The Fourier-Stieltjes Transform and Absolutely Continuous Invariant Measures

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ABSTRACT

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We present some results on the existence of *absolutely continuous invariant measures* (*acim*'s) using the Fourier-Stieltjes transform. We consider the sequence of Perron Frobenius operators

$$\{f, P_\tau f, P_\tau^2 f, P_\tau^3 f, \ldots, P_\tau^n f, \ldots\},$$

induced by the nonsingular transformation $\tau : I \to I$, with $f \in \mathcal{L}^1$, and the associated sequence of Fourier-Stieltjes transforms

$$\{\mathcal{F}(F_0), \mathcal{F}(F_1), \mathcal{F}(F_2), \ldots, \mathcal{F}(F_n), \ldots\},$$

where $F_n(x) = \int_{-\infty}^{x} P_\tau^n f(u)du$ and $\mathcal{F}(F_n)(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x)$. The main result is: if τ is piecewise monotonic expanding and $\frac{1}{\frac{d}{dx}\tau(\cdot)}$ is a function of bounded variations, then τ has an *acim*. Although this is a known result, the method of proof is new and may allow generalizations needed. Finally, we introduce criteria on the Fourier Stieltjes Transforms needded to ensure the existence of *acim*'s.

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CONTENTS

1 INTRODUCTION

The evolution of a dynamical system is usually represented by the iteration of a function $\tau : I \to I$.

$$\{x, \tau(x), \tau^2(x), \ldots, \tau^n(x), \ldots\}.$$

In a measure-theoretic approach to dynamical systems, we consider the sequence

$$\{f, P_\tau f, P_\tau^2 f, P_\tau^3 f, \ldots, P_\tau^n f, \ldots\} \tag{1}$$

where f is a probability density function and P_τ is the Frobenius Perron operator of τ. This sequence is similar to the sequence of distribution functions

$$\{F_0, F_1, F_2, F_3, \ldots, F_n, \ldots\}.$$

where $F_n(x) = \int_{-\infty}^{x} P_\tau^n f(u) d\lambda(u)$ and $\lambda(\cdot)$ is the Lebesgue measure.

The intention of the present work is to find criteria on τ which will guarantee the existence of an *absolutely continuous invariant measure* (*acim*) for τ. We shall use the Fourier-Stieltjes Transform of the Perron Frobenius operator, and consider the sequence

$$\{\mathcal{F}(F_0), \mathcal{F}(F_1), \mathcal{F}(F_2), \ldots, \mathcal{F}(F_n), \ldots\}. \tag{2}$$

The problem of existence of an *acim* has been studied before. One of the main results is due to Lasota and Yorke [3], where the existence of an *acim* is proved assuming τ is a piecewise continuous expanding transformation.

An important observation is that using the Lasota-Yorke's Theorem. we can conclude that under certain assumptions on τ. there exists an *absolutely continuous invariant distribution function (acidf)* F^* of τ such that

$$\mathcal{F}(F_{n_k}) \rightarrow \mathcal{F}(F^*) \tag{3}$$

uniformly when $k \rightarrow \infty$. with $\{F_{n_k}\}_{k=1}^{\infty}$ subsequence of (2).

Here we prove that under similar assumptions. in our case considering the sequence (2) instead of the sequence (1). results in

$$P_\tau^n f \longrightarrow f^* \quad as\ n \rightarrow \infty. \tag{4}$$

where $F^*(x) = \int_{-\infty}^{x} f^*(u)d\lambda(u)$ an *acidf* of τ.

An overview is shown in Table 1. where in the first column we show the iteration of the transformation $\tau : I \rightarrow I$. The second column shows the iterations of the corresponding density functions represented by the Perron Frobenius operator.

The third column shows the distribution functions and the fourth column shows the transformation of each of the Perron Frobenius operators P_τ^n. using the Fourier-Stieltjes transform.

It is important to remark that the link between the first column and the second. is given by measures of the distribution of the initial data represented here by x. The relation between the second and third columns is given by $F_n(x) = \int_{-\infty}^{x} P_\tau^n f(u)du$.

We are interested in the *acidf*s of τ. To this end we introduce criteria in terms of classes of functions to ensure the existence of such *acidf*s. leading to restrictions

Dynamics $(x \in I)$		Densities $(x \in I)$		Distributions $(x \in I)$		Fourier Stieltjes Transform $(t \in \mathbb{R})$
x	$\longrightarrow$	$f(x)$	$\longleftrightarrow$	$F_0(x)$	$\longleftrightarrow$	$\mathcal{F}(F_0)(t)$
$\tau(x)$	$\longrightarrow$	$P_\tau f(x)$	$\longleftrightarrow$	$F_1(x)$	$\longleftrightarrow$	$\mathcal{F}(F_1)(t)$
$\tau^2(x)$	$\longrightarrow$	$P_\tau^2 f(x)$	$\longleftrightarrow$	$F_2(x)$	$\longleftrightarrow$	$\mathcal{F}(F_2)(t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\tau^n(x)$	$\longrightarrow$	$P_\tau^n f(x)$	$\longleftrightarrow$	$F_n(x)$	$\longleftrightarrow$	$\mathcal{F}(F_n)(t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boxed{?}$	$\longrightarrow$	$\boxed{??}$	$\longleftrightarrow$	$\boxed{???}$	$\longleftrightarrow$	$\boxed{????}$

TABLE 1

imposed on τ in order to obtain the result of $\boxed{??}$, whenever these limit exists.

Since the problem of finding the limit $\boxed{?}$ in many cases is impossible, due to chaotic behavior of the transformation τ, we focus on the following question: what is $\boxed{??}$ and/or $\boxed{????}$ when these limits exists.

The first question to solve is then, under which circumstances do $\boxed{????}$ (and therefore $\boxed{??}$) exists. We deal with this convergence problem in Section 2 of Chapter 3. Then we consider the problem of the existence of *acidf*s in Section 3 of the same Chapter.

Some remarks of interest: in the case of the limit $\boxed{??}$, the Theorem of Lasota and Yorke confirms the existence of an *acim* when τ is piecewise monotonic continuous and expanding transformation. Using this result and the fact that the limit $\boxed{??}$ is a density function, we can see that the limit $\boxed{????}$ has to exist, and therefore under the same assumptions, a subsequence of the sequence

$$\{\mathcal{F}(F_0), \mathcal{F}(F_1), \mathcal{F}(F_2), \ldots, \mathcal{F}(F_n), \ldots\}$$

should converge uniformly to $\mathcal{F}(F^*)$, where the uniform convergence of the Fourier-Stieltjes is due to the Continuity Theorem of Fourier-Stieltjes transforms (see Theorem 2.2.4).

We know that under similar assumptions as in the Theorem of Lasota and Yorke, there must be a way to prove the existence of *acim*'s of τ using the Fourier-Stieltjes Transform of the Perron Frobenius operator. This is the main motivation for the use of this alternative approach to prove the existence of *acim*'s, in our case reduced to the proof of the existence of *acidf*s with respect to τ.

The second motivation, is to consider a possible way of relaxing the assumptions of the Lasota and Yorke Theorem to get a new type of existence theorem for the *acidf*s of τ. In Chapter 4 we present general sufficient criteria on the Fourier-Stieltjes Transform of the Perron Frobenius operator in order to ensure the existence of *acidf*s for the transformation τ.

2 BACKGROUND MATERIAL

Before we start the presentation of the main results in Chapter 3. we introduce in the present Chapter some concepts from measure theory. functional analysis. harmonic analysis (mainly related with the Fourier-Stieltjes transform) and ergodic theory (principally the Perron Frobenius operator).

2.1 MEASURE THEORY AND FUNCTIONAL ANALYSIS

In the following. unless it is specified otherwise. we will consider I to be a closed real interval. We use $\mathcal{B}(I)$ to denote the Borel set of I and λ to denote the Lebesgue measure of any set of $\mathcal{B}(I)$. We will represent a measurable space of X with measure μ by $(X. \mathcal{B}(X). \mu)$.

Definition 2.1.1 (SPACE $\mathcal{L}^p_S$) We define the space $\mathcal{L}^p_S$ on the measurable space $(S. \mathcal{B}(S). \lambda)$ as follows:

$$\mathcal{L}^p_S = \left\{ f : \int_S |f(x)|^p \, d\lambda(x) < \infty \right\}.$$

where p is such that $1 \leq p < \infty$.

Remark 2.1.1 We will use in the sequel the representation of $\mathcal{L}^P$ for the space $\mathcal{L}^p_S$ on $(S. \mathcal{B}(S). \lambda)$. when $S = \mathbb{R}$.

Definition 2.1.2 ($\mathcal{L}^p$ NORM) Let $f \in \mathcal{L}^p$, then we define the norm in $\mathcal{L}^p$ as the functional $\|\cdot\|_{\mathcal{L}^p} : \mathcal{L}^p \longrightarrow \mathbb{R}$. defined by

$$\|f\|_{\mathcal{L}^p} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Definition 2.1.3 (DENSITY) Let $f \in \mathcal{L}^1$. We say that f is a density function if and only if $f(x) \geq 0$ for almost all $x \in \mathbb{R}$. and $\|f\|_{\mathcal{L}^1} = 1$.

Definition 2.1.4 (DISTRIBUTION) A right continuous positive and non decreasing function F on $\mathbb{R}$ with $lim_{k \to -\infty} F(k) = 0$ and $lim_{k \to \infty} F(k) = 1$ is called a distribution function.

Definition 2.1.5 (INVARIANT MEASURE) We say that the measure μ is invariant (or measure preserving) with respect to the transform $\tau : I \to I$ if and only if for any $A \in \mathcal{B}(I)$. follows that

$$\mu(\tau^{-1}A) = \mu(A).$$

Definition 2.1.6 (BOUNDED VARIATION) The function $f : I \to \mathbb{R}$ is said to be of bounded variation if and only if for any $\mathcal{P}_I$ partition of I.

$$\sup_{\mathcal{P}_I} \sum_{i=0}^{n} |f(x_i) - f(x_{i-1})| < \infty$$

where $x_0 < x_1 < \cdots < x_n$ denote the border points of the partition $\mathcal{P}_I$.

Definition 2.1.7 (TOTAL VARIATION) Let $f \in BV_I$. then we define the total variation of f in I by

$$\bigvee_I f = \sup_{\mathcal{P}_I} \sum_{i=0}^{n} |f(x_i) - f(x_{i-1})|.$$

where $\mathcal{P}_I$ represents a partition of I with $x_0 < x_1 < \cdots < x_n$ border points.

Definition 2.1.8 (NON-SINGULAR TRANSFORMATION) Let λ be Lebesgue measure. A function $f : I \to I$ is called non-singular if and only if for any $A \in \mathcal{B}(I)$. such that $\lambda(A) = 0$. then $\lambda(f^{-1}(A)) = 0$.

Definition 2.1.9 (acm) Let λ and μ be two measures on the same measurable space. We say that μ is an Absolutely Continuous Measure (*acm*) with respect to λ (and we write $\mu << \lambda$) if and only if

$$\lambda(A) = 0 \Rightarrow \mu(A) = 0.$$

for all $A \in \mathcal{B}(I)$.

Definition 2.1.10 (acim) Let μ be *acm* with respect to λ. Then μ is an Absolutely Continuous Invariant Measure (*acim*) with respect to λ under the non-singular transformation $\tau : I \to I$ if and only if

$$\mu(\tau^{-1}A) = \mu(A).$$

where $A \in \mathcal{B}(I)$.

Remark 2.1.2 (acdf) We say that F is an Absolutely Continuous Distribution Function (*acdf*) if the measure $\mu([a.b]) = F(b) - F(a)$ is an acm.

Remark 2.1.3 (acidf) We say that F is an Absolutely Continuous Invariant Distribution Function (*acidf*) with respect to a non singular transformation τ if the measure $\mu([a.b]) = F(b) - F(a)$ is an acim with respect to τ.

Definition 2.1.11 (VAGUE CONVERGENCE) A sequence of normed measures $\{\mu_n\}_{n=1}^{\infty}$ is said to converge vaguely to the normed measure μ if and only if there exists a dense subset $J \subset \mathbb{R}$ such that

$$\forall a \in J, b \in J: \ \mu_n((a,b]) \to \mu((a,b]).$$

Theorem 2.1.1 (LEBESGUE DOMINATED CONVERGENCE THEOREM) Let $g : I \to \mathbb{R}$ be an integrable function. $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions such that $|f_n| \le g$ on I for all n. If $f = \lim_{n\to\infty} f_n$ then

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Proof: See [19]. □

Remark 2.1.4 (WEAK CONVERGENCE) A sequence of functions $h_n(x)$ is said to converge weakly to a limiting function $h(x)$ if

$$\lim_{k\to\infty} h_k(x) = h(x)$$

for all continuity points x of $h(x)$.

Theorem 2.1.2 (HELLY'S FIRST THEOREM) Every sequence $\{F_n\}_{n=1}^{\infty}$ of uniformly bounded non-decreasing functions, contains a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ which converges weakly to some non-decreasing bounded function F.

Proof: See [13, page 44, Theorem 3.5.1]. □

Theorem 2.1.3 (HELLY'S SECOND THEOREM) Let $g(x)$ be a continuous function and assume that $\{F_k\}_{k=1}^{\infty}$ is a sequence of uniformly bounded , non-decreasing functions which converge weakly to some function $F(x)$ at all points of the interval $[a,b]$, then

$$\lim_{k\to\infty} \int_a^b g(x)dF_k(x) = \int_a^b g(x)dF(x)$$

Proof: See [13, page 45, Theorem 3.5.2] $\square$

Corollary 2.1.3.1 (EXTENSION OF HELLY'S SECOND THEOREM) Let $g(x)$ be continuous and bounded in the infinite interval $-\infty < x < \infty$ and let $\{F_k(x)\}_{k=1}^{\infty}$ be a sequence of non-decreasing, uniformly bounded functions which converges weakly to some function $F(x)$. Suppose that

$$\lim_{k\to\infty} F_k(-\infty) = F(-\infty) \quad \textit{and} \quad \lim_{k\to\infty} F_k(+\infty) = F(+\infty).$$

then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} g(x)dF_k(x) = \int_{-\infty}^{\infty} g(x)dF(x).$$

Proof: See [13, page 45] $\square$

Theorem 2.1.4 ($\mathcal{L}_S^p \Rightarrow \mathcal{L}_S^1$) Let $p > 1$. $S \subset \mathbb{R}$ bounded and $f \in \mathcal{L}_S^p$. Then $f \in \mathcal{L}_S^1$.

Proof: See [1, page 34 item 23c]. $\square$

Theorem 2.1.5 The sequence of functions $\{f_n\}_{n=1}^{\infty}$, defined on I, converges uniformly on I if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N, n \geq N$, implies

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

for every $x \in I$.

Proof: See [19, page 147, Theorem 7.8]. $\square$

2.2 HARMONIC ANALYSIS

Definition 2.2.1 (FOURIER-STIELTJES TRANSFORM) Consider the measure $F(x)$. We define the Fourier-Stieltjes Transform of F by

$$\mathcal{F}(F)(t) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

and when $F(x) = \int_{-\infty}^{x} f(u)d\lambda(u)$ we also call $\mathcal{F}(F)(t)$ the Fourier-Stieltjes Transform of the $\mathcal{L}^1$ function f.

Properties 2.2.1 (of the Fourier-Stieltjes Transform) Consider the distribution function F. Then we have the following properties of the Fourier-Stieltjes Transform of F:

(*i*) $\mathcal{F}(F) : \mathbb{R} \rightarrow \mathbb{C}$.

(*ii*) $\mathcal{F}(F)(0) = 1$. $|\mathcal{F}(F)(t)| \leq 1$ and $\mathcal{F}(F)(-t) = \overline{\mathcal{F}(F)(t)}$.

(*iii*) $\mathcal{F}(F)(t)$ is uniformly continuous in $\mathbb{R}$.

(*iv*) The function $g(t) = Re\,(\mathcal{F}(F)(t))$ is also a Fourier-Stieltjes Transform function.

(*v*) The function $g(t) = |\mathcal{F}(F)(t)|^2 = \mathcal{F}(F)(t) \cdot \overline{\mathcal{F}(F)(t)}$ is also a Fourier-Stieltjes transform function.

(*vi*) If $g(t) = \mathcal{F}(F_1)(t) \cdot \mathcal{F}(F_2)(t)$ then $g(t)$ is itself a Fourier-Stieltjes Transform function.

Definition 2.2.2 (FAMILY OF FOURIER STIELTJES TRANSFORMS) We define the family of Fourier Stieltjes transforms by

$$\mathbb{H} = \{\mathcal{F}(F) : \mathbb{R} \to \mathbb{C} \mid F \text{ is a distribution function}\} .$$

Remark 2.2.1 There exists different ways to represent a family of Fourier-Stieltjes transforms. One possible way is to consider the properties of the Fourier-Stieltjes Transform given above. See [13] for other representations.

Definition 2.2.3 (LIM) Consider the sequence of $\mathcal{L}^2$ functions $\{f_n\}_{n=1}^{\infty}$. Then f is the limit in $\mathcal{L}^2$ (we denote it by $f(t) = \underset{n\to\infty}{\mathrm{LIM}}\ f_n(t)$) if and only if

$$\|f(t) - f_n(t)\|_{\mathcal{L}^2} \to 0 \qquad when\ n \to \infty.$$

Theorem 2.2.1 (PLANCHEREL'S THEOREM) Let $\mathcal{F}(F) \in \mathcal{L}^2$. Then there exists $f \in \mathcal{L}^2$ such that

$$f(t) = \underset{n\to\infty}{\mathrm{LIM}} \frac{1}{2\pi} \int_{-n}^{n} e^{-iut} \mathcal{F}(F)(u) du$$

and

$$\|f\|_{\mathcal{L}^2} = \|\mathcal{F}(F)\|_{\mathcal{L}^2}$$

where

$$\mathcal{F}(F)(t) = \int_{-\infty}^{\infty} e^{iut} f(u) du.$$

Proof: See [2, page 24] or [9, page 51]. □

Theorem 2.2.2 (LEVY'S INVERSION FORMULA) Let $h \in \mathbb{R}$. we have that

$$F(x+h) - F(x) = \underset{n\to\infty}{\mathrm{LIM}} \frac{1}{2\pi} \int_{-n}^{n} \frac{1-e^{-ith}}{it} e^{-itx} \mathcal{F}(F)(t) dt$$

provided that x and $x+h$ are continuity points of $F(x)$.

Proof: See [13, page 31, Theorem 3.2.1]. □

Theorem 2.2.3 (UNIQUENESS) If two distribution functions F and G have the same Fourier-Stieltjes transform, then $F = G$ almost everywhere.

Proof: See [6, page 143, Theorem 6.2.2]. □

Theorem 2.2.4 (CONTINUITY THEOREM) Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions. Then F_n converges weakly to F^* if and only if $\mathcal{F}(F_n)(t)$ converges to $\mathcal{F}(F^*)(t)$ for all t and $\mathcal{F}(F^*)(t)$ is continuous function in a neighborhood around zero.

Proof: See [13, page 48, Theorem 3.6.1]. □

Theorem 2.2.5 (UNIFORM CONVERGENCE) If $\{\mathcal{F}(F_n)\}_{n=1}^{\infty}$ converges to $\mathcal{F}(F^*)(t)$, then the convergence is uniform in every t-interval $[-T, T]$.

Proof: See [13, page 50, Corollary 1]. □

2.3 ERGODIC THEORY

Definition 2.3.1 (PERRON FROBENIUS OPERATOR) Let $\tau : I \to I$, $f \in \mathcal{L}^1$. The Perron Frobenius operator $P_\tau : \mathcal{L}^1 \to \mathcal{L}^1$, is defined by the equation:

$$\int_A P_\tau f(x) d\lambda(x) = \int_{\tau^{-1}A} f(x) d\lambda(x),$$

where $A \subset \mathcal{B}(I)$.

Definition 2.3.2 (KOOPMANN OPERATOR) Let $\tau : I \to I$. We define the Koopmann operator $U_\tau : \mathcal{L}^\infty \to \mathcal{L}^\infty$, by

$$U_\tau g(x) = g(\tau(x)).$$

Properties 2.3.1 (PERRON FROBENIUS OPERATOR) Consider $f \in \mathcal{L}^1$, then we have the following properties of the Perron Frobenius operator $P_\tau^n f$, assuming $\tau : I \to I$ is a non-singular transformation:

(*i*) **(UNIQUENESS)** $P_\tau f$ is an $\mathcal{L}^1$ function *a.e.* unique such that $\int_A P_\tau f(x)d\lambda(x) = \int_{\tau^{-1}A} f(x)d\lambda(x)$. (The proof is based in the Radon-Nikodym's Theorem)

(*ii*) **(LINEARITY)** $P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g$.

(*iii*) **(POSITIVITY)** If $f \geq 0$ then $P_\tau f \geq 0$.

(*iv*) **(MASS PRESERVATION)** $\int_I P_\tau f(x)d\lambda(x) = \int_I f(x)d\lambda(x)$

(*v*) **(CONTRACTION)** $\|P_\tau f\|_{\mathcal{L}^1} \leq \|f\|_{\mathcal{L}^1}$ for any $f \in \mathcal{L}^1$.

(*vi*) **(CONTINUOUS)** $\|P_\tau f_n - P_\tau f\|_{\mathcal{L}^1} \leq \|f_n - f\|_{\mathcal{L}^1}$.

(*vii*) **(COMPOSITION)** If $\sigma : I \to I$ is non-singular transformation, then $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$ *a.e.* In particular, $P_\tau^n f(x) = P_{\tau^n} f(x)$.

(*viii*) Let $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$. Then $\langle P_\tau f, g\rangle = \langle f, U_\tau g\rangle$, where the brackets represent scalar product and U_τ is the Koopmann operator.

(*ix*) Let $\tau : I \to I$ be non-singular transformation. Then $P_\tau f^* = f^* \Leftrightarrow \mu(A) = \int_A f^*(x)d\lambda(x)$ is τ-invariant.

For a proof any of these properties refer to [4].

Theorem 2.3.1 (LASOTA YORKE, ACIM) Let $I = [0, 1]$. Consider $\tau : I \to I$, piecewise C^2_I and $s = \inf_{x\in I} |\tau'(x)| > 1$ whenever the derivative exists. Then, for any f in $\mathcal{L}^1$,

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k_\tau f \xrightarrow{\mathcal{L}^1} f^*$$

as $n \to \infty$, where $f^* \in \mathcal{L}^1$ and has the following properties:

(i) $f \geq 0 \Rightarrow f^* \geq 0$

(ii) $\int_0^1 f^*(x)d\lambda(x) = \int_0^1 f(x)d\lambda(x)$.

(iii) $P_\tau f^* = f^* \Rightarrow \mu^*(A) = \int_A f^*(x)d\lambda(x)$ is *invariant measure*, with $A \in \mathcal{B}(I)$.

(iv) $f^* \in BV[0, 1]$ and $\exists c$ such that $\bigvee_I f^* \leq c\,\|f\|_{\mathcal{L}^1}$.

Proof: See [12]. □

From this last Theorem, we can conclude under the assumptions that τ is piecewise C^2 and expanding, that $\mu^*(A) = \int_A f^*(x)d\lambda(x)$ is an *acim*.

Proposition 2.3.1 (REPRESENTATION OF $\mathcal{F}(F_n)$) Let $I \subset \mathbb{R}$, $\tau : I \to I$. The Fourier-Stieltjes Transform of the Perron Frobenius operator P^n_τ (also called Fourier-Stieltjes transform of the measure $F_n(x) = \int_{-\infty}^x P^n_\tau f(u)d\lambda(u)$) can be represented as

$$\mathcal{F}(F_n)(t) = \int_I e^{it\tau^n(x)}dF(x),$$

where $F(x) = \int_{-\infty}^x f(u)d\lambda(u)$. In this case, we use the notation

$$\phi^n_f(t) = \int_I e^{it\tau^n(x)}dF(x)$$

to represent $\mathcal{F}(F_n)(t)$.

Proof: We have that

$$\mathcal{F}(F_n)(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x) = \int_{-\infty}^{\infty} e^{itx} P_\tau^n f(x) d\lambda(x)$$

using the property $(viii)$ of the Perron Frobenius operator. we obtain that

$$\langle e^{it\cdot}, P_{\tau^n} f(\cdot)\rangle_{\mathcal{L}^1} = \left\langle e^{it\tau^n(\cdot)}, f(\cdot)\right\rangle_{\mathcal{L}^1} = \int_I e^{it\tau^n(x)} f(x) d\lambda(x) = \int_I e^{it\tau^n(x)} dF(x)$$

then

$$\phi_f^n(t) = \mathcal{F}(F_n)(t) = \int_I e^{it\tau^n(x)} dF(x).$$

□

This last result is very important. because this is the representation used in the sequel for the Fourier-Stieltjes Transform of the Perron Frobenius operator.

3 ANALYSIS OF THE PROBLEM

In this Chapter, we show the existence of *acidf*s for τ using a bound on the Fourier-Stieltjes Transform $\phi_f^n(t)$. We assume τ is piecewise monotonic with $\left|\frac{d}{dx}\tau(x)\right|$ a function of bounded variation. The exposition is divided in four parts. The first Section presents previous results, the second Section shows the convergence of a subsequence of $\{\phi_f^n(t)\}_{n=1}^{\infty}$. In the third Section we present a bound on $\phi_f^n(t)$, which is used to prove the existence of an *acm* F^*. Finally, in the fourth Section we show that F^* is also an invariant measure with respect to τ with density function $f^* \in \mathcal{L}^1$. We notice that this result is similar to the Lasota Yorke Theorem.

3.1 PRELIMINARIES

Lemma 3.1.1 Let $I \subset \mathbb{R}$, $\tau : I \to I$, $\tau^n(x)$ differentiable *a.e.* in I for all $n \geq 1$. If $s_1 = \inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$ and $n > m$, then

$$\inf_{x\in I}\left|\frac{d}{dx}\tau^n(x)\right| > \inf_{x\in I}\left|\frac{d}{dx}\tau^m(x)\right|.$$

where x is a point where both $\frac{d}{dx}\tau^m(x)$ and $\frac{d}{dx}\tau^n(x)$ exist.

Proof: Let us define $s_n = \inf_{x\in I}\left|\frac{d}{dx}\tau^n(x)\right|$. Using the chain rule we have:

$s_2 = \inf_{x \in I} \left|\frac{d}{dx}\tau^2(x)\right| = \inf_{x \in I} |\tau'(\tau(x)) \cdot \tau'(x)| \geq s_1 \inf_{x \in I} |\tau'(x)|$

$\geq s_1 s_1.$

$s_3 = \inf_{x \in I} \left|\frac{d}{dx}\tau^3(x)\right| = \inf_{x \in I} \left|\tau'(\tau^2(x)) \cdot \frac{d}{dx}\tau^2(x)\right|$

$\geq \inf_{y \in I} |\tau'(y)| \cdot \inf_{x \in I} \left|\frac{d}{dx}\tau^2(x)\right| \geq s_2 \inf_{x \in I} |\tau'(x)| \geq s_2 s_1.$

$\vdots$

$s_n = \inf_{x \in I} \left|\frac{d}{dx}\tau^n(x)\right| = \inf_{x \in I} \left|\tau'(\tau^{n-1}(x)) \cdot \frac{d}{dx}\tau^{n-1}(x)\right|$

$\geq \inf_{y \in I} |\tau'(y)| \cdot \inf_{x \in I} \left|\frac{d}{dx}\tau^{n-1}(x)\right|$

$\geq s_{n-1} \inf_{x \in I} |\tau'(x)| \geq s_{n-1} s_1.$

It follows by induction that $s_n > s_m$ for all $n > m$. In other words

$$\inf_{x \in I} \left|\frac{d}{dx}\tau^n(x)\right| > \inf_{x \in I} \left|\frac{d}{dx}\tau^m(x)\right|$$

for all $n > m$. □

Lemma 3.1.2 Let $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$, $f \in C_I^1$ and λ be Lebesgue measure. Then $\int_I |f'(x)| \, d\lambda(x) = \bigvee_I f$.

Proof: See [4, page 19, Theorem 2.3.8], or [15]. □

Lemma 3.1.3 Let $I \subset \mathbb{R}$ interval, $f : I \to \mathbb{R}$ and $h \in BV_I$. Assume also that $\exists \alpha > 0$ such that $|h(x)| \geq \alpha$ for all $x \in I$. Then $\frac{1}{h}$ is a function of BV_I and

$$\bigvee_I \left(\frac{1}{h}\right) \leq \frac{1}{\alpha^2} \bigvee_I h.$$

Proof: See [4, page 18, Theorem 2.3.3]. □

Lemma 3.1.4 Let $I \subset \mathbb{R}$ interval. $\tau : I \to I$. τ piecewise C_I^1. $\inf_{x \in I} \left|\frac{d}{dx}\tau(x)\right| \geq \alpha > 1$. If $\frac{d}{dx}\tau(x)$ is a function of BV_I, then

$$\bigvee_I \left(\frac{1}{\frac{d}{dx}\tau^n(x)}\right) \leq M(\alpha).$$

for all $n \geq 1$ and $M(\alpha)$ a real constant independent of n.

Proof: First notice that $\frac{d}{dx}\tau(\cdot) \in BV_I$ implies that the function $\frac{1}{\frac{d}{dx}\tau(\cdot)}$ is also of bounded variation by Lemma3.1 3. Let $g_n(x) = \frac{1}{\frac{d}{dx}\tau^n(x)}$, then we notice with the help of the chain rule that

$$g_n(x) = \frac{1}{\frac{d}{dx}\left[\tau^{n-1}(\tau(x))\right]} = \frac{1}{\left[\tau^{n-1}(\tau(x))\right]' \cdot \tau'(x)} \tag{5}$$

Now, assume as hypothesis of induction that for some $k \in \mathbb{N}$,

$$\bigvee_I g_k \leq M(\alpha)$$

it is true. Then by definition of total variation we have

$$\bigvee_I g_k = \sum_{j=0}^{m(k)} \left|g_k(x_{j+1}^k) - g_k(x_j^k)\right|$$

where here the *supremum* is taken over all the possible finite partitions of the interval I, resulting in the partition with endpoints given by $\{x_0^k, x_1^k, \ldots, x_{m(k)}^k\}$, and where $m(k)$ represents the number of endpoints of the partition.

Now, consider the total variation of the function g_{k+1},

$$\bigvee_I g_{k+1} = \sum_{j=0}^{m(k)+m(k+1)} \left|g_{k+1}(x_{j+1}^{k+1}) - g_{k+1}(x_j^{k+1})\right|$$

The number of intervals of the partition considering the supremum of all possible partitions of I is given by $\{x_0^{k+1}, x_1^{k+1}, \ldots, x_{m(k+1)}^{k+1}\}$. Since it is greater or equal to the number of elements of the partition $\{x_0^k, x_1^k, \ldots, x_{m(k)}^k\}$ for any $k \geq 1$, then we can consider the variation of g_{k+1}as follows

$$\bigvee_I g_{k+1} = \sum_{j=0}^{m(k+1)} |g_{k+1}(x_{j+1}) - g_{k+1}(x_j)|$$

with $\{x_0, x_1, \ldots, x_{m(k)+m(k+1)}\} = \{x_0^{k+1}, x_1^{k+1}, \ldots, x_{m(k+1)}^{k+1}\} \cup \{x_0^k, x_1^k, \ldots, x_{m(k)}^k\}$

Now, using the expression (5), results

$$\begin{aligned}
\textstyle\bigvee_I g_{k+1} &= \textstyle\sum_{j=0}^{m(k)+m(k+1)} |g_{k+1}(x_{j+1}) - g_{k+1}(x_j)| \\
&= \textstyle\sum_{j=0}^{m(k)+m(k+1)} |g_k(\tau(x_{j+1})) \cdot g_1(x_{j+1}) - g_k(\tau(x_j)) \cdot g_1(x_j)| \\
&\leq \textstyle\sum_{j=0}^{m(k)+m(k+1)} |[g_k(\tau(x_{j+1})) - g_k(\tau(x_j))] \cdot g_1(x_{j+1})| + |g_k(\tau(x_j)) \cdot [g_1(x_{j+1}) - g_1(x_j)]| \\
&\leq \textstyle\sum_{j=0}^{m(k)+m(k+1)} |[g_k(\tau(x_{j+1})) - g_k(\tau(x_j))] \cdot g_1(x_{j+1})| \\
&\quad + \textstyle\sum_{j=0}^{m(k)+m(k+1)} |g_k(\tau(x_j)) \cdot [g_1(x_{j+1}) - g_1(x_j)]| \\
&\leq \tfrac{1}{\alpha^k} \textstyle\bigvee_I g_1 + \tfrac{1}{\alpha} \textstyle\bigvee_I g_k \\
&\leq \left(\tfrac{1}{\alpha^k} + \tfrac{k}{\alpha^k}\right) \textstyle\bigvee_I g_1 \leq \tfrac{k+1}{\alpha^k} \textstyle\bigvee_I g_1 \leq M(\alpha)
\end{aligned}$$

Hence, we have for all $n \geq 1$ that $\bigvee_I g_n$ is bounded by the constant $M(\alpha)$. □

Proposition 3.1.1 (INTEGRAL REPRESENTATION OF AN *acdf*) We say that $F(x)$ is an *acdf* if and only if there exists a function $f \in \mathcal{L}^1$ such that for any $x_1 < x_2$:

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(u)du. \tag{6}$$

where $f(u)$ is the derivative of $F(u)$ (properly speaking, $f(u)$ is the Radon Nikodym derivative of F, $f = \frac{dF}{d\lambda}$)

Proof: See [15], [6, page 10] or [13, page 5]. □

Lemma 3.1.5 Let $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}$. $\lim_{n\to\infty} \alpha_n = \alpha$. and $L = |\alpha|^2$. Then $L = \lim_{n\to\infty} |\alpha_n|^2$.

Proof: Let $\alpha = (u + iv)$ and $\alpha_n = (u_n + iv_n)$. We have that

$$\lim_{n\to\infty} \alpha_n = \alpha \Rightarrow \lim_{n\to\infty} (u_n + iv_n) = u + iv$$

therefore

$$\lim_{n\to\infty} u_n = u \qquad \textit{and} \qquad \lim_{n\to\infty} v_n = v$$

Now consider

$$L = |\alpha|^2 = |u + iv|^2 = u^2 + v^2$$

on the other hand,

$$\lim_{n\to\infty} |\alpha_n|^2 = \lim_{n\to\infty} |u_n + iv_n|^2 = \lim_{n\to\infty} \left(u_n^2 + v_n^2\right)$$

and since $u_n \to u$ and $v_n \to v$, then $u_n^2 \to u^2$ and $v_n^2 \to v^2$. Then $u_n^2 + v_n^2 \to u^2 + v^2$. Hence we obtain that

$$L = \lim_{n\to\infty} |\alpha_n|^2 .$$

□

The next result plays a key role in the proof of the existence of an *acidf* of τ, since the bound we present will be used to show that $o_f^n(t)$ is an $\mathcal{L}^2$ function, leading to the desired result related with the existence of an *acidf*.

Proposition 3.1.2 (BOUND ON $\phi_f^n(t)$) Let $I \subset \mathbb{R}$ bounded. $\tau : I \to I$ piecewise differentiable on I. $\frac{d}{dx}\tau(\cdot) \in BV$. $f : I \to \mathbb{R}$. $f \in \mathcal{L}^1 \cap C_I^1 \cap BV_I$. If $\inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$ then

$$\left|\phi_f^n(t)\right| \leq \omega_n(t). \tag{7}$$

where

$$\omega_n(t) = \begin{cases} 1 & when\ t \in [-1, 1] \\ \frac{M_n}{|t|} & when\ t \notin [-1, 1] \end{cases}.$$

$$M_n = \left[\frac{2}{s_n} + \bigvee_I \left(\frac{1}{\frac{d}{dx}\tau^n(\cdot)}\right)\right] \left[\sup_{b\in I}|f(b)| + \bigvee_I f\right].$$

$$s_n = \frac{1}{\inf_{x\in I}\left|\frac{d}{dx}\tau^n(x)\right|}.$$

Proof: Consider $[a, b] \subset I$ and $u \in [a, b]$. Define the function $T_n(u)$ by

$$T_n(u) = \int_a^u e^{it\tau^n(x)}dx \tag{8}$$

and the differential operators D_n as well as its transpose ${}^tD_n$:

$$D_n f = \frac{\frac{d}{dx}f}{it\frac{d}{dx}\tau^n(x)} \qquad and \qquad {}^tD_n f = \frac{-d}{dx}\left(\frac{f}{it\frac{d}{dx}\tau^n(x)}\right) \tag{9}$$

where it is clear that $D_n(e^{it\tau^n(x)}) = e^{it\tau^n(x)}$.

Finally, consider the transformation I_n defined by:

$$I_n = \int_a^b e^{it\tau^n(x)}f(x)dx.$$

Now, we have that the integral

$$\int_a^u e^{it\tau^n(x)}\, {}^tD_n(1)dx = -\int_a^u e^{it\tau^n(x)}\frac{d}{dx}\left(\frac{1}{it\frac{d}{dx}\tau^n(x)}\right)dx.$$

can be integrated by parts, obtaining

$$\begin{aligned}\int_a^u e^{it\tau^n(x)}\ {}^tD_n(1)dx &= -\int_a^u \frac{d}{dx}\left(e^{it\tau^n(x)}\frac{1}{it\frac{d}{dx}\tau^n(x)}\right)dx + \int_a^u \left(\frac{1}{it\frac{d}{dx}\tau^n(x)}\right)\frac{d}{dx}e^{it\tau^n(x)}dx \\ &= -\left[\frac{e^{it\tau^n(x)}}{it\frac{d}{dx}\tau^n(x)}\right]_{x=a}^u + \int_a^u D_n(e^{it\tau^n(x)})dx \\ &= \int_a^u e^{it\tau^n(x)}dx - \left[\frac{e^{it\tau^n(x)}}{it\frac{d}{dx}\tau^n(x)}\right]_{x=a}^u\end{aligned}$$

Then, it follows that

$$\int_a^u e^{it\tau^n(x)}dx = \left[\frac{e^{it\tau^n(x)}}{it\frac{d}{dx}\tau^n(x)}\right]_{x=a}^u + \int_a^u e^{it\tau^n(x)}\ {}^tD_n(1)dx$$

in other words, using the definition of the function $T_n(u)$ given in the formula (8), we have that

$$T_n(u) = \left[\frac{e^{it\tau^n(x)}}{it\frac{d}{dx}\tau^n(x)}\right]_{x=a}^u + \int_a^u e^{it\tau^n(x)}\ {}^tD_n(1)dx \tag{10}$$

Now, consider the following bound using the result obtained before in the equation (10) together with the fact that $\left|e^{i\tau^n(x)}\right| = 1$:

$$|T_n(u)| \leq \frac{1}{|t|}\left(\left|\frac{1}{\frac{d}{dx}\tau^n(u)}\right| + \left|\frac{1}{\frac{d}{dx}\tau^n(a)}\right| + \int_a^u \left|\frac{d}{dx}\left(\frac{1}{\frac{d}{dx}\tau^n(x)}\right)\right| dx\right) \tag{11}$$

thus

$$|T_n(u)| \leq \frac{1}{|t|}\left(\frac{2}{s_n} + \int_a^u \left|\frac{d}{dx}\left(\frac{1}{\frac{d}{dx}\tau^n(x)}\right)\right| dx\right) \tag{12}$$

Using the Fundamental Theorem of Calculus and the definition given in formula (8), we know that $T_n'(u) = e^{it\tau^n(u)}$. Therefore .

$$I_n = \int_a^b e^{it\tau^n(x)}f(x)dx = \int_a^b T_n'(x)f(x)dx.$$

Integrating by parts we obtain

$$I_n = \int_a^b \frac{d}{dx}\left(T_n(x)\cdot f(x)\right)dx - \int_a^b T_n(x)\cdot\frac{d}{dx}f(x)dx$$

then.

$$I_n = T_n(b)f(b) - \int_a^b T_n(x) \cdot \frac{d}{dx} f(x)dx \tag{13}$$

where in the last line we notice that by definition of the function $T_n(x)$. $T_n(a) = 0$.

Finally. let us prove the inequality (7). presenting a bound on I_n. We use the expression obtained in (13). and noticing that the bound of $|T_n(u)|$ given in formula (12) is an increasing function of u. we have that

$$|I_n| \le \frac{1}{|t|}\left[\frac{2}{s_n} + \int_a^u \left|\frac{d}{dx}\left(\frac{1}{\frac{d}{dx}\tau^n(x)}\right)\right| dx\right]\left[|f(b)| + \int_a^b \left|\frac{d}{dx} f(x)\right| dx\right].$$

With these bounds in mind. we can consider the case for $|t| > 1$:

$$|I_n| \le \frac{\delta_{(-\infty,1)\cup(1,\infty)}(t)}{|t|}\left[\frac{2}{s_n} + \bigvee_{[a,b]}\left(\frac{1}{\frac{d}{dx}\tau^n(\cdot)}\right)\right]\left[|f(b)| + \bigvee_{[a,b]} f\right].$$

When $|t| \le 1$. we simply use the property of the Fourier-Stieltjes Transform:

$$\left|\phi_f^n(t)\right| \le 1$$

Therefore we obtain the stated result. since I_n and $\phi_f^n(t)$ are the same. □

Lemma 3.1.6 Let $I = [a.b]$. $\tau : I \to I$ piecewise differentiable on I. $\frac{d}{dx}\tau(\cdot) \in BV$. $f : I \to \mathbb{R}$. $f \in \mathcal{L}_I^1 \cap \mathcal{C}_I^1 \cap BV_I$ and $|f(b)| < \infty$. If $\inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$. then for all $n \ge 1$. the bound $\omega_n(t)$ given in Proposition 3.1.2 is an $\mathcal{L}^2$ function and $\lim_{n\to\infty}\omega_n(t) < \infty$.

Proof: For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\textstyle\int_{-\infty}^{\infty} |\omega_n(t)|^2 dt &= \textstyle\int_{-\infty}^{\infty}\left|\frac{M_n}{|t|}\right|^2 dt = \int_{-1}^{1} dt + 2\int_1^{\infty}\frac{|M_n|^2}{t^2}dt \\ &= \textstyle 2\left(1 + |M_n|^2 \cdot \int_1^{\infty}\frac{1}{t^2}dt\right) = 2\left(1 - |M_n|^2\right) < \infty\end{aligned}$$

Then $\omega_n \in \mathcal{L}^2$ for all $n \geq 1$. Now, consider $s_n = \inf \left| \frac{d}{dx} \tau^n(x) \right|$, then by Lemma 3.1.1 we have that

$$\lim_{n \to \infty} \frac{1}{s_n} = 0$$

and by Lemma 3.1.4 we have that

$$\lim_{n \to \infty} \bigvee_I \left(\frac{1}{\frac{d}{dx} \tau^n(\cdot)} \right) < M(\alpha) < \infty.$$

Then, since $f(b)$ is bounded, we obtain:

$$\begin{aligned} \lim_{n \to \infty} M_n &= \lim_{n \to \infty} \left[\tfrac{2}{s_n} + \textstyle\bigvee_I \left(\frac{1}{\frac{d}{dx} \tau^n(\cdot)} \right) \right] \left[\sup_{b \in I} |f(b)| + \textstyle\bigvee_I f \right] \\ &= \left[\sup_{b \in I} |f(b)| + \textstyle\bigvee_I f \right] \lim_{n \to \infty} \textstyle\bigvee_I \left(\frac{1}{\frac{d}{dx} \tau^n(\cdot)} \right) < \infty. \end{aligned}$$

Hence $\lim_{n \to \infty} \omega_n(t) < \infty$. □

The next Theorem gives useful criteria to determine if the distribution function of a Fourier Transform is singular.

Theorem 3.1.1 The distribution function $F(x)$ has no atoms if and only if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\mathcal{F}(F)(t)|^2 \, dt = 0.$$

Proof: For more details proceed to see [6, page 145, Corollary to the Theorem 6.2.5] or [13, page 42, Theorem 3.3.4]. □

The next result states that the distribution function $F^*(x)$ of the Fourier-Stieltjes Transform $\phi^*(t) = \lim_{n \to \infty} \phi_f^n(t)$ is an *acdf*, and also that its density function $f^*(x)$ is an $\mathcal{L}^2$ function in I.

Theorem 3.1.2 Let $\mathcal{F}(F)$ be a function in $\mathcal{L}^2$. Then the distribution function $F(x)$ is an *acdf* and

$$f(x) = \underset{n\to\infty}{\mathrm{LIM}} \frac{1}{2\pi} \int_{-n}^{n} e^{-itx} \mathcal{F}(F)(t)dt$$

is an $\mathcal{L}^2$ function, such that $\|f\|_{\mathcal{L}^2} = \|\mathcal{F}(F)\|_{\mathcal{L}^2}$, and with $f(x) = F'(x)$.

Proof: (See [6, page 147, Problem 11])

By Plancherel's Theorem (see Theorem 2.2.1) there exists $f \in \mathcal{L}^2$ such that

$$f(x) = \underset{n\to\infty}{\mathrm{LIM}} \frac{1}{2\pi} \int_{-n}^{n} e^{-itx} \mathcal{F}(F)(t)dt$$

integrating everything we obtain

$$\int_0^u f(x)dx = \frac{1}{2\pi} \int_0^u \underset{n\to\infty}{\mathrm{LIM}} \int_{-n}^{n} e^{-itx} \mathcal{F}(F)(t)dtdx.$$

Now, using Fubini's Theorem,

$\frac{1}{2\pi} \int_0^u \mathrm{LIM}_{n\to\infty} \int_{-n}^{n} e^{-itx} \mathcal{F}(F)(t)dtdx = \frac{1}{2\pi} \mathrm{LIM}_{n\to\infty} \int_{-n}^{n} \int_0^u e^{-itx} dx \mathcal{F}(F)(t)dt$
$= \frac{1}{2\pi} \mathrm{LIM}_{n\to\infty} \int_{-n}^{n} \frac{1-e^{-itu}}{-it} \mathcal{F}(F)(t)dt,$

hence

$$\int_0^u f(x)dx = \frac{1}{2\pi} \underset{n\to\infty}{\mathrm{LIM}} \int_{-n}^{n} \frac{1-e^{-itu}}{it} \mathcal{F}(F)(t)dt. \tag{14}$$

Now, using the Levy's inversion formula (Theorem 2.2.2) on the right hand side of (14) for all $u \in \mathbb{R}$, we have that

$$F(u) - F(0) = \frac{1}{2\pi} \underset{n\to\infty}{\mathrm{LIM}} \int_{-n}^{n} \frac{1-e^{-itu}}{it} \mathcal{F}(F)(t)dt$$

thus

$$\int_0^u f(x)dx = F(u) - F(0)$$

implying by the integral representation of an *acdf* (Proposition 3.1.1) that $F(\cdot)$ is an *acdf*.

Finally, by Plancherel's Theorem, we have that $\|f\|_{\mathcal{L}^2} = \|\mathcal{F}(F)\|_{\mathcal{L}^2}$. $\square$

3.2 CONVERGENCE OF $\mathcal{F}(F_n)$

In this Section we present the results that confirms the convergence of the Fourier-Stieltjes Transform of the Perron Frobenius operator $\mathcal{F}(F_n)(t)$ with $F_n(x) = \int_{-\infty}^{x} P_{\tau}^{n} f(u) d\lambda(u)$. It is important to remark that the existence of the limit of the sequence $\{\mathcal{F}(F_n)\}_{n=1}^{\infty}$ can be obtained using other results, namely the Lasota-Yorke Theorem. The intention here is to obtain this result without the use of Lasota and Yorke Theorem.

Here we use Helly's Theorems to prove the existence of a limit of the sequence of Fourier-Stieltjes transforms of the Perron Frobenius operator $\mathcal{F}(F_n)(t)$. We also prove that this convergence is uniform for t in any finite interval $S = [-T, T]$ of the real line.

Theorem 3.2.1 (CONVERGENCE OF $\mathcal{F}(F_n)$) Let $F_n(x) = \int_{-\infty}^{x} P_{\tau}^{n} f(u) du$. There exists a subsequence of $\{\mathcal{F}(F_n)(t)\}_{n=1}^{\infty}$ that converges for all $t \in \mathbb{R}$ to the Fourier-Stieltjes Transform of F^*. The convergence is uniform for every finite t-interval $[-T, T]$, and $\mathcal{F}(F^*)(t)$ is a continuous function for t in a neighborhood of zero.

Proof:

We have that $F_n(x)$ is non-decreasing and uniformly bounded distribution function. Then by Helly's First Theorem (see Theorem 2.1.2) there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that F_{n_k} converges weakly to F^*, where F^* is a non-decreasing and bounded function.

Since the function e^{itx} is bounded for all x and t real values. then using the extension of the second Helly's Theorem (Corollary 2.1.3.1), we have

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} e^{itx}dF_{n_k}(x) = \int_{-\infty}^{\infty} e^{itx}dF^*(x).$$

for all $t \in \mathbb{R}$ and I a real interval.

Therefore, using the Continuity Theorem 2.2.4. $\mathcal{F}(F^*)(t) = \int_I e^{itx}dF^*(x)$ is the Fourier-Stieltjes Transform of the distribution function F^*. Finally. by the Uniform Convergence Theorem 2.2.5. we have that $\mathcal{F}(F_n)(t)$ converges to $\mathcal{F}(F^*)(t)$ uniformly for every finite t-interval $[-T.T]$. $\square$

3.3 ABSOLUTELY CONTINUOUS MEASURES

Having proved the existence of a limit of the subsequences $\{\phi_f^{n_k}(t)\}_{k=1}^{\infty}$. and with the limit also a Fourier-Stieltjes transform. we proceed to show that the distribution function $F^*(x)$ of the limit $\phi^*(t) = \mathcal{F}(F^*)(t)$ is an *acdf.*

Theorem 3.3.1 Let $I \subset \mathbb{R}$ bounded, $\tau : I \to I$ piecewise differentiable on I. $\frac{d}{dx}\tau(x) \in BV_I$. $f : I \to \mathbb{R}$ such that $f \in \mathcal{L}^1 \cap C_I^1 \cap BV_I$ and $\inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$. Assume that $F^*(t)$ was obtained by the limit

$$\int_{-\infty}^{\infty} e^{itx}dF^*(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{itx}P_\tau^n f(x)dx.$$

Then $F^*(x)$ is a **continuous distribution** function and $\phi^*(t)$ is an $\mathcal{L}^2$ function.

Proof: We will show that $F^*(x)$ has no atoms by proving that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\phi^*(t)|^2\,dt = 0.$$

as stated in the Theorem 3.1.1.

Consider the integral

$$\frac{1}{2T}\int_{-T}^{T}\left|\phi_f^n(t)\right|^2 dt$$

By Proposition 3.1.2, we have that $\left|\phi_f^n(t)\right| \leq \omega_n(t)$, where ω_n is an $\mathcal{L}^2$ function for all $n \in \mathbb{N}$ (see Lemma 3.1.6), defined by

$$\omega_n(t) = \delta_{[-1,1]}(t) + \delta_{(-\infty,-1)\cup(1,\infty)}(t)\frac{M_n}{|t|}$$

where $M_n = \left[\frac{2}{s_n} + \bigvee_I\left(\frac{1}{\frac{d}{dx}\tau^n(\cdot)}\right)\right]\left[\sup_{b\in I}|f(b)| + \bigvee_I f\right]$.

It follows that

$$\int_{-T}^{T}\left|\phi_f^n(t)\right|^2 dt \leq \int_{-T}^{T}|\omega_n(t)|^2\, dt = \int_{-1}^{1} dt + 2\int_{1}^{T}\left|\frac{M_n}{t}\right|^2 dt = 2\left[1 + |M_n|^2\left(1 - \frac{1}{T}\right)\right].$$

Then

$$\begin{aligned}\lim_{T\to\infty}\lim_{n\to\infty}\tfrac{1}{T}\textstyle\int_{-T}^{T}\left|\phi_f^n(t)\right|^2 dt &\leq \lim_{T\to\infty}\lim_{n\to\infty}\tfrac{2}{T}\left[1 + |M_n|^2\left(1-\tfrac{1}{T}\right)\right]\\ &= \lim_{T\to\infty}\frac{1+\lim_{n\to\infty}|M_n|^2\left(1-\frac{1}{T}\right)}{T} = 0.\end{aligned}$$

Now, since the limit $\phi^*(t) = \lim_{n\to\infty}\phi_f^n(t)$ exists (by Theorem 3.2.1), using the Lebesgue Dominated Convergence Theorem 2.1.1 and Lemma 3.1.5, we obtain:

$$\begin{aligned}\lim_{T\to\infty}\tfrac{1}{2T}\textstyle\int_{-T}^{T}|\phi^*(t)|^2\, dt &= \lim_{T\to\infty}\tfrac{1}{2T}\textstyle\int_{-T}^{T}\left|\lim_{n\to\infty}\phi_f^n(t)\right|^2 dt\\ &= \lim_{T\to\infty}\tfrac{1}{2T}\textstyle\int_{-T}^{T}\lim_{n\to\infty}\left|\phi_f^n(t)\right|^2 dt\\ &= \lim_{T\to\infty}\lim_{n\to\infty}\tfrac{1}{2T}\textstyle\int_{-T}^{T}\left|\phi_f^n(t)\right|^2 dt = 0\end{aligned}$$

Then, we have that $F^*(x)$ has no atoms. Thus the measure $F^*(x)$ of the Fourier-Stieltjes Transform $\phi^*(t)$ is continuous.

Finally, by Lemma 3.1.6, ω_n is an $\mathcal{L}^2$ function and $\left|\phi_f^n(t)\right| \leq \omega_n(t)$, thus we can conclude that ϕ_f^n is also an $\mathcal{L}^2$ function for all $n \geq 1$. Moreover, since $\lim_{n\to\infty} \omega_n \in \mathcal{L}^2$, we finish the proof having that $\phi^* \in \mathcal{L}^2$. □

Corollary 3.3.1.1 (ACM) Let $I \subset \mathbb{R}$ bounded, $\tau : I \to I$ piecewise differentiable on I, $\frac{d}{dx}\tau(x) \in BV_I$, $f : I \to \mathbb{R}$ such that $f \in \mathcal{L}^1 \cap \mathcal{C}_I^1 \cap BV_I$ and $\inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$. Assume that $F^*(t)$ was obtained by the limit:

$$\int_{-\infty}^{\infty} e^{itx} dF^*(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{itx} P_\tau^n f(x) dx.$$

Then $F^*(x)$ is an *acdf.*

Proof: By Theorem 3.3.1, we have that $\phi^*(t)$ is an $\mathcal{L}^2$ function. Then, using Theorem 3.1.2, we obtain that F^* is *acdf.* □

3.4 ABSOLUTELY CONTINUOUS INVARIANT MEASURES

Theorem 3.4.1 (acidf) Let $I \subset \mathbb{R}$ bounded, $\tau : I \to I$ piecewise differentiable on I, $\frac{d}{dx}\tau(x) \in BV$, $f : I \to \mathbb{R}$, $f \in \mathcal{L}^1 \cap \mathcal{C}_I^1 \cap BV_I$ and $\inf_{x\in I}\left|\frac{d}{dx}\tau(x)\right| > 1$. Then $F^*(x)$ is an *acidf.* obtained by the limit process:

$$\int_{-\infty}^{\infty} e^{itx} dF^*(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{itx} P_\tau^n f(x) dx.$$

with $f^* = \frac{dF^*}{d\lambda}$ an $\mathcal{L}_I^1$ function.

Proof: By the Corollary 3.3.1.1, F^* is an *acdf.*

By Theorem 3.3.1, $\phi^* \in \mathcal{L}^2$. By Plancherel Theorem 2.2.1 we have that $f^* \in \mathcal{L}^2$. Because ϕ^* is an integral on I, then we can consider f^* having its support on I. Applying the Theorem 2.2.1, results that $f^* \in \mathcal{L}^1_I$.

Rest to prove that $F^*(x)$ is an invariant distribution function. Using the Continuity Theorem 2.2.4, we have that $F_n(x) = \int_{-\infty}^{x} P_\tau^n f(u) d\lambda(u)$ converges weakly to $F^*(x) = \int_{-\infty}^{x} f^*(u) d\lambda(u)$, then

$$lim_{k \to \infty} F_n(x) = F^*(x)$$

for all continuity points x of $F^*(x)$. Now, let $\mu_n([a,b]) = F_n(b) - F_n(a)$, then

$$lim_{k \to \infty} \mu_n([a,b]) = lim_{k \to \infty} F_n(b) - F_n(a) = F^*(b) - F^*(a) = \mu^*([a,b])$$

and

$$lim_{k \to \infty} \mu_n(\tau^{-1}[a,b]) = lim_{k \to \infty} \int_{\tau^{-1}[a,b]} P_\tau^n f(u) d\lambda(u)$$

$$= lim_{k \to \infty} \int_a^b P_\tau^{n-1} f(u) d\lambda(u)$$

$$= lim_{k \to \infty} F_{n-1}(b) - F_{n-1}(a) = F^*(b) - F^*(a) = \mu^*([a,b])$$

Hence $lim_{k \to \infty} \mu_n(\tau^{-1}[a,b]) = \mu^*(\tau^{-1}[a,b]) = \mu^*([a,b])$, and therefore F^* is an invariant distribution function with respect to τ. $\square$

4 CRITERIA FOR ACM'S

So far, we have assumed that τ is a piecewise expanding transformation. Although we know that there exists *acim*'s for some other families of transformations not necessarily piecewise expanding, for example the logistic function in $I = [0, 1]$ given by the transformation

$$\begin{aligned} \tau : I \rightarrow I, \\ \tau(x) = 4x(1-x) \end{aligned} \tag{15}$$

For this transformation we know there exists an *acim*, in this case also an *acidf* of τ, given by the density function:

$$f^* : [0, 1] \rightarrow \mathbb{R},$$
$$f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

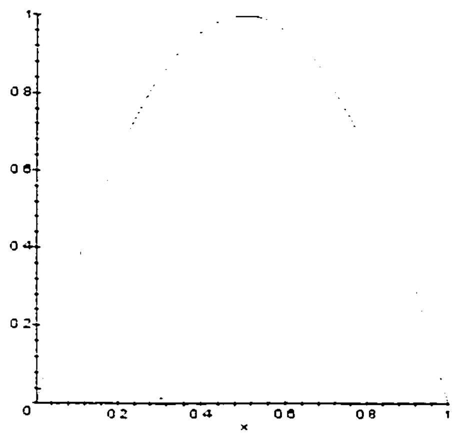

Figure 1. $\tau(x) = 4x(1-x)$

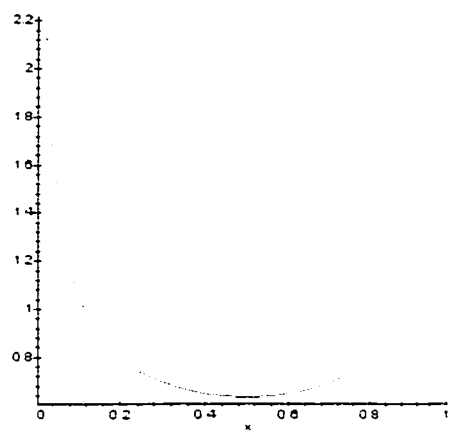

Figure 2. $f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$

This is why we presume the Lasota and Yorke Theorem can be relaxed. In this Chapter we present some criteria on the Fourier-Stieltjes Transform ensuring that it corresponds to an *acdf*. The problem we consider here can be stated as follows:

having the Fourier-Stieltjes transforms

$$\phi_f^n(t) = \int_{-\infty}^{\infty} e^{it\tau^n(x)} dF(x)$$

and its limit

$$\phi^*(t),$$

under what conditions (imposed on τ) is the distribution function F^* of ϕ^* an *acdf*?

The answer to this question can be seen as a generalization of the criteria used before in the Chapter 3 to prove the existence of *acidf*'s of τ.

We presume this can lead to future existence results with less restrictions on τ.

4.1 Λ FAMILIES

Recall from Theorem 3.2.1 that for any $f \in \mathcal{L}^1$ there exists a subsequence $\{\phi_f^{n_k}\}_{k=1}^{\infty}$ of the sequence $\{\phi_f^n\}_{n=1}^{\infty}$ which converges uniformly to some Fourier-Stieltjes Transform $\phi^*(t)$, for all $t \in \mathbb{R}$.

We will introduce a family of complex functions Λ which will represent the Fourier-Stieltjes transforms of *acm*'s. With this in mind, we will present some results which may be a way to relax the assumptions on τ in the Lasota and Yorke Theorem.

Definition 4.1.1 (Λ FAMILY) We define Λ to be the family of Fourier-Stieltjes transforms of *acm*'s. That is,

$$\Lambda = \{\mathcal{F}(F) \in \mathbb{H} \mid F \ acm\}.$$

Therefore, our goal in this Section is to present some criteria on ϕ^* in order to prove that ϕ^* belongs to the family Λ. In other words, we are looking for some criteria on the Fourier-Stieltjes Transform to guarantee the existence of an *acm* of ϕ^*.

4.1.1 Λ FAMILIES FOR FUNCTIONS IN $\mathcal{L}^1$ AND $\mathcal{L}^2$

Let us begin considering the case when the Fourier-Stieltjes Transform $\mathcal{F}(F)$ is a function in $\mathcal{L}^1$ (or $\mathcal{L}^2$). In this case we have that the distribution function F results from an *acm* in $\mathcal{L}^1$ (or $\mathcal{L}^2$). We will represent this by the families of functions Λ_1 and Λ_2.

Definition 4.1.2 (FAMILY Λ_1) The family of functions Λ_1 is defined as:

$$\Lambda_1 = \left\{\mathcal{H} \in \mathbb{H} \mid \mathcal{H} \in \mathcal{L}^1\right\}.$$

Definition 4.1.3 (FAMILY Λ_2) The family of functions Λ_2 is defined as:

$$\Lambda_2 = \left\{\mathcal{H} \in \mathbb{H} \mid \mathcal{H} \in \mathcal{L}^2\right\}.$$

Theorem 4.1.1 If $\mathcal{H} \in \Lambda_1$ then the distribution function F is an *acm*, where $F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\mathcal{H}(t)dt$.

Proof: See [13, page 33, Theorem 3.2.2]. □

Theorem 4.1.2 If $\mathcal{H} \in \Lambda_2$ then the distribution function F is an *acm*, where $F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\mathcal{H}(t)dt$.

Proof: See Theorem 3.1.2. □

Corollary 4.1.2.1 $\Lambda_1 \subset \Lambda$ and $\Lambda_2 \subset \Lambda$.

With this last results. we can consider the restrictions on τ necessary to have that ϕ^* belongs to either Λ_1 or Λ_2; this way we can confirm the existence of a distribution function F^* of ϕ^* which is an *acm*. In our case in Chapter 3 we used a bound on ϕ^* to prove that $\phi^* \in \Lambda_2$. This result leads us to the next Corollary:

Corollary 4.1.2.2 If we assume $I \subset \mathbb{R}$ bounded. $\tau : I \longrightarrow I$ piecewise differentiable on I. $\frac{d}{dx}\tau(x) \in BV_I$. $f : I \rightarrow \mathbb{R}$ such that $f \in \mathcal{L}^1 \cap \mathcal{C}_I^1 \cap BV_I$ and $\inf_{x\in I} \left|\frac{d}{dx}\tau(x)\right| > 1$. then $\phi^* \in \Lambda_2$.

4.1.2 Λ FAMILIES WITH CONTINUOUS MEASURES

We can consider continuous measures to be either absolutely continuous measures. singular measures or mixed. We present a criteria on ϕ^* which shows when the distribution function F^* of the Fourier-Stieltjes Transform ϕ^* is a continuous distribution function.

Definition 4.1.4 (BOHR'S INNER PRODUCT) The inner product (also called Bohr's inner product) is defined by:

$$\langle\cdot,\cdot\rangle_B : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{R}.$$

$$\langle x, y\rangle_B = \lim_{T\to\infty} \tfrac{1}{2T} \textstyle\int_{-T}^{T} x(t)\overline{y(t)}dt.$$

where $\mathbb{F}$ is a linear functional space (instead of $\mathbb{F}$. it is usually considered $\mathbb{F}_n$ the space of periodic functions with period n).

Remark 4.1.1 The inner product $\langle\cdot,\cdot\rangle_B$ on the space of periodic functions $\mathbb{F}_n$ with period n defines a non-separable Hilbert space. See [2, page 29, item 19].

Definition 4.1.5 (FAMILY Λ_C) The family Λ_c is defined by means of the Bohr's inner product as:

$$\Lambda_C = \{\mathcal{H} \in \mathbb{H} \mid \langle \mathcal{H}, \mathcal{H} \rangle_B = 0\}.$$

Theorem 4.1.3 If $\mathcal{F}(F) \in \Lambda_C$ then F is a continuous distribution function.

Proof: See [13, page 42, Theorem 3.3.4]. □

Remark 4.1.2 If we assume $I \subset \mathbb{R}$ bounded, $\tau : I \to I$ piecewise differentiable on I, $\frac{d}{dx}\tau(x) \in BV_I$, $f : I \to \mathbb{R}$ such that $f \in \mathcal{L}^1 \cap \mathcal{C}_I^1 \cap BV_I$ and $\inf_{x\in I} \left|\frac{d}{dx}\tau(x)\right| > 1$, then $\phi^* \in \Lambda_C$.

4.1.3 Λ FAMILIES WITH SUMMABILITY OF INTEGRALS

The following approach uses what is called summability of integrals in the real line. This approach is of common use in harmonic analysis to obtain an inversion of the Fourier and Fourier-Stieltjes transforms. Here we use this method as a technique to ensure the existence of *acidf*'s of τ.

Definition 4.1.6 (θ-FACTOR) An even function $\theta \in \mathcal{L}^1$ is called θ-factor (on the real line) if its Fourier Transform

$$\hat{\theta}(t) = \int_{-\infty}^{\infty} e^{itx}\theta(x)dx$$

is in $\mathcal{L}^1$ and satisfies

$$\int_{-\infty}^{\infty} \hat{\theta}(t)dt = 2\pi.$$

If θ is continuous, we call it a continuous θ-factor.

Example 4.1.1 Some typical examples of θ-factors are:

(i) **(Cesàro Factor)** $\theta(x) = \begin{cases} (1-|x|). & when \quad x \in [-1.1] \\ 0 & when \quad x \notin [-1.1] \end{cases}$.

(ii) **(Abel Factor)** $\theta(x) = e^{-|x|}$. for all $x \in \mathbb{R}$.

(iii) **(Gauss Factor)** $\theta(x) = e^{-x^2}$. for all $x \in \mathbb{R}$.

Remark 4.1.3 Reference [1, page 118, item 64] considers other types of θ-factors.

Definition 4.1.7 (FAMILY $\Lambda_I(\theta)$) Consider a continuous θ-factor with Fourier Transform

$$\hat{\theta}(t) = \int_{-\infty}^{\infty} e^{itx}\theta(x)dx$$

positive and monotonic decreasing on $[0.\infty)$. then

$$\Lambda_I(\theta) = \left\{ \mathcal{H} \in \mathbb{H} \;\middle|\; \begin{array}{l} \exists p \text{ a density function. such that:} \\ p(x) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\theta\left(\frac{t}{T}\right)\mathcal{H}(t)dt \\ \text{whose distribution function } P \text{ is such that:} \\ \mathcal{F}(P)(t) = \mathcal{H}(t). \end{array} \right\}.$$

defines the family of functions $\Lambda_I(\theta)$.

Proposition 4.1.1 For a θ-factor the integral

$$U(\mathcal{F}(\mu); x; T) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta\left(\frac{t}{T}\right) e^{-itx}\mathcal{F}(\mu)(t)dt$$

exists for all $x \in \mathbb{R}$, belong to $\mathcal{L}^1$ and for $T > 0$ satisfy

$$\|U(\mathcal{F}(\mu); \cdot; T)\|_{\mathcal{L}^1} \leq \left\|\hat{\theta}\right\|_{\mathcal{L}^1} \bigvee_{-\infty}^{\infty} \mu.$$

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} g(x)U(\mathcal{F}(\mu); x; T)dx = \int_{-\infty}^{\infty} g(x)d\mu(x)$$

for every continuous function $g(x)$.

Proof: See [5, page 222, Proposition 5.3.7]. $\square$

Theorem 4.1.4 Let $\mu \in BV_{\mathbf{R}}$. If a θ-factor has Fourier Transform $\hat{\theta}(t)$ positive and is monotonely decreasing on $[0, \infty)$, then

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{-itx} \theta\left(\frac{t}{T}\right) \mathcal{F}(\mu)(t) dt = \mu'(x) \qquad a.e.$$

Proof: See [5, page 222, Proposition 5.3.8] . $\square$

Remark 4.1.4 Instead of using θ-factors with $\hat{\theta}(t)$ positive and monotonic decreasing on $[0, \infty)$, it is possible to consider θ to be a Fourier Transform of an *absolutely continuous distribution* function with density function $q(x) = O(x^{-2})$ as $|x| \to \infty$ and θ an $\mathcal{L}^1$ function. See [13, page 38] and [1, page 120 item 65].

Theorem 4.1.5 Let $\mathcal{F}(F) \in \Lambda_f(\theta)$, with θ-factor with Fourier Transform $\hat{\theta}$ positive and monotonic decreasing on $[0, \infty)$. Then F is an *acm*.

Proof: We have that for all $x \in \mathbb{R}$, $\exists p(x)$ density function such that

$$p(x) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{-itx} \theta\left(\frac{t}{T}\right) \mathcal{F}(F)(t) dt.$$

Then by considering the function of bounded variation $Q(x) = \int_{-\infty}^{x} p(u)du$, and by Theorem 4.1.4, $Q(x)$ is a distribution function such that

$$Q'(x) = p(x) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{-itx} \theta(\frac{t}{T}) \mathcal{F}(Q)(t) dt \quad a.e.$$

Since $\mathcal{F}(Q) = \mathcal{F}(F)$, then by the Uniqueness of the Fourier-Stieltjes Transform Theorem 2.2.3, results that $F(x) = Q(x)$ *a.e.*, hence $F \in BV_{\mathbf{R}}$. Applying Theorem

4.1.4 to F. results that

$$F(x) = \int_{-\infty}^{x} p(u)du.$$

and using the integral representation of an *acm* (see Proposition 3.1.1). we have that $F(x)$ is an *acdf* for all $x \in \mathbb{R}$. $\square$

Corollary 4.1.5.1 Consider a θ-factor with $\hat{\theta}$ positive and monotonic decreasing on $[0, \infty)$. Then $\Lambda_I(\theta) \subset \Lambda$.

Corollary 4.1.5.2 Let $\mathcal{F}(F) \in \Lambda_I(\theta)$ with the θ-factor defined as one of the following cases:

$$(i)\ \theta(x) = \begin{cases} (1-|x|), & when \quad x \in [-1, 1] \\ 0 & when \quad x \notin [-1, 1] \end{cases};$$

(ii) $\theta(x) = e^{-|x|}$. for all $x \in \mathbb{R}$.

(iii) $\theta(x) = e^{-x^2}$. for all $x \in \mathbb{R}$.

Then F is an *acm*.

Proof: See [1. page 120. item 65], [13, page 40 Corollary 3 to Theorem 3.3.2] and [5. page 222. Proposition 5.3.8]. $\square$

Other θ-factors can be seen on [1, page 118 item 64], where they are represented as Fejér integral kernels.

Hence. we have that imposing restrictions on τ in order to have that ϕ^* belongs to Λ_1, Λ_2 or $\Lambda_I(\theta)$, the distribution function F^* of ϕ^* will be an *acdf*.

5 EXAMPLES

5.1 THE TENT MAP

Consider the Dynamical System on the measurable space $(I, \mathcal{B}(I), \lambda)$ with $I = [0, 1]$, $\tau : I \longrightarrow I$, $\tau(x) = 2x \ (mod\ 1)$ and f uniform density function on the interval $[0, 1]$ $(f(x) = \delta_{[0,1]}(x))$.

We have that the Fourier-Stieltjes Transform of the Perron Frobenius operator $P_\tau^n f(x)$ is given by

$$\phi_f^n(t) = \int_I e^{it\tau^n(x)} f(x)dx = \int_0^1 e^{it\tau^n(x)}dx.$$

Also, we notice that the n-th iteration of τ, given by

$$\tau^n(x) = 2^n x \ (mod\ 1),$$

can be expressed as follows:

$$\tau^n(x) = 2^n x - k + 1$$

where $x \in [\frac{k-1}{2^n}, \frac{k}{2^n})$ and $k = 1, 2, 3, \ldots, 2^n$.

Then

$$\phi_f^n(t) = \sum_{k=1}^{2^n} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} e^{it(2^n x - k + 1)}dx, \qquad for\ all\ n.$$

which gets reduced to

$$\phi_f^n(t) = \frac{1}{it}\left(e^{it} - 1\right), \qquad for\ all\ n.$$

We notice that $\phi^*(t) = \phi_f^n(t)$ for all n. Also, we have that $\phi^* \in \Lambda_2$ (i.e. $\phi^* \in \mathcal{L}^2$):

$$\int_{-\infty}^{\infty} |\phi^*(t)|^2 \, dt \leq 2 + 2\int_1^{\infty} \frac{1}{t^2} dt < \infty$$

Therefore, by Theorem 4.1.1, there exists an *acm* $F^*(x)$ of τ such that

$$\frac{d}{dx}F^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi^*(t)dt.$$

in this case given by the density function $f^*(x) = \delta_{[0,1]}(x)$. See figure 4.

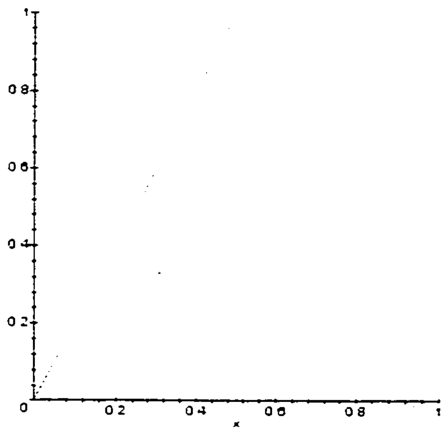

Figure 3. $\tau(x)$

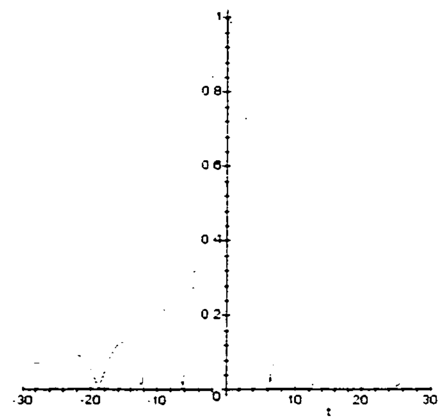

Figure 4. $\left|\phi_f^*(t)\right| = \left|\phi_f^n(t)\right|$ for all n

5.2 THE LOGISTIC MAP

Consider now the Dynamical System on the measurable space $(I, \mathcal{B}(I), \lambda)$, given by:

- $I = [0, 1]$,
- $\tau : I \longrightarrow I$, $\tau(x) = 4x(1-x)$, and
- f uniform density function on the interval $[0, 1]$, i.e. $f(x) = \delta_{[0,1]}(x)$.

In this case, τ is called the "Logistic Transformation", and the iteration of τ induces a dynamical system which is a classical example of chaotic behavior.

We have that the Fourier-Stieltjes Transform of the Perron Frobenius operator $P_\tau^n f(x)$ is given by

$$\phi_f^n(t) = \int_I e^{it\tau^n(x)} f(x)dx = \int_0^1 e^{it\tau^n(x)}dx$$

As we said before on Chapter 4, τ has an *acidf* $F^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, even though τ is not a piecewise monotonic expanding function. However, we can see from the behavior of $\phi_f^n(t)$ that in this case it is possible to search for a proof of existence of *acidf*s, perhaps considering Λ families of functions as approach.

Figures 6 to 11 show different stages of the Fourier-Stieltjes Transform $\phi_f^n(t)$. Figure 12 shows the Fourier-Stieltjes Transform of the *acidf* of τ.

Perhaps a possible way to prove the existence of *acidf*s of τ may be obtained by considering a bound $\omega_n(t)$ on $|\phi^n(t)|$. In any case, we do not solve this problem which remains for future research, but we consider that some of the Λ subfamilies $(\Lambda_1, \Lambda_2, \Lambda_C, \Lambda_I(\theta))$ may be a possible route to solve it.

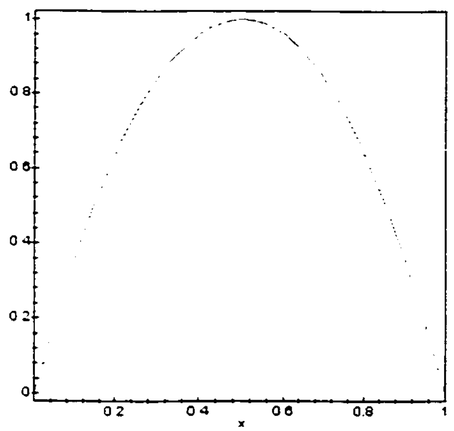

Figure 5. $\tau(x)$

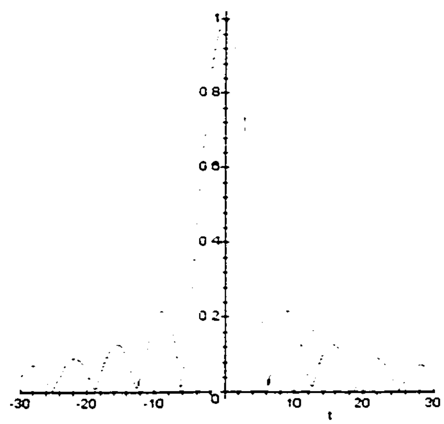

Figure 6. $\left|\phi_f^0(t)\right|$

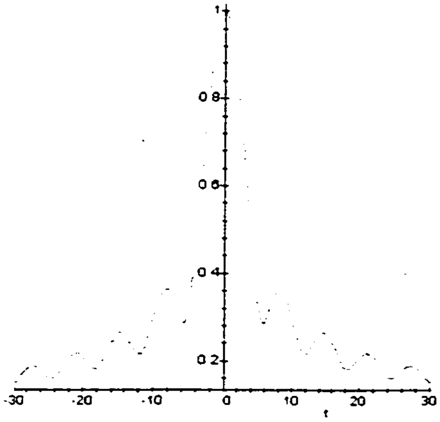

Figure 7. $\left|\varphi_f^1(t)\right|$

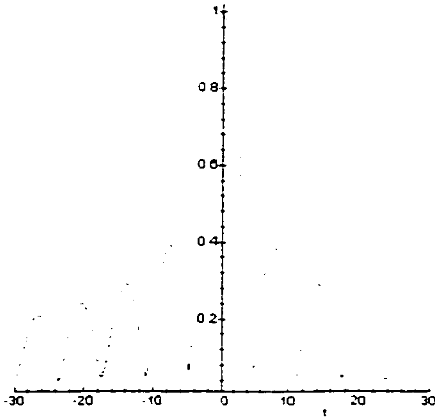

Figure 8. $\left|\varphi_f^2(t)\right|$

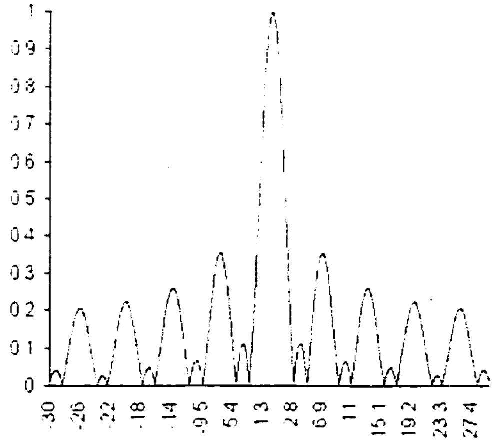

Figure 9. $\left|\varphi_f^3(t)\right|$

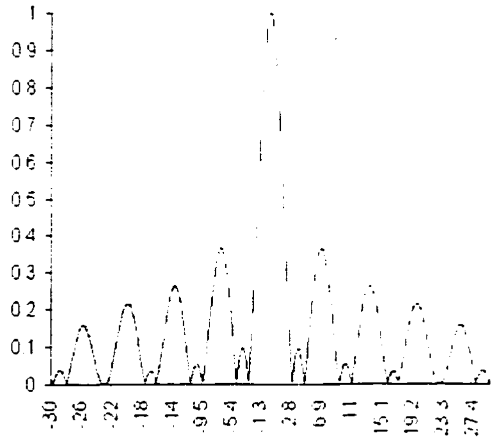

Figure 10. $\left|\varphi_f^4(t)\right|$

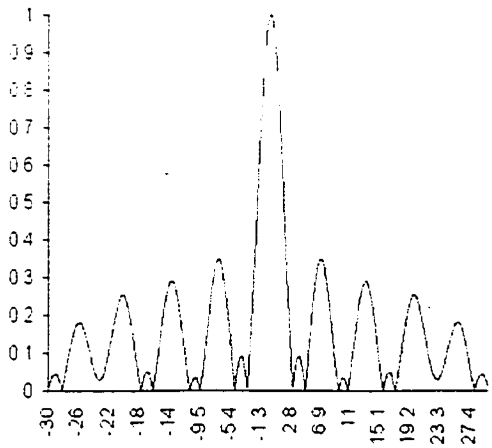

Figure 11. $\left|\varphi_f^5(t)\right|$

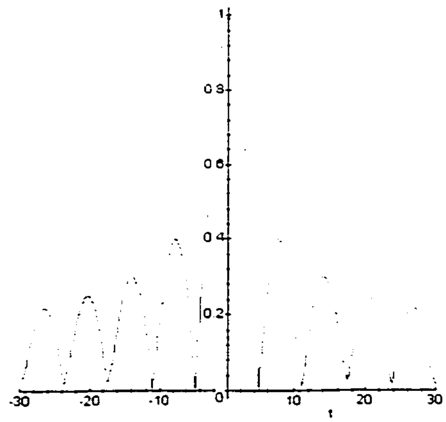

Figure 12. $|\varphi^*(t)|$

6 CONCLUSIONS

The results obtained here represent an alternative method to prove the existence of absolutely continuous invariant measures (*acim*'s) by using the Fourier-Stieltjes Transform under the assumptions that τ is a piecewise monotonic expanding transformation.

The idea of using a bijective transformation to analyze the original problem in another space is not new. However. the analysis of the existence of *acim*'s using the Fourier-Stieltjes transform, we believe, can give more existence results.

We consider important to present criteria that the Fourier-Stieltjes Transform should satisfy in order to ensure that it correspond to an *acdf* . The use of bounds of the Fourier Stieltjes transforms ϕ_f^n and the Λ-families. represent the first step in this direction using the Fourier-Stieltjes transform. We believe that there are more possibilities not yet explored using this approach.

As a final note. the Fourier-Stieltjes Transform represents one type of bijective transformation, and it is possible to consider other kinds of bijective transformations in order to study existence problems of *acim*'s. Here we did use the Fourier-Stieltjes Transform successfully, but there are more ways to attack this problem in the future using similar techniques.

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7 GLOSSARY OF SYMBOLS

- $\langle \cdot, \cdot \rangle_B$: Bohr's inner product, which defines a non separable Hilbert space.
- *acdf*: an Absolutely Continuous Distribution Function.
- *acidf*: an Absolutely Continuous Invariant Distribution Function.
- *acim*: an Absolutely Continuous Invariant Measure.
- *acm*: an Absolutely Continuous Measure.
- *a.e.*: Almost everywhere (i.e. except a set of zero measure).
- $\mathcal{B}(I)$: Borel set in the interval I.
- BV_I: Family of functions of Bounded Variation on I.
- $\mathbb{C}$: Set of Complex numbers
- C_I^1: Family of differentiable functions on I, with continuous derivative.
- C_I^2: Family of twice differentiable functions on I, with continuous second derivative.
- $\delta_I(t)$: The delta function, $\delta_I(t) = \begin{cases} 1 & , if\ t \in I \\ 0 & , if\ t \notin I \end{cases}$
- $D_n f = \frac{\frac{d}{dx} f}{it \frac{d}{dx} \tau^n(x)}$
- ${}^t D_n f = \frac{-d}{dx} \left(\frac{f}{it \frac{d}{dx} \tau^n(x)} \right)$
- $\mathbb{F}$: Space of linear functions of the form $f : \mathbb{R} \to \mathbb{C}$.
- $\mathbb{F}_n$: Space of linear periodic functions with period n.
- $\phi_f^n(t)$: Representation of the Fourier Stieltjes Transform of the n-th iterate of the Perron Frobenius Operator $(\phi_f^n(t) = \int_I e^{itx} P_\tau^n f(x) dx)$.
- $\phi^*(t)$: Function limit of the sequence $\{\phi_f^n(t)\}_{n=1}^{\infty}$. $(\phi^*(t) = \lim_{n \to \infty} \int_I e^{it\tau^n(x)} dF(x))$.
- $f^*(x)$: Density function of the Fourier-Stieltjes Transform $\phi^*(t)$.

- $F^*(x)$: Distribution function of the Fourier-Stieltjes Transform $\phi^*(t)$.
- $f(x)$: Density function of the distribution function $F(x)$.
- $F(x)$: Distribution function.
- $F_n(x)$: Distribution function of the n-th iterate of the Perron Frobenius operator. $(F_n(x) = \int_{-\infty}^{x} P_\tau^n f(u)du)$
- $\mathcal{F}(F)(t)$: Fourier-Stieltjes Transform of the measure F. $(\mathcal{F}(F)(t) = \int_{-\infty}^{\infty} e^{itx} dF(x))$
- $\mathbb{H}$: Family of Fourier-Stieltjes transforms.
- $\mathcal{H}$: The Fourier Stieltjes Transform of some distribution function.
- I: Subset of $\mathbb{R}$, usually considered as the interval $[0, 1]$.
- λ: Lebesgue measure.
- Λ: Family of Fourier-Stieltjes transforms $\mathcal{F}(F)$ with F *acm.*
- Λ_1: Family of Fourier-Stieltjes transforms $\mathcal{F}(F)$ with $\mathcal{F}(F) \in \mathcal{L}^1$.
- Λ_2: Family of Fourier-Stieltjes transforms $\mathcal{F}(F)$ with $\mathcal{F}(F) \in \mathcal{L}^2$.
- $\Lambda_I(g)$: Family of Fourier-Stieltjes transforms $\mathcal{F}(F)$ invertible by the inversion formula $(\lim_{T\to\infty} \int_{-T}^{T} e^{-itx} g(\frac{t}{T})\mathcal{F}(F)(x)dx)$.
- Λ_C: Family of Fourier-Stieltjes transforms $\mathcal{F}(F)$ with Bohr's inner product. $\langle \mathcal{F}(F), \mathcal{F}(F) \rangle_B = 0$.
- $\mathcal{L}_S^p$: Set of integrable functions such that $\int_S |f(x)|^p \, dx < \infty$.
- $\mu << \lambda$: The measure μ is absolutely continuous with respect to λ.
- M_n: Part of the bound $\omega_n(t)$. $\left(M_n = \left[\frac{2}{s_n} + \bigvee_I \left(\frac{1}{\frac{d}{dx}\tau^n(\cdot)}\right)\right] [\sup_{b\in I} |f(b)| + \bigvee_I f]\right)$
- $M(\alpha)$: The upper bound of $\bigvee_I \left(\frac{1}{\frac{d}{dx}\tau^n(x)}\right)$.
- $\|\cdot\|_{\mathcal{L}^p}$: Norm in $\mathcal{L}^p$.
- $\omega_n(t)$: Bound of $\phi_f^n(t)$:

$$\omega_n(t) = \begin{cases} 1 & . \ |t| \leq 1 \\ \frac{1}{|t|}\left[\frac{2}{s_n} + \bigvee_I \left(\frac{1}{\frac{d}{dx}\tau^n(\cdot)}\right)\right]\left[\sup_{b\in I}|f(b)| + \bigvee_I f\right] & . \ |t| > 1 \end{cases}$$

- $P_\tau f(x)$: Perron Frobenius operator applied to the $\mathcal{L}^1$ function f.

  under the maping $\tau : I \longrightarrow I$.

- $\mathbb{R}$: Set of real numbers.

- θ: θ-factor.

- $\hat{\theta}$: Fourier Transform of θ.$(\hat{\theta}(t) = \int_{-\infty}^{\infty} e^{itx}\theta(x)dx)$.

- $\bar{T}_n(u) = \int_a^u e^{|\tau^n(x)|}dx$

- $T_n(u) = \int_a^u e^{it\tau^n(x)}dx$

- U_τ: Koopmann operator. $(U_\tau f(x) = f(\tau(x)))$.

- $V_a^b f$: Total Variation of the function f in the interval $[a.b]$. $(V_a^b f = V_{[a.b]} f)$

- $s_n = \inf_{x\in I}\left|\frac{d}{dx}\tau^n(x)\right|$