal constraints as strict inequalities.

$b) \Rightarrow c)$ If D_f is not empty then (23) is feasible and therefore so is (19). Thus (18) cannot be unbounded because it is easy to verify that the primal objective is never lower than the dual one (see also Luenberger, 1969).

$c) \Rightarrow a)$ Obvious.

Moreover, in the affirmative case (25) provides sufficient optimality conditions because (18) is a convex problem, and these conditions are also necessary because, as shown in the implication $a) \Rightarrow b)$, the Slater Qualification holds (Luenberger, 1969). Finally, this Qualification also implies that the dual maximum is attained and equals the primal infimum. $\square$

Assumption 1. Hereafter we will assume the existence of $P_0 \in \mathbb{R}$ and $g_0 \in L^p$ such that (18) is not unbounded. Thus Conditions b) and c) in the theorem above also hold. □

Remark 3. *Since Condition b) holds D_f (see (24)) is not empty, and its elements will be called "Stochastic Discount Factors (SDF) of (π, ρ)". Notice that*

$$E(yz) = \pi(y) e^{r_f T} \tag{26}$$

holds for every $y \in Y$ and every $z \in D_f$ if the market is perfect, since $-y \in Y$ for every $y \in Y$ and consequently

$$-\pi(y) e^{r_f T} + E(yz) = \pi(-y) e^{r_f T} - E(-yz) \geq 0$$

must also hold. Expression (26) leads to

$$\pi(y) = e^{-r_f T} E(yz) = e^{-r_f T} E_{\mu_z}(y), \tag{27}$$

i.e., the current price of any asset equals the present value of its expected pay-off once modified with the "distortion variable" z, or the present value of its expected pay-off if the expectation is computed with the "risk-neutral probability measure" μ_z such that

$$z = \frac{d\mu_z}{d\mu}.$$

(27) is closely related to the First Fundamental Theorem of Asset Pricing. Notice that μ_z is actually a probability owing to (8), and will be equivalent to μ as long as

$$\mu(z > 0) = 1.$$

See Jouini and Kallal (1995) and De Wagenaere and Wakker (2001), among many others, for further details about the Fundamental Theorem of Asset Pricing and risk-neutral or martingale measures in both perfect and imperfect markets. □

4. Pricing rules

This section will be devoted to extend the pricing rule π to the whole space L^p. According to Theorem 4 and Assumption 1, we can define the function $F : \mathbb{R} \times L^p \longrightarrow \mathbb{R}$ by considering that $F(P, g)$ is the optimal value of (18) and (23) for every $P \in \mathbb{R}$ and $g \in L^p$.

Proposition 5. *The equality*

$$F(P, g) = F(0, g) - P e^{r_f T} \tag{28}$$

holds for every $P \in \mathbb{R}$ and $g \in L^p$.

Proof. Proposition 3 and Theorem 4 point out that the Lagrange multiplier $\lambda = e^{r_f T}$ of (18) continuously depends on P, so the classical sensitivity results of Convex Programming imply that $\dfrac{\partial F}{\partial P} = -e^{r_f T}$ holds for every $P \in \mathbb{R}$ and $g \in L^p$. Hence, (28) is obvious. □

As a consequence of the previous proposition we can introduce the first pricing rule we are going to deal with. Indeed, we will define

$$\pi_\rho (g) = F(P, g)\, e^{-r_f T} + P = F(0, g)\, e^{-r_f T}. \tag{29}$$

Next let us see that the independence of π_ρ with respect to P, pointed out by (28) and (29), and the independence of the solution of (23) with respect to P, obvious consequence of the form of (23), is also fulfilled by the optimal hedging portfolios, *i.e.*, by the solution of (18).

Proposition 6. *Suppose that* (y^*, θ) *solves (18) for* $P \in \mathbb{R}$ *and* $g \in L^p$. *Then* $\left(y^* + \alpha e^{r_f T}, \theta - \alpha e^{r_f T}\right)$ *solves (18) for* $P + \alpha \in \mathbb{R}$ *and* $g \in L^p$.[4]

Proof. The proof is quite easy and consequently we will simplify the exposition. Just consider a dual solution z^*, that does not depend on P as pointed out by (23), and bear in mind that (y^*, θ) and z^* satisfy (25) for (P, g). Then use (5) and (8) so as to verify that $\left(y^* + \alpha e^{r_f T}, \theta - \alpha e^{r_f T}\right)$ and z^* satisfy (25) for $(P + \alpha, g)$. □

Next let us present some interesting properties of the extension π_ρ above.

Theorem 7. *A)* $\pi_\rho (g) \le \rho(-g)\, e^{-r_f T}$ *for every* $g \in L^p$.
B) π_ρ *is sub-additive and positively homogeneous (and therefore convex).*
C) π_ρ *is continuous.*
D) $\pi_\rho (y) \le \pi(y)$ *for every* $y \in Y$.
E) If ρ *is a coherent risk measure then* π_ρ *is increasing.*

Proof. *A)* (29) implies that

$$\pi_\rho (g) = F(0, g)\, e^{-r_f T} = Inf\ \left\{\rho(y - g)\, e^{-r_f T}; \pi(y) \le 0,\ y \in Y\right\}.$$

Hence, for $y = 0$, $\pi_\rho (g) \le \rho(-g)\, e^{-r_f T}$.

[4] Notice that this fact simplifies (25), in the sense that the equation $\pi(y^*) = P$ may be removed.

B) Taking $P = 0$, the absence of duality gap between (18) and (23) and the existence of dual solutions (Theorem 4) show that

$$\pi_\rho (g_1 + g_2) = Max \ \left\{ E\left((g_1 + g_2) z\right) e^{-r_f T}; z \in D_f \right\}.$$

If $z_{g_1+g_2} \in D_f$ denotes the dual feasible solution where the maximum is attained, then

$$\pi_\rho (g_1 + g_2) = E\left((g_1 + g_2) z_{g_1+g_2}\right) e^{-r_f T} = E\left((g_1) z_{g_1+g_2}\right) e^{-r_f T} + E\left((g_2) z_{g_1+g_2}\right) e^{-r_f T}.$$

If $z_{g_1} \in D_f$ and $z_{g_2} \in D_f$ are the obvious, bearing in mind that D_f does not depend on g (see (24)) we have

$$\begin{aligned} E\left((g_1) z_{g_1+g_2}\right) e^{-r_f T} + E\left((g_2) z_{g_1+g_2}\right) e^{-r_f T} &\leq E\left(g_1 z_{g_1}\right) e^{-r_f T} + E\left(g_2 z_{g_2}\right) e^{-r_f T} \\ &= \pi_\rho (g_1) + \pi_\rho (g_2). \end{aligned}$$

On the other hand, if $\alpha > 0$ we have

$$\pi_\rho (\alpha g) = E\left(\alpha g z_{\alpha g}\right) = \alpha E\left(g z_{\alpha g}\right) \leq \alpha E\left(g z_g\right) = \alpha \pi_\rho (g).$$

Analogously,

$$\pi_\rho (g) = \pi_\rho \left(\frac{1}{\alpha} \alpha g\right) \leq \frac{1}{\alpha} \pi_\rho (\alpha g)$$

leads to $\alpha \pi_\rho (g) \leq \pi_\rho (\alpha g)$. For $\alpha = 0$ we only have to prove that $\pi_\rho (0) = 0$, but this equality is clear because otherwise

$$\pi_\rho (0) = \pi_\rho (2 \times 0) = 2\pi_\rho (0)$$

would lead to the contradiction $1 = 2$.

C) Being π_ρ a convex function on L^p it is sufficient to see that π_ρ is continuous at $g = 0$ (Luenberger, 1969). Since ρ is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g\| \leq \delta \Longrightarrow \rho(-g) \leq \varepsilon e^{r_f T}$ and therefore $\pi_\rho (g) \leq \varepsilon$ follows from Statement A). Besides, bearing in mind B) we have that

$$-\pi_\rho (g) \leq \pi_\rho (-g) \leq \varepsilon$$

because $\|-g\| \leq \delta$. Hence, $|\pi_\rho (g)| \leq \varepsilon$.

D) If $y \in Y$ with the notations above we have $\pi_\rho (y) = E\left(y z_y\right) e^{-r_f T}$, and $E\left(y z_y\right) \leq \pi (y) e^{r_f T}$ because $z_y \in D_f$.

E) If $g_1 \leq g_2 \in L^p$ then $y - g_1 \geq y - g_2$ and therefore $\rho(y - g_1) \leq \rho(y - g_2)$ for every $y \in Y$ because ρ is coherent and therefore decreasing. Consequently,

$$Inf \left\{\rho(y - g_1); y \in Y, \pi(y) \leq 0\right\} \leq Inf \left\{\rho(y - g_1); y \in Y, \pi(y) \leq 0\right\},$$

so the conclusion trivially holds. □

As a consequence of the previous properties π_ρ "improves" the bid-ask spread (or the transaction costs) of π. Indeed, if we consider that $-\pi_\rho(-g)$ is the bid price of $g \in L^p$ and $\pi_\rho(g)$ is its ask price, then Theorem $7B$ shows that the bid-ask spread

$$\pi_\rho(g) + \pi_\rho(-g) \geq 0 \tag{30}$$

cannot be negative. Analogously, (2) and (3) lead to $\pi(y) + \pi(-y) \geq 0$ for every $y \in Y$ such that $-y \in Y$. Furthermore we have:

Theorem 8. *A) $\pi_\rho(y) + \pi_\rho(-y) \leq \pi(y) + \pi(-y)$ holds for every $y \in Y$ such that $-y \in Y$.*

B) If g and $-g$ belong to Y and $\pi(-g) = -\pi(g)$ then $\pi_\rho(g) = \pi(g)$. In particular, $\pi_\rho(k) = e^{-r_f T}$ for every $k \in \mathbb{R}$. If the market is perfect then π_ρ extends π to the whole space L^p.

Proof. A) It immediately follows from Theorem $7D$.

B) The assumptions lead to

$$\pi(g) + \pi(-g) = 0,$$

so Statement A) and (30) imply that

$$\pi_\rho(g) + \pi_\rho(-g) = \pi(g) + \pi(-g) = 0.$$

Since Theorem $7D$ shows that $\pi_\rho(g) \leq \pi(g)$ and $\pi_\rho(-g) \leq \pi(-g)$ the equality above can only hold if both inequalities become equalities. □

For imperfect markets π_ρ may strictly reduce the spread, and consequently it does not necessarily equal π on Y. Next let us characterize the equality $\pi(g) = \pi_\rho(g)$ and provide a very simple counter-example.

Proposition 9. *Consider $g \in Y$ and a dual solution z^*.*[5] *$\pi(g) = \pi_\rho(g)$ holds if and only if $E(z^* g) = \pi(g) e^{r_f T}$.*

Proof. The result trivially follows from

$$\pi_\rho(g) = F(0, g) e^{-r_f T} = E(z^* g) e^{-r_f T}.$$

□

[5] Recall that z^* must exist according to Theorem 4 and does not depend on P according to (23).

Remark 4. *(Counter-example illustrating that $\pi_\rho(g) < \pi(g)$ may hold). Consider* $\Omega = \{\omega_1 \omega_2\}$, $\mu(\omega_1) = 0.1$, $\mu(\omega_2) = 0.9$, *and*

$$\pi(\alpha(1,1) + \beta(1,0)) = \begin{cases} \alpha + 0.7\beta, & if\ \beta \geq 0 \\ \alpha + 0.4\beta, & if\ \beta \geq 0 \end{cases}.$$

The example indicates that the risk-free rate vanishes and the risky asset with pay-off $(1,0)$ *has a bid price equal to* 0.4 *and an ask price equal to* 0.7*. Suppose that*

$$\Delta_\rho = \{(z_1, z_2);\ 0.1z_1 + 0.9z_2 = 1 \text{ and } 0 \leq z_i \leq 5,\ i = 1,2\}.$$

It will be seen in Section 6 *that* Δ_ρ *corresponds to the Conditional Value at Risk with* $0.8 = 80\%$ *as the level of confidence. It follows from Theorem* 4 *that* $\pi_\rho(1,0)$ *is the optimal value of*

$$\begin{cases} Max\ 0.1z_1 \\ 0.4 \leq 0.1z_1 \leq 0.7 \\ 0.1z_1 + 0.9z_2 = 1 \\ 0 \leq z_i \leq 5, & i = 1,2 \end{cases}.$$

Obviously, $\pi_\rho(1,0) = 0.5 < 0.7 = \pi(1,0)$. □

Since π_ρ reduces the spread and satisfies the same properties as π (Theorems 7 and 8), one could use π_ρ to generate a new pricing rule π_ρ^* by applying the same method used to construct π_ρ from π. Next we will prove that $\pi_\rho^* = \pi_\rho$, so it is useless to extend the pricing rule two times. However, the equality $\pi_\rho^* = \pi_\rho$ shows that π_ρ may be an exact extension of π in particular situations, even if the market is imperfect.

Theorem 10. *The Stochastic Discount Factors of* (π, ρ) *and* (π_ρ, ρ) *coincide.*[6] *Consequently,*

$$\begin{array}{l} Max\ \{E(gz);\ z \in D_f\} = \\ Max\ \left\{E(gz);\ z \in \Delta_\rho,\ E(yz) \leq \pi_\rho(y)\, e^{r_f T} \text{ for every } y \in L^p\right\} = \\ Max\ \left\{E(gz);\ z \in \Delta_\rho,\ E(yz) \leq \pi_\rho(y)\, e^{r_f T} \text{ for every } y \in Y\right\}, \end{array}$$

i.e., taking $P = 0$*, if we construct a new pricing rule* π_ρ^* *from* π_ρ *then* $\pi_\rho^* = \pi_\rho$.

[6] In other words: If $z \in \Delta_\rho$ then

$$E(yz) \leq \pi(y)\, e^{r_f T}$$

for every $y \in Y$ if and only if

$$E(gz) \leq \pi_\rho(g)\, e^{r_f T}$$

for every $g \in L^p$.

Proof. If $z \in \Delta_\rho$ and $E(yz) \le \pi_\rho(y)\, e^{r_f T}$ for every $y \in L^p$ (or just for every $y \in Y$) then $z \in D_f$ owing to Theorem $7D$. Conversely, suppose that $z \in D_f$ and take $y \in L^p$. Then $\pi_\rho(y)\, e^{r_f T}$ is the maximum value of $E(yz')$ with $z' \in D_f$, so $E(yz) \le \pi_\rho(y)\, e^{r_f T}$. □

A very important consequence of the latter theorem is that natural assumptions on π prevent the existence of arbitrage for π_ρ.

Corollary 11. *Suppose that there exists $z^* \in D_f$ which is strictly positive, i.e.,*

$$E(yz^*) > 0$$

for every $y \in L^p$ such that $y \ge 0$ and $y \ne 0$.[7] *Then π_ρ does not generate arbitrage opportunities, i.e., $g \ge 0$ and $\pi_\rho(g) \le 0$ imply that $g = 0$ and $\pi_\rho(g) = 0$.*[8]

Proof. Suppose that $g \ge 0$ and $\pi_\rho(g) \le 0$. Then $E(gz^*) \ge 0$, with equality if and only if $g = 0$. Besides, the latter theorem implies that

$$E(gz^*) \le \pi_\rho(g)\, e^{r_f T} \le 0,$$

so the equality holds. □

Finally let us show that the proposed extension π_ρ also "improves" the "classical extension", usual in incomplete markets. So, consider $g \in L^p$ and the optimization problem

$$\begin{cases} Min\ \pi(y) \\ y \ge g \\ y \in Y \end{cases}$$

and denotes by $\pi^*(g)$ the infimum of the problem above ($\pi^*(g) = \infty$ if the problem is not feasible). Then we have:

Proposition 12. *If ρ is coherent then $\pi^*(g) \ge \pi_\rho(g)$ holds for every $g \in L^p$.*

Proof. The conclusion is obvious if $\pi^*(g) = \infty$, so assume that $\pi^*(g) < \infty$. Take $n \in \mathbb{N}$ and $y_n \in Y$, $y_n \ge g$ such that

$$\pi^*(g) \ge \pi(y_n) - \frac{1}{n}.$$

Then, $y_n \ge g$ and Theorems $7D$ and $7E$ lead to

$$\pi^*(g) \ge \pi(y_n) - \frac{1}{n} \ge \pi_\rho(y_n) - \frac{1}{n} \ge \pi_\rho(g) - \frac{1}{n},$$

and the result trivially follows because $n \in \mathbb{N}$ is arbitrary. □

[7] or equivalently, $z^* > 0$ almost surely.

[8] Bearing in mind (13) with a similar proof one can see that if ρ is coherent then Assumption 1 prevents the existence of the so called "strong" or "second type" arbitrage (Jaschke and Küchler, 2001), *i.e.*, the existence of $g \in L^p$ such that $g \ge 0$ and $\pi_\rho(g) < 0$.

5. DEALING WITH DEVIATIONS

If we consider a general lower semi-continuous deviation measure σ, *i.e.*, a sub-additive and homogeneous function satisfying (15) and (16), then, as indicated in the second section, (14) establishes a relationship between σ and a risk measure ρ for which we can construct the pricing rule π_ρ, denoted by $\pi_{\sigma-E}$ in this section owing to (14).

A particular interesting case, very used in finance, arises if $p = 2$ and $\sigma = \sigma_2$ is the standard deviation given by

$$\sigma_2(y) = \left(\int_\Omega (y - E(y))^2 \, d\mu \right)^{1/2}$$

for every $y \in L^2$. In such a case L^2 is a Hilbert space so, if we assume that the market is perfect, Y is closed and π is continuous, the Riesz Representation Theorem guarantees the existence of a unique $y_0 \in Y$ such that

$$\pi(y) = E(y_0 y) \tag{31}$$

holds for every $y \in Y$. The literature has often proposed extensions of π to the whole space L^2 by considering an element y_1 orthogonal to Y and defining

$$\pi_{y_0+y_1}(y) = E\left[(y_0 + y_1)\, y\right]$$

for every $y \in L^2$.[9] A particular interesting example arises if $y_1 = 0$ since π_{y_0} becomes the composition of the orthogonal projection on Y and π, or, in other words, $\pi_{y_0}(y)$ coincides with $\pi(\Pi(y))$ for every $y \in L^2$, $\Pi(y)$ denoting the element in Y closest to y.

Obviously, the extensions above are specially useful when there exists y_1 orthogonal to Y and such that $y_0 + y_1 > 0$ almost surely (respectively, $y_0 > 0$ almost surely) because this inequality guarantees the absence of arbitrage for the pricing rule $\pi_{y_0+y_1}$ (respectively, π_{y_0}).

Actually, under the general assumptions above, as far as we were able to analyze the problem there were no clear relationships between the (non necessarily linear) extension π_{σ_2-E} and the extension $\pi_{y_0+y_1}$. However, for those cases such that both extensions generate arbitrage free pricing rules (see Corollary 11) π_{σ_2-E} will be larger than $\pi_{y_0+y_1}$.

Proposition 13. *Suppose that the market is perfect, Y is closed and π is continuous. Consider the unique $y_0 \in Y$ such that (31) holds for every $y \in Y$. Suppose finally that there exists $z^* \in L^2$ such that*

[9] See, among others, Schweizer (1995) and Luenberger (2001).

a) $E(z^*) = 0$, $\sigma_2(z^*) \leq 1$ *and* $1 + z^* > 0$ *almost surely.*
b) $(1 + z^*) e^{-r_f T} - y_0$ *is orthogonal to* Y*.*
Then $\pi_{\sigma_2 - E}$ *and* $\pi_{(1+z^*)e^{-r_f T}}$ *do not generate arbitrage opportunities and*

$$\pi_{\sigma_2 - E}(y) \geq \pi_{(1+z^*)e^{-r_f T}}(y)$$

holds for every $y \in L^2$*.*

Proof. It is shown in Rockafellar *et al.* (2006) that [10]

$$\Delta_{\sigma_2 - E} = \left\{1 + z;\ z \in L^2,\ E(z) = 0 \text{ and } \sigma_2(z) \leq 1\right\}.$$

Hence Condition *a)* imposes that $1 + z^*$ is strictly positive and belongs to $\Delta_{\sigma_2 - E}$. Moreover Condition *b)* leads to

$$E((1 + z^*) y) = E\left(y_0 e^{r_f T} y\right) = e^{r_f T} \pi(y),$$

for every $y \in Y$, which implies that $1 + z^*$ is in D_f (see (24)). Consequently Corollary 11 implies that $\pi_{\sigma_2 - E}$ does not generate arbitrage opportunities. Similarly, $1 + z^* > 0$ almost surely leads to the absence of arbitrage opportunities for $\pi_{(1+z^*)e^{-r_f T}}$. Finally, taking $P = 0$ (29) and (23) lead to

$$\begin{aligned} &\pi_{\sigma_2 - E}(y) \\ &= e^{-r_f T} Max\ \{E(yz);\ z \in D_f\} \\ &\geq e^{-r_f T} E((1 + z^*) y) \\ &= \pi_{(1+z^*)e^{-r_f T}}(y) \end{aligned}$$

for every $y \in L^2$. □

Remark 5. *A very particular case arises if* $(1 + z^*) e^{-r_f T} = y_0$*, i.e., if* $\pi_{(1+z^*)e^{-r_f T}}$ *is the composition of* π *and the orthogonal projection* Π*. This situation appears if* $y_0 > 0$ *almost surely,* $E(y_0) = e^{-r_f T}$ *and* $\sigma_2(y_0) \leq e^{-r_f T}$*, in which case* $\pi_{\sigma_2 - E}$ *and* π_{y_0} *do not generate arbitrage opportunities and* $\pi_{\sigma_2 - E} \geq \pi_{y_0}$ *holds.* □

6. Using the Conditional Value at Risk

In this section we will focus on the Conditional Value at Risk, since it is becoming a very well-known Coherent and Expectation Bounded Risk Measure that respects the second order Stochastic Dominance (Ogryczak and Ruszczynski, 2002). In particular, this risk function has been used, amongst many others, by Wang (2000) in some insurance-linked problems, Alexander *et al.* (2006) in portfolio choice problems

[10] see also Balbás *et al.* (2008*a*).

involving derivatives, Mansini *et al.* (2007) in portfolio choice problems involving bonds and shares, or Balbás *et al.* (2008*a*) in optimal reinsurance problems.

If $0 < 1 - \mu_0 < 1$ represents the level of confidence then the $CVaR_{\mu_0}$ may be defined in L^1 and Rockafellar *et al.* (2006) showed that

$$\Delta_{CVaR_{\mu_0}} = \left\{ z \in L^\infty;\ 0 \leq z \leq \frac{1}{\mu_0} \text{ and } E(z) = 1 \right\}.$$

Suppose the same hypotheses as in the second section as well as Assumption 1, *i.e.*, the existence of $P_0 \in \mathbb{R}$ and $g_0 \in L^1$ such that (17) is bounded, *i.e.*, the value of $CVaR_{\mu_0}(y)$ cannot tend to $-\infty$. According to Theorem 4 there are SDF of $\left(\pi, CVaR_{\mu_0}\right)$, *i.e.*, D_f is non void.

The following result characterizes primal and dual solutions for $\rho = CVaR_{\mu_0}$, as well as it allows us to compute the value $\pi_{CVaR_{\mu_0}}(g)$ for $g \in L^1$ in practical applications.

Theorem 14. *Consider $g \in L^1$ and suppose that (18) attains it optimal value for g.*[11] *Consider also $z^* \in D_f$. Then, z^* solves (23) if and only if there exist a partition*

$$\Omega = \Omega_0 \cup \Omega^* \cup \Omega_{\mu_0}$$

of Ω composed of measurable sets and $y^* \in Y$ such that:
A) $z^* = 0$ on Ω_0 and $z^* = \frac{1}{\mu_0}$ on Ω_{μ_0}.
B) $y^* \leq g$ on Ω_0, $y^* = g$ on Ω^* and $y^* \geq g$ on Ω_{μ_0}.
C) $E(z^* y^*) = \pi(y^*) e^{r_f T}$.
Furthermore, in the affirmative case we have that y^* solves (17) and

$$\pi_{CVaR_{\mu_0}}(g) = E(z^* g)\, e^{-r_f T}. \tag{32}$$

Proof. Fix $P_1 \in \mathbb{R}$ and take (θ, y^*) solving (18) for P_1. If z^* solves (23) then (25) shows that $C)$ must hold and z^* must solve

$$\left\{ \begin{array}{l} Min\ E(y^* z) - E(gz) \\ E(z) = 1 \\ z \leq \frac{1}{\mu_0} \\ -z \leq 0 \\ z \in L^\infty \end{array} \right. . \tag{33}$$

[11] As in Proposition 6, if this property holds for a given $P_1 \in \mathbb{R}$ then it also holds for every $P \in \mathbb{R}$.

The Slater Qualification holds since $z = 1$ belongs to $\Delta_{CVaR_{\mu_0}}$ and satisfies the two inequalities in strict terms. Then z^* must satisfy the optimality conditions. Bearing in mind that the dual space of L^∞ is composed of $\mu-$continuous finitely additive measures on $\mathcal{F}$ with bounded variation (Horvàth, 1966), there exists a couple of non negative such a measures (α_1, α_2) and a real number α such that

$$\begin{cases} y^* - g = \alpha + \alpha_1 - \alpha_2 \\ \int_\Omega \left(z^* - \frac{1}{\mu_0}\right) d\alpha_1 = 0 \\ \int_\Omega z^* d\alpha_2 = 0 \end{cases} .$$

Denote by Ω_0 the set where z^* vanishes and by Ω_{μ_0} the set where $z^* = \frac{1}{\mu_0}$. The second and the third condition, along with $0 \leq z^* \leq \frac{1}{\mu_0}$, lead to $\alpha_1 = 0$ out of Ω_{μ_0} and $\alpha_2 = 0$ out of Ω_0. Thus, $\alpha_1 = y^* - g - \alpha$ on Ω_{μ_0} and $\alpha_2 = -y^* + g + \alpha$ on Ω_0, which shows that $\alpha_i \in L^1$, $i = 1, 2$.

If $\Omega^* = \Omega \setminus \left(\Omega_0 \cup \Omega_{\mu_0}\right)$ then $A)$ is obvious and $B)$ holds as long as $\alpha = 0$. If $\alpha \neq 0$ then Proposition 6 guarantees that $y^* - \alpha$ solves (17) for $P_2 = P_1 - \alpha e^{-r_f T}$, so take this new value for the P variable and rename $y^* - \alpha$ as y^*.

It only remains to prove (32). According to (29), and bearing in mind the objective function of (23),

$$\pi_{CVaR_{\mu_0}}(g) = P_2 + F(P_2, g) e^{-r_f T} = P_2 + \left(E(z^* g) - P_2 e^{r_f T}\right) e^{-r_f T},$$

and (32) holds.

Conversely, suppose that the existence of the partition and $y^* \in Y$ is fulfilled. Take

$$\begin{cases} \alpha_1 = y^* - g & \textit{on } \Omega_{\mu_0} \\ \alpha_1 = 0 & \textit{otherwise} \end{cases}$$

and

$$\begin{cases} \alpha_2 = -y^* + g & \textit{on } \Omega_0 \\ \alpha_2 = 0 & \textit{otherwise} \end{cases}$$

and it is clear that z^* satisfies the optimality conditions of (33). Since this problem is linear z^* is optimal. Hence

$$E(y^* z) - E(gz) \geq E(y^* z^*) - E(g z^*)$$

if $z \in \Delta_{CVaR_{\mu_0}}$ leads to the fulfillment of the first and the second expressions in (25) if $\theta = E(g z^*) - E(y^* z^*)$.

Take $P = \pi(y^*)$ so as to guarantee the fulfillment of the third expression in (25). Then $C)$ and $z^* \in D_f$ show that all the expressions in (25) hold and thus z^* solves (23) for P. $\square$

Another particular interesting case arises if (33) attains "bang-bang" solutions. More accurately we have:

Corollary 15. *Consider $g \in L^1$ and suppose that (18) attains it optimal value for g. Consider $z^* \in D_f$ and suppose the existence of a partition $\Omega = \Omega_0 \cup \Omega_{\mu_0}$ such that $z^* = 0$ on Ω_0 and $z^* = \frac{1}{\mu_0}$ on Ω_{μ_0}. Then, z^* solves (23) if and only there exists $y^* \in Y$ such that:*

A) $y^* \leq g$ on Ω_0 and $y^* \geq g$ on Ω_{μ_0}.

B) $E(z^* y^*) = \pi(y^*) e^{r_f T}$.

Furthermore, in the affirmative case we have that y^* solves (17) and (32) holds.

Proof. It immediately follows from the theorem above. □

Notice that the corollary above may be easily applied in practice. Indeed, on the one hand $E(z^*) = 1$ leads to

$$\mu\left(\Omega_{\mu_0}\right) = \mu_0, \tag{34}$$

and $E(z^* g)$ must be maximized on the other hand. Thus one must look for those measurable subsets Ω_{μ_0} satisfying (34) and making g as large as possible, and then check the fulfillment of A) and B) for some $y^* \in Y$.

7. CONCLUSIONS

This paper has proposed a new method to extend pricing rules in both incomplete and imperfect markets by using general risk functions, with special focus on Expectation Bounded Risk Measures and General Deviation Measures. It is easy to prevent the existence of arbitrage for the proposed extensions. These are continuous, bounded from above by the used risk function and reduce the bid/ask spread in the imperfect market case. Furthermore, the hedging strategy has been also studied and characterized. Some concrete examples and relationships with another extensions presented in the literature have been also analyzed.

The developed theory strongly depends on the duality properties of the Convex Optimization Theory in Banach Spaces, so the paper points out once again how Mathematical Programming may play a crucial role in Asset Pricing and Hedging, two major topics in Finance.

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