SOME NEW APPLICATIONS OF CLASSICAL R-MATRIX THEORY TO INTEGRABLE SYSTEMS AND 2-D FLUID DYNAMICS

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Abstract.

Some New Applications of Classical R-matrix Theory to Integrable Systems and 2-D Fluid Dynamics.

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This thesis combines three different problems related to integrable systems and 2D fluid dynamics, united through the R-matrix approach to integrable systems.

The first part is devoted to the Krall-Sheffer theory of two-dimensional classical orthogonal polynomials. This problem is deeply connected with the classification of second order linear differential operators in 2D having polynomial eigenvectors with certain degeneracies of eigenvalues. The original classification problem is related to the theory of symplectic reductions of Hamiltonian systems. Using the R-matrix theory on loop algebras, the reduction scheme sheds new light on the superintegrability of Krall-Sheffer operators. It is also used to deduce explicit results on separation of coordinates and commuting invariants in both the classical and quantum settings.

In the second part of the thesis we study Poisson structures on spaces of rational maps. Such spaces have natural Poisson structures discovered by Atiyah and Hitchin. The approach developed here relates those to the classical R-matrix scheme and further generalizes the Atiyah-Hitchin construction to spaces of meromorphic functions in different geometries using the quadratic r-matrix construction of Sklyanin. More precisely, any classical R-matrix may be viewed as a special linear automorphism of a

Lie algebra. In the Sklyanin theory, the r-matrix defines simultaneously a quadratic Poisson structure on a Poisson-Lie group and a linear one on the corresponding dual space to the Lie algebra. The quadratic structures appeared first as Poisson brackets between elements of the scattering matrix in the case of rapidly decreasing boundary conditions. The Atiyah-Hitchin structure is also defined as brackets on a space of scattering matrices for solutions of Bogomolny equations governing interactions of monopoles in nonabelian gauge theories. Using this analogy, we obtain the desired generalizations.

The third project relates to the problem of Laplacian Growth (LG) in two-dimensional viscous fluid dynamics. It describes the dynamics of the boundaries in various processes (e.g. propagation of oil-water interfaces, crystal growth etc), and is of great practical importance. Recently established connections with integrable systems has brought the problem into the domain of modern mathematical physics. In this thesis, the Hamiltonian nature of rational conformal maps describing so-called "multi-finger" solutions of the LG problem is investigated. These, in turn, are related to finite-dimensional reductions of 2D dispersionless Toda hierarchy. Such reductions satisfy an extra condition known as the "String" equation, which appears as a consistent constraint on the 2D Toda dynamics, closely related to the fluid dynamical (Laplacian Growth) equation. In the case of dispersionless 1D Toda systems, the Hamiltonian dynamics are also related to the classical R-matrix theory.

Contents

Chapter 1

Introduction

1.1 R-matrix approach: historical remark

This chapter will present a description of an approach to integrable equations, based on the use of a simple and a powerful algebraic tool, the so-called classical R-structure (or R-matrix) [1], [3].

This approach gives a simple and effective method for analysis of the multi-Hamiltonian structure of integrable systems. The crucial point of this approach is the observation that the Lax equation can be treated as an abstract Hamiltonian dynamical system from which the physically interesting dynamical systems may be obtained by introducing suitable reductions. The phase space for most of these equations can be regarded as given by a set of Lax operators taking values in a Lie algebra. The evolution of the Lax operators in this setting is usually isospectral, and in that case is sometimes referred to as the *Lax dynamics*. The multi-Hamiltonian construction of the integrable equations becomes quite transparent within the framework of the R-matrix structure. This approach has its origins in the results of Gelfand and Dikii [4] and Adler [5], who used a Lie algebraic setting to describe integrable partial differential equations via their Lax representations. Besides such integrable differential dynamical systems integrable systems discrete (lattice) systems may also be constructed in a similar manner using Lie algebra techniques.

The Adler-Gelfand-Dikii (AGD) scheme, which was specific to algebras of pseudo-differential operators and its Lie algebraic variants, starts with a dual space to a Lie algebra as a natural phase space for integrable systems. (Later, Semenov-Tian-Shansky showed that the notion of classical R-structures leads to an algebraic construction of integrable systems, generalizing the AGD-scheme [1].)

The Lie-Poisson bracket associated with the Lie algebra structure provides a natural Hamiltonian structure, in which the coadjoint invariants form the center of the Poisson algebra (Casimirs), and hence generate no flows. The classical R-matrix theory, which generalizes the AGD-approach (and the subsequent, so called AKS (Adler-Kostant-Symes) scheme) is based upon introduction of new, modified Lie algebra structure, derived from endomorphisms of the algebra satisfying an auxiliary system of equations - the "classical (modified) Yang-Baxter equation". These assure that the corresponding Poisson brackets on the dual space satisfy the Jacobi identity.

Under this modified Poisson bracket structure, the Casimir invariants for the unmodified one still Poisson commute amongst themselves, but no longer are central elements. They therefore generate commuting Hamiltonian flows and, in many cases, there are sufficiently many of them that remain functionally independent on the symplectic leaves to generate completely integrable systems.

In the first part of this chapter we collect definitions and results necessary to clarify the concept of Lax dynamics. In the second part we present the basic construction of the R-matrix. For proofs and details we refer the reader to literature (e.g. [8], [3]). The application of the theory to the infinite algebra of shift operators is done in the third chapter of the thesis.

1.2 Some basic properties of integrable systems

The study of completely integrable systems goes back to the classical papers of Euler, Lagrange, Jacobi, Liouville and others on analytical mechanics. By the end of the XIX-th century all interesting examples seemed to have been exhausted.

In fact, further studies of integrability of Hamiltonian dynamical systems continued till the end of the 19th century and beyond (especially in Russia through the works of Steklov, Chaplygin, Lyapunov and others, as well as related developments on commuting rings of differential operators (Baker) and quasi-periodic solutions of systems of ODE's (Garnier, etc.)). However, following the results of Poincare (1905) on planetary motion, which revealed that integrability was a very non-generic feature of Hamiltonian dynamical systems, the latter theory developed very different directions in the subsequent 60 years, with emphasis on "qualitative dynamics", stability, long-time behaviour, etc.

A revival of interest in the study of integrable systems began with discovery of the integrable partial differential equations of evolution type, such as the KdV equation in one-dimensional fluid dynamics whose integrable properties were discovered at the end of the 60's. This led to the discovery of the Inverse Scattering Method. Subsequent investigations included the development of the basic geometric ideas of the theory and provided a unified basis for different examples.

The general geometric construction (we follow Semenov-Tian Shansky, [1]) allows one to identify the following features typical of many known examples of integrable systems:

1. The equations of motion are compatibility conditions for a certain auxiliary system of linear equations.
2. They are Hamiltonian with respect to a natural Lie algebraic Poisson bracket.
3. The integrals of motion are spectral invariants of the auxiliary linear operator. They are in involution with respect to the Poisson bracket referred to above.

The flows therefore leave the spectrum of the linear operator invariant.

Depending on the nature of the auxiliary linear problem, the associated nonlinear equations may be divided into the following three groups:

(i) Finite-dimensional systems,

(ii) Infinite-dimensional systems on function spaces in one or two spatial variables,

(iii) Integrable systems on lattices.

In the sequel we shall deal with the first and second groups of problems. In case (i) the auxiliary linear problem is the eigenvalue problem for a finite-dimensional matrix (possibly depending on an additional parameter). In case (ii) the associated linear operator is a formal pseudo-differential operator.

As it happens, the key properties 1–3 referred to above are corollaries of a single general theorem which may be adapted to numerous concrete applications. The original idea of this theorem is due to M.Adler, B.Kostant and W.Symes. The statement and proof of this theorem are particularly simple for systems of types (i) and (ii).

However, there is one important shortcoming in this approach. For any dynamical system (even if it is known to be completely integrable!) it is very difficult to tell *a priori*, what is the underlying Lie group, or Lie algebra. The practical way around this difficulty is to look at various examples associated with different Lie algebras. The list of interesting Lie algebras includes:

1. Finite dimensional semisimple Lie algebras. The associated integrable systems include the open Toda lattice and other finite dimensional Hamiltonian systems, which may be integrated in *elementary functions* .

2. Loop algebras, or affine Lie algebras. The associated integrable systems may be finite dimensional Hamiltonian systems, e.g. when they satisfy a condition of periodicity or quasi-periodicity, and may be integrated in *Abelian functions*. (This includes, e.g. integrable tops, and almost all classical examples from the XIXth century Analytical Mechanics, as well as periodic Toda lattice and various fi-

nite dimensional integrable spin systems, such as the Gaudin and Heisenberg models.)

3. Double loop algebras and their central extensions. This class of algebras accounts for integrable PDE's admitting a zero curvature representation (such as the Nonlinear Schroedinger equation, the Sine-Gordon equation and many others, [12]).

4. Algebras of pseudo differential operators. The KdV equation, its higher analogs and the KP hierarchy are included in this example, [69].

5. The algebra of vector fields on the line (or circle). This algebra, or rather its central extension *(the Virasoro algebra)* and the associated loop algebra again are related to the KP hierarchy, and the Miura transformation relating e.g. the KdV and mKdV (modified KdV) hierarchies.

1.3 Poisson Brackets, Coadjoint Orbits, Lax representation

Typically, the phase space of an individual dynamical system should be a symplectic manifold. The appropriate geometrical setting for the study of integrable systems associated with auxiliary linear problems is provided by the theory of Poisson manifolds. Recall that a Poisson bracket on a smooth manifold M is a Lie algebra structure on the space $C^\infty(M)$, which satisfies the Leibnitz rule. In local coordinates, a Poisson bracket is written as

$$\{\varphi,\psi\}(x)=\sum_{i,j}\pi_{ij}(x)\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j}, \tag{1.1}$$

where π_{ij} is an antisymmetric tensor *(Poisson tensor)* satisfying a quadratic differential equation which assures the Jacobi identity:

$$\frac{\partial\pi_{ij}}{\partial x_s}\pi_{sk}+\frac{\partial\pi_{ki}}{\partial x_s}\pi_{sj}+\frac{\partial\pi_{jk}}{\partial x_s}\pi_{si}=0. \tag{1.2}$$

When the manifold M is symplectic, i.e., admits a nondegenerate closed 2-form $\omega = \sum \omega^{ij} dx_i \wedge dx_j$, the associated Poisson tensor is $\pi_{ij} = (\omega^{ij})^{-1}$. Conversely, whenever the Poisson tensor is nondegenerate, its inverse is a symplectic form. In general, a Poisson manifold is not symplectic, however, it always admits a decomposition into symplectic leaves. (Although this may be a singular, stratified foliation in general). The geometrical meaning of this decomposition is very simple. Any $H \in C^\infty(M)$ defines a Hamiltonian vector field on M which acts on $\varphi \in C^\infty(M)$ via

$$X_H \varphi = \{\varphi, H\}. \tag{1.3}$$

For a given point $x \in M$ the tangent vectors $X_H(x)$ span a linear subspace in the tangent space $T_x M$; this is precisely the tangent space to the symplectic leaf passing through x. By construction, Hamiltonian vector fields are tangent to symplectic leaves, and hence the Hamiltonian flows preserve the leaves. A closely related property of general Poisson manifolds is the existence of *Casimir functions.* By definition, a function $H \in C^\infty(M)$ is called a Casimir function if it lies in the center of the Poisson bracket Lie algebra. Equivalently, Casimir functions define trivial Hamiltonian equations on M. The restrictions of Casimir functions to symplectic leaves in M are constants.

A very typical example of a Poisson manifold is the *dual space of a Lie algebra.*

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual space; for $X, Y \in \mathfrak{g}$ let us denote by $\tilde{X}, \tilde{Y}$ the corresponding linear functions on the dual space $\mathfrak{g}^*$

$$\left\{\tilde{X}, \tilde{Y}\right\}(L) = \langle L, [X, Y]\rangle\,, \ ; L \in \mathfrak{g}^*. \tag{1.4}$$

Since, $\{,\}$ is a biderivation, this uniquely determines the Poisson bracket for arbitrary differentiable functions φ_1, φ_2

$$\{\varphi_1, \varphi_2\}(L) = \langle L, [d\varphi_1(L), d\varphi_2(L)]\rangle\,. \tag{1.5}$$

(Note that $d\varphi_i(L) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$, and hence the Lie bracket is well defined). The bracket (1.5) is usually called the *Lie–Poisson bracket*, [13]. Its properties are closely related to a distinguished representation of the associated Lie group, the *coadjoint representation.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$; $\exp : \mathfrak{g} \longrightarrow G$ the exponential map. The adjoint and coadjoint representations of G acting in $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, are defined by

$$Ad\ g \cdot X = \left(\frac{d}{dt}\right)_{t=0} g \cdot e^{tX} \cdot g^{-1}, \qquad X \in \mathfrak{g}, \tag{1.6}$$

$$\langle Ad^* g \cdot L, X\rangle = \left\langle L, Ad\ g^{-1} \cdot X\right\rangle, \ X \in \mathfrak{g}, \qquad L \in \mathfrak{g}^*. \tag{1.7}$$

Set

$$ad\, X \cdot Y = \left(\frac{d}{dt}\right)_{t=0} Ad\, e^{tX} \cdot Y, \qquad ad^*\, X \cdot L = \left(\frac{d}{dt}\right)_{t=0} Ad^* e^{tX} \cdot L. \tag{1.8}$$

Clearly, one has

$$ad\, X \cdot Y = [X, Y]\,, \qquad ad^*\, X = -\left(ad\, X\right)^*. \tag{1.9}$$

The following fundamental theorem again goes back to Lie; it was rediscovered by Kirillov and Kostant in the 1960's:

Theorem 1.1 *The symplectic leaves of the Lie-Poisson bracket coincide with the G-orbits in* $\mathfrak{g}^*$ *(coadjoint orbits). The Casimir functions of the Lie-Poisson bracket are precisely the coadjoint invariant functions on* $\mathfrak{g}^*$.

It is very easy to verify a somewhat weaker property.

Proposition 1.1 *Let* $\varphi \in C^\infty(\mathfrak{g}^*)$ *be an arbitrary function; the Hamiltonian equation of motion defined by* φ *with respect to the Lie-Poisson bracket admits the following form:*

$$\frac{dL}{dt} = -ad^*\, d\varphi\,(L) \cdot L, \qquad L \in \mathfrak{g}^*; \tag{1.10}$$

It is clear from (1.10) that the velocity vector associated with any Hamiltonian equation on $\mathfrak{g}^*$ is automatically tangent to the coadjoint orbit passing through L.

In the context of integrable systems, coadjoint orbits are of particular importance. In many applications, the phase spaces of integrable systems *are* coadjoint orbits for some appropriate Lie group. However, in some cases, an orbit is too *big* for the purpose of identifying the integrable systems of interest, but it may be possible to cancel out some degrees of freedom by passing to the quotient space under some symmetry group. On the other hand, in some cases, an orbit is too *small* to identify the integrals directly, but it may be possible to use a bigger phase space which is mapped onto the orbit in a way that is compatible with its Poisson structure.

Corollary 1.1 *Assume that* $\mathfrak{g} = \mathfrak{gl}(n)$ *is identified with its dual space through the trace pairing* $<\mu, X>= Tr(\mu X)$ *and equipped with the Lie–Poisson bracket. For any* $\varphi \in C^{\infty}(\mathfrak{g})$ *the Hamiltonian equation of motion has the form*

$$\frac{dL}{dt} = -\left[d\varphi\left(L\right), L\right]; \tag{1.11}$$

hence all Hamiltonian flows on $\mathfrak{g}$ *preserve the spectral invariants of* L.

Here, spectral invariants of matrices are *Casimir functions* for the Lie-Poisson bracket; their conservation is a trivial fact which has nothing to do with integrability of equation 1.11, since they generate *trivial* flows.

1.4 Classical Yang-Baxter equation and split R-matrices

As follows from the preceding section, instead of the initial Lie-Poisson bracket, which shows that the set of spectral invariants does not yield any non-trivial dynamics, we must use a different one to get nontrivial commuting flows. This is provided by introducing the classical R-matrix structure. (The following development closely follows that in [3]).

Let $\mathfrak{g}$ be a Lie algebra, and suppose that we have a linear endomorphism $R : \mathfrak{g} \mapsto \mathfrak{g}$ such that the skew symmetric bilinear operator $[,]_R : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ defined by

$$[\xi, \eta]_R := \frac{1}{2}[R\xi, \eta] + \frac{1}{2}[\xi, R\eta] \tag{1.12}$$

satisfies the Jacobi identity

$$[\xi, [\eta, \zeta]_R]_R + [\eta, [\zeta, \xi]_R]_R + [\zeta, [\xi, \eta]_R]_R = 0, \qquad \forall\, \xi, \eta, \zeta \in \mathfrak{g}. \tag{1.13}$$

This defines a new Lie algebra structure on the space $\mathfrak{g}$ which will be denoted $\mathfrak{g}_R$. We shall say that $R \in End(\mathfrak{g})$ is a *classical R-matrix* and the pair $(\mathfrak{g}, \mathfrak{g}_R)$ is a *double Lie algebra.*

Now let us define a skew symmetric map $R^{\wedge} : \mathfrak{g} \wedge \mathfrak{g} \mapsto \mathfrak{g}$ by

$$R^{\wedge}(\xi, \eta) := [R\xi, R\eta] - 2R[\xi, \eta]_R \tag{1.14}$$

Lemma 1.1 *A sufficient condition that the bracket* $[,]_R$ *satisfy the Jacobi identity is*

$$R^{\wedge}(\xi, \eta) = \alpha[\xi, \eta] \tag{1.15}$$

where α *is any proportionality constant.*

Relation (1.15) is called the *modified classical Yang-Baxter equation.* The unmodified case corresponds to $\alpha = 0$. There exists a simple class of such cases, called *split* R-matrices, that correspond to the original Adler-Kostant-Symes theorem.

Assume that $\mathfrak{g}$ admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{l} \tag{1.16}$$

as the sum of two complementary subalgebras $\mathfrak{h}, \mathfrak{l} \subset \mathfrak{g}$, and define the associated split classical R-matrices as the difference

$$R := P_h - P_l \tag{1.17}$$

between the two projection maps:

$$P_h : \xi_h + \xi_l \mapsto \xi_h \qquad P_l : \xi_h + \xi_l \mapsto \xi_l \tag{1.18}$$

This may be simply verified to satisfy (1.13) and (1.15) with $\alpha = -1$.

The resulting modified bracket is just

$$[\xi, \eta]_R = [\xi_h, \eta_h] - [\xi_l, \eta_l] \tag{1.19}$$

This implies that the modified coadjoint action with respect to this bracket is

$$ad^*_{(X_h+X_l)}(L_{h^*} + L_{l^*}) = ad^*_{X_h} L_{l^*} - ad^*_{X_l} L_{h^*} \tag{1.20}$$

where we use the identification $l^* = h^0$ (annihilator of h) and $h^* = l^0$ (annihilator of l).

Linear brackets

Now, given an R-matrix on $\mathfrak{g}$, we can define the linear Poisson bracket on $\mathfrak{g}^*$ associated with $[,]_R$

$$\{f, g\}^{lin}(\xi) = < \xi, [df, dg]_R >, \qquad \xi \in \mathfrak{g}^*, \qquad f, g \in \mathbb{C}^\infty(\mathfrak{g}^*) \tag{1.21}$$

If we define a Poisson tensor $P_{lin}(\xi) : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ by

$$\{f, g\}^{lin}(\xi) = < P_{lin}(\xi) dg, df >, \tag{1.22}$$

then the Hamiltonian vector field on $\mathfrak{g}^*$ associated to a Hamiltonian H is given by

$$\frac{d\xi}{dt} = P_{lin}(\xi) dH \tag{1.23}$$

Amongst all possible Hamiltonian functions on $\mathfrak{g}^*$ a particular role is played by the functions invariant under the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Adler-Kostant-Symes Theorem

Let $I(\mathfrak{g}^*)$ be the ring of Casimir functions on $\mathfrak{g}^*$ (with respect to the original Lie-Poisson bracket), i.e. the set of coadjoint invariants.

Theorem 1.2 *Functions in $I(\mathfrak{g}^*)$ are in involution with respect to the split R-bracket (1.19) on $\mathfrak{g}^*$. The equations of motion induced by $h \in I(\mathfrak{g}^*)$ with respect to the R-bracket have the form*

$$\frac{dL}{dt} = -ad^*_{\mathfrak{g}} M \cdot L, \; M = R(dh(L)) \;, \qquad L \in \mathfrak{g} \sim \mathfrak{g}^* \tag{1.24}$$

If $\mathfrak{g}$ *admits a nondegenerate ad-invariant bilinear form, so that* $\mathfrak{g}^* \simeq \mathfrak{g}$, $ad^*_{\mathfrak{g}} \simeq ad_{\mathfrak{g}}$, *equations (1.24) have the* Lax form

$$\frac{dL}{dt} = [L, M] \,. \tag{1.25}$$

This theorem shows that the trajectories of the dynamical systems with Hamiltonians $h \in I(\mathfrak{g}^*)$ lie in the intersection of *two families of orbits* in $\mathfrak{g}^*$, the coadjoint orbits of $\mathfrak{g}$ and $\mathfrak{g}_r$. Indeed, the coadjoint orbits of $\mathfrak{g}_r$ are preserved by all Hamiltonian flows in $\mathfrak{g}_r$. On the other hand, because of (1.24), the flow is always tangent to the $\mathfrak{g}$-orbits in $\mathfrak{g}^*$.

In many applications the intersections of orbits are precisely the "Liouville tori" for our dynamical systems.

Quadratic brackets

The quadratic Poisson bracket are naturally defined on a Lie group (the so-called Sklyanin brackets, [36]) rather than on a Lie algebra. Namely, suppose that there exists an ad-invariant nonsingular bilinear form on $\mathfrak{g}$, which allows to make an identification between the space $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, and assume that the classical R-matrix is skew-symmetric with respect to this scalar product and satisfies the modified Yang-Baxter equation (we follow here the lecture notes [10]). In this case there exist an analogue of the Theorem 1.2 above concerning commutative Hamiltonian flows on the Poisson Lie group $\mathbf{G}$.

Let $I(\mathbf{G})$ denote the space of functions on $\mathbf{G}$ that are invariant under conjugation. The condition that a function $f \in \mathbb{C}^\infty(\mathbf{G})$ be in $I(\mathbf{G})$ is thus

$$f \circ L_\xi \circ R_\xi^{-1} = f \tag{1.26}$$

for any $\xi \in \mathbf{G}$ Taking the differentials gives the equivalent condition in infinitesimal form

$$L^*_\xi(df) = R^*_\xi(df) \tag{1.27}$$

for any $\xi \in \mathbf{G}$.

Now, let $\omega \in \mathfrak{g} \wedge \mathfrak{g}$ be defined by

$$\omega(\sigma, \eta) := (\sigma, R\eta). \tag{1.28}$$

Then, for any pair $f, g \in \mathbf{G}$, the bracket

$$\{f, g\}^{skl}(\xi) = \frac{1}{2}\omega(R_{\xi^*}(df), R_{\xi^*}(dg)) - \frac{1}{2}\omega(L_{\xi^*}(df), L_{\xi^*}(dg)) \tag{1.29}$$

defines the Lie Poisson structure on $\mathbf{G}$.

Theorem 1.3 *Suppose that the Lie algebra $\mathfrak{g}$ is identified with its dual space $\mathfrak{g}^*$ via a symmetric ad-invariant bilinear form $(,)$ and that $\omega \in \mathfrak{g} \wedge \mathfrak{g}$ determines a coboundary bialgebra $(\mathfrak{g}, \mathfrak{g}^*_\omega)$ that is the tangent bialgebra of the Poisson Lie group $\mathbf{G}$. Let $\{,\}$ denote the associated Poisson bracket on $\mathbf{G}$. Then:*

(i) If f, g is any pair of functions belonging to $I(\mathbf{G})$, their Poisson bracket vanishes

$$\{f, g\}_\omega = 0 \tag{1.30}$$

(ii) Hamilton's equations on $\mathbf{G}$ generated by $f \in I(\mathbf{G})$ are of the Lax form

$$\frac{d\xi}{dt} = \frac{1}{2}[\xi, M] \tag{1.31}$$

where $M = R(R^*_\xi(df))$, $\xi \in \mathbf{G}$ and the left and right multiplications by ξ signify left and right translations to the point $\xi \in \mathbf{G}$.

1.5 Loop algebras associated with split R-matrix structures

In typical applications the Lie algebra $\mathfrak{g}$ is a *loop algebra*, i.e., an algebra of matrix-valued functions on the circle, and $\mathfrak{g}_\pm$ the subalgebras consisting of functions which admit analytic continuation inside $(+)$ (respectively, outside $(-)$) the circle. Now let us introduce a few definitions concerning loop algebras associated with split R-matrix structures. Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra and let

$$\tilde{\mathfrak{g}} = C^{\omega}(\Gamma, \mathfrak{g}) = \{\xi = \xi(\lambda), \lambda \in \Gamma\} \tag{1.32}$$

denote the space of analytic maps from a positively oriented, simple, closed curve $\Gamma \subset C$ surrounding the origin in the complex λ-plane to $\mathfrak{g}$. We give $\tilde{\mathfrak{g}}$ the structure of a Lie algebra, called the loop algebra associated to $\mathfrak{g}$, by defining the Lie product through pointwise evaluation:

$$[\xi, \eta](\lambda) := [\xi(\lambda), \eta(\lambda)] \qquad \forall \lambda \in \Gamma \tag{1.33}$$

We now consider the split R-matrix associated to the decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ + \tilde{\mathfrak{g}}_-$

$$R = P_+ - P_- \tag{1.34}$$

Denoting the corresponding Lie algebra structure on $\tilde{\mathfrak{g}}$ as $\tilde{\mathfrak{g}}_R$, we have

$$[\xi, \eta]_R = [\xi_+, \eta_+] - [\xi_-, \eta_-]. \tag{1.35}$$

For $\mathfrak{g} = \tilde{\mathfrak{gl}}(N, \mathbb{C})$ with Ad-invariant symmetric form

$$(\xi, \eta) := tr(\xi\eta) \qquad \xi, \eta \in \mathfrak{gl}(n) \tag{1.36}$$

choosing for ξ and η a pair of basis elements consisting of the elementary matrices $\xi = e_{ij}$, $\eta = e_{lk}$ define the corresponding linear functions $\mu_{ij}(\sigma)$ on $\mathfrak{g}^*$ by:

$$\xi(\sigma)\mid_{\mu} = \mu_{ij}(\sigma), \qquad \eta(\tau)\mid_{\mu} = \mu_{kl}(\tau) \qquad \mu \in \mathfrak{g}^*. \tag{1.37}$$

It follows that the Poisson brackets of these elements is given by

$$\{\mu_{ij}(\sigma), \mu_{kl}(\tau)\}_R = \frac{\mu_{il}(\sigma) - \mu_{il}(\tau)}{\sigma - \tau}\delta_{jk} - \frac{\mu_{kj}(\sigma) - \mu_{kj}(\tau)}{\sigma - \tau}\delta_{il} \tag{1.38}$$

This can be expressed in tensorial notation as

$$\{\mu(\sigma) \overset{\otimes}{,} \mu(\tau)\} = [r(\sigma - \tau), \mu(\sigma) \otimes \mathbb{I} + \mathbb{I} \otimes \mu(\tau)] \tag{1.39}$$

where both sides are interpreted as elements of $End(\mathbb{C}^n \otimes \mathbb{C}^n)$. The symbol $\{\ \overset{\otimes}{,}\ \}$ signifies a simultaneous tensor product in $End(\mathbb{C}^n) \sim \mathfrak{gl}(n)$ and Poisson brackets in the components, and $r(\sigma - \tau)$ denotes the classical $R-$matrix. The simplest case is the rational R-matrix:

$$r(\lambda) := \frac{P_{12}}{\lambda}, \tag{1.40}$$

$$P_{12}(\mathbf{u} \otimes \mathbf{v}) := \mathbf{v} \otimes \mathbf{u}. \tag{1.41}$$

Here $P_{12} \in End(\mathbb{C}^n \otimes \mathbb{C}^n)$ signifies the permutation of the first and second factors of decomposable elements of $\mathbb{C}^n \times \mathbb{C}^n$, extended linearly to the whole space, [12].

The corresponding quadratic Poisson bracket structure on the loop group $\tilde{\mathfrak{Gl}}(n)$ in the tensor product notation may be written as

$$\{\mu(\sigma) \overset{\otimes}{,} \mu(\tau)\} = [r(\sigma - \tau), \mu(\sigma) \otimes \mu(\tau)] \tag{1.42}$$

or, equivalently,

$$\{\mu_{ij}(\sigma), \mu_{kl}(\tau)\} = \frac{\mu_{il}(\sigma)\mu_{kj}(\tau) - \mu_{il}(\tau)\mu_{kj}(\sigma)}{\sigma - \tau} \tag{1.43}$$

Remark. The Poisson bracket on the loop group is defined as above (1.29), where now $\omega \in \tilde{\mathfrak{gl}}(n) \wedge \tilde{\mathfrak{gl}}(n)$ is defined by

$$\omega(\mu, \nu) =< \mu_+, \nu_- > - < \mu_-, \nu_+ > \tag{1.44}$$

and $\mu, \nu \in \tilde{\mathfrak{gl}}^*(n) \sim \tilde{\mathfrak{gl}}(n), \quad \mu_+, \nu_+ \in \tilde{\mathfrak{gl}}_+(n), \quad \mu_-, \nu_- \in \tilde{\mathfrak{gl}}_-(n)$.

Now, f and g are understood to be functions on the loop group space, belonging to some suitably restricted smoothness category. (It is sufficient, e.g., to take polynomial functions of the coefficients when restricting to subspaces of rational group elements with given pole support). The differentials df and dg are evaluated at the loop group element ξ and are understood as belonging to the cotangent space to $\tilde{\mathfrak{Gl}}(n)$ at the point ξ, and L^*_ξ, R^*_ξ denote the pullbacks of the left- and right translation maps under the element ξ, which when applied to these differentials give elements of the tangent space at the identity. The latter are, by the trace-residue loop algebra pairing defined on $\tilde{\mathfrak{g}}$ by pointwise evaluation and integration around Γ

$$< \xi, \eta >:= \frac{1}{2\pi i} \oint_{\Gamma} (\xi(\lambda), \eta(\lambda)) d\lambda \tag{1.45}$$

(see also 1.36), identified with elements of the Lie algebra $\tilde{\mathfrak{gl}}(n)$, to which the bivector ω is applied.

1.6 Layout of the thesis.

The layout of the thesis is the following.

In Chapter 1 we present the Krall-Sheffer theory of two-dimensional classical orthogonal polynomials and its relation with integrable systems on the plane or on surfaces of constant curvature. We show that all admissible second-order linear differential operators in the Krall-Sheffer classification list are reducible by gauge transformations to the form of Laplace-Beltrami operators on spaces of constant curvature, plus some potential. In Theorem 2.2 we prove superintegrability of the corresponding operators by giving sets of commuting invariants for each case. Then we provide another approach to the problem, based on classical R-matrix theory on the loop algebra $\tilde{sl}(2)_R$. This allows us to present an explicit construction of integrals of motion both in a classical and a quantum setting as trace invariants of the rational 2×2 Lax matrix, and give separation variables for each case. The parametric dependence of invariants on a pole location of the Lax matrix is used to interpret the phenomenon of superintegrability.

In Chapter 2 we study Poisson structures on the space of rational maps originally introduced in the context of the geometric theory of monopoles constructed by Atiyah and Hitchin. We use the quadratic r-matrix construction of Sklyanin and show how the Atiyah-Hitchin Poisson brackets relate under embedding to the Sklyanin brackets on elements of the corresponding scattering matrix. We then provide a trigonometric generalization of the Atiyah-Hitchin brackets and discuss associated multi-Poisson structures for both cases.

In Chapter 3 we describe the Laplacian growth problem and its statement in the framework of the modern theory of the dispersionless integrable hierarchies. This part of the thesis is devoted to a study of finite-dimensional reductions of the two-dimensional dispersionless Toda hierarchy constrained by the string equation and the Hamiltonian structure of such reductions. In Propositions 4.1-4.4 we prove the consistency of polynomial, rational and logarithmic solutions of the $2D$ dispersionless Toda (dToda) provided the string equation holds. We also introduce additional invariant flows associated to the pole structure of the rational solutions compatible with the $2D$ dToda system, calling them the Krichever-Toda flows in reference to the KP-Krichever case. In the propositions 4.5-4.6 we show that the new Krichever-Toda "time"-parameters play the role of linearization variables for the reductions obtained, and provide a complete set of integrals of motion for them.

The independent integrals of motion are interpreted as functions of the harmonic moments of the boundary curve for the Laplacian growth problem, while the only Casimir function plays the role of the area growing with the unit speed.

We continue with a derivation of the bi-hamiltonian structure of the $2D$ Toda. We describe the R-matrix construction on the direct sum of two algebras of difference operators, corresponding to "halves" of the $2D$ Toda hierarchy. Here we provide explicit formulas for linear and quadratic Poisson brackets given for the arbitrary gauge and derive their dispersionless limit. Then we use a part of these brackets to construct a Poisson structure of rational reductions of the $1D$ dToda. In the final section we discuss open problems and further directions of investigations.

Chapter 2

The Krall-Sheffer problem and superintegrability on spaces of constant curvature.

2.1 Classification of two-dimensional analogs of the classical orthogonal polynomials in the plane

The problem of describing analogues of classical orthogonal polynomials in two dimensions is naturally connected with the study of partial differential equations of hypergeometric type. The main difference in two dimensions in comparison with the one-dimensional case is that for each differential equation one has to construct two families of polynomials that are biorthogonal to each other. In a sense, one deals not with orthogonal polynomials, but with bases for linear spaces of polynomials, which are mutually orthogonal for different eigenvalues. This problem was studied by P.Appel (see [16]), who first constructed four higher hypergeometric series with a region of convergence on a plane. From these he derived two families of orthogonal polynomials satisfying the same linear partial differential equation of second order in two variables, which are two-dimensional analogues of Jacobi polynomials. These polynomials are orthogonal on a triangular domain in the plane, and are now called Appel polynomials. Other contributors to the study of higher hypergeometric series and orthogonal polynomials related to them were Picard, J.Kampe de Feriet, Horn, Hermite and others (see [17], [18]). In particular, Hermite studied higher dimensional spaces of polynomials orthogonal with respect to exponential measures. In two dimensions more then 40 different types of pde's corresponding to the various hypergeometric functions were derived, with eleven regions of support in the plane.

Krall and Sheffer, [20], studied second order pde's in two variables subject to the imposition of a condition of orthogonality on the eigenfunctions of the corresponding linear second-order differential operators. They gave some restrictions on their coefficients that allow one to reduce the problem to the study of a finite number of such operators. The existence of a nondegenerate moment functional and a recurrence property of moments were also taken into account, making the construction closer to the case of classical one-dimensional orthogonal polynomials. Krall and Sheffer reduced the classification of all such admissible operators to exactly nine cases. This result was further analyzed in detail by Suetin [26] and by Engelis [22].

In [28] we have shown that all the operators in the Krall-Sheffer list are reducible by gauge transformations to the form of a Laplace-Beltrami operator on a space of constant curvature, plus some scalar potential, the magnetic field being absent. Moreover, they all were related to two-dimensional superintegrable systems on spaces of constant curvature [28].

This approach is based on Lax matrixes satisfying the rational R-matrix structure and gives a systematic way to derive the Hamiltonians and commuting invariants for all nine cases corresponding to Krall-Sheffer operators on quadrics and on the plane. It also provides a prescription for determining the separating coordinates, both in the classical and quantum cases. The presence of additional parameters (α, β, γ) in the cases of non-zero curvature provides an explanation for their superintegrability.

A similar analysis was made for the cases of Euclidean spaces arising in the Krall-Sheffer problem, which may be obtained as limiting cases of the above, providing an R-matrix approach to the remaining Krall-Sheffer operators.

2.2 Two-dimensional Krall-Sheffer polynomials and Integrable systems

2.2.1 The classification scheme of Engelis and Suetin.

Assume that $P_n(x, y)$ are polynomials in two variables x, y. As usual, the degree n is the maximal value $n = max\{i + j\}$ among all possible monomials $x^i y^j$ in the expansion of the polynomial $P_n(x)$. Krall and Sheffer considered ([20]) the problem of finding all polynomials $P_n(x, y)$ with the following properties:

1. The polynomials $P_n(x, y)$ are eigenfunctions of a second-order admissible differential operator L (to be fully defined later)

$$LP_n(x, y) = \lambda_n P_n(x, y) \tag{2.1}$$

   with polynomial coefficients:

$$L = A(x, y)\partial_{xx} + 2B(x, y)\partial_{xy} + C(x, y)\partial_{yy} + D(x, y)\partial_x + E(x, y)\partial_y \tag{2.2}$$

   where $A(x, y), ..., E(x, y)$ are polynomials in x and y with real coefficients (the degree of polynomials will be defined later in this section). Note that the eigenvalue λ_n depends only on the degree of the polynomial $P_n(x, y)$.

2. There exists a non-degenerate linear functional σ defined on the space of all polynomials in two variables such that the orthogonality property

$$\langle \sigma, P_n(x, y)q(x, y)\rangle = 0 \tag{2.3}$$

holds with $q(x, y)$ any polynomial of degree less than n. Here, $<,>$ stays for a linear non-degenerate scalar product, which may have an integral representation, on the space of two-dimensional polynomials.

The functional σ can be defined through its moments

$$\langle \sigma, x^n y^m \rangle = c_{nm}, \qquad n, m \in \mathbb{N} \tag{2.4}$$

The orthogonality property (2.3) is closely connected with symmetrizability of the operator L. Recall that the Lagrange adjoint of L in equation (2.2) is defined as [21]

$$L^+ = \partial_{xx} A(x, y) + 2\partial_{xy} B(x, y) + \partial_{yy} C(x, y) - \partial_x D(x, y) - \partial_y E(x, y). \tag{2.5}$$

The operator L is called *symmetric* if

$$L^+ = L. \tag{2.6}$$

The operator L is *symmetrizable* if there exists a real function $\rho(x, y)$ such that the operator $\rho(x, y)L$ is symmetric. As shown in [22], the properties (1), (2) (given that the functional σ is non-degenerate) imply the symmetrizability of the operator L.

Engelis [22] independently considered the same problem from a slightly different point of view and found the same classification scheme. In what follows we will use the Engelis scheme which is more convenient for our purposes. Before presenting the classification scheme given in [22], we recall some facts concerning admissible differential operators L [24], [25], [26].

The differential operator L in equation (2.2) is called *admissible* if for any positive integer n there exist $n + 1$ linearly independent polynomial eigenvalue solutions of degree n:

$$LQ_n^{(i)}(x, y) = \lambda_n Q_n^{(i)} \quad i = 0, 1, ..., n \tag{2.7}$$

and there are no polynomial solutions having degree less than n for the same value λ_n. It can be easily shown that the operator L is admissible if and only if the coefficients

$A(x,y),...,E(x,y)$ are of the form [26]

$$A(x,y) = \alpha x^2 + a_{10}x + a_{01}y + a_{00} \tag{2.8}$$

$$B(x,y) = \alpha xy + b_{10}x + b_{01}y + b_{00} \tag{2.9}$$

$$C(x,y) = \alpha y^2 + c_{10}x + c_{01}y + c_{00} \tag{2.10}$$

$$D(x,y) = \beta x + d_0 \quad E(x,y) = \beta y + e_0 \tag{2.11}$$

where $\alpha, \beta, a_{ik}, b_{ik}, c_{ik}, d_0, e_0; (i,k = 0,1)$ are arbitrary real parameters with the only restriction that $\alpha p + \beta \neq 0$ for $p = 0,1,2,....$ The eigenvalues are then

$$\lambda_n = n(\alpha(n-1) + \beta). \tag{2.12}$$

Note that for admissible polynomials, the eigenvalues are non-degenerate, i.e. $\lambda_n \neq \lambda_m$ for $n \neq m$.

2.2.2 Admissible operators and Integrable systems on spaces of constant curvature

There is an obvious geometrical interpretation of admissible operators (we follow [27]). First of all, perform a similarity transformation of the operator L with some function $\Phi(x,y)$:

$$\begin{aligned} \tilde{L} &= \Phi^{-1}(x,y)L\Phi(x,y) = A(x,y)\partial_{xx} + 2B(x,y)\partial_{xy} + \\ &\quad + C(x,y)\partial_{yy} + \tilde{D}(x,y)\partial_x + \tilde{E}(x,y)\partial_y + U(x,y) \end{aligned} \tag{2.13}$$

where

$$\tilde{D}(x,y) = D(x,y) + 2\frac{A(x,y)\Phi_x(x,y) + B(x,y)\Phi_y(x,y)}{\Phi(x,y)} \tag{2.14}$$

$$\tilde{E}(x,y) = E(x,y) + 2\frac{C(x,y)\Phi_y(x,y) + B(x,y)\Phi_x(x,y)}{\Phi(x,y)} \tag{2.15}$$

and

$$U(x,y) = \frac{A\Phi_{xx} + C\Phi_{yy} + 2B\Phi_{xy} + D\Phi_x + E\Phi_y}{\Phi(x,y)} \tag{2.16}$$

The operator $\tilde{L}(2.5)$ can be expressed in a form close to that of the Laplace-Beltrami operator associated with a metric $g_{ik}(x, y)$. Indeed, assume that a two-dimensional metric tensor $g_{ik}(x, y)$ is given.

The Laplace-Beltrami operator Δ_{LB} is defined as [20]

$$\Delta_{LB} = f(x, y)^{1/2}\partial_i g^{ik}(x, y) f(x, y)^{-1/2}\partial_k \tag{2.17}$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and

$$f(x, y) = det \parallel g^{ik}(x, y) \parallel = det \parallel g_{ik}(x, y) \parallel^{-1} \tag{2.18}$$

. From (2.17) we have

$$\Delta_{LB} = g^{11}\partial_{xx} + g^{22}\partial_{yy} + 2g^{12}\partial_{xy} + S_1(x, y)\partial_x + S_2(x, y)\partial_y \tag{2.19}$$

where

$$S_1(x, y) = \frac{\partial g^{11}}{\partial x} + \frac{\partial g^{21}}{\partial y} - \frac{1}{2} f^{-1}(x, y)(g^{11}\frac{\partial f(x, y)}{\partial x} + g^{12}\frac{\partial f(x, y)}{\partial y}) \tag{2.20}$$

and

$$S_2(x, y) = \frac{\partial g^{12}}{\partial x} + \frac{\partial g^{22}}{\partial y} - \frac{1}{2} f^{-1}(x, y)(g^{12}\frac{\partial f(x, y)}{\partial x} + g^{22}\frac{\partial f(x, y)}{\partial y}). \tag{2.21}$$

Comparing expression (2.5) for the operator $\tilde{L}$ with expression (2.19), it is natural to make the following identifications:

$$g^{11} = A(x, y), \qquad g^{12} = B(x, y), \qquad g^{22} = C(x, y). \tag{2.22}$$

on the complement of the locus

$$\{f(x, y) = A(x, y)C(x, y) - B^2(x, y) = 0\} \tag{2.23}$$

we have

$$\tilde{L} = \Delta_{LB} + T_1(x, y)\partial_x + T_2(x, y)\partial_y + U(x, y), \tag{2.24}$$

where

$$T_1(x,y) = \tilde{D}(x,y) - A_x(x,y) - B_y(x,y) + \frac{Bf_y(x,y) + Af_x(x,y)}{2f(x,y)} \tag{2.25}$$

and

$$T_2(x,y) = \tilde{E}(x,y) - B_x(x,y) - C_y(x,y) + \frac{Bf_x(x,y) + Cf_y(x,y)}{2f(x,y)}. \tag{2.26}$$

So, under the identification (2.22) we see that the operator $\tilde{L}$ coincides with the Laplace-Beltrami operator Δ_{LB} up to terms containing only first derivatives and the "potential" $U(x,y)$.

It is natural to ask whether a function $\Phi(x,y)$ exists such that the condition

$$T_1(x,y) = T_2(x,y) \equiv 0 \tag{2.27}$$

holds, and hence such that

$$\tilde{L} = \Delta_{LB} + U(x,y). \tag{2.28}$$

It is well known [20] that the Laplace-Beltrami operator Δ_{LB} plays the role of the Hamiltonian for free-motion of a quantum mechanical particle on a Riemannian space with the metric $g_{ik}(x,y)$. Condition (2.27) would mean that the operator $\tilde{L}$ coincides with the Schroedinger operator on this Riemannian space with the potential $U(x,y)$.

If condition (2.27) cannot be satisfied, the operator $\tilde{L}$ can be interpreted quantum mechanically as containing *a magnetic field* term. Therefore, (2.27) reflects the absence of magnetic fields. It is easily seen [27] that it is equivalent to the condition:

$$\partial_y \left(\frac{CK_1 - BK_2}{f} \right) = \partial_x \left(\frac{AK_2 - BK_1}{f} \right) \tag{2.29}$$

where

$$K_1(x,y) = \frac{4A_x f + 4B_y f - Af_x - Bf_y - D}{4f} \tag{2.30}$$

$$K_2(x,y) = \frac{4B_x f + 4C_y f - Bf_x - Cf_y - E}{4f} \tag{2.31}$$

We call the ten parameters $\alpha, a_{ik}, b_{ik}, c_{ik}$ *internal parameters*, since these parameters define the metric tensor $g_{ik}(x,y)$ and hence describe the geometrical properties of the operator L. The remaining three parameters β, d_0, e_0 will be called *external parameters.* These parameters describe the interaction of our system with external fields.

Of course, it is possible to further reduce the number of independent internal parameters by means of affine transformations of the independent arguments x, y. We will describe this procedure following [26].

Consider all invertible affine transformations of the form

$$x = q_{11}\xi + q_{12}\eta + q_{10} \tag{2.32}$$

$$y = q_{21}\xi + q_{22}\eta + q_{20} \tag{2.33}$$

with some coefficients q_{ik}. It is easily shown [26] that if L is admissible in coordinates x, y, then L remains admissible in the new coordinates ξ, η. Moreover, property (2) and the symmetrizability of the operator L are also preserved under the transformation (2.32). Property (2.27) is also preserved under affine transformations. This means that if the operator L is reduced to the form (2.28) without magnetic field then the affine-transformed operator L can also be reduced to the same form.

The parameter α is preserved under affine transformations. Hence we can make a separation into two cases: $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$ we can put $\alpha = 1$ without loss of generality, by just dividing both sides of equation (2.1) by α and redefining the eigenvalue accordingly. This just rescales the remaining nine internal and three external parameters. Since the affine transformation (2.32) contains six independent parameters, it is possible to reduce the nine internal parameters a_{ik}, b_{ik}, c_{ik} to three independent ones. We thus obtain a separation of the admissible operators L into two classes: those with $\alpha \neq 0$ and those with $\alpha = 0$; with each class containing six independent parameters, three of them internal and three external.

The characteristic determinant

$$f(x,y) = det \parallel g^{ik}(x,y) \parallel = A(x,y)C(x,y) - B^2(x,y) \tag{2.34}$$

plays a crucial role in the classification of all possible distinct cases of admissible operators (for details see, e.g., [26]).

We can formulate the main result of [22] as follows.

Theorem 2.1 *If the operator L is admissible and there exists a nondegenerate functional σ, then the operator L is symmetrizable. Moreover, up to affine transformation, there exist nine distinct types of L:*

(I) $A(x,y) = x^2 - x, B(x,y) = xy, C(x,y) = y^2 - y, D(x,y) = \beta x + d_0,$
$E(x,y) = \beta y + e_0,$ *the characteristic determinant is* $f(x,y) = xy(1-x-y)$.

(II) $A(x,y) = x^2, B(x,y) = xy, C(x,y) = y^2 - y,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = x^2 y.$

(III) $A(x,y) = x^2, B(x,y) = xy, C(x,y) = y^2 + x,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = x^3.$

(IV) $A(x,y) = x, B(x,y) = 0, C(x,y) = y,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = xy.$

(V) $A(x,y) = 0, B(x,y) = x, C(x,y) = y,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = x^2.$

(VI) $A(x,y) = x, B(x,y) = 0, C(x,y) = 1,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = x.$

(VII) $A(x,y) = 1, B(x,y) = 0, C(x,y) = 1,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = 1.$

(VIII) $A(x,y) = y, B(x,y) = 1, C(x,y) = 0,$
$D(x,y) = \beta x + d_0, E(x,y) = \beta y + e_0, f(x,y) = 1.$

(IX) $A(x,y) = x^2 - 1, B(x,y) = xy, C(x,y) = y^2 - 1,$
$D(x,y) = \beta x, E(x,y) = \beta y, f(x,y) = 1 - x^2 - y^2.$

As was shown in [27], in all nine cases the Krall-Sheffer-Engelis (KSE) operators L can be transformed into a form in which they describe integrable quantum mechanical systems on spaces of constant curvature without magnetic field.

Direct computation [27] yields the following

Proposition 2.1 *Condition (2.29) holds for every case (I)-(IX) of the KSE classification scheme. Hence every case can be transformed to a quantum system on two-dimensional manifolds with some potential $U(x, y)$ without magnetic field.*

Our next step will be to find the mean Riemannian curvature $\kappa(x, y)$ corresponding to the metric $g_{ik}(x, y)$. The Riemannian curvature $\kappa(x, y)$ is calculated from the components $g_{ik}(x, y)$ of the metric using standard formulae, see [23] :

$$\kappa = g^{\lambda\mu} R_{\lambda\mu} = g^{\lambda\mu} R^{\alpha}_{\lambda\mu\alpha} \tag{2.35}$$

where $R_{\lambda\mu} = R^{\alpha}_{\lambda\mu\alpha}$ is the Ricci tensor, and $R^{\alpha}_{\beta\lambda\mu}$ is the Riemann tensor which is expressed through Cristoffel symbols of the second kind as follows:

$$R^{\alpha}_{\beta\lambda\mu} = \partial_{\lambda}\Gamma^{\alpha}_{\beta\mu} - \partial_{\mu}\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\delta}_{\beta\mu}\Gamma^{\alpha}_{\delta\lambda} - \Gamma^{\delta}_{\beta\lambda}\Gamma^{\alpha}_{\delta\mu} \tag{2.36}$$

which in turn are defined by components of the inverse metric tensor:

$$\Gamma^{\alpha}_{\beta\lambda} = \frac{1}{2} g^{\sigma\alpha}(\partial_{\lambda} g^{\beta\sigma} + \partial_{\beta} g^{\lambda\sigma} - \partial_{\sigma} g^{\beta\lambda}) \tag{2.37}$$

For the case of the above metrics (2.22) we arrive at the following:

Proposition 2.2 *The mean Riemannian curvature is constant for every case (I)-(IX) of the KSE classification scheme. More precisely, the cases (IV)-(VIII) correspond to zero curvature, whereas the cases (I)-(III) and (IX) correspond to a nonzero curvature.*

For details and examples of the corresponding quantum systems see [27]. Thus all nine types correspond to some quantum mechanical systems describing the motion of a particle in the presence of some scalar potentials on two-dimensional spaces of constant curvature, without magnetic field present.

2.2.3 Krall-Sheffer classification and Superintegrability

We now present the main result of this section, which is that all nine types in the KSE classification correspond to *superintegrable* systems (see [28], [29]). This means that there exist two algebraically independent operators I_1 and I_2 commuting with the operator $L : [L, I_1] = [L, I_2] = 0$. The operators $I_{1,2}$, which are exhibited in Theorem 2.2 below act on the space of polynomials in two variables and preserve their degree.

Theorem 2.2 *For the nine types in the KSE classification scheme the algebraically independent integrals I_1, I_2 commuting with the operator L are*

Case (I).

$$I_1 = x(1-x-y)\partial_{xx} + (d_0(y-1)(\beta+e_0)x)\partial_x$$

$$I_2 = y(1-x-y)\partial_{yy} + (e_0(x-1)(\beta+d_0)y)\partial_y.$$

Case (II).

$$I_1 = x^2\partial_{xx} + ((\beta+e_0)x + d_0(1-y))\partial_x$$

$$I_2 = xy\partial_{yy} + (d_0y - e_0x)\partial_y.$$

Case (III).

$$I_1 = x^2\partial_{yy} + (e_0x - d_0y)\partial_y$$

$$I_2 = 2x^2\partial_{xy} + xy\partial_{yy} + (e_0x - d_0y)\partial_x + (\beta x + d_0)\partial_y.$$

Case (IV).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$

$$I_2 = -xy(\partial_x - \partial_y)^2 + (d_0y - e_0x)\partial_x + (e_0x - d_0y)\partial_y.$$

Case (V).

$$I_1 = x^2\partial_{xx} + (e_0x - d_0y)\partial_x$$

$$I_2 = x\partial_{yy} + (\beta x + d_0)\partial_y.$$

Case (VI).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$

$$I_2 = \partial_{yy} - (e_0 + \beta y)\partial_y.$$

Case (VII).

$$I_1 = -\partial_{xx} + (\beta x + d_0)\partial_x$$

$$I_2 = (d_0 + \beta x)\partial_y - (\beta y + e_0)\partial_x.$$

Case (VIII).

$$I_1 = \partial_{xx} + (\beta y + e_0)\partial_x$$

$$I_2 = (x - y^2)\partial_{xx} - 2y\partial_{xy} - \partial_{yy} + (e_0 x - d_0 y)\partial_x - (\beta x + d_0)\partial_y.$$

Case (IX).

$$I_1 = x\partial_y - y\partial_x$$

$$I_2 = (1 - x^2 - y^2)\partial_{xy} + (1 - \beta)x\partial_y.$$

Note that in cases (VII) and (IX) there exist integrals of first order. In all other cases we have integrals of second order with respect to the derivatives. Thus all types (I)-(IX) correspond to superintegrable systems. Recall that a two-dimensional system is called integrable if there exists at least one integral commuting with the Hamiltonian. Superintegrable systems (with two algebraically independent integrals) form a subclass of integrable systems.

Our next statement is based on a direct computation of second-order integrals for admissible operators. It appears that the existence of such integrals imposes additional restrictions on the values of intrinsic parameters. We saw that affine transformations allow one to reduce the number of intrinsic parameters to three. Direct calculations show that the existence of commuting integrals reduces this number to zero. That is, there are three additional conditions fixing these three parameters. This leads (up to affine transformations) to the same coefficients $A(x, y), B(x, y), C(x, y)$ as in the Krall-Sheffer classification scheme. So, we conclude:

Conclusion. *The existence of a nondegenerate orthogonality functional for an admissible operator L is equivalent to the existence of a second-order integral I commuting with $L : [L, I] = 0$.*

The meaning of this is that all nine cases in the KSE scheme can be equally characterized by the existence of at least one second-order integral. Note that superintegrability (i.e. the existence of a second independent integral) is obtained as a by-product during the proof of this result.

2.3 R-matrix approach to the Krall-Sheffer problem.

In this section we show how to construct a complete set of commuting invariants to the integrable systems arising in the Krall-Sheffer framework using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. We give both the quantum and classical formulations in terms of Lax matrices depending on a loop parameter. The main construction is based on the well-known procedure of symmetry reduction from a free system in a higher dimension space (in particular, quadrics in $\mathbb{R}^6$ or $\mathbb{C}^6$). Classically this corresponds to reduction of geodesic flow, while quantum mechanically it involves reduction of the Laplacian. The reduction process leaves a residue of the original system, providing a complete set of commuting integrals.

We show how the superintegrability of the systems obtained can be interpreted from the presence of multiple parameters in the rational Lax matrix representation. This is also related to the fact that such systems admit a separation of variables in parametric families of coordinate systems.

2.3.1 Scheme of symplectic reduction.

We begin with a parametrization for general rational Lax matrix $\mathcal{N}(\lambda)$ vanishing at $\lambda = \infty$, having n-poles of degree m_i at n points $\alpha_1, \dots \alpha_n$ as follows:

$$\mathcal{N}(\lambda) := \frac{1}{2}(Y^T, -X^T J)(\lambda - A)^{-1}(X, JY) = \sum_{i=1}^{n} \sum_{a=1}^{m_i} \frac{N^a{}_i}{(\lambda - \alpha_i)^a} \tag{2.38}$$

where X and Y are a pair of $\sum_{i=1}^{n} m_i$ dimensional vectors (either real or complex) whose components $(x_i, y_i)_{i=1\ldots\sum_{i=1}^{n} m_i}$ are canonically conjugate variables.

A and J are fixed $\sum_{i=1}^{n} m_i \times \sum_{i=1}^{n} m_i$ matrices with A having either n distinct eigenvalues $\{\alpha_i\}_{i=1..n}$ and minimal polynomial

$$\prod_{i=1}^{n}(\lambda - \alpha_i)^{m_i} \tag{2.39}$$

and J is a symmetric real matrix with antidiagonal blocks of the form

$$\begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & \ldots & 1 & 0 \\ \ldots & & & \\ 1 & \ldots & & \end{pmatrix}$$

for each Jordan block of A. The cases that will be of interest below will be when $n = 1, 2$ or 3 and $\sum_{i=1}^{n} m_i = 6$.

The dynamics are generated by Hamiltonians chosen from the algebra of spectral invariants of $N(\lambda)$. Classically, these Poisson commute and hence generate isospectral flows satisfying a Lax equation:

$$\frac{d\mathcal{N}}{dt} = [B, \mathcal{N}] \tag{2.40}$$

It is easily verified that $N(\lambda)$ satisfies the standard rational R-matrix Poisson bracket relations:

$$\{\mathcal{N}(\lambda) \otimes \mathcal{N}(\mu)\} = [r(\lambda), \mathcal{N}(\lambda) \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{N}(\mu)] \tag{2.41}$$

where both sides are viewed, for fixed $\lambda \neq \mu$ as elements of $End(\mathbb{C}^6 \otimes \mathbb{C}^6)$ and

$$r(\lambda) = \frac{P_{1,2}}{(\lambda - \mu)} \qquad P_{1,2}(u \otimes v) = v \otimes u \tag{2.42}$$

In the cases considered below, we only study Hamiltonians that are $O(6, J)$ invariant and restrict to the quadric defined by

$$X^T J X = 1. \tag{2.43}$$

Quotienting by the stabilizer $G_A \subset O(6,J)$ of A we reduce to a 2-dimensional configuration space, however the reduced system is no longer free. Note, that this R-matrix scheme also leads to separation of variables for special "sets of spectral Darboux coordinates", see [32].

The algebra of spectral invariants is generated by the coefficients of:

$$-\frac{1}{2}Tr\mathcal{N}(\lambda)^2 = \sum_{i=1}^{n}\sum_{d=1}^{2m_i}\frac{H_{i,d}}{(\lambda-\alpha_i)^d} \tag{2.44}$$

with $2m_i \leq n_i$.The numerators $H_{i,d}$ of this partial fraction expansion all Poisson commute and generate the algebra of spectral invariants. They are not all independent, however, since:

$$\sum_{i=1}^{n} H_{i,d} = 0 \tag{2.45}$$

and $H_{i,d}$ with $m_i < d \leq 2m_i$ are Casimir invariants.

The connection between configuration space coordinates in 6-dimensional space and the separating coordinates λ_1, λ_2 in the reduced 2-dimensional space is given by

$$X^T J(\lambda - A)^{-1}X = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{a(\lambda)} \tag{2.46}$$

where $a(\lambda)$ is the minimal polynomial of the matrix A.

The quantum version of this approach is simply obtained through canonical quantization with conjugate (momentum) variables y_j replaced by the partial derivatives $-i\ \partial/\partial x_j$, see [38], [39]. The relation between the quantum integrals and the ones in the corresponding Krall-Sheffer cases is obtained applying a suitable gauge transformation.

2.3.2 2×2 rational Lax matrices

In the following, we shall limit our discussion to the case of 2×2 Lax matrices, although most of the considerations that follow are easily extended to higher rank, see [30]-[35]. We may without loss of generality take $\mathcal{N}(\lambda)$ to be traceless (since the trace coefficients are Casimirs)

$$\mathcal{N}(\lambda) = \begin{pmatrix} h(\lambda) & e(\lambda) \\ f(\lambda) & -h(\lambda) \end{pmatrix} , \tag{2.47}$$

where the rational functions $e(\lambda), f(\lambda), h(\lambda)$ satisfy the Poisson bracket relations

$$\begin{aligned} \{h(\lambda),\ e(\mu)\} &= \frac{e(\lambda) - e(\mu)}{\lambda - \mu} \\ \{h(\lambda),\ f(\mu)\} &= -\frac{f(\lambda) - f(\mu)}{\lambda - \mu} \\ \{e(\lambda),\ f(\mu)\} &= -2\frac{h(\lambda) - h(\mu)}{\lambda - \mu} . \end{aligned} \tag{2.48}$$

For this case, the ring $\mathcal{I}(\tilde{gl}(2))$ of spectral invariants, when restricted to the symplectic leaves of the R-matrix Poisson structure, is generated by the quadratic trace invariants; i.e., the coefficients determining the numerator of the rational function

$$\Delta(\lambda) := -\frac{1}{2}Tr(\mathcal{N}^2(\lambda)) = h^2(\lambda) - \frac{1}{2}\left(e(\lambda)f(\lambda) + f(\lambda)e(\lambda)\right) . \tag{2.49}$$

(The order in the last two terms is irrelevant of course, but it is written here in a form that will also be valid in the quantum version below.) If, for example, the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ is taken to vanish, and only first order poles appear in $\mathcal{N}(\lambda)$, we have

$$\begin{aligned} e(\lambda) &:= \sum_{i=1}^{n} \frac{e_i}{\lambda - \alpha_i} \\ f(\lambda) &:= \sum_{i=1}^{n} \frac{f_i}{\lambda - \alpha_i} \\ h(\lambda) &:= \sum_{i=1}^{n} \frac{h_i}{\lambda - \alpha_i}, \end{aligned} \tag{2.50}$$

where the quantities $\{e_i, f_i, h_i\}_{i=1\ldots n}$ are a set of n $\mathbf{sl}(2)$ generators, which may be canonically coordinatized as:

$$e_i := \frac{1}{2}\left(y_i^2 + \frac{\mu_i^2}{x_i^2}\right)$$

$$\begin{aligned} f_i &:= \frac{1}{2}x_i^2 \\ h_i &:= \frac{1}{2}x_i y_i, \qquad i = 1, \ldots n, \end{aligned} \tag{2.51}$$

where $\{\mu_i^2\}_{i=1\ldots n}$ are the values of the $\mathbf{sl}(2)$ Casimir invariants and $\{x_i, y_i\}_{i=1\ldots n}$ form a set of canonical coordinates on the symplectic leaves .

2.3.3 Parametric dependence of invariants and superintegrability

Again, taking the case when the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ vanishes (but not necessarily just first order poles), a complete set of generators is given by

$$\phi_{ia} := Res_{\lambda=\alpha_i}(\lambda - \alpha_i)^a Tr(\mathcal{N}^2(\lambda)), \qquad i = 1, \ldots n, \ a = 0, \ldots n_i - 1 \ . \tag{2.52}$$

These Poisson commute amongst themselves, but they each depend upon the pole locations $\{\alpha_i\}_{i=1\ldots n}$ in $\mathcal{N}(\lambda)$. However, the linear combination:

$$\phi_{SI} := \sum_{i=1}^{n} \alpha_i \phi_{i0} = Res_{\lambda=\infty} Tr(\mathcal{N}^2(\lambda)) \tag{2.53}$$

does not depend on the α_i's. In general, there is no reason for the invariants $\phi_{ia}(\alpha_i)$ to commute with each other for different choices of the α_i's. But, regardless of the values chosen, they will commute with ϕ_{SI}. Since the $\phi_{ia}(\alpha_i)$'s for different choices of α_i's in general do not generate the same algebra of functions, we may conclude that, taken together, for different evaluations of the parameters $\{\alpha_i\}$, there are more functionally independent integrals that Poisson commute with ϕ_{SI} than half the dimension of the symplectic leaf, and hence the Hamiltonian system it generates is superintegrable. (In fact, in most cases, it may be shown to be maximally superintegrable; see the examples below.)

In particular, if we take the case of purely simple poles as above in (2.50), the resulting Hamiltonian is:

$$\phi_{SI} = \frac{1}{2}\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2 - \frac{1}{2}(\sum_{i=1}^{n} x_i y_i)^2 + \frac{1}{2}\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \frac{\mu_i^2}{x_i^2} \ , \tag{2.54}$$

which, when constrained to the (co)tangent bundle of the $n-1$ sphere S^{n-1}

$$\sum_{i=1}^{n} x_i^2 = 1, \qquad \sum_{i=1}^{n} x_i y_i = 0 \ , \tag{2.55}$$

yields the superintegrable system

$$h_{\mathrm{Ros}} = \frac{1}{2}\sum_{j=1}^{n} y_j^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{\mu_i^2}{x_i^2} \ , \tag{2.56}$$

which is the trivial case of the Rosochatius system (without a harmonic oscillator potential).

2.3.4 Separation of variables

Another viewpoint that helps to explain the superintegrability of systems arising in this way is to note that they may be completely separated in a canonical coordinate system determined by the values of the pole parameters $\{\alpha_i\}$ which, for the $\mathbf{sl}(2)$ case with simple poles, with the phase space constrained to S^{n-1} as above, reduces to the sphero-conical system $\{\lambda_i, \zeta_i\}_{i=1\ldots n-1}$ defined by:

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{j=1}^{n-1}(\lambda - \lambda_j)}{\prod_{i=1}^{n}(\lambda - \alpha_i)}, \qquad \zeta_i := \frac{1}{2}\sum_{i=1}^{n} \frac{x_i y_i}{(\lambda - \alpha_i)} \ . \tag{2.57}$$

These are just the points (λ_i, ζ_i) on the invariant spectral curve

$$\zeta^2 + \frac{1}{2}\Delta(\lambda) = 0 \tag{2.58}$$

where the matrix element $f(\lambda)$ vanishes and $\zeta_i = h(\lambda_i)$ are the eigenvalues at these points. These are particular cases of the spectral *Darboux coordinates* of [10], [11]. (Note that these become hyperellipsoidal coordinates if there is a constant term added in the definition (2.50) of $f(\lambda)$.)

The point to note is that the separation of variables occurs in these coordinates simultaneously for *all* the invariants ϕ_{ia}, viewed as generators of Hamiltonian flows. But again, since the leading term spectral invariant ϕ_{SI} does not depend on the values of the parameters α_i, it admits a separation of variables in *any* of the family of spheroconical (or hyperellipsoidal) coordinates obtained by varying these parameters. This simultaneous separability in multiple coordinates may be viewed as an alternative explanation of the origin of the superintegrability of such systems. (In fact, both these viewpoints are a result of the classical r-matrix setting, and in a sense may be considered as equivalent.)

In the examples given below in the following section, the same principle is used to deduce superintegrable systems from $\mathbf{sl}(2)$ Lax matrices satisfying the Poisson bracket relations (2.41).

2.3.5 Quantum integrable systems

The above discussion is easily extended to the canonically quantized version of such systems. All that must be done is to replace the matrix elements defining $\mathcal{N}(\lambda)$ by their quantized forms $\hat{e}(\lambda), \hat{f}(\lambda), \hat{h}(\lambda)$, which must satisfy the commutator analogs of the Poisson bracket relations (2.48)

$$\begin{aligned} [\hat{h}(\lambda),\ \hat{e}(\mu)] &= \frac{\hat{e}(\lambda)-\hat{e}(\mu)}{\lambda-\mu} \\ [\hat{h}(\lambda),\ \hat{f}(\mu)] &= -\frac{\hat{f}(\lambda)-\hat{f}(\mu)}{\lambda-\mu} \\ [\hat{e}(\lambda),\ \hat{f}(\mu)] &= -2\frac{\hat{h}(\lambda)-\hat{h}(\mu)}{\lambda-\mu}, . \end{aligned} \tag{2.59}$$

These can be realized by canonical quantization of the underlying classical phase space variables. For example, in the case of simple poles only, with vanishing polynomial term $\mathcal{B}(\lambda)$, we have:

$$\begin{aligned} \hat{e}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{e}_i}{\lambda-\alpha_i} \\ \hat{f}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{f}_i}{\lambda-\alpha_i} \\ \hat{h}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{h}_i}{\lambda-\alpha_i}, \end{aligned} \tag{2.60}$$

where the $\mathbf{sl}(2)$ generators $\{\hat{e}_i, \hat{f}_i, \hat{h}_i\}$ may be represented by the operators

$$\begin{aligned} \hat{e}_i &:= \frac{1}{2}\left(\frac{\partial^2}{\partial x_i^2} - \frac{\mu_i^2}{x_i^2}\right) \\ \hat{f}_i &:= \frac{1}{2}x_i^2 \\ \hat{h}_i &:= \frac{1}{2}\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right), \qquad i = 1, \dots n, \end{aligned} \tag{2.61}$$

and the commuting invariants are similarly given by the coefficients of the numerator polynomial of the quantum spectral invariant:

$$\hat{\Delta}(\lambda) := \hat{h}^2(\lambda) - \frac{1}{2}\left(\hat{e}(\lambda)\hat{f}(\lambda) + \hat{f}(\lambda)\hat{e}(\lambda)\right) . \tag{2.62}$$

The resulting systems are similarly quantum integrable, and separable in the same coordinates as the classical ones [33] and, for the same reasons as above, the quantum version of the Hamiltonian ϕ_{SI} is superintegrable.

In the following section, a number of examples of such classical and quantum superintegrable systems will be given and related to the Krall-Sheffer classification of the previous sections.

2.4 Examples of superintegrable classical and quantum systems

2.4.1 Case 1. Sphere. Neumann-Rosochatius system

Classical Lax Matrix

In the case of a sphere in $\mathbb{R}^6$, the matrices A and J are just:

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix} \qquad J = id$$

with $\alpha \neq \beta \neq \gamma$. The symmetry algebra g_A corresponding to the stabilizer $G_A \subset O(6, \mathbb{R}))$ is a maximal torus with generators

$$\{x_1 y_2 - x_2 y_1,\ x_3 y_4 - x_4 y_3,\ x_5 y_6 - x_6 y_5\}. \tag{2.63}$$

The Lax matrix in the first case has three simple poles and vanishing $B(\lambda)$:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \beta)} + \frac{N_3}{(\lambda - \gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix} \tag{2.64}$$

where the N_i are elements of $sl(2)$

$$N_1 = \frac{1}{2} \begin{pmatrix} x_1 y_1 + x_2 y_2 & {y_1}^2 + {y_2}^2 \\ -{x_1}^2 - {x_2}^2 & -x_1 y_1 - x_2 y_2 \end{pmatrix}$$

$$N_2 = \frac{1}{2} \begin{pmatrix} x_3 y_3 + x_4 y_4 & {y_3}^2 + {y_4}^2 \\ -{x_3}^2 - {x_4}^2 & -x_3 y_3 - x_4 y_4 \end{pmatrix}$$

$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & {y_5}^2 + {y_6}^2 \\ -{x_5}^2 - {x_6}^2 & -x_5y_5 - x_6y_6 \end{pmatrix}$$

In terms of the reduced ambient coordinates the matrices N_i read:

$$N_1 = \frac{1}{2}\begin{pmatrix} s_1p_1 & {p_1}^2 + \frac{\mu_1^2}{s_1^2} \\ -{s_1}^2 & -s_1p_1 \end{pmatrix} \tag{2.65}$$

$$N_2 = \frac{1}{2}\begin{pmatrix} s_2p_2 & {p_2}^2 + \frac{\mu_2^2}{s_2^2} \\ -{s_2}^2 & -s_2p_2 \end{pmatrix} \tag{2.66}$$

$$N_3 = \frac{1}{2}\begin{pmatrix} s_3p_3 & {p_3}^2 + \frac{\mu_3^2}{s_3^2} \\ -{s_3}^2 & -s_3p_3 \end{pmatrix} \tag{2.67}$$

Their matrix elements as before generate a Poisson bracket realization of $(\mathbf{sl}(2))^3$.

Here (p_1, p_2, p_3) are canonically conjugate to (s_1, s_2, s_3) (and these coincide with the coordinates $\{x_i, y_i\}_{i=1...n}$ above).

Commuting invariants

The invariants are the coefficients of:

$$-\frac{1}{2}TrN(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{H_3}{(\lambda-\gamma)} + \frac{{\mu_1}^2}{(\lambda-\alpha)^2} + \frac{{\mu_2}^2}{(\lambda-\beta)^2} + \frac{{\mu_3}^2}{(\lambda-\gamma)^2} \tag{2.68}$$

Here μ_1, μ_2 and μ_3 are constants defining the restriction to level sets of invariants of motion under the reduction procedure (the components of the moment map generating the torus action), namely:

$$\mu_1 = x_1y_2 - x_2y_1, \ \mu_2 = x_3y_4 - x_4y_3, \ \mu_3 = x_5y_6 - x_6y_5 \tag{2.69}$$

The integrals H_1, H_2 and H_3 are not all independent, since their sum is equal to zero. The Hamiltonian of the problem is given by the linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3. \tag{2.70}$$

The constraint to a sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ is given by $X^T J X = 1$:

$$x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 + x_5{}^2 + x_6{}^2 = 1. \tag{2.71}$$

The reduced ambient coordinates are given by the radial distance in three planes (X_1, X_2), (X_3, X_4) and (X_5, X_6):

$$s_1{}^2 = x_1{}^2 + x_2{}^2 \tag{2.72}$$

$$s_2{}^2 = x_3{}^2 + x_4{}^2 \tag{2.73}$$

$$s_3{}^2 = x_5{}^2 + x_6{}^2 \tag{2.74}$$

The reduction of the constraint gives

$$s_1{}^2 + s_2{}^2 + s_3{}^2 = 1. \tag{2.75}$$

The reduced Hamiltonian is:

$$H = \frac{1}{2}(p_1{}^2 + p_2{}^2 + p_3{}^2) + \frac{\mu_1{}^2}{s_1{}^2} + \frac{\mu_2{}^2}{s_2{}^2} + \frac{\mu_3{}^2}{s_3{}^2}. \tag{2.76}$$

This describes the Rosochatius system, with harmonic oscillator terms absent, on the cotangent bundle of a two-sphere in $\mathbb{R}^3$. Here (p_1, p_2, p_3) are canonical conjugate to (s_1, s_2, s_3)

In terms of the reduced ambient space coordinates the integrals H_1, H_2 and H_3 are:

$$H_1 = -\frac{1}{2}\frac{L_{13}{}^2 + \mu_3{}^2 s_1{}^2/s_3{}^2 + \mu_1{}^2 s_3{}^2/s_1{}^2}{\alpha - \gamma} - \frac{1}{2}\frac{L_{12}{}^2 + \mu_1{}^2 s_2{}^2/s_1{}^2 + \mu_2{}^2 s_1{}^2/s_2{}^2}{\alpha - \beta} \tag{2.77}$$

$$H_2 = -\frac{1}{2}\frac{L_{23}{}^2 + \mu_3{}^2 s_2{}^2/s_3{}^2 + \mu_2{}^2 s_3{}^2/s_2{}^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{12}{}^2 + \mu_1{}^2 s_2{}^2/s_1{}^2 + \mu_2{}^2 s_1{}^2/s_2{}^2}{\alpha - \beta} \tag{2.78}$$

$$H_3 = \frac{1}{2}\frac{L_{23}{}^2 + \mu_3{}^2 s_2{}^2/s_3{}^2 + \mu_2{}^2 s_3{}^2/s_2{}^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{13}{}^2 + \mu_3{}^2 s_1{}^2/s_3{}^2 + \mu_1{}^2 s_3{}^2/s_1{}^2}{\alpha - \gamma} \tag{2.79}$$

Where $L_{ij} = s_1 p_2 - s_2 p_1$.

Separating coordinates

The reduced separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates. The corresponding momenta are denoted (ξ_1, ξ_2) . They are related to (s_1, s_2, s_3) and (p_1, p_2, p_3) by :

$$ {s_1}^2 = \frac{(\alpha - \lambda_1)(\alpha - \lambda_2)}{(\alpha - \beta)(\alpha - \gamma)} \qquad \xi_1 = -\frac{1}{2}\left(\frac{s_1 p_1}{\lambda_1 - \alpha} + \frac{s_2 p_2}{\lambda_1 - \beta} + \frac{-s_1 p_1 - s_2 p_2}{\lambda_1 - \gamma}\right) \tag{2.80} $$

$$ {s_2}^2 = \frac{(\beta - \lambda_1)(\beta - \lambda_2)}{(\beta - \alpha)(\beta - \gamma)} \qquad \xi_2 = -\frac{1}{2}\left(\frac{s_1 p_1}{\lambda_2 - \alpha} + \frac{s_2 p_2}{\lambda_2 - \beta} + \frac{-s_1 p_1 - s_2 p_2}{\lambda_2 - \gamma}\right) \tag{2.81} $$

$$ {s_3}^2 = \frac{(\gamma - \lambda_1)(\gamma - \lambda_2)}{(\gamma - \alpha)(\gamma - \beta)} \tag{2.82} $$

Quantum system

The quantum versions of the integrals above, denoted $\hat{H}_1, \hat{H}_2, \hat{H}_3$, are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators, $\hat{e}(\lambda), \hat{f}(\lambda), \hat{h}(\lambda)$, which in the case of simple poles are as in (2.60)-(2.62).

The quantization procedure leads to replacing the L_{ij}'s by their quantum version:

$$ \hat{L}_{ij} = \sqrt{-1}(s_i \partial/\partial s_j - s_j \partial/\partial s_i) \tag{2.83} $$

Introducing the functions

$$ \omega_{jk}^2 := {\mu_j}^2 {s_k}^2/{s_j}^2 + {\mu_k}^2 {s_j}^2/{s_k}^2 \qquad j, k = 1..3 \tag{2.84} $$

and denoting $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \alpha_3$, we can present the quantum integrals as

$$ \hat{H}_i = -\frac{1}{2} \sum_{k \neq i} \frac{\hat{L}_{ik} + \omega_{ik}^2}{\alpha_i - \alpha_k} \qquad i, k = 1..3 \tag{2.85} $$

The quantum Hamiltonian is

$$ \hat{H} = -\frac{1}{2}({\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2} + \frac{{\mu_3}^2}{{s_3}^2}. \tag{2.86} $$

The separating coordinates are the configuration space part of the ones for the classical case (λ_1, λ_2).

Note that whereas the Hamiltonian H is independent of the parameters (α, β, γ), which only serve to determine the separating coordinate system. the invariants H_1, H_2 individually do depend on those. Therefore, different choices for these parameters give distinct integrals that commute with H, but do not commute with each other. This provides an explanation for the superintegrability of this system.

To relate the invariants to the ones obtained in [28] for the corresponding Krall-Sheffer case we apply the gauge transformation consisting of conjugation by the function:

$$\Phi = x^{d_1} y^{d_2} (1 - x - y)^{d_3} \tag{2.87}$$

where

$$d_1 = \frac{1}{2}(d_{00} + 1/2) \qquad d_2 = \frac{1}{2}(e_{00} + 1/2) \qquad d_3 = \frac{1}{2}(1/2 - d_{00} - e_{00} - B) \tag{2.88}$$

and d_{00}, e_{00}, B are the parameters appearing in Krall-Sheffer setting. (See [28]). The following are the relations between the integrals constructed in these two approaches:

$$\tilde{H}_1 = 4\frac{\alpha_1 - \gamma_1}{\beta_1 - \gamma_1}\hat{I}_x + 4\hat{I}_y - 4\hat{L} - c_0, \qquad \tilde{H}_2 = 4\frac{\gamma_1 - \beta_1}{\gamma_1 - \alpha_1}\hat{I}_y + 4\hat{I}_x - 4\hat{L} - c_1, \tag{2.89}$$

where $\tilde{H}_i = \Phi \hat{H} \Phi^{-1}$ and $\hat{L}$ is the Krall-Sheffer operator corresponding to case I, c_0 and c_1 depend on $\alpha, \beta, \gamma, d_{00}, e_{00}, B$.

2.4.2 Case 2. Hyperboloid.

Classical Lax Matrix

For the case of a hyperboloid embedded in $\mathbb{R}^6$, the matrices (A, J) may be taken as

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that J has an antidiagonal block corresponding to each jordan block of A and a diagonal block corresponding to the diagonal part of A.

The symmetry algebra g_A again has three generators

$$\{x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4,\ x_2y_1 - x_4y_3,\ x_5y_6 - x_6y_5\} \tag{2.90}$$

but the Lax matrix now has one first order and one second order pole, the latter at $\lambda = \alpha$:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \alpha)^2} + \frac{N_3}{(\lambda - \beta)}. \tag{2.91}$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 & 2y_1y_4 + 2y_2y_3 \\ -2x_1x_4 - 2x_2x_3 & -x_1y_1 - x_2y_2 1 x_3y_3 - x_4y_4 \end{pmatrix} \tag{2.92}$$

$$N_2 = \frac{1}{2}\begin{pmatrix} -x_4y_3 + x_2y_1 & -2y_3y_1 \\ 2x_2x_4 & -x_2y_1 + x_4y_3 \end{pmatrix} \tag{2.93}$$

$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & {y_5}^2 + {y_6}^2 \\ -{x_5}^2 - {x_6}^2 & -x_5y_5 - x_6y_6 \end{pmatrix} \tag{2.94}$$

Symplectic reduction transforms these into:

$$N_1 = \frac{1}{2}\begin{pmatrix} s_1p_1 + s_2p_2 & 2p_1p_2 + 2\gamma_1\gamma_2 \\ -2s_1s_2 & -s_1p_1 - s_2p_2 \end{pmatrix} \tag{2.95}$$

$$N_2 = \frac{1}{2}\begin{pmatrix} -s_2p_1 & -{p_1}^2 - \gamma_2^2 \\ {s_2}^2 & s_2p_1 \end{pmatrix} \tag{2.96}$$

$$N_3 = \frac{1}{2}\begin{pmatrix} s_3p_3 & {p_3}^2 + \gamma_3^2 \\ -{s_3}^2 & -s_3p_3 \end{pmatrix} \tag{2.97}$$

Here we have introduced the following notations

$$2\gamma_1\gamma_2 := \frac{2\mu_2^2 s_1}{s_2^3} - \frac{2\mu_1\mu_2}{s_2^2}\,,\quad \gamma_2^2 := -\frac{{\mu_2}^2}{s_2^2}\,,\quad \gamma_3^2 := \frac{\mu_3^2}{s_3^2}. \tag{2.98}$$

The matrix elements of (N_1, N_2) generate a Poisson bracket realization of the jet extension $\mathbf{sl}(2)^{(1)*}$ while those of N_3 generate a second $\mathbf{sl}(2)$.

Commuting invariants

The invariants again give us only two independent H_1 and H_2

$$-\frac{1}{2}TrN(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\alpha)^2} - \frac{\mu_1\mu_2}{(\lambda-\alpha)^3} + \frac{{\mu_2}^2}{2(\lambda-\alpha)^4} + \frac{H_3}{(\lambda-\beta)} - \frac{{\mu_3}^2}{2(\lambda-\beta)^2} \tag{2.99}$$

where

$$H_1 + H_3 = 0 \tag{2.100}$$

The Hamiltonian is now defined by:

$$H = (\alpha-\beta)H_1 + H_2 - \frac{1}{2}{\mu_3}^2 \tag{2.101}$$

The reduced ambient space coordinates (s_1, s_2, s_3) are now defined by:

$${s_1}^2 = \frac{(x_1x_4 + x_2x_3)^2}{2x_2x_4} \tag{2.102}$$

$${s_2}^2 = 2x_2x_4 \tag{2.103}$$

$${s_3}^2 = {x_5}^2 + {x_6}^2 \tag{2.104}$$

(2.105)

The constraint to the quadric $X^TJX = 1$ reduces to defining a hyperboloid in $\mathbb{R}^3$

$$2s_1s_2 + {s_3}^2 = 1 \tag{2.106}$$

In the ambient coordinates the integrals H_1 and H_2 are

$$\begin{aligned} H_1 &= \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \gamma_3^2 s_1s_2 - 2\gamma_1\gamma_2 s_3^2}{\alpha - \beta} - \\ &\quad - \frac{((s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha-\beta)^2} \end{aligned} \tag{2.107}$$

$$H_2 = \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\gamma_1\gamma_2 s_1 + \frac{(s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha - \beta)}. \tag{2.108}$$

Again, whereas the Hamiltonian H does not depend on the parameters (α, β) the integrals H_1, H_2 do, which provides an explanation for the superintegrability in this case.

Separating coordinates

These are determined by the relations:

$${s_3}^2 = \frac{(\beta - \lambda_1)(\beta - \lambda_2)}{(\alpha - \beta)^2} \qquad \xi_1 = -\frac{1}{2}(\frac{s_1p_1 + s_2p_2}{\lambda_1 - \alpha} - \frac{s_2p_1}{(\lambda_1 - \alpha)^2} + \frac{s_3p_3}{\lambda_1 - \beta}) \tag{2.109}$$

$${s_2}^2 = -\frac{(\alpha - \lambda_1)(\alpha - \lambda_2)}{(\alpha - \beta)} \qquad \xi_2 = -\frac{1}{2}(\frac{s_1p_1 + s_2p_2}{\lambda_2 - \alpha} - \frac{s_2p_1}{(\lambda_2 - \alpha)^2} + \frac{s_3p_3}{\lambda_2 - \beta}) \tag{2.110}$$

$$s_1s_2 = -\frac{1}{2}(\frac{(\beta - \lambda_1)(\beta - \lambda_2)}{(\alpha - \beta)^2} - 1). \tag{2.111}$$

Quantum system

The quantized integrals $\hat{H}_1, \hat{H}_2, \hat{H}_3$ are obtained as before by replacing all conjugate variables by the corresponding differential operators. The quantum integrals may then be expressed as

$$\begin{aligned} \hat{H}_1 &= -\frac{(s_1\partial_3 - s_3\partial_2)(s_3\partial_1 - s_2\partial_3) + \gamma_3^2 s_1s_2 + 2\gamma_1\gamma_2 s_3^2}{\alpha - \beta} + \\ &\quad + \frac{(s_3\partial_1 - s_2\partial_3)^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha - \beta)^2} \end{aligned} \tag{2.112}$$

$$\hat{H}_2 = \frac{1}{2}\hat{L}_{12}^2 - 2\gamma_1\gamma_2 s_1 - \frac{(s_3\partial_{s_1} - s_2\partial_{s_3})^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha - \beta)}, \tag{2.113}$$

where $\partial_k := \partial/\partial s_k$.

The quantum Hamiltonian is

$$\hat{H} = 2\partial_1\partial_2 - \partial_1{}^2 + \partial_3{}^2 + 2\gamma_1\gamma_2 - \gamma_2^2 + \gamma_3^2, \tag{2.114}$$

and this again separates in the configuration space coordinates (λ_1, λ_2).

2.4.3 Case 3. Pseudoeuclidean plane.

Classical Lax matrix

The matrix A in this case has only one degenerate eigenvalue:

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

and J is antidiagonal.

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The symmetry algebra g_A is generated by

$$\{-x_1y_4 - x_2y_5 - x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3,\ x_6y_1 - x_3y_4,\ -x_2y_4 - x_3y_5 + x_5y_1 + x_6y_2\} \tag{2.115}$$

and the Lax matrix is of the form:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\alpha)^3} \tag{2.116}$$

where

$$\begin{aligned}
N_1 &= \frac{1}{2}\begin{pmatrix} x_1y_1+x_2y_2+x_3y_3+x_4y_4+x_5y_5+x_6y_6, & 2y_1y_3+{y_2}^2+2y_4y_6+{y_5}^2 \\ -2x_1x_3-{x_2}^2-2x_4x_6-{x_5}^2, & -x_1y_1-x_2y_2-x_3y_3-x_4y_4-x_5y_5-x_6y_6 \end{pmatrix} \\
N_2 &= \frac{1}{2}\begin{pmatrix} -x_3y_2-x_2y_1-x_6y_5-x_5y_4 & -2y_2y_1-2y_4y_5 \\ 2x_2x_3+2x_5x_6 & x_3y_2+x_2y_1+x_6y_5+x_5y_4 \end{pmatrix} \\
N_3 &= \frac{1}{2}\begin{pmatrix} x_3y_1+x_6y_4 & {y_1}^2+{y_4}^2 \\ -{x_3}^2-{x_5}^2 & -x_3y_1-x_6y_4 \end{pmatrix}
\end{aligned} \tag{2.117}$$

After reduction they become

$$\begin{aligned}
N_1 &= \frac{1}{2}\begin{pmatrix} s_1p_1+s_2p_2+s_3p_3 & -2p_1p_3+2\gamma_1\gamma_3+{\gamma_2}^2 \\ -2s_1s_3-{s_2}^2 & -s_1p_1-s_2p_2-s_3p_3 \end{pmatrix} \\
N_2 &= \frac{1}{2}\begin{pmatrix} -p_1s_2-p_2s_3 & -2p_2p_1+2\gamma_1\gamma_2 \\ 2s_2s_3 & p_1s_2+p_2s_3 \end{pmatrix} \\
N_3 &= \frac{1}{2}\begin{pmatrix} s_3p_1 & -{p_1}^2+{\gamma_1}^2 \\ -{s_3}^2 & -s_3p_1 \end{pmatrix}
\end{aligned} \tag{2.118}$$

Here,

$$\gamma_1 = \frac{\mu_1}{s_1} - \frac{\mu_2 s_2}{{s_1}^2} - \frac{\mu_3 {s_2}^2}{{s_1}^2} - \frac{\mu_3 s_3}{{s_1}^2}, \tag{2.119}$$

$$\gamma_2 = \frac{\mu_2}{s_1} - \frac{\mu_3 s_2}{{s_1}^2}, \qquad \gamma_3 = \frac{\mu_3}{s_1} \tag{2.120}$$

Commuting invariants

The trace formula again gives only two independent integrals H_1 and H_2

$$-\frac{1}{2}TrN(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)^2} + \frac{H_2}{(\lambda-\alpha)^3} - \frac{2\mu_1\mu_2-{\mu_3}^2}{2(\lambda-\alpha)^4} + \frac{\mu_2\mu_3}{2(\lambda-\alpha)^5} - \frac{{\mu_2}^2}{2(\lambda-\alpha)^6} \tag{2.121}$$

The Hamiltonian of the problem is:

$$H = -2p_1p_3 - {p_2}^2 + 2\gamma_1\gamma_3 + {\gamma_2}^2, \tag{2.122}$$

In this case the parameter α may be absorbed in the definition of λ and therefore no parameter dependence appears in the integrals H_1 and H_2:

$$\begin{aligned} H_1 &= (p_2s_3 - s_2p_1)(s_1p_1 - s_3p_3) - 2s_2s_3({p_2}^2 + 2p_1p_3) - \frac{\mu_1\mu_2}{{s_1}^2} - \frac{3\mu_3\mu_2s_3}{{s_1}^3} \\ &- \frac{\mu_3\mu_2s_1}{{s_2}^2} - \frac{4s_3\mu_3\mu_1(1-2s_1s_2)}{{s_1}^4} - \frac{\mu_3{s_2}^2}{{s_1}^2} - \frac{({\mu_2}^2 + \mu_3\mu_1)s_2}{{s_1}^3} \end{aligned} \quad (2.123)$$

$$\begin{aligned} H_2 &= ({p_2}^2 + 2p_1p_3)({s_2}^2 + 2s_1s_3) + \frac{2{\mu_3}^2s_1}{{s_2}^2} + \frac{4{\mu_3}^2{s_3}^2}{{s_1}^2} + \\ &+ \frac{4\mu_3\mu_2s_2}{{s_1}^3} - \frac{{\mu_2}^2 - 2\mu_3\mu_1}{{s_1}^2} + \frac{{\mu_3}^2(1-2{s_2}^2)}{{s_1}^4} \end{aligned} \quad (2.124)$$

Reduced coordinates in $\mathbb{R}^3$ are:

$$\begin{aligned} {s_1}^2 &= -\frac{(x_1x_3 + x_4x_6)^2}{{x_3}^2 + {x_6}^2} && (2.125) \\ {s_2}^2 &= {x_2}^2 + {x_5}^2 && (2.126) \\ {s_3}^2 &= -({x_3}^2 + {x_6}^2) && (2.127) \\ & && (2.128) \end{aligned}$$

The constraint to the quadric $X^TJX = 1$ reduces to $2s_1s_3 + {s_2}^2 = 1$.

2.4.4 Case 4. Euclidean plane.

For the all 'flat' cases we begin with a phase space $\mathbb{M}$ of $\dim \mathbb{M} = 8$, with canonical variables $(x_i, y_i)_{i=1..4}$ subject to no constraint. From these we form a Lax matrix $N(\lambda)$, depending on a spectral parameter $\lambda \in \mathbb{C}$:

$$N(\lambda) := B + \frac{1}{2}(Y^T, -X^TJ)(\lambda - A)^{-1}(X, JY) \quad (2.129)$$

with A being a 4×4 matrix. For the case (IV) A is reconstructed from a minimal polynomial

$$(\lambda - \alpha)(\lambda - \beta) \quad (2.130)$$

that corresponds to the characteristic polynomial for the case (IV) in the Krall-Sheffer classification and J is an identity 4×4 matrix:

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \qquad J = id, \tag{2.131}$$

Classical Lax Matrix

For the cases with zero curvature like the example to follow, the polynomial part $\mathcal{B}(\lambda)$ of the Lax matrix does not vanish. The simplest case involves two distinct finite poles in $N(\lambda)$ and constant $B(\lambda)$ So, $N(\lambda)$ has the form:

$$N(\lambda) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} + \frac{1}{2(\lambda-\alpha)} \begin{pmatrix} x_1y_1 + x_2y_2 & {y_1}^2 + {y_2}^2 \\ -{x_1}^2 - {x_2}^2 & -x_1y_1 - x_2y_2 \end{pmatrix} +$$

$$+\frac{1}{2(\lambda-\beta)} \begin{pmatrix} x_3y_3 + x_4y_4 & {y_3}^2 + {y_4}^2 \\ -{x_3}^2 - {x_4}^2 & -x_3y_3 - x_4y_4 \end{pmatrix} \tag{2.132}$$

the N_i are elements of $sl(2)$ as in the case (1).

The symmetry algebra g_A corresponding to the stabilizer $G_A \subset O(4, \mathbb{R}))$ is a maximal torus with just two generators

$$\{x_1y_2 - x_2y_1,\ x_3y_4 - x_4y_3\} \tag{2.133}$$

which form the components of the moment map. We restrict them to level sets

$$\mu_1 = x_1y_2 - x_2y_1,\ \mu_2 = x_3y_4 - x_4y_3, \tag{2.134}$$

Performing the usual procedure of the symplectic reduction we get:

$$N(\lambda) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} + \frac{1}{2(\lambda-\alpha)} \begin{pmatrix} s_1p_1 & {p_1}^2 + \frac{\mu_1^2}{s_1^2} \\ -{s_1}^2 & -s_1p_1 \end{pmatrix} \tag{2.135}$$

$$+\frac{1}{2(\lambda-\beta)} \begin{pmatrix} s_2p_2 & {p_2}^2 + \frac{\mu_2^2}{s_2^2} \\ -{s_2}^2 & -s_2p_2 \end{pmatrix}. \tag{2.136}$$

The matrix elements of the residues N_1, N_2 generate two copies of $\mathbf{sl}(2)$.

Commuting invariants

The invariants of motion are defined by:

$$-\frac{1}{2}TrN(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{{\mu_1}^2}{(\lambda-\alpha)^2} + \frac{{\mu_2}^2}{(\lambda-\beta)^2} - a. \tag{2.137}$$

The superintegrable Hamiltonian in this case is given by

$$H = \frac{1}{4}Res_\infty TrN(\lambda)^2 = \frac{1}{4}{p_1}^2 + {p_2}^2 + a({s_1}^2 + {s_2}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2}), \tag{2.138}$$

which gives an isotropic oscillator together with Rosochatius terms. As before (p_1, p_2) are canonically conjugate to (s_1, s_2).

In terms of the ambient space coordinates the integrals H_1 and H_2 are given by:

$$\begin{aligned} H_1 &= p_1^2 + as_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \\ H_2 &= p_2^2 + as_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}). \end{aligned} \tag{2.139}$$

and the reduced ambient coordinates are expressed by the following formulas:

$${s_1}^2 = {x_1}^2 + {x_2}^2 \tag{2.140}$$

$${s_2}^2 = {x_3}^2 + {x_4}^2 \tag{2.141}$$

Moreover $L_{12} := s_1 p_2 - s_2 p_1$ and $H = \frac{1}{4}(H_1 + H_2)$. Here the additional integral results from the parametric dependence on $(\alpha - \beta)$.

Separating coordinates

The separating coordinates $(\lambda_1, \lambda_2, \xi_1, \xi_2)$ in this case are defined by

$${s_1}^2 = 2\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)} \qquad \xi_1 = -\frac{1}{2}(\frac{s_1 p_1}{\lambda_1-\alpha} + \frac{s_2 p_2}{\lambda_1-\beta}) \tag{2.142}$$

$${s_2}^2 = -2\frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)} \qquad \xi_2 = -\frac{1}{2}(\frac{s_1 p_1}{\lambda_2-\alpha} + \frac{s_2 p_2}{\lambda_2-\beta}) \tag{2.143}$$

Quantum system

The Hamiltonian of the corresponding quantum problem is

$$\hat{H} = \frac{1}{4} Res_\infty Tr\hat{N}(\lambda)^2 = \frac{1}{4}({\partial_1}^2+{\partial_2}^2+a({s_1}^2+{s_2}^2)+\frac{{\mu_1}^2}{{s_1}^2}+\frac{{\mu_2}^2}{{s_2}^2}) = \frac{1}{4}(\hat{H}_1+\hat{H}_2). \quad (2.144)$$

The quantum integrals $\hat{H}_1$ and $\hat{H}_2$ are:

$$\begin{aligned} \hat{H}_1 &= \partial_1^2 + as_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \\ \hat{H}_2 &= \partial_2^2 + as_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \end{aligned} \quad (2.145)$$

and the separating coordinates are again (λ_1, λ_2), which depend on the additional parameter $(\alpha - \beta)$.

To relate the invariants to the ones obtained in [28] for the corresponding Krall-Sheffer case we apply the gauge transformation consisting of conjugation by the function:

$$\Phi = x^{d_1} y^{d_2} (1 - x - y)^{d_3} \quad (2.146)$$

where

$$d_1 = \frac{1}{2}(d_{00}+1/2) \qquad d_2 = \frac{1}{2}(e_{00}+1/2) \qquad d_3 = \frac{1}{2}(1/2 - d_{00} - e_{00} - B) \quad (2.147)$$

and d_{00}, e_{00}, B are the parameters appearing in Krall-Sheffer setting. (See [28]). The following are the relations between the integrals constructed in these two approaches:

$$\tilde{H}_1 = 4\frac{\alpha_1 - \gamma_1}{\beta_1 - \gamma_1}\hat{I}_x + 4\hat{I}_y - 4\hat{L} - c_0, \qquad \tilde{H}_2 = 4\frac{\gamma_1 - \beta_1}{\gamma_1 - \alpha_1}\hat{I}_y + 4\hat{I}_x - 4\hat{L} - c_1, \quad (2.148)$$

where $\tilde{H}_i = \Phi\hat{H}\Phi^{-1}$ and $\hat{L}$ is the Krall-Sheffer operator corresponding to case I, c_0 and c_1 depend on $\alpha, \beta, \gamma, d_{00}, e_{00}, B$.

$$\mu_1 = (d_0 + \frac{1}{2})(d_0 + \frac{3}{2}) \quad (2.149)$$

$$\mu_2 = (e_0 + \frac{1}{2})(e_0 + \frac{3}{2}) \quad (2.150)$$

2.4.5 Degenerate cases of the Krall-Sheffer scheme.

Case 5. Euclidean plane

With respect to the preceding case the characteristic polynomial of the Krall-Sheffer operator (V) gets a further degeneration. It now has only a double root :

$$(\lambda-\alpha)^2 \tag{2.151}$$

Considering it as a minimal polynomial for the matrix $A \in Mat_{4\times 4}$ we easily reconstruct:

$$A=\begin{pmatrix} \alpha & 1 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \alpha & 1\\ 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad J=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.152}$$

The usual procedure gives the Lax matrix, which after reduction takes the form:

$$N(\lambda)=\begin{pmatrix} 0 & -a\\ 1 & 0 \end{pmatrix}+\frac{1}{2(\lambda-\alpha)}\begin{pmatrix} p_1q_1+p_2q_2 & 2p_1p_2+\frac{2\mu_1^2q_2}{q_1^3}-\frac{2\mu_1\mu_2}{q_1^2}\\ -2q_1q_2 & -p_1q_1-p_2q_2 \end{pmatrix}+$$
$$+\frac{1}{2(\lambda-\alpha)^2}\begin{pmatrix} -q_2p_1 & -p_1^2+\frac{\mu_2^2}{q_2^2}\\ -q_2{}^2 & q_2p_1 \end{pmatrix} \tag{2.153}$$

The Hamiltonian in reduced coordinates is:

$$H=\frac{1}{2}res_{\infty}TrN(\lambda)^2=p_1p_2+aq_1q_2+\frac{\mu_1^2q_2}{q_1^3}-\frac{\mu_1\mu_2}{q_1^2} \tag{2.154}$$

Case 7. Euclidean plane.

Case VII gives rise to case IV . It corresponds to the same isotropic oscillator on the euclidean plane as in (VII), however, with no singular terms. In other words, the invariants μ_1 and μ_2 are restricted to zero level sets. In reduced coordinates the Lax matrix takes the form:

$$N(\lambda)=\begin{pmatrix} 0 & -a\\ 1 & 0 \end{pmatrix}+\frac{1}{2(\lambda-\alpha)}\begin{pmatrix} p_1q_1 & p_1{}^2\\ -q_1{}^2 & -p_1q_1 \end{pmatrix}+\frac{1}{2(\lambda-\beta)}\begin{pmatrix} p_2q_2 & p_2{}^2\\ -q_2{}^2 & -p_2q_2 \end{pmatrix} \tag{2.155}$$

The Hamiltonian of the problem is

$$H = res_{\infty} Tr N(\lambda)^2 = {p_1}^2 + {p_2}^2 + a({q_1}^2 + {q_2}^2) \tag{2.156}$$

This also has realization in pseudo-euclidean plane. After canonical change of coordinates:

$$q_2 \longrightarrow i q_2 \tag{2.157}$$

$$p_2 \longrightarrow -i p_2 \tag{2.158}$$

and rescaling:

$$q_i \longrightarrow \sqrt{2} q_i \tag{2.159}$$

$$p_i \longrightarrow p_i/\sqrt{2} \tag{2.160}$$

one gets

$$N(\lambda) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} + \frac{1}{2(\lambda - \alpha)} \begin{pmatrix} p_1 q_1 & {p_1}^2/2 \\ -2{q_1}^2 & -p_1 q_1 \end{pmatrix} + \frac{1}{2(\lambda - \beta)} \begin{pmatrix} p_2 q_2 & -{p_2}^2/2 \\ -2{q_2}^2 & -p_2 q_2 \end{pmatrix} \tag{2.161}$$

The Hamiltonian is then:

$$H = res_{\infty} Tr N(\lambda)^2 = \frac{1}{2}({p_1}^2 - {p_2}^2) + 2a({q_1}^2 + {q_2}^2) \tag{2.162}$$

Case 9. Sphere.

The ninth case is the last one in the Krall-Sheffer list that corresponds to integrable quantum-mechanical systems on curved spaces. It is a degenerate form of the first case and corresponds to a particular choice of constants defining the restriction to level sets for invariants of motion under the reduction procedure. As we already have seen these values μ_1, μ_2 and μ_3 are closely related to the so called "external" parameters of the Krall-Sheffer operators β, d_0, e_0. The latter two are equal to zero for case IX. It, therefore, corresponds to the Rosochatius potential on the sphere with $\mu_2 = \mu_3 = 0$.

In the reduced space the Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \beta)} + \frac{N_3}{(\lambda - \gamma)} \tag{2.163}$$

where the N_i are elements of $sl(2)$:

$$N_1 = \frac{1}{2} \begin{pmatrix} s_1 p_1 & {p_1}^2 + \frac{\mu_1^2}{s_1^2} \\ -{s_1}^2 & -s_1 p_1 \end{pmatrix} \tag{2.164}$$

$$N_2 = \frac{1}{2} \begin{pmatrix} s_2 p_2 & {p_2}^2 \\ -{s_2}^2 & -s_2 p_2 \end{pmatrix} \tag{2.165}$$

$$N_3 = \frac{1}{2} \begin{pmatrix} s_2 p_2 & {p_2}^2 \\ -{s_2}^2 & -s_2 p_2 \end{pmatrix} \tag{2.166}$$

The reduced Hamiltonian is:

$$H = \frac{1}{2}({p_1}^2 + {p_2}^2 + {p_3}^2) + \frac{{\mu_1}^2}{{s_1}^2} \tag{2.167}$$

which is the kinetic energy on the sphere in $\mathbb{R}^3$ plus the Rosochatius potential. The reduced separating coordinates are the same as in (I). The gauge transformation consisting of conjugation by the function:

$$\Phi(x, y) = (1 - x^2 - y^2)^{1/2 - \beta/4} \tag{2.168}$$

leads us to the original operator $\Phi \hat{H} \Phi^{-1} = \hat{L}$ in the Krall-Sheffer setting. (See [28]).

Chapter 3

On a trigonometric analogue of the Atiyah-Hitchin bracket

The direction of this investigation originates in the fundamental work of Atiyah and Hitchin [46] on the geometry and dynamics of monopoles, which led to a result on the Poisson structure on the space of rational functions.

The rational map associated to the monopole is represented by a ratio of two polynomials which are interpreted as scattering coefficients of the solution of a corresponding scattering problem.

We consider the derivation of the Atiyah-Hitchin Poisson structure on the space of rational functions by means of quadratic (Sklyanin) r-matrix structure which naturally arises for the corresponding scattering problem and we provide a trigonometric analogue.

3.1 The concept of Weyl function

In the theory of the Toda lattice the rational function

$$w(\lambda) = (R(\lambda)e_0, e_0), \tag{3.1}$$

where $R(\lambda) = (L - \lambda I)^{-1}$ is the resolvent of the Toda Lax-matrix and e_0 is a basis vector, plays a key role in the reconstruction of the Lax matrix L from its spectral data. From its continued fraction expansion one can recover the entries of the tri-diagonal Lax matrix L. This was the main idea used in Moser's solution of the nonperiodic Toda lattice [53].

More properties of this function were discovered in [44], [43], where an algebro-geometric approach to the inverse spectral problem for finite Jacobi matrices was developed. It was found that the poles of $w(\lambda)$ determine the spectral curve, while its zeros specify the divisor of the Baker-Akhiezer function. This allows one (see [43]) to obtain an explicit formula for the solution, the symplectic structure and two systems of canonical coordinates for nonperiodic Toda by another approach.

This function is referred to as the Weyl function when encountered in the spectral theory of Sturm-Liouville operators developed by H.Weyl [42]. By definition, the Weyl solution is a linear combination of the couple of fundamental solutions (suitably normalized) of the Sturm-Liouville operator eigenvalue problem with some coefficients chosen so that it belongs to $L^2([y, +\infty])$. The Weyl function is formed by taking the ratio of its coefficients.

3.2 Weyl function in the theory of monopoles

To any N-monopole solution of the Bogomolny equation [46] one can associate a scattering function $w_m(\lambda)$, which is a rational function of degree N such that $w_m(\infty) = 0$. A theorem of S. Donaldson [45] implies that the map from the space of solutions to the space of rational functions is injective. The function $w_m(\lambda)$ has the form:

$$w(\lambda) = \frac{\sum_{i=0}^{N-1} a_i \lambda^i}{\lambda^N + \sum_{j=0}^{N-1} b_j \lambda^j} = -\frac{q(\lambda)}{p(\lambda)} \tag{3.2}$$

The space of all such functions we denote by Rat_N. Coordinates on Rat_N are defined by the roots $\lambda_0, ..., \lambda_{N-1}$ of the monic polynomial $p(\lambda)$ and the values of the polynomial $q(\lambda)$ at the roots of the denominator $q(\lambda_0), ..., q(\lambda_{N-1})$. (These are valid on the dense, open set, where the roots of $p(\lambda)$ are distinct).

Atiyah and Hitchin [46] defined a symplectic structure on Rat_N by the formula:

$$\Omega = \sum_{i=1}^{N} \frac{dq(\lambda_i) \wedge d\lambda_i}{q(\lambda_i)} \tag{3.3}$$

The corresponding Poisson brackets are:

$$\{p(\beta_m), p(\beta_n)\} = 0, \qquad \{\beta_m, \beta_n\} = 0 \tag{3.4}$$

$$\{p(\beta_m), \beta_n\} = p(\beta_m)\delta_m{}^n \tag{3.5}$$

A. Faybusovich and M. Gekhtman recently proposed [47] a compact coordinate-free form of this structure on the space of rational functions:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{3.6}$$

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu}. \tag{3.7}$$

These imply the following Poisson brackets for the rational function evaluated at different points:

$$\{w(\lambda), w(\mu)\} = \frac{(w(\lambda) - w(\mu))^2}{\lambda - \mu} \tag{3.8}$$

"Multi-Poisson" structure of Gekhtman and Faybusovich

Following [47] we define a skew-symmetric bracket $\{\ ,\ \}_k$ on polynomials of $p(\lambda), q(\lambda)$ by setting

$$\{p(\lambda), p(\mu)\}_k = \{q(\lambda), q(\mu)\}_k = 0 \tag{3.9}$$

and

$$\{p(\lambda), q(\mu)\}_k = \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu}. \tag{3.10}$$

where

$$q^{[k]}(\lambda) = \lambda^k q(\lambda) \pmod{p(\lambda)} \tag{3.11}$$

and $k = 0, \dots, n$

These brackets form a *compatible* family of Poisson structures on the space of rational maps of degree n. It means that any linear combination of above brackets is again a Poisson bracket:

Proposition 3.1 *$\{\ ,\ \}_k$ $(k = 0, \dots, n)$ are compatible Poisson structures on Rat_{n+1}.*

(for details of proof see [47].)

3.3 Quadratic r-matrix structure. The relation between the brackets.

In fact, the Poisson bracket (3.8) can be obviously extended to the space of all rational functions that are the ratio of two polynomials of arbitrary degrees. However, the requirement of distinct roots remains essential. As was mentioned by K. Takasaki (see also [47]), the Gekhtman-Faybusovich bracket (3.8) is naturally related to the rational quadratic Sklyanin bracket, [12]

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = [r(\lambda - \mu), T(\lambda) \otimes T(\mu)] \tag{3.12}$$

where

$$r(\lambda) = \frac{P_{12}}{\lambda} = \frac{I \otimes I + \sigma_\alpha \otimes \sigma_\alpha}{2\lambda} \tag{3.13}$$

is the classical rational r-matrix, $P_{12} \in End(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is defined as in (1.41), see Introduction, and σ_α are the Pauli matrices.

If we parameterize the matrix $T(\lambda)$ as:

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix} \tag{3.14}$$

then the Sklyanin bracket (3.12) implies the following Poisson brackets between its matrix elements:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{3.15}$$

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu} \tag{3.16}$$

$$\{\tilde{q}(\lambda), \tilde{q}(\mu)\} = 0, \qquad \{\tilde{p}(\lambda), \tilde{p}(\mu)\} = 0 \tag{3.17}$$

$$\{\tilde{p}(\lambda), \tilde{q}(\mu)\} = \frac{\tilde{p}(\lambda)\tilde{q}(\mu) - \tilde{q}(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{3.18}$$

$$\{\tilde{q}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)q(\mu) - q(\lambda)\tilde{q}(\mu)}{\lambda - \mu}, \qquad \{\tilde{p}(\lambda), p(\mu)\} = \frac{\tilde{p}(\lambda)p(\mu) - p(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{3.19}$$

$$\{p(\lambda), \tilde{q}(\mu)\} = \frac{q(\lambda)\tilde{p}(\mu) - \tilde{p}(\lambda)q(\mu)}{\lambda - \mu} \tag{3.20}$$

$$\{\tilde{p}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)p(\mu) - p(\lambda)\tilde{q}(\mu)}{\lambda - \mu} \tag{3.21}$$

From this, it is clear that we may define Poisson subspaces consisting of matrices T for which $p(\lambda) = \tilde{q}(\lambda)$ and $p(\lambda) = \tilde{q}(\lambda)$. Under this identification, the brackets (3.15) and (3.16) coincide with the Gekhtman-Faybusovich brackets (3.6, 3.7). Therefore, the Atiyah-Hitchin Poisson structure is embedded into the Sklyanin R-matrix structure for polynomial dependence of the matrix T on λ.

This observation leads to several natural questions. The Sklyanin bracket (3.12) arises in the inverse scattering method as a Poisson structure on scattering matrix $T(\lambda)$ implied by fundamental the Poisson structure between physical fields in models of non-linear Schrödinger type, [12]. On the other hand, the Atiyah-Hitchin bracket is also defined as a bracket on the space of scattering matrices related to solutions of the Bogomolny equations [46].

However, in the Atiyah-Hitchin framework it remains unclear how this Poisson structure is related to the fundamental Poisson brackets between the physical fields A and ϕ (gauge and Higgs fields, [46]). This relationship between Atiyah-Hitchin and Sklyanin brackets suggests that there should exist some kind of derivation of the Atiyah-Hitchin structure from the brackets on A and ϕ; however, so far we were unable to find it. Instead in the following section we shall extend this observation to the trigonometric case and show how to derive a natural Poisson structure on the space of trigonometric rational functions starting from the Sklyanin bracket with a trigonometric r-matrix.

3.4 Trigonometric generalizations of Atiyah-Hitchin and Gekhtman-Faybusovich brackets

Consider a space consisting of meromorphic functions $p(\lambda)/q(\lambda)$ being the ratio of a pair of trigonometric polynomials

$$p(\lambda) = \prod_{i=1}^{N-1} sin(\lambda - \alpha_i) \tag{3.22}$$

$$q(\lambda) = \prod_{k=1}^{N} sin(\lambda - \lambda_k) \tag{3.23}$$

The symplectic structure here is chosen to be of the same form as before

$$\Omega = \sum_{i=1}^{N} \frac{dp(\lambda_i) \wedge d\lambda_i}{p(\lambda_i)}. \tag{3.24}$$

Consider the following 2×2 matrix

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix} \tag{3.25}$$

where the polynomials $p(\lambda)$ and $q(\lambda)$ are as above and $\tilde{p}$ and $\tilde{q}$ are also trigonometric polynomials of the same form as q and p respectively. As it follows from the R-matrix theory, the determinant of $T(\lambda)$ is a Casimir (coefficients in $I(T(\lambda))$ Poisson commute). Then the replacement of the matrix T above and a trigonometric r-matrix defined as

$$r(\lambda) = \frac{1}{2sin(\lambda)}[\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + cos(\lambda)\sigma_3 \otimes \sigma_3] \tag{3.26}$$

into (3.12) now give the relations:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{3.27}$$

$$\{p(\lambda), q(\mu)\} = 2\frac{q(\lambda)p(\mu) - cos(\lambda-\mu)p(\lambda)q(\mu)}{sin(\lambda-\mu)} \tag{3.28}$$

$$\{\tilde{q}(\lambda), \tilde{q}(\mu)\} = 0, \qquad \{\tilde{p}(\lambda), \tilde{p}(\mu)\} = 0 \tag{3.29}$$

$$\{\tilde{p}(\lambda), \tilde{q}(\mu)\} = 2\frac{\tilde{q}(\lambda)\tilde{p}(\mu) - cos(\lambda-\mu)\tilde{p}(\lambda)\tilde{q}(\mu)}{sin(\lambda-\mu)} \tag{3.30}$$

$$\{\tilde{q}(\lambda), q(\mu)\} = -2\frac{q(\lambda)\tilde{q}(\mu) - cos(\lambda-\mu)\tilde{q}(\lambda)q(\mu)}{sin(\lambda-\mu)} \tag{3.31}$$

$$\{\tilde{p}(\lambda), p(\mu)\} = -2\frac{p(\lambda)\tilde{p}(\mu) - cos(\lambda-\mu)\tilde{p}(\lambda)p(\mu)}{sin(\lambda-\mu)} \tag{3.32}$$

$$\{p(\lambda), \tilde{q}(\mu)\} = 2\frac{q(\lambda)\tilde{p}(\mu) - \tilde{p}(\lambda)q(\mu)}{sin(\lambda-\mu)} \tag{3.33}$$

$$\{\tilde{p}(\lambda), q(\mu)\} = 2\frac{\tilde{q}(\lambda)p(\mu) - p(\lambda)\tilde{q}(\mu)}{sin(\lambda-\mu)} \tag{3.34}$$

Again the form of these Poisson bracket relations allows a consistent reduction such that $q = -\tilde{p}$ and $p = -\tilde{q}$. The quadratic Sklyanin Poisson bracket on matrices $T(\lambda)$ and $T(\mu)$ then reduce to the follows:

$$\{p(\lambda), p(\mu)\} = 0 \tag{3.35}$$

$$\{q(\lambda), q(\mu)\} = 0 \tag{3.36}$$

$$\{p(\lambda), q(\mu)\} = -2\frac{q(\lambda)p(\mu) - cos(\lambda-\mu)q(\lambda)p(\mu)}{sin(\lambda-\mu)}. \tag{3.37}$$

This leads in turn to the following generalization of the Gekhtman-Faybusovich bracket on the space of rational trigonometric functions:

$$\{w(\lambda), w(\mu)\} = \frac{(w(\lambda) - w(\mu))^2}{sin(\lambda-\mu)} + 2w(\lambda)w(\mu)(ctg(\lambda-\mu) - 1). \tag{3.38}$$

It is easy to see that in rational limit this bracket coincide with (3.8).

We conjecture here that this bracket also admits an extension to "multi-Poisson" family of compatible brackets as in rational case (3.9), (3.10), under the following considerations. Define polynomials $q^{[k]}(\lambda)$ in trigonometric case to be

$$q^{[k]}(\lambda) = tan^k(\lambda)q(\lambda) \tag{3.39}$$

and $k = 0, \ldots, n$. Then the brackets:

$$\{p(\lambda), q(\mu)\}_k = \frac{p(\lambda)q^{[k]}(\mu) - cos(\lambda - \mu)p(\mu)q^{[k]}(\lambda)}{sin(\lambda - \mu)}. \tag{3.40}$$

and

$$\{p(\lambda), p(\mu)\}_k = \{q(\lambda), q(\mu)\}_k = 0 \tag{3.41}$$

form a compatible family of Poisson structures on the space of trigonometric polynomials of n-th degree.

Chapter 4

Constrained Reductions of 2D dispersionless Toda Hierarchy, Hamiltonian Structure and Interface Dynamics

4.1 Laplacian Growth Problem.

4.1.1 Background and motivation.

Many non-equilibrium processes in hydrodynamics, combustion, statistical and condensed matter physics involve the dynamics of the boundary between adjoining domains containing different types of matter (e.g. oil and water in Hele-Shaw cells, different electron states in the quantum Hall effect in nano-structures, solid and liquid regions in crystal growth etc. [57]. Mathematical models describing such processes are of great practical value. The main goal of research dealing with such phenomena is to predict the dynamics of the boundary and formation of patterns between different regions. In the last twenty years much effort has been made in this direction. The problem of understanding interface dynamics and pattern formation is one of the most rapidly developing current themes in nonlinear science. If the motion of the interface is slow in comparison with the processes that take place in the bulk of both regions (such as heat transfer, diffusion etc.) the scalar field governing the evolution of the interface is a harmonic function, [55],[56].

The term Laplacian Growth (LG) refers to a class of growth problems in which there is a harmonic scalar field, representing either a temperature (e.g. in the freezing of a liquid or Stefan problem), a concentration (in the solidification of a supersaturated solution), an electrostatic potential (in electro-deposition), a pressure (in flows through porous media) etc., depending on the system , [57].

The investigation of the LG problem in viscous fluids may be traced back to the Saffman-Taylor plane-wave "finger" solution, [71], and the problem of the "finger" width selection in channels, [77]. It was widely accepted that the inclusion of surface tension (neglected in the LG approximation) at the boundary is a physically justified way to select the most stable patterns. Important recent developments in this direction include the work of groups at LANL,[74]-[79] and of OCIAM (Oxford), [72],[73], where a mechanism of pattern selection has been given without introducing surface tension. In this work N-"finger" solutions to the LG problem were found.

In [79] the authors showed that there is an intimate relation between the Laplacian growth problem and the dispersionless 2D Toda integrable hierarchy. Explicitly, they identified the area and the set of the exterior moments with the times of the dispersionless 2D Toda hierarchy. They showed also that the conformal maps taking the exterior region complementary to simply connected domains with an analytic boundary (representing a moving interface) to the exterior of a unit disk are determined by a particular solution to the hierarchy singled out by conditions known as "string equations". This places the problem within the framework of dispersionless integrable hierarchies, which may be viewed as multi-dimensional extensions of the hierarchies of hydrodynamic type [58], [63]. Thus the LG problems, which have considerable practical importance, have been brought into contact with developments in this domain of integrable hierarchies.

In connection with the LG problem, there remain many important questions to be addressed. For instance, the interface develops finite-time singularities for generic initial data. In reality, surface tension effects lead to formation of a fractal (instead of singular) interfaces with universal characteristics [61], [62]. The problem of describing such fractal interfaces is of great importance in the study of universal asymptotic behaviour of non-linear dynamical systems. Other important questions involve the LG approximation itself. They deal with the underlying Hamiltonian structure of multi-"finger" solutions or with a problem of finding "higher genus" analogs of such solutions (describing, for instance, the motion of oil with several water spots in it or other kinds of multiregion flows).

Another topic directly related to interface dynamics is the 2D Inverse Potential Problem (IPP), [58],[64]. This deals with determining a curve (e.g. representing a boundary between two different media) from a set of data (such as a subset of its harmonic moments, either interior or exterior) relating to the effective electrostatic potential. The latter form a set of variables in which the interface dynamics is analytically tractable and the Inverse Potential transformation, therefore, makes a connection to a simpler problem. The theory of the inverse potential problems uses methods of complex analysis such as the theory of conformal mappings, Green functions etc, [58], [64]. In contrast to these approaches, our line of approach is to attack the problem exploiting connections with the theory of integrable systems. In particular, recent advances, relating the analytic methods to the modern theory of integrable hierarchies, provide a possibility for further development in IP and LG problems. For example, the authors of [58],[79] have established relations between the Hadamard-Dirichlet boundary value problem, inverse potential theory and Laplacian growth problems. Summarizing the description of the topic we would like to stress that exploiting the multi-disciplinary nature of the problem we expect significant advances in its solution due to new approaches using the theory of integrable systems.

4.1.2 Statement of the problem

This section concerns the study of rational and logarithmic reductions of the 2D dispersionless Toda hierarchy of integrable equations (henceforth dToda).

Laplacian growth is a process that governs the dynamics of the boundary in the plane separating two disjoint adjacent open regions in which harmonic (scalar) fields are defined. These may be interpreted as the pressure fields for two incompressible viscous fluids. The movement of the boundary is determined (according to d'Arcy's law, in the case of viscous fluids) by equating the normal velocity to the difference between the boundary values of the gradients of the fields. In particular, one region (say, the "interior" region) may be chosen to be bounded and have constant harmonic field (corresponding to zero viscosity) with the boundary condition for the "exterior" region such that there is a unit sink at infinity, implying that the area of the interior region grows linearly in time [73], [78]. Denoting the harmonic field (e.g. the pressure) in the exterior region by $P(X, Y)$, this satisfies the conditions (see, for example, [59], [60]):

$$\Delta P(X,Y) = 0 \tag{4.1}$$

$$P \to (4\pi)^{-1} \ln(X^2+Y^2) \quad \text{as} \quad X^2+Y^2 \to \infty \tag{4.2}$$

With the exterior normal velocity at the boundary given by:

$$v_n = -\nabla P \tag{4.3}$$

(normalized so that the proportionality constant is -1). Here x denotes time and X, Y are the Cartesian coordinates of a point on the boundary.

In the case where the boundary is an analytic curve it is usual to introduce a time-dependent conformal mapping from the exterior of the unit circle in the "mathematical" (w) plane

$$z = z(w, x), \quad w = \exp(\sqrt{-1}\phi), \quad 0 < \phi < 2\pi \tag{4.4}$$

to the exterior of the region determined by the boundary curve $z = X + \sqrt{-1}Y$ in the "physical" plane, with the unit circle mapping to the boundary. Simple considerations [57], [55], [78] show that equations (4.1, 4.3) are equivalent to the following equation

$$\mathrm{Im}\left(\frac{\partial z}{\partial \phi}\frac{\partial \bar{z}}{\partial x}\right) = w\left(\frac{\partial z(w,x)}{\partial w}\frac{\partial \bar{z}(1/w,x)}{\partial x} - \frac{\partial z(w,x)}{\partial x}\frac{\partial \bar{z}(1/w,x)}{\partial w}\right) = 1, \tag{4.5}$$

where bar stands for complex conjugation (and $\bar{w} = w^{-1}$ on the boundary curve). Equation (4.5) is called the Laplacian growth equation [56], [55].

It turns out that (4.5) plays an essential role in the theory of infinite-dimensional integrable hierarchies. The relation between the contour dynamics above and the dispersionless limit of the integrable Toda hierarchy constrained by (4.5) was established in [78].

Equation (4.5) may be interpreted as a constraint on an infinite commuting set of dynamical systems defined in the space of one-parameter families of conformal maps. This constraint characterizes the fixed points of an "additional symmetry" ([82]). The most interesting aspect of such constrained dToda flows is that they admit finite-dimensional reductions, including the so-called "multi-finger" solutions ([77],[71]). These solutions are of great importance in practical applications and describe numerous phenomena, such as viscous fingering in a Hele-Shaw cell (see [71],[72],[77]), pattern formation in the quantum Hall effect [74] etc.

In what follows, we consider finite-dimensional solutions of (4.5) in the context of the dToda hierarchy. We study formal algebraic solutions of the problem, ignoring the real structure, treating z, $\bar{z}$ as independent functions, and w as a formal variable. To return to the original problem one has to identify bar with complex conjugation and $\bar{z}$ with $S(z^{-1})$, where $S(z)$ is the Schwarz function (see [88]) associated with a conformal map $z = z(w)$.

4.2 2D dispersionless Toda hierarchy and the string equation.

In this section, we summarize the main points of the deduction of the 2D-disprsionless Toda hierarchy, through a suitable asymptotic limit in a small parameter $\hbar$, together with the string equation. We mostly follow the exposition of Takasaki and Takebe given in [68], [70].

4.2.1 2D Toda hierarchy

The $2D$ Toda hierarchy with dispersion is described in the "Lax-Sato" gauge in terms of two difference Lax operators:

$$L = r(x)e^{\hbar\partial_x} + \sum_{k=1}^{\infty} u_k(x)e^{-k\hbar\partial_x} \tag{4.6}$$

$$\bar{L} = r(x-\hbar)e^{-\hbar\partial_x} + \sum_{k=1}^{\infty} \bar{u}_k(x)e^{k\hbar\partial_x} \tag{4.7}$$

where $e^{n\hbar\partial_x}$ is a difference operator of a continuous variable x with spacing unit $\hbar$ acting as follows

$$e^{n\hbar\partial_x}f(x) = f(x+n\hbar) \tag{4.8}$$

These evolve under an infinite number of flows according to the Lax-Sato equations:

$$\hbar\partial_{t_k}L = [H_k, L], \quad \hbar\partial_{\bar{t}_k}\bar{L} = [-\bar{H}_k, \bar{L}] \tag{4.9}$$

$$\hbar\partial_{t_k}\bar{L} = [H_k, \bar{L}], \quad \hbar\partial_{\bar{t}_k}L = [-\bar{H}_k, L] \tag{4.10}$$

where the square brackets $[,]$ stand for the commutator and

$$H_k = (L^k)_+ + 1/2(L^k)_0, \quad \bar{H}_k = (\bar{L}^k)_- + 1/2(\bar{L}^k)_0 \tag{4.11}$$

The subscripts $\pm, 0$ denote, respectively, the positive/negative and zero part of operators. In other words, if

$$T = \sum_n f_n(t)e^{n\hbar\partial_t}, \tag{4.12}$$

then

$$T_- = \sum_{n<0} f_n(t)e^{n\hbar\partial_t}, \qquad T_0 = f_0(t), \qquad T_+ = \sum_{n>0} f_n(t)e^{n\hbar\partial_t} \tag{4.13}$$

In (4.9), we have a system of evolution PDE's for the scalar coefficients $r = r(x, t_k), u = u(x, t_k), \bar{u} = \bar{u}(x, t_k)$ in each of the time variables t_k. The compatibility conditions for (4.9) ensuring the commutativity of the flows are equivalent to the zero-curvature equations

$$[H_i, H_j] - \hbar\partial_{t_i} H_j + \hbar\partial_{t_j} H_i = 0, \qquad [\bar{H}_i, \bar{H}_j] - \hbar\partial_{\bar{t}_i} \bar{H}_j + \hbar\partial_{\bar{t}_j} \bar{H}_i = 0, \tag{4.14}$$

$$[H_i, \bar{H}_j] - \hbar\partial_{t_i} \bar{H}_j + \hbar\partial_{\bar{t}_j} H_i = 0. \tag{4.15}$$

Since all flows (4.9) commute, we can consider "field variables" $r, u_i, \bar{u}_i$ as functions of x, T where $T = \{t_i, \bar{t}_i, i = 1, 2, ...\}$, and (4.9) constitutes an initial-value problem with given $r(x, 0), u_i(x, 0), \bar{u}_i(x, 0)$.

4.2.2 Orlov-Shulmann symmetries and the string equation

The string equation appears as a stationary point of one of the Shulman-Orlov "additional symmetries" of the Toda hierarchy described by the following equations:

$$\delta_{F,\hat{F}} L = [F_- + 1/2F_0 - \hat{F}_- - 1/2\hat{F}_0, L] \tag{4.16}$$

$$\delta_{F,\hat{F}} M = [F_- + 1/2F_0 - \hat{F}_- - 1/2\hat{F}_0, M] \tag{4.17}$$

$$\delta_{F,\hat{F}} \bar{L} = [\hat{F}_+ + 1/2\hat{F}_0 - F_+ - 1/2F_0, \bar{L}] \tag{4.18}$$

$$\delta_{F,\hat{F}} \bar{M} = [\hat{F}_+ + 1/2\hat{F}_0 - F_+ - 1/2F_0, \bar{M}] \tag{4.19}$$

where $M, \bar{M}$ stand for Orlov operators:

$$M = \sum_{n=1}^{\infty} nt_n L^n + x + \sum_{n=1}^{\infty} v_n(t, \bar{t}, x) L^{-n} \tag{4.20}$$

$$\bar{M} = -\sum_{n=1}^{\infty} n\bar{t}_n \bar{L}^{-n} + x + \sum_{n=1}^{\infty} \bar{v}_n(t, \bar{t}, x) \bar{L}^n \tag{4.21}$$

satisfying the same evolution equations (4.9), (4.10) as L and $\bar{L}$, along with the following equations:

$$[L, M] = \hbar L, \qquad [\bar{L}, \bar{M}] = -\hbar \bar{L} \tag{4.22}$$

Here $F = F(L, M), \hat{F} = \hat{F}(\bar{L}, \bar{M})$ are arbitrary functions in two arguments, having formal power expansions in $L^n M^k$ and $\bar{L}^n \bar{M}^k$ respectively. Their positive/negative and zero parts are defined with respect to powers of the difference operator, as it defined in (4.13)

Under the special choice of $F, \hat{F}$:

$$F(L, M) = L^{-1}M, \quad \hat{F}(\bar{L}, \bar{M}) = \bar{M} \tag{4.23}$$

the condition of stationarity

$$F - \hat{F} = 0 \tag{4.24}$$

leads to the string equation:

$$[L, \bar{L}] = \hbar. \tag{4.25}$$

Since the Orlov-Shulman symmetries commute with the Toda flows, the manifold of fixed points of these symmetries and (as a consequence) the string equation (4.25) is invariant under the 2D Toda flows.

4.2.3 Dispersionless limit

In the dispersionless limit $\hbar \to 0$, the Lax operators $L, \bar{L}$ become functions $z, \bar{z}$ of a classical variable w, which replaces the operator $e^{\hbar\partial_x}$,

$$L \to z(w, x) = r(x)w + \sum_{k=0}^{\infty} u_k(x)w^{-k} \tag{4.26}$$

$$\bar{L} \to \bar{z}(w^{-1}, x) = r(x)w^{-1} + \sum_{k=0}^{\infty} \bar{u}_k(x)w^k \tag{4.27}$$

The resulting equations will be referred to as the 2D-dispersionless Toda hierarchy, or simply, dToda hierarchy. The dToda flow equations are

$$\begin{aligned} \partial_{t_k} z &= \{H_k, z\} & \partial_{\bar{t}_k} \bar{z} &= \{\bar{H}_k, \bar{z}\} \\ \partial_{t_k} \bar{z} &= \{H_k, \bar{z}\} & \partial_{\bar{t}_k} z &= \{\bar{H}_k, z\}, \qquad k = 1, \cdots, \infty. \end{aligned} \tag{4.28}$$

where the Poisson-Lax brackets are defined as

$$\{f, g\} := w \frac{\partial f}{\partial w} \frac{\partial g}{\partial x} - w \frac{\partial f}{\partial x} \frac{\partial g}{\partial w} \tag{4.29}$$

The coefficients $r(x), u_k(x), \bar{u}_k(x)$ in (4.26,4.27) are viewed as coordinate functionals on the phase space consisting of such pairs $z(w, x), \bar{z}(w^{-1}, x)$, and

$$H_k := (z^k)_+ + 1/2(z^k)_0, \quad \bar{H}_k := (\bar{z}^k)_- + 1/2(\bar{z}^k)_0 \tag{4.30}$$

with subscripts $\pm, 0$ denoting the negative/positive and zero parts of the formal Laurent expansion in w.

Remark. The Poisson bracket (4.29) should not be interpreted as defined on this phase space. It is just a convenient notation for expressing a certain bi-derivation in terms of a Poisson bracket on the auxiliary 2-dimensional space coordinatized by the variables (w, x), on which $(log \quad w, x)$ are canonical variables.

The dispersionless limit of the "string equation" is the constraint

$$\{z(w, x), \bar{z}(w^{-1}, x)\} = 1, \tag{4.31}$$

This is invariant under the dToda flows (4.28) and defines a consistent reduction of the full dToda hierarchy.

4.2.4 Ueno-Takasaki formulation and gauge equivalence

In the Ueno-Takasaki formulation, [83], the dToda hierarchy is defined in slightly different conventions in terms of two functions $z(w, x)$ and $\bar{z}(w^{-1}, x)$ of the form:

$$z(w, x) = w + \sum_{k=0}^{\infty} u_{k+1}(x) w^{-k} \tag{4.32}$$

$$\bar{z}(w^{-1}, x) = \bar{u}_0(x)w^{-1} + \sum_{k=0}^{\infty} \bar{u}_{k+1}(x)w^k, \tag{4.33}$$

which should be viewed as a gauge transformed version of (4.7), (4.7) in the quasi-classical limit. Its Lax representation is

$$\begin{aligned} \partial_{t_k} z &= \{B_k, z\} & \partial_{\bar{t}_k} \bar{z} &= \{\bar{B}_k, \bar{z}\} \\ \partial_{t_k} \bar{z} &= \{B_k, \bar{z}\} & \partial_{\bar{t}_k} z &= \{\bar{B}_k, z\}, \end{aligned} \tag{4.34}$$

where the Poisson-Lax bracket is defined as in (4.29) and B_k and $\bar{B}_k$ are given by

$$B_k = (z^k)_{\geq 0}, \qquad \bar{B}_k = (\bar{z}^k)_{\leq -1} \tag{4.35}$$

Following T.Takebe, [68], we may introduce the dispersionless Toda lattice hierarchy as the following system:

$$\partial_{t_n} Z_{\pm} = \{B_n^{(\alpha)}, Z_{\pm}\} \tag{4.36}$$

$$\{\partial_{t_n} - B_n^{(\alpha)}, \partial_{t_m} - B_m^{(\alpha)}\} = \partial_{t_m} B_n^{(\alpha)} - \partial_{t_n} B_m^{(\alpha)} + \{B_n^{(\alpha)}, B_m^{(\alpha)}\} = 0 \tag{4.37}$$

for all $m, n = \pm 1, \pm 2, \ldots$. Here, we have used the notations, $Z_+ = z$ and $Z_- = \bar{z}$. The relation between the different formulations of the Toda hierarchy may be expressed by defining

$$B_n^{(\alpha)} = (Z_+^n)_+ + (\frac{1}{2} + \alpha)(Z_+^n)_0 \tag{4.38}$$

$$B_{-n}^{(\alpha)} = (Z_-^n)_- + (\frac{1}{2} - \alpha)(Z_-^n)_0, \qquad n = 1, 2, \ldots, \tag{4.39}$$

where the value of the parameter α determines the gauge. If $Z_{\pm}^{(\alpha)}$ are solutions of the dToda lattice hierarchy in the α-gauge, then, for each β, there exist a gauge transformation for corresponding operators $\hat{Z}_{\pm}^{(\alpha)}$ such that

$$\hat{Z}_{\pm}^{(\beta)} := g^{-1} \hat{Z}_{\pm}^{(\alpha)} g \tag{4.40}$$

is a solution of the Toda lattice hierarchy of β-gauge, and

$$\hat{B}_n^{(\beta)} := g^{-1} \hat{B}_n^{(\alpha)} g + \partial_{t_n} g^{-1} g \tag{4.41}$$

with some gauge function $g = g_{\alpha\beta}$.

In the sequel we will use only transformations of the flows from the Ueno-Takasaki to the Lax-Sato gauge followed by semi-classical limit as needed for our purposes. We present here the transition formulas between the different gauges explicitly. In general, the Lax functions $z^{(\alpha)}$ and $\bar{z}^{(\alpha)}$ in an arbitrary α-gauge are expressed through the Lax-Sato gauge, which corresponds to $\alpha = 0, z = z^{(0)}, \bar{z} = \bar{z}^{(0)}$ as follows

$$\begin{aligned} z^{(\alpha)}(w) &= z(w/r^{2\alpha}) = r(x)^{1-2\alpha}w + \textstyle\sum_{i=-\infty}^{0} u_i^{(\alpha)}(x)w^i, \\ \bar{z}^{(\alpha)}(w^{-1}) &= \bar{z}(r^{2\alpha}/w) = r(x)^{1+2\alpha}w + \textstyle\sum_{i=0}^{\infty} \bar{u}_i^{(\alpha)}(x)w^i \end{aligned} \tag{4.42}$$

while, (omitting evident subscripts)

$$\begin{aligned} H^{(\alpha)}(w) &= B^{(\alpha)}(w) - \left(\tfrac{1}{2} - \alpha\right) B^{(\alpha)}(w = 0), \\ \bar{H}^{(\alpha)}(1/w) &= \bar{B}^{(\alpha)}(1/w) - \left(\tfrac{1}{2} + \alpha\right) \bar{B}^{(\alpha)}(1/w = 0) \end{aligned} \tag{4.43}$$

In particular, the relation between the evolution operators in the Ueno-Takasaki ($\alpha = 1/2$) and Lax-Sato ($\alpha = 0$) gauges is found to be:

$$H_k(w) = B_k(w) - \frac{1}{2}B_k(w = 0), \quad \bar{H}_k(y) = \bar{B}_k(y) - \frac{1}{2}\bar{B}_k(y = 0). \tag{4.44}$$

4.3 Reductions of dToda hierarchy constrained by the string equation.

The reduction of the dToda hierarchy by the string equation is still an infinite, compatible set of infinite-dimensional dynamical systems. In what follows we will be interested in further "functional" reductions where $z, \bar{z}$ are polynomial, rational or logarithmic functions of w.

For these reductions, it may be seen that the string equation (4.31) is equivalent to a finite system of ODE's determining the x-derivatives of an independent finite set of coordinate functions on the reduced spaces. (For the polynomial case, these may be chosen as r together with the finite set of non-vanishing u_k, $\bar{u}_k$ themselves; for rational functions z and $\bar{z}$, they are the location of the poles and the values of their residues; for logarithmic z and $\bar{z}$'s they are the location of the branch points and the residues of $\frac{dz}{dw}$, $\frac{d\bar{z}}{dw}$ at these points.)

As shown below, such reductions are consistent with (4.28) (i.e. they are preserved by the dToda flows) if the string equation (4.31) holds. Thus, for consistency we need a double ("functional" plus "string") reduction. This pair of reductions defines a finite-dimensional invariant submanifold in the phase space of the general dToda hierarchy. Indeed, functional reduction leaves a finite number of discrete indices in the ansatz for z as a function of w, while the string equation determines the dependence of z, $\bar{z}$ on x leaving a finite number of degrees of freedom. These degrees of freedom are connected with the integration constants of the equations determining the x-dependence of the coordinate functions on a phase space that becomes finite-dimensional after the functional reduction.

4.3.1 Polynomial reductions.

We begin with polynomial reductions of the dToda chain

$$z(w) = rw + \sum_{i=0}^{N} u_i w^{-i} \tag{4.45}$$

$$\bar{z}(w^{-1}) = rw^{-1} + \sum_{i=0}^{N} \bar{u}_i w^{i} \tag{4.46}$$

constrained by the string equation (4.31).

The following proposition states the consistency of such polynomial reductions under the dToda flows:

Proposition 4.1 *If the string equation (4.31) holds, then (4.45), (4.46) belong to a manifold invariant under the dToda flows* $\partial/\partial x$, $\partial/\partial t_i$, $\partial/\partial \bar{t}_i, 0 < i < N+2$ *determined by (4.28). This manifold has dimension* $2N+3$ *and the flows are expressed in terms of the functions* $r = r(x,T)$, $u_i = u_i(x,T)$, $\bar{u}_i = \bar{u}_i(x,T)$, $(T = \{t_i, \bar{t}_i\}, i =$

$1, .., N+1$). *The x-dependence is determined by the string equation, in terms of $2N+1$ initial values.*

Proof: see the proof of Proposition 1 given in the paper [84] attached.

Remark. The string equation, in this case, reduces to a generically non-singular system of linear equations for the derivatives $\{\partial u_k/\partial x, \partial \bar{u}_k/\partial x\}_{k=1..N+1}; \partial r/\partial x$ (i.e. for a dense, open set of initial conditions the system is non-singular). These can therefore (on such a dense, open set) be uniquely solved to determine the x-dependence in $u_k(x), \bar{u}_k(x), r(x)$ in terms of $2N+1$ integration constants.

The evolution generated by the $(t_k, \bar{t}_k)_{k=1..N+1}$ flows may then be interpreted as flows on this $2N+1$-dimensional phase space. It would, however be more symmetrical to simply view the components $u_k, \bar{u}_k, r$ defining z and $\bar{z}$ as functions of $2N+3$ flow variables $\{t_0, t_k, \bar{t}_k\}_{k=1..N+1}$ with $t_0 = x$, since the string equation just reduces to an additional first order equation in the t_0 flow variable compatible with the rest.

4.3.2 Rational reductions.

Consider now the space of rational functions $z(w)$ and $\bar{z}(w)$ of the form

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{rw^{N+1} + \sum_{i=0}^{N} a_i w^i}{w^N + \sum_{i=0}^{N-1} b_i w^i} \tag{4.47}$$

$$\bar{z}(w^{-1}) = \frac{\bar{q}_{N+1}(w^{-1})}{\bar{p}_N(w^{-1})} = \frac{rw^{-(N+1)} + \sum_{i=0}^{N} \bar{a}_i w^{-i}}{w^{-N} + \sum_{i=0}^{N-1} \bar{b}_i w^{-i}}, \tag{4.48}$$

where the $4N+3$ coefficients $r, a_i, \bar{a}_i, b_i, \bar{b}_i$ are functions of x.

The following Lemma states the invariance of each such rational reduction of $z(w)$ and $\bar{z}(w)$ under one half of the dToda flows.

Lemma 4.1 *The space of functions $z(w)$ of the form (4.47) is invariant under the dToda flows $\partial_{t_i}, i > 0$. and, similarly, the space of functions $\bar{z}(w^{-1})$ of the form (4.48) is invariant under the $\partial_{\bar{t}_i}, i > 0$ flows.*

Proof. See the proof to Lemma 1 given in the paper [84] attached.

The consistency of rational solutions for the whole Toda hierarchy requires some extra restrictions. This is the point where the string equation plays an essential role.

Proposition 4.2 *The string equation (4.31) is a sufficient condition for the rational functions $z(w)$ and $\bar{z}(w)$ ((4.47), (4.48)) to belong to a manifold that is invariant under the first two-Toda flows $\partial/\partial t_1$, $\partial/\partial \bar{t}_1$ (see (4.28)).*

Proof. See the proof of Proposition 2 given in the paper [84] attached.

It may be seen from the proof of Proposition 4.2, that there are only two flows compatible with the rational ansatz.

Remark. In fact, there is no reason to expect any higher finite number of independent Toda flows to preserve the rational forms (4.47), (4.48), since these flow variables are, in fact, interpretable, locally, as constant translates of the exterior harmonic moments of the corresponding conformal mappings (in the case where z and $\bar{z}$ are in fact complex variables.)

The latter, for the case of rational maps of the forms (4.47), (4.48) are constrained by an infinite number of relations, which are not merely the vanishing of some set of moments past a finite number of them. Hence, the t_k parameters cannot be independently varied while retaining the rational form (4.47), (4.48).

In the polynomial case, the number of invariant flows was equal to the number of variables (polynomial coefficients), since one can associate n invariant flows to the pole of the n-th order, and poles at zero and infinity are immovable.

Below, we introduce additional flows, related to moveable poles in $z(w)$ and $\bar{z}(w^{-1})$ extending a result of Krichever [58] for the case of KP flows, which reduce to the Benney systems.

4.3.3 Additional flows for rational reductions of the dKP hierarchy.

As mentioned in [65], on the phase space of extended Benney systems, i.e. rational dKP reductions admitting poles of arbitrary degree, there arise some new flows related to the pole structure of the corresponding maps. These additional flows were introduced by Krichever (see [58]).

Consider a representation of rational maps in partial fraction form assuming only simple poles along with the $\{t_i\}$ flows only. This amounts to considering the 1D-dToda or dKP hierarchy. In the Takasaki gauge [68], [65]

$$z(w) = w + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha} \tag{4.49}$$

The new flows attached to the poles are determined as before by the equations

$$\partial_{t_{k,\alpha}} z = \{B_{k,\alpha}, z\}, \quad \alpha = \infty, 1, 2, \ldots \quad k = 0, 1, 2, \ldots \tag{4.50}$$

using the same Lax-Poisson bracket (4.29) as above for a canonical pair (w, x), with the evolution determined as before, by:

$$B_{k,\infty} = (z(w)^k)_{\geq 0} \tag{4.51}$$

for the $t_{k,\infty}$ flows ($t_k \longrightarrow t_{k,\infty}$ in this notation), while for each finite pole there appear additional flows with the evolution determined by:

$$B_{k,\alpha} = (z(w)^k)_\alpha, \quad B_{0,\alpha} = \log(w_\alpha - w) \tag{4.52}$$

Here, $f(w)_\alpha$ denotes the negative part of a formal expansion of $f(w)$ near the poles w_α, or, in the analytic setting, the principal part:

$$f(w)_\alpha = \sum_{i>0} \frac{f_i}{(w - w_\alpha)^i}, \quad \text{if} \quad f = \sum_{i \in Z} \frac{f_i}{(w - w_\alpha)^i} \tag{4.53}$$

These additional flows commute amongst themselves and with the ordinary (associated with poles at infinity) 1dToda or dKP flows ([65]).

4.3.4 Additional invariant flows of the 2dToda system.

In analogy to Krichever's extension of the dKP hierarchy in the case of rational z's (Benney system), we shall extend the 2dToda hierarchy to one applicable to rational $(z, \bar{z})$'s of the form (4.47),(4.48). As we have seen, even if the string equation is imposed, only the $(t_1, \bar{t}_1)$ flows preserve this reduction. However, still assuming first order poles only, we may make a partial fraction expansion

$$z(w) = rw + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha}, \tag{4.54}$$

$$\bar{z}(1/w) = r/w + \bar{u}_0 + \sum_{\beta=1}^{N} \frac{\bar{u}_\beta}{1/w - \bar{w}_\beta} \tag{4.55}$$

and introduce new flows generated by the equations:

$$\partial_{1,\alpha} z = \{H_{1,\alpha}, z\}, \qquad \partial_{1,\alpha} \bar{z} = \{\bar{H}_{1,\alpha}, \bar{z}\} \tag{4.56}$$

$$\partial_{1,\alpha} \bar{z} = \{H_{1,\alpha}, \bar{z}\}, \qquad \partial_{1,\beta} z = \{\bar{H}_{1,\beta}, z\} \tag{4.57}$$

$$\partial_{0,\alpha} z = \{H_{0,\alpha}, z\}, \qquad \partial_{0,\beta} \bar{z} = \{\bar{H}_{0,\beta}, \bar{z}\} \tag{4.58}$$

$$\partial_{0,\alpha} \bar{z} = \{H_{0,\alpha}, \bar{z}\}, \qquad \partial_{0,\beta} z = \{\bar{H}_{0,\beta}, z\} \tag{4.59}$$

Here, we transformed the flows from the Takasaki to the Lax-Sato gauge needed for our purposes:

$$H_{k,\alpha}(w) = B_{k,\alpha}(w) - \frac{1}{2} B_{k,\alpha}(w = 0), \tag{4.60}$$

$$\bar{H}_{k,\beta}(y) = \bar{B}_{k,\beta}(y) - \frac{1}{2} \bar{B}_{k,\beta}(y = 0), \qquad k = 0, 1. \tag{4.61}$$

where

$$\begin{aligned} &B_{1,\infty}(w) = z(w)_{\geq 0}, \quad B_{0,\alpha} = \log\left(r(w_\alpha - w)\right), \quad B_{1,\alpha}(w) = z(w)_\alpha \\ &\bar{B}_{1,\infty}(y) = \bar{z}(y)_{\geq 0}, \quad \bar{B}_{0,\beta} = \log\left(r(\bar{w}_\beta - y)\right), \quad \bar{B}_{1,\beta}(y) = \bar{z}(y)_\beta, \quad y = 1/w. \end{aligned} \tag{4.62}$$

As we have seen before, to be consistent, these systems must be restricted by a string equation, which makes (4.58) a finite-dimensional dynamical system.

Proposition 4.3 *The* $4N+2$ *commuting Toda-Krichever flows*

$$\begin{array}{c} \partial_{\tau_i} z = \{h_i, z\}, \quad \partial_{\bar{\tau}_i} z = \{\bar{h}_i, z\} \\ \partial_{\tau_i} \bar{z} = \{h_i, \bar{z}\}, \quad \partial_{\bar{\tau}_i} \bar{z} = \{\bar{h}_i, \bar{z}\} \end{array}, \qquad i = 0..2N \tag{4.63}$$

where

$$\begin{array}{ll} h_0 = H_{1,\infty} = rw + u_0/2, & \bar{h}_0 = \bar{H}_{1,\infty} = r/w + \bar{u}_0/2 \\ h_{2i-1} = H_{1,i} = \frac{u_i}{w - w_i} + \frac{u_i}{2w_i}, & \bar{h}_{2i-1} = \bar{H}_{1,i} = \frac{\bar{u}_i}{\bar{w}_i - 1/w} + \frac{1}{2}\bar{u}_i/\bar{w}_i \\ h_{2i} = H_{0,i} = \log r(w_i - w) + 1/2 \log(r/w_i), & \\ \bar{h}_{2i} = \bar{H}_{0,i} = \log r(\bar{w}_i - 1/w) + 1/2 \log(r/\bar{w}_i) & \end{array} \tag{4.64}$$

and

$$\tau_0 = t_{1,\infty}, \quad \tau_{2i-1} = t_{1,i}, \quad \tau_{2i} = t_{0,i}, \quad \bar{\tau}_0 = \bar{t}_{1,\infty}, \quad \bar{\tau}_{2i-1} = \bar{t}_{1,i}, \quad \bar{\tau}_{2i} = \bar{t}_{0,i} \tag{4.65}$$

preserve the rational form of $z(w)$ *(4.54) and* $\bar{z}(1/w)$ *(4.55) (or equivalently (4.48, 4.47)) provided the string equation (4.31) holds.*

Proof: We prove this proposition as a corollary to a similar statement for the logarithmic solutions below. See the proof of Corollary 1 given in the paper [84] attached.

Lemma 4.2 *All vector fields attached to the pole structure of rational maps formally commute, i.e. the evolution functions given in (4.60), (4.61) satisfy zero-curvature conditions.*

Proof: See the proof of Proposition 4 given in the paper [84] attached.

As in the polynomial case, the total number of invariant flows equals the dimension of the dynamical system minus one. In what follows we show that these flows are Hamiltonian. Since the dimension of the phase space is odd and equal to $4N+3$, it is, in fact a Poisson manifold where the dimension of the symplectic leaf is $4N+2$. The latter is exactly equal to the number of commuting Toda-Krichever flows.

Below we formulate these results for the more general setting of logarithmic maps, from which the above may be deduced as a limiting case. We introduce a logarithmic ansatz for the 2dToda hierarchy and prove lemma 4.2 and proposition 4.3 for logarithms followed by the limiting procedure.

4.3.5 Logarithmic flows.

As was mentioned above, it is easier to prove consistency of the rational ansatz with the dynamics of the 2dToda system using the more general logarithmic solutions. Let us set:

$$\begin{array}{ll} z = r(x)w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w), & \sum_{i=1}^{n+1} a_i = 0 \\ \bar{z} = r(x)w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log(\bar{w}_i(x) - w^{-1}), & \sum_{i=1}^{n+1} \bar{a}_i = 0 \end{array}, \quad i = 1..n+1 \tag{4.66}$$

where $a_i, \bar{a}_i$ are arbitrary constants, subject to conditions $\sum_{i=1}^{n+1} a_i = 0, \sum_{i=1}^{n+1} \bar{a}_i = 0$ which ensure the absence of logarithmic singularities at infinity. For the introduced ansatz (4.66) we have the following result:

Proposition 4.4 *Define the evolution functions as follows:*

$$\begin{array}{l} \mathcal{H}_0 = r(x)w + \frac{1}{2}u(x), \quad \bar{\mathcal{H}}_0 = \bar{r}(x)w^{-1} + \frac{1}{2}\bar{u}(x) \\ \mathcal{H}_i = \log(w_i(x) - w) + \frac{1}{2}\log(r(x)/w_i(x)) \qquad i = 1..n+1 \\ \bar{\mathcal{H}}_i = \log(\bar{w}_i(x) - w^{-1}) + \frac{1}{2}\log(r(x)/\bar{w}_i(x)) \end{array} \tag{4.67}$$

Then, the $2n+4$ flows generated by the Lax equations

$$\begin{array}{l} \partial_{T_i} z = \{\mathcal{H}_i, z\}, \quad \partial_{\bar{T}_i} z = \{\bar{\mathcal{H}}_i, z\} \\ \partial_{T_i} \bar{z} = \{\mathcal{H}_i, \bar{z}\}, \quad \partial_{\bar{T}_i} \bar{z} = \{\bar{\mathcal{H}}_i, \bar{z}\} \end{array}, i = 0..n+1 \tag{4.68}$$

commute. They preserve the logarithmic ansatz (4.66) provided the string equation (4.31) holds.

Proof: See the proof of Proposition 4 given in the paper [84] attached.

In other words, the 2dToda flows are tangent to the manifold of logarithmic functions if the string condition is imposed and we again have $2n+4$ flows leaving invariant a $2n+5$ dimensional submanifold of the 2dToda system.

Corollary 4.1 *The consistency of the rational ansatz follows from the consistency of the logarithmic ansatz.*

Proof: See the proof of Corollary 1 given in the paper [84] attached.

Thus we have obtained a rational ansatz as a limiting case of the logarithmic ansatz, merging pairs of logarithmic singularities together, in which case they become simple poles. In a similar way one may deduce any kind of rational maps containing a combination of poles of any degrees absorbing different numbers of logarithmic singularities.

4.3.6 Hamiltonian structure of rational and logarithmic reductions of the 2D dToda hierarchy.

In this section we study the Poisson structure of rational and logarithmic reductions of the 2dToda hierarchy, which, as we have seen from the above considerations, form a finite-dimensional completely integrable system. We find explicit expressions leading to a canonical Hamiltonian structure on the phase space of rational reductions of the 2D dToda system.

Hamiltonians and linearization variables.

Since the Toda flows commute, the Toda times are actually the linearization variables of the system.

Thus we have only to express the times through coordinates of the phase space. The times $\tau_i, \bar{\tau}_i$ of equation (4.68) are expressed in terms of z and $\bar{z}$ (4.66) as follows:

$$\begin{aligned} \tau_0 &= \frac{1}{2\pi\sqrt{-1}} \oint_\infty \bar{z} d\ln z + \text{const}, \quad \bar{\tau}_0 = \frac{1}{2\pi\sqrt{-1}} \oint_0 z d\ln \bar{z} + \text{const}, \\ \tau_i &= \frac{1}{2\pi\sqrt{-1}} \oint_{w_i} \bar{z} dz + \text{const}, \quad \bar{\tau}_i = \frac{1}{2\pi\sqrt{-1}} \oint_{1/\bar{w}_i} z d\bar{z} + \text{const}, \quad i = 1..n+1 \end{aligned} \tag{4.69}$$

Now, for the logarithmic ansatz (4.4), consider the following values:

$$\begin{aligned} I_0 &= \bar{z}(1/w=0) = \bar{u} + \textstyle\sum_{i=1}^{n+1} \bar{a}_i \ln(\bar{w}_i), \quad \bar{I}_0 = z(0) = u + \sum_{i=1}^{n+1} a_i \ln(w_i) \\ I_i &= a_i \bar{z}(w_i^{-1}) = a_i \left(r w_i^{-1} + \bar{u} + \textstyle\sum_{j=1}^{n+1} \bar{a}_j \ln(\bar{w}_j - w_i^{-1}) \right) \\ \bar{I}_i &= \bar{a}_i z(\bar{w}_i^{-1}) = \bar{a}_i \left(r \bar{w}_i^{-1} + u + \textstyle\sum_{j=1}^{n+1} a_j \ln(w_j - \bar{w}_i^{-1}) \right), \quad i = 1..n+1, \end{aligned} \tag{4.70}$$

and

$$\begin{aligned} Q &= \frac{1}{4\pi\sqrt{-1}} \textstyle\sum_{i=0}^{n+1} \oint_{1/\bar{w}_i} z d\bar{z} + \oint_{w_i} \bar{z} dz \\ &= \frac{1}{2} r \left(\left(\frac{\partial z(w)}{\partial w} \right)_{w=0} + \left(\frac{\partial \bar{z}(1/w)}{\partial (1/w)} \right)_{1/w=0} \right) - \frac{1}{2} \textstyle\sum_{i=1}^{n+1} (I_i + \bar{I}_i) \\ &= r^2 - \frac{1}{2} \textstyle\sum_{i=1}^{n+1} \left(r \left(\frac{a_i}{w_i} + \frac{\bar{a}_i}{\bar{w}_i} \right) + I_i + \bar{I}_i \right) \end{aligned} \tag{4.71}$$

In what follows we show that the values $I, \bar{I}$ form a set of the linearization variables of the system with Q being a Casimir for (4.68).

Proposition 4.5 *The values (4.70) form a set of integrals of the string equation (4.31), i.e., provided the string equation (4.31) holds.*

$$dI_i/dx = 0 \tag{4.72}$$

while for the Casimir (4.71) we have

$$dQ/dx = 1 \tag{4.73}$$

Proof: See proofs to Propositions 5 and 6 given in the paper [84] attached.

This proposition reduces the phase space to $(2n+5)$-dimension and implies that the quantities $I, \bar{I}$ form a coordinate system on this phase space.

We should mention here, that the values (4.70) are similar to those appearing in the work [75] in connection with the Laplacian Growth problem as integrals of the Laplacian Growth (string) equation. The following proposition (although not assuming a complex structure) confirms this fact.

Proposition 4.6 *The equations of motion (4.68) have the following form in $I, \bar{I}$*

$$\begin{aligned} \partial_{\tau_j} Q &= 0, \quad \partial_{\bar{\tau}_j} Q = 0 \\ \partial_{\tau_j} I_i &= \delta_{ij}, \quad \partial_{\bar{\tau}_j} I_i = 0 \\ \partial_{\tau_j} \bar{I}_i &= 0, \quad \partial_{\bar{\tau}_j} \bar{I}_i = -\delta_{ij}. \end{aligned} \tag{4.74}$$

These variables are canonical. The following Poisson structure holds

$$\{I_i, \bar{I}_j\}_p = \delta_{ij}, \quad \{I_i, I_j\}_p = \{\bar{I}_i, \bar{I}_j\}_p = 0, \quad \{Q, \bar{I}_j\}_p = \{Q, I_j\}_p = 0 \tag{4.75}$$

with Hamiltonians

$$\mathrm{H}_i = \bar{I}_i, \quad \bar{\mathrm{H}}_i = I_i \tag{4.76}$$

Proof: See the proof of Proposition 6 given in the paper [84] attached.

As was mentioned in the Statement of the problem, the Laplacian growth problem is endowed with a complex structure, with $\bar{z}$ being the complex conjugate of z (and bar denoting complex conjugation). Then, as may be seen from the proposition, the Casimir Q plays the role of the area (which grows with the unit speed), while I are functions of the harmonic moments of the boundary curve.

It is important to note that the Poisson brackets $\{,\}_p$ in (4.75) are different from $\{,\}$ in (4.29): The former defines the Poisson structure on the space of rational solutions of the 2dToda hierarchy, while the latter (Lax-Poisson bracket) is a dispersionless limit of the commutator, and only plays the role of a Poisson bracket on an auxiliary 2-dimensional phase space.

To prove the corresponding results for the rational case we have to take a limit as in Corollary 4.1.

Corollary 4.2 *Let z , $\bar{z}$ be represented as in (4.54)-(4.55). The following $2N+1$ values ($i = 1..N$)*

$$\begin{aligned}
I_0 &= \bar{u}_0 - \textstyle\sum_{i=1}^N \bar{u}_i/\bar{w}_i, & \bar{I}_0 &= u_0 - \textstyle\sum_{i=1}^N u_i/w_i \\
I_{2i-1} &= rw_i^{-1} + \bar{u}_0 + \textstyle\sum_{j=1}^N \frac{\bar{u}_j}{1/w_i - \bar{w}_j}, & \bar{I}_{2i-1} &= r\bar{w}_i^{-1} + u_0 + \textstyle\sum_{j=1}^N \frac{u_j}{1/\bar{w}_i - w_j} \\
I_{2i} &= \left(r - \textstyle\sum_{j=1}^N \frac{\bar{u}_j}{(1/w_i - \bar{w}_j)^2}\right) \frac{u_i}{w_i^2}, & \bar{I}_{2i} &= \left(r - \textstyle\sum_{j=1}^N \frac{u_j}{(1/\bar{w}_i - w_j)^2}\right) \frac{\bar{u}_i}{\bar{w}_i^2}
\end{aligned}$$

$$Q = r^2 - \frac{1}{2}\sum_{i=1}^{N}\left(r\left(\frac{\bar{u}_i}{\bar{w}_i^2} + \frac{u_i}{w_i^2}\right) + \bar{I}_{2i} + I_{2i}\right)$$

are action-angle variables in the rational case (proposition 4.3), i.e. the variables for which the equations of Proposition (4.3) have the form (4.74).

Again, in the rational limit we may choose the Poisson structure as in (4.75), (4.76).

4.4 Bihamiltonian structure of 2D dispersionless Toda.

4.4.1 Algebras of difference operators

We will consider the following linear spaces of formal difference operators with coefficients in $C^\infty(\mathcal{R})$:

$$A^+ = \{ \sum_{k<<+\infty} u_k(x)\Lambda^k\} \tag{4.77}$$

$$A^- = \{ \sum_{k>>-\infty} u_k(x)\Lambda^k\} \tag{4.78}$$

and
$$A^0 = A^+ \cap A^-$$

The spaces A^+, A^-, A^0 are associative algebras with the usual multiplication defined by $\Lambda f(x) = f(x+\hbar)\Lambda$; hence they are Lie algebras with a Lie bracket given by the commutator. On these spaces we have the natural projections on the positive and negative parts, defined by

$$(\sum_{k\in\mathcal{Z}} u_k(x)\Lambda^k)_+ = \sum_{k\geq 0} u_k(x)\Lambda^k \tag{4.79}$$

$$(\sum_{k\in\mathcal{Z}} u_k(x)\Lambda^k)_- = \sum_{k<0} u_k(x)\Lambda^k \tag{4.80}$$

and

$$X_+ + X_- = X \tag{4.81}$$

for any difference operator X of the form

$$X = \sum_{k\in\mathcal{Z}} X_k(x)\Lambda^k. \tag{4.82}$$

We will also use the notations $X_{>0}$ and $X_{\leq 0}$. The residue is defined by

$$ResX := X_0, \tag{4.83}$$

The trace-form of a difference operator is defined by

$$TrX := \int ResX dx. \tag{4.84}$$

From translation invariance of the integral $\int f dx = \int (\Lambda f) dx$ it follows that $Tr[X,Y] = 0$ The bilinear pairing

$$(X,Y) := TrXY \tag{4.85}$$

gives a non-degenerate symmetric inner product on A^+, A^- and A^0.

4.4.2 R-matrix structure of 2D Toda

The Poisson structures of the 1D Toda hierarchy are related to the splitting of a difference operator in its positive and negative parts. In this case the linear endomorphism $R : A^+ \to A^+$ for $A^+ = (A^+)_+ \oplus (A^-)_-$ of the form

$$R(X) = X_+ - X_- \tag{4.86}$$

automatically satisfies the modified Yang-Baxter equation.

Since the two-dimensional Toda hierarchy is characterized by two Lax operators L and $\bar{L}$ that are respectively elements of A^+ and A^-, it is natural to consider $A^+ \oplus A^-$ as the correct algebra in this case. An introduction of an algebra of difference operators as a direct product of A^+ and A^- for two-dimensional Toda hierarchy (actually, for its generalization, which is the extended bigraded Toda hierarchy) was done in a work by G.Carlet [67].

The natural inner product on $A^+ \oplus A^-$ is defined in the obvious way from the trace form

$$TrX \oplus \bar{X} = TrX + Tr\bar{X} \tag{4.87}$$

where $X \oplus \bar{X} \in A^+ \oplus A^-$. For the Lax-Sato 2D Toda we will define the form of the R-matrix operator on $A^+ \oplus A^-$ in a different way from one given by G.Carlet for the bigraded 2D Toda hierarchy in Takasaki's formulation [67]. In his case the R-matrix is:

$$R_{Carlet}(X, \bar{X}) = (X_+ - X_- + 2\bar{X}_-, \bar{X}_- - \bar{X}_+ + 2X_+) \tag{4.88}$$

We choose a different splitting for the Lax-Sato R-matrix and take the skew-symmetric part of the (4.88), satisfying $R = -R^*$:

$$R(X, \bar{X}) = (X_{>0} - X_{<0} - \bar{X}_0, \bar{X}_{<0} - \bar{X}_{>0} + X_0) \tag{4.89}$$

where $(X, \bar{X}) \in A^+ \ominus A^-$.

Note that this R-matrix is not simply given by a sum of split R-matrices which are usually considered on A^+ or A^- (say, for 1D Toda or KP hierarchy) due to existence of coupling between L and $\bar{L}$ (4.9) that must be taken into account.

Proposition 4.7 *On a direct sum of algebras $A^+ \oplus A^-$, the skew-symmetric R-matrix*

$$R(X, \bar{X}) = (X_{>0} - X_{<0} - \bar{X}_0, \bar{X}_{<0} - \bar{X}_{>0} + X_0)) \tag{4.90}$$

satisfies the modified Yang-Baxter equation.

Proof: Checked by direct substitution to mYBE.

In general, theorems, there exist three compatible Poisson structures on $A^+ \ominus A^-$, see [67], [9]. In the sequel we will consider only the first two structures.

4.4.3 Explicit form of Poisson brackets

In this subsection we provide an explicit form of linear and quadratic Poisson structures for both, Takasaki and Lax-Sato, gauges using R-matrix operator depending on the gauge parameter α.

We start with the linear Poisson structure. Let N be a positive integer. The explicit form of the linear Poisson structure for the two-dimensional Toda hierarchy is calculated in $(L, \bar{L})$ where

$$L = r(x)\Lambda + \sum_{k=1}^{N} u_k(x)\Lambda^{-k} \tag{4.91}$$

$$\bar{L} = r(x-\hbar)\Lambda^{-1} + \sum_{k=1}^{N} \bar{u}_k(x)\Lambda^{k} \tag{4.92}$$

Here, for the Lax-Sato gauge corresponds $\alpha = 0$ and for the Ueno-Takasaki gauge we take $\alpha = 1/2$:

Linear brackets

$$\begin{aligned} \{u_n(x), \bar{u}_m(y)\}_1 &= d_m\left[u_{n+m}(x)(\Lambda^n\delta(x-y) - (\Lambda^{-m}u_{n+m}(x)\delta(x-y))\right] + \\ &+ \bar{b}_n\left[\bar{u}_{n+m}(x)(\Lambda^n\delta(x-y) - (\Lambda^{-m}\bar{u}_{n+m}(x)\delta(x-y))\right] \end{aligned} \tag{4.93}$$

$$\begin{aligned} \{\bar{u}_n(x), u_m(y)\}_1 &= d_n\left[u_{n+m}(x)(\Lambda^n\delta(x-y) - (\Lambda^{-m}u_{n+m}(x)\delta(x-y))\right] + \\ &+ \bar{b}_m\left[\bar{u}_{n+m}(x)(\Lambda^n\delta(x-y) - (\Lambda^{-m}\bar{u}_{n+m}(x)\delta(x-y))\right] \end{aligned} \tag{4.94}$$

$$\begin{aligned} \{\bar{u}_n(x), \bar{u}_m(y)\}_1 &= (\bar{c}_n + \bar{c}_m)\left[\bar{u}_{n+m}(x)(\Lambda^n\delta(x-y) - \right. \\ &\left. -(\Lambda^{-m}\delta(x-y)\bar{u}_{n+m}(x))\right] \end{aligned} \tag{4.95}$$

$$\begin{aligned} \{u_n(x), u_m(y)\}_1 &= (a_n + a_m)\left[u_{n+m}(x)(\Lambda^n\delta(x-y) - \right. \\ &\left. -(\Lambda^{-m}\delta(x-y)u_{n+m}(x))\right] \end{aligned} \tag{4.96}$$

where we have defined:

$$c(n,\alpha) = \begin{cases} 1 & , \text{if } n > 0 \\ 2\alpha & , \text{if } n = 0 \\ 0 & , \text{if } n < 0 \end{cases}$$

$$c(n) = \begin{cases} 1 & , \text{if } n > 0 \\ 1 & , \text{if } n = 0 \\ 0 & , \text{if } n < 0 \end{cases}$$

$$a_n = c(n) + c(n,\alpha) - 1 \tag{4.97}$$

$$\bar{b}_n = c(n,\alpha) - c(n) + 4\alpha(1 - c(n)) \tag{4.98}$$

$$\bar{c}_n = 1 - c(n) - c(n,\alpha) \tag{4.99}$$

$$d_n = c(n,\alpha) + c(n) + (4\alpha - 2)(1 - c(-n)) \tag{4.100}$$

The quadratic R-matrix Poisson structure is given by

Quadratic (Sklyanin) brackets:

$$\begin{aligned}\{u_n(x), u_m(y)\}_2 &= -2u_n(\Lambda^n u_m\delta(x-y) + 2u_m(\Lambda^{-m} u_m\delta(x-y)) + \\ &+2u_n(\Lambda^{n-m} u_m\delta(x-y)) - 2u_m u_n\delta(x-y) + 4(\sum_{k=1}^{1-n}\big[u_{n+m-k} \times \\ &\times(\Lambda^{n-k} u_k\delta(x-y)) - u_k(\Lambda^{k-m} u_{n+m-k}\delta(x-y))\big] \end{aligned} \tag{4.101}$$

$$\begin{aligned}\{u_n(x), \bar{u}_m(y)\}_2 &= \frac{1}{2}u_n(\Lambda^n \bar{u}_m\delta(x-y) + 2u_n(\Lambda^{-m} \bar{u}_m\delta(x-y)) + \\ &+2u_n(\Lambda^{n-m} \bar{u}_m\delta(x-y)) - 2\bar{u}_m u_n\delta(x-y) + 4(\sum_{k=1}^{1-n}\big[u_{n+m-k} \times \\ &\times(\Lambda^{n-k} \bar{u}_k\delta(x-y)) - u_k(\Lambda^{k-m} \bar{u}_{n+m-k}\delta(x-y))\big] \end{aligned} \tag{4.102}$$

$$\begin{aligned}\{\bar{u}_n(x), u_m(y)\}_2 &= -2\bar{u}_n(\Lambda^n u_m\delta(x-y) + 2\bar{u}_n(\Lambda^{-m} u_m\delta(x-y)) + \\ &+2\bar{u}_n(\Lambda^{n-m} u_m\delta(x-y)) - 2u_m \bar{u}_n\delta(x-y) + 4(\sum_{k=1}^{1-n}\big[\bar{u}_{n+m-k} \times \\ &\times(\Lambda^{n-k} u_k\delta(x-y)) - \bar{u}_k(\Lambda^{l-m} u_{n+m-k}\delta(x-y))\big] \end{aligned} \tag{4.103}$$

$$\begin{aligned}\{\bar{u}_n(x), \bar{u}_m(y)\}_2 &= -2\bar{u}_n(\Lambda^n \bar{u}_m\delta(x-y) + 2\bar{u}_n(\Lambda^{-m} \bar{u}_m\delta(x-y)) + \\ &+2\bar{u}_n(\Lambda^{n-m} \bar{u}_m\delta(x-y)) - 2\bar{u}_m \bar{u}_n\delta(x-y) + 4(\sum_{k=1}^{1-n}\big[\bar{u}_{n+m-k} \times \\ &\times(\Lambda^{n-k} \bar{u}_k\delta(x-y)) - \bar{u}_k(\Lambda^{k-m} \bar{u}_{n+m-k}\delta(x-y))\big] \end{aligned} \tag{4.104}$$

4.4.4 Dispersionless limit of Poisson brackets

In this subsection we consider dispersionless limit of the first and the second Poisson structures for the 2D Toda hierarchy. The dispersionless brackets are given by the leading term of the dispersive brackets (4.93)-(4.104) in the $\hbar \to 0$ limit. Namely

$$\{u_n(x), \bar{u}_m(y)\}_{1,2} = \hbar\{u_n(x), \bar{u}_m(y)\}_{1,2}{}^{(disp)} + O(\hbar^2) \tag{4.105}$$

The give here the explicit formulas for the dispersionless brackets.

Linear dispersionless brackets

$$\begin{aligned}\{u_n(x), \bar{u}_m(y)\}_1^{(disp)} &= d_m\left[m u'_{n+m}(x)\delta(x-y) + (n+m)u_{n+m}(x)\delta'(x-y)\right] - \\ &\quad +\bar{b}_n\left[m\bar{u}'_{n+m}(x)\delta(x-y) + \right. \\ &\quad \left. +(n+m)\bar{u}_{n+m}(x)\delta'(x-y)\right]\end{aligned} \tag{4.106}$$

$$\begin{aligned}\{\bar{u}_n(x), u_m(y)\}_1^{(disp)} &= d_n\left[m u'_{n+m}(x)\delta(x-y) + (n+m)u_{n+m}(x)\delta'(x-y)\right] - \\ &\quad +\bar{b}_m\left[m\bar{u}'_{n+m}(x)\delta(x-y) + \right. \\ &\quad \left. +(n+m)\bar{u}_{n+m}(x)\delta'(x-y)\right]\end{aligned} \tag{4.107}$$

$$\begin{aligned}\{\bar{u}_n(x), \bar{u}_m(y)\}_1^{(disp)} &= (\bar{c}_n - \bar{c}_m)\left[m\bar{u}'_{n+m}(x)\delta(x-y) + \right. \\ &\quad \left. +(n+m)\bar{u}_{n+m}(x)\delta'(x-y)\right]\end{aligned} \tag{4.108}$$

$$\begin{aligned}\{u_n(x), u_m(y)\}_1^{(disp)} &= (a_n + a_m)\left[m u'_{n+m}(x)\delta(x-y) + \right. \\ &\quad \left. +(n+m)u_{n+m}(x)\delta'(x-y)\right]\end{aligned} \tag{4.109}$$

Quadratic dispersionless brackets

$$\begin{aligned}\{u_n(x), u_m(y)\}_2^{(disp)} &= (\alpha n - (1-\alpha)m)\{\left[u_n(x)u_m(x) + \sum_{k<m}(n+m-2k) \times \right. \\ &\quad \left. \times u_{n+m-k}(x)u_k(x)\right]\delta'(x-y) + \left[u_n(x)u'_m(x) + \right. \\ &\quad + \sum_{k<m}((n-k)u_{n+m-k}(x)u'_k(x) + (m-k) \times \\ &\quad \left. \times u'_{n+m-k}(x)u_k(x))\right]\delta(x-y)\}\end{aligned} \tag{4.110}$$

$$\{u_n(x),\bar u_m(y)\}_2^{(disp)} = (\alpha n-(1-\alpha)m)\{\big[u_n(x)\bar u_m(x)+\sum_{k<m}(-n-m)\times \times u_{n+m-k}(x)\bar u_k(x)\big]\delta'(x-y)+\big[u_n(x)\bar u'_m(x)+ +\sum_{k<m}((k-m)u'_{n+m-k}(x)\bar u_k(x)-(n+2m-k)\times \times u_{n+m-k}(x)\bar u'_k(x))\big]\delta(x-y)\} \quad (4.111)$$

$$\{\bar u_n(x),u_m(y)\}_2^{(disp)} = (\alpha m-(1-\alpha)n)\{\big[\bar u_n(x)u_m(x)+\sum_{k<m}(n+m)\times \times u_{n+m-k}(x)\bar u_k(x)\big]\delta'(x-y)+\big[\bar u_n(x)u'_m(x)+ +\sum_{k<m}((k+m)u'_{n+m-k}(x)\bar u_k(x)-(n-k)\times \times u_{n+m-k}(x)\bar u'_k(x))\big]\delta(x-y)\} \quad (4.112)$$

$$\{\bar u_n(x),\bar u_m(y)\}_2^{(disp)} = (\alpha m-(1-\alpha)n)\{\big[\bar u_n(x)\bar u_m(x)+\sum_{k<m}(n+m-2k)\times \times \bar u_{n+m-k}(x)\bar u_k(x)\big]\delta'(x-y)+\big[\bar u_n(x)\bar u'_m(x)+ +\sum_{k<m}((n-k)\bar u_{n+m-k}(x)\bar u'_k(x)+(m-k)\times \times \bar u'_{n+m-k}(x)\bar u_k(x))\big]\delta(x-y)\} \quad (4.113)$$

4.4.5 Poisson structure of the 1dToda hierarchy.

In this section we consider the Poisson structure of rational reductions of the 1dToda system.

Recall, that for the 1dToda system one takes into account only a "half" of flows, connected with z (but not $\bar z$, or vice versa).

$$\partial_{t_i} z=\{H_i,z\},\quad H_i=(z(w)^i)_+ + 1/2(z(w)^i)_0,\quad i=0..\infty \quad (4.114)$$

This system is bi-Hamiltonian (for general information e.g. see [69], [80]) with two (linear and quadratic) compatible Poisson structures. As it follows from the preceding section, the dispersionless linear Poisson brackets for the "field variables" $u_i, i = 1..\infty$ (4.26) in the Lax-Sato formulation have the following form

$$\{u_n(x), u_m(y)\}_1 = 2(c_n + c_m - 1)\big[(n+m)u_{n+m}(x)\delta'(x-y) + m u'_{n+m}(x)\delta(x-y)\big] \quad (4.115)$$

where

$$c_k = \begin{cases} 1 & \text{, if } k > 0 \\ 1/2 & \text{, if } k = 0 \\ 0 & \text{, if } k < 0 \end{cases}$$

while the quadratic brackets are

$$\begin{aligned} \{u_n(x), u_m(y)\}_2 \;=\; & \big[\frac{1}{2}(n-m)u_n(x)u'_m(x) + (\sum_{k=1}^{1-n}(n-m+k)u_{n+k}(x)u'_{m-k}(x) + \\ & + k u'_{n+k}(x)u_{m-k}(x))\big]\delta(x-y) + \big[1/2(n-m)u_n u_m + \\ & + (\sum_{k-1}^{1-n}(n-m+2k)u_{n+k}u_{m-k})\big]\delta(x-y) \end{aligned} \quad (4.116)$$

As seen from the lemma 4.1, the rational functions

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{w^{N+1} + \sum_{i=0}^{N} a_i w^i}{\sum_{i=0}^{N} b_i w^i} \quad (4.117)$$

are form-invariant under all of the 1dToda flows ∂_{t_i} (4.114), without any extra restriction (i.e. the string equation is not needed).

We obtain the corresponding Poisson structures for coefficients a_i, b_i, by using result (4.116), expressing u_i in terms of $a_i, b_i, i = 0..N$. Both the linear (4.115) and quadratic (4.116) Poisson structures lead to the quadratic brackets for a_i, b_i . Namely , the second Poisson structure for (4.117) reads as follows

$$\begin{aligned} \{a_k(x), a_l(y)\}_2 = \big[\sum_{n=1}(l+n-k)a_{k-n}(x)a_{l+n}(y) + n a_{k-n}(y)a_{l+n}(x)) \\ + (l-N-1)a_k(x)a_l(y)\big]\delta'(x-y) \end{aligned} \quad (4.118)$$

$$\begin{aligned} \{b_k(x), b_l(y)\}_2 = \big[\sum_{n=1}(k-l-n)b_{k-n}(x)b_{l+n}(y) - n b_{k-n}(y)b_{l+n}(x)) + \\ + \frac{k-l}{2}b_k(x)b_l(y)\big]\delta'(x-y) \end{aligned} \quad (4.119)$$

$$\{a_k(x), b_l(y)\}_2 = \frac{k-N-1}{2} a_k(x) b_l(y) \delta'(x-y) \tag{4.120}$$

The first Poisson structure can be obtained from (4.118 - 4.120) with the help of the linear transformation (shift by a constant)

$$a_i \longrightarrow a_i + \lambda b_i, \quad z(w,x) \longrightarrow z(w,x) + \lambda \tag{4.121}$$

and using the bi-Hamiltonian nature of (4.115), (4.116).

$$\begin{aligned}\{a_k(x), a_l(y)\}_1 = \Big[\sum_{n=1} (k-l-n)(a_{k-n}(x)b_{l+n}(y) + b_{k-n}(x)a_{l+n}(y)) - \\ -n(a_{k-n}(y)b_{l+n}(x) + b_{k-n}(y)a_{l+n}(x))) + \frac{N+1-l}{2} b_k(x)a_l(y) + \\ + \frac{k+N+1-2l}{2} a_k(x)b_l(y)\Big]\delta'(x-y)\end{aligned} \tag{4.122}$$

$$\begin{aligned}\{a_k(x), b_l(y)\}_1 = \Big[\sum_{n=1} ((k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x)) + \\ + \frac{N+1-l}{2} b_k(x)b_l(y)\Big]\delta'(x-y)\end{aligned} \tag{4.123}$$

$$\{b_k(x), b_l(y)\}_1 = 0 \tag{4.124}$$

in all the above expressions $a_{N+1} = 1$ and $a_i = 0$ if i goes beyond the range $i = 0..N+1$ (and $b_j = 0$ if $j \neq 0..N$).

These brackets form a bi-Hamiltonian structure for rational reductions of the 1dToda hierarchy:

$$\partial_{t_i} z = \{\mathrm{H}_i, z\}_1 = \{\mathrm{H}_{i-1}, z\}_2 \tag{4.125}$$

with the following Hamiltonians:

$$\mathrm{H}_i = \frac{1}{i+1} \int (z^{i+1}(x))_0 dx \tag{4.126}$$

4.4.6 Conclusions.

We have established the Hamiltonian structure on the space of rational solutions of the 2dToda hierarchy connected with the problem of ideal interface dynamics. A further challenging application of this result would be to use it in the introduction of a surface tension term for the Laplacian growth process.

The Laplacian growth equation (4.5), (4.31) describes the propagation of the boundary with zero surface tension between the fields. In addressing a number of important questions (e.g. finger width selection problem in [77]), such an idealized model does not account for essential physical features, such as fractal formation, stability etc. Thus, the inclusion of tension effects is important for the solution of the problem.

There are two approaches to look for in such a generalization. The first is to introduce tension terms in the theory, destroying the integrability of the problem. In another approach one might look for integrable deformations of the idealized model, simulating surface effects and stabilizing interface dynamics.

Another feature that gives our result a certain interest lies in the possible investigation of the perturbed (by small surface tension) system given in terms of separated variables (4.75). In other words, is it possible, in some situations, to approximate the perturbed solution by the multi-finger ansatz (4.66) with different dependence of coordinates on time x? If the answer is affirmative we will get a finite-dimensional dynamical system, conveniently written in terms of $I, \bar{I}, Q$.

Finally, returning to the idealized problem, one can further elaborate the theory of rational reductions from the point of view of symmetries. An important observation of our study is that once a rational reduction is compatible with 2dToda dynamics, the string condition is satisfied automatically. In other words *the string (Laplacian growth) equation (4.5), (4.31) turns out to be a consequence of rationality* in the context of the 2dToda hierarchy. On the other hand, it is well known that the string equation is connected with additional (Orlov-Schulman [69],[82]) symmetries of the 2dToda hierarchy [70]. It would be interesting to find whole sets of symmetry constraints defining finite-dimensional solutions of the 2dToda hierarchy and containing (4.31) as a special element.

Another important direction related with a problem of finding "higher genus" analogs of the obtained systems (related, for instance, with the motion of oil with several water spots in it or other kinds of multiregion flows). There appeared some progress in this direction done in the work [97], where the formulation of the dipersionless Toda hierarchy is done on higher genus curves. Another interesting feature of this investigation is the description of many string equations that may be related to a different types of reductions and are related to different Hamiltonian formulation. An application of these results to our case may shed more light on understanding the problem.

Amongst the other open questions and future directions is the following problem: to investigate the Hamiltonian structure of rational reductions in the context of the two-matrix [81] model, whose partition function is a tau-function of the 2dToda system constrained by the string equation. The applicability of our study to the models of normal matrices is another interesting aspect worthy of further analysis [66].

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Appendix 1. Publications

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Two-dimensional Krall–Sheffer polynomials and integrable systems

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Abstract
Two-dimensional Krall–Sheffer polynomials are analogues of the classical orthogonal polynomial. They are eigenfunctions of second-order linear partial differential operators and moreover satisfy orthogonality conditions. We show that all Krall–Sheffer polynomials are connected with two-dimensional superintegrable systems on spaces with constant curvature.

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Assume that $P_n(x, y)$ are polynomials in two variables x, y. As usual, the degree n is the maximal value $n = \max\{i + j\}$ among all possible monomials $x^i y^j$ in the expansion of the polynomial $P_n(x)$.

Krall and Sheffer considered [5] the problem of finding all polynomials $P_n(x, y)$ with the following properties:

(i) The polynomials $P_n(x, y)$ are eigenfunctions of a second-order admissible differential operator L (to be fully defined later)

$$LP_n(x, y) = \lambda_n P_n(x, y) \qquad (1)$$

with polynomial coefficients:

$$L = A(x, y)\partial_{xx} + 2B(x, y)\partial_{xy} + C(x, y)\partial_{yy} + D(x, y)\partial_x + E(x, y)\partial_y \qquad (2)$$

where $A(x, y), \ldots, E(x, y)$ are polynomials in x and y with *real* coefficients. Note that the eigenvalue λ_n depends only on the degree of the polynomial $P_n(x, y)$.

(ii) There exists a nondegenerate linear functional σ defined on the space of all polynomials in two variables such that the orthogonality property

$$\langle \sigma, P_n(x, y) q(x, y) \rangle = 0 \tag{3}$$

holds with $q(x, y)$ any polynomial of degree $<n$. The functional σ can be defined through its moments $\langle \sigma, x^n y^m \rangle = c_{nm}$, where $n, m = 0, 1, 2, \ldots$. By describing a functional as 'nondegenerate' one means that it has the property that if one has $\psi(x, y)\sigma = 0$ for some polynomial $\psi(x, y)$, then $\psi(x, y) \equiv 0$.

The orthogonality property (3) is closely connected with the symmetrizability of the operator L. Recall that the *Lagrange adjoint* of the operator L in equation (2) is defined as [6]

$$L^+ = \partial_{xx} A(x, y) + 2\partial_{xy} B(x, y) + \partial_{yy} C(x, y) - \partial_x D(x, y) - \partial_y E(x, y). \tag{4}$$

The operator L is *symmetric* if $L^+ = L$. The operator L is *symmetrizable* if there exists a real function $\rho(x, y)$ such that the operator $\rho(x, y)L$ is symmetric. As shown in [2], the properties (i), (ii) (given that the functional σ is nondegenerate) imply the symmetrizability of the operator L.

Later on, Engelis [2] independently considered the same problem from a slightly different point of view and found the same classification scheme. In what follows we will use the Engelis scheme which is more convenient for our purposes.

Before presenting the classification scheme given in [2], we recall some facts concerning admissible differential operators L [3, 4, 7].

The differential operator L in equation (2) is called *admissible* if for any positive integer n there exists $n + 1$ linearly independent polynomial eigenvalue solutions of degree n:

$$L Q_n^{(i)}(x, y) = \lambda_n Q_n^{(i)} \qquad i = 0, 1, \ldots, n \tag{5}$$

and there are no polynomial solutions having degree less than n for the same value λ_n.

It can be easily shown that the operator L is admissible if and only if the coefficients $A(x, y), \ldots, E(x, y)$ are of the form [7]

$$\begin{aligned} A(x, y) &= \alpha x^2 + a_{10} x + a_{01} y + a_{00} \\ B(x, y) &= \alpha x y + b_{10} x + b_{01} y + b_{00} \end{aligned} \tag{6}$$

$$\begin{aligned} C(x, y) &= \alpha y^2 + c_{10} x + c_{01} y + c_{00} \\ D(x, y) &= \beta x + d_0 \qquad E(x, y) = \beta y + e_0 \end{aligned} \tag{7}$$

where $\alpha, \beta, a_{ik}, b_{ik}, c_{ik}, d_0, e_0$ $(i, k = 0, 1)$ are arbitrary real parameters with the only restriction $\alpha p + \beta \neq 0$ for $p = 0, 1, 2, \ldots$. The eigenvalues are then

$$\lambda_n = n(\alpha(n - 1) + \beta). \tag{8}$$

Note that for admissible polynomials, eigenvalues are nondegenerate, i.e. $\lambda_n \neq \lambda_m$ for $n \neq m$.

There is an obvious geometrical interpretation of the admissible operators (we follow [8]). First of all, perform a similarity transformation of the operator L with some function $\Phi(x, y)$:

$$\begin{aligned} \tilde{L} = \Phi^{-1}(x, y) L \Phi(x, y) &= A(x, y)\partial_{xx} + 2B(x, y)\partial_{xy} + C(x, y)\partial_{yy} \\ &\quad + \tilde{D}(x, y)\partial_x + \tilde{E}(x, y)\partial_y + U(x, y) \end{aligned} \tag{9}$$

where

$$\tilde{D}(x, y) = D(x, y) + 2\frac{A(x, y)\Phi_x(x, y) + B(x, y)\Phi_y(x, y)}{\Phi(x, y)}$$

$$\tilde{E}(x, y) = E(x, y) + 2\frac{C(x, y)\Phi_y(x, y) + B(x, y)\Phi_x(x, y)}{\Phi(x, y)}$$

and

$$U(x, y) = \frac{A\Phi_{xx} + C\Phi_{yy} + 2B\Phi_{xy} + D\Phi_x + E\Phi_y}{\Phi(x, y)}.$$

The operator $\tilde{L}$ (9) can be presented in a form close to that of the Laplace–Beltrami operator associated with a metric $g_{ik}(x, y)$. Indeed, assume that a two-dimensional metric tensor $g_{ik}(x, y)$ is given. This means that for the length element $\mathrm{d}s$ we have $\mathrm{d}s^2 = g_{11}(x, y)\,\mathrm{d}x^2 + g_{22}(x, y)\,\mathrm{d}y^2 + 2g_{12}(x, y)\,\mathrm{d}x\,\mathrm{d}y$. Then, the Laplace–Beltrami operator Δ_{LB} is defined as [1]

$$\Delta_{\mathrm{LB}} = f(x, y)^{1/2}\partial_i g^{ik}(x, y) f(x, y)^{-1/2}\partial_k \tag{10}$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and $f(x, y) = \det \|g^{ik}(x, y)\| = \det \|g_{ik}(x, y)\|^{-1}$. From (10) we have

$$\Delta_{\mathrm{LB}} = g^{11}\partial_{xx} + g^{22}\partial_{yy} + 2g^{12}\partial_{xy} + S_1(x, y)\partial_x + S_2(x, y)\partial_y \tag{11}$$

where

$$S_1(x, y) = \frac{\partial g^{11}}{\partial x} + \frac{\partial g^{21}}{\partial y} - \frac{1}{2}f^{-1}(x, y)\left(g^{11}\frac{\partial f(x, y)}{\partial x} + g^{12}\frac{\partial f(x, y)}{\partial y}\right)$$

and

$$S_2(x, y) = \frac{\partial g^{12}}{\partial x} + \frac{\partial g^{22}}{\partial y} - \frac{1}{2}f^{-1}(x, y)\left(g^{12}\frac{\partial f(x, y)}{\partial x} + g^{22}\frac{\partial f(x, y)}{\partial y}\right).$$

Let us compare expression (9) for the operator $\tilde{L}$ with expression (11). It is natural to make the following identifications:

$$g^{11} = A(x, y) \qquad g^{12} = B(x, y) \qquad g^{22} = C(x, y). \tag{12}$$

Then $f(x, y) = A(x, y)C(x, y) - B^2(x, y)$ and we have

$$\tilde{L} = \Delta_{\mathrm{LB}} + T_1(x, y)\partial_x + T_2(x, y)\partial_y + U(x, y) \tag{13}$$

where

$$T_1(x, y) = \tilde{D}(x, y) - A_x(x, y) - B_y(x, y) + \frac{Bf_y(x, y) + Af_x(x, y)}{2f(x, y)}$$

and

$$T_2(x, y) = \tilde{E}(x, y) - B_x(x, y) - C_y(x, y) + \frac{Bf_x(x, y) + Cf_y(x, y)}{2f(x, y)}.$$

So, under the identification (12) we see from (13) that the operator $\tilde{L}$ coincides with the Laplace–Beltrami operator Δ_{LB} up to terms containing only the first derivatives and the 'potential' $U(x, y)$.

It is natural to ask whether a function $\Phi(x, y)$ exists such that the condition

$$T_1(x, y) = T_2(x, y) \equiv 0 \tag{14}$$

holds.

If (14) is valid, then we have

$$\tilde{L} = \Delta_{\mathrm{LB}} + U(x, y). \tag{15}$$

On the other hand, it is well known [1] that the Laplace–Beltrami operator Δ_{LB} plays the role of the free-motion Hamiltonian for a quantum mechanical particle on a Riemannian space with the metric $g_{ik}(x, y)$. Hence, condition (14) says that the operator $\tilde{L}$ coincides with the Schrödinger operator on this Riemannian space with the potential $U(x, y)$.

If condition (14) cannot be satisfied then it can be interpreted as indicating the presence of a *magnetic field*. Therefore, condition (14) reflects the absence of magnetic fields.

It is easily seen [8] that condition (14) is equivalent to the condition

$$\partial_y\left(\frac{CK_1 - BK_2}{f}\right) = \partial_x\left(\frac{AK_2 - BK_1}{f}\right) \tag{16}$$

where

$$K_1(x, y) = \frac{4A_x f + 4B_y f - Af_x - Bf_y - D}{4f}$$
$$K_2(x, y) = \frac{4B_x f + 4C_y f - Bf_x - Cf_y - E}{4f}.$$

We see that any admissible operator L depends on 13 parameters. It is natural to call the ten parameters $\alpha, a_{ik}, b_{ik}, c_{ik}$ *internal parameters*. Indeed, these parameters define the metric tensor $g_{ik}(x, y)$ and hence describe the geometrical properties of the operator L. The remaining three parameters β, d_0, e_0 will be called *external parameters*. These parameters describe the interaction of our system with external fields.

Of course, it is possible to reduce the number of independent internal parameters by means of affine transformations of the independent arguments x, y. We will describe this procedure following [7].

Consider all invertible *affine* transformations of the form

$$\begin{aligned} x &= q_{11}\xi + q_{12}\eta + q_{10} \\ y &= q_{21}\xi + q_{22}\eta + q_{20} \end{aligned} \tag{17}$$

with some coefficients q_{ik}. Then it is easily shown [7] that if L is admissible in coordinates x, y, then L remains admissible in the new coordinates ξ, η. Moreover, property (ii) and the symmetrizability of the operator L are also preserved under the transformation (17). Property (14) is also preserved under affine transformations. This means that if the operator L is reduced to the form (15) without magnetic field then the affine-transformed operator L can also be reduced to the same form.

The parameter α is preserved under affine transformations. Hence we can make a division into two cases: $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$ we can put $\alpha = 1$ without loss of generality. Indeed, in this case we can divide the lhs and rhs of equation (1) by α. This will lead only to a renormalization of the remaining nine internal and three external parameters.

As affine transformation (17) contains six independent parameters, it is possible to reduce the nine internal parameters a_{ik}, b_{ik}, c_{ik} to three independent parameters. We have thus achieved a division of the admissible operators L into two classes: those with $\alpha \neq 0$ and those with $\alpha = 0$; and each class contains six independent parameters: three internal and three external ones.

The characteristic determinant $f(x, y) = \det \|g^{ik}(x, y)\| = A(x, y)C(x, y) - B^2(x, y)$ plays a crucial role in the classification of all possible distinct cases of admissible operators (for details see, e.g., [7]).

We can formulate the main result of [2] as follows.

Theorem 1. *If the operator L is admissible and there exists a nondegenerate functional σ, then the operator L is symmetrizable. Moreover, up to affine transformation, there exist nine distinct types of L:*

(I) $A(x, y) = x^2 - x$, $B(x, y) = xy$, $C(x, y) = y^2 - y$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$; the characteristic determinant is $f(x, y) = xy(1 - x - y)$.

(II) $A(x,y) = x^2$, $B(x,y) = xy$, $C(x,y) = y^2 - y$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = -x^2 y$.

(III) $A(x,y) = x^2$, $B(x,y) = xy$, $C(x,y) = y^2 + x$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = x^3$.

(IV) $A(x,y) = -x$, $B(x,y) = 0$, $C(x,y) = -y$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = xy$.

(V) $A(x,y) = 0$, $B(x,y) = x$, $C(x,y) = y$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = -x^2$.

(VI) $A(x,y) = -x$, $B(x,y) = 0$, $C(x,y) = -1$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = x$.

(VII) $A(x,y) = -1$, $B(x,y) = 0$, $C(x,y) = -1$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = 1$.

(VIII) $A(x,y) = y$, $B(x,y) = 1$, $C(x,y) = 0$, $D(x,y) = \beta x + d_0$, $E(x,y) = \beta y + e_0$, $f(x,y) = -1$.

(IX) $A(x,y) = x^2 - 1$, $B(x,y) = xy$, $C(x,y) = y^2 - 1$, $D(x,y) = \beta x$, $E(x,y) = \beta y$, $f(x,y) = 1 - x^2 - y^2$.

As was shown in [8], in all nine cases the Krall–Sheffer–Engelis (KSE) operators L can be transformed into a form in which they describe integrable quantum mechanical systems on spaces of constant curvature without magnetic field.

Direct computation yields [8]:

Proposition 1. *Condition (16) holds for every case (I)–(IX) of the KSE classification scheme. Hence every case can be transformed to a quantum system on two-dimensional manifolds with some potential $U(x,y)$ without magnetic field.*

Our next step will be to find the mean Riemannian curvature $\kappa(x,y)$ corresponding to the metric $g_{ik}(x,y)$. The Riemannian curvature $\kappa(x,y)$ can be calculated from the components $g_{ik}(x,y)$ of the metric using standard formulae from differential geometry. Performing these simple calculations we arrive at the following:

Proposition 2. *The mean Riemannian curvature is constant for every case (I)–(IX) of the KSE classification scheme. More precisely, the cases (IV)–(VIII) correspond to zero curvature, whereas the cases (I)–(III) and (IX) correspond to a nonzero curvature.*

For details and examples of corresponding quantum systems see [8].

Thus all nine types correspond to some quantum mechanical systems describing the motion of a particle in the presence of some potentials on two-dimensional spaces of constant curvature. There are no magnetic fields in any of these cases.

We now present the main result of this paper. That is, we show that all nine types in the KSE classification correspond to superintegrable systems. This means that there exist two algebraically independent operators I_1 and I_2 commuting with the operator L: $[L, I_1] = [L, I_2] = 0$. Operators $I_{1,2}$ act on the space of polynomials of two variables and preserve the degree of a polynomial.

Theorem 2. *For all nine types in the KSE classification scheme the algebraically independent integrals I_1, I_2 commuting with the operator L are*

Case (I).

$$I_1 = x(1-x-y)\partial_{xx} + (d_0(y-1) - (\beta + e_0)x)\partial_x$$
$$I_2 = y(1-x-y)\partial_{yy} + (e_0(x-1) - (\beta + d_0)y)\partial_y.$$

Case (II).

$$I_1 = x^2\partial_{xx} + ((\beta + e_0)x + d_0(1 - y))\partial_x$$
$$I_2 = xy\partial_{yy} + (d_0 y - e_0 x)\partial_y.$$

Case (III).

$$I_1 = x^2\partial_{yy} + (e_0 x - d_0 y)\partial_y$$
$$I_2 = 2x^2\partial_{xy} + xy\partial_{yy} + (e_0 x - d_0 y)\partial_x + (\beta x + d_0)\partial_y.$$

Case (IV).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = -xy(\partial_x - \partial_y)^2 + (d_0 y - e_0 x)\partial_x + (e_0 x - d_0 y)\partial_y.$$

Case (V).

$$I_1 = x^2\partial_{xx} + (e_0 x - d_0 y)\partial_x$$
$$I_2 = x\partial_{yy} + (\beta x + d_0)\partial_y.$$

Case (VI).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = \partial_{yy} - (e_0 + \beta y)\partial_y.$$

Case (VII).

$$I_1 = -\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = (d_0 + \beta x)\partial_y - (\beta y + e_0)\partial_x.$$

Case (VIII).

$$I_1 = \partial_{xx} + (\beta y + e_0)\partial_x$$
$$I_2 = (x - y^2)\partial_{xx} - 2y\partial_{xy} - \partial_{yy} + (e_0 x - d_0 y)\partial_x - (\beta x + d_0)\partial_y.$$

Case (IX).

$$I_1 = x\partial_y - y\partial_x$$
$$I_2 = (1 - x^2 - y^2)\partial_{xy} + (1 - \beta)x\partial_y.$$

Note that in cases (VII) and (IX) there exist integrals of first order. In all other cases we have integrals of second order with respect to the derivatives.

Thus all types (I)–(IX) correspond to superintegrable systems. Recall that a two-dimensional system is called integrable if there exists at least one integral commuting with the Hamiltonian. Superintegrable systems (with two algebraically independent integrals) form a subclass of integrable systems.

Our next main result is a characterization of the existence of a nondegenerate orthogonality functional in terms of integrability.

Theorem 3. *The existence of a nondegenerate orthogonality functional for an admissible operator L is equivalent to the existence of a second-order integral I commuting with L:* $[L, I] = 0$.

The proof of this statement is based on a direct computation of second-order integrals for admissible operators. It appears that the existence of such integrals imposes additional restrictions on the values of intrinsic parameters. We saw that affine transformations allow one to reduce the number of intrinsic parameters to three. Direct calculations show that the existence of commuting integrals reduces this number to zero. That is, there are three additional conditions fixing these three parameters. This leads (up to affine transformations) to the same coefficients $A(x, y)$, $B(x, y)$, $C(x, y)$ as in the KSE classification scheme. Details of the proof will be published separately.

The meaning of this theorem is that all nine cases in the KSE scheme can be equally characterized by the existence of at least one second-order integral. Note that superintegrability (i.e. the existence of the second independent integral) is obtained as a by-product during the proof of this theorem.

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R-Matrix Approach to the Krall–Sheffer Problem

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The complete set of commuting invariants for integrable systems arising in the framework of the Krall–Sheffer problem is derived using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. The separating coordinates are also deduced from this framework.

1 Introduction

Krall and Sheffer studied the problem of finding all polynomial eigenfunctions of second order linear differential operators in two variables having polynomial coefficients of degree equal to the order of derivative under certain further restrictions relating to its symmetrizability and the orthogonality of its eigenfunctions (for details see [2]). They classified all possible normal forms of the operators satisfying the required properties. It was shown in [3] that all the operators in the Krall–Sheffer list are reducible by gauge transformations to the form of a Laplace–Beltrami operator on a space of constant curvature plus some potential, the magnetic field being absent. Moreover, they all are related to two-dimensional superintegrable systems on spaces of constant curvature [2].

In this paper we show how to construct a complete set of commuting invariants to the integrable systems arising in the Krall–Sheffer framework using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. We give both the quantum and classical formulations in terms of Lax matrices depending on a loop parameter. The main construction is based on the well-known procedure of symmetry reduction from a free system in a higher dimension space (in particular, quadrics in $\mathbb{R}^6$ or $\mathbb{C}^6$). Classically this corresponds to reduction of geodesic flow, while quantum mechanically it involves reduction of the Laplacian. The reduction process leaves a residue of the original system, providing a complete set of commuting integrals.

2 General construction scheme

We begin with a phase space $\mathbb{M}$ of $\dim \mathbb{M} = 12$, with canonical variables $(x_i, y_i)_{i=1,\dots,6}$ which form the components of a pair (X, Y) of (either real or complex) column vector.

From these we form a Lax matrix $N(\lambda)$, depending on a spectral parameter $\lambda \in \mathbb{C}$ as follows:

$$N(\lambda) := \frac{1}{2}\left(Y^T, -X^T J\right)(\lambda - A)^{-1}(X, JY) = \sum_{i=1}^{n}\sum_{a=1}^{m_i} \frac{N_i^a}{(\lambda - \alpha_i)^a}.$$

where A, J are fixed 6×6 matrices with A having either $n = 1, 2$ or 3 distinct eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ and minimal polynomial

$$\prod_{i=1}^{n} (\lambda - \alpha_i)^{m_i}$$

and J is a symmetric real matrix with antidiagonal blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & 0 \\ \dots & & & \\ 1 & \dots & & \end{pmatrix}$$

for each Jordan block of A.

The dynamics is generated by Hamiltonians chosen from the algebra of spectral invariants of $N(\lambda)$. Classically, these Poisson commute and hence generate isospectral flows satisfying a Lax equation:

$$\frac{dN}{dt} = [B, N].$$

It is easily verified that $N(\lambda)$ satisfies the standard rational R-matrix Poisson bracket relations:

$$\{N(\lambda) \otimes N(\mu)\} = [r(\lambda), N(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes N(\mu)],$$

where both sides are viewed, for fixed $\lambda \neq \mu$ as elements of End $(\mathbb{C}^6 \otimes \mathbb{C}^6)$ and

$$r(\lambda) = \frac{P_{1,2}}{(\lambda - \mu)}, \qquad P_{1,2}(u \otimes v) = v \otimes u.$$

In the cases considered below, we only study Hamiltonians that are $O(6, J)$ invariant and restrict to the quadric defined by

$$X^T J X = 1.$$

Quotienting by the stabilizer $G_A \subset O(6, J)$ of A we reduce to a 2-dimensional configuration space, however the reduced system is no longer free.

In this case the algebra of spectral invariants is generated by the coefficients of:

$$-\frac{1}{2} \operatorname{Tr} N(\lambda)^2 = \sum_{i=1}^{n} \sum_{d=1}^{2m_i} \frac{H_i}{(\lambda - \alpha_i)^d}$$

with $2m_i \leq n_i$. The numerators H_i of this partial fraction expansion all Poisson commute and generate the algebra of spectral invariants. They are not all independent, however, since:

$$\sum_{i=1}^{n} H_{id} = 0$$

and H_{id} with $m_i < d \leq 2m_i$ are Casimir invariants.

The connection between configuration space coordinates in 6-dimensional space and the separating coordinates λ_1, λ_2 in the reduced 2-dimensional space is given by

$$X^T J (\lambda - A)^{-1} X = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{a(\lambda)},$$

where $a(\lambda)$ is the minimal polynomial of the matrix A.

The quantum version of this approach is simply obtained through canonical quantization with conjugate (momentum) variables y_j replaced by the partial derivatives $i\, \partial / \partial x_j$. The relation between the quantum integrals and the ones in the corresponding Krall–Sheffer cases is obtained applying a suitable gauge transformation.

3 Case 1. Sphere. Neuman–Rosochatius system

In the case of a sphere in $\mathbb{R}^6$, the matrices A and J are just:

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad J = id$$

with $\alpha \neq \beta \neq \gamma$. The symmetry algebra g_A corresponding to the stabilizer $G_A \subset O(6, \mathbb{R})$ is a maximal torus with generators

$$\{x_1y_2 - x_2y_1,\ x_3y_4 - x_4y_3,\ x_5y_6 - x_6y_5\}$$

and the Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \beta)} + \frac{N_3}{(\lambda - \gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix},$$

where the N_i are elements of $sl(2)$

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 & y_1^2 + y_2^2 \\ -x_1^2 - x_2^2 & -x_1y_1 - x_2y_2 \end{pmatrix},$$
$$N_2 = \frac{1}{2}\begin{pmatrix} x_3y_3 + x_4y_4 & y_3^2 + y_4^2 \\ -x_3^2 - x_4^2 & -x_3y_3 - x_4y_4 \end{pmatrix},$$
$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5y_5 - x_6y_6 \end{pmatrix}.$$

The invariants are the coefficients of:

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda - \alpha)} + \frac{H_2}{(\lambda - \beta)} + \frac{H_3}{(\lambda - \gamma)} + \frac{\mu_1^2}{(\lambda - \alpha)^2} + \frac{\mu_2^2}{(\lambda - \beta)^2} + \frac{\mu_3^2}{(\lambda - \gamma)^2}.$$

Here μ_1, μ_2 and μ_3 are constants defining the restriction to level sets of invariants of motion under the reduction procedure (the components of the moment map generating the torus action), namely:

$$\mu_1 = x_1y_2 - x_2y_1, \qquad \mu_2 = x_3y_4 - x_4y_3, \qquad \mu_3 = x_5y_6 - x_6y_5.$$

Integrals H_1, H_2 and H_3 are not all independent, since their sum is equal to zero. The Hamiltonian of the problem is given by the linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3.$$

The constraint to a sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ is given by $X^TJX = 1$:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1.$$

The reduced ambient coordinates are given by the radial distance in three planes (X_1, X_2), (X_3, X_4) and (X_5, X_6):

$$s_1^2 = x_1^2 + x_2^2, \qquad s_2^2 = x_3^2 + x_4^2, \qquad s_3^2 = x_5^2 + x_6^2.$$

The reduction of the constraint gives

$$s_1^2 + s_2^2 + s_3^2 = 1.$$

The reduced Hamiltonian is:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2} + \frac{\mu_3^2}{s_3^2}.$$

which is the kinetic energy on the sphere in $\mathbb{R}^3$ plus Rosochatius potential. Here (p_1, p_2, p_3) are canonical conjugate to (s_1, s_2, s_3).

The reduced separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates related to (s_1, s_2, s_3) by:

$$s_1^2 = \frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)(\alpha-\gamma)}, \qquad s_2^2 = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\beta-\alpha)(\beta-\gamma)}, \qquad s_3^2 = \frac{(\gamma-\lambda_1)(\gamma-\lambda_2)}{(\gamma-\alpha)(\gamma-\beta)}.$$

In terms of the reduced ambient space coordinates the integrals H_1, H_2 and H_3 are:

$$H_1 = -\frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha-\gamma} - \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha-\beta},$$
$$H_2 = -\frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta-\gamma} + \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha-\beta},$$
$$H_3 = \frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta-\gamma} + \frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha-\gamma},$$

where $L_{ij} = s_1 p_2 - s_2 p_1$. The quantum versions of these integrals are denoted by $(\hat{H}_1, \hat{H}_2, \hat{H}_3)$ and are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators. This leads to replacing L_{ij} by their quantum version:

$$\hat{L}_{ij} = \sqrt{-1}(s_i\partial/\partial s_j - s_j\partial/\partial s_i).$$

Note that whereas the Hamiltonian H is independent of the parameters (α, β, γ), which only serve to determine the separating coordinate system, the invariants H_1, H_2 individually do depend on those. Therefore, different choices for these parameters give distinct integrals that commute with H, but do not commute with each other. This provides an explanation for the superintegrability of this system.

To relate the invariants to the ones obtained in [2] for the corresponding Krall–Sheffer case we apply the gauge transformation consisting of conjugation by the function:

$$\Phi = x^{d_1} y^{d_2} (1-x-y)^{d_3},$$

where

$$d_1 = \frac{1}{2}(d_{00} + 1/2), \qquad d_2 = \frac{1}{2}(e_{00} + 1/2), \qquad d_3 = \frac{1}{2}(1/2 - d_{00} - e_{00} - B)$$

and d_{00}, e_{00}, B are the parameters appearing in Krall–Sheffer setting (see [2]).

The following are the relations between the integrals constructed in these two approaches:

$$\tilde{H}_1 = 4\frac{\alpha_1-\gamma_1}{\beta_1-\gamma_1}\hat{I}_x + 4\hat{I}_y - 4\hat{L} - c_0, \qquad \tilde{H}_2 = 4\frac{\gamma_1-\beta_1}{\gamma_1-\alpha_1}\hat{I}_y + 4\hat{I}_x - 4\hat{L} - c_1,$$

where $\tilde{H}_i = \Phi\hat{H}\Phi^{-1}$ and $\hat{L}$ is the Krall–Sheffer operator corresponding to case I, c_0 and c_1 depend on α, β, γ, d_{00}, e_{00}, B.

4 Case 2. Hyperboloid

For the case of a hyperboloid embedded in $\mathbb{R}^6$, matrices (A, J) may be taken as

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that J has an antidiagonal block corresponding to each Jordan block of A and a diagonal block corresponding to the diagonal part of A.

The symmetry algebra g_A again has three generators

$$\{x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4,\ x_2y_1 - x_4y_3,\ x_5y_6 - x_6y_5\}$$

but the Lax matrix now has a second order pole at $\lambda = \alpha$:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\beta)},$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 & 2y_1y_4 + 2y_2y_3 \\ -2x_1x_4 - 2x_2x_3 & -x_1y_1 - x_2y_2 1 x_3y_3 - x_4y_4 \end{pmatrix},$$

$$N_2 = \frac{1}{2}\begin{pmatrix} -x_4y_3 + x_2y_1 & -2y_3y_1 \\ 2x_2x_4 & -x_2y_1 + x_4y_3 \end{pmatrix},$$

$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5y_5 - x_6y_6 \end{pmatrix}.$$

Here (N_1, N_2) should be viewed as an element of the jet extension $sl(2)^{(1)*}$ while $N_3 \in sl(2)$. The invariants again give us only two independent H_1 and H_2

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\alpha)^2} - \frac{\mu_1\mu_2}{(\lambda-\alpha)^3} + \frac{\mu_2^2}{2(\lambda-\alpha)^4} + \frac{H_3}{(\lambda-\beta)} - \frac{\mu_3^2}{2(\lambda-\beta)^2},$$

where

$$H_1 + H_3 = 0.$$

The Hamiltonian is now defined by:

$$H = (\alpha - \beta)H_1 + H_2 - \frac{1}{2}\mu_3^2.$$

The reduced ambient space coordinates (s_1, s_2, s_3) are now defined by:

$$s_1^2 = \frac{(x_1x_4 + x_2x_3)^2}{2x_2x_4}, \qquad s_2^2 = 2x_2x_4, \qquad s_3^2 = x_5^2 + x_6^2.$$

The constraint to the quadric $X^TJX = 1$ reduces to define a hyperboloid in $\mathbb{R}^3$

$$2s_1s_2 + s_3^2 = 1.$$

In these coordinates the integrals H_1 and H_2 are

$$H_1 = \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \mu_3^2 s_1 s_2/s_3^2 + \mu_1\mu_2 s_3^2/s_2^2 - \mu_2^2 s_1 s_3^2/s_2^2}{\alpha - \beta} - \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)^2},$$

$$H_2 = \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\frac{\mu_2^2 s_1^2}{s_2^2} + 2\frac{\mu_1\mu_2 s_1}{s_2} + \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)}.$$

The quantized operators $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$ are obtained as before by replacing all conjugate variables by corresponding differential operators. And again, whereas Hamiltonian H does depend on the parameters (α, β) the integrals H_1, H_2 do, thereby again providing an explanation for the superintegrability in this case.

5 Case 3. Pseudoeuclidean plane

Matrix A in this case has only one degenerate eigenvalue:

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

J is antidiagonal.

The symmetry algebra g_A is generated by

$$\{-x_1y_4 - x_2y_5 - x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3,\ x_6y_1 - x_3y_4,\ -x_2y_4 - x_3y_5 + x_5y_1 + x_6y_2\}$$

and the Lax matrix is of the form:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\alpha)^3},$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} \begin{array}{l} x_1y_1 + x_2y_2 + x_3y_3 \\ \quad + x_4y_4 + x_5y_5 + x_6y_6 \end{array} & 2y_1y_3 + y_2^2 + 2y_4y_6 + y_5^2 \\ -2x_1x_3 - x_2^2 - 2x_4x_6 - x_5^2 & \begin{array}{l} -x_1y_1 - x_2y_2 - x_3y_3 \\ \quad - x_4y_4 - x_5y_5 - x_6y_6 \end{array} \end{pmatrix},$$

$$N_2 = \frac{1}{2}\begin{pmatrix} -x_3y_2 - x_2y_1 - x_6y_5 - x_5y_4 & -2y_2y_1 - 2y_4y_5 \\ 2x_2x_3 + 2x_5x_6 & x_3y_2 + x_2y_1 + x_6y_5 + x_5y_4 \end{pmatrix},$$

$$N_3 = \frac{1}{2}\begin{pmatrix} x_3y_1 + x_6y_4 & y_1^2 + y_4^2 \\ -x_3^2 - x_5^2 & -x_3y_1 - x_6y_4 \end{pmatrix}.$$

The trace formula again gives only two independent integrals H_1 and H_2

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)^2} + \frac{H_2}{(\lambda-\alpha)^3} - \frac{2\mu_1\mu_2 - \mu_3^2}{2(\lambda-\alpha)^4} + \frac{\mu_2\mu_3}{2(\lambda-\alpha)^5} - \frac{\mu_2^2}{2(\lambda-\alpha)^6}.$$

The Hamiltonian of the problem is:

$$H = -2p_1p_3 - p_2^2 + 2\gamma_1\gamma_3 + \gamma_2^2,$$

$$\gamma_1 = \frac{\mu_1}{s_1} - \frac{\mu_2 s_2}{s_1^2} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{\mu_3 s_3}{s_1^2}, \qquad \gamma_2 = \frac{\mu_2}{s_1} - \frac{\mu_3 s_2}{s_1^2}, \qquad \gamma_3 = \frac{\mu_3}{s_1}.$$

In this case the parameter α may be absorbed in the definition of λ and therefore no parameter dependence appears in the integrals H_1 and H_2:

$$H_1 = (p_2s_3 - s_2p_1)(s_1p_1 - s_3p_3) - 2s_2s_3\left(p_2^2 + 2p_1p_3\right) - \frac{\mu_1\mu_2}{s_1^2} - \frac{3\mu_3\mu_2 s_3}{s_1^3}$$
$$- \frac{\mu_3\mu_2 s_1}{s_2^2} - \frac{4s_3\mu_3\mu_1(1-2s_1s_2)}{s_1^4} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{(\mu_2^2 + \mu_3\mu_1)s_2}{s_1^3},$$

$$H_2 = \left(p_2^2 + 2p_1p_3\right)\left(s_2^2 + 2s_1s_3\right) + \frac{2\mu_3^2 s_1}{s_2^2} + \frac{4\mu_3^2 s_3^2}{s_1^2}$$
$$+ \frac{4\mu_3\mu_2 s_2}{s_1^3} - \frac{\mu_2^2 - 2\mu_3\mu_1}{s_1^2} + \frac{\mu_3^2\left(1 - 2s_2^2\right)}{s_1^4}.$$

Reduced coordinates in $\mathbb{R}^3$

$$s_1^2 = -\frac{(x_1x_3 + x_4x_6)^2}{x_3^2 + x_6^2}, \qquad s_2^2 = x_2^2 + x_5^2, \qquad s_3^2 = -\left(x_3^2 + x_6^2\right).$$

The constraint to the quadric $X^TJX = 1$ reduces to $2s_1s_3 + s_2^2 = 1$.

6 Conclusions

The approach based on Lax matrices satisfying the rational R-matrix structure gives a systematic way to derive the Hamiltonians and commuting invariants for these three cases corresponding to Krall–Sheffer operators on quadrics. This also provides a prescription for the separating coordinates, both in the classical and quantum cases. The presence of the additional parameters (α, β, γ) in the Case I, and (α, β) in the case II provides an explanation for their superintegrability.

A similar analysis may be made for the cases of Euclidean space arised in the Krall–Sheffer problem, they may be obtained as limiting cases of the above, providing an R-matrix approach to the remaining Krall–Sheffer operators. The details for all these cases will be provided elsewhere.

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Superintegrability, Lax Matrices and Separation of Variables

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ABSTRACT. We show how the superintegrability of certain systems can be deduced from the presence of multiple parameters in the rational Lax matrix representation. This is also related to the fact that such systems admit a separation of variables in parametric families of coordinate systems.

1. Rational Lax matrix representations of integrable systems

1.1. Classical R-matrix theory of commuting isospectral flows. In the classical R-matrix approach to finite dimensional integrable systems [**2, 4, 9**], there is a Poisson map from the phase space into a space of $r \times r$ Lax matrices $\mathcal{N}(\lambda)$ depending rationally, trigonometrically or elliptically on a spectral parameter λ. The Poisson bracket is defined by the relation

$$\{\mathcal{N}(\lambda) \overset{\otimes}{,} \mathcal{N}(\mu)\} = [r(\lambda-\mu), \mathcal{N}(\lambda)\otimes \mathbf{I} + \mathbf{I}\otimes\mathcal{N}(\mu)], \tag{1.1}$$

where both sides are interpreted as elements of $\mathrm{End}(\mathbf{C}^r \otimes \mathbf{C}^r)$. The symbol $\{\ \overset{\otimes}{,}\ \}$ signifies a simultaneous tensor product in $\mathrm{End}(\mathbf{C}^r) \sim \mathfrak{gl}(r)$ and Poisson brackets in the components, and $r(\lambda-\mu)$ denotes the classical R-matrix. The simplest case is the rational R-matrix,

$$r(\lambda) := \frac{P_{12}}{\lambda}, \qquad P_{12}(\mathbf{u}\otimes\mathbf{v}) := \mathbf{v}\otimes\mathbf{u}, \tag{1.2}$$

with $\mathcal{N}(\lambda)$ a rational function of λ.

$$\mathcal{N}(\lambda) = \mathcal{B}(\lambda) + \sum_{i=1}^{n}\sum_{a=1}^{n_i} \frac{N_{ia}}{(\lambda-\alpha_i)^i} \tag{1.3}$$

$$\mathcal{B}(\lambda) := \sum_{i=1}^{n_0} B_i\lambda^i, \qquad N_{ia}, B_i \in \mathfrak{gl}(r). \tag{1.4}$$

Equations (1.1), (1.2) define the standard linear, rational R-matrix structure.

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This is the final form of the paper.

It follows from the properties of classical R-matrices [**2, 9**] that elements of the algebra of spectral invariants $\phi(\mathcal{N}) \in \mathcal{I}(\widetilde{\mathfrak{gl}}(r))$ Poisson commute amongst themselves and generate commuting isospectral flows determined by the Lax equations:

$$\dot{\mathcal{N}} = \pm[(d\phi_{\mathcal{N}})_{\pm}, \mathcal{N}], \tag{1.5}$$

where $\mathcal{N}$ is here thought of as an element of the loop algebra $\widetilde{\mathfrak{gl}}(r)$, identified in a standard way with its dual $\widetilde{\mathfrak{gl}}^*(r)$ through the trace-residue pairing, and $(\,\cdot\,)_{\pm}$ denotes projection to the $\pm$ components relative to the usual splitting of the loop algebra into positive and negative components

$$\widetilde{\mathfrak{gl}}(r) = \widetilde{\mathfrak{gl}}(r)_{+} + \widetilde{\mathfrak{gl}}(r)_{-} \tag{1.6}$$

(i.e., those admitting holomorphic continuations to the interior $(+)$ and exterior $(-)$ of the unit circle respectively with the latter normalized to vanish at ∞). The spectral invariants generate a maximal Poisson commuting algebra on generic symplectic leaves, defining completely integrable systems [**3, 4**]; i.e., there are as many functionally independent generators as half the dimension of the leaf.

1.2. 2 × 2 rational Lax matrices. In the following, we shall limit our discussion to the case of 2×2 Lax matrices, although most of the considerations that follow are easily extended to higher rank. We may without loss of generality take $\mathcal{N}(\lambda)$ to be traceless (since the trace coefficients are Casimirs)

$$\mathcal{N}(\lambda) = \begin{pmatrix} h(\lambda) & e(\lambda) \\ f(\lambda) & -h(\lambda) \end{pmatrix}, \tag{1.7}$$

where the rational functions $e(\lambda)$, $f(\lambda)$, $h(\lambda)$ satisfy the Poisson bracket relations

$$\begin{aligned} \{h(\lambda), e(\mu)\} = \frac{e(\lambda) - e(\mu)}{\lambda - \mu}, \qquad & \{h(\lambda), f(\mu)\} = -f(\lambda) - f(\mu)\lambda - \mu, \\ \{e(\lambda), f(\mu)\} = -2h(\lambda) - h(\mu)\lambda - \mu. & \end{aligned} \tag{1.8}$$

For this case, the ring $\mathcal{I}(\widetilde{\mathfrak{gl}}(2))$ of spectral invariants, when restricted to the symplectic leaves of the R-matrix Poisson structure, is generated by the quadratic trace invariants; i.e., the coefficients determining the numerator of the rational function

$$\Delta(\lambda) := -\tfrac{1}{2}\operatorname{tr}(\mathcal{N}^2(\lambda)) = h^2(\lambda) - \tfrac{1}{2}\big(e(\lambda)f(\lambda) + f(\lambda)e(\lambda)\big). \tag{1.9}$$

(The order in the last two terms is irrelevant of course, but it is written here in a form that will also be valid in the quantum version below.) If, for example, the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ is taken to vanish, and only first order poles appear in $\mathcal{N}(\lambda)$, we have

$$e(\lambda) := \sum_{i=1}^{n} \frac{e_i}{\lambda - \alpha_i}, \qquad f(\lambda) := \sum_{i=1}^{n} \frac{f_i}{\lambda - \alpha_i}, \qquad h(\lambda) := \sum_{i=1}^{n} \frac{h_i}{\lambda - \alpha_i}, \tag{1.10}$$

where the quantities $\{e_i, f_i, h_i\}_{i=1,\dots,n}$ are a set of n $\mathbf{sl}(2)$ generators, which may be canonically coordinatized as:

$$e_i := \frac{1}{2}\left(y_i^2 + \frac{\mu_i^2}{x_i^2}\right) \qquad f_i := \frac{1}{2}x_i^2 \qquad h_i := \frac{1}{2}x_i y_i, \qquad i = 1, \dots, n, \tag{1.11}$$

where $\{\mu_i^2\}_{i=1,\dots,n}$ are the values of the $\mathrm{sl}(2)$ Casimir invariants and $\{x_i, y_i\}_{i=1,\dots,n}$ form a set of canonical coordinates on the symplectic leaves.

1.2.1. *Parametric dependence of invariants and superintegrability.* Again, taking the case when the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ vanishes (but not necessarily just first order poles), a complete set of generators is given by

$$\phi_{ia} := \operatorname*{res}_{\lambda=\alpha_i} (\lambda-\alpha_i)^a \operatorname{tr}(\mathcal{N}^2(\lambda)), \quad i=1,\ldots,n, \ a=0,\ldots,n_i-1. \tag{1.12}$$

These commute amongst themselves, but they each depend upon the pole locations $\{\alpha_i\}_{i=1,\ldots,n}$ in $\mathcal{N}(\lambda)$. However, the linear combination:

$$\phi_{\rm SI} := \sum_{i=1}^n \alpha_i \phi_{i0} = \operatorname*{res}_{\lambda=\infty} \operatorname{tr}(\mathcal{N}^2(\lambda)) \tag{1.13}$$

does not depend on the α_i's. In general, there is no reason for the invariants $\phi_{ia}(\alpha_i)$ to commute with each other for different choices of the α_i's. But, regardless of the values chosen, they will commute with $\phi_{\rm SI}$. Since the $\phi_{ia}(\alpha_i)$'s for different choices of α_i's in general do not generate the same algebra of functions, we may conclude that, taken together, for different evaluations of the parameters $\{\alpha_i\}$, there are more functionally independent integrals that Poisson commute with $\phi_{\rm SI}$ than half the dimension of the symplectic leaf, and hence the Hamiltonian system it generates is superintegrable. (In fact, in most cases, it may be shown to be maximally superintegrable; see the examples below.)

In particular, if we take the case of purely simple poles as above in (1.10), the resulting Hamiltonian is:

$$\phi_{\rm SI} = \frac{1}{2}\sum_{i=1}^n x_i^2 \sum_{j=1}^n y_j^2 - \frac{1}{2}\Big(\sum_{i=1}^n x_i y_i\Big)^2 + \frac{1}{2}\sum_{i=1}^n x_i^2 \sum_{i=1}^n \frac{\mu_i^2}{x_i^2}. \tag{1.14}$$

which, when constrained to the (co)tangent bundle of the $n-1$ sphere S^{n-1}

$$\sum_{i=1}^n x_i^2 = 1, \qquad \sum_{i=1}^n x_i y_i = 0, \tag{1.15}$$

yields the superintegrable system

$$h_{\rm Ros} = \frac{1}{2}\sum_{j=1}^n y_j^2 + \frac{1}{2}\sum_{i=1}^n \frac{\mu_i^2}{x_i^2}, \tag{1.16}$$

which is the trivial case of the Rosochatius system (without a harmonic oscillator potential).

1.2.2. *Separation of variables.* Another viewpoint that helps to explain the superintegrability of systems arising in this way is to note that they may be completely separated in a canonical coordinate system determined by the values of the pole parameters $\{\alpha_i\}$ which, for the $\mathbf{sl}(2)$ case with simple poles with the phase space constrained to S^{n-1} as above reduces to the sphero-conical system $\{\lambda_i, \zeta_i\}_{i=1,\ldots,n-1}$ defined by:

$$\sum_{i=1}^n \frac{x_i^2}{\lambda-\alpha_i} = \frac{\prod_{j=1}^{n-1}(\lambda-\lambda_j)}{\prod_{i=1}^n(\lambda-\alpha_i)}, \qquad \zeta_i := \frac{1}{2}\sum_{i=1}^n \frac{x_i y_i}{(\lambda-\alpha_i)}. \tag{1.17}$$

These are just the points (λ_i, ζ_i) on the invariant spectral curve

$$\zeta^2 + \tfrac{1}{2}\Delta(\lambda) = 0 \tag{1.18}$$

where the matrix element $f(\lambda)$ vanishes and $\zeta_i = h(\lambda_i)$ are the eigenvalues at these points. These are particular cases of the spectral *Darboux coordinates* of [**3, 4**].

(Note that these become hyperellipsoidal coordinates if there is a constant term added in the definition (1.10) of $f(\lambda)$.)

The point to note is that the separation of variables occurs in these coordinates simultaneously for *all* the invariants ϕ_{ia}, viewed as generators of Hamiltonian flows. But again, since the leading term spectral invariant ϕ_{SI} does not depend on the values of the parameters α_i, it admits a separation of variables in *any* of the family of sphero-conical (or hyperellipsoidal) coordinates obtained by varying these parameters. This simultaneous separability in multiple coordinates may be viewed as an alternative explanation of the origin of the superintegrability of such systems. (In fact, both these viewpoints are a result of the classical r-matrix setting, and in a sense may be considered as equivalent.)

In the examples given below in the following section, the same principle is used to deduce superintegrable systems from $\mathbf{sl}(2)$ Lax matrices satisfying the Poisson bracket relations (1.1)).

1.2.3. *Quantum integrable systems.* The above discussion is easily extended to the canonically quantized version of such systems. All that must be done is to replace the matrix elements defining $\mathcal{N}(\lambda)$ by their quantized forms $\hat{e}(\lambda)$, $\hat{f}(\lambda)$, $\hat{h}(\lambda)$, which must satisfy the commutator analogs of the Poisson bracket relations (1.8)

$$[\hat{h}(\lambda),\hat{e}(\mu)] = \frac{\hat{e}(\lambda)-\hat{e}(\mu)}{\lambda-\mu}, \qquad [\hat{h}(\lambda),\hat{f}(\mu)] = -\frac{\hat{f}(\lambda)-\hat{f}(\mu)}{\lambda-\mu},$$
$$[\hat{e}(\lambda),\hat{f}(\mu)] = -2\frac{\hat{h}(\lambda)-\hat{h}(\mu)}{\lambda-\mu}, . \tag{1.19}$$

These can be realized by canonical quantization of the underlying classical phase space variables. For example, in the case of simple poles only, with vanishing polynomial term $\mathcal{B}(\lambda)$, we have:

$$\hat{e}(\lambda) := \sum_{i=1}^{n} \frac{\hat{e}_i}{\lambda-\alpha_i} \qquad \hat{f}(\lambda) := \sum_{i=1}^{n} \frac{\hat{f}_i}{\lambda-\alpha_i} \qquad \hat{h}(\lambda) := \sum_{i=1}^{n} \frac{\hat{h}_i}{\lambda-\alpha_i}. \tag{1.20}$$

where the $\mathbf{sl}(2)$ generators $\{\hat{e}_i, \hat{f}_i, \hat{h}_i\}$ may be represented by the operators

$$\hat{e}_i := \frac{1}{2}\left(\frac{\partial^2}{\partial x_i^2} - \frac{\mu_i^2}{x_i^2}\right) \quad \hat{f}_i := \frac{1}{2}x_i^2 \quad \hat{h}_i := \frac{1}{2}\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right), \quad i = 1, \dots n, \tag{1.21}$$

and the commuting invariants are similarly given by the coefficients of the numerator polynomial of the quantum spectral invariant:

$$\hat{\Delta}(\lambda) := \hat{h}^2(\lambda) - \tfrac{1}{2}(\hat{e}(\lambda)\hat{f}(\lambda) + \hat{f}(\lambda)\hat{e}(\lambda)). \tag{1.22}$$

The resulting systems are similarly quantum integrable, and separable in the same coordinates as the classical ones [**6**] and, for the same reasons as above, the quantum version of the Hamiltonian ϕ_{SI} is superintegrable.

In the following section, a number of examples of such classical and quantum superintegrable systems will be given.

2. Examples of superintegrable classical and quantum systems

The examples given below arise in the framework of the so-called Krall–Scheffer problem [**8**] of describing all two-dimensional analogs of classical orthogonal polynomials which result in nine classes of second-order partial differential equations on the plane or on constant curvature surfaces. It was shown in [**5, 7**] that all nine cases are connected with superintegrable systems. The following are some illustrative examples.

2.1. The sphere.

2.1.1. *Classical Lax matrix.* The first case corresponds to three simple poles and vanishing $B(\lambda)$. The Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\beta)} + \frac{N_3}{(\lambda-\gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix}, \tag{2.1}$$

where the matrix elements of the N_i generate a Poisson bracket realization of $(\mathbf{sl}(2))^3$:

$$\begin{gathered} N_1 = \frac{1}{2}\begin{pmatrix} s_1p_1 & p_1^2+\mu_1^2/s_1^2 \\ -s_1^2 & -s_1p_1 \end{pmatrix}, \qquad N_2 = \frac{1}{2}\begin{pmatrix} s_2p_2 & p_2^2+\mu_2^2/s_2^2 \\ -s_2^2 & -s_2p_2 \end{pmatrix}, \\ N_3 = \frac{1}{2}\begin{pmatrix} s_3p_3 & p_3^2+\mu_3^2/s_3^2 \\ -s_3^2 & -s_3p_3 \end{pmatrix}. \end{gathered} \tag{2.2}$$

Here (p_1, p_2, p_3) are canonically conjugate to (s_1, s_2, s_3) (and these coincide with the coordinates $\{x_i, y_i\}_{i=1,\ldots,n}$ above).

2.1.2. *Commuting invariants.* The invariants are the coefficients of:

$$-\frac{1}{2}\operatorname{tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{H_3}{(\lambda-\gamma)} + \frac{\mu_1^2}{(\lambda-\alpha)^2} + \frac{\mu_2^2}{(\lambda-\beta)^2} + \frac{\mu_3^2}{(\lambda-\gamma)^2}. \tag{2.3}$$

Note that only two of the integrals H_1, H_2 and H_3 are independent, since their sum is zero. The Hamiltonian of the problem is given by their linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3 = \frac{1}{2}(p_1^2+p_2^2+p_3^2) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2} + \frac{\mu_3^2}{s_3^2}. \tag{2.4}$$

This describes the Rosochatius system with harmonic oscillator terms absent on the cotangent bundle of a two-sphere in $\mathbb{R}^3$:

$$s_1^2+s_2^2+s_3^2 = 1, \qquad s_1p_1+s_2p_2+s_3p_3 = 0. \tag{2.5}$$

The integrals H_1, H_2 and H_3 are as follows:

$$\begin{aligned} H_1 &= -\frac{1}{2}\frac{L_{13}^2+\mu_3^2s_1^2/s_3^2+\mu_1^2s_3^2/s_1^2}{\alpha-\gamma} - \frac{1}{2}\frac{L_{12}^2+\mu_1^2s_2^2/s_1^2+\mu_2^2s_1^2/s_2^2}{\alpha-\beta} \\ H_2 &= -\frac{1}{2}\frac{L_{23}^2+\mu_3^2s_2^2/s_3^2+\mu_2^2s_3^2/s_2^2}{\beta-\gamma} + \frac{1}{2}\frac{L_{12}^2+\mu_1^2s_2^2/s_1^2+\mu_2^2s_1^2/s_2^2}{\alpha-\beta} \\ H_3 &= \frac{1}{2}\frac{L_{23}^2+\mu_3^2s_2^2/s_3^2+\mu_2^2s_3^2/s_2^2}{\beta-\gamma} + \frac{1}{2}\frac{L_{13}^2+\mu_3^2s_1^2/s_3^2+\mu_1^2s_3^2/s_1^2}{\alpha-\gamma}, \end{aligned} \tag{2.6}$$

where $L_{ij} = s_1p_2 - s_2p_1$.

Note that the Hamiltonian H is independent of the parameters (α, β, γ), whereas the invariants H_1, H_2 do depend on them. Therefore, different choices for the parameters give distinct integrals that commute with H, but do not commute with each other.

2.1.3. *Separating coordinates.* The separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates. The corresponding momenta are denoted (ξ_1, ξ_2). They are related to (s_1, s_2, s_3) and (p_1, p_2, p_3) by:

$$s_1^2 = \frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)(\alpha-\gamma)}, \qquad \xi_1 = -\frac{1}{2}\left(\frac{s_1p_1}{\lambda_1-\alpha} + \frac{s_2p_2}{\lambda_1-\beta} + \frac{-s_1p_1 - s_2p_2}{\lambda_1-\gamma}\right) \tag{2.7}$$

$$s_2^2 = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\beta-\alpha)(\beta-\gamma)}, \qquad \xi_2 = -\frac{1}{2}\left(\frac{s_1p_1}{\lambda_2-\alpha} + \frac{s_2p_2}{\lambda_2-\beta} + \frac{-s_1p_1 - s_2p_2}{\lambda_2-\gamma}\right) \tag{2.8}$$

$$s_3^2 = \frac{(\gamma-\lambda_1)(\gamma-\lambda_2)}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.9}$$

2.1.4. *Quantum system.* The quantum versions of the integrals above, denoted $\widehat{H}_1$, $\widehat{H}_2$, $\widehat{H}_3$, are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators, $\hat{e}(\lambda)$, $\hat{f}(\lambda)$, $\hat{h}(\lambda)$, which in the case of simple poles are as in (1.20)–(1.22).

The quantization procedure leads to replacing the L_{ij}'s by their quantum version:

$$\hat{L}_{ij} = \sqrt{-1}(s_i\partial/\partial s_j - s_j\partial/\partial s_i). \tag{2.10}$$

Introducing the functions

$$\omega_{jk}^2 := \mu_j^2 s_k^2/s_j^2 + \mu_k^2 s_j^2/s_k^2, \quad j,k = 1,\dots,3, \tag{2.11}$$

and denoting $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \alpha_3$, we can present the quantum integrals as

$$\widehat{H}_i = -\frac{1}{2}\sum_{k\neq i}\frac{\hat{L}_{ik} + \omega_{ik}^2}{\alpha_i - \alpha_k}, \quad i,k = 1,\dots,3. \tag{2.12}$$

The quantum Hamiltonian is

$$\widehat{H} = -\frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2} + \frac{\mu_3^2}{s_3^2}. \tag{2.13}$$

The separating coordinates are the configuration space part of the ones for the classical case (λ_1, λ_2).

2.2. The hyperboloid.

2.2.1. *Classical Lax matrix.* Consider now a Lax matrix with one first order and one second order pole:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\beta)}, \tag{2.14}$$

with

$$\begin{gathered} N_1 = \frac{1}{2}\begin{pmatrix} s_1p_1 + s_2p_2 & 2p_1p_2 + 2\gamma_1\gamma_2 \\ -2s_1s_2 & -s_1p_1 - s_2p_2 \end{pmatrix}, \\ N_2 = \frac{1}{2}\begin{pmatrix} -s_2p_1 & -p_1^2 - \gamma_2^2 \\ s_2^2 & s_2p_1 \end{pmatrix}, \qquad N_3 = \frac{1}{2}\begin{pmatrix} s_3p_3 & p_3^2 + \gamma_3^2 \\ -s_3^2 & -s_3p_3 \end{pmatrix}. \end{gathered} \tag{2.15}$$

Here we have introduced the following notations

$$2\gamma_1\gamma_2 := \frac{2\mu_2^2 s_1}{s_2^3} - \frac{2\mu_1\mu_2}{s_2^2}, \qquad \gamma_2^2 := -\frac{\mu_2^2}{s_2^2}, \qquad \gamma_3^2 := \frac{\mu_3^2}{s_3^2}. \tag{2.16}$$

The matrix elements of (N_1, N_2) generate a Poisson bracket realization of the jet extension $\mathbf{sl}(2)^{(1)*}$ while those of N_3 generate a second $\mathbf{sl}(2)$.

2.2.2. *Commuting invariants.* The trace formula again gives us only two independent commuting invariants H_1 and H_2

$$-\frac{1}{2}\operatorname{tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\alpha)^2} - \frac{\mu_1\mu_2}{(\lambda-\alpha)^3} + \frac{\mu_2^2}{2(\lambda-\alpha)^4} + \frac{H_3}{(\lambda-\beta)} - \frac{\mu_3^2}{2(\lambda-\beta)^2} \tag{2.17}$$

since, by taking the residue we obtain

$$H_1 + H_3 = 0. \tag{2.18}$$

The superintegrable Hamiltonian in this case is:

$$H = (\alpha-\beta)H_1 + H_2 - \tfrac{1}{2}\mu_3^2 = 2p_1p_2 - p_1^2 + p_3^2 + 2\gamma_1\gamma_2 - \gamma_2^2 + \gamma_3^2. \tag{2.19}$$

The quadratic constraint now defines a hyperboloid

$$2s_1s_2 + s_3^2 = 1. \tag{2.20}$$

In the ambient coordinates the integrals H_1 and H_2 are

$$\begin{aligned} H_1 &= \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \gamma_3^2 s_1s_2 - 2\gamma_1\gamma_2 s_3^2}{\alpha-\beta} - \frac{(s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha-\beta)^2} \\ H_2 &= \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\gamma_1\gamma_2 s_1 + \frac{(s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha-\beta)}. \end{aligned} \tag{2.21}$$

Again, whereas the Hamiltonian H does not depend on the parameters (α, β) the integrals H_1, H_2 do, which provides an explanation for the superintegrability in this case.

2.2.3. *Separating coordinates.* These are determined by the relations:

$$s_3^2 = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)^2}, \quad \xi_1 = -\frac{1}{2}\left(\frac{s_1p_1+s_2p_2}{\lambda_1-\alpha} - \frac{s_2p_1}{(\lambda_1-\alpha)^2} + \frac{s_3p_3}{\lambda_1-\beta}\right), \tag{2.22}$$

$$s_2^2 = -\frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)}, \quad \xi_2 = -\frac{1}{2}\left(\frac{s_1p_1+s_2p_2}{\lambda_2-\alpha} - \frac{s_2p_1}{(\lambda_2-\alpha)^2} + \frac{s_3p_3}{\lambda_2-\beta}\right), \tag{2.23}$$

$$s_1s_2 = -\frac{1}{2}\left(\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)^2} - 1\right). \tag{2.24}$$

2.2.4. *Quantum system.* The quantized integrals $\hat{H}_1$, $\hat{H}_2$,b $\hat{H}_3$ are obtained as before by replacing all conjugate variables by the corresponding differential operators. The quantum integrals may then be expressed as

$$\begin{aligned} \widehat{H}_1 &= -\frac{(s_1\partial_3 - s_3\partial_2)(s_3\partial_1 - s_2\partial_3) + \gamma_3^2 s_1s_2 + 2\gamma_1\gamma_2 s_3^2}{\alpha-\beta} + \frac{(s_3\partial_1 - s_2\partial_3)^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha-\beta)^2}, \\ \widehat{H}_2 &= \frac{1}{2}\hat{L}_{12}^2 - 2\gamma_1\gamma_2 s_1 - \frac{(s_3\partial_{s_1} - s_2\partial_{s_3})^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha-\beta)}, \end{aligned} \tag{2.25}$$

where $\partial_k := \partial/\partial s_k$.

The quantum Hamiltonian is

$$\widehat{H} = 2\partial_1\partial_2 - \partial_1^2 + \partial_3^2 + 2\gamma_1\gamma_2 - \gamma_2^2 + \gamma_3^2, \tag{2.26}$$

and this again separates in the configuration space coordinates (λ_1, λ_2).

2.3. The plane.

2.3.1. *Classical Lax matrix.* For the cases with zero curvature like the example to follow, the polynomial part $\mathcal{B}(\lambda)$ of the Lax matrix does not vanish. The simplest case involves two distinct finite poles in $N(\lambda)$ and constant $\mathcal{B}(\lambda)$

$$(2.27)\quad N(\lambda) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} + \frac{1}{2(\lambda-\alpha)} \begin{pmatrix} s_1 p_1 & p_1^2 + \mu_1^2/s_1^2 \\ -s_1^2 & -s_1 p_1 \end{pmatrix} + \frac{1}{2(\lambda-\beta)} \begin{pmatrix} s_2 p_2 & p_2^2 + \mu_2^2/s_2^2 \\ -s_2^2 & -s_2 p_2 \end{pmatrix}.$$

The matrix elements of the residues $N_1,\ N_2$ generate two copies of $\mathbf{sl}(2)$.

2.3.2. *Commuting invariants.* The invariants of motion are defined by:

$$(2.28)\qquad -\frac{1}{2}\operatorname{tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{\mu_1^2}{(\lambda-\alpha)^2} + \frac{\mu_2^2}{(\lambda-\beta)^2} - a.$$

The superintegrable Hamiltonian in this case is given by

$$(2.29)\qquad H = \frac{1}{4}\operatorname*{res}_{\infty} \operatorname{tr} N(\lambda)^2 = \frac{1}{4}\left(p_1^2 + p_2^2 + a(s_1^2 + s_2^2) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2}\right),$$

which gives an isotropic oscillator together with Rosochatius terms. As before (p_1, p_2) are canonically conjugate to (s_1, s_2).

In terms of the ambient space coordinates the integrals H_1 and H_2 are:

$$(2.30)\qquad \begin{aligned} H_1 &= p_1^2 + a s_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}\left(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}\right) \\ H_2 &= p_2^2 + a s_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}\left(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}\right). \end{aligned}$$

Where $L_{12} := s_1 p_2 - s_2 p_1$ and $H = \frac{1}{4}(H_1 + H_2)$. Here the additional integral results from the parametric dependence on $(\alpha - \beta)$.

2.3.3. *Separating coordinates.* The separating coordinates $(\lambda_1, \lambda_2, \xi_1, \xi_2)$ in this case are defined by

$$(2.31)\qquad s_1^2 = 2\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)} \qquad \xi_1 = -\frac{1}{2}\left(\frac{s_1 p_1}{\lambda_1-\alpha} + \frac{s_2 p_2}{\lambda_1-\beta}\right)$$

$$(2.32)\qquad s_2^2 = -2\frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)} \qquad \xi_2 = -\frac{1}{2}\left(\frac{s_1 p_1}{\lambda_2-\alpha} + \frac{s_2 p_2}{\lambda_2-\beta}\right)$$

2.3.4. *Quantum system.* The Hamiltonian of the corresponding quantum problem is

$$(2.33)\ \widehat{H} = \frac{1}{4}\operatorname*{res}_{\infty} \operatorname{tr} \hat{N}(\lambda)^2 = \frac{1}{4}\left(\partial_1^2 + \partial_2^2 + a(s_1^2 + s_2^2) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2}\right) = \frac{1}{4}(\widehat{H}_1 + \widehat{H}_2).$$

The quantum integrals $\widehat{H}_1$ and $\widehat{H}_2$ are:

$$(2.34)\qquad \begin{aligned} \widehat{H}_1 &= \partial_1^2 + a s_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}\left(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}\right) \\ \widehat{H}_2 &= \partial_2^2 + a s_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}\left(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}\right) \end{aligned}$$

and the separating coordinates are again (λ_1, λ_2), which depend on the additional parameter $(\alpha - \beta)$.

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On a Trigonometric Analogue of Atiyah–Hitchin Bracket

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ABSTRACT. The space of rational functions has a natural Poisson structure discovered by Atiyah and Hitchin. We consider its derivation by means of a quadratic r-matrix structure which naturally arises for the corresponding scattering problem and provide a trigonometric analogue.

1. Poisson brackets on space of rational maps

In the work by Atiyah and Hitchin [1] a symplectic structure on the space R_N of rational functions of the form

$$S(\lambda) = \sum_{i=0}^{N-1} \frac{\rho_i}{\lambda - \beta_i} \tag{1.1}$$

was introduced.

To describe this symplectic structure we represent function $S(\lambda)$ as a ratio of two polynomials

$$S(\lambda) = \frac{p(\lambda)}{q(\lambda)} \tag{1.2}$$

where $q(\lambda) = (\lambda - \beta_1)\cdots(\lambda - \beta_N)$. Then the Atiyah–Hitchin symplectic form looks as follows:

$$\omega = \sum_{i=1}^{N} \frac{dp(\beta_i) \wedge d\beta_i}{p(\beta_i)} \tag{1.3}$$

The symplectic structure implies the following Poisson brackets

$$\{p(\beta_m), p(\beta_n)\} = 0, \qquad \{\beta_m, \beta_n\} = 0 \tag{1.4}$$

$$\{p(\beta_m), \beta_n\} = p(\beta_m)\delta_m{}^n \tag{1.5}$$

These Poisson brackets in turn imply brackets between the polynomials $p(\lambda)$ and $q(\lambda)$:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{1.6}$$

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This is the final form of the paper.

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu} \tag{1.7}$$

It was shown by Gekhtman and Faybusovich in [3] that the relations (1.6) and (1.7) are equivalent to the following brackets on the space of rational functions R_N (1.3):

$$\{S(\lambda), S(\mu)\} = \frac{\left(S(\lambda) - S(\mu)\right)^2}{\lambda - \mu} \tag{1.8}$$

In fact, the bracket (1.8) can obviously be extended to the space of all rational functions, i.e., the ratio of two polynomials of arbitrary degree. However the requirement of distinct roots remains essential. As was pointed out by K. Takasaki [4], the Gekhtman–Faybusovich bracket (1.8) is naturally related to the rational quadratic Sklyanin bracket

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\mu)\} = [\overset{12}{r}(\lambda - \mu), \overset{1}{T}(\lambda)\overset{2}{T}(\mu)] \tag{1.9}$$

where

$$\overset{12}{r}(\lambda) = \frac{\Pi}{\lambda} = \frac{I \otimes I + \sigma_\alpha \otimes \sigma_\alpha}{2\lambda} \tag{1.10}$$

is the classical rational r-matrix and $\Pi(u \otimes v) = v \otimes u$ is a permutation matrix.

If we parametrize matrix $T(\lambda)$ to be:

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix} \tag{1.11}$$

then the Sklyanin bracket (1.9) implies the following Poisson brackets between the matrix elements of $T(\lambda)$:

$$\{q(\lambda), q(\mu)\} = 0, \qquad \{p(\lambda), p(\mu)\} = 0 \tag{1.12}$$

$$\{p(\lambda), q(\mu)\} = \frac{p(\lambda)q(\mu) - q(\lambda)p(\mu)}{\lambda - \mu} \tag{1.13}$$

$$\{\tilde{q}(\lambda), \tilde{q}(\mu)\} = 0, \qquad \{\tilde{p}(\lambda), \tilde{p}(\mu)\} = 0 \tag{1.14}$$

$$\{\tilde{p}(\lambda), \tilde{q}(\mu)\} = \frac{\tilde{p}(\lambda)\tilde{q}(\mu) - \tilde{q}(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{1.15}$$

$$\{\tilde{q}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)q(\mu) - q(\lambda)\tilde{q}(\mu)}{\lambda - \mu}, \quad \{\tilde{p}(\lambda), p(\mu)\} = \frac{\tilde{p}(\lambda)p(\mu) - p(\lambda)\tilde{p}(\mu)}{\lambda - \mu} \tag{1.16}$$

$$\{p(\lambda), \tilde{q}(\mu)\} = \frac{q(\lambda)\tilde{p}(\mu) - \tilde{p}(\lambda)q(\mu)}{\lambda - \mu} \tag{1.17}$$

$$\{\tilde{p}(\lambda), q(\mu)\} = \frac{\tilde{q}(\lambda)p(\mu) - p(\lambda)\tilde{q}(\mu)}{\lambda - \mu} \tag{1.18}$$

Notice, that the brackets (1.12)) and (1.13) coincide with the Gekhtman–Faybusovich brackets (1.6), (1.7). Therefore, the Atiyah–Hitchin Poisson structure coincides with the Sklyanin bracket between elements T_{11} and T_{21} for polynomial dependence of matrix T on λ.

This coincidence leads to several natural questions. The Sklyanin bracket (1.9) arises in inverse scattering method as Poisson structure on scattering matrix $T(\lambda)$ implied by fundamental Poisson structure between physical field in models of non-linear Schrödinger type. On the other hand, the Atiyah-Hitchin bracket is also defined as a bracket on the space of scattering matrices related to solutions of the Bogomolny equations [1].

However, in Atiyah–Hitchin framework it remains unclear how this Poisson structure is related to fundamental Poisson bracket between physical fields A and ϕ; close relationship between Atiyah-Hitchin and Sklyanin brackets suggests that there should exist some kind of derivation of the Atiyah-Hitchin structure from the brackets on A and ϕ; however, so far we were unable to find it. Instead in these note we shall extend this observation to trigonometric case and show how to derive natural Poisson structure on the space of trigonometric rational functions starting from Sklyanin bracket with trigonometric r-matrix.

2. Trigonometric generalizations of Atiyah–Hitchin and Gekhtman–Faybusovich brackets

Now, consider a space of meromorphic functions $p(\lambda)/q(\lambda)$ being the ratio of trigonometric polynomials

$$p(\lambda) = \prod_{i=1}^{N-1} \sin(\lambda - \alpha_i) \tag{2.1}$$

$$q(\lambda) = \prod_{k=1}^{N} \sin(\lambda - \lambda_k) \tag{2.2}$$

The symplectic structure here is of the same form as before

$$\Omega = \sum\_i = 1^N \frac{dp(\lambda_i) \wedge d\lambda_i}{p(\lambda_i)} \tag{2.3}$$

Consider the following 2×2 matrix

$$T(\lambda) = \begin{pmatrix} p(\lambda) & \tilde{p}(\lambda) \\ q(\lambda) & \tilde{q}(\lambda) \end{pmatrix}$$

with polynomials $p(\lambda)$ and $q(\lambda)$ are of the form as above. Then with respect to trigonometric r-matrix defined as

$$r(\lambda) = \frac{1}{2\sin(\lambda)}[\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \cos(\lambda)\sigma_3 \otimes \sigma_3] \tag{2.4}$$

the quadratic Sklyanin Poisson bracket on matrices $T(\lambda)$ and $T(\mu)$ provides the following relations on the space of trigonometric polynomials

$$\{p(\lambda), p(\mu)\} = 0 \tag{2.5}$$

$$\{q(\lambda), q(\mu)\} = 0 \tag{2.6}$$

$$\{q(\mu), p(\lambda)\} = \frac{q(\lambda)p(\mu) - \cos(\lambda - \mu)q(\lambda)p(\mu)}{\sin(\lambda - \mu)} \tag{2.7}$$

which lead in their turn to the following generalization of the Gekhtman–Faybusovich bracket on the space of rational trigonometric functions:

$$\{S(\lambda), S(\mu)\} = \frac{\left(S(\lambda) - S(\mu)\right)^2}{\sin(\lambda - \mu)} + 2S(\lambda)S(\mu)(\mathrm{ctg}(\lambda - \mu) - 1) \tag{2.8}$$

It is easy to see that in the rational limit this bracket coincide with (1.8).

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Constrained Reductions of 2D dispersionless Toda Hierarchy, Hamiltonian Structure and Interface Dynamics.

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Abstract

Finite-dimensional reductions of the 2D dispersionless Toda hierarchy, constrained by the "string equation" are studied. These include solutions determined by polynomial, rational or logarithmic functions, which are of interest in relation to the "Laplacian growth" problem governing interface dynamics. The consistency of such reductions is proved, and the Hamiltonian structure of the reduced dynamics is derived. The Poisson structure of the rationally reduced dispersionless Toda hierarchies is also derived.

1 Introduction.

This paper concerns the study of rational and logarithmic reductions of the 2D dispersionless Toda hierarchy of integrable equations (henceforth dToda). The subject is motivated by important applications to problems in interface dynamics and statistical physics.

Laplacian growth is a process that governs the dynamics of the boundary in the plane separating two disjoint, open regions in which harmonic (scalar) fields are defined. These may be interpreted as the pressure fields for two incompressible viscous fluids. The movement of the boundary is determined (according to d'Arcy's law, in the case of viscous fluids) by equating the normal velocity to the difference between the boundary values of the gradients of the fields. In particular, one region (say, the "interior" region) may be chosen to be bounded and have constant harmonic field (corresponding to zero viscosity) with the boundary condition for the "exterior" region at infinity to be such that there is a unit sink, implying that the area of the interior region grows linearly in time [12]. Denoting the harmonic field (e.g. the pressure) in the exterior region by $P(X,Y)$, this satisfies the conditions:

$$\Delta P(X,Y) = 0 \tag{1}$$

$$P \to (4\pi)^{-1}\ln(X^2+Y^2) \quad \text{as} \quad X^2+Y^2 \to \infty \tag{2}$$

With the exterior normal velocity at the boundary given by:

$$v_n = -\nabla P \tag{3}$$

(normalized so that the proportionality constant is -1); where x denotes time and X, Y are the Cartesian coordinates of a point on the boundary.

In the case where the boundary is an analytic curve it is usual to introduce a time-dependent conformal mapping from the exterior of the unit circle in the "mathematical" (w) plane

$$z = z(w,x), \quad w = \exp(\sqrt{-1}\phi), \quad 0 < \phi < 2\pi \tag{4}$$

to the exterior of the region determined by the boundary curve $z = X + \sqrt{-1}Y$ in the "physical" plane with the unit circle mapping to the boundary). Simple considerations [15], [16], [12] show that equations (1, 3) are equivalent to the following equation

$$\mathrm{Im}\left(\frac{\partial z}{\partial \phi}\frac{\partial \bar{z}}{\partial x}\right) = w\left(\frac{\partial z(w,x)}{\partial w}\frac{\partial \bar{z}(1/w,x)}{\partial x} - \frac{\partial z(w,x)}{\partial x}\frac{\partial \bar{z}(1/w,x)}{\partial w}\right) = 1 \tag{5}$$

where bar stands for complex conjugation (and $\bar{w} = w^{-1}$ on the boundary curve).

It turns out that (5) plays an essential role in the theory of infinite-dimensional integrable hierarchies. The relation between the contour dynamics above and the dispersionless limit of the integrable Toda hierarchy constrained by (5) was established in [12].

Equation (5) may be interpreted as a constraint on an infinite commuting set of dynamical systems defined in the space of one-parameter families of conformal maps. This constraint characterizes the fixed points of an "additional symmetry" ([14]). The most interesting aspect of such constrained dToda flows is that they admit finite-dimensional reductions, including the so-called "multi-finger" solutions ([10],[18]).

These solutions are of great importance in practical applications and describe numerous phenomena, such as viscous fingering in a Hele-Shaw cell [18],[7],[10] , pattern formation in the quantum Hall effect [1] etc.

In what follows, we consider finite-dimensional solutions of (5) in the context of the dToda hierarchy. We study formal algebraic solutions of the problem, ignoring the real structure, treating z, $\bar{z}$ as independent functions, and w as a formal variable. To return to the original problem one has to identify bar with complex conjugation and $\bar{z}$ with $S(z^{-1})$, where $S(z)$ is the Schwarz function associated with the conformal map.

2 dToda hierarchy and string equation.

The dToda hierarchy is defined in terms of two functions $z(w,x)$ and $\bar{z}(w^{-1},x)$ of the form:

$$z(w,x) = r(x)w + \sum_{k=0}^{\infty} u_k(x)w^{-k} \tag{6}$$

$$\bar{z}(w^{-1},x) = r(x)w^{-1} + \sum_{k=0}^{\infty} \bar{u}_k(x)w^{k} \tag{7}$$

The dToda flow equations are

$$\begin{array}{ll} \partial_{t_k} z = \{H_k, z\} & \partial_{\bar{t}_k} \bar{z} = \{\bar{H}_k, \bar{z}\} \\ \partial_{t_k} \bar{z} = \{H_k, \bar{z}\} & \partial_{\bar{t}_k} z = \{\bar{H}_k, z\} \end{array} \tag{8}$$

where the coefficients $r(x), u_k(x), \bar{u}_k(x)$ are viewed as coordinate functionals on the phase space consisting of such pairs $z(w,x), \bar{z}(w^{-1},x)$ with

$$H_k = (z^k)_+ + 1/2(z^k)_0, \quad \bar{H}_k = (\bar{z}^k)_- + 1/2(\bar{z}^k)_0 \tag{9}$$

with subscripts $\pm, 0$ denoting the negative/positive and zero parts of the formal Laurent expansion in w. The Poisson-Lax bracket notation here denotes

$$\{f,g\} := w\frac{\partial f}{\partial w}\frac{\partial g}{\partial x} - w\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} \tag{10}$$

It is important to note that (10) is not the bracket defining Poisson structure on the infinite-dimensional phase space of the dToda system (8), but rather a "quasiclassical" $\hbar \to 0$ limit of commutators in the Lax representation of a dispersionful analog of (8) (where $\hbar$ stands for lattice spacing [19]

The dispersionless limit of the "string equation" is the constraint

$$\{z(w,x), \bar{z}(w^{-1},x)\} = 1, \tag{11}$$

Equation (11) is invariant under (8), since

$$\partial_{t_k}\{z,\bar{z}\} = \{\partial_{t_k} z, \bar{z}\} + \{z, \partial_{t_k}\bar{z}\} = \{\{H_k, z\}, \bar{z}\} + \{z, \{H_k, \bar{z}\}\} = \{\{\bar{z}, z\}, H_k\} = 0$$

Thus the string equation (11) defines an invariant under the dToda flows (8) manifold and a reduction of the full dToda hierarchy.

3 Reductions of dToda hierarchy constrained by the string equation.

The reduction of the dToda hierarchy by the string equation is still an infinite, compatible set of infinite-dimensional dynamical systems. In what follows we will be interested in further "functional" reductions where $z, \bar{z}$ are polynomial, rational or logarithmic functions of w.

For these reductions, it may be seen that the string equation (11) is equivalent to a finite system of ODE's determining the x-derivatives of an independent finite set of coordinate functions on the reduced spaces. (For the polynomial case, these may be chosen as r together with the finite set of non-vanishing u_k, $\bar{u}_k$ themselves; for rational functions z and $\bar{z}$, they are the location of the poles and the values of their residues; for logarithmic z and $\bar{z}$'s they are the location of the branch points and the residues of $\frac{dz}{dw}$, $\frac{d\bar{z}}{dw}$ at these points.

As shown below such reductions are consistent with (8) (i.e. they are preserved by the dToda flows) if the string equation (11) holds. Thus, for consistency we need a double ("functional" plus "string") reduction. This pair of reductions defines a finite-dimensional invariant sub-manifold in the phase space of the general dToda hierarchy. Indeed, functional reduction leaves a finite number of discrete indices in the ansatz for z as a function of w, while the string equation determines the dependence of z, $\bar{z}$ on x leaving a finite number of degrees of freedom. These degrees of freedom are connected with the integration constants of the equations determining the x-dependence of the coordinate functions, which becomes finite-dimensional after the functional reduction.

3.1 Polynomial reductions.

We begin with polynomial reductions of the dToda chain

$$z(w) = rw + \sum_{i=0}^{N} u_i w^{-i} \tag{12}$$

$$\bar{z}(w^{-1}) = rw^{-1} + \sum_{i=0}^{N} \bar{u}_i w^{i} \tag{13}$$

constrained by the string equation (11).

The following proposition states the consistency of such polynomial reductions under the dToda flows:

Proposition 1 *If the string equation (11) holds, then (12), (13) belong to a manifold invariant under the dToda flows* $\partial/\partial x$, $\partial/\partial t_i$, $\partial/\partial \bar{t}_i, 0 < i < N+2$ *(8). This manifold has dimension* $2N+3$ *with* $r = r(x,T)$, $u_i = u_i(x,T)$, $\bar{u}_i = \bar{u}_i(x,T)$, $T = \{t_i, \bar{t}_i\}, i = 1,.., N+1$ *being solutions of the dynamical system induced by the dToda flows (8), the x-dependence determined by the string equation, in terms of* $2N+1$ *initial values.*

Proof: We must prove that z remains of the form (12) under the flows generated by $H_k, \bar{H}_k$ provided the string equation (11) holds. That is, we have to show that the evolution does not change the highest and lowest degrees of the Laurent polynomial (12). The proof for $\bar{z}$ is analogous.

1. First we proceed with the flows generated by H_k. The lowest degree of z is $-N$. Since

$$H_k = (z^k)_+ + 1/2(z^k)_0 = h_k w^k + h_{k-1} w^{k-1} + \ldots + h_0 \tag{14}$$

is a Laurent polynomial of positive degree, the lowest degree term in the bracket $\{z, H_k\}$ is not less than $-N$. Therefore the lowest degree term in z remains $\geq -N$ under the flows.

On the other hand, the complement $z^k - H_k$ of (14) is a polynomial in $1/w$ and so,

$$\{H_k, z\} = -\{z^k - H_k, z\}$$

is a Laurent polynomial with the highest degree in w not exceeding 1. Therefore the highest degree term in z remains ≤ 1.

2. Unlike the evolution under the H_k-flows, the form-invariance (12) of z under the flows generated by the $\bar{H}_k$'s requires extra restrictions on the derivatives $\partial_{t_k} u_i$, stemming from the string equation.

Since

$$\bar{H}_k = (\bar{z}^k)_- + 1/2(\bar{z}^k)_0$$

is a Laurent polynomial of non-positive degree, we again see that the highest degree in $\{z, H_k\}$ is 1, and hence this degree cannot increase under the flow.

However, since

$$\bar{H}_k = \bar{z}^k - ((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0)$$

is the difference between $\bar{z}^k$ and a polynomial of nonnegative degree,

$$\{\bar{H}_k, z\} = \{\bar{z}^k, z\} - \{((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0), z\},$$

the second bracket in this expression has lowest degree $\geq -N$, but not, in general, the first. However, since

$$\{\bar{z}^k, z\} = k\bar{z}^{k-1}\{\bar{z}, z\},$$

imposing the extra restriction (11) (string equation), again implies that the lowest degree term in this bracket does not exceed $-N$, provided $k \leq N+1$

Therefore, the form of (12) is preserved under all t_i-flows, but only those $\bar{t}_i$-flows for which $i \leq N+1$. A similar argument shows the analogous statement to hold for $\bar{z}$ under the first $N+1$ t_i-flows, and all $\bar{t}_i$-flows. Therefore the polynomial reductions (12, 13) are preserved simultaneously under both the t_i- and $\bar{t}_i$-flows for $i \leq N+1$.

This completes the proof.

Remark. The string equation, in this case, reduces to generically non-singular system of linear equations for the derivatives $\{\partial u_k/\partial x, \partial \bar{u}_k/\partial x\}_{k=1..N+1}; \partial r/\partial x$ (i.e. for a dense, open set of initial conditions the system is non-singular). These can therefore (on such dense, open set) be uniquely solved to determine the x-dependence in $u_k(x), \bar{u}_k(x), r(x)$ in terms of $2N+1$ integration constants.

The evolution generated by the $(t_k, \bar{t}_k)_{k=1..N+1}$ flows may then be interpreted as flows on this $2N+1$-dimensional phase space. It would, however be more symmetrical to simply view the components $u_k, \bar{u}_k, r$ defining z and $\bar{z}$ as functions of $2N+3$ flow variables $\{t_0, t_k, \bar{t}_k\}_{k=1..N+1}$ with $t_0 = x$, since the string equation just reduces to an additional first order equation in the t_0 flow variable compatible with the rest.

3.2 Rational reductions.

Consider now the space of rational functions $z(w)$ and $\bar{z}(w)$ of the form

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{rw^{N+1} + \sum_{i=0}^N a_i w^i}{w^N + \sum_{i=0}^{N-1} b_i w^i} \tag{15}$$

$$\bar{z}(w^{-1}) = \frac{\bar{q}_{N+1}(w^{-1})}{\bar{p}_N(w^{-1})} = \frac{rw^{-(N+1)} + \sum_{i=0}^N \bar{a}_i w^{-i}}{w^{-N} + \sum_{i=0}^{N-1} \bar{b}_i w^{-i}}, \tag{16}$$

where the $4N+3$ coefficients $r, a_i, \bar{a}_i, b_i, \bar{b}_i$ are functions of x.

The following Lemma concerns the invariance of such rational reductions of $z(w)$ and $\bar{z}(w)$ under the dToda flows.

Lemma 1 *The space of functions $z(w)$ of the form (15) is invariant under the dToda flows $\partial_{t_i}, i > 0$. and, similarly, the space of functions $\bar{z}(w^{-1})$ of the form (16) is invariant under the $\partial_{\bar{t}_i}, i > 0$ flows.*

Proof. Consider the flows generated by H_k. The proof for $\bar{z}$ is analogous. The function

$$H_k = (z^k)_+ + 1/2(z^k)_0 \tag{17}$$

is a polynomial in w.

Therefore, its Lax bracket (10), where Q is a polynomial, is of the form

$$\{H_k, z\} = Q(w)/p_N(w)^2,$$

Since this bracket may also be expressed as:

$$\{H_k, z\} = -\{z^k - H_k, z\} = k_1(T)w + k_0(T) + k_{-1}(T)w^{-1} + ... \quad (18)$$

It follows from (18) that the highest degree in polynomial $Q(w)$ does not exceed $2N + 1$.

On the other hand, since

$$\partial_{t_k} z = P(w)/p_N{}^2(w), \quad P(w) = p_N \partial_{t_k} q_{N+1} - q_{N+1} \partial_{t_k} p_N$$

where the highest degree in the polynomial $P(w)$ also does not exceed $2N + 1$, we may therefore equate the coefficients of polynomials $P(w)$ and $Q(w)$. This gives a first order system of $2N + 1$ equations for the coefficients r, a_k, b_k, which are linear in derivatives. For general initial conditions these may be solved to obtain $2N + 1$ equations of the form:

$$\frac{dr}{dt_k} = R_k(r, \vec{a}, \vec{b}), \qquad \frac{da_l}{dt_k} = A_{kl}(r, \vec{a}, \vec{b}), \qquad \frac{db_l}{dt_k} = B_{kl}(r, \vec{a}, \vec{b}), \quad (19)$$

for each t_k, where the expressions on the right are rational in the arguments. The rational form of $z(w)$ (15) is thus consistently preserved under the integrated t_k-flows, and the compatibility for different values of the flow parameters follows from the compatibility conditions of the dToda flows equations (8) prior to the reduction.

A similar argument shows the invariance of the rational form of $\bar{z}(w)$ (16) under the $\bar{t}_k$-flows.

This completes the proof.

The consistency of invariance of the set of rational pairs $(z, \bar{z})$ under both the t_i and $\bar{t}_i$ flows would require some extra restrictions, like the string equation for the polynomial case. However, this turns out to only imply invariance under the pair of flows $t_1, \bar{t}_1$.

Proposition 2 *The string equation (11) is a sufficient condition for the set of rational functions $z(w)$ and $\bar{z}(w)$ ((15), (16)) to be invariant under the first two-Toda flows $\partial/\partial t_1$, $\partial/\partial \bar{t}_1$ (see (8)).*

Proof.

In order to prove that z remains of the same form (15) under the flows generated by $\bar{H}_k$, we have to find a compatibility condition for a system of differential equations for $r(T)$, $a(T)$, $b(T)$, induced by (8). We now show that for each flow, the number of equations does not exceed the number of unknowns.

Again we write

$$\partial z/\partial \bar{t}_k = S(w)/p_N{}^2(w), \quad S(w) = p_N \partial_{\bar{t}_k} q_{N+1} - q_{N+1} \partial_{\bar{t}_k} p_N,$$

$S(w)$ being a polynomial in w of order at most $2N + 1$.

Since

$$\bar{H}_k = (\bar{z}^k)_- + 1/2(\bar{z}^k)_0 = \bar{h}_k w^{-k} + \bar{h}_{k-1} w^{-k+1} + \ldots + \bar{h}_0 \quad (20)$$

is a polynomial in w^{-1}, the Laurent expansion near $w = \infty$ of the corresponding bracket has the following form

$$\{\bar{H}_k, z\} = k_1(T)w + k_0(T) + k_{-1}(T)w^{-1} + ...$$

The definition of the Lax bracket (10) and (20) implies that

$$\{\bar{H}_k, z\} = (U(w) + R(1/w))/p_N(w)^2$$

where $U(w)$ is a polynomial of degree at most $2N + 1$ and $R = w^{-1}(\eta_{k-1}(T)w^{-k+1} + \ldots + \eta_0$ is a polynomial in w^{-1}

If $R(1/w) = 0$, the number of equations will not exceed the number of unknowns. But, as we show below, the string equation does not allow the vanishing of $R(w)$.

The condition $R(1/w) = 0$ means that $\{\bar{H}_k, z\}$ has a convergent Taylor expansion about $w = 0$. Imposing the string equation, however, gives

$$\begin{aligned}\{\bar{H}_k, z\} &= \{\bar{z}^k, z\} - \{((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0), z\} = \\ &= k\bar{z}^{k-1}\{z, \bar{z}\} - \{((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0), z\}\end{aligned} \tag{21}$$

which implies that there is a pole of order $k-1$ at $w = 0$. For $k = 1$, however, this shows that $\{\bar{H}_1, z\}$ has no pole at $w = 0$ and hence (provided $b_0 \neq 0$), $R(1/w) = 0$.

A similar argument applies to the rational form (16) for $\bar{z}$ under the t_1 flow. This completes the proof.

As shown in the above proof, there are only two flows compatible with the rational ansatz. In fact, we could not expect more invariant flows associated to the two simple poles at $w = 0$ and $w = \infty$. In the polynomial case, the number of invariant flows was equal to the number of variables (polynomial coefficients), since one can associate n invariant flows to the pole of the n th order, and poles at zero and infinity are fixed.

Below, we introduce additional flows, related to movable singularities of $z(w)$ and $\bar{z}(w^{-1})$ using a result by Krichever.

3.3 Additional flows for rational reductions of the dKP hierarchy.

As mentioned in [4], on the phase space of extended Benney systems, i.e. rational dKP reductions admitting poles of arbitrary degree, there arise some new flows related to the pole structure of the corresponding maps. These additional flows were introduced by Krichever (see [9]).

Consider a representation of rational maps in pole-residue form for simple poles along with a half (1dToda or dKP) of flows. In the Takasaki gauge [20], [4]

$$z(w) = w + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha} \tag{22}$$

The new flows attached to the poles are defined as before

$$\partial_{t_{k,\alpha}} z = \{B_{k,\alpha}, z\}, \quad \alpha = \infty, 1, 2, \ldots \quad k = 0, 1, 2, \ldots \tag{23}$$

with the evolution operators being associated with the pole structure of z as follows :

$$B_{k,\infty} = (z(w)^k)_{\geq 0}$$

for an immovable pole at infinity, while for each finite-distance pole there appear additional flows with evolution operators as follows:

$$B_{k,\alpha} = (z(w)^k)_\alpha, \quad B_{0.\alpha} = \log(w_\alpha - w)$$

Here, $z(w)_\alpha$ stays for the negative part of a formal expansion of $z(w)$ near its poles w_α:

$$f(w)_\alpha = \sum_{i>0} \frac{f_i}{(w - w_\alpha)^i}, \quad \text{if} \quad f = \sum_{i \in Z} \frac{f_i}{(w - w_\alpha)^i} \tag{24}$$

These additional flows commute among themselves and with the ordinary (associated with poles at infinity) 1dToda or dKP flows ([4]).

3.4 Additional invariant flows of the 2dToda system.

The dKP hierarchy can be extended to the 2dToda system by introducing an infinite set of $\bar{t}$-flows associated to the poles at $w = 0$ and $w = \infty$, similarly the Krichever-Benney system has the following 2dToda extension:

$$\begin{array}{c} z(w) = rw + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w-w_\alpha}, \\ \bar{z}(1/w) = r/w + \bar{u}_0 + \sum_{\beta=1}^{N} \frac{\bar{u}_\beta}{1/w-\bar{w}_\beta} \end{array} \tag{25}$$

$$\begin{array}{ll} \partial_{t_{k,\alpha}} z = \{H_{k,\alpha}, z\} & \partial_{\bar{t}_{k,\beta}} \bar{z} = \{\bar{H}_{k,\beta}, \bar{z}\} \\ \partial_{t_{k,\alpha}} \bar{z} = \{H_{k,\alpha}, \bar{z}\} & \partial_{\bar{t}_{k,\beta}} z = \{\bar{H}_{k,\beta}, z\} \end{array} \tag{26}$$

In (26) we transformed the flows from the Takasaki to the Lax-Sato gauge needed for our purposes. The relation between the evolution operators in different gauges is found to be:

$$H_{k,\alpha}(w) = B_{k,\alpha}(w) - \frac{1}{2}B_{k,\alpha}(w=0), \quad \bar{H}_{k,\beta}(y) = \bar{B}_{k,\beta}(y) - \frac{1}{2}\bar{B}_{k,\beta}(y=0) \tag{27}$$

where

$$\begin{array}{lll} B_{k,\infty}(w) = (z(w)^k)_{\geq 0}, & B_{0,\alpha} = \log\left(r(w_\alpha - w)\right), & B_{k,\alpha}(w) = (z(w)^k)_\alpha \\ \bar{B}_{k,\infty}(y) = (\bar{z}(y)^k)_{\geq 0}, & \bar{B}_{0,\beta} = \log\left(r(\bar{w}_\beta - y)\right), & \bar{B}_{k,\beta}(y) = (\bar{z}(y)^k)_\beta, \quad y = 1/w. \end{array} \tag{28}$$

Lemma 2 *All vector fields attached to the pole structure of rational maps formally commute, i.e. the introduced evolution operators (27) satisfy zero-curvature conditions.*

It is important to note that in the system (26), the equations for the flows $\partial_{t_k}\bar{z}, \partial_{\bar{t}_k} z$ do not make a sense fully, since these flows do not preserve the rational ansatz for $z, \bar{z}$ in (22) and (25) until the additional reduction is made. As we have seen before, to be consistent, these systems must be restricted by a string equation, which makes (26) a finite-dimensional dynamical system.

Proposition 3 *The $4N+2$ commuting Toda-Krichever flows*

$$\begin{array}{l} \partial_{\tau_i} z = \{h_i, z\}, \quad \partial_{\bar{\tau}_i} z = \{\bar{h}_i, z\} \\ \partial_{\tau_i} \bar{z} = \{h_i, \bar{z}\}, \quad \partial_{\bar{\tau}_i} \bar{z} = \{\bar{h}_i, \bar{z}\} \end{array}, \qquad i = 0..2N \tag{29}$$

where

$$\begin{array}{ll} h_0 = H_{1,\infty} = rw + u_0/2, & \bar{h}_0 = \bar{H}_{1,\infty} = r/w + \bar{u}_0/2 \\ h_{2i-1} = H_{1,i} = \frac{u_i}{w-w_i} + \frac{u_i}{2w_i}, & \bar{h}_{2i-1} = \bar{H}_{1,i} = \frac{\bar{u}_i}{\bar{w}_i - 1/w} + \frac{1}{2}\bar{u}_i/\bar{w}_i \\ h_{2i} = H_{0,i} = \log(w_i - w) + 1/2\log(r/w_i), & \bar{h}_{2i} = \bar{H}_{0,i} = \log(\bar{w}_i - 1/w) + 1/2\log(r/\bar{w}_i) \end{array} \tag{30}$$

and

$$\tau_0 = t_{1,\infty}, \quad \tau_{2i-1} = t_{1,i}, \quad \tau_{2i} = t_{0,i}, \quad \bar{\tau}_0 = \bar{t}_{1,\infty}, \quad \bar{\tau}_{2i-1} = \bar{t}_{1,i}, \quad \bar{\tau}_{2i} = \bar{t}_{0,i} \tag{31}$$

preserve the rational form of $z(w)$ (22) and $\bar{z}(1/w)$ (25) (or equally (16, 15)) provided the string equation (11) holds.

As in the polynomial case, the total number of invariant flows equals the dimension of the dynamical system minus one. In what follows we show that these flows are Hamiltonian. Since the dimension of the phase space is odd and is equal to $4N+3$, it is, in fact a Poisson manifold where the dimension of the symplectic leaf is $4N+2$. This last number is exactly equal to the number of commuting Toda-Krichever flows.

Now we conjecture that the stated result is true for the more general setting. We introduce a logarithmic ansatz for the 2dToda hierarchy and prove lemma 2 and proposition 3 for logarithms followed by a limiting procedure.

3.5 Logarithmic flows.

As was mentioned above, it is easier to prove consistency of the rational ansatz with the dynamics of the 2dToda system using the more general Logarithmic solutions. Let us set:

$$\begin{array}{ll} z = r(x)w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w), & \sum_{i=1}^{n+1} a_i = 0 \\ \bar{z} = r(x)w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log(\bar{w}_i(x) - w^{-1}), & \sum_{i=1}^{n+1} \bar{a}_i = 0 \end{array}, \quad i = 1..n+1 \tag{32}$$

where $a_i, \bar{a}_i$ are arbitrary constants, subject to conditions $\sum_{i=1}^{n+1} a_i = 0, \sum_{i=1}^{n+1} \bar{a}_i = 0$ which ensure the absence of logarithmic singularities at infinity. For the introduced ansatz (32) we claim the following result:

Proposition 4 *Let us generalize the evolution operators to be as follows:*

$$\begin{array}{ll} \mathcal{H}_0 = r(x)w + \frac{1}{2}u(x), \quad \bar{\mathcal{H}}_0 = \bar{r}(x)w^{-1} + \frac{1}{2}\bar{u}(x) & \\ \mathcal{H}_i = \log(w_i(x) - w) + \frac{1}{2}\log(r(x)/w_i(x)) & i = 1..n+1 \\ \bar{\mathcal{H}}_i = \log(\bar{w}_i(x) - w^{-1}) + \frac{1}{2}\log(r(x)/\bar{w}_i(x)) & \end{array} \tag{33}$$

Then, the $2n+4$ *flows generated by the Lax equations*

$$\begin{array}{ll} \partial_{\tau_i} z = \{\mathcal{H}_i, z\}, & \partial_{\bar{\tau}_i} z = \{\bar{\mathcal{H}}_i, z\} \\ \partial_{\tau_i} \bar{z} = \{\mathcal{H}_i, \bar{z}\}, & \partial_{\bar{\tau}_i} \bar{z} = \{\bar{\mathcal{H}}_i, \bar{z}\} \end{array}, i = 0..n+1 \tag{34}$$

commute. They preserve the logarithmic ansatz (32) provided the string equation (11) holds.

In other words, the 2dToda flows are tangent to the manifold of logarithmic functions if the string condition is imposed and we again have $2n+4$ flows leaving invariant a $2n+5$ dimensional sub-manifold of the 2dToda system.

It is easier to prove the proposition in the Takasaki gauge. For this purpose we give the transition formulas between different gauges before the proof. For the general 2dToda hierarchy, the Lax functions $z^{(g)}$ and $\bar{z}^{(g)}$ in an arbitrary g-gauge are expressed through the Lax-Sato gauge which corresponds to $g = 0, z = z^{(0)}, \bar{z} = \bar{z}^{(0)}$ as follows

$$\begin{array}{l} z^{(g)}(w) = z(w/r^{2g}) = r(x)^{1-2g}w + \sum_{i=-\infty}^{0} u_i^{(g)}(x)w^i, \\ \bar{z}^{(g)}(w^{-1}) = \bar{z}(r^{2g}/w) = r(x)^{1+2g}w + \sum_{i=0}^{\infty} \bar{u}_i^{(g)}(x)w^i \end{array}$$

while, (omitting evident subscripts)

$$\begin{array}{l} H^{(g)}(w) = B^{(g)}(w) - \left(\frac{1}{2} - g\right) B^{(g)}(w=0), \\ \bar{H}^{(g)}(1/w) = \bar{B}^{(g)}(1/w) - \left(\frac{1}{2} + g\right) \bar{B}^{(g)}(1/w=0) \end{array} \tag{35}$$

Here, $B^{(g)}$ and $\bar{B}^{(g)}$ are specified similarly to those in (28) . In particular, in the Takasaki gauge $g = 1/2$ (below we omit superscripts $(1/2)$)

$$z = w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w), \quad \bar{z} = r(x)^2 w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log(\bar{w}_i(x) - w^{-1}) \tag{36}$$

$$\begin{array}{ll} \mathcal{H}_0 = w + u(x), & \bar{\mathcal{H}}_0 = r(x)^2 w^{-1} \\ \mathcal{H}_i = \log(w_i(x) - w), & \bar{\mathcal{H}}_i = \log(\bar{w}_i(x) - w^{-1}) + \log(w_i(x)/r(x)) \end{array} \tag{37}$$

Proof of Proposition 4

1.*Formal commutativity of flows*: We demonstrate the commutativity of $\partial_{\tau_i} z$, and $\partial_{\tau_j} z$. The proof is similar for the rest of the flows.

The commutativity of the flows is equivalent to the zero-curvature equation

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{\partial \mathcal{H}_j}{\partial \tau_i} + \frac{\partial \mathcal{H}_i}{\partial \tau_j} = 0 \tag{38}$$

Equations (37), (36) and (34) imply that

$$\frac{\partial \mathcal{H}_j}{\partial \tau_i} = \frac{\partial_{\tau_i} w_j}{w_j - w} = \frac{1}{a_j}(\partial_{\tau_i} z)_j = \frac{1}{a_j}\{\mathcal{H}_i, z\}_j$$

where subscript j stands for the singular part of the expansion around w_j (see (24)). Thus instead of the lhs of (38) we have

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{1}{a_j}\{\mathcal{H}_i, \mathcal{H}_j\}_j + \frac{1}{a_i}\{\mathcal{H}_i, \mathcal{H}_j\}_i$$

which is zero by direct calculation applying (37) and (10). This proves the formal commutativity of the flows.

2. *Consistency of (36) with equations of motion (34)*: In a way similar to that of the polynomial and rational cases the consistency of the half-flows $\partial_{\tau_i} z$, and $\partial_{\bar{\tau}_i} \bar{z}$ for logarithmic solutions is automatic and does not require any extra restrictions on the coefficients (the proof is also similar). In the logarithmic ansatz the main obstacle appears for the flows $\partial_{\tau_i} \bar{z}$ and $\partial_{\bar{\tau}_i} z$. We prove the consistency for $\partial_{\bar{\tau}_i} z$. The proof for $\partial_{\tau_i} \bar{z}$ is similar.

Differentiating z directly with respect to $\bar{\tau}_i$ and using the equation of motion (34) we get

$$\partial_{\bar{\tau}_i} z = \partial_{\bar{\tau}_i} u + \sum_{j=1}^{n+1} \frac{a_i \partial_{\bar{\tau}_i} w_j}{w - w_j} = \{\bar{\mathcal{H}}_i, z\}$$

The lhs of the equation contains only singularities of z. Using definition (10) and equations (36), (37), we see that the rhs contains both singularities of z and one singularity of $\bar{\mathcal{H}}_i$. In order for the equation of motion to be consistent with the logarithmic ansatz, the rhs , $\{\bar{\mathcal{H}}_i, z\}$, must contain the same singularities as those in the lhs, i.e. it must not contain a singularity of $\bar{\mathcal{H}}_i$ at $w = 1/\bar{w}_i$

Let us exploit the fact that in the Takasaki gauge (36), (37), z and $\bar{z}$ can be represented as sums over the corresponding $\mathcal{H}$:

$$z = \mathcal{H}_0 + \sum_{i=1}^{n+1} a_i \mathcal{H}_i, \quad \bar{z} = \bar{\mathcal{H}}_0 + \sum_{i=1}^{n+1} \bar{a}_i \bar{\mathcal{H}}_i + f(r, w_1, .., w_{n+1}) \tag{39}$$

where f is a w-independent function. Note that the conditions $\sum_{i=1}^{n+1} a_i = 0, \sum_{i=1}^{n+1} \bar{a}_i = 0$ of Proposition 4 ensure that there are no logarithmic singularities at infinity.

Now, using (39) we obtain the following expression

$$\bar{a}_i\{\bar{\mathcal{H}}_i, z\} = \{\bar{z} - f - \sum_{j\neq i} \bar{a}_j \bar{\mathcal{H}}_j, z\} = \{\bar{z}, z\} - \{f, z\} - \sum_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}$$

The term $\sum_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}$ of the rhs of the last expression does not contain singularities of $\bar{H}_i$ since it does not contain the index i in the sum. The next term $\{f, z\}$ does not contain them either, since f is independent of w. The only term which may contain undesired singularities is $\{\bar{z}, z\}$, however the string equation (11) holds and therefore $\{\bar{\mathcal{H}}_i, z\}$ is free of singularities of $\bar{z}$. Thus the equation of motion is consistent with the logarithmic ansatz due to the string equation.

This completes the proof.

Corollary 1 *The consistency of the rational ansatz follows from the consistency of the logarithmic ansatz. (Proposition 3 is a consequence of proposition 4)*

Proof: Choosing $n = 2N - 1$ and substituting the values

$$\begin{array}{lll} a_{2i-1} = 1/\epsilon, & a_{2i} = -1/\epsilon & w_{2i} = w_{2i-1} + \epsilon u_i \\ \bar{a}_{2i-1} = 1/\epsilon, & \bar{a}_{2i} = -1/\epsilon & \bar{w}_{2i} = \bar{w}_{2i-1} + \epsilon \bar{u}_i \end{array} \tag{40}$$

for z and $\bar{z}$ into (32) we get rational solutions (25), (22) in the $\epsilon \to 0$ limit.

The Hamiltonians are then related as follows:

$$\begin{array}{ll} h_0 = \lim_{\epsilon=0}\mathcal{H}_0, & \bar{h}_0 = \lim_{\epsilon=0}\bar{\mathcal{H}}_0 \\ h_{2i-1} = \lim_{\epsilon=0}\frac{1}{\epsilon}(\mathcal{H}_{2i}-\mathcal{H}_{2i-1}), & \bar{h}_{2i} = \lim_{\epsilon=0}\frac{1}{\epsilon}(\bar{\mathcal{H}}_{2i}-\bar{\mathcal{H}}_{2i-1}) \\ h_{2i} = \lim_{\epsilon=0}\mathcal{H}_{2i-1}, & \bar{h}_{2i-1} = \lim_{\epsilon=0}\bar{\mathcal{H}}_{2i-1} \end{array}$$

End of proof

Thus we have obtained a rational ansatz as a limiting case of the logarithmic ansatz, merging pairs of logarithmic singularities together, in which case they become simple poles. In a similar way one may deduce any kind of rational maps containing a combination of poles of any degrees absorbing different numbers of logarithmic singularities.

4 Poisson structure of rational reductions of the 2D dToda hierarchy.

In this section we study the Poisson structure of rational reductions of the 2dToda hierarchy, which, as we have seen from the above considerations, form a finite-dimensional completely integrable system. We find explicit expressions leading to a canonical Hamiltonian structure on the phase space of rational reductions of the 2D dToda system. Such reductions are interesting from the point of view of the applications mentioned in the Introduction.

A reader interested in the Poisson structure of rational reductions of the 1dToda hierarchy (which are infinite-dimensional, unrestricted by string equation systems) may refer to the Appendix. This structure, which turns out to be a quadratic algebra, is interesting by itself.

4.1 Hamiltonians and action-angle variables.

Since the Toda flows commute, the Toda times are actually the action-angle variables of the system.

Thus we have only to express the times through coordinates of the phase space. For the logarithmic ansatz (4) the following result holds:

Proposition 5 *The times $\tau_i, \bar{\tau}_i$ of equation (34) are expressed in terms of z and $\bar{z}$ (32) as*

$$\tau_0 = \frac{1}{2\pi\sqrt{-1}}\oint_\infty \bar{z}\, d\ln z + \text{const}, \quad \bar{\tau}_0 = \frac{1}{2\pi\sqrt{-1}}\oint_0 z\, d\ln\bar{z} + \text{const},$$
$$\tau_i = \frac{1}{2\pi\sqrt{-1}}\oint_{w_i}\bar{z}\,dz + \text{const}, \quad \bar{\tau}_i = \frac{1}{2\pi\sqrt{-1}}\oint_{1/\bar{w}_i} z\,d\bar{z} + \text{const}, \quad i = 1..n+1$$

or, in other words, the following values

$$\begin{array}{l} I_0 = \bar{z}(1/w=0) = \bar{u} + \sum_{i=1}^{n+1}\bar{a}_i\ln(\bar{w}_i), \quad \bar{I}_0 = z(0) = u + \sum_{i=1}^{n+1} a_i\ln(w_i) \\ I_i = a_i\bar{z}(w_i^{-1}) = a_i\left(rw_i^{-1} + \bar{u} + \sum_{j=1}^{n+1}\bar{a}_j\ln(\bar{w}_j - w_i^{-1})\right) \\ \bar{I}_i = \bar{a}_i z(\bar{w}_i^{-1}) = \bar{a}_i\left(r\bar{w}_i^{-1} + u + \sum_{j=1}^{n+1} a_j\ln(w_j - \bar{w}_i^{-1})\right), \quad i = 1..n+1, \end{array} \tag{41}$$

are the action-angle variables and

$$\begin{array}{l} Q = \frac{1}{4\pi\sqrt{-1}}\sum_{i=0}^{n+1}\oint_{1/\bar{w}_i} z\,d\bar{z} + \oint_{w_i}\bar{z}\,dz \\ = \frac{1}{2}r\left(\left(\frac{\partial z(w)}{\partial w}\right)_{w=0} + \left(\frac{\partial\bar{z}(1/w)}{\partial(1/w)}\right)_{1/w=0}\right) - \frac{1}{2}\sum_{i=1}^{n+1}(I_i + \bar{I}_i) \\ = r^2 - \frac{1}{2}\sum_{i=1}^{n+1}\left(r\left(\frac{a_i}{w_i} + \frac{\bar{a}_i}{\bar{w}_i}\right) + I_i + \bar{I}_i\right) \end{array} \tag{42}$$

is a Casimir for (34)

Proof: We first calculate the derivatives of each I_i with respect to the times $\tau_i, \bar{\tau}_i, i = 1..n$. Using integration by parts, we get

$$2\pi\sqrt{-1}\frac{\partial I_i}{\partial\tau_j} = \frac{\partial}{\partial\tau_j}\oint_{w_i}\bar{z}\frac{\partial z}{\partial w}dw = \oint_{w_i}\frac{\partial}{\partial w}\left(\bar{z}\frac{\partial z}{\partial\tau_j}\right)dw + \oint_{w_i}\left(\frac{\partial\bar{z}}{\partial\tau_j}\frac{\partial z}{\partial w} - \frac{\partial z}{\partial\tau_j}\frac{\partial\bar{z}}{\partial w}\right)dw$$

By equations of motion (34) and definition of the Lax-Poisson brackets (10) this value is equal to

$$\oint_{w_i} \frac{\partial}{\partial w}\left(\bar{z}\frac{\partial z}{\partial \tau_j}\right) dw + \oint_{w_i}\left(\{\mathcal{H}_j, \bar{z}\}\frac{\partial z}{\partial w} - \{\mathcal{H}_j, z\}\frac{\partial \bar{z}}{\partial w}\right) dw$$

$$= \oint_{w_i} \frac{\partial}{\partial w}\left(\bar{z}\frac{\partial z}{\partial \tau_j}\right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} w \left(\frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial x} - \frac{\partial z}{\partial x}\frac{\partial \bar{z}}{\partial w}\right) dw$$

which by the string equation (11) reduces to:

$$\oint_{w_i} \frac{\partial}{\partial w}\left(\bar{z}\frac{\partial z}{\partial \tau_j}\right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} dw = 2\pi\sqrt{-1}\delta_{ij}$$

We get the Kronecker δ-symbol as the final answer, since the first term in the last line vanishes. Indeed, this occurs because $\bar{z}\frac{\partial z}{\partial \tau_j}$ is a meromorphic in w in the vicinity of any singularity w_i of z and the second term gives the δ_{ij} contribution by definition (33) of $\mathcal{H}_j$.

Using similar reasoning we prove the rest of the proposition.

The values (41) are similar to those appearing in the work [11] in connection with the Laplacian Growth problem as integrals of the Laplacian Growth (string) equation.The following proposition (although not assuming complex structure) confirms this fact.

Proposition 6 *The values (41) form a set of integrals of the string equation (11), i.e., provided the string equation (11) holds.*

$$dI_i/dx = 0$$

while for the Casimir (42) we have

$$dQ/dx = 1$$

Proof: This is similar to the proof of proposition (5); one has to take the same steps substituting x for τ_i.

As was mentioned in the Introduction the Laplacian growth problem is endowed with a complex structure, with $\bar{z}$ being the complex conjugate of z (and bar denoting complex conjugation). Then, as may be seen from the proposition, the Casimir Q plays the role of the area (which grows with the unit speed), while I are functions of the harmonic moments of the boundary curve.

Since the equations of motion (34) have the following form in $I, \bar{I}$

$$\begin{array}{ll} \partial_{\tau_j} Q = 0, & \partial_{\bar{\tau}_j} Q = 0 \\ \partial_{\tau_j} I_i = \delta_{ij}, & \partial_{\bar{\tau}_j} I_i = 0 \\ \partial_{\tau_j} \bar{I}_i = 0, & \partial_{\bar{\tau}_j} \bar{I}_i = -\delta_{ij} \end{array} \tag{43}$$

these variables are canonical. For instance, we may choose the following Poisson structure

$$\{I_i, \bar{I}_j\}_p = \delta_{ij}, \quad \{I_i, I_j\}_p = \{\bar{I}_i, \bar{I}_j\}_p = 0, \quad \{Q, \bar{I}_j\}_p = \{Q, I_j\}_p = 0 \tag{44}$$

$$\mathrm{H}_i = \bar{I}_i, \quad \bar{\mathrm{H}}_i = I_i \tag{45}$$

It is important to note that the Poisson brackets $\{,\}_p$ in (44) are different from $\{,\}$ in (10): The former defines the Poisson structure on the space of rational solutions of the 2dToda hierarchy, while the latter (Lax-Poisson bracket) is a dispersionless limit of the commutator.

To get similar results for the rational case we have to take a limit as in Corollary 1.

Corollary 2 *Let z , $\bar{z}$ be represented as in (22)-(25). The following $2N+1$ values ($i = 1..N$)*

$$\begin{array}{ll} I_0 = \bar{u}_0 - \sum_{i=1}^N \bar{u}_i/\bar{w}_i, & \bar{I}_0 = u_0 - \sum_{i=1}^N u_i/w_i \\ I_{2i-1} = rw_i^{-1} + \bar{u}_0 + \sum_{j=1}^N \frac{\bar{u}_j}{1/w_i - \bar{w}_j}, & \bar{I}_{2i-1} = r\bar{w}_i^{-1} + u_0 + \sum_{j=1}^N \frac{u_j}{1/\bar{w}_i - w_j} \\ I_{2i} = \left(r - \sum_{j=1}^N \frac{\bar{u}_j}{(1/w_i - \bar{w}_j)^2}\right)\frac{u_i}{w_i^2}, & \bar{I}_{2i} = \left(r - \sum_{j=1}^N \frac{u_j}{(1/\bar{w}_i - w_j)^2}\right)\frac{\bar{u}_i}{\bar{w}_i^2} \end{array}$$

$$Q = r^2 - \frac{1}{2}\sum_{i=1}^{N}\left(r\left(\frac{\bar{u}_i}{\bar{w}_i^2} + \frac{u_i}{w_i^2}\right) + \bar{I}_{2i} + I_{2i}\right)$$

are the action-angle variables in the rational case (proposition 3), i.e. the variables for which equations in the Proposition (3) have the form (43).

Again, in the rational limit we may choose the Poisson structure as in (44), (45).

5 Conclusions.

We have established the Hamiltonian structure on the space of rational solutions of the 2dToda hierarchy connected with the problem of ideal interface dynamics. A further application of this result can become quite challenging in dealing with the problem of encountering a surface tension for the Laplacian growth process.

The Laplacian growth equation (5), (11) describes the propagation of the boundary with zero surface tension between the fields. Although, addressing a number of important questions (e.g. finger width selection problem in [10]), such an idealized model does not account for essential physical features, such as fractal formation, stability etc. Thus, the inclusion of tension effects is important for the solution of the problem.

There are two approaches to look for in such a generalization. The first one is to introduce tension terms in the theory, destroying the integrability of the problem. In another approach one might look for integrable deformations of the idealized model, simulating surface effects and stabilizing interface dynamics.

Another feature that gives our result a certain interest lies in the possible investigation of the perturbed (by small surface tension) system given in terms of separated variables (44). In other words, is it possible, in some situations, to approximate the perturbed solution by the multi-finger ansatz (32) with different dependence of coordinates on time x? If the answer is affirmative we will get a finite-dimensional dynamical system, conveniently written in terms of $I, \bar{I}, Q$.

Finally, returning to the idealized problem, one can further elaborate the theory of rational reductions from the point of view of symmetries. An important observation of our study is that once a rational reduction is compatible with 2dToda dynamics, the string condition is satisfied automatically. In other words *the string (Laplacian growth) equation (5), (11) turns out to be a consequence of rationality* in the context of the 2dToda hierarchy. On the other hand, it is well known that the string equation is connected with additional (Orlov-Schulman [3],[14]) symmetries of the 2dToda hierarchy [19]. It would be interesting to find whole sets of symmetry constraints defining finite-dimensional solutions of the 2dToda hierarchy and containing (11) as a special element.

Amongst the other open questions and future directions is the following problem: to investigate Hamiltonian structure of rational reductions in the context of the two-matrix [6] model, whose partition function is a tau-function of the 2dToda system constrained by the string equation. The applicability of our study to the models of normal matrices is another interesting aspect worthy of further analysis [8].

6 Appendix: Poisson structure of the 1dToda hierarchy.

In this section we consider the Poisson structure of rational reductions of the 1dToda system.

Recall, that for the 1dToda system one takes into account only a "half" of flows, connected with z (but not $\bar{z}$, or vice verse).

$$\partial_{t_i} z = \{H_i, z\}, \quad H_i = (z(w)^i)_+ + 1/2(z(w)^i)_0, \quad i = 0..\infty \tag{46}$$

This system is bi-Hamiltonian (for general information e.g. see [3], [13]) with two (linear and quadratic) compatible Poisson structures. As it was first found in [2] for the generic Toda system (6), the dispersionless linear Poisson brackets for the "field variables" $u_i, i = 1..\infty$ (6) have the following form

$$\{u_n(x), u_m(y)\}_1 = 2(c_n + c_m - 1)\left[(n+m)u_{n+m}(x)\delta'(x-y) + m u'_{n+m}(x)\delta(x-y)\right] \tag{47}$$

where

$$c_k = \begin{cases} 1 & , \text{if } k > 0 \\ 1/2 & , \text{if } k = 0 \\ 0 & , \text{if } k < 0 \end{cases}$$

while the quadratic brackets are

$$\{u_n(x), u_m(y)\}_2 = \big[\frac{1}{2}(n-m)u_n(x)u'_m(x) + (\sum_{k=1}^{1-n}(n-m+k)u_{n+k}(x)u'_{m-k}(x) + \\ + ku'_{n+k}(x)u_{m-k}(x))\big]\delta(x-y) + \big[1/2(n-m)u_n u_m + \\ +(\sum_{k-1}^{1-n}(n-m+2k)u_{n+k}u_{m-k})\big]\delta(x-y) \tag{48}$$

As seen from the lemma 1, the rational functions

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{w^{N+1} + \sum_{i=0}^N a_i w^i}{\sum_{i=0}^N b_i w^i} \tag{49}$$

are form-invariant under all of the 1dToda flows ∂_{t_i} (46), without any extra restriction (i.e. the string equation is not needed).

We obtain the corresponding Poisson structures for coefficients a_i, b_i, by using result (48), expressing u_i in terms of $a_i, b_i, i = 0..N$. Both the linear (47) and quadratic (48) Poisson structures lead to the quadratic brackets for a_i, b_i . Namely , the second Poisson structure for (49) reads as follows

$$\{a_k(x), a_l(y)\}_2 = \big[\sum_{n=1}(l+n-k)a_{k-n}(x)a_{l+n}(y) + na_{k-n}(y)a_{l+n}(x)) \\ +(l-N-1)a_k(x)a_l(y)\big]\delta'(x-y) \tag{50}$$

$$\{b_k(x), b_l(y)\}_2 = \big[\sum_{n=1}(k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x)) + \\ +\frac{k-l}{2}b_k(x)b_l(y)\big]\delta'(x-y) \tag{51}$$

$$\{a_k(x), b_l(y)\}_2 = \frac{k-N-1}{2}a_k(x)b_l(y)\delta'(x-y) \tag{52}$$

The first Poisson structure can be obtained from (50 - 52) with the help of the linear transformation (shift by a constant)

$$a_i \to a_i + \lambda b_i, \quad z(w,x) \to z(w,x) + \lambda$$

and using the bi-Hamiltonian nature of (47), (48).

$$\{a_k(x), a_l(y)\}_1 = \big[\sum_{n=1}(k-l-n)(a_{k-n}(x)b_{l+n}(y) + b_{k-n}(x)a_{l+n}(y)) - \\ -n(a_{k-n}(y)b_{l+n}(x) + b_{k-n}(y)a_{l+n}(x))) + \frac{N+1-l}{2}b_k(x)a_l(y) + \\ +\frac{k+N+1-2l}{2}a_k(x)b_l(y)\big]\delta'(x-y) \tag{53}$$

$$\{a_k(x), b_l(y)\}_1 = \big[\sum_{n=1}((k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x)) + \\ +\frac{N+1-l}{2}b_k(x)b_l(y)\big]\delta'(x-y) \tag{54}$$

$$\{b_k(x), b_l(y)\}_1 = 0 \tag{55}$$

in all the above expressions $a_{N+1} = 1$ and $a_i = 0$ if i goes beyond the range $i = 0..N+1$ (and $b_j = 0$ if $j \neq 0..N$).

These brackets form a bi-Hamiltonian structure for rational reductions of the 1dToda hierarchy:

$$\partial_{t_i} z = \{\mathrm{H}_i, z\}_1 = \{\mathrm{H}_{i-1}, z\}_2 \tag{56}$$

with the following Hamiltonians:

$$\mathrm{H}_i = \frac{1}{i+1} \int (z^{i+1}(x))_0 dx \tag{57}$$

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Appendix 2. Statement on collaboration

- **Harnad, J.; Loutsenko, I.; Yermolayeva, O. Constrained reductions of 2D dispersionless Toda hierarchy, Hamiltonian structure and Interface Dynamics.**

  The paper concerns with the study of rational solutions of the dispersionless 2D-Toda hierarchy constrained by a string equation. The subject was motivated by numerous applications originating in the Laplacian growth problem. The Laplacian growth describes dynamics of unstable interfaces in different applications and is of great practical importance. In the paper we investigate Hamiltonian nature of rational and, more generally, logarithmic conformal maps describing so-called "multi-finger" solutions of the Laplacian growth problem. An application of the R-matrix methods reveals an integrable structure of such solutions, which are, in turn, finite-dimensional reductions of 2D dispersionless Toda hierarchy. Such reductions satisfy an extra condition known as a "string" equation in the string theory and as Laplacian growth equation in two-dimensional fluid dynamics.

  My contribution to this work consists in performing analytic proofs to the statements concerning the consistency of the functional reductions of the constrained 2dToda; in finding in literature and suggesting to extend the notion of the

Krichever flows for the Benney systems (rational KP reductions) to the 2D-Toda case. The observation on additional Krichever flows related to the pole structure of rational solutions of 2dToda allowed us to complete the construction of finite-dimensional reductions and gave the idea of construction of Hamiltonians of the system. As well, I performed the calculation of the linear and quadratic Poisson brackets for the Lax-Sato case using the R-matrix scheme proposed in the work by G.Carlet, [67], where it first appeared for the Takasaki 2D-Toda.

- **Yermolayeva O. On trigonometric analogue of Atiyah-Hitchin bracket.**

  The paper is done without coauthors.

  The main idea of the paper is to apply R-matrix theory to the fundamental Atiyah-Hitchin structure on the moduli spaces of the rational functions.

  In the theory of monopoles, the fundamental Atiyah-Hitchin structure is naturally defined on a space of scattering matrices of physical fields representing solutions of the Bogomolny equation. We express this structure through the r-matrix construction (of Sklyanin).

  A generalization of the Atiyah-Hitchin structures derived in the paper might be considered as a step towards a theory of Poisson structures for meromorphic functions on the Riemann surfaces of non-zero genus (and, in particular, for tau-functions on Hurwitz spaces, introduced recently by D.Korotkin et al.). Alternatively, a translation of our construction into the language of scattering matrices develops another important direction, namely, theory of brackets on an algebra of Weyl functions. This fact emphasizes the potential of the work.

- **Harnad, J.; Yermolayeva, O.; Zhedanov A. R-Matrix Approach to the Krall-Sheffer problem.**

This paper is devoted to a problem of classification of "admissible" differential equations on the plane. It establishes a correspondence between classical orthogonal polynomials in two variables and 2D integrable systems.

Admissibility (after Krall and Sheffer), is a property of existence of a linear functional which determines the orthogonality of eigenvectors of the operators. In the article, we found a non-trivial correlation between the admissibility (or classicity) of linear differential operators and their integrability.

In more detail: The Krall-Sheffer list of admissible operators consists of nine classes. As a crucial observation, the first class in the list, turns out, to represent the so-called Appel systems (with eigenvectors being Appel polynomials) . On the other hand, this class is related with well-known Neumann-Rosochatius problem, for which a construction of a special Lax matrix was proposed by J.Harnad. I found that the resolvent construction for the Lax matrix is, in fact, an extension of the notion of Stieltjes function to higher dimensions. The classical works of J.Moser underlines the importance of the latter fact in the theory of integrable systems (and especially for the Toda lattice). Performing numerous computer experiments and analytic work, I have shown integrability of the remaining eight classes in the admissibility list. In particular, encountering a difficulty in proving integrability by a standard scheme of symplectic reduction (which works only for three classes in the admissibility list), I have proposed a scheme based on topological properties of roots of a characteristic determinant of admissible operators. And so, the work was completed for all nine cases.

In conclusion, the theory of integrable systems influences various areas of the modern mathematical physics and applied mathematics, such as nonlinear PDE's and others. In our work, we revealed (so far unknown) aspects of an interplay between the theories of integrable systems and classical orthogonal polynomials. To our knowledge, the only work (on this topic) preceding it was a classical paper on one-dimensional Toda lattice by J.Moser.

- **Harnad, J., Vinet, L., Yermolayeva, O., Zhedanov, A., Two-dimensional Krall-Sheffer polynomials and integrable systems**

  This is the first paper on the Krall-Sheffer polynomials and integrability, where we state their connection with two-dimensional superintegrable systems on spaces of constant curvature. It was proposed by A. Zhedanov, to perform a similarity transformation to present the linear second-order differential operators in the Krall-Sheffer list in a form of Laplace-Beltrami operators with some metric g_{ik} plus some potential. Considering coefficients of the operators in the Krall-Sheffer list as matrix elements of the inverse metric, I have calculated the Riemannian curvature and performing some further symbolic computations showed a close dependence of the admissibility of the operators, symmetrizability, existence of a nondegenerate functional leading to orthogonality of their eigenfunctions, their integrability with the condition that the Riemannian curvature should be constant. Thus we have got a list of superintegrable systems on surfaces of constant curvature and on the plane and showed an equivalence of the existence of a nondegenerate orthogonality functional for an admissible Krall-Sheffer operators with existence of a commuting second-order differential operator in each case.

- **Harnad,J., Yermolayeva, O. Superintegrability, Lax matrices and separation variables**

  In this paper we show how the superintegrability of certain systems can be deduced from the presence of multiple parameters in the rational Lax matrix representation. This is also related to the fact that such systems admit a separation of variables in parametric families of coordinate systems.

  My contribution here consisted in providing explicit examples stemming from the Krall-Sheffer problem framework both on spaces of constant curvature and in the plane. I have performed an explicit construction of integrals of motion both in a classical and a quantum setting as trace invariants of the rational 2×2 Lax matrix, and gave separation variables. We have used the parametric dependence of invariants on the pole location of the Lax matrix to interpret the phenomenon of superintegrability.