Wavelet-Based Linearization Method in Nonlinear Systems

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ABSTRACT

The mathematical theories for linear systems provide a unified framework for understanding all possible linear system behaviors but no correspondingly general theoretical ones exist for treating non-linear problems. Due to the inherent nonlinearities of almost physical systems, the subject of nonlinear control is an important area of automatic control. Among many methods to deal with a nonlinear system, the most commonly used one is to linearize it which allows taking advantage of mature linear system control techniques to control the nonlinear systems approximately.

This thesis introduces a wavelet-based linearization method to estimate the nonlinear system response based on the traditional equivalent linearization technique. The mechanism by which the signal is decomposed and reconstructed using the wavelet transform is investigated. The properties and characteristics of some famous harmonic wavelet transforms are analyzed. Since the wavelet analysis can capture temporal variations in the energy and frequency content, a nonlinear system can be approximated as a time dependent linear system by combining the wavelet analysis technique with well-known traditional equivalent linearization method. The nonlinear systems including Duffing oscillator and bilinear hysteresis system are used as examples to verify the effectiveness of the proposed wavelet-based linearization method. Results from this study demonstrate that the wavelet linearization approach is more accurate and promising tool compared with traditional equivalent linearization methods in nonlinear control systems in the real world.

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Symbol	Definition
FFT	Fast Fourier Transform
IFFT	Inverse Fast Fourier Transform
STFT	Short-Time Fourier Transform
WVD	Wigner-Ville Distribution
DWT	Discrete Wavelet Transform
LTI	Linear Time Invariant
SDOF	Single Degree Of Freedom

1 INTRODUCTION

1.1 Motivation

In engineering applications, the use of a linear model for the system under consideration leads to fairly simple and often useful results when the system behaves linearly and excitation is stationary. The evaluation of the system response is not difficult. However, no real system is exactly linear. In mechanical and structural systems nonlinearities can arise in various forms and usually becomes progressively more significant as the amplitude of motion increases to excite nonlinear behavior requiring techniques which go beyond linear approach. The mathematical theories for linear systems provide a unified framework for understanding all possible linear system behaviors but no correspondingly general theoretical ones exist for treating non-linear problems. Due to the inherent nonlinearities of almost physical systems, the subject of nonlinear control is an important area of automatic control. Among many methods to deal with a nonlinear system, the most commonly used one is to linearize it which allows taking advantage of mature linear system control techniques to control the nonlinear systems approximately.

This dissertation introduces a wavelet-based method to linearize nonlinear systems to its equivalent linearization systems described by their wavelet coefficients. Time-frequency localization is the main motivation for using wavelet theory to approximate the nonlinear response as represented in this thesis.

In this dissertation, the harmonic wavelet function is chosen as the basis function in the linearization procedure due to its explicit form in mathematical expression and its non-overlapping characteristic in the frequency domain which can be interchanged from the scale to frequency band. This thesis also demonstrates that the harmonic wavelet function is used for its computational simplicity where there is a simple algorithm for harmonic wavelet analysis which uses the Fast Fourier Transform (FFT) and which is generally faster than the dilation wavelet analysis algorithm. The mechanism by which the signal is decomposed and reconstructed using the wavelet transform is investigated. The properties and characteristics of some famous harmonic wavelet transforms are analyzed.

In order to investigate the validity of this proposed method, the Duffing oscillator system and the bilinear hysteresis system are used as the examples to illustrate that the wavelet-based linearization method is a quite promising tools to find the equivalent linear parameters of the nonlinear systems in terms of time-frequency localization.

1.2 Objectives of the thesis

In this research, the main goal is to develop wavelet-based linearization method to linearize nonlinear dynamic systems. The objectives are listed as follows:

- To investigate the mechanism by which the signal is decomposed and reconstructed using the wavelet transform and to analyze the properties and characteristics of some famous harmonic wavelet transforms,
- To implement a multi-frequency linearization and wavelet-based linearization methods for Duffing oscillator system when the system is subject to multiple harmonic inputs,
- To develop a wavelet-based linearization method for the nonlinear system-bilinear hysteresis system and to develop numerical algorithm to find the equivalent linear parameters.

1.3 Thesis Outline

The thesis is organized as follows. Chapter 1 introduces the motivation, objective and contributions of the thesis. The thesis outline is given as well. Chapter 2 reviews time frequency signal analysis and linearization technique as the foundation of this dissertation. Chapter 3 focuses on the wavelet transform including its mechanism by which a signal is decomposed and reconstructed using wavelet basis function, and how to obtain the wavelet function based on dilation equation method. Also some famous wavelet families are displayed as some examples to illustrate the signal decomposition at different levels and time locations. Finally, the harmonic wavelets and their significant properties are discussed. As the most important features, these characteristics play very critical roles in the wavelet-based linearization method.

Chapter 4 gives the detailed discussion about the statistical linearization procedure for a nonlinear system including the traditional equivalent method and the multi-frequency linearization. The linearization methods are applied to Duffing oscillator system. The wavelet-based linearization approach is implemented. The effectiveness of the proposed method is verified by numerical method.

Chapter 5 addresses the wavelet-based linearization method with application in the bilinear hysteresis system and the comparison with statistical linearization method. The simulation results demonstrate that the wavelet-based linearization is a promising and applicable technique in linearization of bilinear hysteresis systems.

Chapter 6 concludes the thesis by reviewing the results. In addition, it points out the future research work.

2 LITERATURE REVIEW

In this chapter, we will review the methods related to the time-frequency analysis of signal processing. The properties and limitations of each method will be given. Also, various kinds of linearization techniques in different engineering areas will be reviewed.

2.1 Time –Frequency Analysis

The organization of this section is as follows: Section 2.1.1 starts with Fourier transform and its properties as the basic of signal analysis and then extends to Time-Frequency analysis. In Section 2.1.2， the short- time Fourier transform (STFT) and the Wigner-Ville（WV） transform are emphasized and their properties and limitations are discussed. Next, Section 2.1.3 introduces the concept of wavelets transform, its advantages and definitions as the foundation of this thesis.

2.1.1 Fourier Transform

Most engineers are familiar with the idea of frequency analysis by which a periodic function can be broken down into its harmonic components. And such a periodic function may be synthesized by adding together its harmonic components, namely, the Fourier series of function $f(t)$ defined on the interval $[-p, p]$ is given by:[27]

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} t + b_n \sin \frac{n\pi}{p} t \tag{2.1}$$

where:

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(t)dt \ (1.2) \tag{2.2}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(t)\cos \frac{n\pi}{p} tdt \tag{2.3}$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(t)\sin \frac{n\pi}{p} tdt \tag{2.4}$$

In order to obtain a representation for a non-periodic function defined for all real *t*, it seems desirable to take the limits as $p \rightarrow \infty$, *t* leads to the formulation of the famous Fourier integral theorem. Mathematically, this is continuous version of the completeness property of Fourier series. Physically, form (2.1) can be resolved into an infinite number of harmonic components with continuously varying frequency ($\omega/2\pi$), and amplitude $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega\tau}d\tau$, called Fourier transform whereas the ordinary Fourier series represents a resolution of a given function into an infinite but discrete set of harmonic components.

The Fourier transform originated from the Fourier integral theorem is believed to be one of the most remarkable discoveries in the mathematical sciences and engineering applications including the analysis of stationary signals and real-time signal processing which make an effective use of the Fourier transform in time and frequency domains. The success of Fourier transform is due to the fact that under some conditions, the function *f(t)* can be reconstructed by the Fourier inversion formula that defines:

$$f(t) = F^{-1}\{f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(\omega) d\omega = \frac{1}{2\pi}\left(f, \left(e^{-i\omega t}\right)\right) \qquad (2.5)$$

Thus, the Fourier transform theory has been very useful for analyzing harmonic signal. On the other hand, in spite of some remarkable successes, Fourier transform analysis seems to be inadequate for studying non-stationary signal that the spectral content of the signal changes with time [51]. First, the Fourier transform of a signal does not contain any local information in the sense that reflects the change of wave number with space. Second, the Fourier transform method can be used to investigate problems either in time domain or in frequency domain, but not simultaneously in both domains. There are probably the major weaknesses of the Fourier transform analysis. It is necessary to define a single transform of time and frequency that can be used to describe the energy density of a signal simultaneously in both time and frequency domains. Such a single transform would give complete time and frequency information of signal.

2.1.2 Short Time Fourier Transform

It is well-known that the Fourier transform analysis is a very effective tool for studying stationary signal. However, signals are, in general, non-stationary, and hence cannot be analyzed completely by Fourier transform. Therefore, a complete analysis of non-stationary signal requires both time-frequency.

In order to get information about local frequency spectrum, a lot of research work have been carried out to cut the signal to segments with a short data windows centered at time *t*,[70] and spectral coefficients are computed for this short length of data. The window is then moved to a new position and the calculation is repeated. This transform is called the short time Fourier transform (STFT). Based on such a procedure, Dennis Gabor [24] first introduced the windowed Fourier transform (the Gabor transform) to measure localized frequency component of signal and used a Gaussian distribution function as window function. The main idea was to use a time-localization window function $g_a(t-b)$ for extracting local information from the Fourier transform of a signal where the parameter *a* measures the width of the window and the parameter *b* is used to translate the window in order to cover the whole time domain. The property of the Gabor transform provides the local aspect of the Fourier with time resolution equal to the size of the window.[24]

$$g_{t,\omega}(\tau) = \exp(i\omega\tau)g(t-\tau) \tag{2.6}$$

As the window function by translating, the Gabor transform of *f* with the respect to *g*, is denoted by

$$\tilde{f}_g(t,\omega) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-i\omega\tau}d\tau = \left(f, \overline{g}_{t,\omega}\right) \tag{2.7}$$

And the inversion formula for the Gabor transform is given by Equation (2.6).

The Gabor transform $\tilde{f}_g(t,\omega)$ of a given signal *f(t)* depends on both time and frequency, thus the Gabor transform completely describes the spectral signal varying with time and changes an one dimensional signal into a complex value function of two real parameters: time and frequency in the two dimensional time-frequency space.

The short time Fourier Transform (STFT) remains the time information but has strong time-frequency resolution limitations. The problem with STFT can be demonstrated by Heisenberg`s Uncertainty Principle [38], [71]. According to this principle, it is impossible to know the exact time -frequency representation of a signal, but one can obtain the frequency bands existing in a time interval which is a resolution problem and once the window is chosen, the frequency and time resolution are fixed for all frequencies and all times.[77]

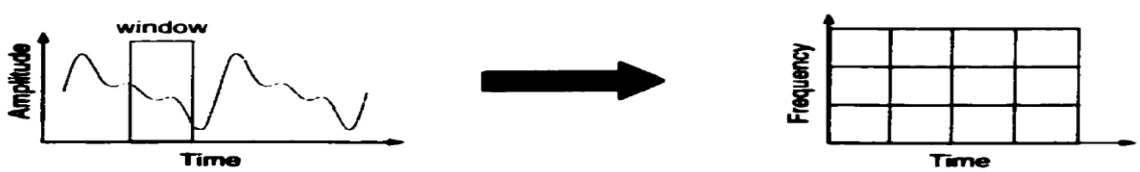

Figure 2. 1 Time-frequency plane partition by using STFT [69]

2.1.3 Wigner-Ville Distribution

Wigner-Ville Distribution (WVD) is another popular tool for time-frequency analysis. It can also be thought of as a short time Fourier transform (STFT) where windowing function is time-scaled, time-reversed copy of the original signal. It is the Fourier transform of the signal's autocorrelation function with respect to the delay variable. It has much better resolution than STFT.[72]

In 1948, Ville [65] proposed the Wigner Distribution of a function or a signal $f(t)$ in the form:

$$W_{f(t,\omega)} = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\bar{f}\left(t-\frac{\tau}{2}\right)e^{-iw\tau}d\tau \qquad (2.8)$$

For analysis of the time-frequency structure of non-stationary signal, where $\bar{f}(t)$ is the complex conjugate of $f(t)$, this time-frequency representation of a signal $f(t)$ is known as the Wigner-Ville Distribution [68] which is one of the fundamental methods which have been developed over the years for the time- frequency signal analysis. In the view of its remarkable mathematical structure and properties, the Wigner-Ville Distribution is now well-recognized as an effective method for the time-frequency analysis of non-stationary signals and non-stationary random process based on the generalizing relationship between the power spectrum and autocorrelation function.

Suppose that it is possible to calculate an instantaneous correlation function:

$$R_x(\tau,t) = E\left[x\left(t-\frac{\tau}{2}\right)x\left(t+\frac{\tau}{2}\right)\right] \qquad (2.9)$$

At time *t*, the operator *E* denotes the average of a statistical ensemble which is assumed to be available for the analysis. The Fourier transform of $R_x(\tau, t)$ gives:

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau, t)\exp(-i\omega\tau)d\tau \tag{2.10}$$

$S_x(\omega)$ is called the spectral density of the x process and is a function of angular frequency ω. It provides information on how the mean square of the ensemble is distributed over frequency and time. But in practice, it is never possible to compute ensemble average correlation function of $R_x(\tau, t)$. Instead, the ensemble averaging operation is replaced by:

$$\phi_x(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x\left(t - \frac{\tau}{2}\right)x^*\left(t + \frac{2}{\tau}\right)\exp(-i\omega\tau)d\tau \tag{2.11}$$

$\phi_x(\omega, t)$ is called the Wigner-Ville Distribution of x and * denotes complex conjugate. From theoretical and application points of view, the Wigner-Ville Distribution [53] plays a central role and has several important structures and properties.

First, it provides a high-resolution representation in time and in frequency for some non-stationary signals. Second, it has the special property of satisfying the time and frequency analysis in terms of the instantaneous power in time and energy spectrum in frequency. In spite of these desired features, its disadvantage is that although the integral is center at time t, it covers an infinite range of τ and thus depends on the characteristic of x far away from the local t. Therefore, it does not describe the truly local behavior of x at time t, this is the fundamental uncertainty principle since time-dependent spectral high resolution cannot be obtained simultaneously in time and frequency. Also it often possesses severe cross-terms or interference terms between different time-frequency regions leading to undesirable properties.

2.1.4 Wavelets Transform

In order to overcome some of the inherent weaknesses of the short time Fourier transform and the Wigner-Ville Distribution, there has been considerable recent interest in looking for a general time-frequency distribution analysis as a mathematical method for time-frequency signal.

Wavelet Transform as the new concept can be viewed as a synthesis of various ideas originating from different disciplines including mathematic, physics and engineering [27]. The pioneer research work in wavelet transform can be found in [28], [41], [19]. Morlet (1984)[28] first introduced the idea of wavelet as a family of function constructed from translation and dilation of a single function called “mother wavelet” $\varphi(t)$ defined by:

$$\varphi_{a,b}(t) = {}^{1}\!\big/_{\sqrt{|a|}}\,\varphi\left[\frac{(t-b)}{a}\right]; \quad a, b \in R, a \neq 0 \tag{2.12}$$

where a is called a scaling parameter which measures the degree of compression or scale, and b is a translation parameter which determines the time location of the wavelet.

If |a|<1, the wavelet (2.12) is the compressed version (smaller support in time domain) of mother wavelet and corresponds mainly to higher frequencies. On the other hand, where |a|>1, $\varphi_{a,b}(t)$ has a bigger time width than $\varphi(t)$ and corresponds to lower frequencies. Thus wavelets have time-width adapted to their frequencies. This is the main reason for the success of wavelet in signal processing and time-frequency signal analysis.

It may be noted that the resolution of wavelets at different scales varies in time and frequency domains as governed by the Heisenberg uncertainty principle[71][38], that is, at large scale, the resolution is coarse in time domain and fine in the frequency domain. As the scale a decrease, the resolution in time domain becomes fine whereas the resolution in frequency domain becomes coarse.

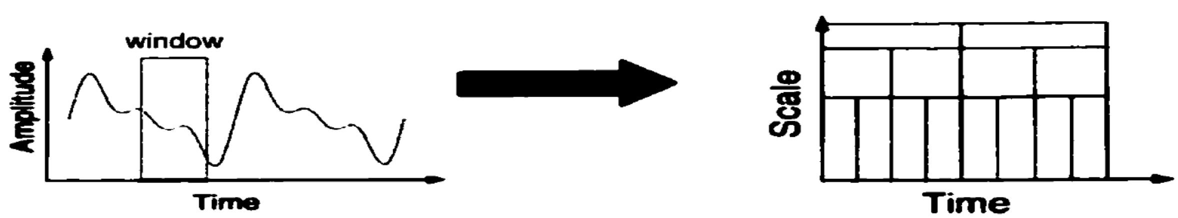

Figure 2. 2 Partition of the time-scale plane due to the wavelet transform [69]

A more extensive study has been carried out by Grossman, [29] and the wavelet transform of function $f(t)$ is defined by:

$$W_{\varphi}[f](a,b) = (f, \varphi_{a,b}) = \int_{-\infty}^{\infty} f(t)\overline{\varphi}_{a,b}(t)dt \tag{2.13}$$

where $\varphi_{a,b}(t)$ plays the same role as the kernel $\exp(i\omega t)$ in the Fourier transform. However, unlike the Fourier transform, the continuous wavelet transform is not a single transform, but any transform obtained in this way. The inverse wavelet transform can be defined so that $f(t)$ can be reconstructed by means of the formula:

$$f(t) = \frac{1}{C_{\varphi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{a^2} W_{\varphi}[f](a,b)\varphi\left(\frac{t-b}{a}\right) dadb \tag{2.14}$$

where parameter C_{φ} satisfies the so-called admissibility condition;

$$C_{\varphi} = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\omega)|^2}{\omega} d\omega < \infty \tag{2.15}$$

where parameter $\hat{\varphi}(\omega)$ is the Fourier transform of the mother wavelet $\varphi(t)$.

In practical applications involving fast numerical algorithms, the continuous wavelet can be computed at discrete grid points. To do this, a general function $\varphi_{a,b}(t)$ can be defined by replacing a with a_0^m $(a_0 \neq 0,1)$, b with $nb_0a_0^m(b_0 \neq 0)$, where (m,n) are integers and making

$$\varphi_{m,n}(t) = a_0^{-m/2}\varphi(a_0^{-m}t - nb_0) \tag{2.16}$$

The discrete wavelet transform of $f(t)$ is defined as the

$$W_{\varphi}[f](m,n) = (f, \varphi_{m,n}) = \int_{-\infty}^{\infty} f(t)\varphi_{m,n}(t)dt \tag{2.17}$$

If the wavelets form an orthogonal basis and complete, $f(t)$ can be reconstructed from its discrete wavelet transform $\{f(m,n)=(f,\varphi_{m,n})\}$ by means of the formula $W_{\varphi}[f](m,n)$

$$f(t)=\sum(f,\varphi_{m,n})\varphi_{m,n}(t) \tag{2.18}$$

provided that wavelets form an orthogonal basis.

For some very special choices of φ and a_0, b_0, the $\varphi_{m,n}$ constitute an orthogonal basis for L®. In fact, if $a_0=2$ and $b_0=1$, then there exists a function φ with good time-frequency localization properties:

$$\varphi_{m,n}(t)=2^{-m/2}\varphi(2^{-m}t-n) \tag{2.19}$$

It forms an orthogonal basis. These different orthogonal basis functions have been founded to be very useful in application to speech processing [73], image processing [76], computer vision and so on [66].

Generally speaking, comparing with the short time Fourier transform and the Wigner-Ville Distribution, wavelet transform has more advantages in time-frequency signal analysis due to its localization in time and frequency domains and flexible resolution to a certain extent. Meanwhile, wavelet allow complex information such as music, speech, images and patterns to be decomposed into elementary forms at different positions and scales that are generated from a single function called “mother wavelet” by translation and dilation operations. This information is subsequently reconstructed with higher precision. Table 2.1 summarizes the comparison of four methods in signal analysis.

Table 2.1 Comparison of Four Methods in Signal Analysis

	Signal Analysis	Basis Function	Dimension	Resolution
Fourier Transform(FT)	stationary signal	triangle function	only one, either in the time domain or in the frequency domains	Global function without any local information in time and frequency domain.
Short time Fourier transform(STFT)	non-stationary signal	time -localization window function(a Gaussian distribution function)	two, both time and frequency domains	Local information in time and frequency but their resolutions are fixed and no any exact amount because of uncertainty principle.
Wiger-Ville Distribution(WVD)	non-stationary signal	autocorrelation function	two, both time and frequency domaisn	Better than STFT but similar with STFT in the uncertainty principle
Wavelet Transform(WV)	non-stationary signal	wavelet functions which are constructed from translation and dilation mother wavelet	two, both time and frequency domains	More flexible resolution and location in time and frequency due to the wavelets have time-width adapted to their frequency

2.2 Linearization Techniques

In many practical applications, if one wants to predict the system performance accurately, the system nonlinearity must be taken into account.

As it is mentioned previously, the importance of studying nonlinear systems in automatic control area lies in:

- No model of real system is truly linear even if the operating range of a control system is small to study its linear approximation by a linearized method.
- Nonlinear equations cannot be solved analytically and the typical analysis tools like Laplace and Fourier transform are not suitable for analyzing the nonlinear systems. As a result, nonlinear systems may demonstrate complex effects, such as limit cycles, bifurcations and even chaos which cannot be anticipated.
- In order to design an accurate and desirable controller to meet specified control performance, we need some powerful techniques to analyze such system's behaviour which must be predicted and properly compensated for.

Since the linear control is a mature subject with a plenty of powerful methods and great achievements in the industrial and engineering applications, it is natural to adopt linear control techniques to analyze nonlinear problems.

In the following sections, some typical linearization methods are introduced and, first the

Lyapunov's linearization is reviewed in Section 2.2.1. Section 2.2.2 introduces the method of feedback linearization. The describing function method is discussed in the Section 2.2.3 and the statistical linearization method is introduced in the last section.

2.2.1 Lyapunov's Linearization

The main idea of the Lyapunov's linearization technique [30] is to approximate a nonlinear system by a linear one that is around the equilibrium point, and do hope the behaviour of the solutions of the linear system will be the same as the nonlinear one.

We study Lyapunov linearization for autonomous system, and for non-autonomous systems that the parameters vary with time, we extend the research of the linearization method for non-autonomous nonlinear system.

Consider an autonomous system of second order differential equation has the form:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y) \tag{2.20}$$

The constant solution to this system is called equilibrium point that satisfies:

$$f(x_0, y_0) = 0\,,\ g(x_0, y_0) = 0\,, \tag{2.21}$$

If the system is linear with constant coefficients, we can obtain the solution of this system. However, for the nonlinear system, no method is developed for deriving a general solution to this equation.

To deal with a nonlinear problem, we must use an approximation method which is called linearization approach. As (x_0, y_0) is an equilibrium point, we try to find the linear system when (x, y) is close to (x_0, y_0), that is, to approximate $f(x, y)$ and $g(x, y)$ when (x, y) is close to (x_0, y_0). This is to calculate the functions $f(x, y)$ and $g(x, y)$ by the Taylor expansion approximation at a point (x_0, y_0).

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$g(x, y) \approx g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (2.22)$$

By replacing $f(x, y)$ and $g(x, y)$ in (2.20) with their linear approximations near (x_0, y_0), we obtain:

$$\frac{dx}{dt} = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\frac{dy}{dt} = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (2.23)$$

Since (x_0, y_0) is an equilibrium point,

$$f(x_0, y_0) = g(x_0, y_0) = 0 .$$

$$\frac{dx}{dt} = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\frac{dy}{dt} = \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \tag{2.24}$$

This is a linear system and its coefficients matrix is:

$$\begin{vmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{vmatrix} \tag{2.25}$$

This matrix is called Jacobian matrix of system at the point (x_0, y_0), and the eigenvalues of the Jacobian matrix tell the information about the stability of the linear system and that of original nonlinear system.

- If the eigenvalues are negative or complex with negative real part, then the equilibrium point is stable, i. e., all solutions are convergent at the equilibrium point or if the eigenvalues are complex, then the solution will spiral around the equilibrium point. Hence, the linearized system is stable and so is the original nonlinear system.
- If the eigenvalues are positive or complex with positive real part, then the equilibrium point is unstable, i.e., all solutions will move away from the equilibrium point or if eigenvalues are complex, then the solutions will spiral away from equilibrium point. The linearized system is unstable and so is the nonlinear one.
- If the eigenvalues are real number with different sign (one positive and one

negative), then the equilibrium point is a saddle. Both of two systems are unstable.

- If the eigenvalues are complex with real part equal to zero, then the equilibrium point is center point, and all solutions form ellipses around the equilibrium point in center or at least, there is one on the imaginary axis and another is complex with negative part. The equilibrium point may be stable or unstable and hence one cannot draw any conclusion about the stability of the nonlinear system.

Note that the stability characteristics of linear systems are determined by their equilibrium point, therefore, if the nonlinear system is linearized, the local behaviour of this nonlinear system can be approximated by its linear replacing one.

The Lyapunov linearization method for non-autonomous nonlinear system involves more conditions and the powerful autonomous theorem does not apply to non-autonomous nonlinear system. However, there still exist some rules about instability for non-autonomous nonlinear system from that of its linear approximation as long as its linear approximation is time invariant.

In summary, Lyapunov's linearization method both for autonomous and non-autonomous nonlinear system provide the linearization method to analyze the nonlinear system stability issue which is critical and essential in nonlinear control system design.

2.2.2 Feedback Linearization Method

Feedback linearization is a common approach [26] used in nonlinear control systems. The idea of this method is to transform original system model into equivalent linear one which is changed to the state variables and a suitable input instead.

This method is different from the Jacobian linearization mentioned previously, which is approximated by its linearized system, actually, feedback linearization method transforms the exact state variables in some algebraic forms so that the nonlinear system is changed into a linear one and linear control techniques can be applied.

As one of the most important control techniques, the feedback linearization is applied successfully in the control of high performance aircraft, industrial robots and so on.

This part provides a procedure of feedback linearization including the basic concepts of this method and mathematical tools to transform nonlinear components into linear systems.

First, the basic concept of feedback linearization is to cancel the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form [37].

The nonlinear system has the forms:

$$x^{(n)} = f(X) + b(X)u \tag{2.26}$$

where u is the scalar control input, x is the scalar output, and $X = [x, \dot{x}, ..., x^{(n-1)}]^T$ is the state vector, and $f(X)$ and $b(X)$ are nonlinear function of the states. The Equation (2.26) can be written as the form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ ... \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ ... \\ x_n \\ f(X) + b(X)u \end{bmatrix} \tag{2.27}$$

Thus the system is expressed in the controllability canonical form, and the input u is designed as:

$$u = \frac{1}{b}[v - f] \quad b \neq 0, \tag{2.28}$$

We cancel the nonlinearities and obtain the simple input –output relationship in the form that is:

$$x^{(n)} = v, \tag{2.29}$$

We can design the control law that is:

$$v = -c_0 x - c_1 \dot{x} - ... - c_{n-1} x^{(n-1)} \tag{2.30}$$

With the parameters c_i chosen so that the polynomial $p^n + c_{n-1} p^{n-1} + ... + c_0$ has all its roots strictly in the left-half complex plane, the nonlinear system leads to the exponentially stable dynamics.

$$x^{(n)} + c_{n-1} x^{(n-1)} + ... + c_0 x = 0 \tag{2.31}$$

If the control of the tracking problem is considered, the control law can be:

$$v = x_d^{(n)} - c_0 e - c_1 \dot{e} - \ldots - c_{n-1} e^{(n-1)} \tag{2.32}$$

where $e(t) = x(t) - x_d(t), x_d(t)$ is a desired output.

Second, some mathematical tool related to feedback linearization technique is introduced.

Lie derivatives:

Consider nonlinear system:

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x) \tag{2.33}$$

where $x \in R^n$ is the state vector, $u \in R^P$ is the vector of inputs, and $y \in R^m$ is the vector of outputs.

$$\dot{y} = \frac{dh(x)}{dx} \cdot \frac{dx}{dt} = \frac{dh(x)}{dx} \cdot \dot{x} = \frac{dh(x)}{dx} f(x) + \frac{dh(x)}{dx} g(x)u \tag{2.34}$$

Defining the lie derivative $h(x)$ along $f(x)$ as

$$L_f h(x) = \frac{dh(x)}{dx} f(x), \tag{2.35}$$

And similarly, the lie derivative $h(x)$ along $g(x)$ as

$$L_g h(x) = \frac{dh(x)}{dx} g(x) \tag{2.36}$$

Thus, $\dot{y} = L_f h(x) + L_g h(x)u$ (2.37)

The goal of the feedback linearization is to find a direct and simple relationship between the system output y and the control input u by differentiating the output y until the input u

is expressed in the same equation. Thus we obtain:

$$z = T(x) = \begin{bmatrix} z_1(x) \\ z_2(x) \\ \cdots \\ z_n(x) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \cdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \cdots\cdot \\ L_f^{n-1} h(x) \end{bmatrix} \tag{2.38}$$

$$\begin{Bmatrix} \dot{z}_1 = L_f h(x) = z_2(x) \\ \dot{z}_2 = L_f^2 h(x) = z_3(x) \\ \cdots \\ \dot{z}_n = L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{Bmatrix} \tag{2.39}$$

We assumed the relative degree of system is *n*, the lie derivative of the form $L_g L_f^i h(x)$ for *i=1…n-2* are all zeros. And the original X coordinate is transformed into the Z coordinate so that the new linearized system with z variables is obtained as well as the relationship between input and output. Therefore, the control law is that:

$$u = \frac{1}{L_g L_f^{n\;1} h(x)} \left(\;\; L_f^n h(x) + v \right) \tag{2.40}$$

Also the input-output relationship from v to $z_1 = y$ is that

$$\begin{vmatrix} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \cdots \\ \dot{z}_n = v \end{vmatrix} \tag{2.41}$$

Choosing the suitable C in $v = -Cz$ with standard linear system methodology, we get the linear system with the linearizing state *z* and linearizing control law *v*.

In summary, feedback linearization method is based on the idea of transforming nonlinear system into a linear one by using state feedback with input-state linearization or input-output linearization method which can be used for both stabilization and tracking control problems and has been applied in many practical nonlinear controlling systems to achieve successful results. However, this linearization method needs the exact nonlinear function in the system dynamics which is usually difficult to obtain in the real practice. In addition, the controller design based on the linearized model will cause infinity control action if $L_g L_f^{n-1} h(x)$ in (2.40) becomes very small.

2.2.3 Describing Function Method

Basically, the describing function method is the statistical extension of [40] linearization technique in the electrical engineering literature which is an approximation procedure for analyzing certain nonlinear control problems that were originally developed by [12] and used as a tool in control engineering. Subsequent developments of this method in this field have described in [56] and [25].

The describing function method is an approximation procedure for analyzing certain nonlinear control problems. It is based on replacing each nonlinear element with a quasi-linear descriptor which is the approximation of nonlinear system by a linear transfer function whose gain is determined by the input amplitude while a true linear system transfer function does not depend on that amplitude. And describing function can also be treated as an extended version of the frequency response method to predict nonlinear behavior, especially, for limit cycle prediction in nonlinear systems. The describing function is a powerful tool to discover the existence of limit cycles and determine their stability. [62]

Before discussing the property of describing function, some assumptions must be given to satisfy some conditions:

- There is only a single nonlinear component.
- The nonlinear component is the time invariant.
- Corresponding to a sinusoidal input $x = \sin(\omega t)$, only the fundamental component $y_1(t)$ in the output $y(t)$ has been considered. This assumption implies that higher frequency harmonics can be neglected, and the approximated linear one has low-pass properties.
- The nonlinearity is odd that means the relationship with input and output of nonlinear element is symmetric about origin so that the static term in the Fourier expansion of output can be ignored.

A. Basic Definitions

Consider a sinusoidal input $x = A\sin(\omega t)$ to nonlinear element.

where A is amplitude and ω is frequency, as show in Figure 2.3.

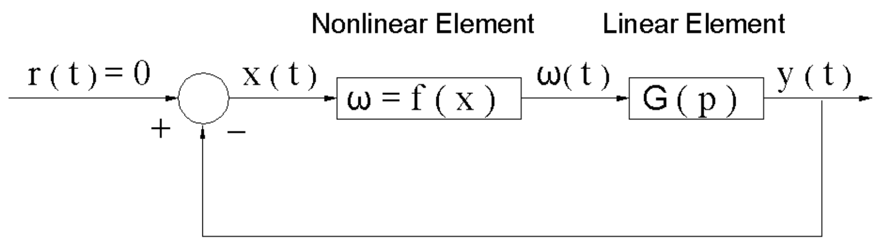

Figure 2. 3 A nonlinear system

The output *y(t)* is always a periodic function that is extended by Fourier series expressed as:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \tag{2.42}$$

where the Fourier coefficients a_n, b_n are determined by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)d(\omega t) \tag{2.43}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\cos(n\omega t)d(\omega t) \tag{2.44}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\sin(n\omega t)d(\omega t) \tag{2.45}$$

Due to the fourth assumption implying that $a_0 = 0$, the third assumption implies that only the fundamental component $y_1(t)$ is considered:

$$y(t) \approx y_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t) = M\sin(\omega t + \phi) \tag{2.46}$$

where $M(A,\omega)=\sqrt{a_1^2+b_1^2}$ and $\phi(A,\omega)=\arctan\left(a_1/b_1\right)$.

And also this output can be written as :

$y_1(t)=Me^{j(\omega t+\phi)}=(b_1+ja_1)e^{j(\omega t+\phi)}$.

Since the concept of frequency response function is the frequency ratio of the sinusoidal input and the sinusoidal output of system, we define the describing function of nonlinear element to be complex ratio of the fundamental component of nonlinear element by input sinusoidal, that is:

$$N(A,\omega)=\frac{Me^{j(\omega t+\phi)}}{Ae^{j\omega t}}=\frac{M}{A}e^{j\phi}=\frac{1}{A}(b_1+ja_1) \qquad (2.47)$$

Thus, the describing function [64] can be treated as the approximation of linear one with a frequency response function $N(A,\omega)$, which is considered as an extension of the concept of frequency response function. However, the describing function of nonlinear element differs from linear system frequency response function in that it depends on the input amplitude A.

B. Describing Function Analysis Nonlinear System

As we know how to get the describing function for the nonlinear element, we use this tool to predict the limit cycles that is based on the linear approach of the famous Nyquist criterion in control system to investigate the stability.

Consider a self-sustained oscillation of amplitude A and frequency ω in the system of

Figure 2.4.

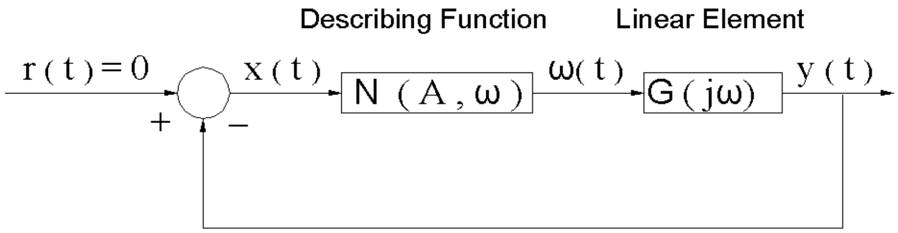

Figure 2. 4 A linearization system

The variables in the loop must satisfy the following relations:

$$x = y$$
$$w = N(A,\omega)x$$
$$y = G(j\omega)w$$

Therefore, $y = G(j\omega)N(A,\omega)(-y)$. And $y \neq 0$. (2.48)

So, $G(j\omega)N(A,\omega)+1=0$ and it can be written as:

$$G(j\omega) = -1/N(A,\omega) \tag{2.49}$$

The amplitude A and frequency ω of the limit cycles must satisfy (2.49), if the above equation has no solution and the nonlinear system has no limit cycles.

However, it is difficult to get the solution by this analytical approach, especially, for higher order system, thus the graphical way is usually used. The idea is to plot both sides of (2.49) in complex plane and find the intersection points of two curves. And also according to Nyquist criterion, we can determine the stability of limit cycles as shown in Figure 2.5.

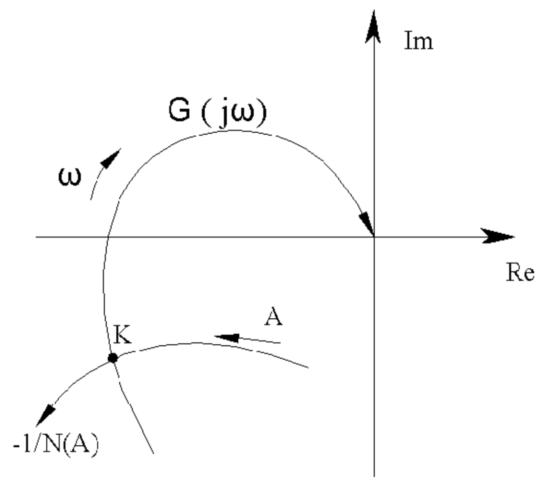

Figure 2. 5 Detection of limit cycle

Generally speaking, the describing function method which is an extension of the frequency response method of linear control can be used as the approximate analysis to predict some important characteristics of nonlinear system including systems with hard nonlinearities such as, saturation, backlash and hysteresis [1]. Particularly, it is the main tool to predict the limit cycles in the nonlinear system in the graphical nature of frequency domain [52].

2.2.4 Statistical Linearization Method

Among those various possible approaches which are available, the statistical method or equivalent linearization method has proven to be very useful approximate technique in structure dynamics and earthquake engineering [6].

Methods of predicting the random vibration response of mechanical and structural system to random external forces has grown rapidly. Since the excitation and response of such system are always non-periodic, highly irregular and no repeatability, linear method is not suitable to apply, and new techniques are required.

However, in engineering application, because the linear system is fairly simple and there a lot of resources to obtain the useful and desired solution, the systems can be modeled in terms of the linear differential equations of motion. Although no real system is exactly linear, in mechanical and structural systems, non-linearity can arise in various forms and usually become more significant as the amplitude of vibration increase. In order to predict the response of this kind of system or to get an approximation solution of non-linear equation, the statistical method or equivalent linearization method is applied to estimate the accurate equivalent linear parameters.

The statistical equivalent linearization method is based on the idea that the non-linear system is replaced by an equivalent linear equation by minimizing the difference between the two systems in some appropriate sense. This technique has been used for deterministic non-linear problems for many years, for instance, referred to as "describing function" method [40], and the adaptation of approach to deal with stochastic problem was first developed by [12] to use as a tool in control engineering And subsequent developments of this field have been described by[56],[25],[57]. This method known as statistical linearization or equivalent linearization was proposed by [14], [15]as a way to solve nonlinear vibration problems, furthermore, the statistical method is applied and served as the fundamental concept[58],[59] to propose the novel wavelet-based linearization method.[8],[9]

In summary, this part reviews some typical linearization methods and their applications in the nonlinear system control design and analysis. Comparing with their properties and limits, it provides fundamental background knowledge for the proposed linearization method that is based on the statistical linearization approach. The details about this method will be discussed in the Chapter 4 & 5 by using some typical nonlinear systems as the examples.

2.3 Conclusion

In this chapter, comprehensive review and introduction on time-frequency analysis of signal processing and linearization techniques are given. The properties and limitations of each method have been analyzed.

3 WAVELET TRANSFORM

This chapter presents the mechanism by which the signal is decomposed and reconstructed using the wavelet transform and introduces some famous wavelet basis functions. The properties and characteristics of the harmonic wavelet transform are discussed. The discrete harmonic wavelet transform will be also described.

3.1 Signal Decomposition and Reconstruction Using the Wavelets

The discrete wavelet scheme is used for this presentation due to its simplicity and ease of implementation. In discrete wavelet transform (DWT), a time-scale representation of a digital signal is obtained using digital filtering techniques [44]. At each of filtering process, a signal is decomposed in high scale-low frequency components which are called approximations (A) and in low scale-high frequency components which are called details (D). This signal is passed through a series of low- pass filters to analyze the low frequencies and pass through a series of high- pass filters to analyze the high frequencies. During the decomposition process, twice the amount of the original signal data is generated at each stage. To avoid this result, the data has to been down-sampled by a factor of 2 that is shown as:

$$f(t) \rightarrow \underset{\text{down sampled}}{\boxed{\downarrow 2}} \rightarrow g(t) \;=\; f(2t)$$

That means a discrete time signal *f(t)* consists of omitting every other value and we can think of a system whose input is *f(t)* and whose output is g(t)=f(2t). The same calculation holds for the detail coefficients.

The filter process is demonstrated in Figure 3.1

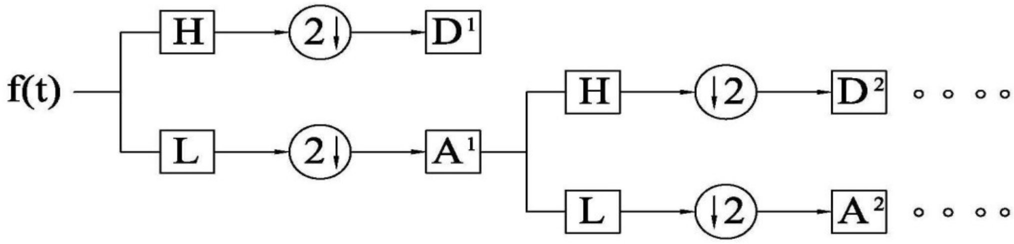

Figure 3. 1 Two-level wavelet decomposition using high-pass and low-pass filters.

The decomposition process can be iterated with successive approximations being decomposed in turn [42] so that one signal is broken down into many lower resolution components. This is called wavelet decomposition tree which is shown in Figure 3.2

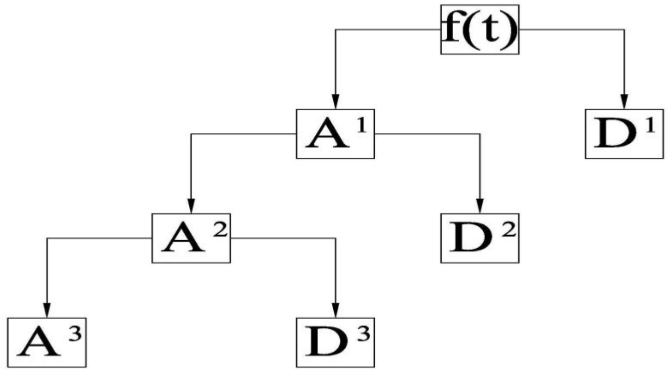

Figure 3. 2 The wavelet decomposition tree

The decomposition lasts until the detail coefficients consist of repetitions of a single sample. To synthesize the signal, the inverse process of up sampling and filtering is followed by using the low (L') and high (H') pass reconstruction filters. Up sampling of discrete time signal *f(t)* consists of inserting zeros between the values. We can think about a system with input *f(t)* and output g(t)=f(t/2) for even values of *t*, and g(t)=0 for odd values of *t* shown as:

$$f(n) \rightarrow \underset{\text{up sampled}}{\boxed{\uparrow 2}} \rightarrow g(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{for n even} \\ 0 & \text{for n odd} \end{cases}$$

The design of the low and high pass filter for the decomposition and reconstruction process is very important because it determines the shape of the wavelet function used in analysis.[61],[20],[44],[74],[75] In fact, the filter design and wavelet function choice develop at the same time.

The filters used in the decomposition and reconstruction process are different in that they must be satisfied some conditions:[20]

Condition 1:

$$H\left(\omega + \tfrac{1}{2}\right)H'(\omega) + L\left(\omega + \tfrac{1}{2}\right)L'(\omega) = 0 \qquad (3.1)$$

where $\mathrm{H}(\omega)$ and $L(\omega)$ are Fourier transform of high-pass filter *h* and low-pass filter *l*, also H' and L' are conjugate to H and L and the same as h and l to h' and l'. It is defined by,

$$H(\omega) = \sum_n h(n)e^{-2\pi n\omega} \qquad (3.2)$$

And

$$L(\omega) = \sum_n g(n)e^{-2\pi n\omega} \qquad (3.3)$$

Condition 2:

$$H(\omega)H'(\omega) + L(\omega)L'(\omega) = 1 \qquad (3.\ 4)$$

In the signal decomposed or reconstructed process, the function is involved into successive two filters L (low frequency) and H(high frequency) or their conjugate filter L' and H' that so-called quadric mirror filters.

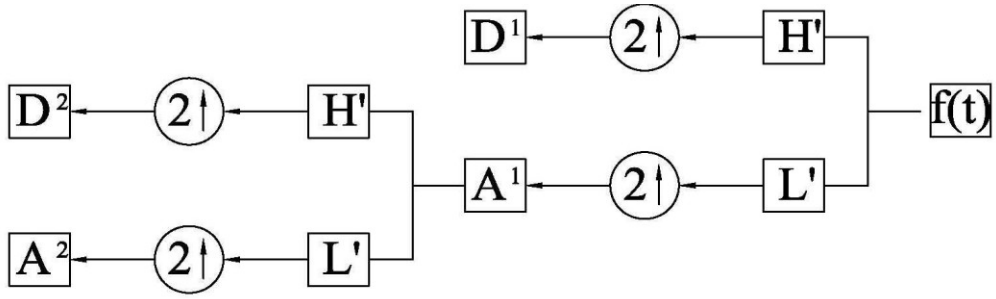

Figure 3. 3 Two-level wavelet reconstruction using high-pass and low-pass filters.

There are infinite numbers of possibility to choose the analyzing wavelet. However, only a small group of these meets the conditions that are necessary if the wavelets are given an accurate decomposition and reconstruction and also be orthogonal to each other[45]. This is extremely critical and efficient for computing wavelet transform which does for wavelet analysis the same as what FFT (fast Fourier transform) does for Fourier analysis.

3.2 Wavelet Function Design

As mentioned previously, the wavelet function design is based on accuracy and calculating efficiency. There are two classes of orthogonal wavelets-- Dilation wavelet and harmonic wavelet.

First we consider the theory of dilation wavelets and later discuss harmonic wavelet.

The successful development of dilation wavelet in mathematical community has been led by Daubechies[19], [20], [21] who invented so-called dilation schemes and for first time gave the method to calculate the dilation wavelet coefficients.

3.2.1 Dilation Equations

To dilate is to spread out so that dilation means expansion [61]. The basis function φ (x) is a dilated horizontally version φ (2x) that has the same height but is stretched over twice the horizontal scale of x, where x is a independent variable that may represent time or position depending on the applications. In dilation equation, φ (x) is expressed as a finite series of terms, each of which involves φ (2x). Each of these φ (2x) terms is positioned at different places on the horizontal axis by making the translation (2x-k) instead of 2x, where *k* is an integer(positive or negative).

The dilation equation has a form:

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3) \tag{3.5}$$

where c's are numerical constants. It is impossible to solve (3.5) directly to find function φ (x) except for very simple case. But indirectly setting up an iterative algorithm in which each new approximation $\phi_j(x)$ is calculated from the previous approximation $\phi_{j-1}(x)$ by the scheme:

$$\phi_j(x) = c_0\phi_{j-1}(2x) + c_1\phi_{j-1}(2x-1) + c_2\phi_{j-1}(2x-2) + c_3\phi_{j-1}(2x-3) \tag{3.6}$$

The iterative process continues until $\phi_j(x)$ becomes the same as $\phi_{j-1}(x)$.

Consider starting from a box function:

$\phi_0 = 1 \quad 0 \le x < 1$;

$\phi_0 = 0$, *elsewhere* .

After one iteration the box function over interval x=0 to 1 has expanded into a staircase function over the interval x=0 to 2 also shown in the Figure 3.4

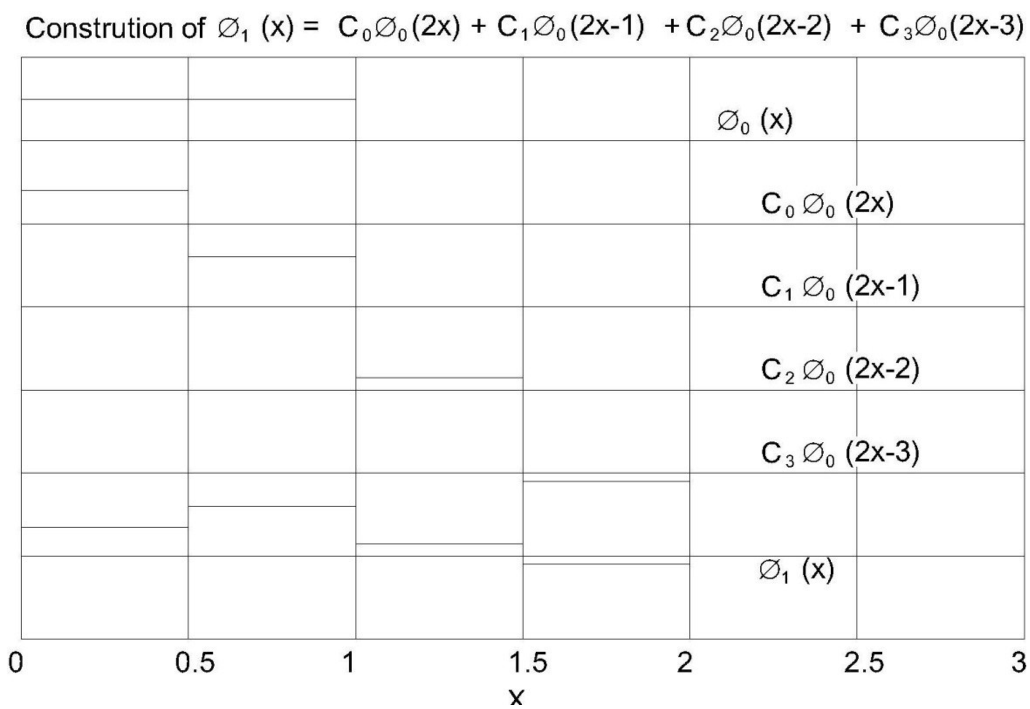

Figure 3. 4 Construction of the scaling function

Each contribution to $\phi_1(x)$ is shown separately and then four contributions are shown added together. A particular set of the coefficients $c_{0,} c_1, c_2, c_3$ have been obtained as follows:

$$c_0 = \left(1+\sqrt{3}\right)/4;\ c_1 = \left(3+\sqrt{3}\right)/4;$$

$$c_2 = \left(3-\sqrt{3}\right)/4;\ c_3 = \left(\sqrt{3}-1\right)/4; \tag{3.7}$$

This is the orthogonal D4 wavelet. The D stands for Daubechies.

The function $\phi(x)$ coming from a unit box (unit height and length) is called scaling function and a corresponding wavelet function will be obtained from it. When the iterative process continues, it has been seen that the first iteration produces the four coefficients $c_{0,}c_1, c_2, c_{3,}$ at x=0, 0.5, 1, 1.5. At the second process, each coefficient produces another four coefficients. For example, c_0 at x=0 produces four new coefficients which are $c_0^2, c_0c_1,\ c_0c_{2,}c_0c_{3,}$ at x=0, 0.25, 0.5, 0.75; and c_1 at x=0.5 contributes to four new coefficients $c_1c_{0,}c_1^2, c_1c_2, c_1c_3$ at x=0.5, 0.75, 1, 1.25; and so on. After second iteration, all the resulting ordinates corresponding to the coefficients are obtained. Figure 3.5 displays the first eight iterations of the development of the D4 scaling function. Figure 3.6 shows scaling function for D4 wavelet calculated for 12288 points.

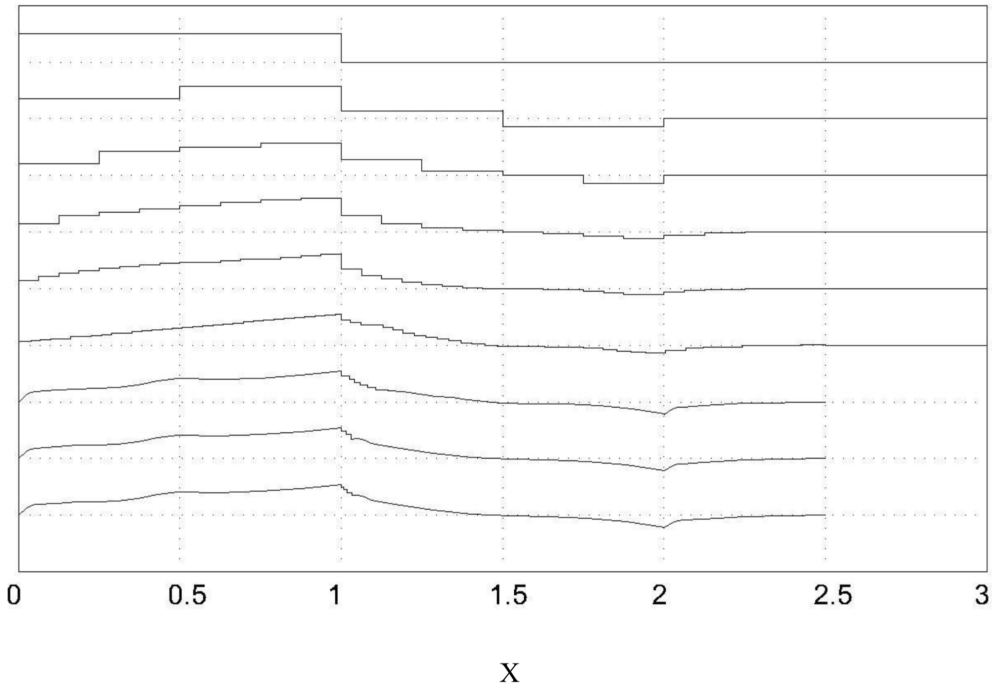

Figure 3. 5 Construction of D4 scaling function by iteration from a box function over the

interval x=0 to 1

From this process we find the matrix scheme

$$[\phi_2]=\begin{bmatrix} c_0 & & & \\ c_1 & & & \\ c_2 & c_0 & & \\ c_3 & c_1 & & \\ & c_2 & c_0 & \\ & c_3 & c_1 & \\ & & c_2 & c_0 \\ & & c_3 & c_1 \\ & & & c_3 \\ & & & c_3 \\ & & & \\ & & & \end{bmatrix}\begin{matrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{matrix}[1]=M_2M_1[1]$$

where M_r denotes a matrix of order $\left(2^{r+1}+2^{r}-2\right)\times\left(2^{r}+2^{r-1}-2\right)$ in which each column has a sub-matrix of coefficients $c_{0,}c_1,c_2,c_3$ positioned two places below the sub-matrix to left.

The number of points increases in the sequence and after eight iterations it reaches 766 points and each of them spaced $1/2^8$=1/256. Although the iteration method is not an efficient way, it is simple to understand and easily to be programmed.

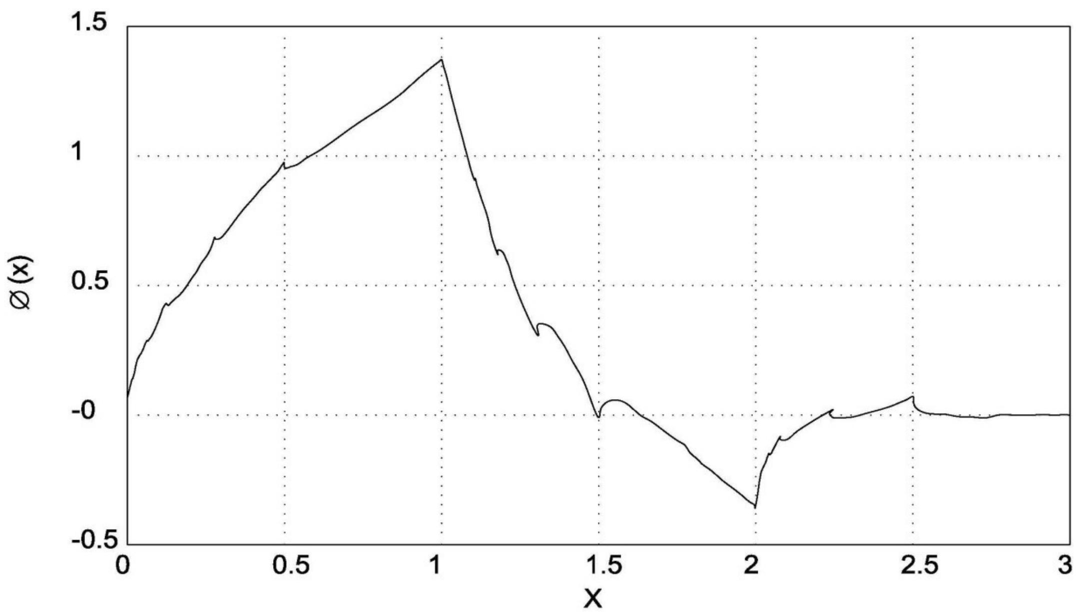

Figure 3. 6 Scaling function for D4 wavelet calculated for 12288 points

3.2.2 Dilation Wavelets

The wavelet function W(x) is derived from scaling function by taking differences. For the four –coefficient scaling function defined by (3.5) the dilation wavelet function is $w(x) = -c_3\phi(2x) + c_2\phi(2x-1) - c_1\phi(2x-2) + c_0\phi(2x-3)$ (3.8)

The same coefficients are used as scaling function $\phi(x)$ but in reverse order and with alternate terms having their signs changing from plus to minus.

The results of making the calculation (3.8) for D4 scaling function in Figure 3.6 are shown in Figure 3.8 and the iteration of development of the D4 wavelet is shown in Figure 3.7.

So far using the scaling function $\phi(x)$ to construct orthogonal wavelet function which has been developed by Daubeches[19] [20], [21] who first computed the values of required number N of coefficients $c_0, c_1, \ldots c_{N-1}$ in order to achieve accurate level expansion for signal decomposition and reconstruction

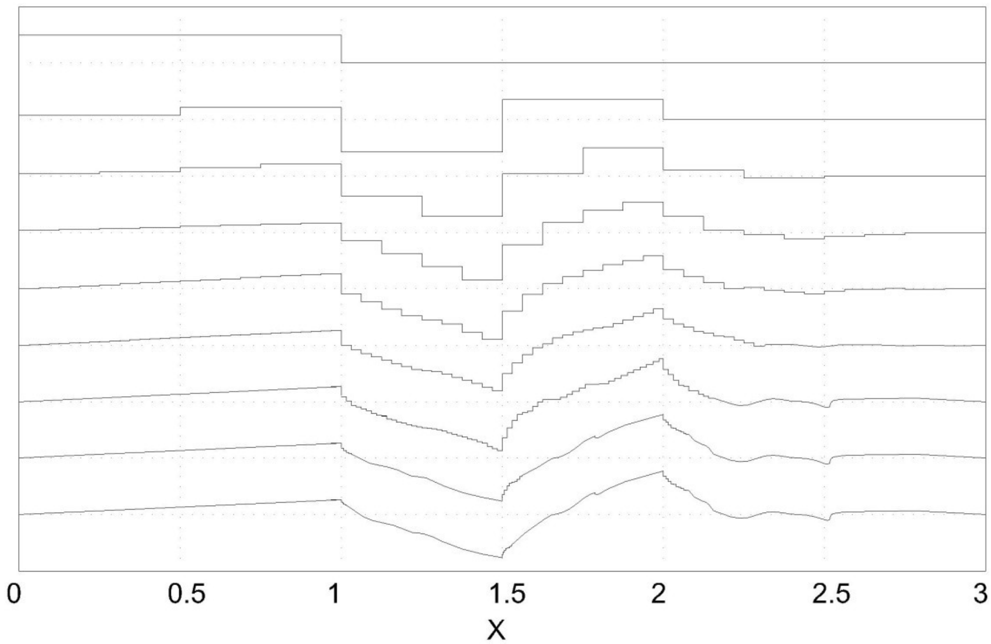

Figure 3. 7 D4 wavelet calculated by iteration

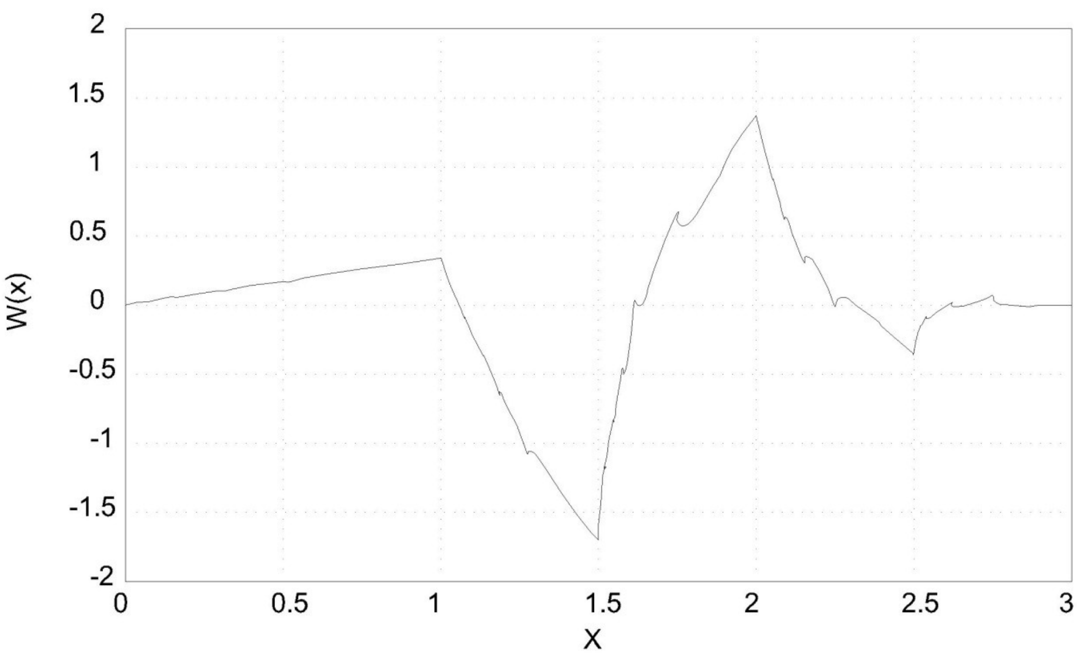

Figure 3. 8 D4 wavelet according to (2.8) from scaling function in Figure 3.6

3.2.3 Wavelet Family Introduction

There is a lot of wavelet functions which can be used as the mother wavelet function to achieve a signal's wavelet transform. Among those choices we introduce some very commonly used wavelets in engineering applications. [38]

First, it is the Haar wavelet which is the simplest orthogonal basis function. Figure 3.9 shows how a signal is decomposed in levels.

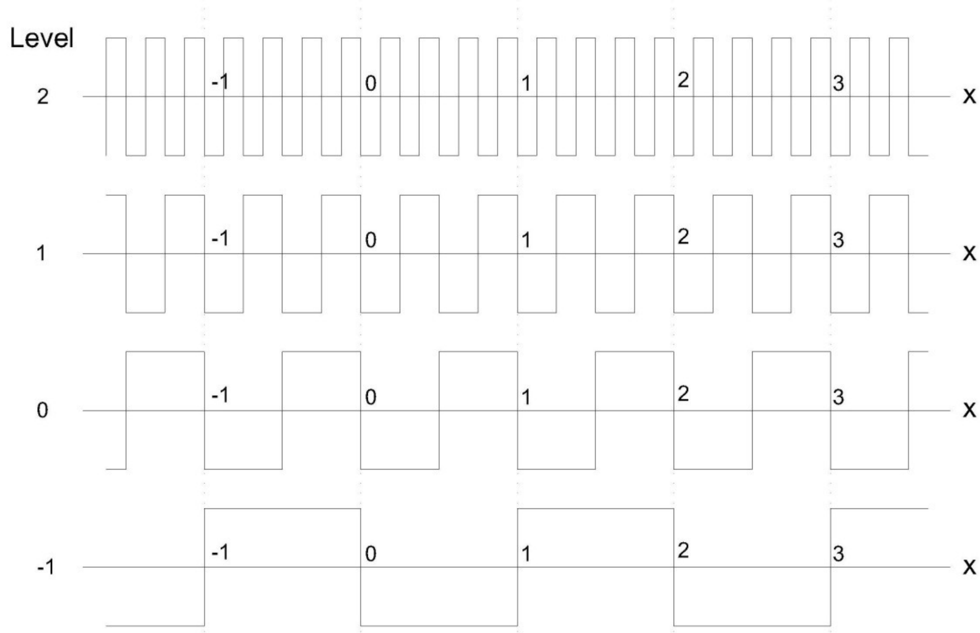

Figure 3. 9 Haar wavelet with unit amplitude plotted for level -1, 0, 1, 2

The Haar wavelet function is: $\begin{Bmatrix} w(x) = -1 \, for \, 0 \le x \le 1/2 \\ w(x) = 1 \, for \, 1/2 \le x \le 1 \end{Bmatrix}$ (3. 9)

The time or position x is plotted along the horizontal axis and wavelet function $w(x)$ is 0 to 1 which are located at different position along x axis.

There are two half length wavelets represented by $w(2x)$ and $w(2x-1)$ and four-quarter length wavelets represented by $w(4x), w(4x-1), w(4x-2), w(4x-3)$, as shown in Figure 3.9. The scale and position can be obtained from its argument. For example, $w(2x-1)$ is the same as $w(x)$ except that it is compressed into half the horizontal length and it begins at x=1/2 instead of x=0.

The wavelet's level is determined by how many wavelets fit into the unit interval x=0 to1. At level 0 there is $2^0 = 1$ wavelet like the third top view of Figure3.9. And there are $2^1 = 2$ wavelets which fit between 0 to 1and at level 2, there are $2^2 = 4$ wavelets which fit between 0 to1 like the top view Figure 3.9 and so on.

The decomposition of a signal or function f(x) is the same as its Fourier transform in which the sequence length N of signal being analyzed determines how many separate frequencies can be represented. In the wavelet transform, the sequence length determines how many wavelet levels there are with different scales and positions in which the sequence of N=2^n points consist of n+1 levels (scales) running from -1 to n-1 and 2^{n-1} position.

$$f(x)=a_0+a_1w(x)+a_2w(2x)+a_3w(2x-1)+a_4w(4x)+a_5w(4x-1)$$
$$+a_6w(4x-2)+a_7w(4x-3)+\ldots\ldots \tag{3.10}$$
$$=a_0+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}a_{2^j+k}w\left(2^jx-k\right)$$

There are some other wavelet family functions such as Mexcian hat wavelet whose analytical expression is:[21]

$$\varphi(x)=\left(\frac{2}{\sqrt{3}}\pi^{1/4}\right)\left(1-x^2\right)e^{-x^2/2} \tag{3.11}$$

And Morlet wavelet is defined as:

$$\varphi(x)=e^{-x^2/2}e^{j\omega_0 t} \quad \text{where } \omega_0=1.75\pi \tag{3.12}$$

Both basis wavelet functions shown in Figure 3.10 are applied in different signals analysis and other engineering applications. Therefore mother wavelets choice should take into account the details and particular application so that the wavelet transform can represent function characteristics effectively.

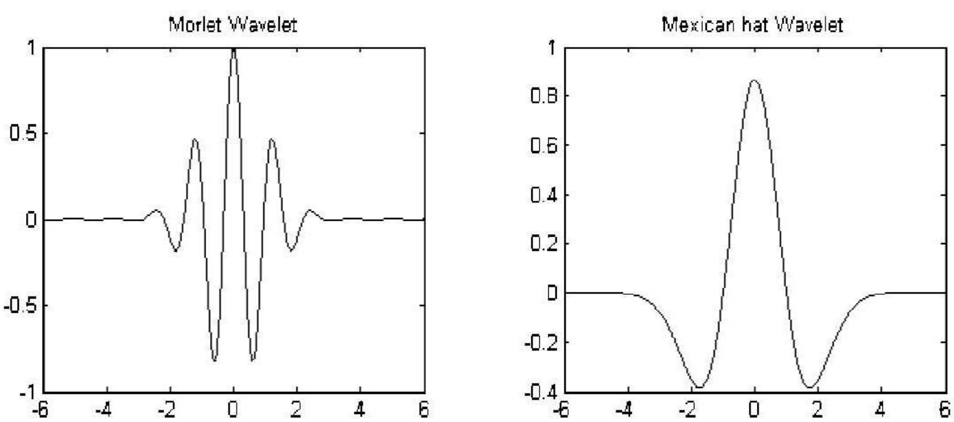

Figure 3. 10 Morlet and Mexican hat Wavelet forms

3.3 Harmonic Wavelet

In this section harmonic wavelet is introduced and its properties and characteristics are discussed as the fundamental concept to construct the rest of this thesis. As the number N of wavelet coefficients is increased, the wavelet's Fourier transform becomes compact. Only the values of N up to 20 are included in the program. However the number of wavelet coefficients for N> 20 is required to be computed and it turns out that the spectrum of a wavelet with N coefficients are more like box-like as N increases. This fact lead Newland [46] to seek a wavelet $w(x)$ whose spectrum is exactly like a box so that the magnitude of Fourier transform $w(\omega)$ is zero except for an octave band of frequencies. A simple way to define $|w(x)|$ is shown in Figure 3.11.

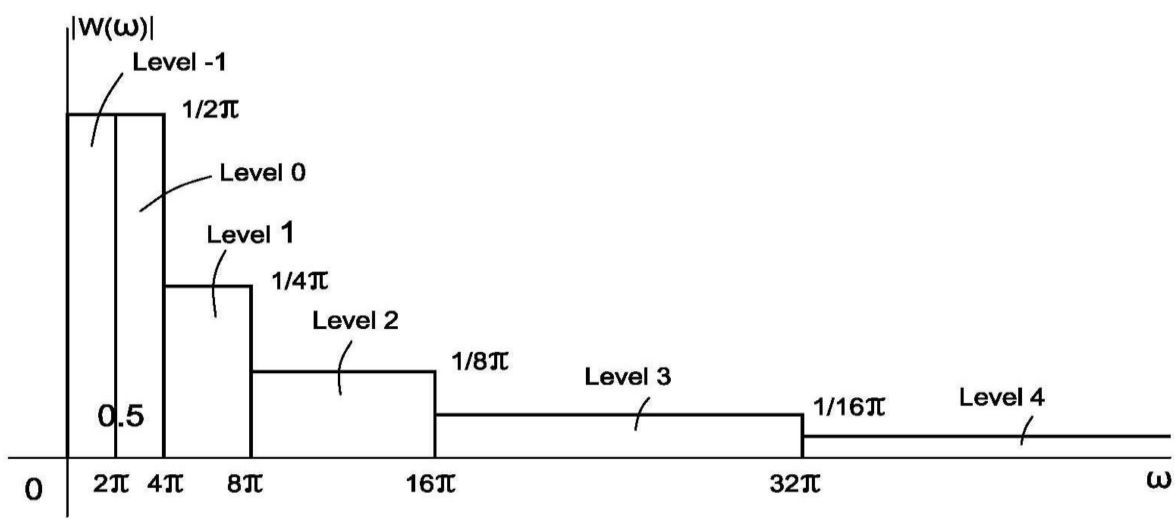

Figure 3. 11 Magnitudes of the Fourier transforms of harmonic wavelets of different level

All the Fourier transforms are identically zero for $\omega < 0$, and for level zero the definition of $|w(x)|$ is :

$$|w(\omega)| = \frac{1}{2\pi} \; for \; 2\pi \le \omega < 4\pi \qquad (3.13)$$
$$= 0 \; elsewhere$$

By calculating the inverse Fourier transform of $w(\omega)$, we obtain the corresponding complex wavelet as:

$$w(x) = \left(e^{i4\pi x} - e^{i2\pi x}\right) \Big/ i2\pi x \qquad (3.14)$$

whose real and imaginary parts are shown in Figure 3.12 and Figure 3.13

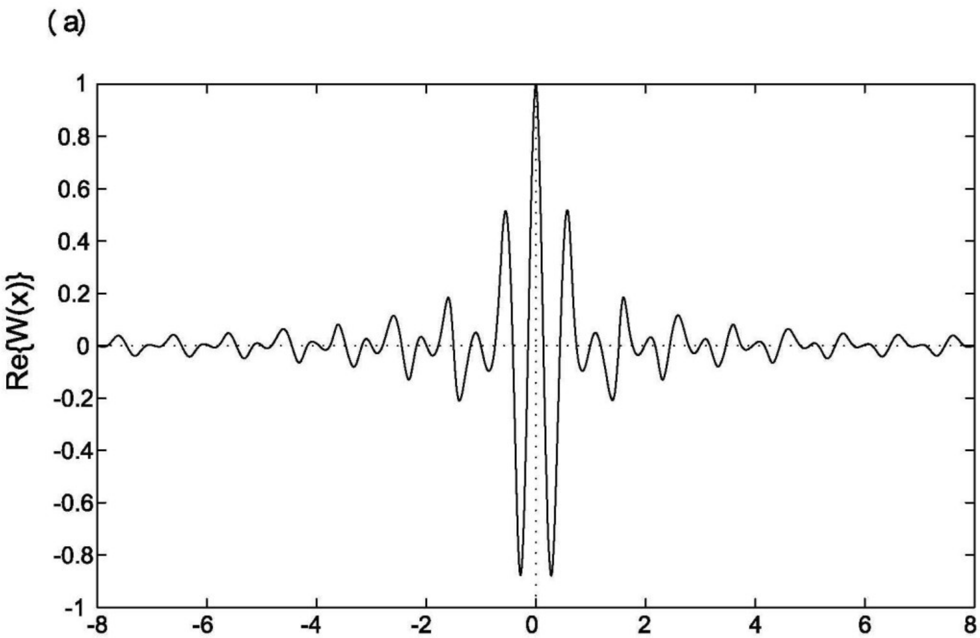

Figure 3. 12 Real part of the harmonic wavelet (3.14)

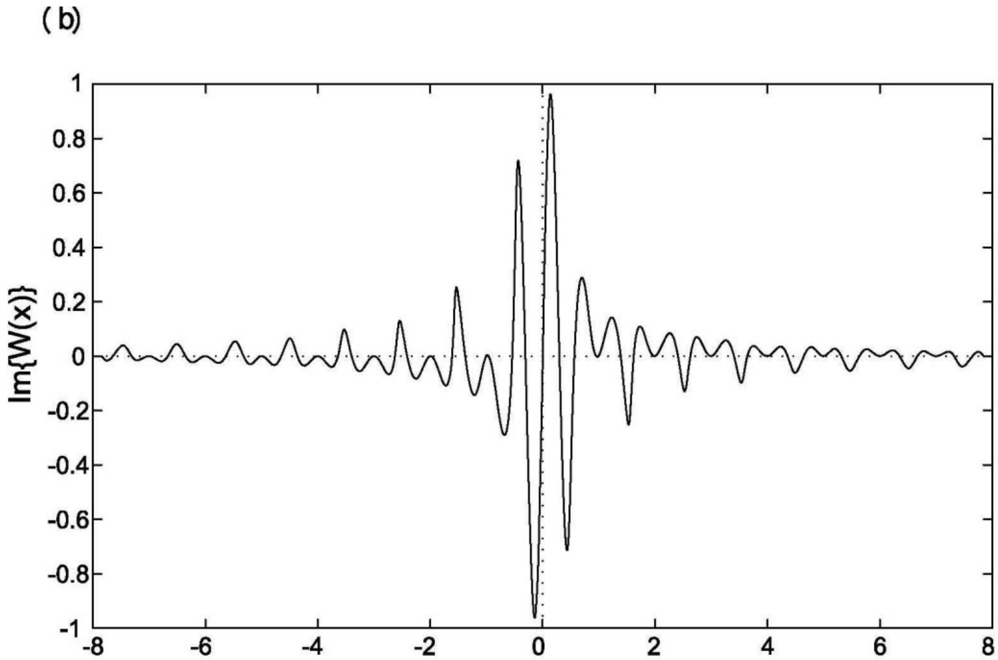

Figure 3. 13 Imaginary part of the harmonic wavelet (3.14)

For the general complex wavelet, at level j and translated by k step of size $1/2^j$ we define:

$$w(\omega) = \left(1/2\pi\right) 2^{-j} e^{i\omega k/2^j} \; for \, 2\pi 2^j \le \omega < 4\pi 2^j$$
$$= 0 \, elsewhere \tag{3. 15}$$

By calculating the inverse Fourier transform, this gives:

$$w\left(2^j x - k\right) = \left(e^{i4\pi\left(2^j x-k\right)} - e^{i2\pi\left(2^j x-k\right)}\right) \Big/ i2\pi\left(2^j x - k\right) \tag{3.16}$$

where $j = 0 \; to \; \infty, k = \;\; \infty to \; \infty$. For level -1 with the frequency band $0 \le \omega < 2\pi$ in Figure 3.11 the wavelet is defined as

$$w(\omega) = \left(1/_{2\pi} \right) e^{-i\omega k} \, for \, 0 \leq \omega < 2\pi$$
$$= 0 elsewhere \tag{3. 17}$$

The scaling function is defined as:

$$\phi(x) = \left(e^{i2\pi(x-k)} - 1 \right) / _{i2\pi(x-k)} \tag{3. 18}$$

$k = \ \infty to\ \infty$, the real and imaginary parts of $\phi(x)$ are shown in Figure 3.14 and Figure 3.15.

(a)

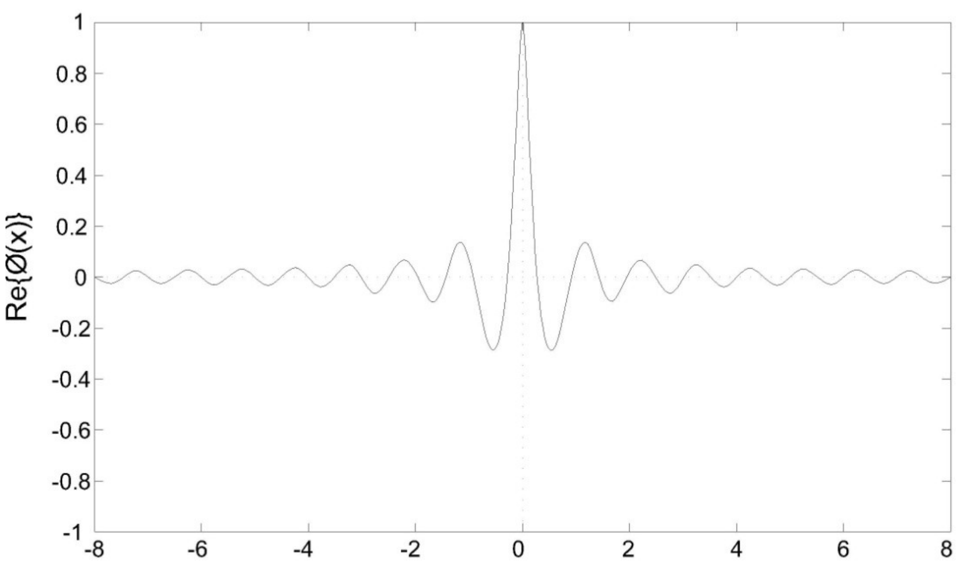

Figure 3. 14 Real part of the harmonic scaling function (3.18)

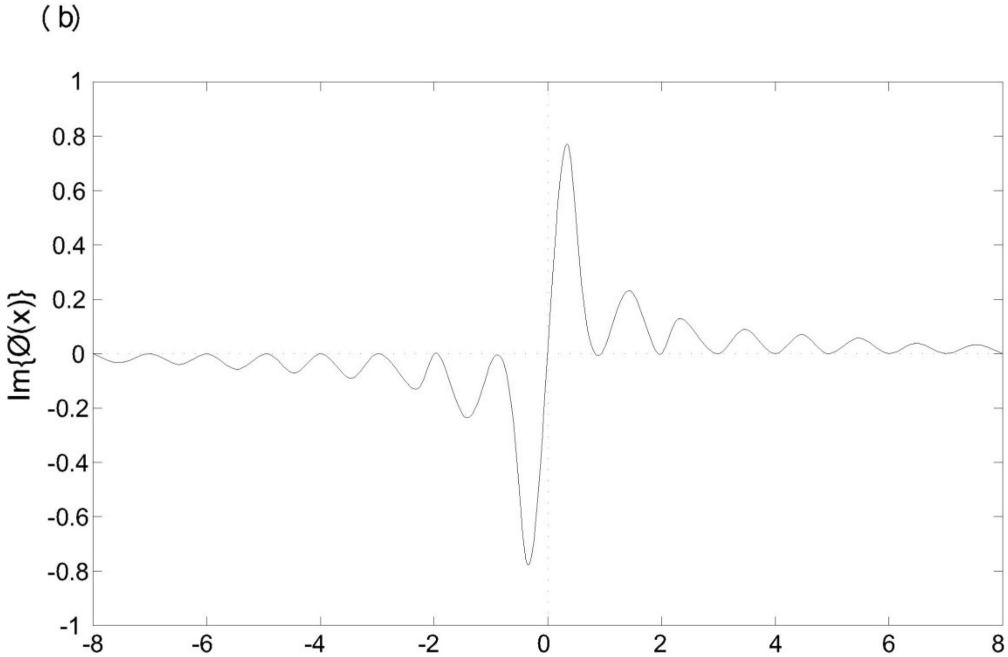

Figure 3. 15 Imaginary part of the harmonic scaling function (3.18)

The idea to choose the wavelet and scaling function is that they form the orthogonal set and have the properties:

$\int_{-\infty}^{\infty} w(2^j x-k)w^*(2^r x-s)dx = 0$ for all j, k, s; j, r≥ 0 (3.19)

$\int_{-\infty}^{\infty} w(2^j x-k)w(2^r x-s)dx = 0$ for all j, k, r, s; j, r≥ 0 (3.20)

Except for the case when r=j, and s=k. when $\int_{-\infty}^{\infty} \left|w(2^j x-k)\right|^2 dx = 1/2^j$

Also $\int_{-\infty}^{\infty} \phi(x-k)\phi(x-s)dx = 0$ for all k, s (3. 21)

$\int_{-\infty}^{\infty} \phi(x-k)\phi^*(x-s)dx = 0$ for all k, s (3. 22)

Except s=k, when $\int_{-\infty}^{\infty} \left|\phi(x-k)\right|^2 dx = 1$

And finally, $\int_{-\infty}^{\infty} w(2^j x-k)\phi(x-s)dx = 0$ for all j, k, s; (j≥ 0) (3. 23)

$$\int_{-\infty}^{\infty} w\left(2^j x-k\right)\phi^*(x-s)dx=0 \quad \text{for all j, k, s; (j} \geq 0) \qquad (3.24)$$

The proof of these results depends on the properties of their Fourier transform and derives from two results from Fourier transform theory of w(x) and v(x) respectively, then one has [48]

$$\int_{-\infty}^{\infty} w(x)v(x)dx = 2\pi\int_{-\infty}^{\infty} W(\omega)V(-\omega)d\omega \qquad (3.25)$$

$$\int_{-\infty}^{\infty} w(x)v^*(x)dx = 2\pi\int_{-\infty}^{\infty} W(\omega)V^*(\omega)d\omega \qquad (3.26)$$

From Equations(3.15),(3.16),(3.17), the wavelets of different of levels have Fourier transform that occupy different frequency band while for the wavelets in the same frequency band, each wavelet is only orthogonal to the complex conjugate of another wavelet.

The result of all this is that the $w\left(2^j x-k\right)$ defined by (3.16) together with $\phi(x-k)$ defined by (3.18) construct an orthogonal family of wavelet that offer an alternative to wavelet family from dilation equation.

The differences of two wavelet functions are that Harmonic wavelet can be described by a simple analytical formula which are compact in frequency domain and can be described by a complex function so that there are two real wavelets for each j, k pair. While Dilation wavelets can not be expressed in functional form. They are compact in x-domain and there is one real wavelet for each j, k pair.

Harmonic wavelet has an advantage that is computation simplicity which uses FFT (fast

Fourier transform) to harmonic wavelet analysis to a signal or function.

3.4 Discrete Harmonic Wavelet Transform

Because of complex wavelets, two amplitude coefficients have to be defined as follows:[46]

$$a_{j,k} = 2^j \int_{-\infty}^{\infty} f(x) w^*\left(2^j x - k\right) dx \tag{3.27}$$

$$\tilde{a}_{j,k} = 2^j \int_{-\infty}^{\infty} f(x) w\left(2^j x - k\right) dx \tag{3.28}$$

If f(x) is a real function, $\tilde{a}_{j,k}$ is the complex conjugate of $a_{j,k}$. However, if f(x) is complex $\tilde{a}_{j,k}$ and $a_{j,k}$ are different.

Similarly,

$$a_{\phi,k} = \int_{-\infty}^{\infty} f(x) \phi^*(x-k) dx \tag{3.29}$$

$$\tilde{a}_{\phi,k} = \int_{-\infty}^{\infty} f(x) \phi(x-k) dx \tag{3.30}$$

In terms of these coefficients, the wavelet expansion of a general function f(x) for which

$$\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx < \infty \qquad (3.31) \qquad \text{is given by:}$$

$$f(x) = \sum_{k=-\infty}^{\infty} \left(a_{\phi,k} \phi(x) + \tilde{a}_{\phi,k} \phi^*(x-k) \right) + \sum_{j=0}^{\infty} \sum_{k=\infty}^{\infty} \left(a_{j,k} w\left(2^j x - k\right) + \tilde{a}_{j,k} w^*\left(2^j x - k\right) \right) (3.32)$$

Now we try to find out an algorithm to compute the coefficients $a_{\phi,k}$, $\tilde{a}_{\phi,k}$, $a_{j,k}$, $\tilde{a}_{j,k}$ in this expansion. As an example, consider $a_{j,k}$ in (3.27). First substitute for

$w^*\left(2^j x - k\right)$ in terms of its Fourier transform where from (3.15)

$$w^*\left(2^j x - k\right) = \int_{2\pi 2^j}^{4\pi 2^j} \left(1/2\pi\right) 2^{-j} e^{i\omega k/2^j} e^{-i\omega x} d\omega \qquad (3.33)$$

So that $a_{j,k}$ in (2.27) becomes

$$a_{j,k} = 1/2\pi \int_{2\pi 2^j}^{4\pi 2^j} d\omega e^{i\omega k/2^j} \int_{-\infty}^{\infty} dx f(x) e^{-i\omega x} dx \qquad (3.34)$$

Since the Fourier transform of $w^*\left(2^j x - k\right)$ is zero outside $2\pi 2^j \le w < 4\pi 2^j$ and second term integral over x is just the Fourier transform of f(x) multiplied by 2π so that

$$a_{j,k} = \int_{2\pi 2^j}^{4\pi 2^j} F(\omega) e^{i\omega k/2^j} d\omega \qquad (3.35)$$

Now we replace integral by summation to get the discrete form of the wavelet transform. In this case, we step samples in increment of 2π in frequency band so that there 2^j steps in level *j*. Then $F(\omega)$ is replaced by the discrete coefficient F_{2^j+s} when $F_{2^j+s} = 2\pi F\left(\omega = 2\pi\left(2^j + s\right)\right)$.

And the integral in (3.35) becomes the summation

$$a_{2^j+k} = \sum_{s=0}^{2^j-1} F_{2^j+s} e^{i2\pi\left(2^j+s\right)k/2^j} \qquad (3.36)$$

Because $\nabla\omega = 2\pi$ cancels with $1/2\pi$ from (3.27) and $e^{i2\pi k} = 1$ for all integer k we obtain the formula:

$$a_{2^j+k} = \sum_{s=0}^{2^j-1} F_{2^j+s} e^{i2\pi sk/2^j} \quad \text{k=0 to } 2^j - 1 \qquad (3.37)$$

This is the inverse discrete Fourier transform for the sequence of frequency coefficients F_{2^j+s}, s=0 to $2^j - 1$.

To compute the wavelet amplitude coefficients by a discrete algorithm, we begin by representing f(x) by the discrete sequence f(r), r=0 to N-1(where N is power of 2). Next FFT is used to compute the set of complex frequency coefficients F(p), p=0 to N-1, then octave blocks of F(p) are processed by IFFT to generate the amplitudes of the harmonic wavelet expansion of f(r) with the coefficients:

$$\tilde{a}_{2^j+k} = \sum_{s=0}^{2^j-1} F_{-(2^j+s)} e^{-i2\pi sk/2^j} \quad \text{k=0 to } 2^j - 1 \tag{3.38}$$

Because the discrete Fourier transform does not have negative indices but instead of F_{-s} by F_{N-S} that becomes:

$$\tilde{a}_{2^j+k} = \sum_{s=0}^{2^j-1} F_{N-(2^j+s)} e^{-i2\pi sk/2^j} \tag{3.39}$$

As an example, for N=16, Figure 3.16 shows how the algorithm works in diagram. For (a) f(r) is a real function and $\tilde{a}_{2^j+k}$ are complex conjugates of a_{2^j+k}. It is not necessary to calculate these coefficients. And in Figure 3.17 for (b) f(r) is complex as shown. And the computation of a_0 and $a_{N/2}$ in algorithm involves special cases.

First, for a_0 consider

$$\phi(x-k) = \int_{-\infty}^{\infty} \phi(\omega) e^{-i\omega k} e^{i\omega x} d\omega \tag{3.40}$$

$$a_{\phi,k} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega \phi^*(\omega) e^{i\omega k} f(x) e^{-i\omega x} = \int_0^{2\pi} d\omega F(\omega) e^{i\omega k} \tag{3.41}$$

Because $\phi(\omega)$ is zero except for $0 \le \omega < 2\pi$ when it is $1/2\pi$, the integral is covered by a single of 2π. If the value of $F(\omega = 0)$ is replaced by F_0, then

$F_0 = 4\pi F(\omega = 0), a_{\phi,0} = {F_0}/{2}, \widetilde{a}_{\phi,0} = {F_0}/{2}$. Hence, $a_0 = a_{\phi,0} + \widetilde{a}_{\phi,0} = F_0$.

Second, for $a_{N/2}$, the initial FFT operation which transform f(r) to F(p) preserves the mean square so that $\sum_{p=0}^{N-1 1} |F(p)|^2 = 1/N \sum_{r=0}^{N-1} |f(r)|^2$ holds. Each subsequent IFFT operation on octave bands of $F(p)$ for p<N/2 or its complex conjugate on octave band for p>N/2 similarly preserves the mean square in octave bands so that $1/2^j = \sum_{k=0}^{2^j\,1} |a_{2^j+k}|^2 = \sum_{k=0}^{2^j\,1} |F_{2^j+k}|^2$ holds. Hence one has $a_{N/2} = F_{N/2}$.

The Parseval’s theory in form is:

$$|a_0|^2 + \sum_{j=0}^{N-2} 1/2^j \sum_{k=0}^{2^j-1} \left(|a_{2^j+k}|^2 + |a_{N-2^j-k}|^2 \right) + |a_{N/2}|^2 = 1/N \sum_{r=0}^{N-1} |f(r)|^2 \tag{3.42}$$

The summation is made over all levels j=0 to n-2, the highest level n-2 being the maximum that can reach with initial sequence of length 2^n. For example an original sequence length of $2^4 = 16$ can provide harmonic wavelet amplitudes at level -1, 0, 1, and n-2=2, comparing with the levels that obtained from the dilation wavelet, the levels from -1, 0, 1, 2 and n-1=3. Since for each pair of j, k, the dilation method has only one real dilation wavelet whereas harmonic wavelet has two real ones that are even and odd which are given by the real and imaginary parts of the complex harmonic wavelet [49].

The algorithm illustrated in Figure 3.16 is a fast way to compute the discrete harmonic wavelet transform and for the inverse transform, the same algorithm works in reverse starting at the bottom of diagrams in Figure 3.17.

In summary, the wavelet determined from the dilation equation is a sequential algorithm in the sense that successive wavelets levels are calculated in sequence. In contrast, the FFT algorithm for harmonic wavelets are parallel method that FFT is only used once to get full sequence and then in the second time in octave block all of which are computed simultaneously. Therefore, for application which is involved very long sequence or complicate signal processing, harmonic wavelet transform algorithm is a very significant time-saving method.

(a)

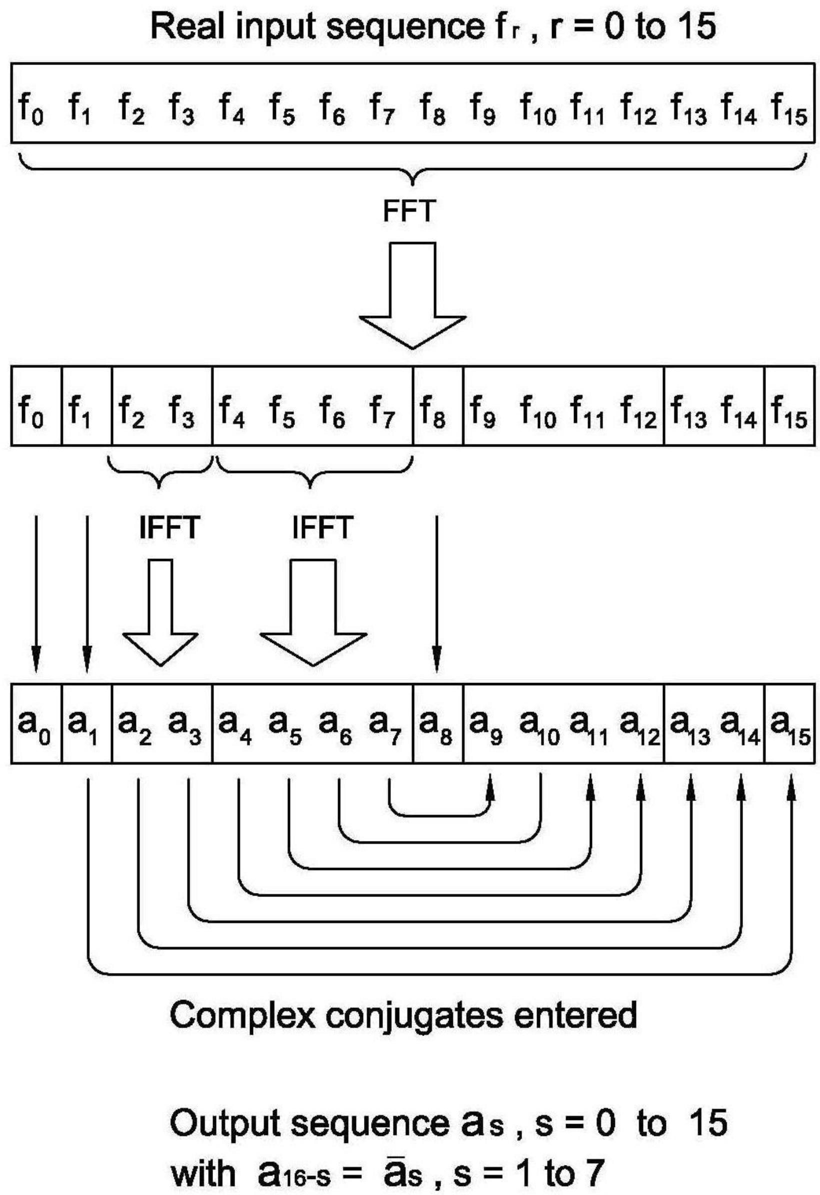

Figure 3. 16 FFT algorithm to compute the harmonic wavelet transform: (a) for a sequence of 16 real elements;[46]

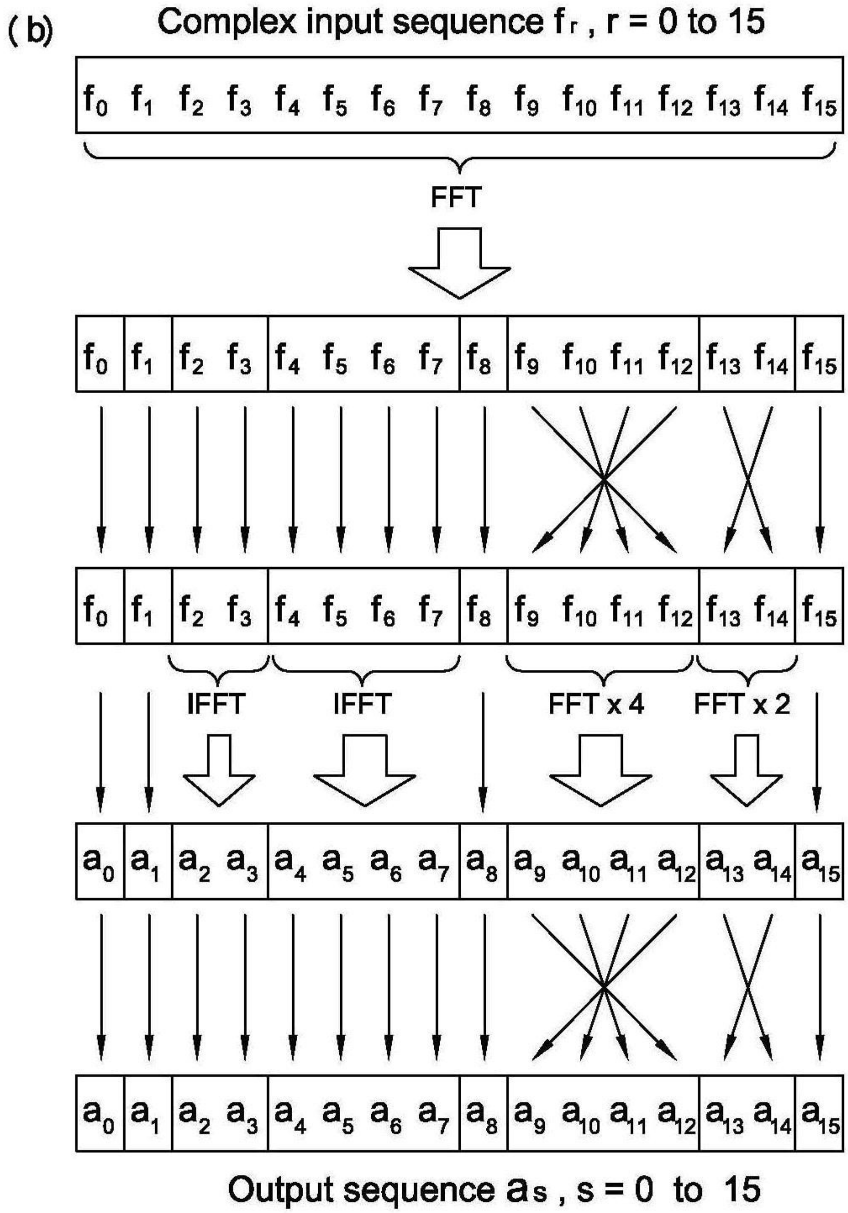

Figure 3. 17 FFT algorithm to compute the harmonic wavelet transform: (b) for a sequence of 16 complex elements [46]

3.5 Conclusion

In this chapter, the mechanism of a signal decomposition and reconstruction is presented based on the wavelet filter processing. The properties and characteristics of some typical harmonic wavelet transform are discussed with emphasis on two types of wavelet basis functions i.e. dilation wavelet function and harmonic wavelet function. Comparing these two wavelet functions, the conclusion is drawn that the harmonic wavelet function is more suitable to be the basis function in the wavelet-based linearization method which will be introduced in Chapters 4 & 5.

4 WAVELET-BASED LINEARIZATION FOR DUFFING OSCILLATOR SYSTEM

In this chapter, we will first introduce one kind of nonlinear system-Duffing oscillator system with non-linear stiffness. A multi-frequency linearization and wavelet-based linearization methods are implemented and compared for Duffing oscillator system when the system is subject to multiple harmonic inputs.

4.1 Duffing Oscillator System Model

To illustrate the procedure of equivalent linearization theory, first we consider the following oscillator with non-linear stiffness [54], as shown in Figure 4.1:

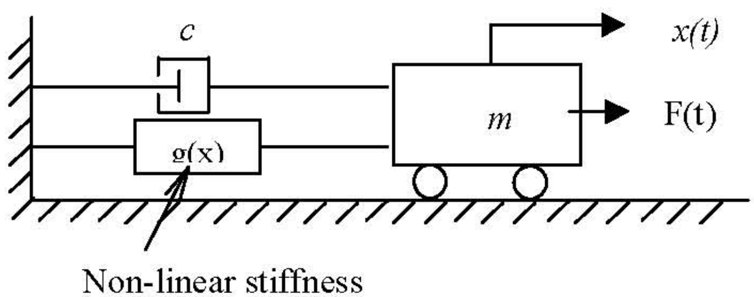

Figure 4. 1 SDOF mass-spring-damper system [54]

The ordinary differential equation of motion can be written as:

$$m\ddot{x}(t)+c\dot{x}(t)+g(x)=F(t) \tag{4.1}$$

where m is the mass, c is the viscous damping coefficient, F(t) is the external excitation, and $x(t)$ is the displacement response of system.

Dividing the equation by m, the equation becomes:

$$\ddot{x}(t) + \beta\dot{x}(t) + h(x) = f(t) \tag{4.2}$$

where β is the damping parameter, $h(x)$ is the non-linear restoring force that could depend on displacement, and $f(t)$ is a zero mean stationary random excitation.

We can always find a way to decompose the non-linear restoring force to one linear component plus a non-linear component, that is:

$$h(x) = \omega_n^2\left[x + \lambda H(x)\right] \tag{4.3}$$

where λ is the non-linear factor that presents the type and degree of non-linearity in the system, and ω_n is the un-damped natural frequency for linear system.

The idea of linearization is replacing the equation (4.2) by the following linear system:

$$\ddot{x} + \beta_{eq}\dot{x} + \omega_{eq}^2 x = f(t) \tag{4.4}$$

where β_{eq} is the equivalent linear damping coefficient per unit mass, ω_{eq}^2 is the equivalent linear stiffness coefficient per unit mass, and $\beta_{eq} = \dfrac{\omega_n}{\omega_{eq}}\beta$, to find an expression for ω_{eq}^2 is to minimize the expected value of the mean square error which is:

$$\varepsilon = h(x) - \omega_{eq}^2 x \tag{4.5}$$

The way that minimizes the expected value of mean square error, is shown:

$$\frac{\partial}{\partial\omega_{eq}^2} E\left[\varepsilon^2\right] = 0 \tag{4.6}$$

By substituting (4.5) into equation (4.6), and carrying out the differentiation, the following equations for ω_{eq}^2 is shown as:

$$\omega_{eq}^2 = \frac{E\{xh(x)\}}{E\{x^2\}} = \frac{E\{xh(x)\}}{\sigma_x^2} \tag{4.7}$$

where σ_x is the standard deviation of x(t). By substituting the expression for $h(x)$, given by (4.3), into equation(4.7), the result is:

$$\omega_{eq}^2 = \omega_n^2\left(1 + \lambda \frac{E\{xH(x)\}}{\sigma_x^2}\right) \tag{4.8}$$

This expression shows, very clearly, how the non-linear component of the stiffness element influences the value of ω_{eq}^2. In equation (4.8), the exact evaluation of $E\left\{\frac{xH(x)}{\sigma_x^2}\right\}$ requires a knowledge of the first order density function of the response process *x(t)* which is unknown [54]. However the displacement *x(t)* is assumed to be Gaussian and Equation (4.7) becomes:

$$\omega_{eq}^2 = E\left\{\frac{\partial h}{\partial x}\right\} = \omega_n^2\left(1 + \lambda E\left\{\frac{dH}{dx}\right\}\right) \tag{4.9}$$

For example, the Duffing oscillator is used to illustrate this procedure by which the non-linear restoring force is written as:

$$h(x) = \omega_n^2\left(x + \lambda x^3\right) \tag{4.10}$$

In this case, ω_{eq}^2 can be expressed:

$$\omega_{eq}^2 = \omega_n^2\left(1 + 3\lambda E\{x^2\}\right) \tag{4.11}$$

And by defining:

$$A = E\{x^2\} \tag{4.12}$$

Then assume the density function of *x(t)* to be the Gaussian form:

$$f_x(x) = \frac{1}{(2\pi)^{1/2}\sigma_x} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \tag{4.13}$$

To combine (4.12) and (4.13) together, one has

$$A = E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x) dx \tag{4.14}$$

And by performing integration the equation (4.14) with the Gamma function, the result is obtained:

$$A = \frac{2}{\sqrt{\pi}} \sigma_x^2 \Gamma\left(\frac{3}{2}\right) = \sigma_x^2 \tag{4.15}$$

The next step is to evaluate σ_x and the procedure is to use the frequency domain. The input-output formula $S_f(\omega)$ is the spectral density for $f(t)$ and $S_x(\omega)$ is the spectral density for output $x(t)$. Use the well-known relationship in the linear vibration :

$$S_x(\omega) = |\alpha(\omega)|^2 S_f(\omega) \tag{4.16}$$

where $\alpha(\omega)$ is the frequency response function given as:

$$\alpha(\omega) = \frac{1}{(\omega_{eq}^2 - \omega^2 + 2i\xi_{eq}\omega\omega_{eq})} \tag{4.17}$$

Once $S_x(\omega)$ is determined, σ_x is found from the equation:

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_x(\omega) d\omega \tag{4.18}$$

The equations (4.16), (4.17), and (4.18) yield a relationship between ω_{eq} and σ_x. Another independent relationship between two quantities can be found from Equations (4.11) (4.14), (4.13) and (4.15). Hence, two algebraic relationship are obtained for two unknowns, ω_{eq} and σ_x, and the formulas for ω_{eq} and σ_x expression are formed. By iterative process the equivalent coefficient finally is computed. The procedure can be described as follows:

1. Assign an initial value of equivalent coefficient ω_{eq} .

2. Use equation (4.16), (4.17), (4.18) to get σ_x .

3. Solve equation (4.11) and (4.14) for the new ω_{eq}.

4. Repeat step 2 and step 3 until the new ω_{eq} converge to the old ω_{eq} within a pre-defined threshold.

4.2 Multi-Frequency Linearization for Stationary Inputs

In this section, we will discuss the multi-frequency-linearization method based on the equivalent linearization procedure and use Duffing system as an example.

For this kind of system, the non-linear factor only depends on the displacement so that the nonlinearity restoring force can be decomposed by linear one term and another non-linear one that mentioned in previous part. That is:

$$h(x) = kx + h_{nl}(x) \tag{4.19}$$

The equivalent method is to replace the original non-linear system given by:

$$\ddot{x} + \beta\dot{x} + h(x) = f(t) \tag{4.20}$$

with the linear system given by

$$\ddot{x} + \beta\dot{x} + kx + k_{nl}x = f(t) \tag{4.21}$$

where k_{nl} is the equivalent coefficient which needs to be found out. Following the method proposed by Caughey [13][14], it is assumed that the damping β is very small and $f(t)$ is a wide-band process. Under these circumstances the solution of equation (4.21) is the narrow-band process and one can write

$$x(t) = A(t)\cos(\omega_{eq}t + \theta(t)) \tag{4.22}$$

$$x(t) = -\omega_{eq}A(t)\sin(\omega_{eq}t + \theta(t)) \tag{4.23}$$

where $\omega_{eq}^2 = k + k_{nl}$ (4.24)

When β is small, $A(t)$ and $\theta(t)$ will be slowly varying function of time over one cycle of response in period of $T = \dfrac{2\pi}{\omega_{eq}}$. Thus the expected value in the equation (4.7) can be treated as average value over one cycle. On minimizing the mean square of error one cycle of response, k_{nl} is obtained as:

$$\omega_{eq}^2 = \frac{\oint h(x)x\,dx}{\oint x^2 dx} \tag{4.25}$$

For Duffing oscillator, the non-linear term is given by

$$h_{nl}(x) = k_{nl}x = \lambda k x^3 \tag{4.26}$$

And equation (4.20) becomes:

$$\ddot{x} + \beta\dot{x} + kx + \lambda k x^3 = f(t) \tag{4.27}$$

And the equivalent stiffness is obtained according to equation (4.25)

$$k_{nl} = \frac{\lambda k \int_0^T x^4 dx}{\int_0^T x^2 dx} \tag{4.28}$$

From the linear vibration theory, the input signal is a sum of sinusoids:

$$f(t) = F_1 \sin(\omega_1 t) + F_2 \sin(\omega_2 t) + \ldots + F_N \sin(\omega_N t) \tag{4.29}$$

An then the output will be:

$$x(t) = X_1 \sin(\omega_1 t + \phi_1) + X_2 \sin(\omega_2 t + \phi_2) + \ldots + X_N \sin(\omega_N t + \phi_N) \tag{4.30}$$

where $X_i^2 = \dfrac{F_i^2}{(k-\omega_i^2)^2 + (\beta\omega_i)^2}$ and $\phi_i = \tan^{-1} \dfrac{\beta\omega_i}{k-\omega_i^2}$ (4.31)

Substituting $x(t)$ from (4.30) into (4.28), the additional stiffness due to the nonlinearity found for each frequency of excitation is calculated as

$$k_{nl,1} = \frac{3}{4}\lambda k X_1^2 \left[1 + 2\left(\frac{X_2}{X_1}\right)^2 + 2\left(\frac{X_3}{X_1}\right)^2 + \ldots + 2\left(\frac{X_N}{X_1}\right)^2\right]$$

$$k_{nl,2} = \frac{3}{4}\lambda k X_2^2 \left[2\left(\frac{X_1}{X_2}\right)^2 + 1 + 2\left(\frac{X_3}{X_2}\right)^2 + \ldots + 2\left(\frac{X_N}{X_2}\right)^2\right]$$

…

$$k_{nl,N} = \frac{3}{4}\lambda k X_N^2\left[2\left(\frac{X_1}{X_N}\right)^2 + 2\left(\frac{X_2}{X_N}\right)^2 + 2\left(\frac{X_3}{X_N}\right)^2 + \ldots + 1\right] \tag{4.32}$$

To obtain the solution for this equation, we need the iterative scheme. Starting with

$$k_{nl,i} = 0 \quad \text{and} \quad X_i = \frac{F_i}{\sqrt{\left(-\omega_i^2 + k + k_{nl,i}\right)^2 + \left(\beta\omega_i\right)^2}} \quad \text{for i=1:N} \tag{4.33}$$

The coefficient k_{nl} is computed by equation (4.32) until it is converged and the result for above system only produces the same frequency as the excitation one. It has been shown that (Stoker 1965) [63] when the harmonic solution is known, the response of the Duffing oscillator to a single frequency can be expressed by

$$x_i(t) = X_i \sin(\omega_i t) - \frac{k\lambda X_i^3}{36\omega_i^2}\sin(3\omega_i t) + \frac{k\lambda^2 X_i^5}{1296\omega_i^4}\sin(5\omega_i t) \tag{4.34}$$

The whole response can be calculated by summing up all the response components as

$$x(t) = \sum_{i=1}^{N} x_i(t) \tag{4.35}$$

The example is given to display this procedure. The system is:

$\ddot{x} + \beta\dot{x} + kx + \lambda kx^3 = f(t)$, $x(0) = \dot{x}(0) = 0$, $\beta = 0.5, k = 16\pi^2, \lambda = 1$ and

$$f(t) = 15\sin(4t) + 10\sin(8t) + 5\sin(12t) + 5\sin(16t) \tag{4.36}$$

The input and output process are shown as in Figure 4.2

.

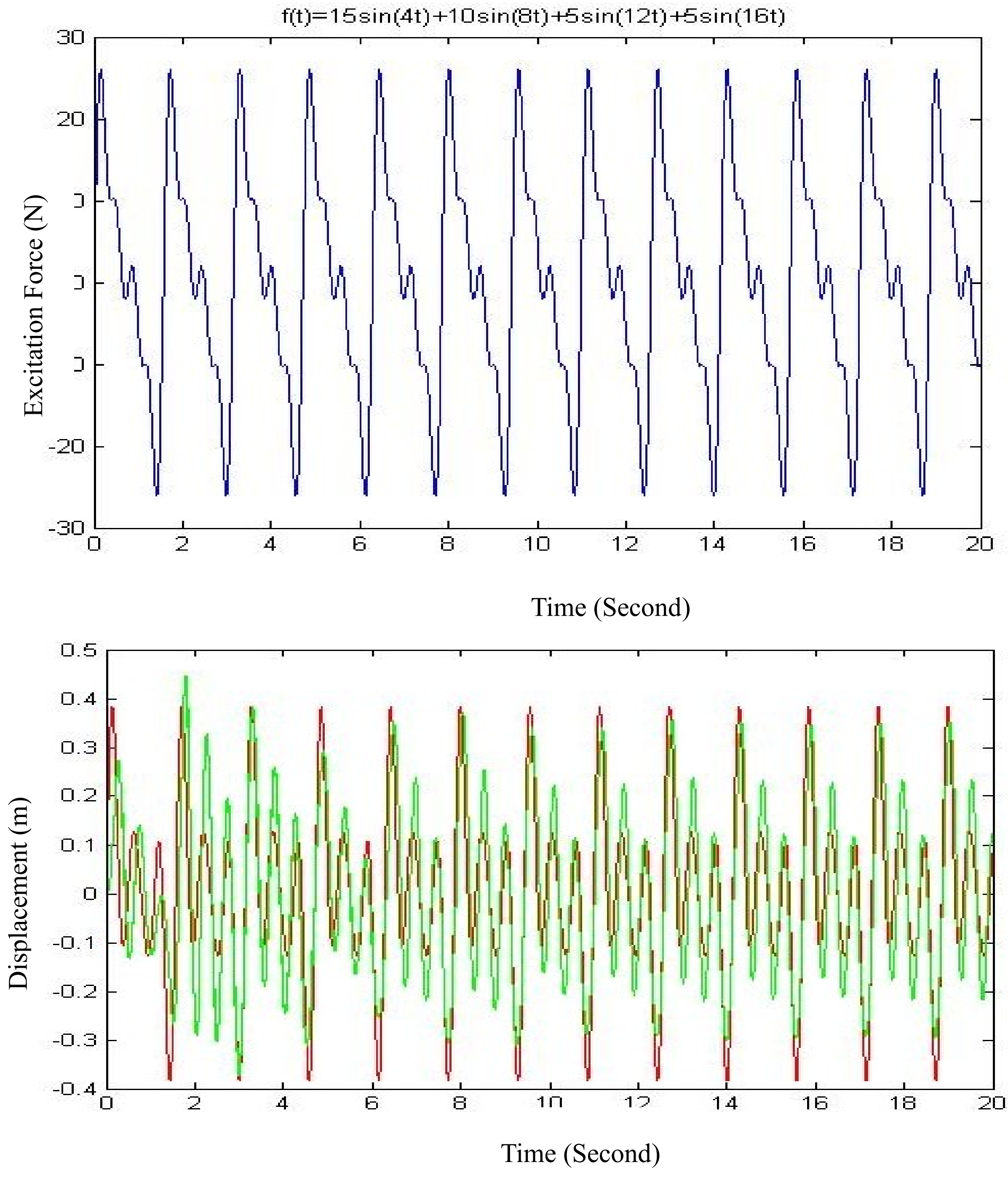

Figure 4. 2 Equivalent linearization for stationary input

The Figure 4.2 shows that the linearization method yields a nice approximation of the original system response because of the stationary excitation. In the next section, a wavelet-based linearization method is proposed for Duffing oscillator.

4.3 Wavelet-Based Linearization for Duffing Oscillator

Although the method of equivalent linearization to analyze the non-linear system response is quite successful in engineering applications, it is still limited at some aspects that the response must be harmonic in nature. Thus if the response energy fluctuates temporally, a time –frequency representation may approximate the response of the system better.

Since the wavelet analysis provide a powerful way for response of system subjected to the non-stationary signal, especially, wavelet transformation can be used to obtain the time-frequency local information of functions very well, to develop a wavelet-based method might be a good alternative to traditional equivalent linearization method.

In this section we introduce the harmonic wavelet function as the basis function to develop the wavelet-based linearization method.

The harmonic wavelet has some typical features different with others. First, it has explicit form function and the definition of a harmonic wavelet at a particular scale is quite easy, unlike the dilation wavelets that must be calculated by iterative process. Second, due to its compact supported form in frequency domain that builds up the non-overlapping frequency bands to form an orthogonal basis, the level of a wavelet is interchangeable with its frequency band. And also because of its frequency characteristic, the fast algorithm can be implemented to reduce the burden of computation.

As any function can be expressed by a series of linear combination components with decomposed scales and positions of harmonic wavelet functions, one has:

$$x(t) = \sum_{j=0}^{n-2} x_j(t) \tag{4.37}$$

with

$$x_j(t) = \sum_{k=0}^{2^j-1} a_{j,k} w(2^j t - k) + a^*_{j,k} w^*(2^j t - k) \tag{4.38}$$

where the symbol * denotes complex conjugate. For zero mean and real value $x(t)$

$$x_j(t) = 2\,\mathrm{Re}\left[\sum_{k=0}^{2^j-1} a_{j,k} w(2^j t - k)\right] \tag{4.39}$$

where Re[] represents the real part of a complex number. The harmonic wavelet coefficients are defined as [46],

$$a_{j,k} = \int_{-\infty}^{\infty} x(t) w^*(2^j t - k) dt \tag{4.40}$$

In the frequency domain, equation(4.40) becomes

$$A_{j,k}(\omega) = W_{j,k}(\omega) X(\omega) \quad \text{for} \quad 2^j 2\pi \le \omega < 2^j 4\pi$$

$$A_{j,k}(\omega) = 0\text{, else.} \tag{4.41}$$

For the level *j* and translated by *k* steps of size $1/2^j$, the wavelet transform is:

$$W_{j,k} = \left(1/2\pi\right) 2^{-j} e^{-i\omega k/2^j} \quad \text{for} \quad 2^j 2\pi \le \omega < 2^j 4\pi$$

$$W_{j,k} = 0\text{, elsewhere} \tag{4.42}$$

And for a discrete sequence x_n, n=1 to N-1, discrete Fourier coefficients of harmonic wavelet transform for level *j* can be expressed as

$$A_k = X_k W_k^* \quad \text{for} \quad k = 0 to 2^j - 1 \tag{4.43}$$

where X_k for $k = 0 to 2^j - 1$ are the Fourier coefficients of x_n for n=1 to N-1, and the square of the absolute values are:

$$A_k^2 = X_k^2 W_k^2 \quad \text{for} \quad k = 0 \; to \; 2^j - 1 \tag{4.44}$$

From equation (4.42), the absolute value of harmonic wavelet for scale j is:

$$\left|W_{j,k}(\omega)\right| = \frac{1}{2\pi} 2^{-j} \qquad 2^j 2\pi \le \omega < 2^j 4\pi$$

$$\left|W_{j,k}(\omega)\right| = 0 \text{, else.} \tag{4.45}$$

Substituting equation (4.45) into (4.44), a relationship between the absolute values of Fourier coefficients of response and its wavelet coefficients is clearly displayed.

$$A_k^2 = \frac{1}{4\pi^2 \left(2^j\right)^2} X_k^2 \qquad \text{for} \quad k = 0 to 2^j - 1.$$

$$A_k^2 = 0 \text{, else.} \tag{4.46}$$

Or changed to:

$$X_k^2 = 4\pi^2 \left(2^j\right)^2 A_k^2 \qquad \text{for} \quad k = 0 to 2^j - 1.$$

$$X_k^2 = 0 \text{, else.} \tag{4.47}$$

Using equation (4.47) in equation (4.32), the scale dependent equivalent stiffness values are calculated as

$$k_{nl,0} = 3\pi^2 \left(2^j\right)^2 \lambda k A_{2^j}^2 \left[1 + 2\left(\frac{A_{2^j+1}}{A_{2^j}}\right)^2 + 2\left(\frac{A_{2^j+2}}{A_{2^j}}\right)^2 + \ldots + 2\left(\frac{A_{2^j+k}}{A_{2^j}}\right)^2\right]$$

$$k_{nl,1} = 3\pi^2 \left(2^j\right)^2 \lambda k A_{2^j+1}^2 \left[2\left(\frac{A_{2^j}}{A_{2^j+1}}\right)^2 + 1 + 2\left(\frac{A_{2^j+2}}{A_{2^j+1}}\right)^2 + \ldots + 2\left(\frac{A_{2^j+k}}{A_{2^j+1}}\right)^2\right]$$

….

$$k_{nl,k} = 3\pi^2 \left(2^j\right)^2 \lambda k A_{2^j+k}^2 \left[2\left(\frac{A_{2^j}}{A_{2^j+k}}\right)^2 + 2\left(\frac{A_{2^j+1}}{A_{2^j+k}}\right)^2 + 2\left(\frac{A_{2^j+2}}{A_{2^j+k}}\right)^2 + \ldots + 1 \right] \tag{4.48}$$

and the transfer function is updated. The equation (4.48) expresses that for the level j, there are 2^j equivalent stiffness coefficients, each $1/2^j$ units apart on the time axis, the process is iterated until the solution converges.

The wavelet-based linearization results are displayed and compared with numerical method of the 4th Runge-Kutta as shown in Figure 4.3.

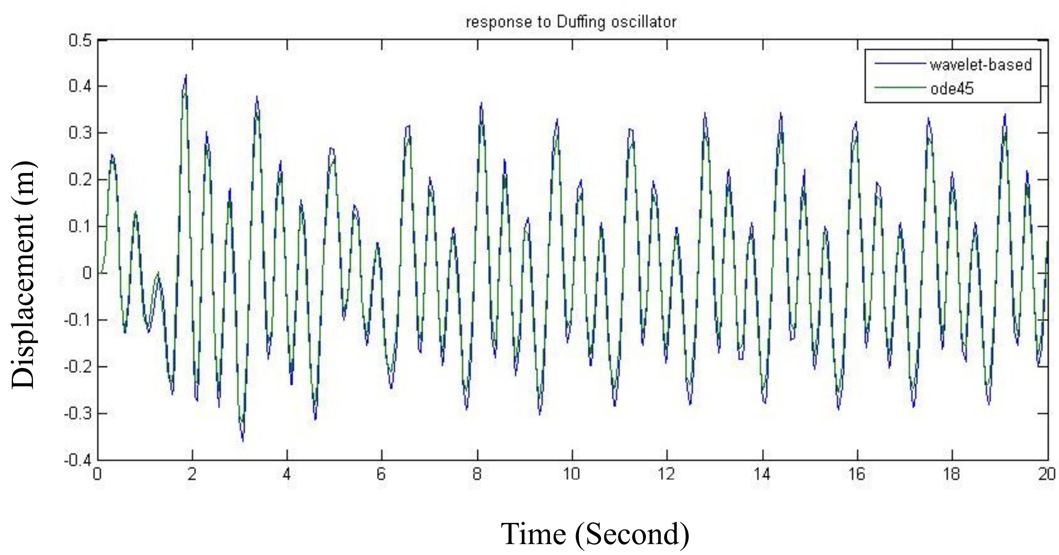

Figure 4. 3 Wavelet-based linearization solution

For a given system and the input signal with the length N, the response is obtained by using the Fourier transform of the excitation multiplying the frequency response function but ignoring the non-linearity. Next, every level (scale) of the wavelet coefficient of the response in the frequency domain is calculated with the Newland dyadic harmonic [46] wavelet as the basis function. At last, the equivalent parameters are obtained when the iterative process converges.

In order to illustrate the procedure validity, the wavelet-based linearization method is applied to the system defined in equation (4.36). Figure 4.2 shows the excitation and system response by using equivalent linearization method. Figure 4.3 shows the response of non-linear system using the wavelet-based method comparing with the numerical solution of ordinary differential equation using 4^{th} order Runge-Kutta algorithm. From figures, the wavelet-based algorithm developed above yields a good approximation of the exact response.

In summary, the statistical linearization method provides the way that obtains the parameters in the equivalent linear model by minimizing the mean square of the equation deficiency, but this procedure needs the knowledge of the probability distribution function. In order to avoid the complication of computing, an alternative approach is required called multi-frequency linearization method with some assumptions, the parameters are obtained by iterative algorithm. Since the harmonic wavelets function is compact in the frequency domain, its Fourier transform offers explicit form to the wavelet coefficients with the algebraic transformation of the response in frequency domain. The equivalent linearization parameters are obtained in both time and frequency domain by using wavelet-based linearization technique.

4.4 Conclusion

This chapter presents the procedure of linearzing the Duffing oscillator system by using traditional and wavelet-based equivalent linearization methods. First of all, it introduces the statistical linearization method and then extends to the multi-frequency approach which is based on this statistical method to derive the equivalent linearization parameters in this Duffing system. The parameters can be determined by multi-frequency Fourier transform of responses according to the multi-frequency system excitations respectively. By using the harmonic wavelet basis function which provides the non-over lapping frequency bands and scale, a wavelet-based linearization method for the Duffing system is proposed and the simulation results are compared with the numerical method Runge-Kutta to show that the proposed linearization method is an appealing approach to linearize the nonlinear Duffing systems.

5 WAVELET-BASED LINEARIZATION FOR BILINEAR HYSTERETIC SYSTEMS

In this chapter, the wavelet-based linearization method is developed for another type of nonlinear system- bilinear hysteresis system. The organization of this chapter is: Section 5.1 introduces bilinear hysteresis system model. Section 5.2 describes the statistic linearization equivalent method for the bilinear hysteresis system. The basic concept of wavelet transform for the relationship of input and output in the linear system is introduced in Section 5.3. In the Section 5.4, a wavelet-based linearization method is proposed to obtain equivalent linear parameters based on both frequency and time information of the nonlinear system. The numerical results are provided in Section 5.5.

5.1 Bilinear Hysteretic System Model

For the system of bilinear hysteresis, the system model is

$$\ddot{x} + \beta\dot{x} + h(\dot{x}, x, t) = f(t) \quad x(0) = \dot{x}(0) = 0 \tag{5.1}$$

where the hysteretic force is

$$h(x, \dot{x}, t) = \alpha x + (1-\alpha)h_0(\dot{x}, y; x_y) \tag{5.2}$$

$$\dot{y} = h_1(\dot{x}, y; x_y) \tag{5.3}$$

where $h_0 = y\{u(y + x_y) - u(y - x_y)\} + x_y\{u(y - x_y)u(\dot{x}) - u(-y - x_y)u(-\dot{x})\}$

$$h_1 = \dot{x}\{u(y + x_y) - u(y - x_y) + u(y - x_y)u(-\dot{x}) + u(-y - x_y)u(\dot{x})\} \tag{5.4}$$

where y is the relative displacement of the purely elasto-plastic component h_0, the hysteresis force of $h(x, \dot{x}, t)$ is assumed to have the characteristic as shown in Figures 5.1-5.3, $\alpha = 0.5$ and $x_y = 1$ are respectively the second slope ratio and the yield displacement, $u(\ )$ is a unit step function.

The bilinear hysteresis and elasto-plastic hysteresis loops are shown in Figures 5.4 and 5.5.

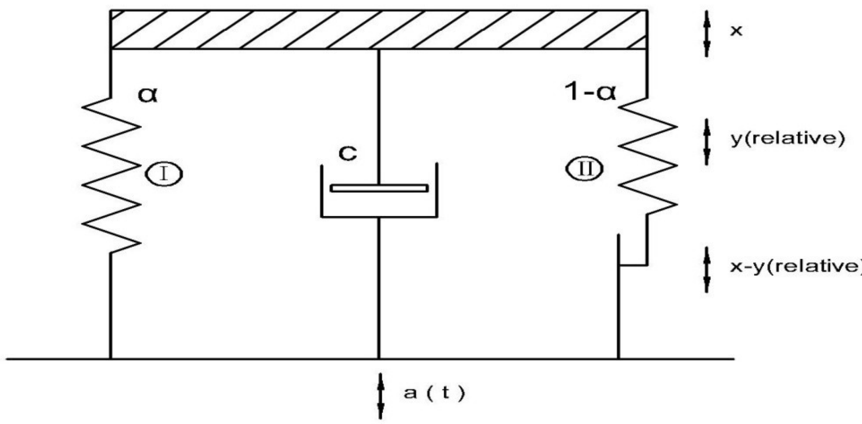

Figure 5. 1 The bilinear system model

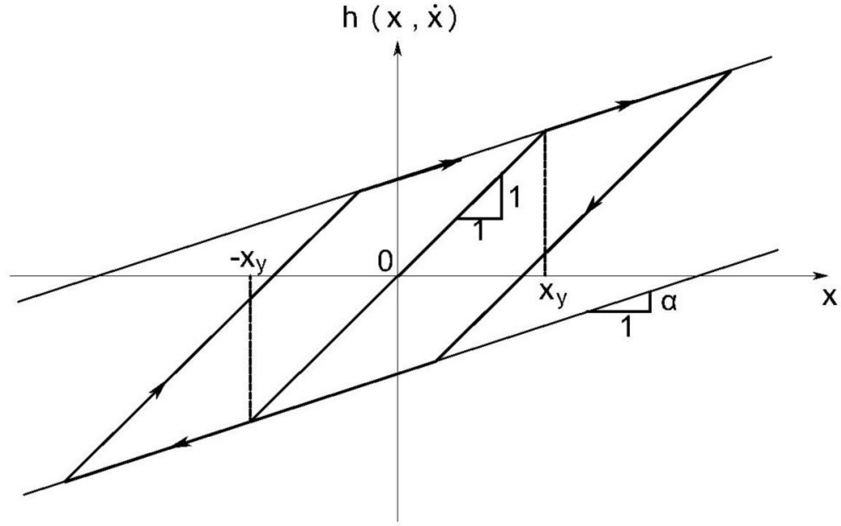

Figure 5. 2 The bilinear hysteresis force h characteristic $\alpha = \tan\varphi$

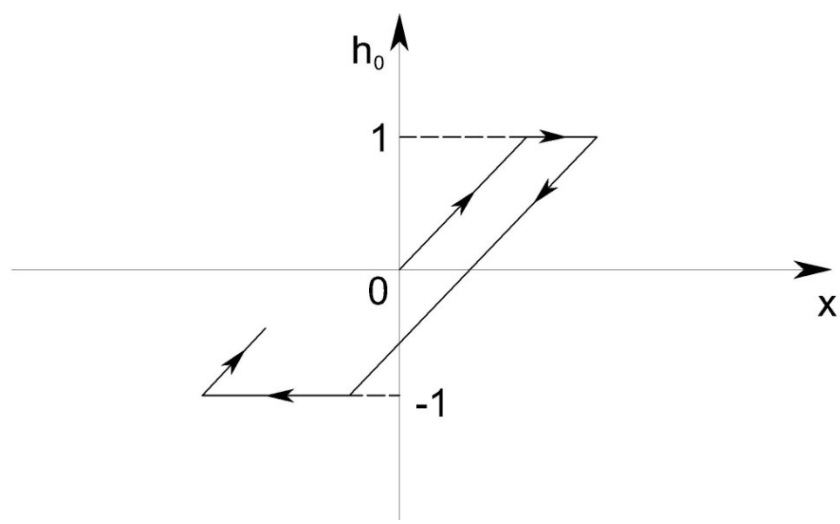

Figure 5. 3 Elasto-plastic hysteresis h_0

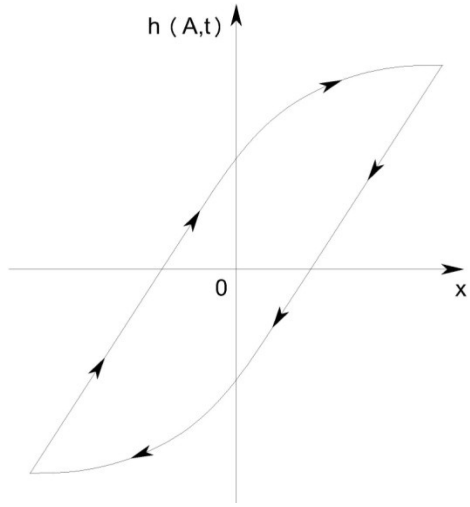

Figure 5. 4 The bilinear hysteresis loop

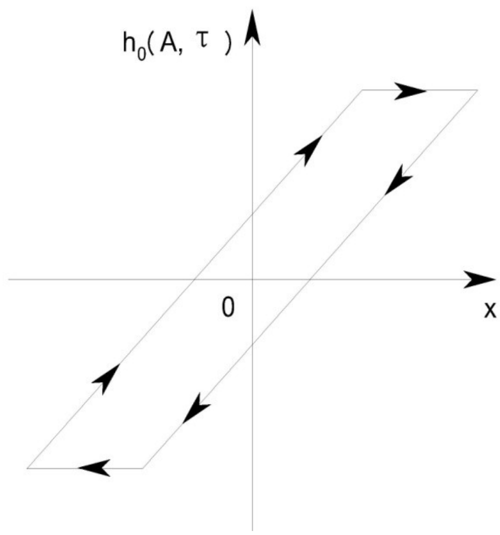

Figure 5. 5 The elaso-plastic hysteresis loop

The equivalent linearization model to the system is given by the formula:

$$\ddot{x} + \beta\dot{x} + \alpha x + (1-\alpha)(c_1\dot{x} + c_2 x) = f(t) \tag{5.5}$$

$$\dot{y} = c_3\dot{x} + c_4 y \tag{5.6}$$

Equation (5.5) can be written as:

$$\ddot{x}+\left(\beta+c_1(1-\alpha)\right)\dot{x}+\left(\alpha+(1-\alpha)c_2\right)x=f(t) \tag{5.7}$$

$\ddot{x}+\beta_{eq}\dot{x}+\omega_{eq}^2 x=f(t)$,(5.7a) $\beta_{eq}=\left(\beta+c_1(1-\alpha)\right)$,(5.7b) $\omega_{eq}^2=\left(\alpha+(1\quad\alpha)c_2\right)$(5.7c)

where β_{eq} and ω_{eq}^2 are the equivalent linearized parameters.

5.2 Statistical Linearization for Bilinear Hysteretic System

The statistical linearization technique is used to investigate this proposal method [58], [59]. Consider the same bilinear hysteresis system given by

$$\ddot{x}+\beta\dot{x}+h(x,\dot{x},t)=f(t) \quad x(0)=\dot{x}(0)=0 \tag{5.8}$$

where $\beta=0.02$ and $f(t)=15\sin(4t)+10\sin(8t)+5\sin(12t)+5\sin(16t)$ (5.9)

The hysteretic force $h(x,\dot{x},t)=\alpha x+(1-\alpha)h_0(\dot{x},y;x_y)$ (5.10)

$$\dot{y}=h_1(\dot{x},y;x_y) \tag{5.11}$$

where $h_0=y\{u(y+x_y)-u(y-x_y)\}+x_y\{u(y-x_y)u(\dot{x})-u(-y-x_y)u(-\dot{x})\}$

$$h_1=\dot{x}\{u(y+x_y)-u(y-x_y)+u(y-x_y)u(-\dot{x})+u(-y-x_y)u(\dot{x})\} \tag{5.12}$$

where $\alpha=0.5$ and $x_y=1$ are respectively the second slope ratio and the yield displacement, $u(\ )$ is a unit step function. The numerical method is used to obtain solution of h_0 and displacement of x.

Meanwhile, we use the statistical linearization method [7] that replaces the nonlinear system in equation (5.8) with the linear one as:

$$\ddot{x} + \beta\dot{x} + \alpha x + (1-\alpha)(C_1\dot{x} + C_2 x) = f(t)$$
$$\dot{y} = C_3\dot{x} + C_4 y \tag{5.13}$$

where the equivalent coefficients C_1, C_2, C_3, C_4 can be calculated by assuming a Gaussian probability distribution of the multi-dimensional response process which are

$$C_1 = E\left[\partial h_0 \big/ \dot{x}\right] = \frac{x_y}{\sqrt{2\pi}\sigma_{\dot{x}}} erfc\left(\frac{x_y}{\alpha_y\sqrt{2(1-\rho^2)}}\right)$$

$$C_2 = E\left[\partial h_0 \big/ y\right] = erf\left(\frac{x_y}{\sqrt{2}\sigma_y}\right) - \frac{x_y}{\sqrt{2\pi}\sigma_y}\exp\left(\frac{-x_y^2}{2\sigma_y^2}\right) erfc\left(\frac{\rho x_y}{\sigma_y\sqrt{(2(1-\rho^2))}}\right)$$

$$C_3 = E\left[\partial h_1 \big/ \dot{x}\right] = \frac{1}{2}\left\{1 + erf\left(\frac{x_y}{\sqrt{2}\sigma_y}\right)\right\} - \frac{1}{2\pi}\exp\left(\frac{-\rho x_y^2}{2\sigma_y^2\sqrt{(1-\rho^2)}}\right)$$

$$C_4 = E\left[\partial h_1 \big/ \partial y\right] = -\frac{\rho x_y \sigma_{\dot{x}}}{\sqrt{2\pi}\sigma_y^2}\exp\left(\frac{-x_y^2}{2\sigma_y^2}\right)\left[1 + erf\left(\frac{\rho x_y}{\sigma_y\sqrt{2(1-\rho^2)}}\right)\right] -$$
$$\frac{\sigma_{\dot{x}}\sqrt{(1-\rho^2)}}{\pi\sigma_y}\exp\left(-\frac{x_y^2}{2(1-\rho^2)\sigma_y^2}\right) \tag{5.14}$$

In which $\sigma_{\dot{x}}$ and σ_y are respectively the root mean square value of $\dot{x}$, the parameters y, ρ are the coefficient of correlation of $\dot{x}$ and y, and $erf(\ )$ and $erfc(\ )$ are respectively the error function and complementary error function.

5.3 Input - Output Relationship for a Linear System

Since the wavelet transform is the convolution operation in time, where the signal to the wavelet transform is decomposed in the distinct scales and translation values, the wavelet coefficients calculation is quite similar to the calculation of the response of linear system that the convolution operation is implemented using impulse response function in the LTI system. In the LTI system, the response calculation in the wavelet basis is obtained to replace the impulse function with the wavelet function.

Consider a linear SDOF system with the equation motion:

$$\ddot{x}(t)+\beta\dot{x}(t)+kx(t)=f(t) \tag{5.15}$$

where β is damping parameter, k is stiffness parameter. By performing the wavelet transform both sides of (5.15), one has [9], [10]:

$$W_\varphi\ddot{x}(a,b)+\beta W_\varphi\dot{x}(a,b)+kW_\varphi x(a,b)=W_\varphi f(a,b) \tag{5.16}$$

where $W_\varphi\ddot{x}, W_\varphi\dot{x}, W_\varphi x, W_\varphi f$ denote the wavelet transform of $\ddot{x}, \dot{x}, x, f$ respectively, performing the integration by part on $W_\varphi\ddot{x}(a,b)$ according to the characteristic of fast decaying for wavelet basis, the following expression can be obtained:

$$W_\varphi\ddot{x}(a,b)=\frac{1}{a^2}W_{\ddot{\varphi}}x(a,b) \tag{5.17}$$

where the term on the right hand side is the wavelet transform with the wavelet basis $\ddot{\varphi}$, the second order partial differential equation of $W_\varphi x(a,b)$ with respect b can be expressed as:

$$\frac{\partial^2}{\partial b^2}W_\varphi x(a,b)=\frac{\partial^2}{\partial b^2}\int_{-\infty}^{\infty}x(t)\varphi\left(\frac{t-b}{a}\right)dt \tag{5.18}$$

Exchanging the differential operator with the integral operator on the right hand side the equation (5.18) can be written as

$$\frac{\partial^2}{\partial b^2} W_\varphi x(a,b) = \frac{1}{a^2} W_{\ddot{\varphi}} x(a,b) \tag{5.19}$$

Thus, one has

$$\frac{\partial^2}{\partial b^2} W_\varphi x(a,b) = W_\varphi \ddot{x}(a,b) \tag{5.20}$$

Similarly, the following is obtained

$$\frac{\partial}{\partial b} W_\varphi x(a,b) = W_\varphi \dot{x}(a,b) \tag{5.21}$$

The equation (5.16) can be expressed by:[8]

$$\frac{\partial^2}{\partial b^2} W_\varphi x(a,b) + \beta \frac{\partial}{\partial b} W_\varphi x(a,b) + k W_\varphi x(a,b) = W_\varphi f(a,b) \tag{5.22}$$

From this equation, the wavelet coefficient of the output can be obtained from the wavelet coefficient of the input. Because the parameter b contains time information, the information of response output in time domain is achieved as well.

It is noted that in the LTI linear system, the wavelet coefficients of response output calculation is obtained through the way that replaces impulse response function in time domain convolution operation in the response calculation by the wavelet basis function. Thus, we can have the response information about time-frequency combination.

5.4 Wavelet-Based Equivalent Linearization

Considering the bilinear system (5.1), by using the input-output relationship introduced previously and performing the wavelet transform on both sides of the equation (5.8) and

(5.7a), the following form can be obtained:[8][9]

$$\frac{\partial^2}{\partial b^2} W_\varphi x(a,b) + \beta \frac{\partial}{\partial b} W_\varphi x(a,b) + W_\varphi h(a,b) = W_\varphi f(a,b) \tag{5.23}$$

$$\frac{\partial^2}{\partial b^2} W_\varphi x(a,b) + \beta \frac{\partial}{\partial b} W_\varphi x(a,b) + \omega_{eq}^2 W_\varphi x(a,b) = W_\varphi f(a,b) \tag{5.24}$$

Following the basic procedure of the traditional equivalent linearization approach, one can first find the mean square least error between the equations (5.23) and (5.24)

$$\varepsilon^2 = (\omega_{eq}^2 W_\varphi x(a,b) - W_\varphi h(a,b))^2 \tag{5.25}$$

In fact, the wavelet coefficients are calculated numerically. In this case, parameters a, b will be discretized as $a = 2^j$ and $b = k2^j$. By summing the discretized error over all j values and by multiplying by the factor $1/2^j$, which represents the error in the instantaneous energy of the response at a instant time interval k, then minimizing the square error with respect ω_{eq}^2, the instantaneous equivalent natural frequency will be obtained [62].

$$\frac{\partial}{\partial \omega_{eq,k}^2} \sum_{allj} \varepsilon_{j,k}^2 = 0 \tag{5.26}$$

$$\omega_{eq,k}^2 = \frac{\sum_{allj} W_{j,k} x W_{j,k} h}{\sum_{allj} W_{j,k}^2 x} \tag{5.27}$$

Also by summing the discretized error for all the time positions over k which represents the frequency dependent equivalent parameter, one obtains

$$\frac{\partial}{\partial \omega_{eq,j}^2} \sum_{allk} \varepsilon_{j,k}^2 = 0 \tag{5.28}$$

$$\omega_{eq,j}^2 = \frac{\sum_{allk} W_{j,k} x W_{j,k} h}{\sum_{allk} W_{j,k}^2 x} \tag{5.29}$$

Thus, the wavelet method can examine the error between the nonlinear and linearized system either at each time interval or each frequency band corresponding to wavelet scale. This flexible feature is very important to solve the nonlinear response of systems.

One can compare with the same equivalent linearization model:

$$\ddot{x} + \beta_{eq}\dot{x} + \omega_{eq}^2 x = f(t) \tag{5.30}$$

with

$\beta_{eq} = (\beta + c_1(1-\alpha))$(5.31), and $\omega_{eq}^2 = (\alpha + (1-\alpha)c_2)$ (5.32)

The output $x(t)$ and $y(t)$ can be decomposed by using the harmonic wavelet transform to obtain:

$$x(t) = \sum_{j=0}^{n-2} x_j(t) \approx \sum_{j=0}^{n-2} \left(2\,\mathrm{Re}\left(\sum_{k=0}^{2^j-1} a_{j,k} w\left(2^j t - k\right) \right) \right) \tag{5.33}$$

where Re[] represents the real part of a complex number. The harmonic wavelet coefficients are defined as [46],

$$a_{j,k} = \int_{-\infty}^{\infty} x(t) w^*\left(2^j t - k\right) dt \tag{5.34}$$

Similarly,

$$y(t)=\sum_{j=0}^{n-2} y_j(t) \approx \sum_{j=0}^{n-2}\left(2\,\mathrm{Re}\left(\sum_{k=0}^{2^j-1} a'_{j,k}\, w(2^j t-k)\right)\right) \quad (5.35)$$

where

$$a'_{j,k}=\int_{-\infty}^{\infty} y(t) w^*(2^j t-k) dt \quad (5.36)$$

where $w(2^j t-k)$ is the harmonic wavelet transform function represented by the level (scale) j and time position of k . $a_{j,k}$ and $a'_{j,k}$ are the wavelet coefficients of $x(t)$ and $y(t)$ respectively, also giving the assumption that $x(t)$ and $y(t)$ are both real functions.

In order to obtain the equivalent liner damping and stiffness parameters, also the equivalent linear parameters c_3 and c_4, we use the method discussed previously that is called wavelet-based linearization method.

Performing the wavelet transform on both sides of the equations (5.9) and (5.30), the following equations can be obtained:

$$\frac{\partial^2}{\partial k^2} W_w x(2^j t-k)+\beta \frac{\partial}{\partial k} W_w x(2^j t-k)+W_w h(2^j t-k)=W_w f(2^j t-k) \quad (5.37)$$

$$\frac{\partial^2}{\partial k^2} W_w x(2^j t-k)+\beta_{eq} \frac{\partial}{\partial k} W_w x(2^j t-k)+\omega_{eq}^2 W_w x(2^j t-k)=W_w f(2^j t-k) \quad (5.38)$$

The square of error between the nonlinear and linearized formulation at each scale j and time position k of the wavelet transform is:

$$\varepsilon_{j,k}^2=[\beta_{eq} \frac{\partial}{\partial k} W_w x(2^j t-k)+\omega_{eq}^2 W_w x(2^j t-k)-\beta \frac{\partial}{\partial k} W_w x(2^j t-k)-W_w h(2^j t-k)]^2 \quad (5.39)$$

The above error, when multiplying with reciprocal of the scale $1/2^j$ and summing over all that scale values, represents the error in the instantaneous energy of the response at the time t=k. By minimizing this error of the instantaneous energy with respect to the equivalent damping parameter β_{eq} and stiffness parameter ω_{eq}^2, the instantaneous equivalent damping and stiffness parameters of the linearization can be obtained.

Hence, a time variant linear system model is derived. The minimization conditions are:

$$\frac{\partial}{\partial \beta_{eq,k}} \sum_{allj} \frac{1}{2^j} \varepsilon_{j,k}^2 = 0 \tag{5.40}$$

$$\frac{\partial}{\partial \omega_{eq,k}^2} \sum_{allj} \frac{1}{2^j} \varepsilon_{j,k}^2 = 0 \tag{5.41}$$

Substituting equation (5.39) to (5.40) and (5.41) respectively, one has:

$$\beta_{eq,k} = \frac{\sum_{allj} \frac{\partial}{\partial k} W_w x(j,k) W_w h(j,k)}{\sum_{allj} \left(\frac{\partial}{\partial k} W_w x(j,k) \right)^2} \tag{5.42}$$

$$\omega_{eq,k}^2 = \frac{\sum_{allj} W_w x(j,k) W_w h(j,k)}{\sum_{allj} \left(W_w x(j,k) \right)^2} \tag{5.43}$$

Similarly, when summing the error for all the time positions, the error represents the energy of the process at each frequency band corresponding to a wavelet scale. Thus, the equivalent damping and stiffness parameters of this linearized system are determined as shown:

$$\frac{\partial}{\partial \beta_{eq,j}} \sum_{allk} \varepsilon_{j,k}^2 = 0 \tag{5.44}$$

$$\frac{\partial}{\partial \omega_{eq,j}^2} \sum_{allk} \varepsilon_{eq}^2 = 0 \tag{5.45}$$

Thus, one has

$$\beta_{eq,j} = \frac{\sum_{allk} \frac{\partial}{\partial k} W_w x(j,k) W_w h(j,k)}{\sum_{allk} \left(\frac{\partial}{\partial k} W_w x(j,k) \right)^2} \tag{5.46}$$

$$\omega_{eq,j}^2 = \frac{\sum_{allk} W_w x(j,k) W_w h(j,k)}{\sum_{allk} (W_w x(j,k))^2} \tag{5.47}$$

For the equations (5.3) and (5.6), taking the wavelet transform on both sides of two equations, we obtain the formulations:

$$\frac{\partial}{\partial k} W_w y\left(2^j t - k\right) = W_w h_1\left(2^j t - k\right) \tag{5.48}$$

$$\frac{\partial}{\partial k} W_w y\left(2^j t - k\right) = c_3 \frac{\partial}{\partial k} W_w x\left(2^j t - k\right) + c_4 W_w y\left(2^j t - k\right) \tag{5.49}$$

Similarly, taking the square of error between the equation (5.48) and (5.49) at every scale j and time position k of their wavelet basis function, one obtains:

$$\varepsilon_{j,k,y}^2 = \left[c_3 \frac{\partial}{\partial k} W_w x\left(2^j t - k\right) + c_4 W_w y\left(2^j t - k\right) - W_w h_1\left(2^j t - k\right) \right]^2 \tag{5.50}$$

By using the same technique of finding the equivalent damping and stiffness parameters to obtain the parameters c_3 and c_4 under the minimization conditions, we obtain the following formulations:

$$\frac{\partial}{\partial c_{3,k}} \sum_{allj} \frac{1}{2^j} \varepsilon_{j,k,y}^2 = 0 \tag{5.51}$$

$$\frac{\partial}{\partial c_{4,k}} \sum_{allj} \frac{1}{2^j} \varepsilon_{j,k,y}^2 = 0 \quad (5.52)$$

These two formulations are defined by summing the square of errors over all the scale values j to represent the energy of the process at the instantaneous k, so that the information about linear time variant systems are obtained with equivalent parameters c_3 and c_4 at time locations. The same ideas are taken to sum the square of errors over all the time positions k to get the equivalent parameters c_3 and c_4 at the frequency dependent representations.

$$\frac{\partial}{\partial c_{3,j}} \sum_{allk} \frac{1}{2^j} \varepsilon_{j,k,y}^2 = 0 \quad (5.53)$$

$$\frac{\partial}{\partial c_{4,j}} \sum_{allk} \frac{1}{2^j} \varepsilon_{j,k,y}^2 = 0 \quad (5.54)$$

From equations (5.51) and (5.52), we obtain the equivalent parameters c_3 and c_4 at the time location presentations:

$$c_{3,k} = \frac{\sum_{allj} \frac{\partial}{\partial k} W_w x\left(2^j t - k\right) W_w h_1\left(2^j t - k\right)}{\sum_{allj} \left(\frac{\partial}{\partial k} W_w x\left(2^j t - k\right)\right)^2} \quad (5.55)$$

$$c_{4,k} = \frac{\sum_{allj} W_w y\left(2^j t - k\right) W_w h_1\left(2^j t - k\right)}{\sum_{allj} \left(W_w y\left(2^j t - k\right)\right)^2} \quad (5.56)$$

Similarly, from equations (5.53) and (5.54), we obtain the equivalent parameters c_3 and c_4 at the frequency band presentations:

$$c_{3,j} = \frac{\sum_{allk} \frac{\partial}{\partial k} W_w x\left(2^j t - k\right) W_w h_1\left(2^j t - k\right)}{\sum_{allk} \left(\frac{\partial}{\partial k} W_w x\left(2^j t - k\right) \right)^2} \tag{5.57}$$

$$c_{4,j} = \frac{\sum_{allk} W_w y\left(2^j t - k\right) W_w h_1\left(2^j t - k\right)}{\sum_{allk} \left(W_w y\left(2^j t - k\right) \right)^2} \tag{5.58}$$

In the case of a stochastic system, the equivalent parameters are obtained by minimizing the expected value of the mean square of the error between the nonlinear system and the linearized system [33], [34].

For β_{eq} and ω_{eq}^2, one has

$$\frac{\partial}{\partial \beta_{eq,j}} \sum_{allk} E\left[\varepsilon_{j,k,x}^2\right] = 0 \tag{5.59}$$

$$\frac{\partial}{\partial \omega_{eq,j}^2} \sum_{allk} E\left[\varepsilon_{j,k,x}^2\right] = 0 \tag{5.60}$$

For c_3 and c_4, one has

$$\frac{\partial}{\partial c_{3,j}} \sum_{allk} E\left[\varepsilon_{j,k,y}^2\right] = 0 \tag{5.61}$$

$$\frac{\partial}{\partial c_{4,j}} \sum_{allk} E\left[\varepsilon_{j,k,y}^2\right] = 0 \tag{5.62}$$

Using these three methods, we can obtain the responses of the hysteresis system. In the next section, we will show the linearization results by using traditional equivalent and wavelet-based linearization method.

5.5 Linearization Results

The main objective of this part is to compare the wavelet-based linearization method with the statistical equivalent technique and verify the feasibility of this method. The displacement response of the bilinear hysteresis system under the multi-frequency excitation is compared to numerical solution of the same system response to check the results as well.

We set the system to be a unit mass system with $\beta = 0.02$, $\alpha = 0.5$ and the initial equivalent parameters $\beta_{eq} = 0, \omega_{eq}^2 = 0$, the sampling time is 0.01 sec. and time period $t = 10.23\sec$, N=2^{12}, $x_y = 1$, the hysteretic force: $h(x, \dot{x}, t) = \alpha x + (1-\alpha)h_0(\dot{x}, y; x_y)$

$$\dot{y} = h_1(\dot{x}, y; x_y)$$

where $h_0 = y\{u(y+x_y) - u(y-x_y)\} + x_y\{u(y-x_y)u(\dot{x}) - u(-y-x_y)u(-\dot{x})\}$

$$h_1 = \dot{x}\{u(y+x_y) - u(y-x_y) + u(y-x_y)u(-\dot{x}) + u(-y-x_y)u(\dot{x})\}$$

The system is excited by the signal $f(t) = 15\sin(4t) + 10\sin(8t) + 5\sin(12t) + 5\sin(16t)$ and the initial conditions are $\beta_{eq} = 0, \omega_{eq}^2 = 0$.

Assume that the system is a linear one to obtain the responses at the different frequencies. By using the procedure demonstrated above and substituting the response to the expressions of equivalent parameters β_{eq} and ω_{eq}^2 in equations (5.42) and (5.43), the parameters converged after 4 times iteration. Therefore, the time varying linear system is obtained and the similar procedure can be applied to the equivalent parameters c_3 and c_4 as well. Thus, the bilinear hysteresis system can be linearzed as this time varying linear system and these results are compared with those responses which are obtained by the statistical method and numerical method. These results are shown in Figures 5.6 and 5.7.

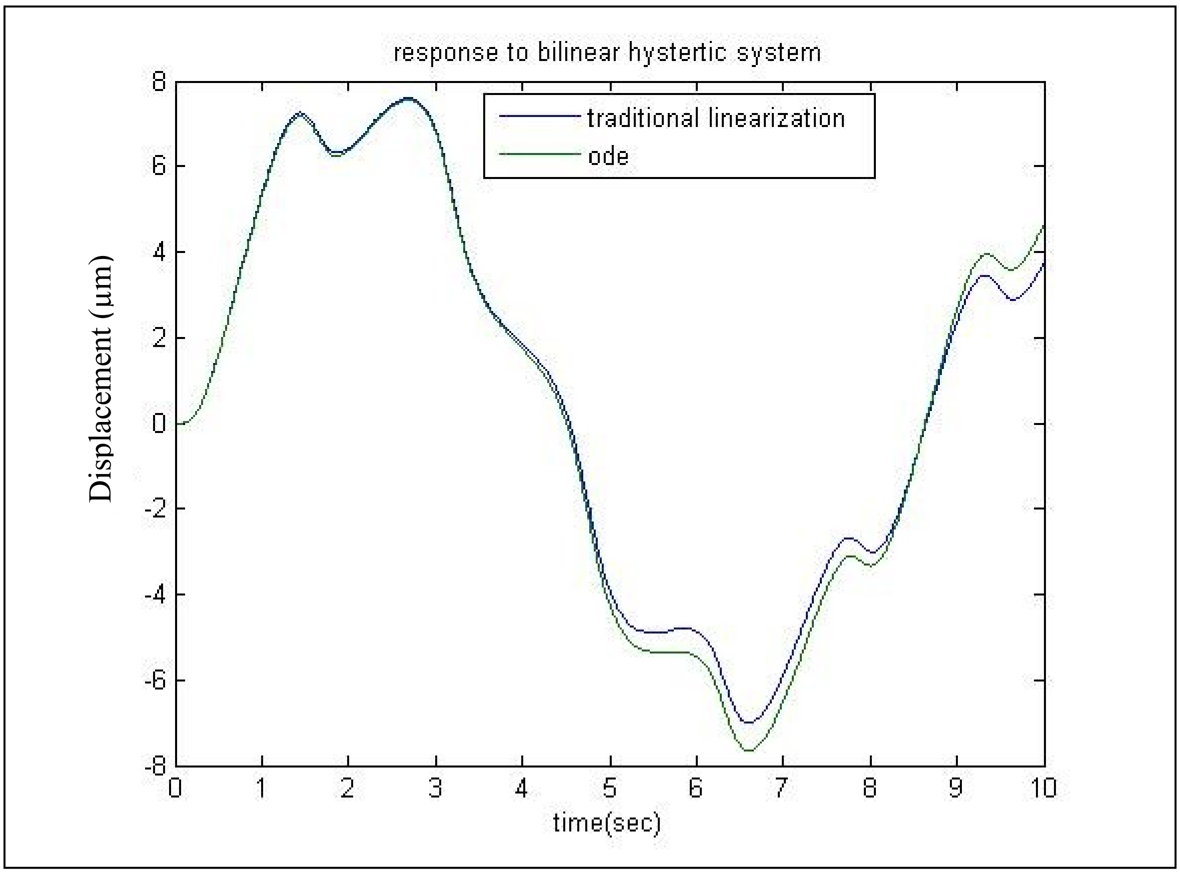

Figure 5. 6 Numerical solution of bilinear hysteretic system to stationary excitation

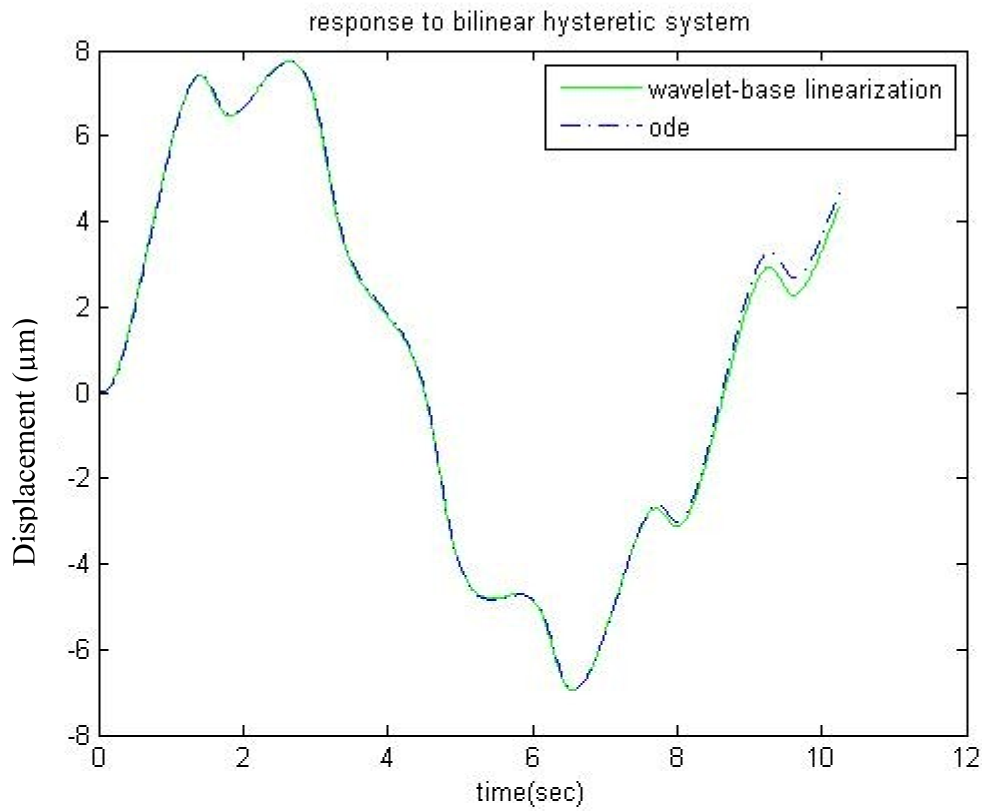

Figure 5. 7 Wavelet-based linearization method comparing with numerical solution

5.6 Conclusion

In this chapter, a wavelet-based linearization method is developed for bilinear hysteresis system. The equivalent parameters in this bilinear hysteresis systems can be obtained in time dependent and frequency dependent respectively, thus, the nonlinear system is linearized to be the time varying linear system. The simulation results are compared with those of numerical method and the traditional equivalent method to demonstrate this wavelet-based linearization method is a promising tool in linearizing nonlinear systems to their approximate linear ones.

6 CONCLUSIONS

6.1 Contribution of the Thesis

In this dissertation, a wavelet-based linearization method for approximating the nonlinear system behavior to its equivalent linearized system is presented. This method has been compared with the traditional equivalent technique. The whole procedure for this novel approach has been demonstrated by using the SDOF Duffing oscillator and bilinear hysteresis nonlinear models as examples to verify the feasibility of this method.

The contributions of the thesis are summarized as:

- A comprehensive literature survey on time-frequency analysis and linearization techniques has been carried out.
- The mechanism by which the signal is decomposed and reconstructed using the wavelet transform is presented. The properties and characteristics of some famous harmonic wavelet transforms are investigated.
- A multi-frequency linearization and wavelet-based linearization methods are implemented and compared for Duffing oscillator system when the system is subject to multiple harmonic inputs.
- A wavelet-based linearization method is developed for another nonlinear system-bilinear hysteresis system. The wavelet transform for the relationship of input and

output of the linear system is implemented and the differential equation of nonlinear system is expressed in terms of time-frequency localization. The comparison between wavelet-based linearization and statistical equivalent linearization for bilinear hysteresis system has been carried out.

6.2 Future Work

Considering the results of the research project, the author summarizes the following research topics to be investigated in the future:

- It will be worthwhile investigating some other types of nonlinear system to see the feasibilities of wavelet linearization method on different cases. Especially, for some multi-degree freedom system. If the general form of wavelet-based equivalent linearization method can be obtained, it will be a valuable contribution.
- Another possible extension is to find different wavelet basis functions in this approach because so many mother wavelets can be chosen. It might be worthwhile checking some different wavelets for different applications.
- The very promising application is to use this technique to design the nonlinear control system to meet control requirement.

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