Smooth Estimation of Survival and MRL Functions under Mean Residual Life Order

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Abstract

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Let X and Y be two random variables denoting life times having finite means. Let, S_1, S_2 and M_1, M_2 denote their survival and MRL functions, respectively . X is said to be smaller than Y in mean residual life order, if and only if

1. $M_1(x) \leq M_2(x)$ in all x; or equivalently,
2. $\dfrac{\int_t^\infty S_1(x)\ dx}{\int_t^\infty S_2(x)\ dx}$ does not increase in$tover\{t: \displaystyle\int_t^\infty S_2(x)\ dx > 0\}$.

In this thesis smooth estimators for the survival and MRL functions under the above ordering are studied. Nonparametric method given by Hu *et al.* (2002) has shown good properties, but it is not smooth enough, when the true function is continuous.

Chaubey and Sen(1996) have proposed a new approach to smooth survival and density function in stead of the popular kernel method. Following their approach, we introduce two methods for smooth estimation of a survival function based on the two criteria of mean residual life ordering . The strong uniform consistency of the estimators has also been shown here. Numerical studies based on simulation indicate both smooth estimators to be superior to the estimator due to Hu *et al.* (2002) in terms of bias and MSE in majority of cases.

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Chapter 1

Introduction

In the last two decades, there has been an intensified interest in mean residual life (**MRL**). This concept relates to the aging property of an individual or a component, whose survival time X is a random variable. Given survival up to the time x, the average remaining life among the population members who have survived until time x is called the **Mean Residual Life** at time x. Denoting this function as $M(x)$, we therefore have,

$$M(x) = E[X|X > x] - x.$$

For life tables, it is called the life expectancy at age x, or more generally the biometric function [see Chiang (1986) Chap 10]. The MRL function gives a different picture of survival of aging than that seen through the more commonly studied survival function $S(x)$ or hazard function $\lambda(x) = \frac{f(x)}{S(x)}$, where $f(x)$ is the density function corresponding to $S(x)$. Here, note that for a non-negative random variable X having the probability density function $f(x)$, the survival function is defined as

$$S(x) = P[X > x] = \begin{cases} 0 & \text{for} \quad x \leq 0, \\ \int_x^\infty f(t)dt & \text{for} \quad x > 0. \end{cases}$$

Many times, we may have a situation in which, logically, MRL functions of two populations must be ordered. For example, MRL function should often be ordered after the patients have been treated with different kinds of therapies. Also, if a mechanical device is improved, the MRL function for the improved device should not be less than that for the original one. Therefore, the consequent problem of interest is that of estimating the MRL function or survival function under this order restriction.

In 1978, Yang studied the properties of the empirical version of MRL function on a fixed interval $[0, T]$, $T < \infty$. In spite of its good properties, it may not preserve the ordering condition. Nonparametric maximum likelihood estimators (NPMLE) on two distribution have been derived and studied under stochastic ordering (Brunk *et al.*, 1966; Huang and Praestgaard, 1996), and uniformly stochastic ordering (Dykstra *et al.*, 1991; Rojo and Samaniego, 1991). Also, there is a lot of literature on the projection type estimators for stochastic ordering (Rojo and Ma, 1996; Rojo, 1995) and uniformly stochastic ordering (Rojo and Samaniego, 1993; Mukerjee, 1996; Arcones and Samaniego, 2000), the latter often proving to be superior to the NPMLEs.

Ebrahimi (1993) considered estimators for MRL functions subject to the constraint $M_1(x) \leq M_2(x)$, for known as well as unknown M_2, on an interval $[t_1, t_2]$. He provided an excellent life example in his paper. Hu *et al.* (2002) improved on these estimators providing a rigorous proof of asymptotic unbiasedness. This paper also established the weak convergence of their estimators which allows computation of asymptotic confidence interval. Beiger *et al.* (1998) have considered the problem of testing the hypothesis $M_1(x) \leq M_2(x)$, but they did not consider the estimation problem.

When the distributions are assumed to be continuous, many applied practitioners would prefer to have smooth estimators and as such there is a lot of interest in smooth estimations of the MRL and Survival functions. Chaubey and Sen (1996) formulated a new technique based on the classical Hille theorem (1948) in real analysis and obtained smooth estimators for survival function and density function, which may have some advantage over their counterparts based on the usual kernel method of smoothing. Their further research work includes the use of resulting estimators in smooth estimation of the hazard, cumulative hazard functions (Chaubey and Sen, 1997) and MRL function (Chaubey and Sen, 1999). Chaubey and Kochar (2000) also considered modified estimators of survival functions that are stochastically ordered.

In this thesis, we propose smooth estimators of MRL and survival functions under MRL ordering, using the ideas developed in Chaubey and Kochar (2000) by considering the two equivalent conditions of MRL order and the smoothing method of Chaubey and Sen (1996).

We will review some properties of MRL and survival functions, especially under MRL order in Chapter 2. Chapter 3 presents the modification of Chaubey and Sen (1996) given in Chaubey and Kochar (2000) and Chaubey and Kochar (2001) for estimation of survival functions under stochastic order, and uniform stochastic order, respectively. These papers have been useful in proposing two estimators of the survival function under the mean residual life ordering which are given in Chapter 4. The first is based on the estimators of Hu *et al.* (2002), and the other one is based on an equivalent definition of MRL order. In Chapter 5, we will provide the proof for strong consistency of both the estimation methods and a discussion of the asymptotic properties. In Chapter 6, we conduct a simulation study for a variety of MRL functions to compare the MSE of these two estimators along with that of Hu *et. al.* (2002). The last chapter, Chapter 7 provides some conclusions and further remarks.

Chapter 2

Preliminaries

2.1 Mean Residual Life

In this section we formally define the mean residual life and catalogue some of its basic properties.

Definition 2.1. *Let X be a non-negative random variable with survival function $S(x)$ and finite mean μ, then mean residual life of X is defined as*

$$M(x) = E[X - x | X > x] = \begin{cases} \dfrac{\int_x^\infty S(t)dt}{S(x)} & S(x) > 0 \\ 0 & S(x) = 0 \end{cases} \tag{2.1}$$

Note that $M(0) = \mu$, and since we assume that $\mu = \int_0^\infty f(x)dx$ is finite, $M(x) < \infty$, for $x < \infty$. However, it is possible that $M(\infty) = \lim_{x \to \infty} M(x) = \infty$. Guess and Proschan (1998) provide a nice summary of the theory of MRL and an extensive bibliography.

Clearly, $M(x)$ is a non-negative function but not every nonnegative function corresponds to a mean residual life function of some random variable. In fact, the following properties characterize a function $M(x)$ to be a MRL function given by Shaked and Shanthikumar (1991):

1. $0 \leq M(x) < \infty$ for all $x \geq 0$,
2. $M(0) > 0$,
3. M is continuous,
4. $M(x) + x$ is nondecreasing on $[0, \infty)$, and
5. When there exits a x_0 such that $M(x_0) = 0$, then $M(x) = 0$ for all $x \geq x_0$. Otherwise,

$$\int_0^\infty \frac{1}{M(x)} dx = \infty.$$

Furthermore, like the probability density function or the characteristic function, for a distribution with finite mean, the MRL function completely determines the distribution. The survival function can be determined from the MRL function according to the following formula,

$$S(x) = \frac{M(0)}{M(x)} \, exp \left\{ - \int_0^x \frac{1}{M(t)} \, dt \right\} I_{(M(x)>0)},$$

where I_A denotes the indicator function of A.

2.2 The Mean Residual Life (MRL) Order

Here we provide the definition of mean residual life (MRL) order and its relation with some other stochastic orderings.

Definition 2.2. *Let X and Y be two nonnegative random variables with finite means whose corresponding survival functions and MRL functions are S_1 and S_2, M_1 and*

M_2, *respectively. The random variable X is said to be smaller than Y in the mean residual life order (denoted as $X \leq_{mrl} Y$), if*

$$M_1(x) \leq M_2(x), \quad \textit{for all } x > 0. \tag{2.2}$$

We note that

$$M_1(x) \leq M_2(x) \Leftrightarrow \frac{d}{dx}\left(\frac{W_1(x)}{W_2(x)}\right) \leq 0,$$

where $W_i(x) = \int_x^\infty S_i(u)du$, $i = 1, 2$. This is easy to see, since

$$\frac{1}{S_1(x)S_2(x)}\left[\frac{d}{dx}\left(\frac{W_1(x)}{W_2(x)}\right)\right] = \frac{1}{W_2^2(x)}(M_1(x) - M_2(x)).$$

This provides an alternative equivalent definition of MRL order, similar to the characterization of uniform stochastic order as given by Lehmann (1955):

Definition 2.3. *Under the conditions of Def. 2.2, $X \leq_{mrl} Y$ if, and only if,*

$$\frac{\int_t^\infty S_1(x) \ dx}{\int_t^\infty S_2(x) \ dx} \ \textit{does not increase in } t \textit{ over } \left\{t : \int_t^\infty S_2(x) \ dx > 0\right\}. \tag{2.3}$$

This alternative definition gives us a new idea to estimate the survival functions under the mean residual life order.

Instead of comparing the two MRL functions in order to define a partial ordering between two random variables, we may compare the distribution or hazard rate functions, which gives give two different partial orderings known as, stochastic ordering and hazard rate ordering (or uniform stochastic ordering)respectively. The definitions are given below:

Definition 2.4. *Let X and Y be two random variables such that*

$$P\{X > x\} \leq P\{Y > x\} \ \textit{for all } \ x \in (-\infty, \infty), \tag{2.4}$$

then X is said to be stochastically smaller than Y in the (denoted by $X \leq_{st} Y$).

Definition 2.5. *Let X and Y be two nonnegative random variables with absolutely continuous survival functions $S_1(x)$, $S_2(x)$, and with hazard rate function $h_1(x)$ and $h_2(x)$, respectively, such that,*

$$h_1(x) \geq h_2(x), \quad x \geq 0 \tag{2.5}$$

Then X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$).

Similar to the definition of the mean residual life order given in Def (2.3), Lehmann (1955) observed that since $h(x) = -\frac{d}{dx} \log S(x)$,

$$h_1(x) \geq h_2(x) \Leftrightarrow \frac{d}{dx} \log \frac{S_1(x)}{S_2(x)} \leq 0,$$

therefore, $X \leq_{hr} Y$ holds, if and only if,

$$\frac{S_1(x)}{S_2(x)} \text{ does not increase in } x. \tag{2.6}$$

Hazard rate ordering is stronger than stochastic ordering, hence it is also called uniform stochastic ordering [see Lehmann (1955)].

Since, the MRL function M for a random variable X can be written in terms of its hazard rate function h as

$$M(x) = \int_x^\infty \exp\{-\int_x^u h(t) \ dt\}du, \tag{2.7}$$

under the condition of Eq. (2.5), we have $M_1(x) \leq M_2(x) \ \forall x \geq 0$. Thus, we can easily obtain the following theorem giving relation between the uniform stochastic ordering and MRL ordering.

Theorem 2.1. *If X and Y are two random variables such that $X \leq_{hr} Y$, then $X \leq_{mrl} Y$.*

There is no direct relation between mean residual life order and stochastic order, however, under some particular conditions, MRL ordering implies stochastic ordering as given in the following theorem.

Theorem 2.2. *Let X and Y be two random variables with mrl functions $M_1(x)$ and $M_2(x)$ respectively. Suppose that $\frac{M_1(x)}{M_2(x)}$ is increasing in x. Then, $X \leq_{mrl} Y \implies X \leq_{hr} Y$ which subsequently implies that, $X \leq_{st} Y$.*

Proof. [Shaked and Shanthkumar (1994)] Since $M_1(x)$ is differentiable on $\{x : P\{X > x\} > 0\}$, it is easy to see that

$$h_1(x) = \frac{M_1'(x) + 1}{M_1(x)} \tag{2.8}$$

where $M_1'(x)$ denotes the derivative of $M_1(x)$. Similarly, we have

$$h_2(x) = \frac{M_2'(x) + 1}{M_2(x)} \tag{2.9}$$

Hence, we have,

$$\begin{aligned} h_1(x) - h_2(x) &= \frac{M_1'(x) + 1}{M_1(x)} - \frac{M_2'(x) + 1}{M_2(x)} && (2.10) \\ &= \frac{M_1'(x)M_2(x) + M_2(x) - M_1(x)M_2'(x) - M_1(x)}{M_1(x)M_2(x)} && (2.11) \\ &= \frac{(M_1'(x)M_2(x) - M_1(x)M_2'(x)) + (M_2(x) - M_1(x)}{M_1(x)M_2(x)} && (2.12) \end{aligned}$$

Since $M_1(x) \leq M_2(x)$ and $\frac{M_1(x)}{M_2(x)}$ is increasing in x, we have $h_1(x) - h_2(x) \geq 0$, that is $X \leq_{hr} Y$, furthermore, $X \leq_{st} Y$.

□

Chapter 3

Estimation of a Survival Function under Stochastic Order and Hazard Rate Order

In this chapter, we will outline the smooth estimators for survival function preserving the stochastic order and Hazard Rate order, developed in Chaubey and Kochar (2000) and Chaubey and Kochar (2001), respectively. These papers provide the motivation for developing the estimators under the mean residual life ordering. Here we consider only one sample case, *i.e.* the estimation of a survival function S_1 of a random variable X given that $X \leq_{st} Y$, or $X \leq_{hr} Y$, where we know the survival function S_2 corresponding to the random variable Y.

3.1 Estimation under Stochastic Order

Let S_1 and S_2 be absolutely continuous survival function such that $S_1(x) \geq S_2(x)\ \forall x$ and S_2 is known. That means X is stochastically larger than Y. Ma (1991) and Puri and Singh (1992) considered the following estimator of S_1 by modifying the empirical distribution function of random variable X :

$$\hat{S}_{1n} = \max(S_2, S_{1n}) \tag{3.1}$$

where S_{1n} denotes the empirical survival function based on a random sample (X_1, X_2, ..., X_n), *i.e.*

$$S_{1n}(x) = \frac{1}{m}\sum_{i=1}^{n} I_{(X_i > x)}. \tag{3.2}$$

Strong uniform consistency of the above estimator is established by Rojo (1995) and Rojo and Ma (1996). Rojo and Ma (1996) also showed that the above estimator has uniformly smaller bias than the corresponding NPMLE and simulation indicates that it has smaller MSE (mean squared error) for a variety of distributions. Lo (1987) considered the problem for two sample case, deriving similar results, and Rojo and Samaniego (1993) considered the alternative case that $S_1 \leq S_2$.

Chaubey and Kochar (2000) adapt the method of smoothing of a empirical survival function given in Chaubey and Sen (1996) which preserves the stochastic ordering. Their estimator is described in through the following steps:

1. Smooth the $U_n(x) = \hat{S}_{1n}(x) - S_2(x)$ by technique in Chaubey and Sen (1996);

$$\tilde{U}_n(x) = \sum_{k=0}^{\infty} U_n(\frac{k}{\lambda_n}) p_k(\lambda_n x) \tag{3.3}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of constants such that $\lambda_n \to \infty$ as $n \to \infty$, but $\frac{\lambda_n}{n} \to 0$ and here,

$$p_k(u) = e^{-u}\frac{u^k}{k!}, k = 0, 1, 2, \cdots \tag{3.4}$$

2. A smooth estimator for $S_1(x)$ is proposed as

$$\tilde{S}_1(x) = \tilde{U}_n(x) + S_2(x) \tag{3.5}$$

Chaubey and Kochar (2000) establish the strong consistency of the smooth estimator as given in the following theorem:

Theorem 3.1. *: Let $\lambda_n \to \infty$ almost surely as $n \to \infty$. Then*

$$\sup_{x\in\mathbb{R}^+} \left|\tilde{S}_1(x) - S_1(x)\right| \to 0 \text{ almost surely as } n \to \infty. \tag{3.6}$$

They further use the celebrated Bahadur (1996) representation of quantiles as in Chaubey and Sen (1996), and provide the weak convergence of the smooth estimator $\tilde{S}_1$ similar to that obtained in Rojo (1995).

Theorem 3.2. *If $\lambda_n \to \infty$ and $\frac{\lambda_n}{n} \to 0$, then for $S_1(x) \geq S_2(x)\ \forall x$,*

$$\sqrt{n}\,(\tilde{S}_1 - S_1) \to W^0 \tag{3.7}$$

where W^0 donotes a Brownian bridge. However, if $S_1(x) = S_2(x)$ for some x_0 with $S_1 \neq S_2$, then, $\tilde{S}_1$ does not converge weakly.

3.2 Estimation under Hazard Rate Order

Consider X to be smaller than Y in uniform stochastic ordering then we know that

$$\theta(x) = \frac{S_1(x)}{S_2(x)} \text{ is nonincreasing with respect to } x. \tag{3.8}$$

Rojo and Samaniego (1993) used this characterization of hazard ordering and proposed the following estimators $\hat{\theta}$ and $\hat{S}_1$, for θ and S_1 respectively, if S_2 is known:

$$\hat{\theta}(x) = inf_{0\leq y\leq x}\frac{S_{1n}(y)}{S_2(y)}, \tag{3.9}$$

$$\hat{S}(x) = \hat{\theta}(x)S_2(x.) \tag{3.10}$$

This estimator is shown to be strongly consistent with an optimal rate of convergence. Mukerjee (1996) extended this estimator to the case with S_2 is unknown by introducing a convex class of estimators, linearly ordered by uniformly stochastic order.

Also, since $\hat{S}_1$ is not smooth, Chaubey and Kochar (2001) in an unpublished manuscript, have proposed to modify it by a smooth version which maintains the property of uniform stochastic ordering. Their proposal is outlined below.

First smooth the $\hat{\theta}$ by the following method.

$$\tilde{\theta}(x) = \sum_{k=0}^{\infty} \hat{\theta}\left(\frac{k}{\lambda_n}\right) p_k(\lambda_n x) \tag{3.11}$$

with $p_k(u)$ as before denoting the weights given by Poisson probabilities. Then the smooth version of $S_1(x)$ here is given by

$$\tilde{S}_1(x) = \tilde{\theta}(x) S_2(x) \tag{3.12}$$

They also claim the strong convergence and other asymptotic properties for this estimator which are inherent in $\hat{S}_1(x)$.

Theorem 3.3. *Let S_1 and S_2 be* continuous survival functions with support $[0, \infty)$ such that $\theta(x) = \frac{S_1(x)}{S_2(x)}$ is bounded, non-increasing function and λ_n be a sequence of constants tending to ∞, then

$$\sup_{x \geq 0} \left| \tilde{S}_1(x) - S_1(x) \right| \to 0 \textit{ almost surely as } n \to \infty. \tag{3.13}$$

Chapter 4

Estimation of Survival Function under Mean Residual Life Order

Assume we have $X_1, X_2, \cdots, X_n$ random samples of size n from a survival distribution S_1, then the empirical survival function corresponding to S_1 is defined by

$$S_{1n}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(X_i > x)},$$

hence,

$$\int_x^\infty S_{1n}(t)dt = \frac{1}{n}\sum_{i=1}^{n} I_{(X_i - x)}(X_i - x)$$

Thus, plugging in S_{1n} in place of S_1 for defining the mean residual life, following Yang (1978) we get the following estimator of the mean residual life $M_1(x)$:

$$\hat{M}_1(x) = \frac{T_{1n}(x)}{nS_{1n}(x)} \tag{4.1}$$

$$= \frac{\sum_{i=1}^{n} I_{(X_i - x)}(X_i - x)}{\sum_{i=1}^{n} I_{(X_i - x)}} \tag{4.2}$$

where

$$T_{1n}(x) = n\int_x^\infty S_{1n}(t)dt = \frac{1}{n}\sum_{i=1}^n I_{(X_i - x)}(X_i - x).$$

Now, suppose that X and Y are two nonnegative random variables with survival functions S_1, S_2 , and MRL functions M_1 , M_2, both having finite means and supports $[0, b_1)$, $[0, b_2)$.

Given $M_1 \le M_2$ for all x, where M_2 is known, Hu *et al.*(2002) gave the following estimator:

$$\hat{M}_1^\star(x) = \begin{cases} \hat{M}_1(x) \wedge M_2(x) & \text{for all x in } [0, B), \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

where, $B = \min(b_1, b_2)$. It is a right continuous function, with only some up jump points.

For the reverse order restriction, we can change minimum by maximum in (4.3) , such that,

$$\hat{M}_1^\star(x) = \begin{cases} \hat{M}_1(x) \vee M_2(x) & \text{for all x in } [0, B), \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

We use the above estimator as the building block for obtaining the smooth estimator of $S_1(x)$ under the MRL order.

4.1 Estimation Method One

Using the smoothing method given in Chaubey and Sen(1996) for estimating survival function under stochastic ordering $M_1 \le M_2$, we obtain the following smooth version of $\hat{M}_1^\star(x)$,

$$\tilde{M}_{1,1}(x) = M_2(x) - \tilde{V}_n(x) \tag{4.5}$$

$$\tilde{V}_n(x) = \sum_{k=0}^{n} V_n(\frac{k}{\lambda_n})p_k(\lambda_n x) \tag{4.6}$$

where $V_n(x) = M_2(x) - \hat{M}_1(x) \wedge M_2(x)$, $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of constants such that $\lambda_n \to \infty$ as. as $n \to \infty$, but $\frac{\lambda_n}{n} \to 0$ and $p_k(u)$ as in Eq. (3.4).

For the reverse order restriction $M_1 \geq M_2$, we could just easily change $V_n(x) = \hat{M}_1(x) \vee M_2(x) - M_2(x)$, and then,

$$\tilde{M}_{1,1}(x) = M_2(x) + \tilde{V}_n(x) \tag{4.7}$$

$$\tilde{V}_n(x) = \sum_{k=0}^{n} V_n(\frac{k}{\lambda_n})p_k(\lambda_n x) \tag{4.8}$$

Note that, $\tilde{M}_{1,1}(x)$ is nonnegative and infinitely differentiable by definition, hence it is quite smooth. Therefore we could use it to derive a smooth estimator for the survival function, as given by

$$\tilde{S}_{1,1}(x) = \frac{\tilde{M}_{1,1}(0)}{\tilde{M}_{1,1}(x)} \exp\left\{-\int_0^x \frac{1}{\tilde{M}_{1,1}(t)}dt\right\} I_{(\tilde{M}_{1,1}(x)>0)} \tag{4.9}$$

The choice of λ_n is very important in the study of the asymptotic properties of the estimators. If we only consider these estimators for a compact interval $[0, b]$, then $\frac{\lambda_n}{n} \to 0$ and $\lambda_n \to \infty$ as $n \to \infty$ suffice. Chaubey and Sen (1996) considered a stochastic choice $\lambda_n = \frac{n}{X_{n:n}}$, where $X_{n:n}$ is the largest ordered statistic of $X_1, X_2, \cdots, X_n$, so that $S_{1n}(\frac{n}{\lambda_n}) = \hat{S}_1(X_{n:n}) = 0$. Since our estimator is derived from the empirical distribution, we use the same λ_n as theirs.

4.2 Estimation Method Two

Based on the alternative definition, Definition 2.3 for mean residual life order, we will consider the method similar to Chaubey and Kochar (2001) to estimate the survival function under mean residual life order. In this case, we consider smooth estimation of

$$\theta(x) = \frac{\int_x^\infty S_1(u)du}{\int_x^\infty S_2(u)du}.$$

First, we propose to estimate it as in Rojo and Samniego (1993) by

$$\begin{aligned}
\hat{\theta}_n(x) &= \inf_{0\le t\le x} \frac{\int_t^\infty S_{1n}(u)du}{\int_t^\infty S_2(u)du} \\
&= \inf_{0\le t\le x} \frac{T_{1n}(t)}{n\int_t^\infty S_2(u)du} \\
&= \inf_{0\le t\le x} \frac{\sum_{i=1}^n I_{(X_i-t)}(X_i-t)}{n\int_t^\infty S_2(u)du} \quad \text{for all } 0 < x < b_2 \qquad (4.10)
\end{aligned}$$

and

$$\hat{\theta}_n(x) = 0 \quad \text{for all } x \ge b_2. \qquad (4.11)$$

Then, we have

$$\int_x^\infty \widehat{S_1(u)}\, du = \hat{\theta}_n(x) \int_x^\infty S_2(u)du \qquad (4.12)$$

as an estimator of $\int_x^\infty S_1(u)\, du$.

Thus, similar to the Theorem 1 of Rojo and Samaniego (1993) we can establish:

Theorem 4.1. *For $\forall$ survival functions $S_1(x)$, $S_2(x)$, let*

$$\int_x^\infty \tilde{S}_1(u)du = \begin{cases} \inf_{0\le t\le x} \frac{\int_t^\infty S_1(u)du}{\int_t^\infty S_2(u)du} \int_x^\infty S_2(u)du & \text{if } 0 \le x < b_2 \\ 0 & \text{otherwise} \end{cases} \qquad (4.13)$$

Then

1.

$$\frac{\int_x^\infty \tilde{S}_1(u)du}{\int_x^\infty S_2(u)du} \text{ is non-increasing.}$$

2.

$$0 \leq \int_x^\infty \tilde{S}_1(u)du \leq \int_x^\infty S_1(u)du.$$

3.

$$\int_x^\infty \tilde{S}_1(u)du = \int_x^\infty S_1(u)du \Leftrightarrow M_1(x) \leq M_2(x) \text{ i.e. } X \leq_{mrl} Y$$

By construction of $\hat{\theta}_n$ and the Theorem above, for the the estimator proposed here, the estimated mean residual life $\int_x^\infty \widehat{S_1(u)}\, du$ satisfies the required property of mean residual life ordering.

We can also obtain comparable results in the case $X \geq_{mrl} Y$, when S_2 is known by defining $\hat{\theta}_n(x)$ as

$$\hat{\theta}_n(x) = \begin{cases} \sup\limits_{0 \leq t \leq x} \frac{\sum_{i=1}^n I_{(X_i - t)}(X_i - t)}{n \int_t^\infty S_2(s)ds}, & \text{for all } 0 \leq x < b_2 \\ 0, & x \geq b_2 \end{cases} \tag{4.14}$$

Then

$$\int_x^\infty \widehat{S_1(u)}\, du = \hat{\theta}_n(x) \int_x^\infty S_2(u)du \tag{4.15}$$

is what we need and we obtain the following theorem similar to Theorem 4.1.

Theorem 4.2. *For* $\forall$ *survival functions* $S_1(x)$, $S_2(x)$, *let*

$$\int_x^\infty \tilde{S}_1(u)du = \begin{cases} \sup\limits_{0 \leq x \leq t} \frac{\int_t^\infty S_1(u)du}{\int_t^\infty S_2(u)du} \int_x^\infty S_2(u)du & \text{if } 0 \leq x < b_2 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

Then

1.

$$\frac{\int_x^\infty \tilde{S}_1(u)du}{\int_x^\infty S_2(u)du} \text{ is non-decreasing.}$$

2.

$$\int_x^\infty S_1(u)du \geq \int_x^\infty \tilde{S}_1(u)du \geq 0.$$

3.

$$\int_x^\infty \tilde{S}_1(u)du = \int_x^\infty S_1(u)du \Leftrightarrow M_1(x) \geq M_2(x) \text{ i.e. } X \geq_{mrl} Y$$

But $\int_x^\infty \widehat{S_1(u)}du$ may not be smooth enough for taking derivatives, since inherits the nature of $\S_{1n}(x)$ containing flat pieces. Hence we would replace $\hat{\theta}_n$ by a smooth version so that the property of the MRL order is not lost.

We first replace the estimator $\hat{\theta}_n(x)$ by the following smooth version

$$\tilde{\theta}_n(x) = \sum_{k=0}^{\infty} \hat{\theta}_n(\frac{k}{\lambda_n})p_k(\lambda_n x) \tag{4.17}$$

where $p_k(u)$ is defined as before.

Then smooth estimator of $\int_x^\infty S_1(u)\, du$ is given by

$$\int_x^\infty \widetilde{S_1(u)}\, du = \tilde{\theta}_n(x) \int_x^\infty S_2(u)du \tag{4.18}$$

Note $\int_x^\infty \widetilde{S_1(u)}du$ is continuous, bounded and continuously differentiable, so the smooth estimator of $S_1(x)$ is given by

$$\begin{aligned}
\tilde{S}_{1,2}(x) &= -\frac{d}{dx}(\int_x^\infty \widetilde{S_1(s)\, ds}) \\
&= -\frac{d}{dx}[\tilde{\theta}_n(x) \int_x^\infty S_2(s)\, ds] \\
&= -\frac{d}{dx}\tilde{\theta}_n(x) \int_x^\infty S_2(s)\, ds + S_2(x)\tilde{\theta}_n(x) \qquad (4.19)
\end{aligned}$$

Here,

$$\begin{aligned}
\frac{d}{dx}\tilde{\theta}_n(x) &= \frac{d}{dx}\sum_{k=0}^{n}\hat{\theta}_n(\frac{k}{\lambda_n})p_k(\lambda_n x) &(4.20)\\
&= \sum_{k=0}^{n}\hat{\theta}_n(\frac{k}{\lambda_n})[-\lambda_n p_k(x)+\lambda_n\frac{k}{\lambda_n x}p_k(x)] &(4.21)\\
&= -\lambda_n\tilde{\theta}_n(x)+\lambda_n\sum_{k=0}^{n-1}\hat{\theta}_n(\frac{k+1}{\lambda_n})e^{-\lambda_n x}\frac{(\lambda_n x)^k}{k!} &(4.22)\\
&= -\lambda_n\sum_{k=0}^{n-1}[\hat{\theta}_n(\frac{k}{\lambda_n})-\hat{\theta}_n(\frac{k+1}{\lambda_n})]e^{-\lambda_n x}\frac{(\lambda_n x)^k}{k!}\\
&\quad -\lambda_n\hat{\theta}_n(\frac{n}{\lambda_n})e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!} &(4.23)
\end{aligned}$$

Hence we can derive a smooth estimator of MRL function as

$$\begin{aligned}
\tilde{M}_{1,2}(x) &= \frac{\int_x^\infty \widetilde{S_1(s)}\, ds}{\tilde{S}_{1,2}(x)}\\
&= \frac{\tilde{\theta_n}(x)\int_x^\infty S_2(s)\, ds}{S_2(x)\tilde{\theta_n}(x)-[\frac{d}{dx}\tilde{\theta_n}(x)]\int_x^\infty S_2(s)\, ds} &(4.24)\\
&= \frac{M_2(x)}{1-\frac{d}{dx}[\ln\tilde{\theta_n}(x)]M_2(x)}.
\end{aligned}$$

These two estimators will be further studied in the next chapters.

Chapter 5

Consistency of the Estimators

5.1 Consistency for $\tilde{M}_{1,1}$ and $\tilde{S}_{1,1}$

The property of consistency of $\tilde{M}_{1,1}(x)$ may be derived from the results of Yang (1978) and Hu *et al.* (2002). Yang (1978) showed that, when X has finite mean,

$$P[\sup_{0\le x\le b} |\hat{M}_1(x) - M_1(x)| \to 0, \ \ as \ \ n \to \infty] = 1 \tag{5.1}$$

where $b \in \mathbb{R}^+$ and $b < b_1$, b_1 is support of $S_1(x)$. Hu *et al.* (2002) demonstrated such a property for their estimator of mean residual life under the MRL order condition, namely,

$$\sup_{0\le x\le b} |\hat{M}_1^{\star}(x) - M_1(x)| \to 0 \ \ a.s. \ uniformly \ \ as \ \ n \to \infty \tag{5.2}$$

where $b \in \mathbb{R}^+$ and $b < B = \min(b_1, b_2)$, b_i is the support of S_i..

Now let us consider (4.7) $\tilde{M}_{1,1}(x)$. We have a similar result established in the following theorem.

Theorem 5.1. *If $S_1(x)$ is continuous a.e ,$\lambda_n \to \infty$ and $\frac{\lambda_n}{n} \to 0$, then*

$$\sup_{0 \le x \le b} |\tilde{M}_{1,1}(x) - M_1(x)| \to 0 \ \ a.s. \ \ as \ \ n \to \infty. \tag{5.3}$$

here, b in $\mathbb{R}^+$ and $b < B = \min(b_1, b_2)$, b_i is the support of S_i.

Proof. In this case, to derive the results in Theorem 3.1, we make use of the following lemma, known as Hille Theorem.

Lemma 5.1. *(Hille Theorem) Let $U(t)$ be bounded continuous function on $\mathbb{R}^+$, then*

$$e^{-\lambda t} \sum_{k \ge 0} U(\frac{k}{\lambda}) \frac{(\lambda t)^k}{k!} \to U(t) \ \ as \ \ \lambda \to \infty \tag{5.4}$$

uniformly in any finite interval J contained in $\mathbb{R}^+$.

Since, $V(x) = M_2(x) - M_1(x)$ is bounded and continuous on $[0, b]$, by Lemma 5.1 we can claim that

$$\begin{aligned} \tilde{V}(x) &= e^{\lambda_n t} \sum_{k=0}^{n} V(\frac{k}{\lambda_n}) \frac{(\lambda_n t)^k}{k!} \\ &\to V(x) \ \ as \ \ \lambda_n \to \infty \end{aligned} \tag{5.5}$$

uniformly on $[0, b]$. Then we will have

$$\begin{aligned} & \sup_{0 \le x \le b} |\tilde{V}_n(x) - V(x)| \\ = \ & \sup_{0 \le x \le b} |\tilde{V}_n(x) - \tilde{V}_n(x) + \tilde{V}_n(x) - V(x)| \\ \le \ & \sup_{0 \le x \le b} |\tilde{V}_n(x) - \tilde{V}_n(x)| + \sup_{0 \le x \le b} |\tilde{V}_n(x) - V(x)| \\ \le \ & \max_{k \le n} |V_n(\frac{k}{\lambda_n}) - V(\frac{k}{\lambda_n})| + \sup_{0 \le x \le b} |\tilde{V}(x) - V(x)| \\ \le \ & \sup_{0 \le x \le b} |V_n(x) - V(x)| + \sup_{0 \le x \le b} |\tilde{V}(x) - V(x)| \\ \to \ & 0 \ \ almost \ \ surely \ \ as \ \ n \to \infty. \end{aligned}$$

whenever $\lambda_n \to \infty$ but $\frac{\lambda_n}{n} \to 0$. This follows from Eq. (5.5) and the fact that $V_n(x) = M_2(x) - \hat{M}_1^*(x) \to V(x) = M_2(x) - M_1(x)$ uniformly as $n \to \infty$, since $\hat{M}_1^*(x) \to M_1(x)$ uniformly as $n \to \infty$ as established in Hu *et al.* (2000).

Therefore, we have

$$\tilde{M}_{11}(x) = M_2(x) - \tilde{V}_n(x) \to M_1(x) \tag{5.6}$$

uniformly as $n \to \infty$, that finish the proof. □

Remark 5.1.1. *We can also derive the consistency of $\tilde{S}_{1,1}(x)$ under the same conditions.*

Proof. Since $\tilde{M}_{1,1}(x)$ and $M_1(x)$ are bounded on [0,b] and nonzero , so

$$\sup_{0\le x\le b}\left|\frac{1}{\tilde{M}_{1,1}(x)} - \frac{1}{M_1(x)}\right| = \sup_{0\le x\le b}\left|\frac{M_1(x) - \tilde{M}_{1,1}(x)}{\tilde{M}_{1,1}(x)M_1(x)}\right| \to 0$$

a.s. as $n \to 0$, then

$$\begin{aligned}\sup_{0\le x\le b}\left|\int_0^x \frac{1}{\tilde{M}_{1,1}(t)}dt - \int_0^x \frac{1}{M_1(t)dt}\right| &\le \sup_{0\le x\le b}\{\int_0^x \left|\frac{1}{\tilde{M}_{1,1}(x)} - \frac{1}{M_1(x)}\right| dx\} \\ &\to 0 \textit{ almost surely as } n \to \infty.\end{aligned}$$

Then we have

$$\begin{aligned}&\sup_{0\le x\le b}\left|\tilde{S}_{1,1}(x) - S_1(x)\right| \\ = &\sup_{0\le x\le b}\left|\frac{\tilde{M}_{1,1}(0)}{\tilde{M}_{1,1}(x)}\exp\{-\int_0^x \frac{1}{\tilde{M}_{1,1}(x)}dt\} - \frac{M(0)}{M(x)}\exp\{-\int_0^x \frac{1}{M(x)}dt\}\right| \\ \to &\ 0 \textit{ almost surely as } n \to \infty.\end{aligned}$$

□

Hence we have proved the consistency of $\tilde{S}_{1,1}(x)$ and $\tilde{M}_{1,1}(x)$ over $[0, b]$.

5.2 Consistency for $\tilde{M}_{1,2}$ and $\tilde{S}_{1,2}$

Now we will show the consistency for $\tilde{S}_{1,2}(x)$ and $\tilde{M}_{1,2}(x)$ under the condition that $X \leq_{mrl} Y$.

We will need the following lemma, introduced in Rojo and Samaniego (1993) as Lemma 1.

Lemma 5.2. *let h and g be bounded functions on interval $[0, x]$, then*

$$|\inf_{0\leq y\leq x} h(y) - \inf_{0\leq y\leq x} g(y)| \leq \sup_{0\leq y\leq x} |h(y) - g(y)| \tag{5.7}$$

Now, for any $x \in [0, b]$, $\; b < B = min(b_1, b_2)$,

$$\begin{aligned}
& \sup_{0\leq x\leq b} \left|\hat{\theta}_n(x) \int_x^\infty S_2(u)\,du - \theta(x) \int_x^\infty S_2(u)\,du\right| \\
= \; & \sup_{0\leq x\leq b} \left| \inf_{0\leq t\leq x} \frac{\int_t^\infty \hat{S}_1(u)\,du}{\int_t^\infty S_2(u)\,du} \int_x^\infty S_2(u)\,du - \inf_{0\leq t\leq x} \frac{\int_t^\infty S_1(u)\,du}{\int_t^\infty S_2(u)\,du} \int_x^\infty S_2(u)\,du \right| \\
\leq \; & \sup_{0\leq x\leq b} \sup_{0\leq t\leq x} \left| \frac{\int_t^\infty \hat{S}_1(u)\,du}{\int_t^\infty S_2(u)\,du} \int_x^\infty S_2(u)\,du - \frac{\int_t^\infty S_1(u)\,du}{\int_t^\infty S_2(u)\,du} \int_x^\infty S_2(u)\,du \right| \\
= \; & \sup_{0\leq x\leq b} \sup_{0\leq t\leq x} \left| \frac{\int_x^\infty S_2(u)\,du}{\int_t^\infty S_2(u)\,du} \right| \left| \int_t^\infty \hat{S}_1(u)\,du - \int_t^\infty S_1(u)\,du \right| \\
\leq \; & \sup_{0\leq t\leq x} \left| \int_t^\infty \hat{S}_1(u)\,du - \int_t^\infty S_1(u)\,du \right| \\
\to \; & 0 \;\; as \;\; n \to \infty.
\end{aligned}$$

Since $\int_x^\infty S_2(u)\,du$ is bounded and positive, we have

$$\sup_{0\leq x\leq b} \left|\hat{\theta}_n(x) - \theta(x)\right| \to 0 \;\; as \;\; n \to \infty. \tag{5.8}$$

By the definition (4.10), $\hat{\theta}_n(x)$ is bounded by $\frac{\mu_1}{\mu_2}$, hence by using the Lemma 5.1(Hille Lemma), for $\tilde{\theta}_n(x)$, defined as (4.17), we can establish the following theorem by following a parallel proof to that of Thoeorem 5.1.

Theorem 5.2. *If $S_1(x)$ and $S_2(x)$ are continuous and have finite mean and support b_1, b_2, $\lambda_n \to \infty$ and $\frac{\lambda_n}{n} \to 0$, then*

$$\sup_{0 \le x \le b} \left| \tilde{\theta}_n(x) - \theta(x) \right| \to 0 \ a.s. \ as \ n \to 0, \quad for\ all\ x \tag{5.9}$$

where, $b < B = \min(b_1, b_2)$.

For the estimator under the reverse ordering we can establish the similar result as above by the Lemma 5.1 and Lemma 2 in Rojo and Samaniego (1993). A minor difference is that, under this restriction, the $\hat{\theta}_n(x)$ is bounded at b.

Lemma 5.3. *let h and g be bounded functions on interval $[0, x]$, then*

$$|\sup_{0 \le y \le x} h(y) - \sup_{0 \le y \le x} g(y)| \le \sup_{0 \le y \le x} |h(y) - g(y)| \tag{5.10}$$

Now considering $\frac{d}{dx}\tilde{\theta}_n(x)$, defined by (4.20), first, we show the following theorem.

Theorem 5.3. *Under the hypothesis of theorem 5.2, if $\lambda_n = o(n^\alpha)$ for some $\alpha < 3/4$, $\sup_{0 \le x \le b} \left| \hat{\theta}_n(x) - \theta(x) \right| = o(\frac{1}{2\lambda_n})$ and $\frac{d^2}{dx^2}\,\theta(x)$ is bounded almost everywhere on $[0, b]$, we will have*

$$\sup_{0 \le x \le b} \left| \frac{d}{dx}\tilde{\theta}_n(x) - \frac{d}{dx}\theta(x) \right| \to 0 \ a.s. \ as \ n \to \infty. \tag{5.11}$$

Proof. From (4.23) we could write $\frac{d}{dx}\tilde{\theta}_n(x)$ as

$$\begin{aligned}\frac{d}{dx}\tilde{\theta}_n(x) &= -\lambda_n\{\hat{\theta}_n(\frac{n}{\lambda_n})e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!} \\ &\quad +\sum_{k=0}^{n-1}[\theta(\frac{k}{\lambda_n})-\theta(\frac{k+1}{\lambda_n})]e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!} \\ &\quad +\sum_{k=0}^{n-1}[(\hat{\theta}_n(\frac{k}{\lambda_n})-\hat{\theta}_n(\frac{k+1}{\lambda_n}))-(\theta(\frac{k}{\lambda_n})-\theta(\frac{k+1}{\lambda_n}))]e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!}\} \\ &= T_{n1}(x)+T_{n2}(x)+T_{n3}(x).\end{aligned} \tag{5.12}$$

We prove the above theorem step by step as follows.

1. For the first part $T_{n1}(x)$, Chaubey and Sen (1996) has shown that

$$e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!} = (e^{-\frac{1}{n}\lambda_n x}\lambda_n x)^n\frac{1}{\sqrt{2\pi n}}n^{-n}e^{n-\frac{1}{12n}+\cdots} \tag{5.13}$$

$$= \frac{1}{\sqrt{2\pi n}}\{\frac{e}{n}\lambda_n x e^{-\frac{1}{n}\lambda_n x}\}^n\{1+O(\frac{1}{n})\} \tag{5.14}$$

When $\frac{\lambda_n}{n}\to 0$ a.s. as $n\to\infty$, by the stirling approximation. For $\forall x\in[0,b]$, we could find a adequtely large N, whenever $n\geq N$, $\frac{e}{n}\lambda_n x e^{-\frac{1}{n}\lambda_n x}$ can be smaller than 1. Thus, we could write

$$e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!} = \frac{1}{\sqrt{2\pi n}}\{\rho_n(x)\}^n\{1+O(\frac{1}{n})\} \tag{5.15}$$

where $\rho_n(x)\to 0$ as $n\to\infty$. So, $e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!}$ can be made $o(n^{-2})$ as $n\to\infty$. Hence,

$$\sup_{0\leq x\leq b}\left|T_{n1}(x)\left(=-\lambda_n\hat{\theta}_n(\frac{n}{\lambda_n})e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!}\right)\right|\to 0$$

a.s. as $n\to\infty$, while by the assumption $\frac{\lambda_n}{n}\to 0$ a.s. as $n\to\infty$.

2. Second, for $T_{n2}(x)$, we expand $\theta(x)$ as a Taylor Series at $\frac{k}{\lambda_n}$.

$$\theta(x) = \theta(\frac{k}{\lambda_n}) + \frac{d}{dx}\theta(\frac{k}{\lambda_n})(x - \frac{k}{\lambda_n}) + \frac{1}{2}[\frac{d^2}{dx^2}\theta(\frac{k}{\lambda_n})](x - \frac{k}{\lambda_n})^2 + o(\lambda_n^{-2}) \quad (5.16)$$

If we replce x by $\frac{k+1}{\lambda_n}$, we could write

$$-\lambda_n[\theta(\frac{k}{\lambda_n}) - \theta(\frac{k+1}{\lambda_n})] \;=\; \frac{d}{dx}\theta(\frac{k}{\lambda_n}) + \frac{1}{2\lambda_n}[\frac{d^2}{dx^2}\theta(\frac{k}{\lambda_n})] + o(\frac{1}{\lambda_n}) \quad (5.17)$$

so that under the assumed boundedness of $\frac{d^2}{dx^2}\theta(x)$, $\frac{1}{2\lambda_n}[\frac{d^2}{dx^2}\theta(\frac{k}{\lambda_n})] \to 0$ as $n \to \infty$, we may virtually repeat the proof of theorem 3.1 and conclude that

$$\sup_{0 \le x \le b} \left| T_{n2}(x) - \frac{d}{dx}\theta(x) \right| \to 0, \; \textit{almost surely as } n \to \infty, \qquad (5.18)$$

since $\frac{1}{\lambda_n} \to 0$ as $n \to \infty$.

3. Finally , we will show $\sup\limits_{0 \le x \le b} |T_{n3}(x)| \to 0$ as $n \to \infty$.

Since $\sum\limits_{k=0}^{\infty} \frac{x^k}{k!} = e^x$,

$$\begin{aligned}
\sup_{0 \le x \le b} |T_{n3}(x)| &= \sup_{0 \le x \le b} \left| -\lambda_n[(\hat{\theta}_n(\frac{k}{\lambda_n}) - \hat{\theta}_n(\frac{k+1}{\lambda_n})) - (\theta(\frac{k}{\lambda_n}) - \theta(\frac{k+1}{\lambda_n}))]e^{-\lambda_n x}\frac{(\lambda_n x)^n}{n!}\} \right| \\
&\le \lambda_n \max_{k \le n-1} \left| [\hat{\theta}_n(\frac{k}{\lambda_n}) - \hat{\theta}_n(\frac{k+1}{\lambda_n})] - [\theta(\frac{k}{\lambda_n}) - \theta(\frac{k+1}{\lambda_n})] \right| \\
&\le \lambda_n (\max_{k \le n-1} \left| \hat{\theta}_n(\frac{k}{\lambda_n}) - \theta(\frac{k}{\lambda_n}) \right| + \max_{k \le n-1} \left| \hat{\theta}_n(\frac{k+1}{\lambda_n}) - \theta(\frac{k+1}{\lambda_n})] \right|) \\
&\le 2\lambda_n \sup_{0 \le x \le b} \left| \hat{\theta}_n(x) - \theta(x) \right| \\
&\to 0
\end{aligned}$$

as $n \to 0$.

This completes the proof of Theorem 5.3. □

Since we have $\tilde{S}_{12}$ defined as (4.19), writing $S_1(x)$ as

$$S_1(x) = \theta(x)S_2(x) - \frac{d}{dx}\theta(x)\int_x^\infty S_2(u) \tag{5.19}$$

we can show that

$$\begin{aligned}
& \sup_{0\le x\le b}\left|\tilde{S}_{12}(x) - S_1(x)\right| \\
= & \sup_{0\le x\le b}\left|-[\frac{d}{dx}\tilde{\theta}_n(x) - \frac{d}{dx}\theta(x)]\int_x^\infty S_2(u)\,du + S_2(x)[\tilde{\theta}_n(x) - \theta(x)]\right| \\
\le & \sup_{0\le x\le b}\left|\frac{d}{dx}\tilde{\theta}_n(x) - \frac{d}{dx}\theta(x)\right|\int_x^\infty S_2(u)\,du + \sup_{0\le x\le b}\left|\tilde{\theta}_n(x) - \theta(x)\right| S_2(x) \\
\to & \ 0 \ a.s. \ as \ n\to\infty,
\end{aligned}$$

since $S_2(x)$ and μ_2 are finite.

Furthermore, we see that

$$\sup_{0\le x\le b}\left|\tilde{M}_{1,2}(x) - M_1(x)\right| \to 0 \ \textit{almost surely as} \ n\to 0 \tag{5.20}$$

for every b less than the support of $\tilde{S}_{1,2}(x)$, where $\tilde{M}_{1,2}(x)$ is defined as (4.24).

Chapter 6

Simulation

In this chapter, we present some simulation results comparing the estimator of Hu *et al.* (2002) and the estimators proposed here for various quantiles. Since, Hu *et al.* (2002) do not present any results for estimating the survival function we compare the two estimators of survival functions. The following decreasing, constant and increasing MRL functions are used to carry out the simulation:

$$M_i(x) = a_i(1 - \frac{x}{b_i})I[x \le b_i], \ \ b_i > a_i, \ \ \text{with} \ \ S_i(x) = (1 - \frac{x}{b_i})^{\frac{b_i}{a_i}-1}$$

which corresponds to the $U(0,1)$ distribution when $a_i = 0.5, b_i = 1$;

$$M_i(x) = \theta_i \ \ \text{corresponding to the} \exp(\theta_i) \ \ \text{distribution ;} \quad \text{and}$$

$$M_i(x) = a_i x + b_i, \ \ a_i x + b_i \ge 0, \ \ b_i > 0 \ \ \text{with} \ S_i(x) = (\frac{a_i x + b}{b_i})^{-(1+\frac{1}{a_i})}.$$

Our interest interest is in contrasting the simulated bias and MSE. Each model has been treated based on 100 iterations for sample of size 10 and 20.

In Tables 6.1 and 6.2, we compare the bias and MSE for the decreasing MRL functions, where we consider $M_1(x) = \frac{1}{3}(1-x)$, $M_2(x) = \frac{1}{2}(1-x)$, with corresponding survival functions $S_1(x) = (1-x)^2$ and $S_2(x) = (1-x)$ respectively.

In Tables 6.3 and 6.4 we compare the bias and MSE for the exponential distribution, take $M_1(x) = 1$,$M_2(x) = 1.1$, with responding survival functions $S_1(x) = e^{-x}$ and $S_2(x) = e^{-\frac{x}{1.1}}$ respectively.

Finally, Tables 6.5 and 6.6, we compare the bias and MSE for the decreasing MRL functions, take $M_1(x) = \frac{1}{2}x + 1$, $M_2(x) = x + 1$, with responding survival functions $S_1(x) = (\frac{2}{x+2})^3$ and $S_2(x) = (x+1)^{-2}$.

From these tables, it may be seen that , generally, our two estimators have a little bit more bias than the unsmooth estimator of Hu *et al.* (2002), but they have smaller MSE almost always, particularly at the tail of the distribution. We must note from the Table 6.1, for the decreasing MRL function model, $\tilde{M}_{1,1}(x)$ and $\tilde{M}_{1,2}(x)$ even has the smaller estimated bias than $M_1^\star(x)$ with the perfect smaller MSE; and for the increasing one, our estimators do not seem to work as well as under the decreasing model, their MSE increases as q increases from 0 to the 0.5, but after that, it decreases. At the tails our estimators do perform better.

We would also like to be able to differentiate between $\tilde{M}_{1,1}$, $\tilde{S}_{1,1}(x)$ and $\tilde{M}_{1,2}(x)$, $\tilde{S}_{1,2}(x)$. Unfortunately, we can not find any general rules for comparing the two estimators prposed here. It seems though, that the first method generally gives much more bias, especially in the last tow models. However , the difference is very every small. And in general, the $\tilde{M}_{1,2}$ produces much smaller MSE at the tail of the survival function. But the $\tilde{S}_{1,2}$ is not always equal to 1 at 0; it approaches 1 as n becomes large. We also see the pattern that estimated bias decreases with the increase in sample size, which is expected due to strong consistency result.

Table 6.1: Comparison of bias(B) and MSE of $M_1^\star(x)$, $\tilde{M}_{1,1}(x)$ and $\tilde{M}_{1,2}(x)$ at various q-quantiles for $M_i(x) = a_i(1 - \frac{x}{b_i})I[x \leq b_i],\ \ b_i > a_i$

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = \frac{1}{3}(1-x)$, $M_2(x) = \frac{1}{2}(1-x)$, $n = 10$, $\#iterations = 100$					
0.1	-0.0395	-0.0368	-0.03980	1.2178	1.0787
0.2	-0.0367	-0.0340	-0.0355	1.3019	1.1395
0.5	-0.0266	-0.0272	-0.0477	1.2578	0.7249
0.8	-0.0311	-0.0025	-0.0211	4.0131	4.8582
0.9	-0.0268	-0.0181	-0.0070	1.7160	23.3061

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = \frac{1}{3}(1-x)$, $M_2(x) = \frac{1}{2}(1-x)$, $n = 20$, $\#iterations = 100$					
0.1	-0.0271	-0.0260	-0.0260	1.0997	1.0053
0.2	-0.0279	-0.0261	-0.0252	1.1845	0.9160
0.5	-0.0206	-0.0209	-0.0258	1.2919	1.0157
0.8	-0.0112	-0.0036	-0.0032	2.8544	1.0551
0.9	-0.0137	0.0122	-0.0077	2.1733	8.1841

Table 6.2: Comparison of bias(B) and MSE of $\tilde{S}_{1,1}(x)$ and $\tilde{S}_{1,2}(x)$ at various q-quantiles for $M_i(x) = a_i(1 - \frac{x}{b_i})I[x \leq b_i], \;\; b_i > a_i$

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = \frac{1}{2}(1-x)$, $M_2(x) = 1 - x$, $n = 10$, $\#iterations = 100$				
0.1	-0.0126	-0.0091	0.0026	0.0037
0.2	-0.0264	-0.0235	0.0058	0.0062
0.5	-0.0507	-0.0285	0.0096	0.0093
0.8	-0.0652	-0.0820	0.0067	0.0085
0.9	-0.0403	-0.0582	0.0025	0.0037

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = \frac{1}{2}(1-x)$, $M_2(x) = 1 - x$, $n = 20$, $\#iterations = 100$				
0.1	0.0003	0.0008	0.0008	0.0011
0.2	-0.0068	-0.0078	0.0017	0.0019
0.5	-0.0336	-0.0235	0.0044	0.0044
0.8	-0.0474	-0.0519	0.0036	0.0042
0.9	-0.0345	-0.0477	0.0017	0.0026

Table 6.3: Comparison of bias(B) and MSE of $M_1^\star(x)$, $\tilde{M}_{1,1}(x)$ and $\tilde{M}_{1,2}(x)$ at various q-quantiles for $M_i(x) = \theta_i$

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = 1,\ M_2(x) = 1.1,\ n = 10,\ \#iterations = 100$					
0.1	-0.0958	-0.1142	-0.1005	0.9265	0.9779
0.2	-0.1028	-0.1253	-0.1117	1.0145	1.0556
0.5	-0.1440	-0.1686	-0.1601	1.1112	1.1107
0.8	-0.3076	-0.3095	-0.2805	1.5642	1.9126
0.9	-0.5337	-0.4811	-0.3905	1.5824	2.4140

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = 1,\ M_2(x) = 1.1,\ n = 20,\ \#iterations = 100$					
0.1	-0.0637	-0.0718	-0.0680	0.9552	0.9476
0.2	-0.0703	-0.0806	-0.0761	1.0120	0.9930
0.5	-0.1211	-0.1303	-0.1254	1.0824	1.0635
0.8	-0.1917	-0.2207	-0.2101	1.1993	1.2830
0.9	-0.3738	-0.3414	-0.3031	1.6022	2.0353

Table 6.4: Comparison of bias(B) and MSE of $\tilde{S}_{1,1}(x)$ and $\tilde{S}_{1,2}(x)$ at various q-quantiles for $M_i(x) = \theta_i$

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = 1$, $M_2(x) = 1.1$, $n = 10$, $\#iterations = 100$				
0.1	0.0114	-0.0040	0.0047	0.0057
0.2	0.0234	0.0090	0.0081	0.0068
0.5	0.0347	0.0296	0.0102	0.0097
0.8	0.0481	0.0414	0.0069	0.0067
0.9	0.0383	0.0310	0.0050	0.0045

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = 1$, $M_2(x) = 1.1$, $n = 10$, $\#iterations = 100$				
0.1	0.0103	0.0063	0.0019	0.0019
0.2	0.0231	0.0192	0.0038	0.0038
0.5	0.0043	0.0029	0.0042	0.0043
0.8	0.0193	0.0182	0.0029	0.0030
0.9	0.0202	0.0183	0.0018	0.0018

Table 6.5: Comparision of bias(B) and MSE of $M_1^\star(x)$, $\tilde{M}_{1,1}(x)$ and $\tilde{M}_{1,2}(x)$ at various q-quantiles for $M_i(x) = a_i x + b_i$

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = \frac{1}{2}x + 1,\ M_2(x) = x + 1,\ n = 10,\ \#iterations = 100$					
0.1	-0.2131	-0.2278	-0.2262	0.9743	0.9959
0.2	-0.2097	-0.2423	-0.2308	0.9167	0.9506
0.5	-0.2602	-0.2949	-0.2493	0.9238	0.8701
0.8	-0.5984	-0.6266	-0.4877	1.1967	1.1432
0.9	-1.1794	-1.1569	-0.9377	1.2524	1.4528

q	$B(M_1^\star(x))$	$B(\tilde{M}_{1,1}(x))$	$B(\tilde{M}_{1,2}(x))$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,1}(x))}$	$\frac{MSE(M_1^\star(x))}{MSE(\tilde{M}_{1,2}(x))}$
$M_1(x) = \frac{1}{2}x + 1,\ M_2(x) = x + 1,\ n = 10,\ \#iterations = 100$					
0.1	-0.1265	-0.1377	-0.1390	0.9482	0.9004
0.2	-0.1427	-0.1456	-0.1340	0.9882	0.9338
0.5	-0.1701	-0.1912	-0.1530	0.9197	0.8342
0.8	-0.3333	-0.4471	-0.3727	1.0685	1.0292
0.9	-0.7107	-0.7718	-0.6625	1.2238	1.2153

Table 6.6: Comparison of bias(B) and MSE of $\tilde{S}_{1,1}(x)$ and $\tilde{S}_{1,2}(x)$ at various q-quantiles for $M_i(x) = a_i x + b_i, a_i, b_i > 0$

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = \frac{1}{2}x + 1$, $M_2(x) = x + 1$, $n = 10$, $\#iterations = 100$				
0.1	0.0094	0.0052	0.0043	0.0075
0.2	0.0094	-0.0043	0.0075	0.0081
0.5	0.0074	-0.0081	0.0106	0.0101
0.8	0.0388	0.0314	0.0064	0.0058
0.9	0.0377	0.0341	0.0039	0.0035

q	$B(\tilde{S}_{1,1}(x))$	$B(\tilde{S}_{1,2}(x))$	$MSE(\tilde{S}_{1,1}(x))$	$MSE(\tilde{S}_{1,2}(x))$
$M_1(x) = \frac{1}{2}x + 1$, $M_2(x) = x + 1$, $n = 10$, $\#iterations = 100$				
0.1	-0.0003	0.0009	0.0038	0.0050
0.2	-0.0111	-0.0187	0.0060	0.0074
0.5	-0.0059	-0.0153	0.0063	0.0073
0.8	0.0117	0.0100	0.0022	0.0022
0.9	0.0180	0.0184	0.0012	0.0012

Chapter 7

Future Directions and Remarks

7.1 Asymptotic Properties

In this section, we will discuss about the asymptotic properties of the new estimators. While the proof is not complete, we just give some idea and direction that is worth further studies.

Hu *et al.* (2002) gave some assumptions to ensure that the continuous mapping theorem can be applied in their paper:

1. X_1 and X_2 have finite variances;

2. S_1 and S_2 have common discontinuous on the intersection of their supports.

Since we assume continuous survival functions, the second condition is satisfied automatically.

Yang (1978) showed that over the interval $x \in [0, d]$, for $d : F(d) < 1$, $n^{\frac{1}{2}}(\hat{M}_1 - M_1)$ converges in distribution to a zero mean Gaussian process if S_1 has a density.

Hall and Wellner (1979) improved the above result that if the convergence is on the domain of S_1, the density assumption is not essential; and $n^{\frac{1}{2}}(\hat{M}_1 - M_1)$ weakly converges to $U(x)$ on all compound intervals $[0, b]$, $b < b_1$ as before. Based on this result, X. Hu *et al.* (2002) prove the following theorem for their estimator under MRL ordering, as their Theorem 5.1 .

Theorem 7.1. *Let $b < b_1$ be fixed, Consider the estimator $\hat{M}_1^\star$ as 4.3 when M_2 is known.*

1. *If $M_1 < M_2$ on $[0, b]$, then*

$$n^{\frac{1}{2}}(\hat{M}_1^\star - M_1) \to U \ \ on \ \ [0, b]. \tag{7.1}$$

2. *If $M_1(x_0) = M_2(x_0)$ for some $x_0 \in (0, b)$ and $M_1 < M_2$ on $(x_0, s_0]$, $s_0 < b$, or on $[s_0, x_0)$, $s_0 > 0$, then $n^{\frac{1}{2}}(\hat{M}_1^\star - M_1)$ does not converge weakly.*

3. *If $M_1 = M_2$ on $[0, b]$, then*

$$n^{\frac{1}{2}}(\hat{M}_1^\star - M_1) \to U \wedge 0 \ \ on \ \ [0, b]. \tag{7.2}$$

Since we have shown the strong consistency of $\tilde{M}_{1,1}$, if we could show that $||\tilde{M}_{1,1} - \hat{M}_1^\star|| = ||\tilde{V}_n - V_n||$ converge to some $O(n^{-1})$, then we could claim asymptotic Gaussian law for the smooth estimator.

Chaubey and Kochar (2000) has given a similar proof for the asymptotic properties to their estimator $\tilde{S}_1(x)$ as Thoerem 3.2. The only difference is their proof is for the step function, and our $\hat{V}_n(x)$ is not exactly a step function but with some linear curves for interval $[X_i, X_{i+1}]$, we could not apply the theorem of Bahadur [1996]. This could probably modified to prove the required Gaussian limit law, however, we leave it for future research.

For $\tilde{S}_{1,2}(x)$, the situation seems more complicated, since we can not establish some relationship between our estimator and some known asymptotic theory, and the computation of the variance of the resulting estimator seems quite also difficult.

7.2 Estimation for Two-Sample Case

We can easily extend the problems we studied here to the two-sample case, *i.e.* where S_2 is also unknown but estimated from an independent sample. HU *et al.* (2002) used the same technique which was shown by Mukerjee (1996) to be preferred to the two-sample estimator discussed by Rojo and Samaniego (1993) for the uniform stochastic order case. Define $\hat{S} = \frac{n\hat{S}_1 + m\hat{S}_2}{n+m}$ to be an empirical estimator for the survival function $S = \frac{nS_1+mS_2}{n+m}$. Then MRL function of S is given by

$$
\begin{aligned}
M(x) &= \frac{\int_x^\infty S(u)\, du}{S(x)} I(x < b_2) \\
&= \frac{\int_x^\infty [n\hat{S}_1(u) + m\hat{S}_2(u)]\, du}{n\hat{S}_1(x) + m\hat{S}_2(x)} I(x < b) \\
&= \frac{nS_1(x)M_1(x) + mS_2(x)M_2(x)}{nS_1(x) + mS_2(x)} I(x < b)
\end{aligned}
$$

Then, the corresponding nonparametric estimators are given by

$$\hat{M}_1^\star(x) = \hat{M}_1(x) \wedge \hat{M}(x) \tag{7.3}$$

$$\hat{M}_2^\star(x) = \hat{M}_2(x) \vee \hat{M}(x) \tag{7.4}$$

We could apply Chaubey and Sen (1996) approach to these to get the smooth estimators, but we can not guarantee the MRL order in the smooth estimators. Hence the technique similar to the one used by Chaubey and Kochar (2000) could be used resulting in estimators similar properties as in one-sample case.

For the second method, we may define $\tilde{S}(x)$ as a smooth version of $\hat{S}(x)$ by

Chaubey and Sen (1996) technique, and then define

$$\hat{\theta}_1(x) = \inf_{0\le t\le x} \frac{\sum_{i=1}^{n} I_{(X_i-t)}(X_i-t)}{n\int_t^{\infty}\tilde{S}(s)ds} \quad \text{for all } 0<x<b \tag{7.5}$$

$$\hat{\theta}_2(x) = \sup_{0\le t\le y} \frac{\sum_{i=1}^{n} I_{(Y_i-t)}(Y_i-t)}{n\int_t^{\infty}\tilde{S}(s)ds} \quad \text{for all } 0<x<b \tag{7.6}$$

Now based on these estimators we could derive new smooth version of θ_1 and θ_2, and then get the smooth estimator of S_1 and S_2. However, the MRL order property of these estimators need to be established.

These estimators are also expected to have similar properties as in one-sample case.

7.3 Concluding Remarks

This thesis has focused on the problem of estimating the Survival and MRL functions for a life time variable X, under the assumption that it is smaller than another known life variable Y in mean residual life order, which is very common in reliability and survival analysis problems. We have considered two smooth versions of the estimator given by Hu *et al.* (2002) under the MRL order. This estimator has good asymptotic properties, however, its discrete nature makes it unattractive to practitioners. The smooth estimators proposed here show good bias and MSE properties as compared to the non-smooth estimators as demonstrated through some simulation studies. Some asymptotic results are established, however, much remains to be done with respect to smooth estimators. A possible direction for estimation in the case of two samples is also outlined.

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