The Effect of Weekly Quizzes on the Development of Students' Theoretical Thinking

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ABSTRACT

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In this thesis, we evaluate the effectiveness of a basic tool for the teaching of linear algebra at university level: the administration of weekly quizzes. These tests, each made of two short, conceptual questions, are believed to have the potential of developing students' theoretical thinking. At the start of our research we believed that this kind of thinking is necessary for the understanding of linear algebra and that it represents, in general, an important educational aim in university teaching.

To see if, indeed, the quizzes can contribute to the development of students' theoretical thinking, we have analyzed students' solutions to the weekly quizzes administered in a Linear Algebra II course at Concordia University in Montreal, over a one-semester time span. We have used, for this purpose, a model of thinking that characterizes theoretical thinking by certain properties, which contrast practical and theoretical thinking.

Our study shows that the administration of weekly quizzes could not account for the development, in general, of theoretical thinking, with all the characteristics that it encompasses. We have found that the group's engagement with certain features of thinking is directly influenced by the mathematical content of the questions in the quizzes, and thus it is impossible to evaluate their development in time. However, the quizzes do have the potential of developing other features of theoretical thinking, that we have called, more generally "habits of the mind", because they are less conditioned by content specificities.

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INTRODUCTION

THE RESEARCH QUESTION

In the literature it is generally recognized that linear algebra is a very difficult subject to teach and learn (see Dorier et al., 2000; Harel, 2000; Hillel, 2000; Robert and Robinet, 1992, 1996; Sierpinska, 2000) and the problems it poses are sometimes disproportionate with its intrinsic difficulty, causing frustration to both students and instructors. Research into the teaching and learning of linear algebra has revealed that students' difficulties are both conceptual – stemming from the nature of linear algebra as a subject matter, and cognitive – related more to students' ways of thinking (see Dorier and Sierpinska, 2001). In particular, students' tendency to think practically rather than theoretically was found by Sierpinska (2000) to act as an obstacle for their learning of linear algebra.

In this thesis, we want to take a closer look at students' tendency to think theoretically and reflect upon an approach to the teaching of linear algebra that may influence it. Namely, we want to evaluate the effectiveness of a rather basic tool employed in teaching: the administration of weekly quizzes, that is short, up to ten or fifteen-minute tests, as a way to enhance students' theoretical thinking and, consequently, their learning of linear algebra. The following question was thus at the start of our research: **Can weekly quizzes contribute to the development of students' theoretical thinking?** To answer this question we have analyzed, using the model of theoretical thinking proposed by Sierpinska (2000; see also Sierpinska et al., 2002), students' solutions to seven quizzes – each consisting of two questions – administered in a Linear Algebra II (MATH 252) course at the Concordia University in Montreal.

However, along the way, we have developed another interest, related to the aforementioned model of thinking. In our analyses of the data we have often found the categories of this model to be rather general for the purpose of analyzing the "concrete mathematics" of our students. While doing our work of identifying the features of theoretical thinking to which one or another concrete mathematical behavior corresponds, we began to reflect on a meta level on our theoretical framework and to anticipate another possible outcome of this research, namely an **operationalization of the model of theoretical thinking**. This operationalization is done to a certain extent by Sierpinska et al. (2002), but we believed that the quantitative nature of our analyses (we have analyzed around 700 written solutions of students) would perhaps allow us to better see the regularities or the recurrences in our students' problem solving behavior. For a researcher in mathematics education, or maybe even for a teacher of linear algebra, who is not familiar with the theoretical issues of research, the result might be a very practical one: a more comprehensive list of concrete mathematical behaviors corresponding to each feature of theoretical thinking postulated in the model.

OUTLINE OF THE THESIS

We precede the empirical part of our research with our answers to the following questions, describing the positions – theoretical and methodological – from which we undertake this research:

- *What do we mean by "theoretical thinking" and why is it important for learning linear algebra?*
- *What was our motivation for undertaking this research?*

- *How is our research situated in the context of the research into the teaching and learning of linear algebra?*
- *What are our means for measuring the development of theoretical thinking?*

The first chapter of the thesis, containing essentially our theoretical framework, addresses the first two issues above. We will discuss the distinction between theoretical and practical thinking (TT, respectively PT, in the sequel) as defined by Sierpinska (2000, 2002), in the light of other similar distinctions between ways of thinking, made in philosophy, psychology and mathematics education. We will add then some important clarifications fundamental to our understanding of the distinction between TT and PT, in an attempt to avoid some traps that one – including ourselves – may easily fall into when using these categories. One such erroneous understanding is the belief that the TT and PT categories refer to an ontological reality, describing some distinct kinds of human cognitive activity. We also want escape the attachment of general value judgments to either TT or PT, such as "TT is good, while PT is bad".

In Chapter II we depict the current state of the research into the teaching and learning of linear algebra. Our review unfolds along the following three axes of interest, which allow us to situate our research in context:

- the hypotheses advanced with regard to the learning and teaching of linear algebra;
- the characteristics of the experimented teaching designs;
- the methodological means used and the problems encountered at this level.

Chapter III starts with a summary of the features of theoretical thinking, followed by a description of our method of employing it in the empirical analyses. The main body of this chapter consists of our empirical analyses of students' solutions to the quizzes.

In Chapter IV we give a quantitative account of students' development of theoretical thinking through quizzes. Chapter V is dedicated to the conclusions of our research and to the methodological problems we have encountered in our study.

CHAPTER I. THEORETICAL AND PRACTICAL THINKING AND OTHER SIMILAR DISTINCTIONS BETWEEN WAYS OF THINKING

In our research we wanted to see if weekly quizzes can foster theoretical thinking in students. In this chapter we explain just what we mean by "theoretical thinking". Namely, we will present Sierpinska's (2000, 2002) definition of theoretical thinking. We have employed Sierpinska's distinction between theoretical and practical thinking in our research because, for us, it appeared to be the best choice in terms of methodological power. However, we want to give a larger picture of the kind of thinking we envisage the developing of, by discussing some other similar distinctions between ways of thinking made in philosophy, psychology and in mathematics education.

THEORETICAL AND PRACTICAL THINKING DISTINCTION

Inspired by some classical understandings of the notion of theoretical thinking, and particularly by Vygotsky's (1987) distinction between scientific and everyday concepts, Sierpinska et al. (2002) characterize theoretical thinking by contrasting it with practical thinking. For them, the two categories do not refer to an ontological reality; they are not opposite kinds of human cognitive activity, but rather models of ways of knowing or descriptions of epistemological positions. In reality, thinking is not either theoretical or practical, but results from an interaction between the two. Practical thinking identifies problems and aims at action. It is theoretical thinking then that formulates the problem and provides possible courses of action for the practical thinking. They function, the author argues, as epistemological obstacles for one another. In the sense of Bachelard (1983), an epistemological obstacle is a way of thinking that stands in the way of another way of thinking that would not exist though (as an obstacle) without this other way of thinking.

Thus, Sierpinska's distinction is a methodological one, considered to be useful for analytic purposes. Her definition of theoretical thinking reflects such an aim: it is formulated in terms of a certain number of detailed properties of TT. One of them is the *reflective* character. This means that theoretical thinking is critical with respect to its own results. It considers techniques and procedures open to questioning and change, without taking anything for granted.

Secondly, theoretical thinking is *systemic*; its objects are not particular actions but systems of relations between these actions and systems of relations between these relations. Practical thinking, on the other hand, establishes the meanings of concepts by reference to concrete objects and actions. Given its systemic character, theoretical thinking is extremely sensitive to contradictions and it is at all times concerned with the coherence and the validity of the system; in practical thinking validity depends on the context at hand. Theoretical thinking is aware of the conditional character of statements and is interested in exploring all possible logical cases, while practical thinking looks only at those cases that seem plausible.

Finally, theoretical thinking is *analytic*, that is, mediated by semiotic systems of representation. Moreover these systems of signs are objects of reflection and invention. In its analytical approach to signs, theoretical thinking is sensitive to formal symbolic notations, to specialized terminology and to the structure of the languages it uses.

Notwithstanding our interest in the development of these characteristics of thinking in our students, it should be clear that, as a consequence of our understanding of TT and PT as being complementary, we are attaching no general values to either TT or PT, such as "TT is good, while PT is bad", even if we do believe that students' tendency to think in practical rather than in theoretical ways in many instances adversely affects their understanding of linear algebra.

SIMILAR DISTINCTIONS BETWEEN WAYS OF THINKING

In philosophy

Distinctions between ways of thinking have been made in philosophy since Antiquity. The passage from the prehistoric forms of spiritual life to the historic ones is marked by the shift of focus that took place in the human mind from the mystical and magic structures to the rational ones, so well illustrated by the unfolding of the Greek thinking in Antiquity. During those times, not only has reason become prevalent over mystical ways of thinking, but it also became an object of reflection for philosophers. Aristotle's book A of the Metaphysics contains such meta-reflections about the science of philosophy. In arguing that philosophy is a theoretical science par excellence, he distinguishes between two different ways of thinking and the accordingly resulting knowledge. Aristotle's categories have influenced western culture for a long time and they seem to be, to the present day, very pertinent if one wants to talk about how people reason and how they go about knowing. *Men of experience*, in Aristotle's distinction, aim mainly at knowing the *facts*, while *men of art* are principally concerned with the *causes* of the facts. The latter thus produce *universal knowledge*, as opposed to *knowledge of the individuals* that is the result of experience. While he observes that men of experience are more likely to succeed when something is to be done, he expresses his appreciation for men of art that have the *ability to understand* and not just the *ability to do.* That a man of art is able to understand is mainly revealed by his capacity to teach, to transmit his knowledge. This is a mark of wisdom, in Aristotle's view, and it is not a trivial ability: "[...] he who is more accurate and more able to teach the causes in each science is wiser; [...] a wise man can acquire knowledge of what is hard and not easy for any man to know" (Aristotle, Metaphysics, Book A, in Apostle, 1966, p.14). Art develops on the ground of experience: "[And] experience seems to be almost similar to science and art, but science and art come to men through experience [...]" (ibid., p.12), just as practical thinking is the ground against which theoretical thinking develops (Sierpinska et al., 2002).

Kant (1873/1956) contrasts practical and theoretical reason as ways of thinking about or considering something. Practical reason, in the philosopher's system, is related to action, to *what one desires to do*, as opposed to theoretical reason that is related to cognition, to *what one knows*. In his system the two sets of assumptions from which an object can be viewed are pure, rather dichotomous, and what distinguishes them is the involvement or non-involvement of will in judgment. They differ thus from the categories of theoretical and practical thinking which are both somewhat impure, complementary and related to will, emotions and desires: "The difference between a theoretically thinking individual and a practically thinking individual is that the former is able to distance him or herself from what he or she feels, and engage in hypothetical thinking [. . .]. The practical thinker makes one with his or her will and emotions and is unable or reluctant to looking at them from this perspective" (Sierpinska et al., 2002, p.6).

The Romanian philosopher, Blaga (1969/1998), distinguishes several modes of rationalization and discusses how they interacted in the history of science:

- Rationalization based on identity: judgments where the relationship between subject and predicate is either of *partial identity*, for example "Humans are mammals" or *elastic identity*: "Humans are beings capable of progress";
- Rationalization based on equivalence: equivalence is possible between concepts that are logically heterogeneous;
- Rationalization based on dialectics: it seeks identity not only under diverse forms but also under contradictory forms.

The first of the three modes gives birth to generic concepts and it was largely cultivated in Antiquity, while the second and the third mode generate relational concepts, are more specific to mathematics and to the Galileo-Newtonian science and are definitely more fertile in terms of methodological potential. Common sense differs from scientific spirit in that they adopt different modes of rationalization. Common sense, which is close to what we understand as practical thinking, is in Blaga's acceptance "the mentality of men, as beings acting in a concrete world of senses, without essentially going beyond this world" (Blaga 1969/1998, p. 171; my translation from Romanian). Common sense adopts the mode of rationalization based on identity and doesn't trust or it is even adverse to the other two modes. One of its defining characteristics is that it believes in "empirical realism", i.e., that there exists a world independent of the human conscience that is directly accessible to men through their senses. However, when talking about common sense and scientific spirit, Blaga is not referring to some 'pure' categories and in this sense his distinction is closer to that between theoretical and practical thinking: all thinking is to some extent practical or directed by common sense.

In psychology

Dewey's (1933) notion of reflective thinking shares many features with that of theoretical thinking. He argues in favor of reflective thinking as "the better way of thinking" (p. 3) and describes its characteristics by contrast with other kinds of thought. Reflective thinking is, for him, "the kind of thinking that consists in turning a subject over in the mind and giving it serious and consecutive consideration" (p. 3). Reflective thinking differs from the mere *stream of consciousness* which can be daydreaming or dreaming and is composed of sequences of somewhat randomly following ideas. By contrast, reflective thinking is a chain of units linked together, leading to one another. A *succession of imaginative mental pictures* of something not actually present can simulate reflective thought if it is articulate, but unlike reflective thinking it does not aim at a consequence.

The mind does not act upon these kinds of thought; they may lead to emotional commitment but not to intellectual commitment. What impels reflective thinking to inquiry is the existence of *belief*, which reflective thinking voluntarily seeks to establish on a firm ground based on evidence and rationality. Dewey also describes *reflective thinking as a process* having several necessary phases. The origin of reflective thinking lies in some kind of uncertainty. There is then a suggestion of several tentative plans to remove doubt. The data at hand may suggest solutions based on past knowledge and experience, but do not supply them. With these suggestions available, it may still be that reflective thinking does not occur: the necessary condition is for the person to be critical, to engage into searching and to accept doubt. Dewey claims that a person can, if he or she wants, change his or her ways of thinking to become better (and the better way of thinking is reflective thinking). Thus, given the values of reflective thinking, it should and it can be an accessible educational aim.

More interested in the relation between students' ways of knowing and their academic success were studies in cognitive psychology like that of Perry (1970) and Belenky, Clinchy, Goldberger and Tarule (1997). Perry shows how students' conceptions change, as they advance throughout their undergraduate years, and how this evolution is shaping their academic experience. He describes several epistemological positions that students adopt successively, evolving in their interpretations of the sources of knowledge and of themselves as knowers:

- *basic dualism*: the student sees the world in polarities: right/wrong, good/bad, we/they; he or she is dependent on authorities as sources of truth;
- *multiplicity*: there begins a gradual search for personal freedom as the student understands that there exist a multitude of opinions and it may be that authorities do not detain the absolute truth;
- *relativism subordinate*: own ideas are not only developed at this stage but evidence is also added to support them, and an analytical attitude is adopted and cultivated consciously, at least in academic life;

- *relativism*: the student understands completely the relative, context and framework-dependent character of truth, not only in academic life but also in everyday life; knowledge is now understood as constructed and not as given.

Belenky and her collaborators (1997) were also interested in epistemological positions, but the subjects of their study were women. Perry's study had very few women as subjects and only interviews with men were used to illustrate this scheme of evolution in students' epistemological positions. The aim of Belenky et al. was to uncover the specificities of women's ways of knowing. They identified five epistemological positions of women. *Silent* women see the source of knowledge, even of self-knowledge in others; they are passive and they do not intend to be otherwise as they perceive themselves as knowing nothing and as not being able to learn anything. *Received knowers* rely on authority to tell them what is wrong and what is right and think of truth as being validated by authorities and not by one's thinking. Unlike silent women, they do expect to learn but they do not develop any personal opinion whatsoever. *Subjective knowers* rely almost entirely on their intuition as the source of a personal truth. They perceive thus of truth as being relative, i.e. everybody has his or her own truth, they do develop personal opinions but they do not make any attempt to assess their own assumptions nor to subject them to questioning. *Procedural knowers* follow standard procedures learned from experts but they have the capacity to engage in objective thinking and have powerful reasoning abilities. However, they do not perceive themselves as owners of knowledge, and do not take distance from their own framework. *Constructed knowers*, on the other hand, are aware of the fully relative character of knowledge and accept it only after evaluating the implicit and explicit assumptions on which it is based. Therefore in their epistemological approach they integrate both objective and subjective ways of knowing and they perceive themselves as owners of knowledge.

The categories proposed by Belenky et al. are interestingly evaluated by Sierpinska et al.(2002) with regard to the three features of theoretical thinking (reflective, systemic and analytic): silent and received knowers do not exhibit any of the features of theoretical thinking, while subjective knowers are reflective and concerned with meanings and justifications, but not with their formal representations. Procedural knowers can be reflective and analytic, as well as systemic, without being hypothetical (they are rather *systematic*). The closest in their epistemological position to theoretical thinkers are constructed knowers.

In mathematics education

Whether they refer to ways of thinking, to epistemological positions or to forms of knowledge, distinctions like those mentioned above have frequently been dealt with in mathematics education. One of such classic distinctions in mathematics education is that of Skemp (1976). He characterizes instrumental and relational understanding through analogy with different forms of knowledge of the geography of a locality: instrumental understanding is equivalent to knowing large numbers of fixed local plans, whereas relational understanding is equivalent to having a mental map of the town. Relational mathematics involves the development of a conceptual structure from which one can produce an unlimited number of plans for getting from any starting point within one's schema to any finishing point. Referring to the learner's implicit beliefs about the nature of mathematical constructs, Sfard (1992) identifies two different sets of such beliefs manifested in two different approaches to mathematics: the structural approach, where mathematical notions are referred to as object-like entities and the operational approach, where notions are conceived as computational processes rather than as static products. In the same volume, dealing with epistemological and pedagogical aspects of the concept of function, similar observations are made by Dubinsky and Harel (1992) when they distinguish between process and object conceptions, or by Goldenberg, Lewis and O'Keefe (1992) as they talk about dynamic vs. static interpretations. The interpretation of Boero, Pedemonte and Robotti (1997) is slightly different. When they refer to theoretical knowledge they characterize the objects of knowledge, and not the processes or the individual's epistemological ways.

All these interpretations, along with other differently termed distinctions, i.e. formal/intuitive, conceptual/procedural, analytic/synthetic, etc., express similar ideas and refer to the same basic dichotomy. What is interesting, for us, about the distinction between theoretical and practical thinking is that it is tailored for the domain of linear algebra, with each of the features of theoretical thinking thoroughly justified in terms of its relevance for the learning of linear algebra (see Sierpinska et al, 2002, p.17). Thus, the distinction is to some extent operationalized, whence our choice of this model for measuring the development of theoretical thinking.

However, before embarking on a research that envisages the development of theoretical thinking, it is worthwhile perhaps to first ask ourselves a more fundamental (and necessary) question: *Can we*, as educators, address this problem, can we do something to cultivate a certain way of thinking? I strongly believe that the university <u>is</u> the place where no efforts are to be spared to develop thinking, in particular the type of thinking that carries the characteristics that we have discussed above. Moreover, my personal educational path led me to believe that teachers *can* play a role in this enterprise, even if the training students get at the university is by far not merely a result of interactions with their teachers and their textbooks. Of course, an answer to this question from a personal perspective is not doing more than providing *my* motivation to proceed in the research, while a more objective answer with implications for teaching will hopefully be provided as a result of the research.

CHAPTER II. A SURVEY OF THE RESEARCH INTO THE TEACHING AND LEARNING OF LINEAR ALGEBRA

Research in the teaching and learning of linear algebra as compared to that carried out for other domains of undergraduate mathematics – like Calculus for example – is relatively recent, as is the increased attention it receives from the curriculum reform movements in the last decade. It has tried mainly to diagnose students' difficulties, to identify their sources and to conduct teaching experiments offering local remediation.

A research program conducted in France by Robert, Robinet, Rogalski and Dorier in the late 80s (in Dorier 2000, p. 85-124) played an initiating role in the field, by revealing the sources of students' difficulties through a series of diagnostic studies. These studies pointed to a major obstacle for students' learning in linear algebra encountered in all successive generations of students, regardless of the type of teaching: the *obstacle of formalism.* Students expressed this difficulty themselves, as revealed by the responses to a questionnaire: they appeared to be overwhelmed by the highly abstract character of linear algebra and the great number of symbols, definitions theorems and formal manipulations it introduces. Teachers, as well, complained about their students' erratic use of the tools of logic and set theory. The researchers found out through a series of tests administered to undergraduate students studying linear algebra in French universities that a minimum of prior knowledge of logic is indeed a necessary condition for success in linear algebra. Still, it turned out that it is not a sufficient condition: further in-depth analysis showed that students did not just have a problem with formalism in general but with the use of formalism within the theory of vector spaces and with linking their prior and more intuitive elements of knowledge with the new formal concepts introduced by linear algebra. These concepts, as Dorier (2000) argues in his historical survey of the genesis of the theory of vector spaces, are the final products of a long and complex process of generalization and unification, which involved various parts of mathematics. The axiomatic theory of vector spaces did not solve new problems in mathematics, but it imposed itself as a universal approach and language to be used in a variety of contexts (functional analysis, quadratic forms, arithmetic, geometry, etc.).

Three hypotheses were advanced as a result of the diagnostic studies and of the epistemological analysis of the historical development of linear algebra, which then stood at the basis of the experimental teaching carried out at the University of Lille in France.

The first hypothesis stated the necessity of students' awareness of the specific characteristics of linear algebra. The fact that students only have the chance to know the finite products of a long and awkward process that led to the development of the theory of vector spaces cannot be ignored. Another advanced hypothesis was that the interaction between previous related levels of knowledge and the new concepts introduced by linear algebra could be an insuperable obstacle. A third hypothesis referred to the appropriate didactic strategy that would combine the use of the meta lever with a long-term teaching design and the use of changes in points of view as a stimulator.

The meta lever designates the use in the teaching of mathematics of knowledge about mathematics. The long-term strategy refers to a type of teaching that cannot be broken into several smaller parts, which is necessary – in the author's opinion – in order for the changes in the didactical contracts to operate over a long enough period so as to be efficient for the students. The change of settings and points of view means that the teaching is organized so that the problems and the content of the teaching use translations from one setting into another (from formal to numerical, from numerical to geometrical, etc.).

The teaching was then organized comprising several steps and with the coordinates given by the didactic choices described above. The results showed that the overall experiment was satisfactory and did not result into major failures. While good results were obtained regarding the appropriation of some core concepts and of the changes in points of view, it appeared that only a minority of the students had proven capabilities to spontaneously model a problem in the setting of linear algebra.

More interested in the curricular aspects of the teaching of linear algebra, Harel (2000) gives an account of the curricular reform actions in the United States, of which he was an active promoter as a member of the Linear Algebra Curriculum Study Group (LACSG). The LACSG was started in 1990 as a study group whose members, mainly mathematicians with an interest in mathematics education, proposed a set of recommendations for a first course in linear algebra. The main sources of these recommendations were research findings in various areas of mathematics education, the teaching experience of the members of LACSG and the suggestions given by consultants from various client disciplines.

The work of the LACSG was concretized into several communications at national meetings on the teaching and learning of linear algebra and into a series of textbooks written on the basis of the group's recommendations. Giving a personal perspective of the recommendations put forward by LACSG, Harel insists on some of the aspects he considers crucial in these guidelines.

He first suggests that proofs, which represent an important part in the university linear algebra course, be emphasized earlier in the mathematics curricula, so that, by the time they take their first course in linear algebra, students have already reasonably strong justification skills. Moreover, he proposes that linear algebra basic notions be introduced in high-school, to compensate for the insufficient time allocated to linear algebra in the undergraduate mathematics curriculum and thus bridge high-school and university mathematics. Another pedagogical suggestion proposed by Harel is the incorporation of technology in the first linear algebra course to prepare students for a matrix-oriented course, along with a core-syllabus that would follow the LACSG recommendations.

Harel's set of suggestions is theoretically grounded in three principles he postulates for the teaching/learning of linear algebra, inspired by the Piagetian theory of concept development. The Concreteness Principle states that in order for students to abstract mathematical structures, it is necessary that the concepts of linear algebra first become conceptual entities for them. Only this way will students be able to operate on these objects as inputs. In the light of this principle, geometry embodiments are important in the early phase of teaching linear algebra. However, the researcher observes that there exists the risk for the students to be less able to abstract later, as they may regard geometry as the raw material to be studied. Harel illustrates the violation of the Concreteness principle in practice, giving, among other pieces of evidence, the example of students' difficulties to grasp the concept of functions as vectors, as entities in a vector space. The Necessity Principle postulated by Harel suggests that students should see an intellectual need for the building of linear algebra concepts. This is indeed a challenging pedagogical task for the linear algebra teacher. Finally, the Generalizability Principle states that students should be encouraged through appropriate instruction to generalize concepts in the context of linear algebra.

Hillel (2000) observes that students entering their undergraduate program in mathematics are confronted with proof-related difficulties that are not necessarily specific to linear algebra; rather, they are encountered in most undergraduate mathematics courses. He then focuses on describing sources of conceptual difficulties peculiar to linear algebra, which include:

- the existence of several languages and modes of description
- the problem of representations
- the applicability of the general theory

Hillel distinguishes three modes of description used in linear algebra: the *abstract mode* of vectors and of linear operators, the *algebraic mode* of n-tuples and of matrices and the *geometric mode*, of directed line segments, point, lines, planes, etc. He then shows, by observing students at Concordia University in Montreal, how they are used, how they interact and how the passage from one mode to the other can confuse students. He illustrates how students' attempts to link the new formal concepts of linear algebra to previous knowledge of geometry only results into a greater epistemological obstacle for them. On the other hand, difficulties are especially encountered when there is a shift form the algebraic to the abstract mode of representation and this is the source of a second epistemological obstacle for students. It stems from their learning of very specific notions related to $\mathbf{R}^n$. Hillel argues that using these notions to solve problems linked to systems of linear equations involves an algebraic level of description, which then prevents students from switching to an abstract mode of description, which would allow them to understand the general theory, where other kinds of objects (e.g. functions, matrices or polynomials) are to be seen as vectors.

Sierpinska (2000) describes three modes of reasoning in linear algebra corresponding to these three interacting languages: the visual geometric, the arithmetic and the structural language, illustrating with examples students' reluctance to enter into the structural mode of thinking and to move between the three modes. Alves-Dias and Artigue (1995) reach a similar conclusion, indicating that students' lack of cognitive flexibility between settings, thinking modes and points of view and in linear algebra particularly between the parametric and Cartesian points of view is a major source of obstacles for their understanding. They extended Pavolopoulou's work (1993) where students' difficulties are explained through a problem of shifting representations, as an application and verification of Duval's theory about registers of semiotic representations and cognitive processes.

Based on several experiments and teaching projects, Sierpinska (2000) argues that students think in practical rather than in theoretical ways and this creates difficulties for them in linear algebra. This conclusion was used by Sierpinska, Nnadozie and Oktac (2002) to conduct a study about the relevance of theoretical thinking for high achievement in linear algebra. They postulated a model of theoretical thinking and a method of evaluating an individual's or a group's disposition to think theoretically in the sense of the postulated model. The findings of this study show that theoretical thinking was not a condition for high achievement in the courses taken by the students in their study, but it turned out to be very strong in the best of these students, indicating an unused potential.

SYNTHETIC OVERVIEW: WHAT DOES RESEARCH TELL US ABOUT THE TEACHING AND LEARNING OF LINEAR ALGEBRA?

My review of the research in the domain of teaching and learning linear algebra was largely based on the book "On the teaching of linear algebra" (edited by Dorier, 2000). Without claiming to provide an ultimate set of guidelines for the teaching of linear algebra, or to contain an exhaustive collection of research into the teaching and learning of linear algebra, this book presents however, with a high degree of detail, research works that have played an initiating role in the field as well as more recent developments. We give below a synthetic overview of the main results of the research reported in this book. We also situate our research in this context.

The advanced hypotheses: why is linear algebra difficult and what are the possible remedies?

Linear algebra is difficult for students because of <u>its specific nature</u> – its core concepts are the results of a process of generalization and unification, offering a more universal language to deal with mathematical problems in various contexts. However, it is very improbable that such a power of simplification be within the reach of undergraduate students, especially since the notions of linear algebra don't lend themselves to illustrations through particular problems. It is therefore important to make students aware of the epistemological nature of linear algebra and of the theoretical detour it can offer.

Another source of students' difficulties indicated by this research is their <u>lack of prior knowledge in logic and set theory</u>. It is proposed that preliminary training in this area be included as part of the teaching of linear algebra. This is indeed a necessary condition for success in linear algebra, but the obstacle of formalism – as the researchers term the erratic use of the tools of logic and set theory – is not a general problem students have with formalism. It is rather a problem with the specific use of formalism within the theory of vector spaces and has to be dealt with in this particular context.

The interaction between students' prior and more intuitive elements of knowledge and the new formal concepts introduced by linear algebra is also pointed to as a major obstacle for their learning. Since many problems can be solved without making use of the axiomatic theory, most of the times students will use the formal theory only because they are situated in the context of using it and not because they see any meaning of that. Making students reflect on their own learning and on the mathematics being taught to them could help them gain control over their learning.

The change of settings and points of view (geometrical, analytical, logical, formal, etc.) that is very common in linear algebra is another source of confusions for students, especially because in most of the cases it's not made explicitly. Students' moving between the multitude of languages and modes of description specific to the different settings is not natural and it therefore requires open debates on the nature and status of the language in linear algebra (an idea close to that of 'meta-lever').

Students' ways of thinking. Cognitive flexibility, trans-object level of thinking, as well as an inclination to think theoretically are the main characteristics identified as necessary for understanding linear algebra. On the other hand many experiments gave evidence that students, in many cases, do not engage in such ways of thinking.

It is this latter explanation advanced with regard to the causes of students' difficulties that we take a closer look at, namely, investigating students' engagement with theoretical thinking when solving linear algebra problems and trying then to evaluate an approach that could enhance an engagement with TT.

The characteristics of the experimented teaching designs

This section refers mainly to the French research (except for point d) which is shortly addressed in Hillel's paper). The design of the experimental teaching took into account the above hypotheses and the choices of didactical strategies were made along the following axes:

a) **The use of the meta lever**. The meta activities were present in a preliminary (preceding the teaching) phase, when problems were solved to be then unified with the use of vector space theory, pointing at the simplification offered by such an approach. Then the meta was involved throughout the teaching in clarifying the specificity of the concepts and in activities that induced students to reflect on various mathematical concepts and on the methods that could be used to solve problems.

b) **The change of settings and points of view**. The content of the teaching and the problems used translations between settings.

c) **The long-term strategy.** The course was engineered over a one-year period in order to take into account the non linearity of the teaching and to be better evaluated.

d) **The incorporation of dynamic geometry software (Cabri).** It was believed that introducing linear algebra concepts in a synthetic-geometric way with the aid of Cabri will stand as a coordinate-free approach, facilitating the notion of general basis. However, students seemed to perceive the computer screen as an implicit coordinate system and so, the tasks on transformations given in a geometric context, were approached by the students analytically.

The overall experiment in the French universities was appreciated as having good results, but some of the students' problems persisted: they still didn't seem to overcome completely the obstacle of formalism or to be able to engage in autonomous questioning. Also, the capacity for spontaneously approaching a problem in the formal setting of linear algebra was present only in a small minority of students.

We are trying, in this research, to investigate the effectiveness of **a minimalist approach** to the teaching of linear algebra, with no special embodiments. The main component of this approach is the administration **weekly quizzes** as a way to enhance students' theoretical thinking and consequently their learning of linear algebra. We believe that weekly administered tests have the potential to cultivate the disposition for theoretical thinking for at least two reasons:

1. Students are put in the situation of reflecting on their own for a few minutes in every class and therefore they will most probably develop both a reflective attitude and a practice in proving activities.
2. The type of problems that suit such limited time tests are hardly likely to be computational and to require solely the mastery of various procedures; instead they challenge students to involve deeper thinking abilities in order to solve them.

Methodological issues

A vast variety of methodological means was used in these research works and problems encountered at this level were constantly accounted for, in order to provide a more complete and reliable context for the results. A brief list of the methodological tools used in the research work reported in the book includes:

- the administration of specially designed questionnaires and analysis (both qualitative and quantitative) of responses to them in order to evaluate students' knowledge and ideas;
- pretests administered to evaluate students' individual level of knowledge in basic logic and set theory;
- qualitative analysis of usual tests given throughout the year to determine both the characteristics and the effects of the teaching by examining the tasks required by these tests and how these tasks were responded to by students;
- quantitative – through statistical methods – analysis of students' responses to the above mentioned tests, involving coding of their responses and awarding scores according to their manifested behavior in these tests; correlation with the pretest;
- observation of students behavior during tutorials and workshops; descriptions of several case studies.
- engineering of teaching sequences based on the conclusions resulted from the diagnostic studies.

The quantitative analysis of the tests given by the French researchers was limited by the fact that the tests were not all compulsory for students, and therefore only a small sample of sets of tests was finally obtained and analyzed. However, this allowed the researchers to analyze them more in-depth and to draw some hypotheses which will eventually serve to build adequate statistical tests involving larger populations of students.

The main methodological problems were posed by the evaluation of the impact of the engineering; more precisely, it was very difficult to validate the type of hypotheses made in this research. First, it was difficult to choose the time appropriate for the evaluation of long-term teaching, as many significant factors cannot be kept under control – like students' own organization of their time. On the other hand, indicators of students' reflection at the meta level were not easily traceable, especially in their written work; most of the time, they would give the answer, but not the means which led to it. In this context, the researchers stated their next goal to be the finding of better ways to gain access to the didactical variables they want to evaluate.

Harel's paper was special in that it reported on developmental research – research that provided mainly a set of concrete ideas – anchored in theoretical and experimental work, which found their application in curricular developments and textbooks designs.

A crucial problem that is in fact acknowledged in the Conclusion of the book "On the teaching of linear algebra" (Dorier, 2000), as an issue requiring serious attention from research aimed at improving linear algebra teaching, is the transmission to other teachers of the ideas proposed as fundamental for teaching designs. This is because in most of the reported research works, the teacher experimenting the projects was one of the researchers involved in designing it. This brings us to the more general question of the possibility of transfer of knowledge derived from research in mathematics education to regular teachers, who are not necessarily involved in research. Such a fundamental question is at the basis of the endeavor I want to take in my thesis. There is indeed, as a result of very extensive and fruitful work in research, a quite comprehensive range of explanations for students' difficulties in linear algebra along with suggestions for the improvement of teaching. My work will try to contribute to making some of these suggestions – namely students' development of theoretical thinking – more easily applicable to teaching practices by evaluating a simple teaching tool for linear algebra classes. On the other hand I also hope to contribute to another kind of transfer, that from theory in mathematics education to research practices, by employing the theoretical model of TT in an operational manner within my research and, on a meta level, by reflecting on this use of the model in my research.

CHAPTER III. ANALYSIS OF STUDENTS' SOLUTIONS TO THE QUIZZES

This chapter starts with a detailed description of the participants and of the data of our research. Before starting our analyses of the data, we include, for the purpose of quick reference, a schematic overview of the characteristics of theoretical thinking to which we refer in our analyses of the data. We give then a general description of our method for analyzing the quizzes. The main body of this chapter consists of our analyses of the quizzes: for every question (in total fourteen questions), we describe our a-priori scheme for analyzing students' responses, followed by the results of our analyses using this scheme. We include as well, in order to illustrate these results, samples of students' solutions.

PARTICIPANTS AND DATA

We have analyzed students' solutions to the quizzes administered in a Linear Algebra II (MATH 252) course at the Concordia University in Montreal. The quizzes were given on a weekly basis, the exceptions being the first and last week of classes, as well as the week when the midterm exam took place.

There were, in total, nine sets quizzes, but we have analyzed only seven of them. Quiz 3 was one of the quizzes we have excluded from our analysis because we weren't able to photocopy it in the short time that it was in the hands of the instructor for marking. Our reason for not considering quiz 9 was of a more methodological nature. The questions in this quiz required very direct answers, without requiring students' to justify them. For instance, a matrix representation of a nilpotent operator was given and students were asked to find its nullity and its index of nillpotency. Now, this kind of problems did require our students to involve to some extent their theoretical thinking, but their answers were, almost without exception, specific numbers. Thus, our analyses of their answers would have been based on theoretical speculations, with no empirical grounds.

Fifty-seven students were registered for this Linear Algebra II course, but the number of students answering the quizzes varied slightly from week to week. The majority of these students were attending the first year in the Actuarial program offered in the Department of Mathematics and Statistics at Concordia. The clientele of this program usually consists of mathematically strong students, the main criterion for their selection in the program being the high grades they have obtained in their pre-university mathematics courses.

The quizzes were intended to develop students' inclination to think theoretically. However, they were not designed to satisfy certain characteristics explicitly mentioned, but rather reflected an implicit goal of the instructor of the course. The problems were in general not computational or requiring solely the mastery of various procedures; instead they challenged students to involve deeper thinking abilities in order to solve them.

THE FEATURES OF THEORETICAL THINKING – A SCHEMATIC OVERVIEW

In our analyses of the data, we have considered three main features of theoretical thinking: REFLECTIVE, SYSTEMIC and ANALYTIC. We present their characterization below, in a schematic manner, along with the subcategories they subsume.

Theoretical thinking is

REFLECTIVE = is critical with respect to its own products

SYSTEMIC = is concerned with systems of representations of relations between objects, which are relations themselves.

subcategories:

DEFINITIONAL = stabilizes meanings by definitions and theorems
= accepts and understands axiomatic definitions

PROVING = is concerned with the verification of validity; is sensitive to contradictions

HYPOTHETICAL = takes into account all logically possible cases
= is aware of the conditional character of mathematical statements

ANALYTIC = is mediated by semiotic systems of representation; takes systems of signs as an object of reflection and invention

subcategories:

LINGUISTIC SENSITIVITY

- to the syntax of formal symbolic notations
- to the usage of specialized terminology in less formal sentences

META-LINGUISTIC SENSITIVITY

- to the symbolic distance between sign and object
- to the structure and logic of mathematical language
- to parts of "mathematical speech", i.e.: distinguishes among axioms, definitions, theorems, proofs, examples, explanations, etc.

GENERAL DESCRIPTION OF METHOD

We wanted to see how students' theoretical thinking evolved as they were taking the quizzes. For that, we first had to identify in their written answers, instances of behavior could be characterized as corresponding to acts of theoretical thinking. This identification was theory-driven. We first analyzed the questions in the quizzes: for each question we described a set of behaviors that would be, in principle, necessary for solving it correctly. Each of these expected behaviors was then assigned a code made of the following three coordinates: **TT feature / subcategory of the TT feature / concrete theoretical behavior**. For instance, if, to solve a question it was necessary to distinguish between a definition and a theorem, then this act represented the concrete theoretical behavior (TB) requiring analytic thinking, and, more precisely, the meta-linguistic sensitivity. This expected behavior would then be coded as: **Analytic / meta-linguistic / definition-theorem distinction**.

Each student's solution was analyzed using this a-priori identified set of TT features. A fourth coordinate was added to this coding, that resulted this time from our empirical analyses: one of the numbers 1, -1 or 0. The '1' codes signaled an explicitly theoretical behavior, while the '-1' code was awarded when, with respect to the given feature of TT, the student's behavior was practical. In the above example, for instance, if the student, expected to provide the definition of an object, wrote instead a theorem about that object, then his behavior would be coded *Analytic / meta-linguistic / def-thm distinction / -1.* Our focus thus, in the analyses we have performed on students' answers, was on whether or not they engaged in TT when such an engagement was necessary in order to solve the problem at hand. On the other hand, practical thinking was in many cases opportune and even necessary. However, our primary interest being in the necessary features of TT, we only discussed students' practical tendencies by contrast with TT, when students used PT to their detriment.

We coded a behavior with '0' in two situations. Firstly, in some situations a student's behavior was neither explicitly theoretical, nor definitely practical. We have discussed in a previous chapter how closely TT and PT interact in mathematical thinking and how fine the line that separates them may be. The "0 situations", as we called them, illustrate precisely how a definite distinction between TT and PT is sometimes impossible to be made in practice, especially when dealing with written outcomes of thinking. We chose not to include them in our quantitative analysis, yet many of these ambiguous – from the point of view of the TT/PT distinction – were much more interesting for our qualitative analysis.

Secondly, since we coded pieces of students' solutions, in some cases, within a solution, a certain theoretical behavior could only occur if another theoretical behavior was previously enacted. One of the problems in the quizzes, for instance, asked the students to give an example of a product over a vector space satisfying two of the properties of an inner product, but not all three of them. To solve this problem, a student would first have to involve his[1] systemic thinking in order to understand and apply axiomatic definitions. This would enable him to give an example of a function over a vector space. Another feature of TT he would have then to resort to is his analytical thinking, in order to correctly introduce and manipulate formal notations throughout the solution. However, if he did not engage in axiomatic reasoning in order to provide an example in the first place – for instance if his example was a concrete vector – then,

[1] The generic pronoun "he" will be used throughout the thesis to alleviate the text.

obviously, we could not decide whether or not he would have been analytically sensitive in using formal notations. In such cases, where, say TB2 is conditioned by TB1, and TB1 does not occur, we assigned TB2 a '0' code. Even if in this particular problem situation TB2 appeared to be dependent on TB1, one could not conclude that in general the two are dependent and thus code TB2 with '-1' whenever TB1 is coded with '-1'.

In our analyses of the questions in the quizzes we did not intend to investigate all the possibilities for theoretical thinking that a question may evoke in a student's mind. In an a-priori, theoretical analysis of a question there may be an unlimited number of such possibilities for TT. Our intention, however, in this preliminary endeavor, was rather to imagine a probable course of action for the solving of the problem, signaling features of TT that were necessary and sufficient to complete the task. Nevertheless, when looking at the students' solutions, we did acknowledge and discuss instances where students explored theoretical possibilities beyond those required by the mere resolution of the problem in the quiz.

ANALYSIS OF THE QUIZZES

Quiz 1 – Question 1: Example of an "almost" inner product

The first quiz was administered the second week of classes, after the students were introduced to the notion of inner product space. In the first question they were asked to give an example of an "almost" inner product. The question was formulated as follows:

> *An inner product in a vector space is a product of vectors which satisfies three conditions: linearity, symmetry and positive definite property. Give an example of*

a product of vectors in a vector space, which satisfies two of these properties but not all three of them (and therefore is not an inner product).

Analysis of the question

We will use this problem to illustrate how the interplay of acts of theoretical and practical thinking contributes to the construction of the solution to the problem. In our subsequent analyses of the questions, as we have mentioned, we will limit our interest to the set of features of TT that we found necessary and sufficient to complete the task. But for now, in order to give an idea of our understanding of the TT and PT categories, let us propose the following scenario of how one may involve his TT and/or PT when solving the problem above:

PT: *To discover the required object* one would most probably resort to the examples one knows of products in a vector space. This does not imply a limitative process, where the person trying to provide the example would only choose from the space of known examples. By the involvement of practical thinking here, we just mean some kind of intuition of what an "almost inner product" would look like, an intuition which is more likely to be present if some examples of inner products have been previously viewed and, in this context, quickly *re*viewed in the search for an appropriate example. This situation – where previous experience shapes intuition – is often encountered in the process of doing mathematics, according to Fischbein (1987).

PT&TT: *To provide one example* then, one involves both theoretical and practical thinking. Theoretical thinking is necessary for a proper understanding of the definition of an inner product. The acceptance of its axiomatic character will enable the student to give an example of a function over a vector space and not, for instance, an example of two concrete vectors. However, his task in providing such an example is made much easier if he will think practically and choose an example whose properties are easier to check and whose parameters are easier to modify in order to satisfy the requirements of the problem. The presence of TT in this practical behavior would be concretized, however, by reference to the axioms that define the inner product. Students' behaviors resulting from whether or not they involve TT when constructing their example were coded **Systemic / definitional / axiomatic understanding** with the fourth coordinate added as follows:

1, if the inner product is understood as a function over a vector space and reference to the axioms that it has to satisfy is made.

-1, in all cases where the product is not understood as a function (for instance if a multiplication of concrete vectors is given as example) or where there is no reference to the defining axioms of the inner product.

TT: *To correctly represent the object and manipulate it* throughout the solution one has to be linguistically sensitive in using formal notations. One first has to engage in correct discursive procedures to represent the product and then to manage and discriminate among the notations introduced throughout the solution, for instance: $\mathbf{x}, \mathbf{y}$ for vectors, $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{y}_1, \mathbf{y}_2$ for coordinates of these vectors, notations for the constants involved in the statement of the linear property, etc.

With regard the manifestation of TT in this linguistic aspect of their written solutions, we coded students' solutions with the code **Analytic / linguistic / formal notations:**

1, if notations are used correctly, without confusion and control over them is maintained throughout the solution.

-1, if variables or notations are used in a confused way, leading to failure in solving the problem.

TT: *To validate the chosen example* one has to involve systemic thinking and exhibit a concern with the validity of what he is claiming. In order to justify that two of the properties of the inner product hold and one of them doesn't, one has to engage in proving activity. It is this process, of validating the example through proving, that allows the student to make adjustments to the initial example, to "fix" it to satisfy the requirements of the problem. Depending on whether attempts at proving were present or not, the code **Systemic / proving / concern with validity** was awarded to students' solutions, with the values:

1, if there is a justification of the properties being satisfied or not, whether correct or not.

-1, if there is no justification of the properties.

PT: is necessary in all the steps of the solution *to perform the necessary algebraic manipulations.*

We would like to restate, at this point, an assumption of the TT/PT model which is fundamental for our analysis of the question: the two categories represent simplified ways of understanding human cognitive activity and they do not refer to some separate cognitive activities. This being the case, the sequential process that we have described above should be understood as our simplified description of a possible thinking process and not as an attempt to faithfully describe a process of human thinking. Human thinking does not follow such a linear trajectory. Reiterations and alternative paths are most probably possible in a student's mind when solving the problem. Also, TT and PT do not occur separately, each as an individual act of thinking, but most likely, any piece of reasoning would contain traces of both. The figure below, which represents the simplified model of thinking we have described, expresses this idea of simultaneous presence of PT and TT in mathematical constructions in a graphical manner. The acts of TT are enclosed in rectangular shapes, while those of PT in cloudy, rounded shapes; the results of thinking are a combination of both: rectangles with rounded edges.

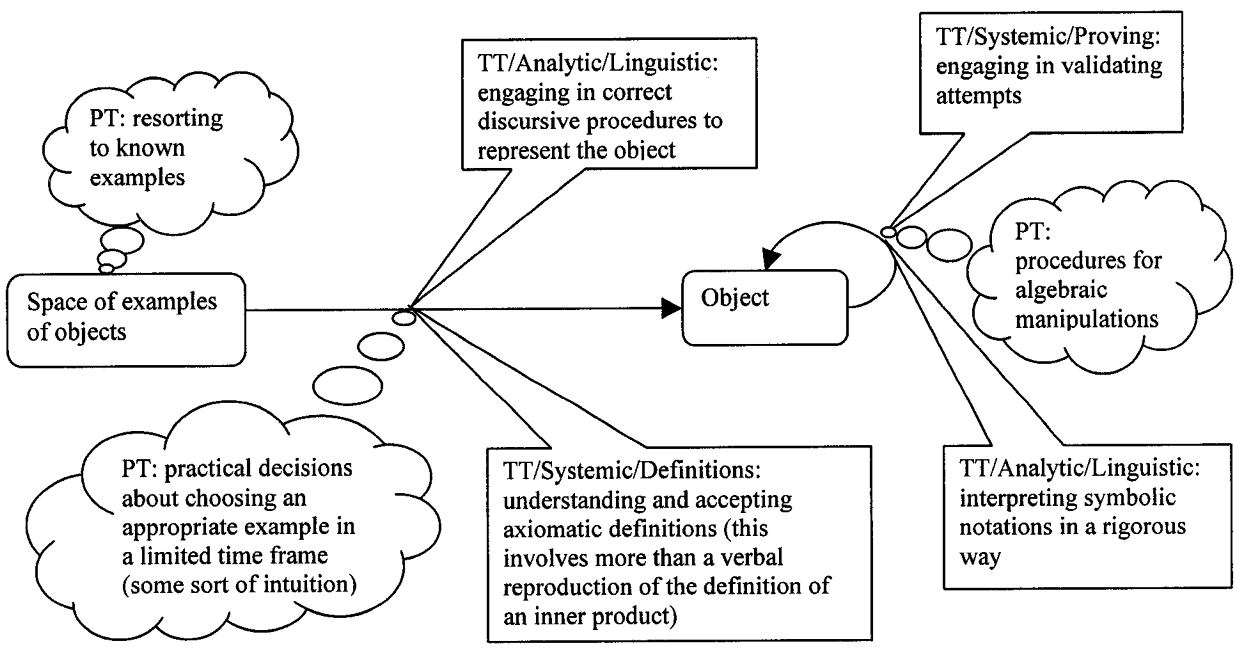


Figure 1: Scenario for a reasoning process – finding an example of an "almost" inner product.

Analysis of students' responses

This was a construction problem and the aim of the instructor was probably to have students engaged in axiomatic reasoning. Such reasoning would encompass not only the understanding of the notion of inner product as an operation, satisfying certain properties, defined over a vector space, but also the undertaking of the freedom to invent any such structure. Sierpinska et al. (2002) argued that, in practical thinking, this freedom, resulting from axiomatic understanding of definitions, is not conceivable: definitions always describe some already existing, real object. In this frame of mind, one could verify successfully whether a given object satisfies the definition, but would find it a much more difficult task to construct such an object. We believe that this problem required at least an acceptance of the axiomatic definition of an inner product: without accepting its "non-real" character, it is impossible to even conceive of constructing a new structure, satisfying two out of three properties.

Most of our students did understand the product as a function defined over a vector space, with only one student out of the 48 having the test that day giving an example of two concrete vectors. Our doubts however came from the fact that almost none of them went beyond the example of the product that was presented in class during the previous lecture, that is: $<\mathbf{x},\mathbf{y}>$ over $\mathbf{R}^2$ defined as $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y}$, where $\mathbf{A}$ is a 2 by 2 matrix. For this to be an inner product, the matrix $\mathbf{A}$ had to meet certain conditions: (1) to be symmetric – thus assuring the symmetry of the product, and (2) to have a positive determinant and positive diagonal entries – to assure the positive definite property. It becomes then almost trivial to give an example of a product satisfying only two properties of an inner product, by simply constructing the matrix accordingly: either not symmetric but satisfying (2), or symmetric but with either a negative determinant or negative diagonal entries. This was a perfectly legitimate solution, and it did show proof of theoretical thinking, as long as those that had this approach linked the properties of the matrix with the properties of the inner product. The corresponding behavior was coded *Systemic / definitional / axiomatic / 1* and below is an example of a student's solution illustrating it:

Solution

$u = (x_1, x_2)$ $v = (y_1, y_2)$

$\langle u, v\rangle = u^T \overset{A}{\begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}} v = -3x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$

$\det A = 2$

We know that for $\langle u, v\rangle$ to be an inner product A must be symmetric, $\det A > 0$ and given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a > 0$ and $d > 0$

Here, we have A such that $a < 0$ and $d < 0$

thus the positive definite property does not hold

i.e. $\langle u, u\rangle > 0$ for all u

example: $\langle [1,1], [1,1]\rangle = -3(1)^2 + 1 + 1 - (1)^2 = -2 < 0$

Yet the other properties hold since A is symmetric.
Indeed, the product is linear for any matrix A and is symmetric when A is symmetric

It is obvious that, even if the student was practical enough to choose an example whose parameters could easily be modified to satisfy the requirements of the problem, she did not do it as if she had memorized the conditions to be met by the matrix without linking them to the axioms defining the inner product. The properties of matrix multiplication and addition assured the linearity of the product, symmetry was linked with the symmetric property to be satisfied by the product, while the negative sign of the diagonal entries would mean that not all vectors multiplied with themselves yield a positive result. In fact she gave a counterexample to prove this last claim.

Her behavior was quite different from that of the student whose solution we present below. He did give an example satisfying the requirements of the problem, yet clearly ignored the axioms that define the inner product. He just checked the properties of the matrix without showing what their relevance is for those of the inner product:

$V = \mathbb{R}^2$. if we view it as a matrix.

Let $\langle u, v\rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = 2x_1y_1 + x_1y_2 + x_2y_1 - 3x_2y_2$

since ① $\begin{bmatrix} 2 & 5 \\ 5 & -1 \end{bmatrix} \rightarrow$ it's symmetric ✓ satisfy.

② $\begin{bmatrix} 2 & 5 \\ 5 & -1 \end{bmatrix} \rightarrow$ det < 0 ✓ satisfy.

③ $\begin{bmatrix} 2 & 5 \\ 5 & -1 \end{bmatrix} \rightarrow$ $d < 0$ ✓ not satisfy

Not only the matrix whose properties he checked wasn't the matrix defining the product he gave as an example, but for him, the three properties of the matrix obviously represent the three properties of the inner product. Linearity, assured by the properties of matrix multiplication, was not even considered in his proof of the validity of the example. Seven out of the 48 students whose solutions we analyzed had a similar behavior focused on the conditions to be imposed on the matrix, without any mention whatsoever of the properties of the inner product. We also categorized the student below, who chose as example of a product the multiplication of two concrete vectors, as lacking a proper understanding of axiomatic definitions:

$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. not conform.

① $(X_1 + X_2) \cdot X_2 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 6 + 3 = 9 = X_1 X_2 + X_2^2 = 2 + 2 + 4 + 1 = 9$

$k=2$ $(kX_1) \cdot X_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 4 + 4 = 8 = 2(X_1 X_2) = 2(2+2) = 8$

② $(X_1, X_2) = X_2 \cdot X_1 = 2 + 2 = 4$

③ $\langle X_1, X_1 \rangle = 1 + 4 = 5$

$\langle X_2 \cdot X_2 \rangle = 1 + 4 = 5$

Yet another student did not even come up with an example, even if he did vaguely remember the properties that had to be satisfied by an inner product. Theoretically, these properties represent everything one needs to know in order to at least attempt to come up with an example as required by the problem. But clearly, for this student, this was just a verbal reproduction of these properties:

Solution

linearity : 1- $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$

$\langle u, kv \rangle = k \langle u, v \rangle$

Simmetry : $\langle u, v \rangle = \langle v, u \rangle$

Positive definite : $\langle u, u \rangle \geq 0$

To summarize, three categories of answers demonstrated a lack of axiomatic understanding of definitions:

- examples constructed based on the properties of a certain matrix, but without justifying how these properties relate to those of the inner product;
- statement of the properties of the inner product without applying them to construct an example;
- examples of concrete vectors.

In total, nine students – so approximately 18% of the sample – showed such a complete indifference or misunderstanding of axiomatic definitions. For us, the others, by simply attempting to name a function and trying to justify its properties were exhibiting, to some degree, axiomatic acceptance. A deeper engagement in axiomatic reasoning, which was probably what the instructor aimed at when she included this problem in the quiz, was a much rarer occurrence. Only two students, like the one whose example we include below, went beyond the example showed in class the previous day and took the freedom to invent a different structure:

Define a product of vectors as follow:

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\rangle = x_1x_2 - y_1y_2 + z_1z_2$$

which satisfies linearity and symmetry properties, but doesn't satisfy the positive definite property.

This solution is in fact, one may argue, the only one really containing a theoretical behavior with regard to the TT feature *Axiomatic understanding of definitions*. It becomes obvious here how fragile the distinction between TT and PT is and how a distinction in absolute terms is impossible to make especially in the case of written outcomes of thinking, where there is little access to the actual thinking process. The identification of TT features within the behavior of one student can only be made within the larger context of the behaviors of the whole population: for instance, the student who referred to the paradigmatic example shown in class but made links with the axiomatic definition is clearly more theoretically inclined than the one who is just applying it without the slightest reflection at an axiomatic level. We often based the decisions we made when characterizing a certain behavior as being theoretical or not on this kind of relative judgments.

The engagement in proving was not a behavior prompted by the requirements of the problem, but it was undertaken by many students, in their attempts to validate why their example was a satisfactory one. In fact, failure to engage in validating attempts, led in many cases to examples of functions that were actually inner products, as it was the case for the student below:

$v = \langle v_1, v_2 \rangle \quad x = \langle x_1, x_2 \rangle$

$w = \langle w_1, w_2 \rangle$

1. lin: $\langle v, w \rangle = 2v_1w_1 + v_1w_2 + v_2w_1 + 3v_2w_2$

$\langle v+x, w \rangle = 2(v_1+x_1)w_1 + (v_1+x_1)w_2 + (v_2+x_2)w_1 + 3(v_2+x_2)w_2$

$2w_1v_1 + w_2v_1 + w_1v_2 + 3w_2v_2 + 2x_1w_1 + w_2x_1 + w_1x_2 + 3w_2x_2$

$\langle v, w \rangle + \langle x, w \rangle$

2. sym $\langle v, w \rangle = \langle w, v \rangle$ It is symmetric

3. Pos def does not hold.

Were he concerned with the validation of his last statement, he would have had the chance, at least, to realize that the positive definite property does hold and reconsider his example.

Examples of products failing to satisfy more than one property were also due to the same lack of concern with validity. This was the case for the student who wrote the following solution; she only proved the symmetry of the product, without bothering to justify the other two properties:

Solution LET $\langle u,v\rangle = 2x_1y_1 + 3x_1y_2 - x_2y_1 - 5x_2y_2$,

$u = (1,2)$ and $v = (2,3)$

BY SYMMETRY:

$\langle u,v\rangle = \langle v,u\rangle$ in a inner product.

$\langle u,v\rangle = 2(1\cdot 2) + 3(1\cdot 3) - (2\cdot 2) - 5(2\cdot 3)$

$= 4 + 9 - 4 - 30 = -21$

$\langle v,u\rangle = 2(2\cdot 1) + 3(2\cdot 2) - (3\cdot 1) - 5(3\cdot 2)$

$4 + 12 - 3 - 30 = -17$

$\therefore \langle u,v\rangle \neq \langle v,u\rangle$

THE SYMMETRY PROPERTY DOES NOT HOLD

$\therefore \langle u,v\rangle = 2x_1y_1 + 3x_1y_2 - x_2y_1 - 5x_2y_2$ is not an inner product

Twenty-one students did not provide justification of why one or another property held and, consequently, ten – so approximately half of them – failed to provide a good example of product, as required by the problem.

For eight students, the misuse of formal notations led in most cases to wrong solutions or to obstacles in providing a good solution. Here is an example where the lack of control over notations prevented the student from providing a good solution:

(3) Positive Definite $\langle u,u\rangle = 2u_1^2 + 1u_1u_2 + 2u_2^2$

is $\langle u,u\rangle \geq 0$?

$$2u_1^2 + u_1u_2 + 2u_2^2 \geq 0 \quad \text{iff}$$

$$4 \cdot u_1^2u_2^2 - 4(2u_1^2)(2u_2^2) \geq 0$$

$$-15u_1^2u_2^2 \not\geq 0$$

$\therefore$ Positive definite fails! Not inner product

He most likely recalled a procedure for checking the positive definite property of a product over $\mathbf{R}^2$ involving the discriminant of a quadratic function. This procedure involves a change of meaning to the initially introduced notations: $\mathbf{u}_1$ and $\mathbf{u}_2$ were first entries of a vector, while for studying the monotony of the quadratic function, one had to understand, for instance $\mathbf{u}_1$ as variable and $\mathbf{u}_2$ as some constant. We believe it was his lack of control over notations that prevented him from applying this knowledge correctly. He proved his lack of linguistic sensitivity in at least two instances. First he had trouble nominating within the quadratic function the variable and the constants: he calculated the discriminant as if $\mathbf{u}_1^2\mathbf{u}_2^2$, $\mathbf{u}_1\mathbf{u}_2$ and $\mathbf{u}_2^2$ were the coefficients in the quadratic. Secondly, even if he had calculated the discriminant correctly, apparently, he would have still obtained the wrong conclusion because he simply associates the positive definite property with the positive sign of the discriminant. This indicates that he performed those algebraic manipulations without having in mind the monotony of the quadratic function. This type of reasoning, where symbolic expressions are manipulated in order to provide the solution to a problem, without understanding the actual problem situation, is specific to what Harel and Sowder (1998) term as symbolic proof schemes.

The confused use of notations is also obvious in the following attempts of a student, and it prevented him from engaging in any kind of justifications:

Solution

$u_1 = (a_1, a_2) \quad v = (c_1, c_2)$

$u_2 = (b_1, b_2)$

$g(u, v) = x_1y_1 - 3x_1y_2 - 3x_2y_1 - x_2y_2 \qquad a>0 \quad b>0$

$= (a_1+b_1)c_1 - 3(a_1+b_1)c_2 - 3(a_2+b_2)c_1 - c_2(a_2+b_2)$

$= a_1c_1 + b_1c_1 - 3a_1c_2 - 3b_1c_2 - 3a_2c_1 - 3b_2c_1 - c_2b_2 - c_2a_2$

In conclusion, 81% of the students in our group did at least attempt to use the axiomatic definition of an inner product in order to come up with an example satisfying only two of its properties, but they failed to produce a good example mainly for two reasons. One of the reasons was that they did not engage in validating their example by proving the holding or non-holding of its properties, which led in most cases to examples of inner products. A second reason, accounting for a smaller number of wrong solutions, was the misuse of notations, exhibited by 17% of the group. We summarize these results in the following table:

TB	Sys/def/axiomatic		Sys/prov/concern		Analytic/ling/formal not		
Score	1	-1	1	-1	1	0	-1
Number of students	39	9	27	21	33	7	8
% of the total	81%	19%	56%	44%	69%	16%	17%

Table 1: Students' behavior in Quiz 1 – Question 1.

Quiz 1 – Question 2: Proof verification

The second problem was also on the subject of inner product spaces. Students were showed a proof that a given product is an inner product and they were asked to verify the validity of this argument. The question was formulated as follows:

In $\mathbf{R}^2$ *we define a product of vectors* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *by the following formula:*

$< \mathbf{x}, \mathbf{y} >= 2x_1y_1 + x_1y_2 + x_2y_1 - 3x_2y_2.$

Below we prove that this product is an inner product. Check if this proof is correct.

Explain why yes or why no.

1. Linearity

Let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$< \mathbf{x}+\mathbf{z}, \mathbf{y} >= 2(x_1+z_1)y_1 + (x_1+z_1)y_2 + y_1(x_2+z_2) - 3(x_2+z_2)y_2$
$= (2x_1y_1 + x_1y_2 + y_2x_1 - 3x_2y_2) + (2z_1y_1 + z_1y_2 + z_2y_1 - 3z_2y_2) =< \mathbf{x}, \mathbf{y} > +$
$< \mathbf{z}, \mathbf{y} >$

Let $\mathbf{k}$ *be a real number.*

$< \mathbf{kx}, \mathbf{y} >= \mathbf{2kx_1y_1 + kx_1y_2 + kx_2y_1 + 3kx_2y_2 = k} < \mathbf{x}, \mathbf{y} >$

So the product is linear.

2. Symmetry

$< \mathbf{x}, \mathbf{y} >= (2\mathrm{y}_1 + \mathrm{y}_2)\mathrm{x}_1 + (\mathrm{y}_1 - 3\mathrm{y}_2)\mathrm{x}_2 = (2\mathrm{x}_1 + \mathrm{x}_2)\mathrm{y}_1 + (\mathrm{x}_1 - 3\mathrm{x}_2)\mathrm{y}_2 =< \mathbf{x}, \mathbf{y} >$

So the product is symmetric.

3.Positive definite property

Let us take any vector, for example $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then $< \mathbf{x}, \mathbf{x} >= 2 \cdot 2 \cdot 2 + 2 \cdot 1 + 1 \cdot 2 - 3 \cdot 1 \cdot 1 = 9 > 0.$

If $\mathbf{x}$ *is a zero vector, then* $x_1 = x_2 = 0$ *and so* $< \mathbf{x}, \mathbf{x} >= 0.$

So the product is positive definite.

Therefore the product is an inner product.

Analysis of the question

The proof of the positive definite property of the inner product was made using an example. We expected students approaching this question with a theoretical mind to notice this mistake in the proof because, in their systemic approach to meanings, they would seek to establish the validity of general statements based on definitions of concepts and not on some concrete representations of these concepts. For them, it would not be acceptable to conclude that for the given product the positive definite property holds because the product of one particular vector with itself gives a positive number.

Consequently, theoretical thinking would be concretized for this question in the behavior of rejecting the last part of the proof, on the grounds that it is not a general proof. We coded with **Systemic / definitional / in proving / 1** this category of answers. If a student does not comment on the mistake in the last part of the proof, regardless of whether or not he concludes that the given product is not an inner product, we would code his answer with **Systemic / definitional / in proving / -1.**

This problem will also provide us with a good opportunity to look at our students' conceptions about proofs, and in general, about validity in mathematics. Works that describe the types of proofs that students produce and/or consider acceptable, refer mainly to two categories of justifications: empirical and analytic (see Harel and Sowder, 1998; Balacheff, 1999; Bell, 1976). Empirical justifications are characterized by the use of examples as basis for conviction, whereas analytical proofs are characterized by the use of deduction to obtain conclusions from the given data. A priori, we would assume that the systemic approach to proving will result in analytic proof schemes, while reference to concrete examples engages empirical proof schemes. Let us now check this hypothesis against our empirical data.

Analysis of students' responses

Eighteen students – approximately one third of the sample – observed that the positive definite property was not proved in general, thus refuting the proof because of its last part. Accordingly, we coded these students' answers with *Systemic / definitional / in proving / 1*. They were all interested also in verifying whether the given product is an inner product, doing that in essentially two ways. The student whose solution we reproduce below showed that the positive definite property doesn't hold by providing a counterexample:

Solution Wrong. we can not prove this property by select a special case.

in fact, if we select another case we can easy to prove.

Lee $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\langle x, x \rangle = -3 < 0$.

positive definition doesn't hold.

To prove the same claim, one could also engage in a direct proof, similar to the one we include below, where the positive definite property is proved not to hold in the general case:

Solution

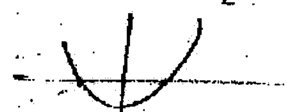
For condition 3. we must show the general proof; not only an example.

$\langle x,x\rangle = 2x_1^2 + 2x_1x_2 + 3x_2^2$. check if $\langle x,x\rangle \geq 0 \;\forall x$

$\Rightarrow$ discriminant $\leq 0 \;\forall x$: $(b^2 - 4ac)$:

$(2x_2)^2 - 4(2)(3x_2^2) = 4x_2^2 + 24x_2^2 \; +28x_2^2 \geq 0$

if we take $2x_1^2 + 2x_1x_2 - 3x_2^2$ as a quadratic equation



it has 2 zeros $\Rightarrow$

it is negative at some point .. so we have

$\langle x,x\rangle < 0$ for some $x \Rightarrow$ not an inner product

Of the 18 students who refuted the proof in the quiz, only 3 students concluded that the last property doesn't hold based on a counterexample. So, even if, clearly, a significant part of our group (about 35%) would not rely on evidence from examples, many of them still do not reach what we would like to call a higher level of mathematical sophistication – understanding that examples *can* be used to disprove a general statement. Proving in a straightforward manner is not, of course, a bad thing in itself, but the almost complete absence of proof by counterexample is an indication that students are stuck with what Harel and Sowder (1998) called "an envisioned step-by-step creation of the result" (p.20). This conception leads, in Harel and Sowder's view, to restrictive proof schemes. These are analytic proof schemes, they may well be correct, but they encompass a narrow conception about proof, limiting students' choices when approaching a proof problem. We believe that this limiting conception was the main culprit in our students' preference for a much longer proof, when a simple example would have done the job.

The other 33 students (representing two thirds of the group) had essentially two approaches to this problem. Seventeen of them accepted the last part of the proof. We coded their answers with *Systemic / definitional / in proving / -1*. The other half concluded, based on a direct proof, that the given product is not an inner product, without making any comment on what exactly was wrong with the proof. We coded their answer with *Systemic / definitional / in proving / 0* since we couldn't assume in a definite manner that they accepted the proof by example done in the last part of the problem. I think that this is a common situation that many instructors encounter and complain about: students haphazardly engaging in solving a problem without the slightest reflection on the actual question they have to answer.

Now, to check another theoretical conjecture we have made regarding the conditioning of analytic proof schemes by the involvement of systemic thinking, we could see that analytical proofs do not necessarily entail theoretical thinking. Let us consider the following solution to the problem, containing solely the proof that the given product is not an inner product without saying exactly what is wrong with the proof:

In R^2 we define a product of vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ by the following formula: $y^T A x = (y_1\ y_2)\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(*) $\langle x, y\rangle = 2x_1y_1 + x_1y_2 + x_2y_1 - 3x_2y_2$

Below we prove that this product is an inner product. Check if this proof is correct. Explain why yes or why no.

1. Linearity

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$\langle x + z, y\rangle = 2(x_1+z_1)y_1 + (x_1+z_1)y_2 + y_1(x_2+z_2) - 3(x_2+z_2)y_2$

$= (2x_1y_1 + x_1y_2 + y_1x_2 - 3x_2y_2) + (2z_1y_1 + z_1y_2 + z_2y_1 - 3z_2y_2) = \langle x, y\rangle + \langle z, y\rangle$

Let k be a real number.

$\langle kx, y\rangle = 2kx_1y_1 + kx_1y_2 + kx_2y_1 - 3kx_2y_2$ $\boxed{\checkmark}$

$= k\langle x, y\rangle$

So the product is linear.

2. Symmetry

? $\langle y, x\rangle = (2y_1 + y_2)x_1 + (y_1 - 3y_2)x_2 = (2x_1 + x_2)y_1 + (x_1 - 3x_2)y_2 = \langle x, y\rangle$ $\boxed{\checkmark}$

So the product is symmetric.

3. Positive definite property

Let us take any vector, for example $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Then $\langle x, x\rangle = 2*2*2 + 2*1 + 1*2 - 3*1*1 = 9 > 0$. $\quad 2x_1^2 + x_1x_2 + x_2x_1 - 3x_2^2$

If x is a zero vector, then $x_1 = x_2 = 0$ and so $\langle x, x\rangle = 0$. $\boxed{\checkmark}$ $\quad 4 + 2\cdot 2\cdot 1 - 3 = 4+4-3 =$

So the product is positive definite.

Therefore the product is an inner product.

Solution

1. Linearity is correct since $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$
$k\langle u, v\rangle = \langle ku, v\rangle$

2. Symmetry $\langle y, x\rangle = 2y_1x_1 + y_1x_2 + y_2x_1 - 3y_2x_2$

$\langle y, x\rangle = y^TAx \quad (y_1, y_2)\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (2y_1 + y_2,\ y_1 - 3y_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

$= (2y_1 + y_2)x_1 + (y_1 - 3y_2)x_2$

$= 2x_1y_1 + y_2x_1 + x_2y_1 - 3y_2x_2$

$= \langle x, y\rangle$

3. Positive definite

$\langle x, y\rangle = 2x_1y_1 + x_1y_2 + x_2y_1 - 3x_2y_2$

$\langle x, x\rangle = 2x_1^2 + 2x_1x_2 - 3x_2^2$

since $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow 2\cdot 2^2 + 2\cdot 2\cdot 1 - 3\cdot 1^2 = 8 + 4 - 3$

$= 9$

$\det(A) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} = -6 - 1 = -7$ is negative

Since the formula (*) is not satisfy the property 3 as $\det(A) \geq 0$ to be a inner product. Therefore it is not inner prod

It appears that this student's single concern was to verify if the product defined in the text is an inner product, without bothering to examine the validity of the reasoning that led to this result. To arrive at his conclusion, he obviously used deduction and did not rely on empirical evidence. His proof scheme is analytical in Harel and Sowder's sense (1998), and, in fact, when making our assessment decisions, we may well give this student the benefit of the doubt, and assume that, since he's refuting the conclusion of the proof, he must be considering the proof by example unacceptable, and give him full points for identifying the proof as flawed. However, the student's solution does not appear to involve much theoretical thinking. Let us point to some "signs" that led us to this conclusion:

- to establish correctness of the linearity of the product he had to rewrite the statement of this property using the notations $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}$, because these were probably the symbols used by the instructor when she first introduced the notion of inner product. This shows his lack of flexibility in using formal notations;
- he studied the symmetry and the positive definite property of the product by rewriting the product in the form of matrix multiplication, and by checking the properties of the matrix. The actions he took were not only superfluous, but also required little, if any, comprehension of the actual meaning of the axioms defining an inner product. The properties of the matrix **are** the properties of the inner product, just as the positive result of the multiplication of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with itself **is** the positive definite property. He put a check mark for all three parts of the given proof, as if he accepted it as correct, and at the same time reached a different conclusion in his own proof. The meanings of both the given proof and of his own proof of the positive definite property were sought in material reference. Had he involved his systemic thinking, his only reference for

establishing validity would have been the definition of the positive definite property, and he would have judged the two "proofs" in relation to this definition, thus sensing the contradiction between the two results. We marked in red the contradicting claims in his solution.

- on the whole, his approach to solving this problem is that of a proactive practical thinker, who prefers undertaking action and checking the properties himself, instead of reflecting on the given proof.

Sixteen students, as we have mentioned, had a similar approach to this problem, exhibiting analytical proof schemes, while not engaging in theoretical thinking. They seemed to have mastered a discourse or a "genre" on a superficial level, but not the underlying intellectual attitude. We believe that the quizzes can prove to be a powerful tool for developing theoretical thinking in precisely this kind of situations, where students have the practical abilities needed to solve the problem, but they do not control them through theoretical thinking. The teacher can transform such situations into opportunities for cultivating certain habits of the mind either through assessment – for instance by subtracting points for such answers that do not give a precise answer to the question – or simply by using them as starters of meta dialogues about the mathematics they contain. This kind of communication can be done through written comments on their solutions, or by commenting, in class, about the most common mistakes done in the quiz.

Students' scores with regard to the behavior *Systemic / definitional / in proving* in this problem is synthesized in the following table:

TB	Sys/def/in proving		
Score	1	0	-1
Number of students	18	16	17
% of the total	35%	31%	33%

Table 2: Students' behavior in Quiz 1 – Question 2.

There appears to be an almost equal share of answers in each category: rejection of the proof on the grounds that it is based on an example (coded *1*), rejection of the inner product, without saying exactly what is wrong with the proof (coded *0*) an acceptance of the proof (coded *-1*). This may already suggest a distribution of our students' tendencies to think theoretically, at least with regard to this feature of TT: one third of the sample has a definite tendency to think theoretically, while one third is rather practically inclined. The remaining third does not exhibit an explicit tendency to think either theoretically or practically.

Quiz 2 – Question 1: Dimension of orthogonal complement

The second quiz had as main subject the notion of orthogonality. In particular, the first question referred to orthogonal subspaces:

> *Let V be an inner product space of dimension* **n**, *and let* **W** *be its subspace of dimension* $\mathbf{k}, \mathbf{k} \leq \mathbf{n}$. *What is the dimension of the orthogonal complement of W? Justify.*

Analysis of the question

To reach the conclusion that $\mathbf{dimW^{\perp} = n - k}$, one could reason as follows:

W is a subspace of **V**, and $\mathbf{W^{\perp}}$ is its orthogonal complement so: **V** is a direct sum of **W** and $\mathbf{W^{\perp}}$. Since **V** is a direct sum of **W** and $\mathbf{W^{\perp}}$, $\mathbf{dimV = dimW + dimW^{\perp}}$ hence $\mathbf{dimW^{\perp} = dimV - dimW = n - k}$.

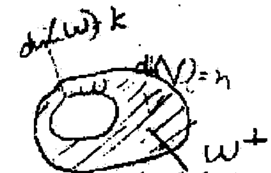
This is a question that requires explicitly the justification of the result. If, when writing their proofs, students referred to definitions in deciding about the meanings of the terms involved in the problem (i.e.: dimension, subspace, orthogonal complement, etc.) their answers would be coded with **Systemic / definitional / in proving / 1.** Otherwise, if their approach to meanings is not theoretical, their solutions would be coded with **-1** on this feature.

The dimension of a vector space is equal to the sum of the dimensions of its subspaces only under the assumption that the vector space is a direct sum of its subspaces. If one thinks hypothetically, one will be aware that under different circumstances, i.e. **V** not being a direct sum of its subspaces, this statement is no longer true and thus one would be careful to make explicit the condition under which this result holds. On the other hand, if one's thinking is not bound by a conceptual system, truths are facts, not conditional statements. In this frame of mind, one remembers a theorem in a practical manner, concerned only with its "final product" (for example, a formula), and not with the conditions under which its conclusions are true. We used the code **Systemic / hypothetical / assumptions in theorems** to classify students' answers with regard to their engagement with hypothetical thinking for solving this problem. Following our empirical analyses a student's solution would be coded **1** with regard to this feature if the dimension of the orthogonal complement was justified by the fact that the vector space is a direct sum of its subspaces and with **-1**, if the result was not justified. Of course, we also expected that some students would not know any of the theorems needed to solve the problem, but for those cases we wouldn't be able to decide whether or not they engaged with hypothetical thinking. Such solutions would be coded **0** on this feature.

We also looked at students' language in their answers to this question, in particular to their correct use of mathematical terms. Sensitivity to terminology is more than just a "definitional" or theoretical approach to meanings. It entails referring to correct definitions and true theorems, not just to any theoretical (as opposed to "everyday") meanings in general. The code **Analytic / linguistic / terminology** was added the value **1** for students' solutions that employed correct terminology and meanings, and **-1** for answers that used incorrect terminology or meanings.

Analysis of students' responses

Indeed, as we had expected, many students – 20 or 40% of the 49 students answering this question – came up with the answer **n – k** for the dimension of the orthogonal complement of **W**, failing, nonetheless, in one way or another, to provide the good justification for it. Let us first look at this student's solution:



Question 1 (1 mark)

Let V be an inner product space of dimension n, and let W be its subspace of dimension k, $k \leq n$. What is the dimension of the orthogonal complement of W? Justify.

Solution

$V, \langle \cdot,\cdot \rangle$ $\dim(V)=n$ W is a subspace of V $\dim(W)=k$ $\boxed{k\leq n}$

$W^{\perp} = \{v \in V \mid \langle v,w \rangle = 0 \text{ for } w \in W\}$

If $\underline{k<n}$ then $\dim(W^{\perp}) = \dim(V) - \dim(W)$

$\dim(W^{\perp}) = \underline{n - k}$

Her diagram is more appropriate for representing a set-theoretic union of sets with an empty intersection. This doesn't mean, however, that she is consciously thinking within the set-theoretical conceptual frame. “Dimension”, for her, could be no different than the everyday notion of size of a collection of objects, perhaps even the number of elements in the collection. This understanding still yields the correct value for the dimension, but not a correct justification of this result. Eight students, out of the twenty giving the result without the correct justification, seemed to have a similar misunderstanding of the notion of dimension of a vector space. We coded their answers with *Systemic / definitional / in proving / -1*, which is a code we have used before when students looked for meaning in some material reference, instead of theorems or definitions. Such material reference can also be found in the following solution of a student, who, besides, doesn't even provide the correct answer:

Solution

$V \rightarrow \dim n$

$W \rightarrow \dim k \Rightarrow k \leq n$

If W is a subspace of V and that $\dim(W) = k \leq \dim(V) = n$

the dim of the orthogonal complement of W has to be

$k \leq \dim W^{\perp} \leq n$. it has to be equal or greater than k since a subspace is the smallest. and $\leq n$ since the inner product is of dim n.

[>
[>
>

Question 2 (2 marks)

This solution reveals the misconception that the dimension of the orthogonal complement must be greater or equal to the dimension of the subspace. However, the discriminating factor between the two solutions we have included above is the linguistic sensitivity of their authors. The first one contains good mathematical "grammar" and the confusion between direct sum and union of sets still makes sense, i.e. it is still grammatically correct (it is thus coded *Analytic / linguistic / terminology / 1*). In the second solution, on the other hand, there are two serious grammatical mistakes:

- the expression "a subspace is the smallest" (probably coming from the definition of a span of a set S, as " the smallest subspace containing S");
- "inner product is of dim n" (confusion between inner product and inner product space).

We coded it thus with *Analytic / linguistic / terminology / -1.* Both solutions, on the other hand, were coded with *0* on *Systemic / hypothetical / assumptions in theorems* since, as we have argued above, they failed to provide the good justification for other reasons than that of the non-involvement with hypothetical thinking.

Solutions similar to the one we include below (given by 12 students in our group), where the equality $\mathbf{dimV} = \mathbf{dimW} + \mathbf{dimW}^{\perp}$ appears as unconditionally true, were coded with *Systemic / hypothetical / assumptions in theorems / -1*:

$$\text{Dim } W^{\perp} = n - k$$
$$\text{Dim } V = \text{Dim } W + \text{Dim } W^{\perp}$$
$$n = k + \text{Dim } W^{\perp}$$
$$\text{Dim } W^{\perp} = n - k$$

Twenty-five students – 51% of the 49 students answering this question – gave the complete answer, similar to the one we have mentioned in our a-priori analysis. A significant part of the group (40%) provided the correct answer, but lacked the correct justification. Three reasons accounted for their incorrect (or absent) justifications:

(1) the "concrete" reference for the meanings of some of the terms involved in the problem (*Sys/def/in proving/-1*);

(2) the using of a result of a theorem as an unconditionally true fact (*Sys/hyp/assumptions in thms/-1*);

(3) the lack of linguistic sensitivity (*Analytic/ling/terminology/-1*);

There were 4 students (9% of the group) who didn't give the correct answer for one of the reasons (1) or (3) above. We present these results in the table below:

TB	Sys/def/in proving			Sys/hyp/assumptions in thms			Analytic/ling/terminology		
Score	1	0	-1	1	0	-1	1	0	-1
Number of students	16	15	18	25	12	12	35	8	6
% of the total	33%	31%	37%	51%	24%	24%	71%	16%	12%

Table 3: Students' behavior in Quiz 2 – Question 1.

Quiz 2 – Question 2: Norm of the sum of two vectors

The second question was formulated as follows:

> *Let* $\mathbf{V}, <.,.>$ *be an inner product space, and let* $\mathbf{u}, \mathbf{v}$ *be in* $\mathbf{V}$. *What is enough to assume for the following equality to hold :*
> $\mathbf{norm(u + v)^2 = norm(u)^2 + norm(v)^2}$
> *Justify your answer.*

Analysis of the question

We expected essentially two kinds of solutions for this problem:

(1) One approach, considered more likely, would be:

(a) to find, through a series of equivalent transformations of the equality, a necessary and sufficient condition, namely: $<\mathbf{u}, \mathbf{v}>= \mathbf{0}$ or **u** orthogonal to **v**.

(b) and then to state, in conclusion, that "it is enough to assume that <**u, v**> = **0** (or **u** is orthogonal to **v**)".

(2) Another approach would be to give some possible sufficient condition, e.g. one of the vectors equal to **0**, and then verify that the equality holds under this assumption.

A theoretical behavior for us in this question would be the association of the word "enough" with the requirement to provide a sufficient condition for the given equality to hold. This ability to "translate" the meaning of everyday words in the mathematical register entails meta-linguistic sensitivity, specific to analytic thinking. It is perhaps not the very translation of "enough" by "sufficient condition" that we call meta-linguistic sensitivity, but rather one's ability to recognize this word as a focal point in the problem, dictating a mathematical endeavor for solving the problem sensibly different then if it were replaced by another adverb. This realization then is completed with the carefulness to use one's own language in the same sensitive manner. For instance, a student having such an analytical approach to language will use implication and equivalence symbols such as: $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$ so that they point to the correct direction of the implication they want to express. We coded students' behavior on this question with **Analytic / meta-linguistic / necessary-sufficient distinction** with a fourth coordinate added as follows:

1 was assigned to solutions of the type 1 or 2.

-1 was assigned to solutions of the type 2a with b replaced by a statement of necessity, such as, "we have to assume that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{0}$ (or **u** is orthogonal to **v**)", or, in general, to solutions stating some necessary condition (not necessarily deduced from a reasoning of the type 2a). Solutions containing only the reasoning 2a with no statement of sufficient or necessary conditions, or, in general, solutions without explicit response to the question were coded **0** with regard to this feature.

We don't believe that the students scoring -1 on this feature would not in general understand the logical distinction between a necessary and sufficient condition, nor that they simply do not associate the word "enough" with "necessary condition". Ferrari (2004) shows that advanced students have different linguistic behaviors depending on the context they are situated in. For instance, in one problem they would interpret correctly the notions of integer and of real numbers, understanding that any integer is a real number as well, while in a more complex context, where other parts of the question are considered more important and difficult, they tend to interpret the notions of "real" and "integer" according to their colloquial use, in which the combined use of the two words may suggest that "real" should mean "non-integer". So, and this is our claim too with regard to our students' interpretation of the word "enough":

"[...] the interpretation of a text is hardly a plain translation (based on vocabulary and grammar), but involves the context the text is produced within. [T]he interpretation of texts is an enterprise which required reader (or hearers) to play an active role, performing some inferences, recognizing some part of the text as essential and focusing on them."(p.386).

The ability to switch between what he calls, with terms borrowed from functional linguistics, the colloquial register and the literate register, as well as to recognize key-words in the organization of the problem is what we characterize as meta-linguistic sensitivity.

Students were also expected, in their solutions, to understand terms such as "inner product space" and "norm" according to their general meanings and not as referring to some concrete examples, e.g. the Euclidean plane. We used the code **Systemic / definitional / in proving** to characterize students' behavior from this point of view, as follows:

1 was a assigned to solutions that contained an understanding of the norm of a vector **u** as **<u, u>**;

-1 was assigned to solutions that used some concrete example of norm.

Analysis of students' responses

Many students failed to interpret the word "enough" as the requirement to provide the sufficient condition for the equality to hold. We coded 26 answers with *Analytic / meta-linguistic / necessary-sufficient distinction / -1.* This student, for instance, proved such a lack of meta-linguistic sensitivity by approaching the problem as if he was required to produce the necessary condition:

We have to assume u is orthogonal to v.

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

$$\langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle$$

$$\cancel{\langle u,u\rangle} + \langle u,v\rangle + \langle v,u\rangle + \cancel{\langle v,v\rangle} = \cancel{\langle u,u\rangle} + \cancel{\langle v,v\rangle}$$

$$\langle u,v\rangle + \langle v,u\rangle = 0$$

$$2\langle u,v\rangle = 0$$

Page 1

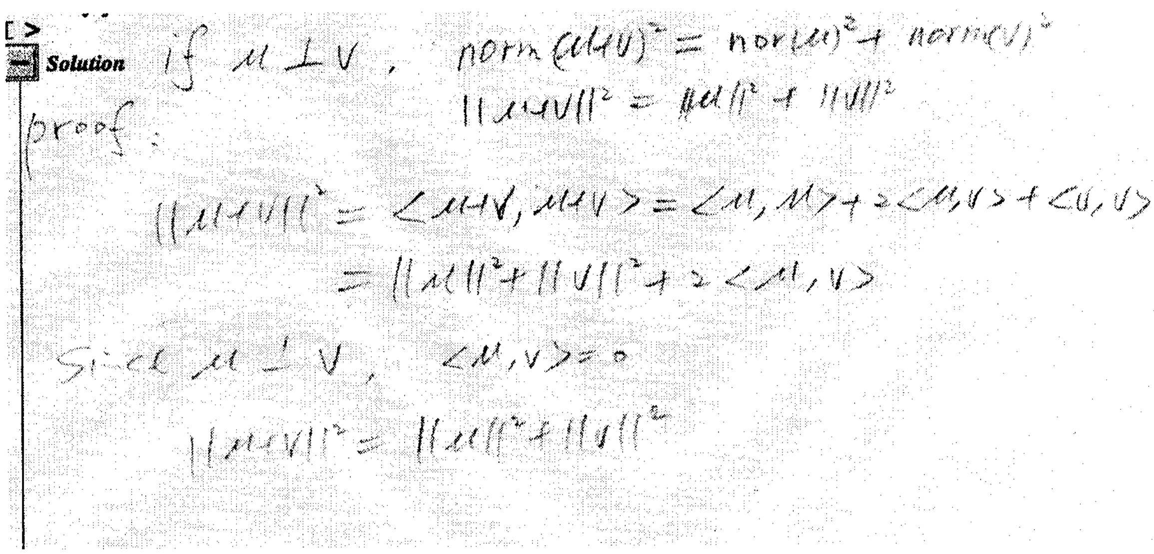
$$\therefore \langle u,v\rangle = 0$$

∴ u is orthogonal to v

The verb "have to" in his proof reveals his misunderstanding. Indeed, it so happened that the condition was both sufficient and necessary but, as we have mentioned, around half of the students who solved the problem did not notice this fact. Their focus in solving the problem was on applying the norm on those vectors and obtaining some condition on **u** and **v**, never mind if it's the necessary or the sufficient one, concerned only with how these vectors "must", "need" or "have" to be.

Eighteen students either provided exactly what the problem asked for – the sufficient condition, or produced a set of equivalent statements to finally conclude that the equality holds if and only if the vectors are orthogonal, hence the orthogonality of the vectors is enough for the equality to hold. Below are examples of these other two approaches undertaken by our students, demonstrating their meta-linguistic sensitivity:

- sufficient condition:



- sufficient condition deduced from equivalence:

[>

Solution

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$
$$\langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle$$

$L.H.S = \langle u,u+v\rangle + \langle v,u+v\rangle$ linearity

$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$ linearity

$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$ symmetry

Thus

$L.H.S. = R.H.S.$ iff $2\langle u,v\rangle = 0 \rightarrow \langle u,v\rangle = 0$

which is true when u and v are orthogonal

Page 1

So it is enough to assume that u & v are $\perp$

Their answers were coded with *Analytic / meta-linguistic / necessary-sufficient distinction / 1.*

As a side remark we wish to note that these two latter approaches were graded as 100% correct, while the approach in the first example received 75% of the points. The instructor also wrote short comments on the returned quizzes and made general remarks in class regarding the most common mistakes she encountered as she corrected them. This way the quizzes acquired their lever-like character for enhancing students' inclination to think theoretically.

Besides the two categories of answers we have described above, for six students we couldn't decide whether they made the correct distinction between the necessary and the sufficient condition and we coded their answers with *0* on the aforementioned theoretical behavior. Only two students had a concrete reference for the conditions under which the equality in the problem holds. Namely, they referred to the Pythagorean theorem, not only failing to realize that they are considering the particular space of $\mathbf{R}^2$ equipped with the dot product, but also not even "translating" the familiar theorem in terms of properties of vectors.

Solution

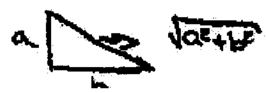
$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

will be true if

we have a rectangular triangle (90° angle)

because this equality is Pythagoras

equality.



We coded these two answers with *Systemic / definitional / in proving/ -1*. Most of the students in our group (90%) appeared to have a general notion of norm, without limiting themselves to the familiar Euclidean plane, or at least not exhibiting explicitly such a limitation. However, only one third of the group noticed the more subtle requirement of the problem, namely that of producing a sufficient condition for the given equality to hold, while half of them explicitly confused it with a necessary condition. In the table below we summarize our students' behavior in this question:

TB	Analytic/meta-ling/N-S			Sys/def/in proving		
Score	1	0	-1	1	0	-1
Number of students	18	8	26	47	3	2
% of the total	35%	15%	50%	90%	6%	4%

Table 4: Students' behavior in Quiz 2 – Question 2.

Quiz 3 – Question 1: Definition and product of orthogonal matrices

The third quiz was on the subject of orthogonal matrices. The first question required one to know the definition of orthogonal matrices: to be reproduced in the first part of the question and to be used for solving the second part of it. We include its text below:

(a) Define the notion: Orthogonal matrix;

(b) Prove that the product of two $\mathbf{n} \times \mathbf{n}$ *orthogonal matrices is an orthogonal matrix.*

Analysis of the question

Either one of the following three conditions on an $\mathbf{n} \times \mathbf{n}$ real matrix could be accepted as defining an orthogonal matrix $\mathbf{A}$:

(Def. 1) $\mathit{transpose}(\mathbf{A}) \cdot \mathbf{A} = \mathbf{I}$.

(Def. 2) $\mathit{transpose}(\mathbf{A}) \cdot \mathbf{A} = \mathbf{A} \cdot \mathit{transpose}(\mathbf{A}) = \mathbf{I}$.

(Def. 3) $\mathbf{A}$ is non-singular and $\mathit{inverse}(\mathbf{A}) = \mathit{transpose}(\mathbf{A})$.

The textbook section on orthogonal matrices starts by assuming that all matrices are real and that "inner product space" and related concepts will always refer to the Euclidean space $\mathbf{R}^n$ with dot product. Since complex matrices are not studied in the course and the notion of orthogonal matrix is not generalized to Hermitian matrix, students are not expected to even notice these assumptions. Therefore, we did not plan to discriminate between solutions where the assumption of the matrix being real was made and those where it wasn't.

The textbook defined an orthogonal matrix using both (Def. 2) and (Def. 3), quoting (Def. 2) as saying the same as (Def. 3), "in other words". (Def. 1) and (Def. 2) do not require the assumption of **A** being non-singular; this property of orthogonal matrices follows directly from the defining condition.

The equivalence of (Def. 1) and (Def. 2) is never really proved in linear algebra courses, but it is used all the time by instructors in practice; in general, they do not bother proving that, say, **BA** = **I**, once they have proved that **AB** = **I**. It would be a significant symptom of theoretical thinking in a student if he or she wondered if this instructor's behavior was actually mathematically well-founded. On the other hand, following the instructor's behavior without questioning its discrepancy with the textbook definition, is a symptom of a (perfectly legitimate and efficient) practical thinking. It would be impossible to tell, just from the students' responding by (Def. 1) to part (a) of the question, if it is based on well-founded knowledge of the equivalence, or just on the observed usage, or simply on poor memory of the conditions given by the textbook. Therefore, (Def. 1) answers have to be accepted as correct but they cannot serve as a basis of any statement about students' theoretical behavior with respect to its reflective or systemic aspects.

On the other hand, the accuracy of quoting (Def. 3) could be used to decide about an aspect of theoretical thinking. If non-singularity of the matrix **A** is mentioned then this can be interpreted as attention being paid to assumptions in definitions and coded as **Systemic / hypothetical / assumptions in definitions / 1**. The suffix **-1** would be assigned to solutions using (Def. 3) without the assumption of non-singularity, and **0** to all other solutions, including solutions using (Def. 1) or (Def. 2), and lack of response.

But we were aware that students could respond to part (a) with neither of the above defining conditions 1 - 3, using, instead, the theorem, which, in the textbook, immediately follows the definition, and characterizes orthogonal matrices as matrices the set of whose rows (or columns) forms an orthonormal set. This could be a symptom of a flaw in theoretical thinking, namely a lack of sensitivity to the distinction between a definition and a theorem in mathematics. Therefore, solutions quoting the "orthonormal set of columns (or rows)" characterization would be assigned the code **Analytic / Meta-linguistic / def-thm distinction / -1**. The **1** suffix would be assigned to responses containing any one of the defining conditions 1 - 3, even if the assumption of singularity were missing. Suffix **0** would be assigned to all other solutions.

Thus, in summary, the following coding scheme is planned to be used in analyzing solutions in part (a):

- depending on the status of the statement given:

Analytic / meta-ling / def-thm / 1 if only a definition is given

Analytic / meta-ling / def-thm / -1 if there is an answer but it does not contain only a definition

- depending on the mentioning of the assumptions in the statement:

Systemic / hyp / assumptions in def / 1 if Def. 3 is given and the assumption of non-singularity is mentioned

Systemic / hyp / assumptions in def / -1 if Def. 3 is given without the assumption of non-singularity.

In part b) of the problem, to prove that the product of two matrices, say **P** and **Q**, is orthogonal, one could multiply **PQ** with its transpose, $(\mathbf{PQ})^T$, and show, based on the orthogonality of **P** and **Q**, that their product is equal to the identity matrix, as follows:

$\mathbf{PQ}\cdot(\mathbf{PQ})^{\mathrm{T}} = \mathbf{PQQ}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}$ and since **P** and **Q** are orthogonal, hence $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ and $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$:
$\mathbf{PQ}\cdot(\mathbf{PQ})^{\mathrm{T}} = \mathbf{PQQ}^{-1}\mathbf{P}^{-1} = \mathbf{I}$. It follows that $(\mathbf{PQ})^{\mathrm{T}} = (\mathbf{PQ})^{-1}$, and thus **PQ** is orthogonal.

In analyzing students' solutions we would first discriminate between the proofs that referred to theory for establishing meanings and those employing concrete examples as basis for justification. We planned to use the following scheme of coding students' answers:

Systemic / definitional / in proving / 1, if the proof is based on reference to a general characterization of orthogonal matrices, whether a definition or a theorem.

Systemic / definitional / in proving/ -1, if the argument is based on a concrete example, but also, more generally, if there is no explicit reference to some characterization of orthogonal matrices.

However, the theorem, which characterizes orthogonal matrices as matrices the set of whose rows (or columns) forms an orthonormal set, while being very practical from the point of students interested mostly in solving computational exercises of verifying or constructing concrete orthogonal matrices, is not very practical in "proof" exercises, such as the one in part (b). Attempting to use this characterization over the definition in part (b) is, therefore, a symptom of an "im-practical understanding of theory". Practical understanding of theory is something quite different than everyday practical thinking, and should count, indeed, as a feature of theoretical thinking. We could say that it involves, in this particular case, an awareness of the existence of alternative ways of talking and writing about matrices and flexibility in choosing the most appropriate language depending on the problem at hand. In Definitions 1 - 3, matrices are written and operated upon as elements of the algebra of matrices (this was called a "structural understanding of matrices" in Sierpinska, 2000); in the "theorem" - as arrays of numbers ("arithmetic" understanding). Choosing to use the "structural" language in the proof of (b) could thus be regarded as a symptom of a meta-linguistic sensitivity. Thus, we decided to discriminate solutions to part (b) also on the basis of the feature **Analytic / meta-linguistic / structural understanding of matrices**, with the suffixes **+1** and **-1** assigned as follows:

+1, if matrices are operated upon as elements of an algebra

-1, if matrices are operated upon as arrays of numbers.

Analysis of students' responses

a) The definition of orthogonal matrices

28 students – representing 56% of the group – when answering the first part of the problem, wrote the theorem characterizing orthogonal matrices, instead of the definition. One category of answers contained only the theorem, in one form or another:

a. as a necessary condition for the orthogonality of matrices:

Solution a) The inner product of 2 rows of an orthogonal matrix must be equal to zero and each row must be orthonormal (the inner product of the row with itself-norm- must be equal to 1)

b. as an equivalent condition:

Solution
a) a matrix P is said to be orthogonal if and only if its rows (or columns) are orthonormal in respect to the dot prod

Yet another typical behavior, symptomatic of the lack of meta-linguistic sensitivity inside this group, was that of writing everything one knows about orthogonal matrices. The defining property of orthogonal matrices, i.e. transpose equal to inverse, and the property stated in the theorem, i.e. columns (or rows) forming orthonormal sets, were not even understood as equivalent conditions, but as independent conditions, both defining orthogonality for matrices. This understanding is revealed in this student's answer:

A matrix is said to be orthogonal if the set of its columns (rows) is orthonormal. Moreover, a matrix P must be non-singular and $P^T = P^{-1}$ for P to be orthogonal.

The above excerpts from students' solutions, representative for the behavior of more than half of the group, demonstrate that our students had the practical knowledge to give a correct answer, but failed to act selectively towards it. They either didn't distinguish between the form of a definition and that of a theorem, or didn't consider this distinction important. Such answers were assigned the code *Analytic / meta-ling / def-thm / -1.* TT approaches knowledge quite differently: it is not only sensitive to the different formats of and roles played by various types of mathematical statements, i.e.: axioms, definitions, theorems, proofs, examples, etc., but it is also careful to express the distinctions it senses through language.

Of the 32 students who referred to the condition "transpose equal to inverse" in their answers to part a), 22 failed to denote, in their attempts to define orthogonal matrices, the class of objects to which this definition applies, namely non-singular matrices. Let us look at this solution of a student, for instance:

Solution

a) An orthogonal $(n \times n)$ matrix is a matrix where if we express the columns of the matrix as vectors (ie $a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$) then each of these vectors are perpendicular to each other. $a_i \perp a_j$ $i \neq j$. Also $A^{-1} = A^T$

We coded it as follows:

Analytic / meta-ling / def-thm / -1, since it contains more than the definition of orthogonal matrices (it also contains the characterizing theorem we have talked about, but only partially, referring to the orthogonality instead of the orthonormality of the set of column vectors).

Systemic / hyp / assumptions in def / -1, since it mentions the condition "transpose equal to inverse" without mentioning the non-singularity assumption.

Only five students managed to correctly define orthogonal matrices, scoring *1* on both codes that we have set to signal theoretical thinking in this question.

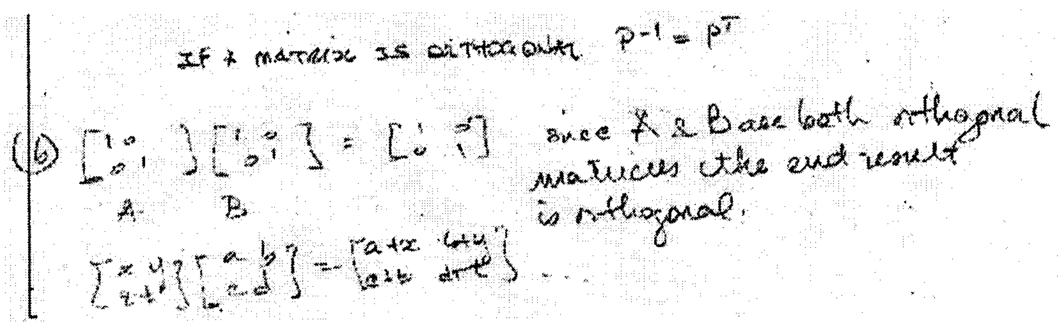
b) Proving that the product of two orthogonal matrices is an orthogonal matrix

The text of the problem, we thought, may trigger in our students' minds the idea of generality because of the reference to $\mathbf{n} \times \mathbf{n}$ matrices, and consequently make them prove the statement in general and not by giving concrete examples of orthogonal matrices. Indeed 31 students, so more than half of the group did not resort to empirical evidence to justify the statement in point b), and their answers were all coded accordingly with *Systemic / definitional / in proving / 1.* 25 of them succeeded in their validating attempts producing the solution that we included in our a-priori analysis of the question. The other six students, while still thinking in general terms, seemed to have encountered a different kind of difficulty more peculiar to the domain of linear algebra. Let us present one of these three almost identical approaches:

(a) A matrix P is orthogonal if its rows (columns) are orthonormal. ($P^{-1} = P^T$; $P^T P = I$)

(b)
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn}\end{bmatrix} = \begin{bmatrix} (a_{11}b_{11}+\cdots+a_{1n}b_{n1}) + \cdots + (a_{11}b_{1n}+\cdots+a_{1n}b_{nn}) \\ \vdots \\ (a_{n1}b_{11}+\cdots+a_{nn}b_{n1}) + \cdots + (a_{n1}b_{1n}+\cdots+a_{nn}b_{nn})\end{bmatrix}$$

This student constructs the two matrices in an $\mathbf{n} \times \mathbf{n}$ dimension and uses notations that do no lose sight of generality, demonstrating systemic thinking, but obviously his approach is not successful. On the other hand as illustrated in point a) of his answer, he has the necessary knowledge to solve the problem, yet he doesn't use it. We think that he understands matrices essentially as tables of numbers, while this problem requires an understanding of matrices rather as elements of the set $\mathbf{M}_{\mathbf{n}\times\mathbf{n}}$ of $\mathbf{n} \times \mathbf{n}$ matrices, satisfying a certain property, i.e.: they are nonsingular matrices with equal transpose and inverse. To solve the problem, one needs to manipulate matrices as such almost concrete entities, having certain properties, without caring much (or at all) about their "internal" appearance. The two different understandings of matrices correspond to two modes of thinking in linear algebra described in Sierpinska (2000): the analytic-arithmetic mode, corresponding to the language of vectors as lists and tables of numbers and the analytic-structural mode of vector spaces and linear transformations. We believe that it was the reluctance to enter in the structural mode of thinking that hindered these students' success in this problem, despite their legitimate approach to solving it. This is an indicator for us of a lack of flexibility to move between the different languages of linear algebra, and we coded it with *Analytic / meta-linguistic / structural understanding of matrices / -1.* We suspect, in fact, a similar obstacle for the seventeen students who used examples to prove the statement in part b) and whose solutions were coded with *Systemic / definitional / in proving / -1.* What leads us to this conclusion is the fact that almost all of them mentioned in their answers to part a) of the problem the defining property of orthogonal matrices – or as they put it: $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ – but failed to use it to construct a relatively simple proof. Instead they all focused on the entries of the matrix, as it was the case for the student whose solution we reproduce below:



From the his attempt to generalize the result to 2 by 2 orthogonal matrices, we could infer that he did have in mind the idea of generality, but he abandoned it and replaced it with the three dots. In other words, he had the knowledge he needed to prove the claim, he also had the "good" intention to prove it in general, yet he ended up by showing an example of orthogonal matrices whose product is orthogonal. We dare to conjecture that his engagement with a proof by example was triggered by his resistance to leaving the arithmetic mode of reasoning about matrices. This means that he doesn't hold an empirical proof scheme (in the sense of Harel and Sowder, 1998): he does not, in general, obtain conviction from empirical evidence. In fact, it was interesting to see that he was one of the students who found the proof by example of the positive definite property in the first quiz unacceptable and provided a deductive proof. Proving by example then in linear algebra is for him perhaps just a coping behavior, prompted by the barriers he encounters in understanding and using linear algebra concepts.

We have two tables containing the statistics of students' engagement with theoretical thinking, corresponding to the two parts of the problem:

TB	Analytic/meta-ling/def-thm			Sys/hyp/assumptions in def		
Score	1	0	-1	1	0	-1
Number of students	28	4	18	22	18	10
% of the total	56%	8%	36%	44%	36%	20%

Table 5: Students' behavior in Quiz 3– Question 1a.

Sys/def/in proving			Analytic/meta-ling/matrices-structural		
1	0	-1	1	0	-1
31	2	17	25	2	23
62%	4%	34%	50%	4%	46%

Table 6: Students' behavior in Quiz 3– Question 1b.

Quiz 3 – Question 2: 2×2 orthogonal matrices

The second problem in the quiz was on the same subject, but "downsized" now to a more particular domain, that of 2×2 orthogonal matrices:

Find the number and exhibit all $\mathbf{2 \times 2}$ *orthogonal matrices of the form* $\begin{bmatrix} x & y \\ z & \frac{1}{2} \end{bmatrix}$.

Analysis of the question

The set of conditions that imposed on *x*, *y* and *z* give their values for which the matrix is orthogonal, can be easily derived from the theorem stating that orthogonal matrices have columns (respectively rows) forming orthonormal sets. Generating this set of conditions thus depends upon the reference to theory and on the understanding of its implications. Relying on theory to establish validity is a feature of systemic thinking, and we characterized the corresponding behavior with the same code we have used before to signal this feature of thinking in students' answers:

Systemic / definitional / in proving / 1, if the conditions to be satisfied by *x*, *y* and *z* are derived from the definition or the properties of orthogonal matrices;

Systemic / definitional / in proving / -1, otherwise, for instance if some examples of such matrices are given.

The following set of conditions is obtained if one uses the fact that the given matrix multiplied with its transpose is equal to the identity matrix:

$$(*)\begin{cases} x^2 + z^2 = 1 \\ xy + \dfrac{1}{2}z = 0 \\ y^2 + \dfrac{1}{4} = 1 \end{cases}$$

One needs to think of these conditions as forming a system, and explore *all* the possible values for x, y and z *within* this system. A correct solution to this system must satisfy two conditions: the matrices found should all be orthogonal, and all possible such matrices should be found. We used the code **Systemic / hypothetical / all possible cases** to discriminate between students who searched systematically for all hypothetically possible cases (suffix **+1**) and those who were satisfied with finding or guessing just a few (suffix **-1**). But once a candidate was found, with whatever approach, theoretical thinking would compel one to verify if the result is, indeed, an orthogonal matrix. This entails a concern, from the part of the student solving the problem, with the validity of the results obtained. The code **Systemic / proving / concern with validity** was added a fourth coordinate, based on students' answers, as follows:

1, if the matrices obtained were orthogonal (regardless if all or only some of them were found;

-1, if at least one of the matrices obtained is not orthogonal.

In other words, the **-1** suffix, added to the two codes we have considered in this question, corresponds to two kinds of wrong solutions:

Systemic / hypothetical / all possible cases / -1 corresponds to too few matrices found;

Systemic / proving / concern with validity / -1 corresponds to too many matrices found.

Analysis of students' responses

Of the 47 students answering the question, 40 tried to obtain the conditions for *x, y* and *z* by deducing them from theory. Only 7 students came up directly with an example, scoring thus *-1* on *Systemic / definitional / in proving*. A typical approach for this type of arguments is the following:

Solution

Let $x = \frac{1}{2}$

$y = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$

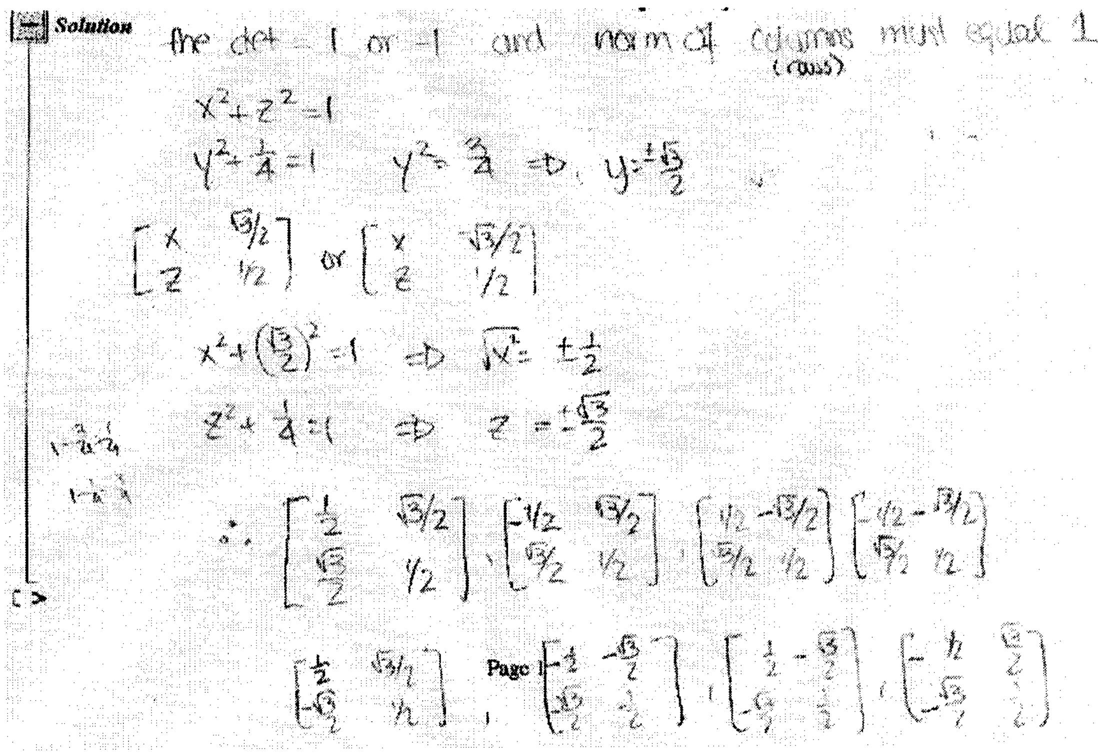
$z = -\frac{\sqrt{3}}{2}$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

They all however, produced at least one orthogonal matrix of the required form, demonstrating that they know, to some extent, the conditions to be satisfied by orthogonal matrices. One reason for this approach could be that it wasn't required of them to justify their results or that they had the practical attitude to find *some* of those matrices by trial and error, without engaging in an approach that would give *all* such matrices.

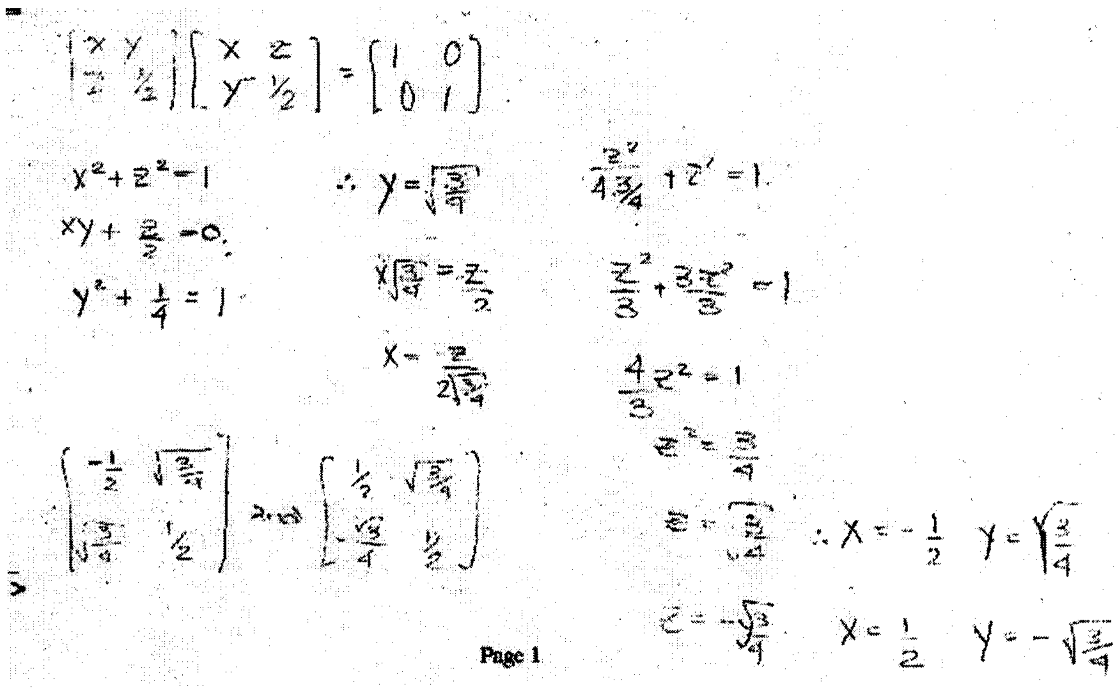
Most of the students, as we have mentioned, referred explicitly to theory and came up with the correct set of conditions to be satisfied by *x, y* and *z*. However, many of them either gave wrong examples of such matrices (answers coded *Systemic / proving /*

concern with validity / -1), or did not obtain all the matrices satisfying the premise of the problem (answers coded *Systemic / hypothetical / all possible cases / -1*). Let us exemplify the two categories of wrong solutions:



This solution does contain the behavior "search for all hypothetically possible solutions" (coded *Systemic / hypothetical / all possible cases / 1*), but shows lack of concern with the validity of the outcome (coded *Systemic / proving / concern with validity / -1*), because the student did not eliminate some of these solutions, by checking their validity (i.e., if they are orthogonal).

Answers in the second category were missing some of the four matrices satisfying the requirements of the problem. This student, for instance, did not produce all the possible (x, y, z)-triplets (behavior coded *Systemic / hypothetical / all possible cases / -1*) and thus missed two orthogonal matrices of the given form. However, both his solutions are valid (his answer is thus coded *Systemic / proving / concern with validity / 1*):


$$\begin{bmatrix} x & y \\ z & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x & z \\ y & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$x^2 + z^2 = 1$

$xy + \frac{z}{2} = 0$

$y^2 + \frac{1}{4} = 1$

$\therefore y = \sqrt{\frac{3}{4}}$

$x\sqrt{\frac{3}{4}} = \frac{z}{2}$

$x = \frac{z}{2\sqrt{\frac{3}{4}}}$

$\frac{z^2}{4\frac{3}{4}} + z^2 = 1$

$\frac{z^2}{3} + \frac{3z^2}{3} = 1$

$\frac{4}{3}z^2 = 1$

$z^2 = \frac{3}{4}$

$z = \sqrt{\frac{3}{4}} \quad \therefore x = -\frac{1}{2} \quad y = \sqrt{\frac{3}{4}}$

$z = -\sqrt{\frac{3}{4}} \quad x = \frac{1}{2} \quad y = -\sqrt{\frac{3}{4}}$

$$\begin{bmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$$

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It turned out that the students in our sample had a high failure rate in solving what we believed to be a simple system of equations. Even if most of them (85% in this question), by now, base their reasoning on definitions and general characterizations of concepts, their engagement with hypothetical thinking is still unsatisfactory: 21 of them, representing 45% of the group, failed in their solutions because they didn't explore all hypothetically possible solutions. A little more than a quarter of the group (28%) showed a lack of concern with validity by giving examples of matrices that were not orthogonal:

TB	Sys/def/in proving		Sys/hyp/all possible cases			Sys/prov/concern with validity		
Score	1	-1	1	0	-1	1	0	-1
Number of students	40	7	24	2	21	32	2	13
% of the total	85%	15%	51%	4%	45%	68%	9	28%

Table 7: Students' behavior in Quiz 3 – Question 2.

Quiz 4 – Question 1: Polynomials with matrices as roots

In the fourth week the instructor discussed and proved the Cayley-Hamilton theorem, after a preliminary part where she explained why it is not obvious at all that any matrix is a root of its characteristic polynomial. Also in relation with the Cayley-Hamilton theorem, she talked about polynomials with matrix coefficients, about matrices with polynomial entries and about matrices as roots of polynomials.

The first question in the quiz was related more to these preliminaries of the Cayley-Hamilton theorem.

> *Let $f(t)$ be a polynomial with real coefficients and let* **A** *be an $n \times n$ real matrix.*
> *Prove :*
> *If* **A** *is a root of $f(t)$ and* **B** *is similar to* **A**, *then* **B** *is also a root of $f(t)$.*

Analysis of the question

To prove that **B** is a root of *f* one could show that $f(\mathbf{B})$ is zero, with **B** written as $\mathbf{P}^{-1}\mathbf{AP}$, where **P** is a non-singular matrix. It is clear that, to solve the problem correctly, the student must:

(1) Pay attention to the assumptions made about the symbol "f(t)". One assumption we predicted our students might wrongly make is that f(t) is the characteristic polynomial of **A**. The assumption that f(t) is the characteristic polynomial of **A** could be caused either by the belief in the converse of the Cayley-Hamilton theorem, in which case there would be a failure of the meta-linguistic sensitivity to logic; or by the conviction that the symbol "f(t)", used in the context of matrices, is a name for the

characteristic polynomial (indeed, "f" was the symbol often used for the characteristic polynomial in the lectures), in which case there would be a failure of the meta-linguistic sensitivity to the symbolic character of mathematical signs. However, the question does not allow us to decide which of these causes is at play. We have only signaled, using the code **Systemic / hypothetical / assumptions of the problem**, whether or not our students assumed that f(t) is the characteristic polynomial of **A**, according to the following scheme:

1, if the student understands f(t) as any polynomial;

-1, if the student assumes f(t) to be the characteristic polynomial of A.

(2) Understand the expression "**A** is a root of f(t)" formally as follows: if $f(t) = a_o + +a_1 t + \ldots + a_n t^n$, then $\mathbf{0} = f(\mathbf{A}) = a_o\mathbf{I} + a_1\mathbf{A} + \ldots + a_n\mathbf{A}^n$, where **0** is the nxn zero matrix and **I** is the nxn identity matrix. This requires **linguistic sensitivity** to both the terminology (the meaning of "matrix as a root of a polynomial") and the formal notation. On the other, one would not even engage in writing this equation, unless one is thinking about matrices structurally. Such and understanding requires **meta-linguistic sensitivity**. This kind of understanding is not concerned with the numerical values that make up a matrix and enables the student to use matrices, in this context, as arguments of polynomial functions. We have referred to such an understanding before, as corresponding to the structural mode of thinking in linear algebra (see Sierpinska, 2000). It is also believed to be at the source of students' difficulties by Harel (2000). He found that students do not easily arrive to treating some of the concepts in linear algebra as concrete entities and this creates problems for them in using some of the abstract structures of linear algebra in problem solving settings. They engage, as a result, in

symbolic manipulations that are entirely mechanical and thus much more prone to mistakes. We believe, thus, that in this particular situation it is the *lack of meta-linguistic sensitivity*, i.e.: the inability to think about matrices structurally, that is at the source of students' *linguistic problems*, i.e. of their incorrect symbolic manipulations.

What we may see, when we analyze students' solutions, are their wrongful symbolic manipulations but underlying them is a deeper conceptual problem, stemming from an inability to move flexibly between the different languages of linear algebra.

Therefore, if we code a student's answer with **Analytic / linguistic / notation and terminology / -1** to signal the inability to manipulate matrices as inputs of polynomial functions, this also means **Analytic / meta-linguistic / structural understanding of matrices / -1.**

(3) Understand "**B** is similar to **A**" with reference to the definitional condition, i.e. the existence of an invertible matrix **P** such that: $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. We believe, however, that this reference does not necessarily require theoretical thinking: students may simply remember it as some relationship that they would usually associate with similar matrices, without being aware of the more subtle theoretical assumptions regarding the existence of **P** and invertibility of **P**. In fact, in this problem, it is enough to use this "formula" encompassing a very practical understanding of the definition of similar matrices, namely: "**A~B** means $\mathbf{P}^{-1}\mathbf{AP}$".

(4) prove that f(**B**) = $\mathbf{P}^{-1}$ f(**A**)**P**, which involves the proof, or, at least, a reference to the non-trivial fact that $(\mathbf{P}^{-1}\mathbf{AP})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}$. A simple algebraic manipulation gives this conclusion, but we were quite sure that not all students would

even feel the need to justify this piece of the argument. To discriminate between those that did and those that didn't, we used the code: **Systemic / proving / concern with validity** with the values **1** and, respectively, **–1**. In general, students do have this habit of proving their claims, but we mostly used this code to characterize situations where the results would seemed obvious to them and they wouldn't feel the need to prove them.

Thus, in all, this question tests 4 features of theoretical thinking:

Systemic / hypothetical / assumptions of the problem

Analytic / linguistic / notation and terminology

Analytic / meta-linguistic / structural understanding of matrices

Systemic / proving / concern with validity

Analysis of students' responses

Our students' first source of failure in this problem was their trying to prove that **B** is also a root of the characteristic polynomial of **A**. Many of them considered this special polynomial of which **A** is a root and tried to make deductions on the basis of the properties of this polynomial. They failed to notice that the assumptions they were making were not there. We coded the solutions of 14 students with **Systemic / hypothetical / assumptions of the problem / -1** for their attempts to prove that **B** is a root of the characteristic polynomial of **A**. Below is one such approach:

If A is a root of $f(t)$ and B is similar to A, then B is also a root of $f(t)$. if A is similar to B

Solution $\Delta_B(t) = \det(tI - B)$ $\quad \therefore B = P^{-1}AP$

$= \det(tI - P^{-1}AP) = \det(P^{-1}tIP - P^{-1}AP) = \det(P^{-1}(tIP - AP))$

$= \det(P^{-1}) \det(tI - A) \det(P) = \det(tI - A) = \Delta_A(t)$

since $\det(P^{-1}) \cdot \det(P) = 1$

$\therefore \Delta_B(t) = \Delta_A(t)$ this implies that

A and B are both roots of $f(t)$

However, only partially were these attempts successful even in this particular frame. Most of these students did not carry them to the end and engaged in some non-sense symbolic manipulations, that always seemed to be caused by their inability to think about matrices structurally (in the sense of Sierpinska, 2000) or concretely (in the sense of Harel, 2000). We wouldn't be able to illustrate this inability with a typical behavior because almost every student having such a difficulty with the understanding of matrices showed it differently in his written language. This student for instance added, in the second line of his solution, a matrix – f(**B**)Q(t) with a polynomial – R(t):

Solution

if $f(A) = 0$ AND $B \sim A$.

$f(t) = f(B)Q(t) + R(t)$

$f(A) = f(B)Q(A) + R(A)$

$0 = f(B)Q(A) + 0.$

$0 = f(B)Q(A)$

$0 = f(B)$ $\quad \therefore$ B is also a root of $f(t)$

Another student exhibits even more serious conceptual misunderstandings, by talking about the roots of the matrices **A** and **B**, and by making some bizarre deductions based on their similarity:

Solution $f(t)$ $a_n t^n + a_{n-1} t^{n-1} \dots + a_1 t + a_0$ $\{ A_{n \times n}$

$\frac{f(t)}{A} = \frac{f(t)}{B}$

if $A \sim B$

A and B are char poly of A.

If B is similar to A

Then $\Rightarrow$ B and A have the same char poly

That means that A and B have the same roots

Now, if A is a root of $f(t)$ $\Rightarrow$ B is also a root of $f(t)$ because A and B have the same (char poly) or (root) by similarity

In the solution below this student's persistence with the understanding of matrices as tables of numbers is even more obvious in his symbolic drawing of **A**, with the mention of its dimension. He then uses some matrices $\mathbf{A}_n, \mathbf{A}_{n-1}, \dots, \mathbf{A}_0$ as coefficients of, apparently, the characteristic polynomial of **A**, without making much notational sense in his tentative:

Solution

$f(t) \rightarrow$ poly. $A = \square\, n$

$B \sim A$

$|tI - A| = t^n + A_{n-1} t^{n-1} + \dots + A t + A_0$

In total, the solutions of 16 students, out of the 51 answering the question, were hindered by such wrongful symbolic manipulations, and we coded them accordingly with

Analytic /meta-linguistic / structural understanding of matrices /-1 as well as with **Analytic / linguistic / notation and terminology / -1.**

The assumption that f(t) is the characteristic polynomial of **A** and the inability to manipulate matrices as concrete entities were the two main causes accounting for students' difficulties in this problem. Thirty-one students encountered none of these obstacles, but twenty of them ignored to prove that $(\mathbf{P}^{-1}\mathbf{AP})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}$. Only eleven students thus produced a complete answer to this question (see Table 8), similar to the one provided by this student:

We know that $f(A)=0$, $B \sim A \rightarrow B=P^{-1}AP$

Prove that $f(B)=0$

$f(B) = f(P^{-1}AP) = a_n(P^{-1}AP)^n + \dots + a_1(P^{-1}AP) + a_0I$

$(P^{-1}AP)^n = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) = P^{-1}A^nP$

$f(B) = a_n P^{-1}A^nP + \dots a_1 P^{-1}AP + a_0 I$

$= P^{-1}(a_nA^n + \dots + a_1A + a_0I)P$

$= P^{-1}f(A)P$

$f(B) = P^{-1}\,0\,P = 0$ so B is also a root of f(t)

TB	Sys/hyp/assumptions of the problem			Analytic/meta-linguistic/struct and Analytic/linguistic/not&ter			Sys/prov/concern with valid		
Score	1	0	-1	1	0	-1	1	0	-1
Number of students	*34*	3	14	33	2	16	10	20	21
% of the total	67%	6%	27%	65%	4%	31%	20%	39%	41%

Table 8: Students' behavior in Quiz 4 – Question 1.

Quiz 4 – Question 2: Converse of the Cayley-Hamilton Theorem

The second problem in the quiz tested more directly students' understanding of the Cayley-Hamilton theorem:

*Assume **A** is an* $n \times n$ *real matrix.*

True or false:

*The Cayley-Hamilton theorem implies that if f(x) is a polynomial of which **A** is a root, then f(x) is the characteristic polynomial of **A**.*

Explain.

Analysis of the question

The answer to this question entailed meta-linguistic sensitivity, namely sensitivity to logic. One has to approach the proposed statement analytically in order to recognize in it the converse of the Cayley-Hamilton theorem. The code **Analytic / meta-linguistic / logic-converse / 1** was awarded to students' solutions when they concluded that the statement is false, while the value **-1** was given if they believed it was true.

The justification of the falsity of the converse, on the other hand, required students to involve their systemic thinking in order to look for a counterexample. It is this kind of thinking that would attempt at disproving a general statement on the basis of even one single instance where the statement is not true. Depending on whether or not a student justified the falsity of the converse by giving a counterexample, his behavior would be coded with **1** or **-1** on the feature **Systemic / proving / refutation by counterexample**.

Analysis of students' responses

Two thirds of the students in our group – in total 33 students – were able to see that the statement in the quiz was not the formulation of the Cayley-Hamilton theorem. We coded their answers with *Analytic / meta-linguistic / logic-converse / 1*. However, only two students explicitly mentioned that the proposed statement is the converse of the Cayley-Hamilton theorem, as this student did:

False:

The Cayley Hamilton theorem implies the "reverse"
NOT $\Rightarrow$ but $\Leftarrow$ If you wish.
IF $f(t)$ is $\Delta_A(t)$ $\Rightarrow$ A is a root OF $f(t)$.

The others were less explicit, but still showed signs of understanding that the given statement contains an implication in a different direction:

Cayley-Hamilton: Every matrix A is a root of its char. polynomial

Solution $f(A)=0$ where $f(x)$ is the char. polyn

False: It is sure that if you put the matrix A in $f(x)$ where $f(x)$ is the char. polyn, the answer will be 0.
But there's a lot of polynomial where you can put a matrix A and the answer will be 0 and the polyn is not the char. poly

Of these 33 students, 22 students justified the falsity of the converse by giving a counterexample, while three students mentioned the idea of existence of counterexamples, as the student whose solution we included above. The remaining 8 students only stated that the given statement is false.

The other third of the class did not notice the logical distinction involved in the problem and accepted the statement as true. Their attempts revealed that they are not only failing to sense the logical distinction involved in the problem, but also that they have deeper conceptual misunderstandings about the Cayley-Hamilton theorem, as it is the case for this student:

true

$f(A) = \Theta$

$\Delta_A(x) = \det(xI - A) = f(x)$

$$
\begin{aligned}
\Delta_A(A) &= \det(A \cdot I - A) \\
&= \det(0) \\
&= \Theta \\
&= f(A)
\end{aligned}
$$

This distribution can be read from the table below:

TB	Analytic/meta-ling/logic-converse		Sys/proving/refutation by counterex		
Score	1	-1	1	0	-1
Number of students	33	17	25	17	8
% of the total	66%	34%	50%	34%	16%

Table 9: Students' behavior in Quiz 4 – Question 2.

Quiz 5 – Question 1: Characteristic polynomial, minimal polynomial

Quiz 5 tested students' knowledge of mainly two notions: the characteristic polynomial and the minimal polynomial of a matrix. The first question included the requirement to provide the definition of the minimal polynomial of a matrix.

(a) Define the notion: Minimal polynomial of a matrix.

(b) Find the characteristic and the minimal polynomial of the matrix A*:*

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Analysis of the question

In part a) we expected our students to encounter the same kind of problems as in the first question of quiz 3, related to the distinction between a definition and a theorem, thus among specific statements within the mathematical discourse. We were afraid that some students would see properties of the minimal polynomial as its definition. This would indicate a lack of meta-linguistic sensitivity in approaching this question. We used the following codes to characterize students' behavior with respect to the "meta-linguistic sensitivity" feature of theoretical thinking:

Analytic / meta-linguistic / def-thm distinction / 1, if they gave the definition of a minimal polynomial of a matrix.

Analytic / meta-linguistic / def-thm distinction / -1, if any other result, whether true or not, involving minimal polynomials was provided.

We have also looked at students' linguistic sensitivity the terminology involved in the definition of the minimal polynomial, using the following coding scheme:

Analytic / linguistic / terminology / 1 was assigned if students used correct terminology and meanings in their definition.

Analytic / linguistic / terminology / -1 was assigned if incorrect terminology or meanings were used.

To solve the second part of the problem one first needed to recognize the matrix **A** as a block diagonal matrix and to recall some results about the characteristic and, respectively, the minimal polynomial of this kind of matrices. The characteristic polynomial of the block matrix **A** is the product of the characterization polynomials of the diagonal blocks: $\mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{A}_3 = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$, while the minimal polynomial is equal to the least common multiple of the minimal polynomials of these matrices. The minimal polynomials of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ can be produced from their respective characteristic polynomials, using the fact that the minimal and the characteristic polynomial of a matrix have the same irreducible factors. While the finding of the characteristic polynomial is almost direct, requiring the student to act practically and apply directly the results we have mentioned above, for obtaining the minimal polynomial, there is one instance where their engagement with TT becomes necessary. To name the possible candidates for the minimal polynomial one needs to engage in hypothetical thinking. This kind of thinking, due to its systemic character is, in general, preoccupied with exploring in an exhaustive manner all cases in a given situation. It is then the feature of TT of being concerned with the validity of the results that compels one to reject all but one of the found candidates based on some theory-driven validating criteria. Adopting this frame of mind means that, for instance, if $(1-x)^2$ is the characteristic polynomial of $\mathbf{A}_1$, one would name $(1-x)^2$ as the minimal polynomial only after verifying that $\mathbf{A}_1$ is not a root of the smaller degree polynomial $(1-x)$.

We coded students' behavior with regard to the hypothetical feature of thinking with **Systemic / hypothetical / all possible cases / 1** if they explored in a systematic manner all the possibilities for the minimal polynomial, and with **Systemic / hypothetical / all possible cases / -1,** if their choice of the minimal polynomial was not based on such an exhaustive evaluation of all possible cases. The feature **Systemic / proving / concern with validity** was added the suffix **1**, if the polynomial proposed as a solution is indeed the minimal polynomial, and the suffix **-1**, otherwise.

Analysis of students' responses

Of the 51 students responding to this question, 29 gave the correct definition of the minimal polynomial of a matrix. Six students did not answer the first part of the question, while 16 students failed to provide the correct definition. The following answer is representative of one category of incorrect answers.

(a) Define the notion: Minimal polynomial of a matrix
minimal polynomial has the same irreducible factors the characteristic polynomial.

This "definition" is very much in the style of practical thinking: it contains a theorem that students most often use when they have to find the minimal polynomial of a given matrix, thus a very "practical" theorem. The definition of the minimal polynomial then, for this student, is *what he does* in a concrete situation in order to find the minimal polynomial of a matrix. We thus coded it with *Analytic / meta-linguistic / def-thm distinction / -1.*

Another category of answers revealed linguistic difficulties, with terms used apparently without any understanding of their meanings. Below are two such examples:

The minimal polynomial of a matrix is the lowest degree of the roots of the matrix that gives 0

(a) Define the notion: Minimal polynomial of a matrix

the minimal polynomial of a matrix is the LCM of the characteristic polynomial that is a root of a given matrix.

They were coded *Analytic / linguistic / terminology / -1.*

In the second part of the problem, for various reasons, many students, after finding the characteristic polynomial of **A**, were not able to find its minimal polynomial. This student for instance, correctly nominated the candidates for the minimal polynomial, but stopped maybe because the limited time she had at her disposal didn't allow her to carry out the lengthy calculations for verifying which of these polynomials has **A** as a root:

characteristic polynomial: $\Delta t = (t-1)^4 (t-3)^2$

candidate of minimal polynomial can be:

$m(t) = (t-1)^4 (t-3)^2$ $\quad$ $m(t) = (t-1)^4 (t-3)$

$m(t) = (t-1)^3 (t-3)^2$ $\quad$ $m(t) = (t-1)^3 (t-3)$

$m(t) = (t-1)^2 (t-3)^2$ $\quad$ $m(t) = (t-1)^2 (t-3)$

$m(t) = (t-1)(t-3)$

In nominating all the candidates for the minimal polynomial, she was thinking theoretically (*Systemic / hypothetical / all possible cases / 1*), but it was perhaps her practical thinking that would have helped her here to look for the minimal polynomials of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ in order to deal with much simpler calculations. We coded her solution with *0* on the feature *Systemic / proving / concern with validity*, because she didn't propose any of these polynomials as the minimal polynomial, and thus we weren't able to decide if she is concerned with the validity of what she is claiming.

This student, on the other hand, gave the wrong answer because he didn't engage in hypothetical thinking in order to nominate the candidates for the minimal polynomial, nor did he check the validity of the given answer by replacing **A** in the polynomial that he had found:

Solution

characteristic: $\Delta_A(t) = (t-1)^4 \cdot (t-3)^2$

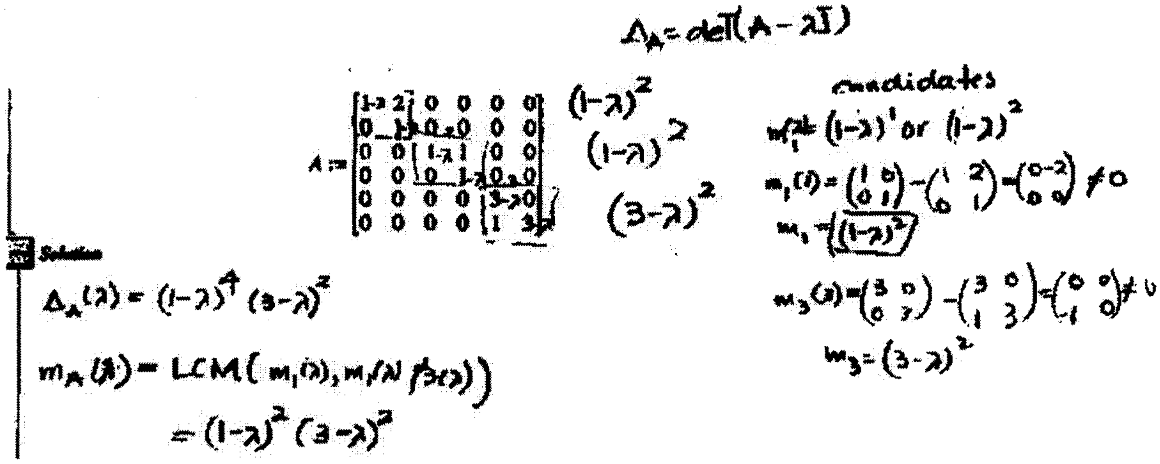
And the minimal polynomial divides $\Delta_A(t)$ and has the same factors

$m_A(t) = (t-1)^2(t-3)$

We coded his answer *Systemic / hypothetical / all possible cases / -1* and *Systemic / proving / concern with validity /-1.*

However, of the 30 students who failed to provide the correct answer, only 16 were not preoccupied with exploring all the possibilities for the minimal polynomial or with verifying the validity of their answer, proving thus a lack of engagement with TT. The remaining 14 were rather not practical enough in order to look for a way of solving the problem that would have involved simpler calculations. We considered their behavior to be theoretical with respect to the hypothetical feature of TT (*Systemic / hypothetical / all possible cases / 1*), but we couldn't decide whether they would be concerned with validating their results (*Systemic / proving / concern with validity /0*).

This student's approach was one of the effective ones for solving the problem in a limited time frame: she looked at the minimal polynomials of the diagonal blocks, which were much easier to validate.



The statistics of students' engagement with theoretical thinking in this question are included in the tables below, separately for part a) and part b):

TB	Analytic/meta-ling/def-thm			Analytic/linguistic/terminology		
Score	1	0	-1	1	0	-1
Number of students	36	2	7	33	3	9
% of the total	80%	4%	16%	73%	7%	20%

Table 10: Students' behavior in Quiz 5– Question 1a.

Sys/hyp/all possible cases			Sys/prov/concern with validity		
1	0	-1	1	0	-1
31	4	16	31	14	6
61%	8%	31%	61%	27%	12%

Table 11: Students' behavior in Quiz 5– Question 1b.

Quiz 5 – Question 2: Candidates of the minimal polynomial

In the second question of quiz 5, a polynomial was given as the characteristic polynomial of a matrix and the students were required to nominate the candidates for the minimal polynomial of this matrix:

If the characteristic polynomial of a matrix M is $(t^2-1)^2(t+3)$, what are the possible candidates for the minimal polynomial of M?

Analysis of the question

This is not a difficult question and also not requiring much engagement with theoretical thinking. One needs to know the theorem stating that the characteristic and the minimal polynomial of a matrix have the same irreducible factors. However, this is not enough, and this is where the instructor probably wanted to test the students' understanding of the theorem. The given characteristic polynomial is already factored, but one has to factor it further, into irreducible polynomials, namely into the form $(t-1)^2(t+1)^2(t+3)$, in order to nominate all the candidates for the minimal polynomial. The irreducibility attribute of these polynomials is thus essential in this theorem. We believe that the identification of the word "irreducible" as a focal point of this theorem requires meta-linguistic sensitivity. This realization dictates further factoring of the polynomial. Students that do not have such a sensitive understanding of the theorem will have $(t-1)^2$ and $(t+3)$ as necessary factors of the minimal polynomial, and will thus fail to provide all the possible candidates for the minimal polynomial. One may argue perhaps that it is not the meta-linguistic ignorance that would induce students into this error, but rather their not knowing of the definition of the notion of irreducibility of polynomials. We believe, however, that this is a less likely explanation, not only because our students are advanced math majors, but also because the teacher specifically insisted on reviewing with the students the notion of irreducible polynomials in the lecture preceding the quiz.

We thus coded students' solutions with **Analytic / Meta-linguistic / essential terms / 1** if they correctly identified the irreducible factors of the characteristic polynomial and with **-1** if they identified only $(t^2-1)^2(t+3)$ and $(t^2-1)(t+3)$ as candidates for the minimal polynomial.

We also expected answers that would nominate, as candidates for the minimal polynomial *the* factors of the characteristic polynomial. For instance one could propose $(t-1)$, $(t+1)$ and $(t+3)$ as candidates, failing to grasp the meaning of "the same irreducible factors" as "*all* the same irreducible factors", missing the general quantifier implicitly present in this phrase. This is for us a sign of meta-linguistic sensitivity, in particular a lack of sensitivity to the logic embedded in the mathematical language. We coded students' answers with regard to this feature of TT with **Analytic / meta-linguistic / logic-Universal Quantifier**, adding the suffix **1** if the candidates for the minimal polynomial contain all the factors of the characteristic polynomial, and the suffix **-1**, if the proposed candidates do not contain all the factors of the characteristic polynomial.

Analysis of students' responses

The most frequently encountered mistake in our students' answers to this problem was their not looking for the irreducible factors of the characteristic polynomial to provide the candidates for the minimal polynomial. We give below a sample of this kind of answers:

$$\Delta_{t_m} = (t^2-1)^2(t+3)$$

$$\text{so } m_t = (t^2-1)(t+3) \text{ or } (t^2-1)^2(t+3)$$

We coded them with *Analytic / Meta-linguistic / essential terms / -1.*

There were only 5 students failing to notice the implicit universal quantifier, present in the above mentioned theorem. This student, for instance, provided all the possible combinations of the irreducible factors of the characteristic polynomial as candidates for the minimal polynomial, failing thus to grasp the meaning of "*the same* irreducible factors of..." as "*all* the irreducible factors of ...":

possible candidates for min polys of M

1. $t+1$
2. $t-1$
3. $t+3$
4. $(t+1)(t-1)$
5. $(t+1)(t+3)$
6. $(t-1)(t+3)$
7. $(t+1)(t-1)(t+3)$

We characterized these students as lacking sensitivity to logic, and their answers were coded, accordingly, with *Analytic / Meta-linguistic / Logic-Universal Quantifier /-1.*

Out of the 52 students who answered the quiz, 34 gave the correct answer to the question (their answers being coded with *1* on both features we have considered), while 2 students gave the wrong answer due to a missed exponent in their writing of the characteristic polynomial (their solutions were coded with *0* on the feature *Analytic / Meta-linguistic / essential terms* , and with *1* on *Analytic / Meta-linguistic / Logic-Universal Quantifier*). The lack of meta-linguistic translated in failure to notice that $(t^2-1)^2$ is not an irreducible factor was the main source of for the students' incorrect answers (it accounted for 27% of the wrong solutions), while only 5 students (representing 10% of the group) came up with the incorrect answer due to their lack of sensitivity to logic (see Table 12).

TB	**Analytic/Meta-ling /essential terms**			**Analytic/Meta-ling/Logic-UQ**	
Score	1	0	-1	1	-1
Number of students	36	2	14	47	5
% of the total	69%	4%	27%	90%	10%

Table 12: Students' behavior in Quiz 5 – Question 2

Quiz 6 – Question 1: Minimal polynomial of a linear operator

The first question in quiz 6 also involved the notion of minimal polynomial, but now the setting has slightly changed: while in quiz 5 there was reference to the "minimal polynomial of a matrix", now the requirement is to find the "minimal polynomial" of a linear operator:

Let $\mathbf{T}:\mathbf{R}^2 \to \mathbf{R}^2$ *be a linear operator such that:*

$\mathbf{T}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$ **and** $\mathbf{T}\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}1\\8\end{bmatrix}$. *What is the minimal polynomial of* $\mathbf{T}$ *?*

Analysis of the question

To find the minimal polynomial of **T** one needed to find a basis-dependent matrix representation for the given operator. Finding the characteristic polynomial then is direct, and since it is a second degree irreducible polynomial, it is equal to the minimal polynomial.

Basis representation problems are a domain where students' difficulties with the theory of vector spaces often surface. Hillel (2000; see also Hillel and Sierpinska, 1994), in his paper about modes of description in linear algebra, discusses how basis-dependent representations of vectors and linear operators represent an area where students' inability to move between the different settings of linear algebra becomes particularly noticeable. He argues that the shift from the *abstract mode* of vectors and of linear operators to the *algebraic mode* of n-tuples and of matrices is done through a choice of basis for the underlying vector space. In a finite dimensional vector space, the basis representation assures the coordination between the two modes often via subtle notational shifts, such as: a vector $\mathbf{v}$ is associated with an n-tuple, denoted by $[\mathbf{v}]_\beta$, and a linear operator **T** with a matrix $[\mathbf{T}]_\beta$ (where $\boldsymbol{\beta}$ is the chosen basis). However, these notational mechanisms don't seem to resolve students' problems related to the distinction between the abstract and the algebraic modes of description, and one symptom of this persisting problem of movement between modes is the confusion students make between a vector and its representation relative to a basis, especially when the underlying space is $\mathbf{R}^n$. In the case of the matrix representation of a linear operator relative to a basis, the confusion lies in the belief that the columns of the matrix representation of an operator are images of the basis vectors, and not the coordinates of these images relative to the given basis.

In our problem, for instance, an answer containing this confusion would be to give the matrix $\begin{bmatrix} -1 & 1 \\ 0 & 8 \end{bmatrix}$ as the representation of **T** in the basis formed of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We will look at students' answers to this problem to see if they possess this or any other kind misunderstandings related to the representation of a linear operator in a basis. Without hoping to capture the entire complexity of students' difficulty to move flexibly between the different "languages" of linear algebra, we will code students' answers with:

Analytic / meta-linguistic / matrix representation / 1, if they perform correctly the operation of representing **T** by a matrix in a basis.

Analytic / meta-linguistic / matrix representation / -1, if they fail to represent **T** in some basis.

We will zoom in on their written speech, so to speak, to read how they cope with this "bilingualism" of linear algebra. At the same time, we realize that, while their mistakes might give us insights into some of their misunderstandings, given the relative simplicity of this problem, there may be misunderstandings that will not be revealed by their solutions to this problem.

Besides the "meta-linguistic sensitivity" code, that we have considered precisely for signaling students' flexibility in moving between the two modes of description of linear algebra, we have also introduced another one, meant to characterize students' sensitivity in using notations (for instance, if they use basis-sensitive notations) and to terminology. The code **Analytic / linguistic / notations and terminology** was used according to the following scheme:

1 was added if correct notations and correct terminology is used.

-1 was added if incorrect notations or terminology is used.

Analysis of students' responses

37 students successfully found a matrix representation for the operator **T**, either in the standard basis or in the basis formed of the vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$. Their answers were coded *Analytic / meta-linguistic / matrix representation / 1.* However, not all of them exhibited the same linguistic sensitivity. Only 23 of them were preoccupied with conveying properly, through sensitive notations, the meanings of the notions involved in the problem (i.e., basis, linear operator, matrix representation, etc.). Here is an example of this approach:

$$w_1 = \begin{bmatrix}1\\2\end{bmatrix} \quad w_2 = \begin{bmatrix}3\\4\end{bmatrix} \qquad B = \{w_1, w_2\} \text{ Basis}$$

$$T(w_1) = \begin{bmatrix}-1\\0\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix} = 2w_1 - w_2$$

$$T(w_2) = \begin{bmatrix}1\\8\end{bmatrix} = 10\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}3\\4\end{bmatrix} = 10w_1 - 3w_2$$

$$[T]_B = \begin{bmatrix}2 & 10\\-1 & -3\end{bmatrix} \qquad \begin{matrix}tr = -1\\ det = 4\end{matrix}$$

$$\Delta_T(t) = t^2 + t + 4$$

$$m_T(t) = t^2 + t + 4$$

Such theoretical behavior with regard to the linguistic aspect of thinking was coded *Analytic / linguistic / notations and terminology/1.*

On the other hand, 14 students, while still successful in coming up with the correct matrix representation for **T**, seemed to have done it without much linguistic concern. For these students, the linear operator and its matrix representation seem to be just two different, equivalent in meaning, arrays of numbers, obtained one from another through some procedure. They didn't mention the basis in which they represented **T** and they also did not distinguish, through some notational means, the notion of linear operator from that of a matrix representation of an operator. The following approach is typical for this group of less linguistically sensitive students:

What is the minimal polynomial of T?

Solution

$$\left[\begin{array}{cc|cc} 1 & 3 & -1 & 1 \\ 2 & 4 & 0 & 8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & -1 & 1 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 10 \\ 0 & 1 & -1 & -3 \end{array}\right]$$

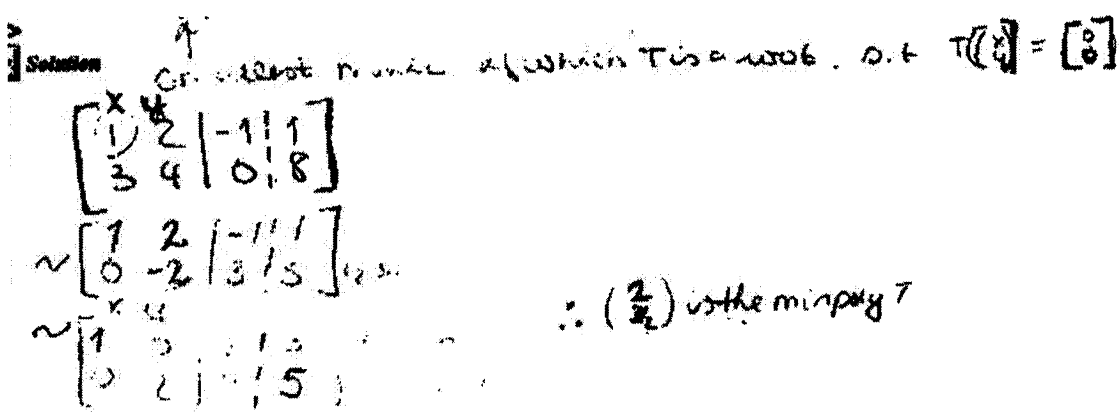
$$\Delta_T(t) = \det \begin{bmatrix} 2-t & 10 \\ -1 & -3-t \end{bmatrix} = -(2-t)(3+t)+10 = t^2+t+4$$

$$\Delta_T(t) = m_T(t) = t^2+t+4$$

This solution does not contain a conceptual mistake, the student might have a good technical understanding of theory, but he is certainly less concerned with the communication of his thoughts. The solutions in this category were coded with code *Analytic / linguistic / notations and terminology/-1.*

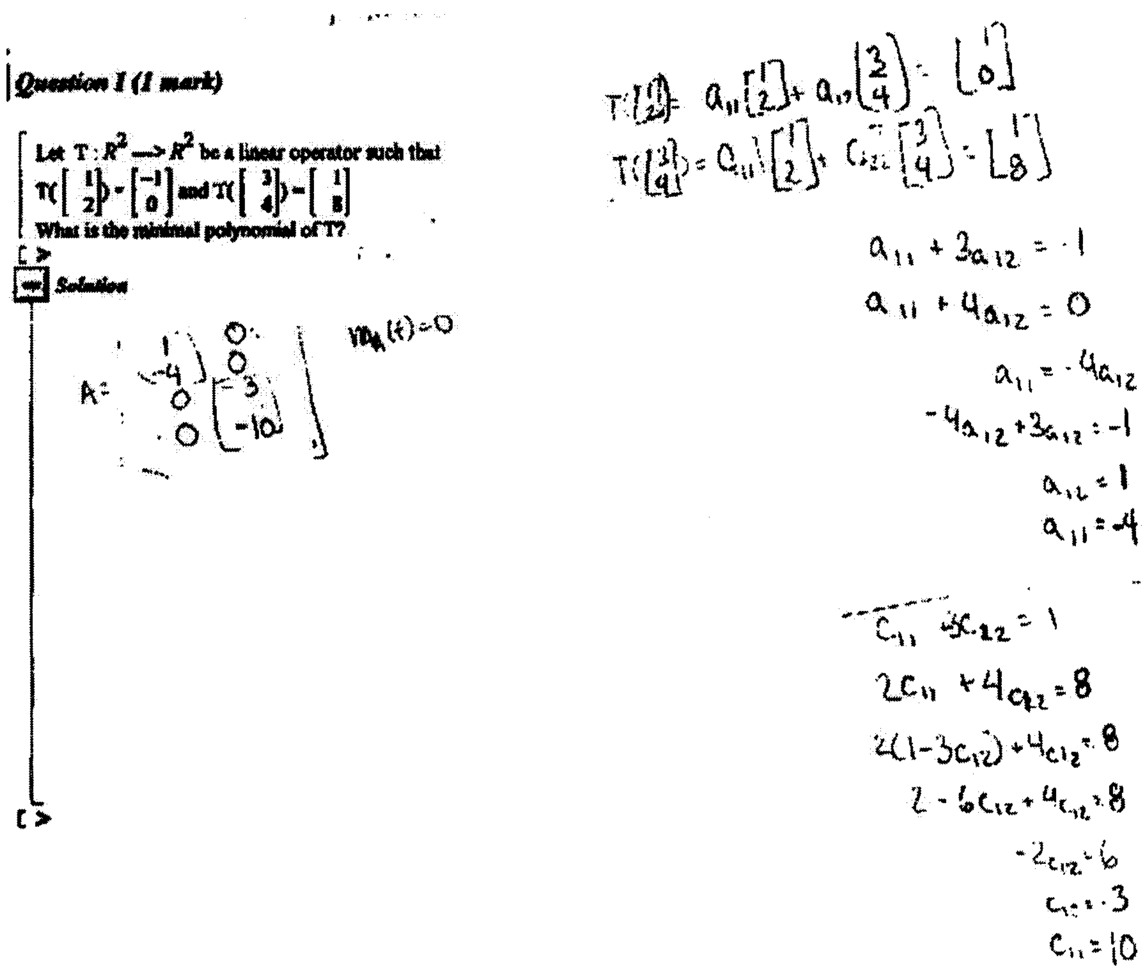
The more specific representational misunderstanding, mentioned by Hillel and Sieprinska in their paper (1994) and consisting of the belief that the columns of the matrix representation of an operator are the images of the basis vectors, was held by 6 students in our group. They all came up with $\begin{bmatrix} -1 & 1 \\ 0 & 8 \end{bmatrix}$ as a matrix representation of the operator T. These solutions were coded *Analytic / meta-linguistic / matrix representation / -1.*

Without exhibiting a particular misconception, 7 students failed to solve the problem seemingly also due to their misunderstanding of the operation of representing a linear mapping by a matrix, manifested in various ways in their written language. Let us look at this student's solution, for instance, which contains more than one instance of linguistically insensitive behavior:



We cannot see in this scanned image the final result of her transformations, but we can nevertheless characterize her behavior by looking at the rest of the solution. Her misunderstanding of the operation of representation is revealed by the fact that she writes the basis vectors as rows and the images of these vectors as columns (being thus awarded the code *Analytic / meta-linguistic / matrix representation / -1*). For her, "the smallest monic of which **T** is a root" is some vector that maps into the zero vector through **T**. She then gives a vector as the minimal polynomial, demonstrating her lack of sensitivity to mathematical terminology (coded *Analytic / linguistic / notations and terminology/-1).*

Another student, whose behavior was also coded *Analytic / linguistic / notations and terminology/-1,* seemed to have the correct approach for finding the coordinates of the images through **T** of the basis vectors, but then apparently, didn't posses the linguistic means to use the results she obtained in order to produce a representation matrix for **T**. As a result, she came up with a 4×4 matrix, and with 0 as the minimal polynomial:



Apparently, the students in our group did not have particular difficulties with representing an operator in a basis. 71% of our group managed to do the operation correctly, even if around 40% did not care much about communicating the meaning of their techniques in a linguistically sensitive manner. Misunderstandings related to the operation of representation of an operator were exhibited by only 21% percent of the group, while around 13% (7 students) of the group made serious grammatical mistakes in their solutions. We include in the table below the statistics of our students' engagement with theoretical thinking in this question:

TB	**Analytic/meta-ling /matrix representation**			**Analytic/linguistic/not&ter**		
Score	**1**	**0**	**-1**	**1**	**0**	**-1**
Number of students	37	4	11	23	6	23
% of the total	71%	8%	21%	44%	12%	44%

Table 13: Students' behavior in Quiz 6 – Question 1.

Quiz 6 – Question 2: Intersection of T-invariant subspaces

The second question in quiz 6 was on the topic of **T**-invariant subspaces:

> *Let* $\mathbf{T}:\mathbf{V}\to\mathbf{V}$ *be a linear operator and let* $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *be* $\mathbf{T}$*- invariant subspaces of* $\mathbf{V}$.
>
> *Prove that the intersection of* $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *is also a* $\mathbf{T}$*- invariant subspace of* $\mathbf{V}$.

Analysis of the question

It may be said that, in theory, to solve this problem one only needs to know (even at a purely verbal level) the definition of the invariance of a subspace relative to a linear operator. Using then basic tools of logic and set theory the proof is trivial: if we take any element of the intersection set, since it belongs to $\mathbf{W}_1$, its image through $\mathbf{T}$ also belongs to

$\mathbf{W}_1$ (by the definition of **T**-invariance), but since it also belongs to $\mathbf{W}_2$, the image through **T** belongs to $\mathbf{W}_2$ as well. Thus, the image of that element through **T** belongs to the intersection of $\mathbf{W}_1$ and $\mathbf{W}_2$, which means that $\mathbf{W}_1$ intersected with $\mathbf{W}_2$ is also **T** invariant. I intentionally tried to avoid as much as possible the terminology of linear algebra ("subspace", "linear operator", "vector", etc.) to show how simple this proof is on the purely formal level. Assuming that one masters such basic logic skills – a quite reasonable assumption to make about first year students majoring in math – then we might expect this problem to create no major difficulty for the students in our group.

We believe however that to solve this problem, one needs to "juxtapose" over this simple logic a more fundamental competence that only comes with a fairly deep understanding of some concepts in linear algebra. Each of the linear algebra notions appearing in this problem needs to be identified with the correct category in set theory, for instance: a vector is an element of a set and a subspace is a set. A subspace, as a subset, *is included* in a vector space, while a vector *belongs* to a subspace or a space. One has to be able to connect the plain language of set theory with the language of linear algebra, and this coupling, we believe, is conditioned by the formation of correct concepts of linear algebra notions.

We will characterize students' behavior in this question from the point of view of their ability to use the language of sets in the context of linear algebra by using the code **Analytic / linguistic / set language / 1** if, throughout the solution, they use correctly the language of sets to denote linear algebra concepts and the relations between them, and the code **Analytic / linguistic / set language / -1,** otherwise. Answers awarded the latter code are actually the manifestation of what Dorier et al. (2000) termed as the *obstacle of* *formalism* or the erratic use of the tools of logic and set theory in the context of linear algebra.

Analysis of students' responses

23 responses to the quizzes (46% of the total) contained an erroneous use of the set theory terminology. As we have predicted, the mistakes were often related to the identification of the linear algebra notions involved in the problem with the corresponding category in set theory: vectors were not operated upon as elements of sets and subspaces were not seen as sets. Inclusion was sometimes confused with equality and the symbols "$\subset$" and "$\in$" were not employed according to their meaning, i.e. "$\subset$" characterizing the relation between two sets and "$\in$" that between an element of a set and a set. Such confusion is exhibited by the student whose solution we include below:

$W_1, W_2 \in V$ and are T-inv

let w_3 be $\in W_1 \cap W_2$ → w_3 is part of the intersection of W_1 and W_2 which are both T-inv, so w_3 has to be also T-inv, and so the inter must be T-inv.

since W_1 (T-inv) and W_2 (T-inv) and $w_3 \in W_1 \cap W_2$ it implies that w_3 is also T-inv.

$W_1 \subset V$, $W_2 \subset V$

$W_1 \cap W_2$ is also T-inv since W_1 and W_2 are T-inv

In the first line she uses the symbol "$\in$" instead of "$\subset$" for the inclusion of the subspaces $\mathbf{W_1}$ and $\mathbf{W_2}$ in the vector space $\mathbf{V}$. However, it is clear that her incorrect use of symbols is more than just some "sloppy" utilization of language: in the next line $\mathbf{w_3}$ appears to be a vector (she uses slightly different notations for vectors and subspaces), but then she characterizes it as being $\mathbf{T}$-invariant, as if it were a subspace. Thus she is clearly mixing up the notions of vector and subspace. This is also obvious in the last line of the solution where she uses the intersection symbol between what appear to be as two vectors ("$\mathbf{w}_1 \in \mathbf{W}_1$" and "$\mathbf{w}_2 \in \mathbf{W}_2$").

We could see many such "grammatical mistakes" (i.e.: following Wittgenstein, 1974, "signs" not being used according to their defined usage) in students' solutions, revealing their conceptual misunderstandings related to the notions of vector and subspace (mostly the confusion between the two) or **T**-invariance of a subspace. We coded these answers with *Analytic / linguistic / set language / -1*.

The analytic approach to language turned out to be of major importance for providing a correct answer to this question. While in other quizzes we could see solutions lacking linguistic sensitivity but still accepted as correct, here, the linguistic component was crucial for success: the 23 students whose answers were coded *Analytic / linguistic / set language / -1* also failed to produce a proof of the statement in the quiz. On the other hand, the 27 students who proved to be linguistically sensitive (answers coded *1* on the above mentioned feature) also succeeded in proving the **T**-invariance of the intersection of the two subspaces. We include these figures in the following table:

TB	**Analytic/meta-ling/ set language**	
Score	1	-1
Number of students	23	27
% of the total	44%	52%

Table 14: Students' behavior in Quiz 6 – Question 2.

Quiz 7 – Question 1: Restriction of a linear operator to a subspace

Quiz 7 was administered in the 9th week of classes following the lecture on the primary decomposition theorem. The first question involved the notion of the restriction of a linear operator to a subspace:

> *Let* $\mathbf{T}:\mathbf{R}^3 \to \mathbf{R}^3$ *such that:*
>
> $$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \to \begin{bmatrix} -\mathbf{y} \\ 2\mathbf{x}+4\mathbf{y}-2\mathbf{z} \\ 4\mathbf{x}+\mathbf{y}+4\mathbf{z} \end{bmatrix}$$
>
> *Find* $\mathbf{W} = \mathbf{Ker}((\mathbf{T}-2\mathbf{I})^2)$ *and a matrix representation of the restriction of* $\mathbf{T}$ *to* $\mathbf{W}$.

Analysis of the question

Having a matrix representation of the operator $\mathbf{T}$ – for instance that in the standard basis – one may easily find the kernel of $(\mathbf{T}-2\mathbf{I})^2$ by finding the solution space of the homogenous system $[(\mathbf{T}-2\mathbf{I})^2]_{St} \cdot \mathbf{X} = \mathbf{0}$, where $[(\mathbf{T}-2\mathbf{I})^2]_{St}$ is the representation, in the usual basis, of $(\mathbf{T}-2\mathbf{I})^2$. Indeed, this subspace of $\mathbf{R}^3$ is the set of elements in $\mathbf{R}^3$ that map into the zero vector through the operator $(\mathbf{T}-2\mathbf{I})^2$, giving thus the kernel of $(\mathbf{T}-2\mathbf{I})^2$.

If, say, $\boldsymbol{\beta}$, is a basis of this subspace (which turns out to be of dimension 2), then a matrix representation relative to $\boldsymbol{\beta}$ of the restriction of $\mathbf{T}$ to $\mathbf{W}$ is a 2 by 2 matrix, having as columns the coordinates of the images, through $\mathbf{T}$, of the basis vectors, relative to $\boldsymbol{\beta}$.

The understanding of two general notions seems to be crucial for solving this problem: that of kernel and that of restriction of an operator to a subspace. However, one also needs to have quite stable understandings of more basic notions such as that of subspace or of basis. It is obvious how, at this point in the course, the complexity that students have to deal with increases, as new concepts and results keep accumulating. This calls for their ability to think in terms of systems of concepts, where newly introduced concepts acquire meaning from their relations with already existing, hopefully internalized concepts. Reference to the definitions of the concepts involved in the problem is inevitable if one wants to solve it. One has to at least know with what kind of object one is dealing with: an operator is a function, a kernel is a set, a restriction of an operator to a subspace is also an operator, not to mention the multitude of relations to establish between these concepts in order to grasp their meaning: kernel *of* an operator, restriction *of* an operator *to* a subspace, etc. This all requires systemic thinking, and we coded students' solutions to this problem with regard to this feature of thinking as follows:

Systemic / definitional / relations between concepts / 1, if correct links between concepts are made. Examples of such correct understandings may include: a kernel of an operator is a set having a certain property with regard to that operator, a restriction of an operator to a subspace is also an operator, etc.

Systemic / definitional / relations between concepts / -1, if the concepts involved in the problem are not correctly understood and/or linked to each other.

Inevitably, one has to be also quite linguistically capable in order to manipulate, in writing, the many concepts involved in the problem. To characterize students' behavior from the linguistic point of view we have used the code **Analytic / linguistic / notations and terminology** with the suffix **1**, for solutions that used correct notations and terminology, and **-1** if incorrect notations or terminology were used.

Analysis of students' responses

One third of the group – 17 students – provided a correct and complete answer to the question. For one category of students who failed to provide the correct answer, the main difficulty surfaced with the finding of the matrix representation of the restriction of **T** to the kernel of $(\mathbf{T}-2\mathbf{I})^2$. Some students stopped after finding the kernel, while others continued but with an incorrect solution. One frequent mistake that they exhibited was that of obtaining a 3x3 matrix representation of the restriction of **T** to **W**. This was the case for the student whose solution we reproduce below:

Solution

$$\begin{bmatrix} -y \\ 2x+4y-2z \\ 4x+y+4z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 4 & -2 \\ 4 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$W = (T-2I)^2 = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 2 \\ -8 & 0 & -8 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ -8 & 0 & -8 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow W = \left\{ \begin{bmatrix} x \\ y \\ -x \end{bmatrix} ;\ x, y \in R \right\}$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is basis of W.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore [T] = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is the matrix rept. of the restricT of T to W

Question 2 (1 mark)

Let $T: V \to V$ be a linear operator with minimal polynomial equal to $(t^2-1)(t+1)$.

He clearly has difficulties with the concept of the restriction of a linear operator to a subspace. His approach is correct for finding the kernel of a linear operator and he also seems to have a fairly good understanding of the operation of representing a linear operator in a given basis. His concepts of basis, subspace and linear operator are correct, but he fails to make the necessary connections between these concepts in order to grasp the meaning of restriction of an operator to a subspace. We coded thus his answer with *Systemic / definitional / relations between concepts / -1.*

Other students exhibited even greater difficulties with the concepts involved in the problem. Let us look at this student's attempted solution for instance:

$$[T]_{sv} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 4 & -2 \\ 4 & 1 & 4 \end{bmatrix} = A.$$

So; $T - 2I = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix}$

$$(T-2I)^2 = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 2 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-2 & 2-2 & 2 \\ -4+4-8 & -2+4-2 & -4+4 \\ -8+2+8 & -4+2+2 & -2+4 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & 0 & 2 \\ -8 & 0 & -8 \\ 2 & 0 & 2 \end{bmatrix}$$

So: [illegible]

basis $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

$T: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}.$

A basis, for him, is a sum of vectors, a subspace is a matrix and a linear operator is a vector. We coded his answer *Analytic / linguistic / notations and terminology/ -1* for his use of incorrect terminology (i.e.: terms used not according to their defined usage), and with *Systemic / definitional / relations between concepts / -1* for his failure to establish any conceptual relations between the concepts involved in the problem. Approximately 18 students (35%) had a linguistically insensitive behavior in this question, while 31 (60%) did not link the concepts involve din the problem correctly or failed to link them at all (see table 15)

TB	**Systemic/def/relations between concepts**			**Analytic/linguistic/not&ter**		
Score	**1**	**0**	**-1**	**1**	**0**	**-1**
Number of students	17	4	31	25	9	18
% of the total	33%	8%	60%	48%	17%	35%

Table 15: Students' behavior in Quiz 7 – Question 1.

Quiz 7 – Question 2: Assumptions of the primary decomposition theorem

The second question in quiz 7 tested more directly students' understanding of the primary decomposition theorem:

> *Let* $\mathbf{T}:\mathbf{V}\to\mathbf{V}$ *be a linear operator with minimal polynomial equal to* $(\mathbf{t}^2-\mathbf{1})\cdot(\mathbf{t}+\mathbf{1})$.
> *Let* $\mathbf{W}_1=\mathbf{Ker}(\mathbf{T}^2-\mathbf{1})$ *and* $\mathbf{W}_2=\mathbf{Ker}(\mathbf{T}+\mathbf{1})$.
> *It is necessarily true that* $\mathbf{V}$ *is a direct sum of* $\mathbf{W}_1$ *and* $\mathbf{W}_2$*?*
> *Explain.*

Analysis of the question

Three features of theoretical thinking would be tested, in principle, by this question:

- hypothetical thinking, namely, the feature **Systemic / hypothetical / assumptions in theorems**: one would give a negative answer to the question in the quiz (i.e.: **V** is not necessarily a direct sum of $\mathbf{W}_1$ and $\mathbf{W}_2$) if one notices that the assumption of the primary decomposition theorem that would assure this result is not satisfied, i.e.: t^2-1 is not irreducible. If a student stated that **V** is not necessarily a direct sum of $\mathbf{W}_1$ and $\mathbf{W}_2$, because t^2-1 is not irreducible (or because t^2-1 and $t-1$ are not relatively prime), then his answer would be assigned the code **Systemic / hypothetical / assumptions in theorems / 1**. Otherwise, if his answer is "yes" to the question stated in the problem, or if he doesn't give the correct reason for his "no" answer, the code **Systemic / hypothetical / assumptions in theorems / -1** would be assigned;

- to provide a counterexample then, one needs to engage in systemic thinking, namely to exhibit the theoretical behavior **Systemic / proving / refutation by counterexample.** A student's solution would score **1** on this feature if he provides a counterexample, and **-1**, otherwise.

To signal students' sensitivity in the linguistic domain we used the code **Analytic / linguistic / notations and terminology**, with the suffix **1** added for solutions that used correct notations and terminology and with the suffix **-1**, for incorrect notations and terminology.

Analysis of students' responses

A significant number of our students behaved theoretically in this question, with regard to the hypothetical feature of thinking. 40 of them, representing 77% of the group, noticed that the statement is not necessarily true because $t^2 - 1$ is not irreducible. Nine of them failed to notice that the assumption is not fulfilled, and gave essentially, an answer similar to the one we include below:

Solution

Yes, because of primary decomposition theorem

where (here) $m_T(t) = (t^2-1)(t-1)$

then $V = \ker(t^2-1) \oplus \ker(t-1)$

Only 10 students (representing 19% of the group), however, were able to come up with a counterexample, while 9 students exhibited serious linguistic difficulties, as this student did:

Explain. YES

Solution: since (t^2-1) and $(t+1)$ are both minimal polynomial of V. The primary decomposition tells us that they should be direct sums.

We include the results of our observations of students' engagement with theoretical thinking in the table below:

TB	**Sys/hyp/assumptions of theorem**			**Analytic/linguistic/not&ter**			**Sys/prov/refutation by counterex**	
Score	1	0	-1	1	0	-1	1	-1
Number of students	40	3	9	26	7	19	10	42
% of the total	77%	6%	17%	50%	13%	37%	19%	81%

Table 16. Students' behavior in Quiz 7- Question 2

CHAPTER IV

A QUANTITATIVE EVALUATION OF STUDENTS' THEORETICAL BEHAVIOR ALONG THE SEMESTER

In this chapter we use the results of our empirical analyses to discuss how our students' behavior has changed along the semester from the point of view of their engagement with theoretical thinking.

INTRODUCTION TO CHAPTER IV

Let us recall, very shortly and for the purposes of explaining our method for measuring the changes in the group's tendency to think theoretically, that for each question in the quizzes, we have identified, a-priori, the features of theoretical thinking we considered, in principle, necessary to solve the problem. We then evaluated each student's answer to a given question with respect to the manifestation, in his behavior, of the respective features of thinking. The tables we have included after the analysis of each question (in Chapter III), contained the numbers and the percentages[2] of students who acted theoretically with respect to a given feature (in the column under the rubric labeled "**1**"), of those who acted practically (under "**-1**"), and of those who didn't have an explicitly theoretical or practical behavior with regard to that feature (under **0**).

At a given time (e.g.: at the time of quiz 1, of quiz 2, etc.), the percentage of students that have behaved theoretically with respect to a given feature, is, for us, the

[2] calculated on the number of students that answered the question

measure of the group's tendency to engage in that particular kind of thinking. For each feature of TT, one such index can be defined and its values calculated for every quiz testing the respective feature. We labeled these indices, corresponding to individual features of TT: "fgt_<respective feature of TT>". For instance, one index would be "the group's tendency to pay sensitivity to assumptions" labeled "fgt_Systemic/hyp/ assumptions". The evolution of the values taken by this index in the quizzes where this feature has been tested expresses how students' sensitivity to assumptions has changed along the semester.

Besides these indices, defined for each feature of TT, we have also considered the index "the group's general tendency to think theoretically", labeled "ggt", whose value for a given quiz *i* – ggt_qi – is given by the average of the values taken by the all the "feature indices" in quiz i.

In the next section we examine the evolution of the group's general tendency to think theoretically, given by the "ggt" index. We look then at the group's tendency to think theoretically with respect to some specific features of theoretical thinking. Based on these evaluations we try then to answer our research question: have weekly quizzes contributed to the development of students' theoretical thinking? At the same time we reflect on the effectiveness of our tools for answering this question.

EVOLUTION OF THE GROUP'S GENERAL TENDENCY TO THINK THEORETICALLY

The following values were taken by the index "group's general TT tendency" in the seven quizzes that we have analyzed:

ggt_q1	ggt_q2	ggt_q3	ggt_q4	ggt_q5	ggt_q6	ggt_q7
52%	57%	58%	54%	72%	60%	45%

Table 17. The group's general TT tendency in the seven quizzes.

Graphically, the variation of this index can be observed from Figure 2 below:

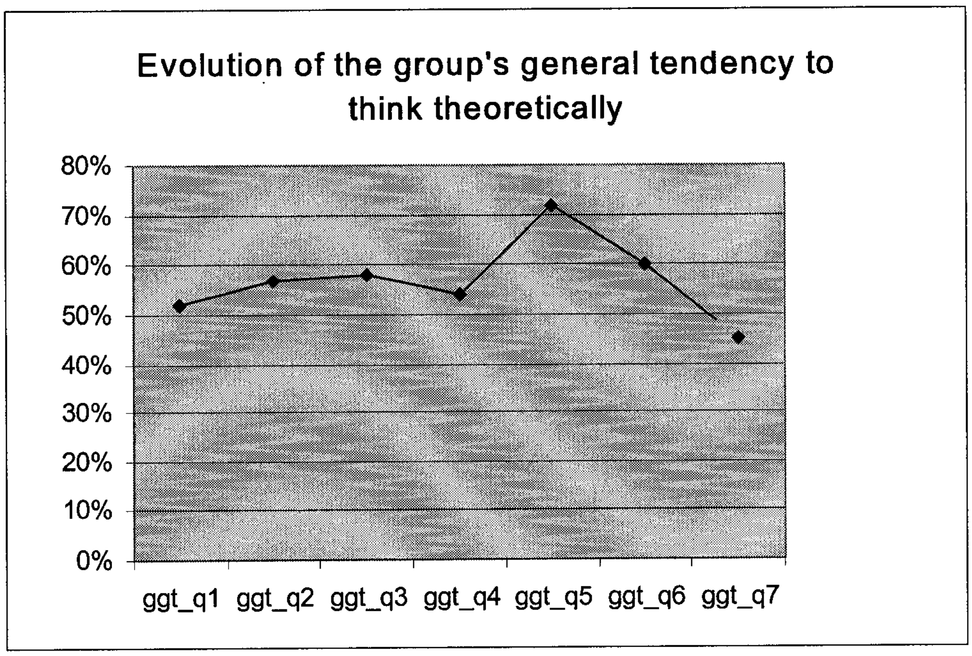


Figure 2. Evolution of the group's general tendency to think theoretically

The first "raw" conclusion we can draw, by looking at these figures, is that the administration of the quizzes didn't really improve our group's inclination to think theoretically; on the contrary: the last value taken by our index (45%) is smaller than the first one (52%). But let us let us have a closer look at these results and deepen our analysis.

Already in our analyses of the quizzes, as we were coding students' solutions, we have "sensed", without necessarily looking at the numbers, that our students' theoretical behavior in a given quiz was strongly influenced by the mathematical content of that quiz. It was not necessarily the level of difficulty (generally speaking) of that problem that seemed to be related to students' engagement with theoretical behavior: some problems seemed to be quite trivial – as it was the case of the proof that the intersection of two **T**-invariant subspaces is also a **T**-invariant subspace – and yet, they engaged a low number of theoretical behaviors from the part of our students. The difficulties of our students were rather related to the nature of concepts involved, and, more specifically, to the complexity of the conceptual systems that they had to deal with at a given time. This was again, just a "feeling" that we had as we were looking through our students' solutions to the quizzes. But it now makes us interested in evaluating the statistical results we have obtained in relation with the content-wise specificities of each quiz. So let us now look at the evolution of the group's general theoretical thinking tendency (ggt in the sequel), but this time, taking into account the mathematical content of each quiz (see next page):

Inner product spaces: - example of "almost inner product" - proof verification: "is <.,.> an inner product?"	Orthogonal vectors: - dimension of orthogonal complement - norm of the sum of two vectors.	Orthogonal matrices: - definition and product; - 2x2 orthogonal matrices.	Preliminaries of + the Cayley-Hamilton theorem - polynomial with matrices as roots; - converse of C.-H.	Characteristic polynomial, minimal polynomial of a matrix: - definition, finding min.poly. of a block diagonal matrix; - candidates of minimal polynomial.	Characteristic polynomial, minimal polynomial of an operator: - finding them for a given op. T-invariant subspaces: - intersection of T-invariant subspaces	Restriction of a linear operator to a subspace: - finding a matrix representation of the restriction of a lin.op. to a subspace (which is the kernel of another lin.op.) Assumptions of PDT

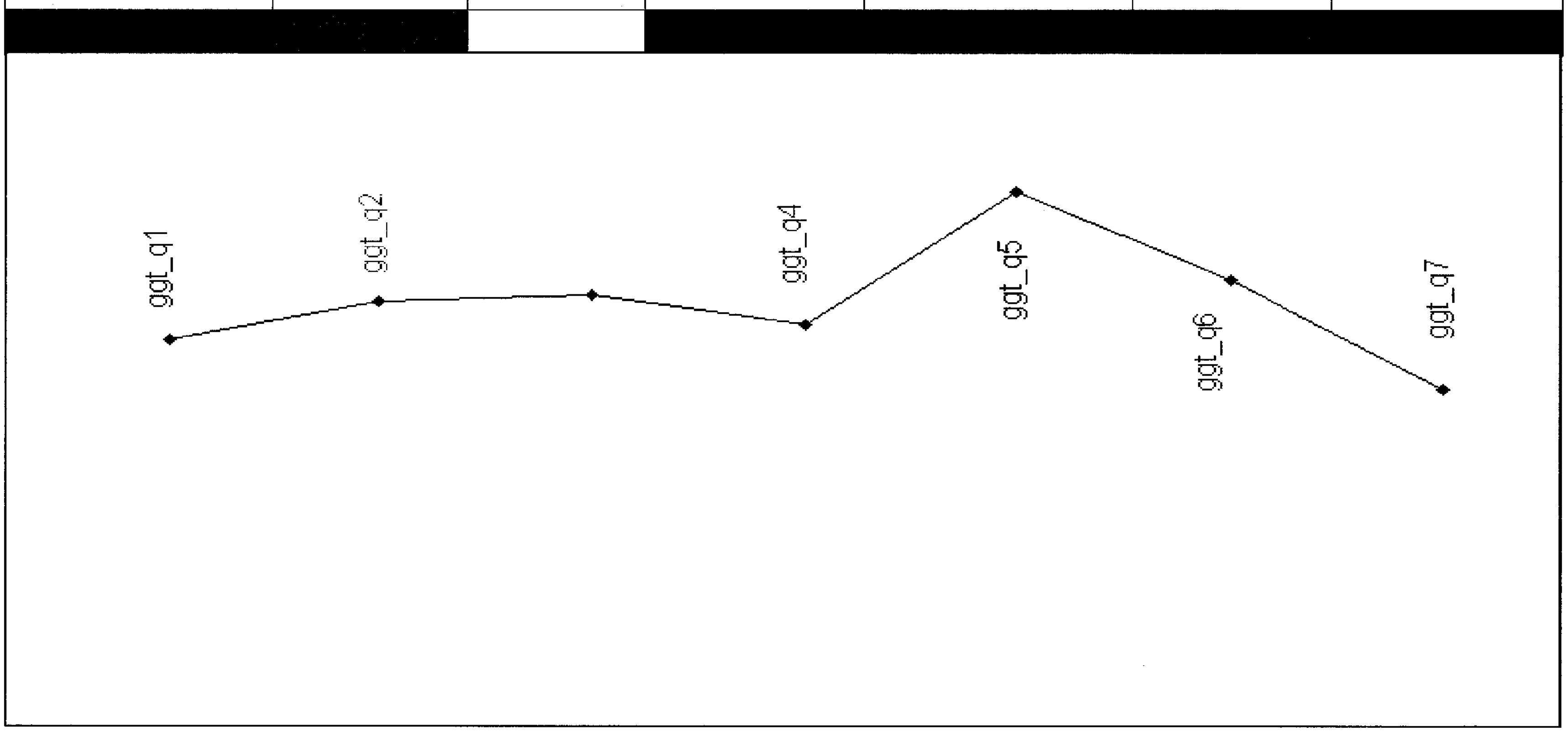


Figure 3: Evolution of the group's general tendency to think theoretically and mathematical content of the quizzes.

We will organize our comments of this evolution by breaking the graph into four segments, corresponding to increasing or decreasing portions of the ggt index:

1. from ggt_q1 to ggt_q3: the group's general tendency to think theoretically increases from 52% to 58%. This segment corresponds to the first three quizzes, which were all on the subject of inner product spaces and orthogonality. The context being somewhat unchanged, we may have a better comparability of results here. The increase observed in the ggt can thus be accounted for by the use of quizzes as a tool for teaching. There is not, however, a dramatic development of our group's inclination to think theoretically.
2. from ggt_q3 to ggt_q4: a drop in the ggt can be observed, from 58% to 54%. The change is again not very significant and the decrease is probably explained by the difficulties that our students encountered in quiz 4, when dealing with matrices structurally.
3. from ggt_q4 to ggt_q5: this time the change is quite important, from 54% in quiz 4 to 72% in quiz 5. We wouldn't however jump to the conclusion that the increase in the ggt is exclusively the result of the use of quizzes. Our reason for saying this is that the problems in quiz 5 involved the notions of characteristic and minimal polynomial of a matrix. Students, perhaps, found themselves in a more familiar domain here: they were required to view matrices as arrays of numbers, thus in an arithmetic-analytic fashion (see Sierpinska, 2000).

4. from ggt_q5 to ggt_q7:

- the group's general tendency to think theoretically first decreases from 72% in quiz 5 to 60% in quiz 6. One reason for this decrease may have been the change of setting that occurred: in one of the problems students had to find the characteristic and the minimal polynomial of a linear operator (as compared to that of a matrix, in the previous quiz). They had to go through the representational problems that this change of mode of description brought along (see Hillel, 2000). Many of them succeeded to find the representation matrix of the given linear operator, but much fewer seemed have had an awareness of the shift in the modes of description. This lack of awareness was reflected in their linguistic behavior, whence the low value of the index fgt_Analytic/linguistic/notations, apparently influencing the low value of ggt_q6.

- the ggt decreases even further, from 60% in quiz 6 to 45% in quiz 7. This is where, we believe, the mathematical content involved in the quizzes can definitely be considered the main factor influencing the decrease. As we have mentioned in our analysis of these last two quizzes, at that point in the course the complexity that students had to deal with increased, as new concepts and results kept accumulating. This called for their ability to think in terms of systems of concepts, where newly introduced concepts acquired meaning from their relations with already existing, hopefully internalized concepts. It was precisely students' engagement with systemic thinking that was low in these quizzes and accounted for the smaller value of the gtt index.

Another, more refined conclusion emerges now from our comments above: the group's tendency to think theoretically is strongly influenced by the specificity of the context at hand in each quiz. At this point, for our question: "Can weekly quizzes contribute to the development of theoretical thinking in students?", we can formulate a more nuanced answer. *An increase in the group's tendency to think theoretically can be attributed to the administration of weekly quizzes only in the first four weeks of classes. For the rest of the period, the mathematical content of the questions in the quizzes, different from week to week, influenced significantly our students' engagement with theoretical thinking, obscuring thus our attempts to evaluate the effect of the administration of quizzes on the development of theoretical thinking in students.*

However another question arises now with regard to the effectiveness of our tools for measuring the evolution of our students' tendency to think theoretically: is the model of TT, as we have employed it in our analyses, at all effective for measuring the development of a group's tendency to think theoretically?

It was again our "qualitative eye" that directed us to an answer to this question. As we were doing our analyses of the data, we had, sometimes, the impression that something *was* improving: our students seemed to have better mathematical "habits", so to speak, in their problem solving behavior: they paid more attention to assumptions, they were more rigorous in their proofs, they were more attentive to the requirements of the problem, etc. And in fact, when we looked at the "feature" indices (labeled "fgt" – "feature" group tendency) our impression was confirmed by the quantitative results. In the next section we examine the evolution of the group's TT tendencies with regard to particular features of TT, an analysis that allowed us to provide an answer to the question we have formulated in the previous paragraph.

EVOLUTION OF THE GROUP'S TENDENCY TO THINK THEORETICALLY WITH REGARD TO SPECIFIC FEATURES OF THINKING

We consider, in this analysis, the fgt indices for all the features of theoretical thinking that were tested at least three times in the quizzes. When a feature appeared twice in the same quiz (i.e., in both questions of the quiz) we took the average of the corresponding fgt indices. The fgt indices corresponding to two features of TT have shown an increasing trend along the semester (see Figure 4a and b, below)

- the sensitivity to assumptions (present in definitions, theorems, or the problem at hand);
- the meta-linguistic sensitivity to logic.

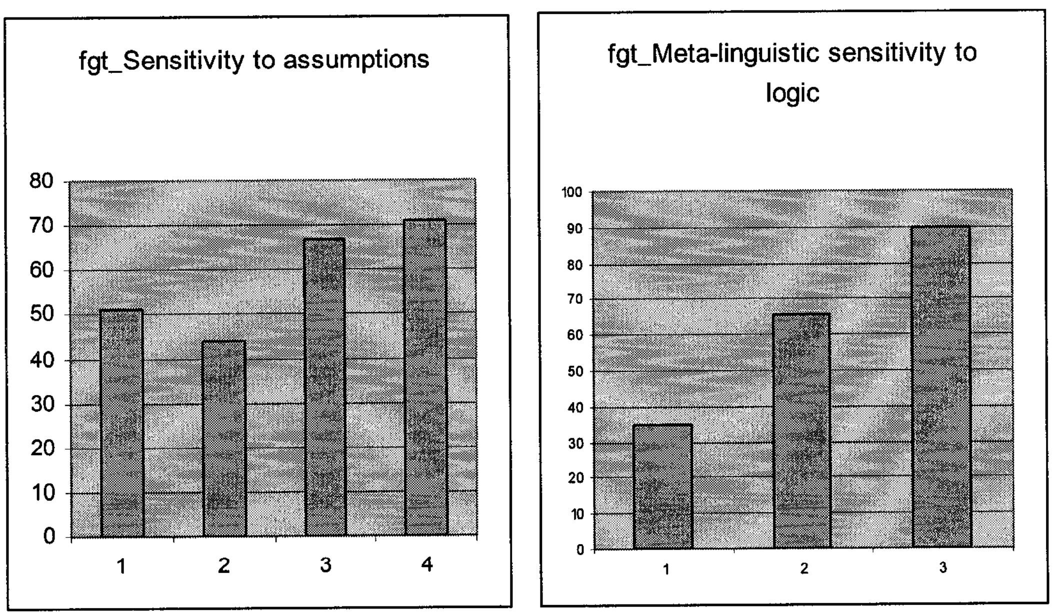


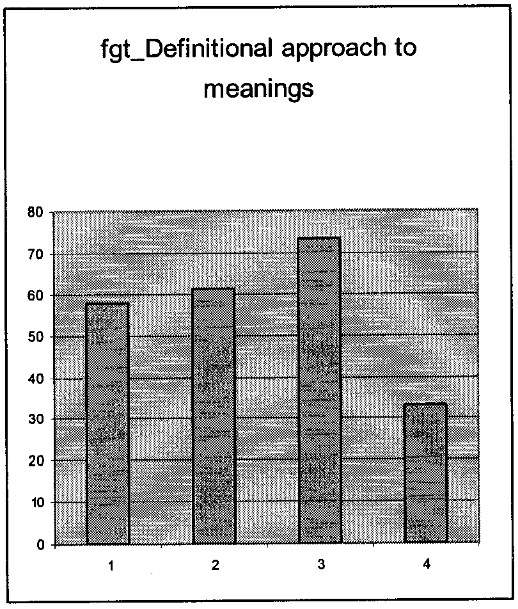
Figure 4. "fgt" indices – increasing.

a. fgt_Sensitivity to assumptions
b. fgt_Meta-linguistic sensitivity to logic

The following features of TT were also tested at least three times through the quizzes, along the semester:

- the definitional approach to meanings;
- the linguistic sensitivity to notations and terminology;
- the concern with validity.

The *fgt_definitional approach to meanings* index followed an increasing trend for the first three quizzes in which it appeared, but in the last quiz (quiz 7) it suffered a dramatic drop (see figure 5a below). The index calculated for the "linguistic sensitivity to notations and terminology" feature varied insignificantly in the beginning (but in the high ranges: between 69 and 71%), to then decrease in a sensible manner towards the end (see figure 5b). The *fgt_concern with validity* index did not seem to have a steady increasing or decreasing trend along the semester (see figure 6).



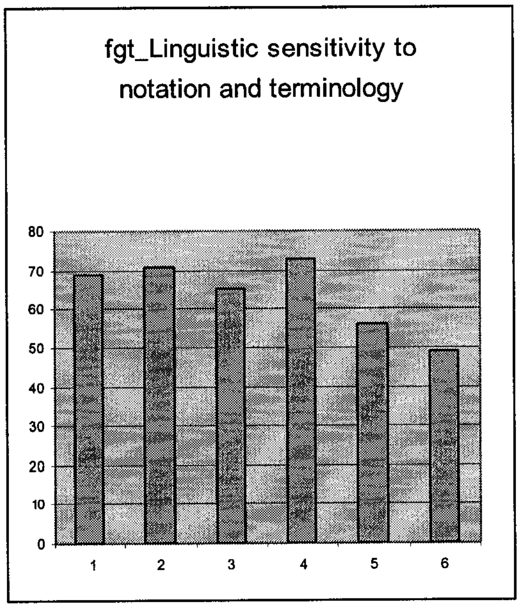


Figure 5. "fgt" indices – decreasing towards the end.

a. fgt_definitional approach to meanings
b. fgt_ linguistic sensitivity to notations and terminology.

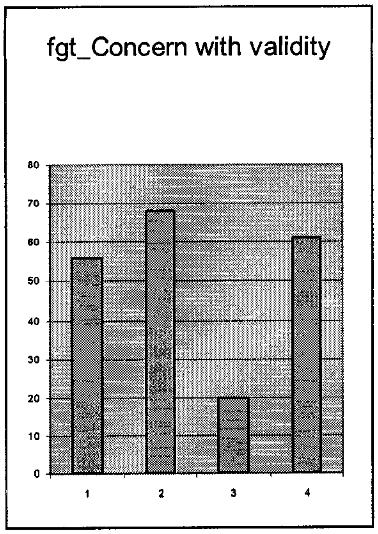


Figure 6. "fgt" index – irregular trend.

The important decrease in Figure 6 is accounted for by a piece of argument in quiz 4, question 1, that students could only arrive at if they performed other several steps in the solution. Thus, many answers were coded with 0 on the feature "concern with validity" because students hadn't performed the steps in the solution that would have put them in the situation of being concerned with the validity of the result.

The two indices: *fgt_definitional approach to meanings* and *fgt_ linguistic sensitivity to notations and terminology* follow largely the same trend as our general index: they reach a peak, to then drop in the last quizzes. We dare to conjecture that this behavior is caused by the fact that the engagement with any of these two features of thinking in a problem solving setting is directly conditioned – compared to other features of theoretical thinking – by a correct understanding of the linear algebra concepts involved. One's definitional approach to the meaning of a notion is conditioned by one having formed a correct concept of that notion. Thus, the nature of the concepts involved influences directly students' engagement with this feature of thinking. It is then of no surprise that when the student has to manage a large number of concepts in a systemic way, with all the relations that such complex systems entail, he will not spontaneously refer to definitions when using one or another linear algebra concept.

The same direct conditioning by the mathematical content characterizes the feature "linguistic sensitivity to notations and terminology". It has been argued by many (Alves Dias&Artigue, 1995; Pavlopoulou, 1993; Hillel, 2000; Sierpinska et al., 2002) how linguistically demanding linear algebra is. The students use incorrect terminology or notation and often form non-sensical sentences because they have not grasped the meanings of the concepts that they want to denote in their written language.

On the other hand, features as: "sensitivity to assumptions", "meta-linguistic sensitivity to logic" or "concern with validity" could be, perhaps, characterized as "habits of the mind". Even if the nature of the concepts involved may obscure students' capability of observing assumptions or their logic abilities, the influence of the content on such abilities is not as direct as for the feature "definitional approach to meanings", for instance. Such "habits of the mind" may be enhanced through a certain class culture, which includes the weekly quizzes as means for cognitive communication between the teacher and the students and our method of measuring the development of theoretical thinking is more likely to succeed in checking if such an enhancement has taken place. *The effectiveness of our method for comparing a group's tendency to think theoretically at different moments in time is limited to only a part of the features included in the model of theoretical thinking, namely the ones that are less dependent on the mathematical content involved.* We could thus evaluate the effect of quizzes on only three of the features of thinking tested by the quiz:

- the students, as a group, developed their hypothetical thinking, in the sense of becoming more sensitive to the assumptions made in various mathematical statements (theorems, definitions, formulations of problems, etc.).

- they have also become more sensitive to the logic of the mathematical language.

- the quizzes, however, did not influence their concern with epistemological validity. The students were, in general, preoccupied with justifying their results, but we have only evaluated their behavior with regard to the "concern with validity" feature when a particular result seemed obvious, thus signaling an authentic concern with validity and not justifications possibly brought about by the didactical contract. This genuine concern did not seem to improve in students as a result of the quizzes. The reason can be sought in the nature of the problems that tested this feature of thinking. We observed that in some of them the concern with validity was either not relevant for obtaining the good result in the problem, or the lack of concern with validity was not "sanctioned" by the grading. For instance, the problem asking students to provide an example of a product that satisfies only two properties of the inner product was possible to solve using the example given in class the previous lecture, containing a "recipe" of how one can construct such a product. The concern with validity was not that relevant then in order to solve the problem.

CONCLUSIONS

We wanted, through this research, to give the teacher of linear algebra a practical tool for bringing about desired ways of thinking in his students. We also wanted to give the researcher in mathematics education a more refined model for analyzing students' pieces of reasoning. What can they learn from the findings of our research?

THE IMPLICATIONS OF OUR STUDY FOR TEACHING

Our study showed that the administration of weekly quizzes could not account for the development, in general, of theoretical thinking, with all the characteristics that it encompasses. *The main factor influencing our group's engagement with theoretical thinking was the mathematical content of the questions in the quizzes.* The engagement with some of the features of theoretical thinking, especially the definitional approach to meanings and the linguistic sensitivity is very strongly conditioned by the specific nature of the linear algebra concepts. When the complexity of the systems of concepts increases students refer less to definitions and become more incoherent in using formal notations and terminology. Dorier et al. (2000) have reached a similar conclusion with regard to the obstacle of formalism that students encounter in linear algebra: students do not just have a problem with formalism in general but with the use of formalism within the theory of vector spaces.

However, we have found that the administration of weekly quizzes contributed to the development of our students' hypothetical thinking, in particular to their awareness of the conditional character of mathematical statements. It has also made them more sensitive to the logic embedded in the mathematical language. These features of thinking belong to a more general domain of mathematical thinking, we believe, and even if they are to some extent conditioned by the mathematical context, the influence is less direct. The quizzes do have the potential of developing these features of theoretical thinking, that we called, more generally, "habits of the mind". This potential was not realized for the "concern with validity" feature of theoretical thinking. As a group, our students were, already at the outset, preoccupied with justifying their statements and also quite able in writing mathematical proofs. They seemed to have a habit of justifying their claims. However, fewer students exhibited a genuine concern with epistemological validity and, the administration of quizzes did not, apparently, change this situation. Some problems in the quizzes did appear to have the potential of making the concern with validity necessary, but somehow, our students were able to avoid it. We have talked about the problem that asked students to give an example of a product satisfying only two properties of the inner product, and how many students solved it almost automatically, without much concern for epistemological validity, by simply making the matrix **A** (a 2x2 matrix) in the product $<\mathbf{x},\mathbf{y}>=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y}$ meet certain conditions. This question also appeared to have the potential of engaging students in axiomatic reasoning, but it did not fulfill this expectation either. Not only with regard to this question, but also regarding other questions in the quizzes we have asked ourselves: how could they be changed to make the engagement with theoretical thinking necessary? For instance, how would our students answer to the following alternative formulation of the question above: "Give an example of a product that is symmetric and positively defined, but it is not linear"? Or, what if the product were required to be defined over a vector space of dimension equal to at least 3? For sure, they would be forced to think of another example than the one whose parameters could so easily be modified to obtain the product that satisfied the requirements of the problem, and this would have made the concern with validity and the engagement with axiomatic reasoning necessary. *These inquiries make us suggest that the value of our study for a teacher does not lie so much in our conclusions with regard to the development of one or another feature of theoretical thinking in students. We think that what is more interesting, for them, is the kind of analysis we have performed, of the questions in the quizzes and of the students' answers, an analysis characterized by the concern for the tasks' potential to enhance desired cognitive behaviors.* Awareness of the potential of a task, or of a "piece of mathematics", to foster a desired students' behavior, can help a teacher have better chances at changing students' ways of thinking, and gaining this access, we believe, is in a way, one of the most desirable goals of teaching, especially of teaching at university level. Thus, our method, rather than our results, is perhaps of more interest for a teacher of linear algebra, and of any mathematics, for that matter.

METHODOLOGICAL REFLECTIONS

We had the illusion that the quantitative nature of our research would allow us to give a quite definite and reliable answer to our research question. It didn't: we could not evaluate, even by analyzing a large amount of data, the impact of quizzes on the development of theoretical thinking. We could advance the hypothesis that the model we have used was not effective for that purpose, and maybe we should look for another model, or alternatively, strive for its improvement. But we do not think that this is the reason of our failure to provide a precise answer to our research question. The quizzes, we realized, were not part of a minimalist approach, as we have initially thought. Many other factors were in action: the teaching has used the meta-lever advocated by Dorier et al.(2000) and was also characterized by the long-term strategy suggested by these researchers. These didactical choices were not explicitly and maybe not consciously integrated in teaching, but reflected implicit goals of the instructor of the course. The quizzes then were only another tool for teaching among others, and therefore their exact impact on the development of theoretical thinking may not even be possible to evaluate.

It was nevertheless the quantitative nature of our research that allowed us to make sensitive analyses of the questions and of the students' answers. By analyzing, for instance, the solutions of 50 students to a given question, we were able to find in the answers of some of the students the explanations of other students' conceptual misunderstandings and also draw more "grounded" conclusions with regard to the cognitive source of an observed mistake.

We think that our way of analyzing the data partially describes a research tool based on task problematization. We say partially, because we only perform the first phase of this process, where we analyze the specific content of the task and observe the impact of its characteristics on students' learning. This should be completed then by a proposal of alternative formulations of the task and perhaps of the study of the changes in learning determined by a different "piece of mathematics". We strongly believe in the importance of detailed epistemological and didactic analyses of the mathematical concepts involved in a teaching or learning situation for evaluating the impact of that teaching or the characteristics of that learning. Our task analyses, always concerned with the relationship between a specific piece of mathematics and its cognitive impact on students, have contributed, we believe, to the endeavor of *keeping the mathematics in mathematics education research*.

REFERENCES

Alves Dias, M. & Artigue, M.: 1995, 'Articulation problems between different systems of symbolic representations in linear algebra', *Proceedings of the 19th Annual Meeting of the International Group for the Psychology of Mathematics Education, Universidade Federal de Pernambuco, Recife, Bresil, Vol. 3*, 34-41.

Aristotle's Metaphysics, 1966, translated with commentaries and glossary by Hippocrates G. Apostle, Indiana University Press, Bloomington and London.

Bachelard, G.: 1983, *La Formation de l'Esprit Scientifique*, Presses Universitaires de France, Paris (First published in 1938).

Balacheff, N. : 1999, *Apprendre la preuve*. In: J. Sallantin & J. J. Szczeciniarz (Eds.), Le concept de preuve de la lumière de l'intelligence artificielle. Paris: Presses Universitaires de France, 197-236.

Belenky, M.F., Clinchy, B. M., Goldberger, N. R., Tarule, J. M.: 1997, *Women's Ways of Knowing, The Development of Self, Voice, and Mind*, BasicBooks, Inc., NewYork.

Bell, A.W.: 1976, 'A study of pupil's proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7(1), 23-40.

Bell, A.W.: 1976, 'A study of pupil's proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7(1), 23-40.

Blaga, L.: 1969 (reprinted in 1998), *Experimentul si Spiritul Matematic*, Editura Humanitas, Bucuresti.

Boero, P., Pedemonte, B., Robotti, E. 1997: 'Approaching Theoretical Knowledge through Voices and Echoes: a Vygotskian Perspective', *Proceedings of the 21st*

Conference of the International Group for the Psychology of Mathematics Education, Lahti, Finland, 1997, Vol.2, pp. 81-88.

Dewey, J.: 1933, *How we think, a Restatement of the Relation of Reflective Thinking to the Educative Process*, D.C. Heath & Co., Boston.

Dorier, J.-L. (Ed.): 2000, *On the Teaching of Linear Algebra.* Dordrecht: Kluwer Academic Publishers.

Dorier, J.-L., Sierpinska, A.: 2001, 'Research into the teaching and learning of linear algebra', in D. Holton (ed.), *The Teaching and Learning of Mathematics at University Level. An ICMI Study*, Kluwer Academic Publishers, Dortrecht/Boston/London, pp. 255-274.

Dubinsky, E., Harel, G..: 1992, 'The Nature of the Process Conception of Function' , in E. Dubinsky, and G. Harel (eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, Mathematical Association of America, MAA Notes Volume 25, pp.85-107.

Ferrari, P.L.: 2004, 'Mathematical language and advanced mathematics learning', *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway, 2004*, Vol.2, pp. 383-390.

Fischbein, E.: 1987, *Intuition in Science and Mathematics. An Educational Approach*, D. Reidel Publishing Company, Dortrecht.

Goldenberg, P., Lewis, P., O'Keefe, J.: 1992,'Dynamic Representation and the Development of a Process Understanding of Function', in E. Dubinsky, and G. Harel (eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, Mathematical Association of America, MAA Notes Volume 25, pp.235-261.

Harel, G.: 2000, 'Three principles of learning and teaching mathematics', in J.-L. Dorier (ed.), *On the Teaching of Linear Algebra,* Kluwer Academic Publishers, Dortrecht/ Boston / London, pp. 177-191.

Harel, G. und Sowder, L.:1998, *Students' proof schemes: Results from exploratory studies.* In: CBMS Issues in Mathematics Education, American Mathematical Society, 7, 234-283.

Hillel, J.: 2000, 'Modes of description and the problem of representation in linear algebra', in J.-L.Dorier (ed.), *On the Teaching of Linear Algebra,* Kluwer Academic Publishers, Dortrecht/ Boston / London, pp. 191-209.

Hillel, J. and Sierpinska, A.: 1994, 'On One Persistent Mistake in Linear Algebra'. In *The Proceedings PME 18*, pp. 65-72, University of Lisbon, Portugal.

Kant, I.: 1873 (reprinted in 1956), *Critique of practical reason*; translated, with an introd., by Lewis White Beck, New York, Liberal Arts Press.

Pavolopoulou, K.: 1993, 'Un probleme decisif pour l'apprentissage de l'algebre lin.aire: la coordination des registres de repr.sentation', *Annales de Didactique et de Sciences Cognitives 5*, Strasbourg, IREM, 67-93.

Perry, W. G.: 1970, *Forms of Intellectual and Ethical Development in the College Years*, Holt, Rinehart & Winston, New York.

Robert, A.: 1992, 'Projets Longs et Ingenierie pour L'Enseignement Universitaire: Questions de Problematique et de Methodologie. Un exemple: un Enseignement Annuel de Licence en Formation Continue'. *Recherches en Didactique des Mathematiques*, 12(2/3), 181-220.

Robert, A. and Robinet, J.: 1996, 'Prise en Compte du Meta en Didactique des Mathematiques'. *Recherches en Didactique des Mathematiques*, 16(2), 145-176.

Sfard, A.: 1992, 'Operational origins of mathematical objects and the quandary of reification – the case of function', in E. Dubinsky, and G. Harel (eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, Mathematical Association of America, MAANotes Volume 25, pp.59-84.

Sierpinska, A.: 2000, 'On Some Aspects of Students' Thinking in Linear Algebra. In J-L. Dorier (Ed.)', *On the Teaching of Linear Algebra*, pp. 209-246. Dordrecht: Kluwer Academic Publishers.

Sierpinska, A., Nnadozie, A., Oktaç, A.: 2002, *A Study of Relationships between Theoretical Thinking and High Achievement in Linear Algebra*. Manuscript. Concordia University.

Skemp, R.: 1971, *The Psychology of Learning Mathematics*, Harmondsworth, Eng.: Penguin.

Vygotsky, L.S.: 1987, The Collected Works of L. S. Vygotsky. Volume 1. Problems of General Psychology, including the volume *Thinking and Speech*, Plenum Press, New York and London.

Wittgenstein, L.: 1974, *Philosophical grammar*, edited by Rush Rhees; translated by Anthony Kenny, Oxford University Press, 1974.