Existence, Approximation and Properties of Absolutely Continuous Invariant Measures for Random Maps

Md. Shafiqul Islam

A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University.
Montreal, Quebec, Canada.

December 2004

Your file Votre référence
ISBN: 0-494-04042-4
Our file Notre référence
ISBN: 0-494-04042-4

Abstract

Existence, Approximation and Properties of Absolutely Continuous Invariant Measures for Random Maps

Md Shafiqul Islam, Ph.D.

Concordia University, 2004

A random map is a discrete-time dynamical system where one of a number of transformations is selected randomly and applied in each iteration of the process. In this thesis we study existence, approximation and properties of absolutely continuous invariant measures (acim) for random maps and obtain several new results. We generalize a result of Straube, which provides a necessary and sufficient condition for existence of an acim of a nonsingular map, to random maps. We approximate absolutely continuous invariant measures for Markov switching position dependent random maps using Ulam's method. For certain random maps, we prove the existence of ergodic infinite acims. Finally, we prove that the invariant density of an acim for random maps is

strictly positive on its support.

Acknowledgments

I am grateful to my supervisors, professor Pawel Góra and professor Abraham Boyarsky of Concordia University, for their guidance, encouragement and valuable suggestions throughout the preparation of this thesis. As well, I wish to thank Dr. Pawel Góra and Dr. Abraham Boyarsky for partial support of this research through their NSERC grant.

I would like to thank professor Hal Proppe and S.T. Ali with whom I took very useful courses in measure theory and functional analysis respectively.

I would also like to thank Dr. Wael Bahsoun with whom I had numerous conversations.

I am grateful to the Department of Mathematics and Statistics, Concordia University, for financial support of this research. Finally, I am grateful to the Department of Mathematics, Shah Jalal University, Bangladesh, for the study leave and support.

Dedication

TO THE MEMORY OF MY FATHER IN LAW

Late Md. Mokhlesar Rahman

Contents

Introduction

The fundamental problem in the ergodic theory of dynamical systems is to describe the asymptotic behavior of trajectories defined by a dynamical system. In general, the long time behavior of trajectories of a chaotic dynamical system is unpredictable. Therefore, it is natural to describe the behavior of the system as a whole by statistical means. In this approach, one attempts to describe the dynamics by proving the existence of an invariant measure and determining its ergodic properties. In particular, the existence of invariant measures which are absolutely continuous with respect to Lebesgue measure is very important from a physical point of view, because computer simulations of orbits of the system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure [15]. The Birkhoff Ergodic Theorem [12] states that if $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and $\mu-$invariant and E is a measurable subset of X then the orbit of almost every point of X occurs in the set E with asymptotic frequency $\mu(E)$. The Birkhoff Ergodic Theorem establishes the dynamical importance of an invariant measure, but it says nothing about the existence of invariant measures. Thus, the existence of absolutely continuous invariant measures is one of the most important problem in the ergodic theory of dynamical systems.

In 1940, Ulam and von Neumann found examples of transformations having absolutely continuous invariant invariant measure (acim). In 1957, Rényi [31] defined a class of transformations that have an acim. Rényi's key idea of using distortion estimates has been used in the more general proofs of Adler and Flatto [1]. In 1973, Lasota and Yorke [26] proved a general sufficient condition for the existence of an absolutely continuous invariant measure for piecewise expanding C^2 transformations. Their result was an important generalization of Renyi's [31] result using the theory of bounded variation and their essential observation was that, for piecewise expanding transformations, the Perron-Frobenius operator is a contraction. Since then the bounded variation techniques has been generalized in a number of directions [22].

Ulam and von Neuman [38] suggested the study of more general dynamical systems, namely random dynamical systems. Random dynamical systems provide a useful framework for modeling and analyzing various physical, social, and economic phenomena [7, 33]. A random dynamical system of special interest is a random map where the process switches from one map to another according to fixed probabilities [30] or, more generally, position dependent probabilities [2, 3, 4, 5, 13, 14]. The existence and properties of invariant measures for random maps reflect their long time behavior and play an important role in understanding their chaotic nature. Such dynamical systems have recently found application in the study of fractals [6], in modelling interference effects in quantum mechanics [7], in computing metric entropy [34], and in forecasting the financial markets [2]. In 1984 Pelikan [30] proved sufficient condition for existence of acim for random maps with constant probabilities. Morita [28] proved a spectral decomposition theorem. Góra and Boyarsky [13] proved sufficient conditions for the existence of acim for random maps with position dependent probabilities. Bahsoun and Góra [4, 5] proved sufficient average expanding conditions for the existence of acim for position dependent random maps in one and higher dimensions, weakly convex and concave position dependent random maps. Recently, Froyland [10] , Bahsoun, Góra and Boyarsky [2] proved the sufficient condition for the existence of Markov switching random map with constant switching matrix and position dependent switching matrix respectively.

The Frobenius-Perron operator P_τ is the main tool for proving the existence of acim of a transformation τ. It is well known that f is the density of an acim μ under a transformation τ if and only if $P_\tau f = f$. Once the existence result is established, the next question is: can we actually find the invariant measure. Unfortunately, the operator P_τ is an infinite dimensional operator and it is difficult to solve the functional equation $P_\tau f = f$ except in some simple cases. However, we can approximate the fixed point of the Frobenius-Perron operator P_τ by the fixed point of a matrix operator. Approximation of invariant measures was suggested by Ulam [37]. For a single transformation, Li [25] first proved convergence of Ulam's approximation. In [10] Froyland extended Ulam's method for a single transformation to random maps with constant probabilities [25]. Góra and Boyarsky in [13] proved the convergence of Ulam's approximation for position dependent random maps. For Markov switching random maps Froyland [10] considered the constant stochastic irreducible matrix W and proved the existence and convergence of Ulam's approximation of invariant measures.

Since absolutely continuous invariant measures (acim) for a dynamical system describes the long time behavior of trajectories of the system, it is also important to study the properties of acim. There are results on properties of the acim for single transformations[12]. Now the question is: do analogous results hold for random maps.

In this thesis first we prove the necessary and sufficient conditions for the existence of absolutely continuous invariant measures (acim) for certain classes of random maps. Then we approximate acim for a class of random maps. Then we study random maps with ergodic infinite acims. Finally, we study properties of acim for random maps.

In Chapter 1, notations, definitions, results from ergodic theory, random maps theory, Frobenius-Perron operators are presented. In Chapter 2, we prove necessary and sufficient condition for the existence of invariant measure for random maps. This new result is a generalization of Straube's Theorem [35]. In Chapter 3, we introduce Markov switching position dependent random maps. We prove the sufficient condition for the existence of an acim of Markov switching random maps using bounded variation method and we approximate the invariant measure using the method of Ulam. In Chapter 4, we study random maps with ergodic infinite acims. In Chapter

5, we prove that the invariant density of certain class of random maps is strictly positive on its support.

Now we present an example of a Markov switching position dependent random map $T = \{\tau_1, \tau_2; p_1, p_2; W\}$, that satisfies the sufficient condition for existence of acim in Theorem 3.1 in chapter 3. τ_1, τ_2 are maps on $I = [0, 1]$ defined by

$$\tau_1(x) = \begin{cases} 4x, & 0 \leq x \leq \frac{1}{4}, \\ 4x - 1, & \frac{1}{4} < x \leq \frac{1}{2}, \\ 4x - 2, & \frac{1}{2} < x \leq \frac{3}{4} \\ 4x - 3, & \frac{3}{4} < x \leq 1 \end{cases} \tag{1}$$

and

$$\tau_2(x) = \begin{cases} \frac{8}{3}x & , 0 \leq x \leq \frac{1}{3}, \\ \frac{8}{3}x - \frac{8}{9} & , \frac{1}{3} < x \leq \frac{2}{3}, \\ \frac{8}{3}x - \frac{16}{9} & , \frac{2}{3} < x \leq 1. \end{cases} \tag{2}$$

and W ia a stochastic switching matrix defined by

$$W = \begin{bmatrix} \frac{1}{2}x + \frac{1}{10} & \frac{9}{10} - \frac{1}{2}x \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix},$$

and p_1, p_2 are initial probabilities. By Theorem 3.1 the random T has an absolutely

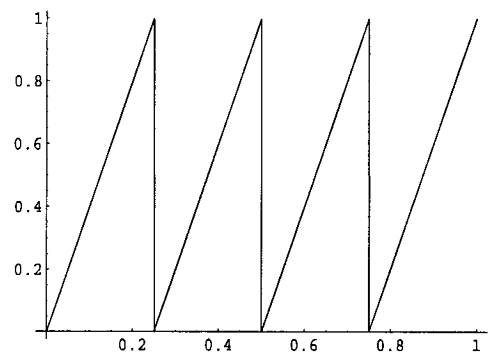


Figure 1: The graph of τ_1.

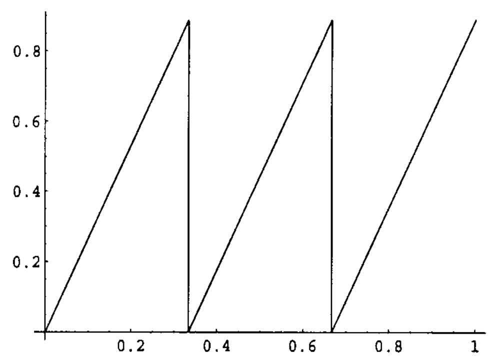


Figure 2: The graph of τ_2.

continuous invariant measure. In Chapter 3 we describe a method of approximation fixed point of the Frobenius-Perron operator of T by fixed point of a matrix operator and obtain piecewise constant densities for T.

Chapter 1

Preliminaries

1.1 Ergodic theory of dynamical systems

1.1.1 Definitions and notations

Let $(X, \mathcal{B}, \mu)$ be a normalized measure space where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a measure such that $\mu(X) = 1$. Let I be an interval of the real line $\mathbb{R}$. Throughout this thesis we denote by $V_I(\cdot)$ the standard one dimensional variation of a function on $[0, 1]$ and $BV(I)$ the space of functions of bounded variations on I equipped with the norm $\| \cdot \|_{BV} = V_I(\cdot) + \| \cdot \|_1$, where $\| \cdot \|_1$ denotes the norm on $L^1(I, \mathcal{B}, \mu)$. A transformation $\tau : X \to X$ is nonsingular if for any $A \in \mathcal{B}$ with $\mu(A) = 0$, we have $\mu(\tau^{-1}(A)) = 0$. Let ν be another measure on the measure space $(X, \mathcal{B}, \mu)$. The measure μ is absolutely continuous with respect to ν if for any $A \in \mathcal{B}$ with $\nu(A) = 0$, we have $\mu(A) = 0$. A measurable transformation $\tau : X \to X$ preserves measure μ or the measure μ is τ-invariant if $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$. In this case the quadraple $(X, \mathcal{B}, \mu, \tau)$ is called a dynamical system. A measure-preserving transformation $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic if for any $B \in \mathcal{B}$ such that $\tau^{-1}B = B$, we have $\mu(B) = 0$ or $\mu(X \setminus B) = 0$. For $x \in X$, the nth iteration of x is defined by

$$\tau^n(x) = \tau^{n-1} \circ \tau^{n-2} \circ \ldots \circ \tau(x).$$

A basic result of ergodic theory is the Birkhoff Ergodic Theorem.

Theorem 1.1 ***Birkhoff's Ergodic Theorem [12]:*** *If μ is an invariant measure under $\tau : (X, \mathcal{B}, \lambda) \to (X, \mathcal{B}, \lambda)$ and $f \in L^1(X, \mathcal{B}, \lambda)$, then there exists a function $f^* \in L^1(X, \mathcal{B}, \lambda)$ such that for $\mu-$almost all $x \in X$ the limit of the time averages $\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x))$ exists and*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)) \to f^*, \tag{1.1}$$

$\mu-$ almost everywhere. Moreover, if τ is ergodic and $\mu(X) = 1$, then f^* is constant μ a.e. and $f^* = \int_X f d\mu$.

In particular, for any $E \in \mathcal{B}$

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_E(\tau^k(x)) \to \mu(E), \tag{1.2}$$

$\mu-$ almost everywhere and thus the orbit of almost every point of X occurs in the set E with asymptotic frequency $\mu(E)$.

The following three theorems will be useful later.

Theorem 1.2 **[Helly's Selection Principle[12]]** *Let $\mathcal{B}$ be a family of functions such that $f \in \mathcal{B} \Rightarrow \bigvee_{[a,b]} f \leqq \alpha$ and $|f(x)| \leqq \beta$, for any $x \in [a,b]$. Then there exists a sequence $\{f_n\} \subset \mathcal{B}$ such that $f_n \to f^* \ \forall x \in [a,b]$ and $f^* \in BV[a,b]$.*

Theorem 1.3 **[Mazur's Theorem[12]]** *Let X be a Banach space with $A \subset X$ relatively compact. Then $\overline{\text{co}}(A)$ is compact where* $\text{co}(A)$ is the convex hull of A and $\overline{\text{co}}(A)$ denotes its closure with respect to the metric topology.

Theorem 1.4 **[Kakutani-Yoshida [12]]** *Let $T : X \to X$ be a bounded linear operator from a Banach sapce X into itself. Assume that there exists $M > 0$ such that $\| T^n \| \leqq M$, $n = 1, 2, \cdots$. Furthermore, if for any $f \in A \subset X$, the sequence $\{f_n\}$, where $f_n = \frac{1}{n}\sum_{k=1}^{n} T^k f$, contains a sub-sequence $\{f_{n_k}\}$ which converges weakly in X, then for any $f \in \bar{A}$, $\frac{1}{n}\sum_{k=1}^{n} T^k f \to f^* \in X$ (norm convergence) and $T(f^*) = f^*$.*

Recall that a set $A \subset X$ of a Banach space X is called relatively compact if every infinite subset of A contains a sequence that convergences to a point of X.

1.1.2 Frobenius-Perron operator: A tool for proving the existence of an absolutely continuous invariant measure

Consider the measure space $([a,b], \mathcal{B}, \lambda)$. Let $\mathcal{M}([a,b])$ denote the space of measures on $([a,b], \mathcal{B})$. Let $\tau : ([a,b], \mathcal{B}, \lambda) \to ([a,b], \mathcal{B}, \lambda)$ be a piecewise monotonic nonsingular transformation on the partition $\mathcal{P}$ of $[a,b]$: $\mathcal{P} = \{I_1, I_2, \ldots, I_N\}$ and $\tau_i = \tau|_{I_i}$. Let μ be a measure absolutely continuous with respect to λ. The transformation τ

induces an operator $\mathcal{O}$ on $\mathcal{M}([a,b])$ defined by

$$\mathcal{O}(\mu)(A) = \mu(\tau^{-1}(A)).$$

Since τ is nonsingular $\mathcal{O}(\mu) << \lambda$. If μ has a density f with respect to λ, then $\mathcal{O}(\mu)$ has a density $P_\tau f$. Thus

$$\mathcal{O}(\mu)(A) = \int_A P_\tau f d\lambda = \mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f d\lambda.$$

Clearly, $P_\tau : L^1([a,b], \mathcal{B}, \lambda) \to L^1([a,b], \mathcal{B}, \lambda)$ is a linear operator. This operator has the following representation [12] :

$$P_\tau f(x) = \sum_{i=1}^{N} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1},a_i)}(x), \tag{1.3}$$

where $\tau_i^{-1}, i = 1, 2, \dots, N$ are inverse branches of τ on I_i.

The Frobenius-Perron operator P_τ has the following properties [12] :

1. **Linearity:** $P_\tau : L^1 \to L^1$ is a linear operator;
2. **Positivity:** If $f \in L^1$ and $f \geq 0$, then $P_\tau f \geq 0$;
3. **Contractivity:** $P_\tau : L^1 \to L^1$ is a contraction, i.e., $\| P_\tau f \|_1 \leq \| f \|_1$ for any $f \in L^1$. Moreover, $P_\tau : L^1 \to \mathrm{L}^1$ is continuous with respect to norm topology;
4. P_τ preserves integrals, i.e., $\int_{[a,b]} f d\lambda = \int_{[a,b]} P_\tau f d\lambda$;
5. If $\tau_1, \tau_2 : X \to X$ are non-singular transformations, then $P_{\tau_1 \circ \tau_2} f = P_{\tau_1} \circ P_{\tau_2} f$. In particular, $P_{\tau^n} f = P_\tau^n f$;

Theorem 1.5 *[12] Let $\tau : ([a, b], \mathcal{B}, \lambda) \to ([a, b], \mathcal{B}, \lambda)$ be a non-singular transformation. Then P_τ has a fixed point f^* if and only if the measure $\mu = f^* \cdot \lambda$ defined by $\mu(A) = \int_A f^* d\lambda$ is $\tau-$invariant.*

The measure μ in the above theorem is absolutely continuous with respect to λ. The Frobenius-Perron operator is one of the main tools for proving the existence of absolutely continuous invariant measures. One of the advantages of studying P_τ is that P_τ is a linear operator and we can apply the powerful tools of functional analysis. Lasota and Yorke [26] proved the following important result for the existence of an acim for a single transformation using bounded variation methods and Frobenius-Perron operator:

Theorem 1.6 [Lasota-Yorke] *Let $\tau : [0, 1] \to [0, 1]$ be a piecewise $\mathcal{C}^2$ transformation such that $\inf |\tau'| > 1$. Then for any $f \in L^1[0, 1]$ the sequence $\frac{1}{n}\sum_{k=1}^{n} \mathcal{P}_\tau^k f$ is convergent in norm to $f^* \in L^1[0, 1]$. The limit function has the following properties:*

(i) $f \geqq 0 \Rightarrow f^* \geqq 0.$

(ii) $\int_0^1 f^* d\lambda = \int_0^1 f d\lambda.$

(iii) $\mathcal{P}_\tau f^* = f^*$ *and consequently* $d\mu^* = f^* d\lambda$ *is invariant under* τ.

(iv) $f^* \in \mathrm{BV}[0, 1]$. *Moreover there exists c independent to the choice of initial f such that*

$\bigvee_{[0,1]} f^* \leqq c \parallel f \parallel_1 .$

The operator P_τ is an infinite dimensional operator and it is difficult to find a solution of the equation $P_\tau f = f$ except in some simple cases, for example, Markov cases. As we have mentioned before, approximation of invariant measures was suggested by Ulam [37]. In one dimension Li [25] first proved convergence of Ulam's approximation. The following is a brief description of his method:

Let $\tau : I = [0, 1] \to I$ be a piecewise C^2 transformation with $\inf_{x\in[0,1]}|\tau'(x)| > 2$. Let $\mathcal{P}^{(n)} = \{I_1, I_2, \dots, I_n\}$ be a partition of $[0, 1]$ into subintervals of equal length and let M_n be the matrix of transition probabilities between the elements of $\mathcal{P}^{(n)}$ for the map $\tau : I \to I$

$$M_n = \left(\frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}\right)_{1\le i,j\le n}.$$

Let $L^{(n)} = \{f \in BV(I) : f = \sum_{i=1}^{n} f_n^i \chi_{I_i} = (f_n^1, f_n^2, \dots, f_n^n)\}$. Define an operator $Q^{(n)} : BV(I) \to L^{(n)}$ by

$$Q^{(n)}(f) = \sum_{i=1}^{n} n \left(\int_{I_i} f d\lambda\right) \chi_{I_i} = \left(n \int_{I_1} f d\lambda, n \int_{I_2} f d\lambda, \dots, n \int_{I_n} f d\lambda\right).$$

Let $f = (f_1, f_2, \dots, f_n) \in L^{(n)}$. Let P_τ be the Frobenius-Perron operator of τ and $P_\tau^{(n)} : L^{(n)} \to L^{(n)}$ be a finite approximation of P_τ, defined by

$$P_\tau^{(n)} f = (M_n)^t f,$$

where where A^t denotes the transpose of A. Li [25] proved the following results:

1. For $f \in L_1$, $Q^{(n)} f \longrightarrow f$ in L_1 as $n \to \infty$;

2. For $f \in L^{(n)}, P_\tau^{(n)} f = Q^{(n)} P_\tau f$;

3. For $f \in L_1, V_I Q^{(n)} f \le V_I f$;

4. For $f \in L^{(n)}$, $P_\tau^{(n)} f \longrightarrow P_\tau f$ in L_1 as $n \to \infty$;

1.2 Random dynamical systems

1.2.1 Skew product

Let $(X, \mathcal{U}, \sigma, \nu)$ be a dynamical system and let $(Y, \mathcal{B}, \tau_\omega, \mu_\omega)_{\omega \in X}$ be a family of dynamical systems such that the functions $\tau_\omega(x)$ are $\mathcal{U} \times \mathcal{B}$ measurable. A skew product of σ and $\{\tau_\omega\}_{\omega \in X}$ is a transformation $S : X \times Y \to X \times Y$ defined by

$$S(\omega, x) = (\sigma(\omega), \tau_\omega(x)),$$

$\omega \in X, x \in Y$.

1.2.2 Random maps with constant probabilities

Random maps with constant probabilities are an important special case of skew products. Let $(X, \mathcal{B}, \lambda)$ be a measure space and $\Omega = \{1, 2, 3, \dots, K\}^{\{0,1,2,\dots\}} = \{\omega = \{\omega_i\}_{i=0}^\infty : \omega_i \in \{1, 2, 3, \dots, K\}\}$. Let $\tau_k : X \to X, k = 1, 2, \dots, K$ be nonsingular piecewise one-to-one transformations and $p_1, p_2, \dots, p_K$ be constant probabilities such that $\sum_{i=1}^K p_i = 1$. The topology on Ω is the product of the discrete topology on $\{1, 2, 3, \dots, n\}$ and the Borel probability measure μ_p on Ω is defined as $\mu_p(\{\omega : \omega_0 = i_0, \omega_1 = i_1, \dots, \omega_n = i_n\}) = p_{i_0} p_{i_1} \dots p_{i_n}$. Let $\sigma : \Omega \to \Omega$ be the left shift. Now consider the skew product $S : \Omega \times X \to \Omega \times X$ defined by

$$S(\omega, x) = (\sigma(\omega), \tau_{\omega_0}(x)), \omega \in \Omega, x \in X.$$

A random map

$$T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\},$$

with constant probabilities $p_1, p_2, \dots, p_K$ is defined as follows: for any $x \in X, T(x) = \tau_k(x)$ with probability p_k and for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \dots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^N p_{k_j}$. $T^N(x)$ can be viewed as the second component of the S^N of a skew product S. It can be easily shown that a measure μ is $T-$invariant if and only if the the measure $\mu_p \times \mu$ is $S-$invariant. Pelikan[30] proved that a $T-$invariant measure μ satisfies the following condition:

$$\mu(E) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{1.4}$$

for any measurable set $E \in \mathcal{B}$. Let f be the density of μ. Then $d\mu = f \cdot d\lambda$. Let $A \times B$ be a measurable subset of $\Omega \times X$. Then

$$\begin{aligned}
(\mu_p \times \mu)(S^{-1}(A \times B)) &= \sum_i p_i \mu_p(A) \mu(\tau_i^{-1}(B)) \\
&= \sum_i p_i \mu_p(A) \int_{\tau_i^{-1}(B))} f d\lambda \\
&= \sum_i p_i \mu_p(A) \int_B P_{\tau_i} f d\lambda \\
&= \mu_p(A) \sum_i p_i \int_B P_{\tau_i} f d\lambda.
\end{aligned}$$

Thus, the density on the second component is $\sum_i p_i P_{\tau_i} f$. Hence the Perron-Frobenius operator P_T for the random map T is given by $P_T f = \sum_i p_i P_{\tau_i} f$. The properties of P_T resemble the properties of the traditional Perron-Frobenius operator. For random maps with constant probabilities where the component maps are Lasota-Yorke maps [26], Pelikan [30] proved the following sufficient condition for the existence of an acim:

$$\sum_k \frac{p_k}{|\tau_k'(x)|} \leq \alpha < 1,$$

for all $x \in [0,1]$.

1.2.3 Random maps with position dependent probabilities

Let $(X, \mathcal{B}, \lambda)$ be a measure space, where λ is an underlying measure. Let $\tau_k : X \to X$, $k = 1, 2, \ldots, K$, be piecewise one-to-one non-singular transformations on a common partition $\mathcal{P}$ of X : $\mathcal{P} = \{I_1, I_2, \ldots, I_q\}$ and $\tau_{k,i} = \tau_k|_{I_i}, i = 1, 2, \ldots, q, k = 1, 2, \ldots, K$. We define the transition function for the random map

$$T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), \ldots, p_K(x)\}$$

as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \chi_A(\tau_k(x)), \tag{1.5}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^K$ is a set of position dependent probabilities, i.e., $\sum_{k=1}^K p_k(x) = 1,\ p_k(x) \geq 0$, for any $x \in X$. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$ with probability

$$p_{k_N}(\tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)) p_{k_{N-1}}(\tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x).$$

The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(X, \mathcal{B})$ defined by

$$
\begin{aligned}
\mathbb{P}_*\mu(A) &= \int \mathbb{P}(x, A)d\mu(x) \\
&= \sum_{k=1}^{K} \int p_k(x)\chi_A(\tau_k(x))d\mu(x) \\
&= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x)d\mu(x).
\end{aligned}
$$

If μ has density f with respect to λ, the $P_*\mu$ also has a density which we denote by $P_T f$. By a change of variables, we obtain

$$
\begin{aligned}
\int_A P_T f(x)d\lambda(x) &= \sum_{k=1}^{K}\sum_{k=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x)f(x)\frac{1}{J_{k,i}(\tau_{k,i}^{-1})}d\lambda(x) \\
&= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{(A)} p_k(\tau_{k,i}^{-1})f(\tau_{k,i}^{-1})\frac{1}{J_{k,i}(\tau_{k,i}^{-1})}d\lambda(x),
\end{aligned}
$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ. Since this holds for any measurable set A, we obtain an almost everywhere equality:

$$
(P_T f)(x) = \sum_{k=1}^{K}\sum_{k=1}^{q} p_k(\tau_{k,i}^{-1})f(\tau_{k,i}^{-1})\frac{1}{J_{k,i}(\tau_{k,i}^{-1})}\chi_{\tau_k(I_i)}(x) = \sum_{k=1}^{K} P_{\tau_k}(p_k f)(x), \tag{1.6}
$$

where P_{τ_k} is the Perron-Frobenius operator corresponding to the transformation τ_k. We call P_T the Perron-Fronenius operator of the random map T. As before the properties of P_T resemble the properties of the traditional Perron-Frobenius operator. For random maps $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1(x), p_2(x), \dots, p_K(x)\}$ where $\tau_k : X \to X, k = 1, 2, \dots, K$, with $|\tau_k'| \geq \alpha > 1$ for all k, are piecewise one-to-one non-singular transformations, Góra and Boyarsky [13] proved the following sufficient condition for the existence of an acim

$$\frac{K \cdot \sum_{k=1}^{K} \sup p_k(x)}{\alpha} < 1.$$

For piecewise monotonic transformations, Bahsoun and Góra [5] proved a weaker sufficient condition for the existence of an acim: for all $x \in X$,

$$\sum_{k=1}^{K} \frac{p_k(x)}{|\tau_k'(x)|} < 1$$

1.2.4 Markov switching position dependent random maps

Let $X = ([a,b], \mathcal{B}, \lambda)$ be a measure space where λ is Lebesgue measure on $[a,b]$. Let $\tau_k : X \to X, k = 1, 2, \ldots, K$, be piecewise one-to-one continuous non-singular transformations on a common partition $\mathcal{P}$ of $[a,b] : P = \{J_1, J_2, \ldots, J_q\}$ and $\tau_{k,i} = \tau_k | J_i$, $i = 1, 2, \ldots, q$, $k = 1, 2, \ldots, K$. A Markov switching position dependent random map T is a Markov process which is defined as follows: at time $n = 1$, we select a transformation τ_k randomly according to initial probabilities $p_k, k = 1, 2, \ldots, K$. The probability of switching from transformation τ_k to transformation τ_l is given by $W_{k,l}$, the $(k,l)^{th}$ element of a position dependent stochastic matrix $W = W(x)$. Therefore, if we choose τ_{k_1} at time $n = 1$ when we are at position x, the Markov process at time N is given by

$$T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$$

with probability

$$W_{k_{N-1},k_N}(\tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)) \cdot W_{k_{N-2},k_{N-1}}(\tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1}(x)) \cdot \ldots \cdot W_{k_1,k_2}(x).$$

We assume that the probabilities $W_{k,l}(x)$ are defined on the same partition $\mathcal{P}$. Let $\Omega = \{1, 2, \dots, K\}$. We define the transition function of the Markov process on $\Omega \times X$ as follows:

$$\mathbb{P}((k,x), \{l\} \times A) = W_{k,l}(x)\chi_A(\tau_k(x)),$$

where A is any measurable set and χ_A denotes the characteristic function of the set A. The random map T is the projection of the process we defined on the space X. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures μ on $\Omega \times X$ as follows:

$$\begin{aligned}
\mathbb{P}_*\mu(\{l\} \times A) &= \int_{\Omega \times X} \mathbb{P}((k,x), \{l\} \times A) d\mu(k,x) \\
&= \int_{\Omega \times X} W_{k,l}(x)\chi_A(\tau_k(x)) d\mu(k,x).
\end{aligned}$$

Let ν be a measure on $\Omega \times X$ such that $\nu(\{s\} \times A) = \lambda(A)$. If μ has density f with respect to ν, $f(s,x) = \sum_{k=1}^{K} f_k(x)\chi_{\{k\}\times X}(s,x)$, where $\sum_{k=1}^{K} \int_X f_k(x) = 1$, then $\mathbb{P}_*\mu$ also has a density which we denote by $P_T f$. By a change of variables, we obtain

$$\begin{aligned}
\int_{\{l\}\times A} P_T f(s,x) d\nu(s,x) &= \sum_{k=1}^{K} \int_X W_{k,l}(x)\chi_A(\tau_k(x)) f_k(x) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} W_{k,l}(x) f_k(x) d\lambda(x). \qquad (1.7)
\end{aligned}$$

Using the definition of P_{τ_k}, the Frobenius-Perron operator associated with transformation τ_k [12] and 1.7, we obtain

$$\int_A \hat{f}_l(x) d\lambda(x) = \sum_{k=1}^{K} \int_A P_{\tau_k}(W_{k,l} f_k)(x) d\lambda(x), \qquad (1.8)$$

where $P_T f(s,x) = \sum_{l=1}^{K} \hat{f}_l \chi_{\{l\}\times X}(s,x)$. Since (1.8) is true for any $A \in \mathcal{B}$, we obtain an a.e. equality

$$\hat{f}_l(x) = \sum_{k=1}^{K} P_{\tau_k}(W_{k,l} f_k)(x). \tag{1.9}$$

Thus, the density $f^*(s,x) = \sum_{l=1}^{K} f_l^*(x)\chi_{\{l\}\times X}(s,x)$ is $T-$invariant if

$$f_l^*(x) = \sum_{k=1}^{K} P_{\tau_k}(W_{k,l} f_k^*)(x). \tag{1.10}$$

for $l = 1, 2, \dots, K$. If we denote

$$w_l = \int_X f_l^*(x) d\lambda(x), \quad l = 1, 2, \dots, K,$$

then integrating (1.10) with respect to λ, we obtain

$$w_l = \sum_{k=1}^{K} w_k \int_X W_{k,l}(x) \frac{f_l^*(x)}{\int_X f_k^*(x) d\lambda(x)} d\lambda(x). \tag{1.11}$$

Note that, in the special case when $W_{k,l}$'s are constant, (1.11) reduces to $w_l = \sum_{k=1}^{K} w_k W_{k,l}$, i.e., to the case when $(w_1, w_2, \dots, w_K)$ is a left invariant eigenvector of the matrix W.

As before, denote by $V(\cdot)$ the standard one dimensional variation of a function, and $BV([a,b])$ the space of functions of bounded variations on $[a,b]$ equiped with the norm $\| \cdot \|_{BV} = V(\cdot) + \| \cdot \|_1$, where $\| \cdot \|_1$ denotes the norm on $L^1([a,b], \mathcal{B}, \lambda)$. Let $\widehat{BV} = \prod_{k=1}^{K} BV$ denote the $K-$fold product of the space BV of functions of bounded variation and we define a norm on $\widehat{BV}$ as $\| f_1, f_2, \dots, f_K \|_{\widehat{BV}} = \sum_{k=1}^{K} \| f_k \|_{BV}$. We also define L^1 norm on $\widehat{BV}$: $\| f_1, f_2, \dots, f_K \|_1 = \sum_{k=1}^{K} \| f_k \|_1$. We define an operator $\widehat{P_T} : \widehat{BV} \to \widehat{BV}$ by

$$\widehat{P_T}(f_1, f_2, \dots, f_K) = \left(\sum_{k=1}^{K} P_{\tau_k}(W_{k,1} f_k), \sum_{k=1}^{K} P_{\tau_k}(W_{k,2} f_k), \dots, \sum_{k=1}^{K} P_{\tau_k}(W_{k,K} f_k) \right). \tag{1.12}$$

If $(f_1^*, f_2^*, \dots, f_K^*)$ is fixed point of $\widehat{P_T}$, we call

$$f^* = \sum_{k=1}^{K} f_k^*$$

an invariant density of the Markov switching position dependent random map T. For more details about $\widehat{P_T}$ see [2].

Chapter 2

On the Existence of Absolutely Continuous Invariant Measures for Random Maps : A Generalization of Straube's Theorem

2.1 Introduction

It is well known that if a map $\tau : I \to I$, $I = [0,1]$, is piecewise expanding then it possesses an absolutely continuous invariant measure (acim) [12]. This result can be generalized to random maps where the condition of piecewise expanding is replaced by an average expanding condition where the weighting coefficients are the probabilities of switching [30, 13, 14, 5]. Such results have been generalized in [10, 2]. There are a number of interesting examples which do not fall into the average expanding condition for which the conditions of this chapter may present a possible approach.

Consider the following simple random maps on I :

$$\tau_1(x) = \frac{x}{2}, \ \tau_2(x) = \frac{(x+1)}{2}$$

with constant probabilities p_1 and p_2. τ_1 has an attracting fixed point at 0 while τ_2 has an attracting fixed point at 1. Thus, neither τ_1 nor τ_2 has an acim. Applying the constant L^1 function $\mathbf{1}$ on the Perron-Frobenius operator of the random map $T = \{tau_1, \tau 2; p_1, p_2\}$ it can be shown that T has Lebesgue measure as its unique acim. This shows that a random map does not necessarily inherit the properties of the underlying maps.

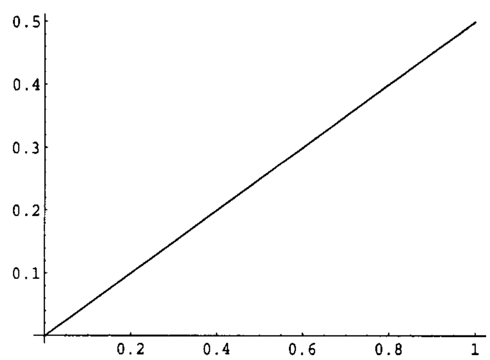


Figure 2.1: The graph of τ_1.

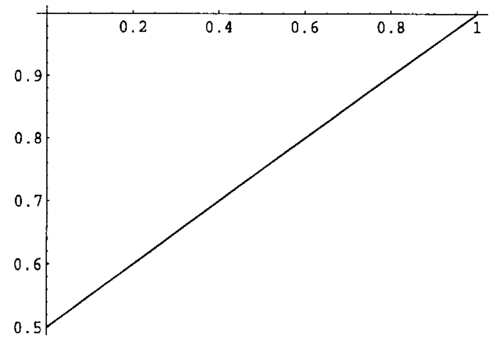


Figure 2.2: The graph of τ_2.

Consider now an expanding map τ_3 on I and the logistic map τ_4 on I. Both maps have an acim, but the *average expanding* sufficiency condition for existence of an acim for the random map based on τ_3 and τ_4 fails since τ_4 has regions of arbitrarily small slope.

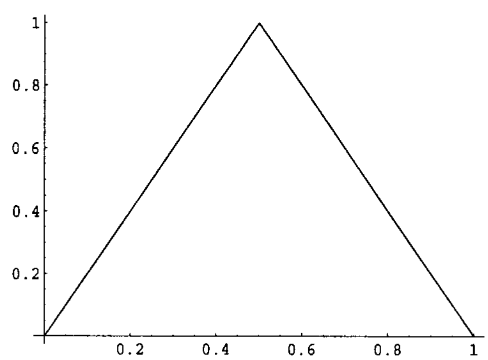


Figure 2.3: The graph of τ_3.

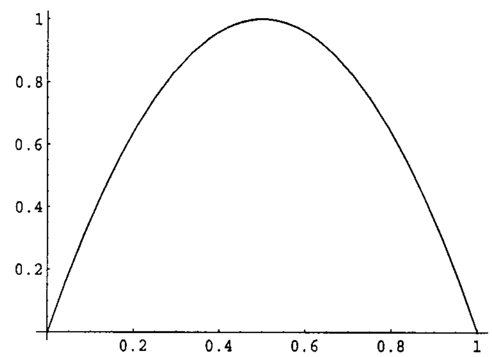


Figure 2.4: The graph of τ_4.

Hence, in general, we cannot conclude that even such a simple random map admits an acim.

The foregoing suggests the need for results that can establish existence of an acim directly for random maps. To this end we generalize a theorem of Straube [35], which provides a necessary and sufficient condition for existence of an acim of a nonsingular map, to random maps. We consider both random maps with constant probabilities and random maps with position dependent probabilities.

In Section 2 we present the notation and summarize results we shall need in the sequel. In Section 3 we prove the main result.

2.2 Preliminaries

We now recall some definitions and results from [35, 36] which will be used to prove our main results in Section 3.

Definition 2.1 *A set function $\phi : \mathcal{B} \to R$ is a finitely additive measure if*

(i) $-\infty < \phi(E) < \infty$, *for all* $E \in \mathcal{B}$;

(ii) $\phi(\emptyset) = 0$;

(iii) $\sup_{E\in\mathcal{B}}|\phi(E)| < \infty$;

(iv) $\phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2)$, *for all* $E_1, E_2 \in \mathcal{B}$ *such that* $E_1 \cap E_2 = \emptyset$.

Definition 2.2 *A finitely additive positive measure μ is a purely additive measure if every countably additive measure such that $\nu \geq 0$, $\nu \leq \mu$ is identically zero.*

Theorem 2.1 *[36] Let ϕ be a finitely additive (positive) measure. Then ϕ has a unique representation $\phi = \phi_c + \phi_p$, where ϕ_c is countably additive ($\phi_c \geq 0$) and ϕ_p is purely additive ($\phi_p \geq 0$).*

Lemma 2.1 *[36] If μ is a finitely additive positive measure on $\mathcal{B}$, then μ_c is the greatest measure among countably additive measures ν with $0 \leq \nu \leq \mu$.*

Theorem 2.2 *[36] Let ϕ be a finitely additive positive measure on a $\sigma-$algebra $\mathcal{B}$ and ν be a countably additive positive measure on $\mathcal{B}$. Then there exists a decreasing sequence $\{E_n\}_{n\geq 1}$ of elements of $\mathcal{B}$ such that $\lim_{n\to\infty} \nu(E_n) = 0$ and $\phi(E_n) = \phi(X)$.*

Theorem 2.3 *[35] Let (X, B, λ) be a measure space with normalized measure λ, $f : X \to X$ be a nonsingular transformation. Then the following conditions are equivalent:*

(i) there exists an $f-$invariant normalized measure μ which is absolutely continuous with respect to λ;

(ii) there exists $\delta > 0$, and $\alpha, 0 < \alpha < 1$ such that

$$\lambda(E) < \delta \Rightarrow \sup_{k \in N} \lambda(f^{-k}(E)) < \alpha, E \in \mathcal{B}. \tag{2.1}$$

2.3 Existence of absolutely continuous invariant measures

In this section we prove necessary and sufficient conditions for existence of an absolutely continuous invariant measure for random maps. For notational convenience, we consider $K = 2$; that is, we consider only two transformations τ_1, τ_2. The proof for a larger number of maps is analogous. We consider first random maps with constant probabilities, then random maps with position dependent probabilities.

Theorem 2.4 *Let $(X, \mathcal{B}, \lambda)$ be a measure space with normalized measure λ and $\tau_i : X \to X$, $i = 1, 2$ be nonsingular transformations. Consider the random map $T = \{\tau_1, \tau_2; p_1, p_2\}$ with constant probabilities p_1, p_2. Then, there exists a normalized absolutely continuous (w.r.t. λ) $T-$invariant measure μ if and only if there exists $\delta > 0$ and $0 < \alpha < 1$ such that for any measurable set E and any positive integer k, $\lambda(E) < \delta$*

implies

$$p_1\lambda\left(\tau_1^{-1}(E)\right) + p_2\lambda\left(\tau_2^{-1}(E)\right) < \alpha;$$

$$p_1^2\lambda\left(\tau_1^{-2}(E)\right) + p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + p_1p_2\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2^2\lambda\left(\tau_2^{-2}(E)\right) < \alpha;$$

$$\vdots$$

$$\sum_{(i_1,i_2,i_3,\dots,i_k)} p_{i_1}p_{i_2}\dots p_{i_k}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_k}^{-1}(E)\right) < \alpha. \tag{2.2}$$

To prove this theorem, we first prove the following two lemmas:

Lemma 2.2 *Let $(X, \mathcal{B}, \lambda)$ be a probability measure space and μ be absolutely continuous with respect to λ, $\mu = f \cdot \lambda$, for f an $L^1(X, \mathcal{B}, \lambda)$ function. Then there exists a constant $M \geq 0$ and a measurable set A_0 such that $\mu(A_0) \leq \frac{1}{10}$ and $f \leq M$ on $X \setminus A_0$.*

Proof. Consider the following sets :

$$B_n = \{x \in X : n \leq f(x) < n+1\},\ n = 0, 1, \dots . \tag{2.3}$$

Clearly, $\{B_n\}$ are disjoint measurable sets and $X = \cup_{n=0}^{\infty} B_n$ and $1 = \mu(X) = \sum_{n=0}^{\infty} \mu(B_n)$. Thus, there exists an $M \geq 0$ such that $\sum_{n=M}^{\infty} \mu(B_n) < \frac{1}{10}$. Let $A_0 = \cup_{n=M}^{\infty} B_n$. Then on $X \setminus \cup_{n=M}^{\infty} B_n$, $f(x) \leq M$. ■

For any measure ϕ, any integer k, and any measurable set E, define

$$\Lambda_k^{\phi}(E) := \sum_{(i_1,i_2,i_3,\dots,i_k)} p_{i_1}p_{i_2}\dots p_{i_k}\phi\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_k}^{-1}(E)\right). \tag{2.4}$$

It can be easily shown that Λ_k^{λ} and Λ_k^{μ} are normalized measures and Λ_k^{μ} are measures absolutely continuous with respect to Λ_k^{λ}.

Lemma 2.3 *Let M be the constant from the previous lemma and δ be such that $M\delta + \frac{1}{10} < \frac{1}{4}$. Then, for any $n \geq 1$, and any measurable set A, we have $\Lambda_n^\lambda(A) < \delta \Rightarrow \Lambda_n^\mu(A) < \frac{1}{4}$.*

Proof. Let M and A_0 be as in the previous lemma. We have

$$\begin{aligned}
\Lambda_n^\mu(A) &= \sum_{(i_1,i_2,i_3,\ldots,i_n)} p_{i_1}p_{i_2}\ldots p_{i_n}\mu\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\ldots\tau_{i_n}^{-1}(A)\right) \\
&= \sum_{(i_1,i_2,i_3,\ldots,i_n)} p_{i_1}p_{i_2}\ldots p_{i_n}\mu\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\ldots\tau_{i_n}^{-1}(A)\cap A_0\right) \\
&\quad + \sum_{(i_1,i_2,i_3,\ldots,i_n)} p_{i_1}p_{i_2}\ldots p_{i_n}\mu\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\ldots\tau_{i_n}^{-1}(A)\cap (X\setminus A_0)\right) \\
&\leq \sum_{(i_1,i_2,i_3,\ldots,i_n)} p_{i_1}p_{i_2}\ldots p_{i_n}\frac{1}{10} + \sum_{(i_1,i_2,i_3,\ldots,i_n)} p_{i_1}p_{i_2}\ldots p_{i_n}M\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\ldots\tau_{i_n}^{-1}(A)\right) \\
&\leq \frac{1}{10} + M\Lambda_n^\lambda(A) < \frac{1}{10} + M\delta < \frac{1}{4}.
\end{aligned}$$

■

Proof of Theorem 2.4:

Suppose

$$\mu(E) = \sum_{i=1}^{2} p_i\mu(\tau_i^{-1}(E)),\ E \in B,\ \mu(X) = 1,\ \mu << \lambda.$$

We want to prove that there exist $\delta > 0$, $0 < \alpha < 1$, such that for any $E \in \mathcal{B}$ and for any positive integer k

$$\lambda(E) < \delta \Rightarrow \Lambda_k^\lambda(E) < \alpha. \tag{2.5}$$

Suppose not. Then, for any α, $0 < \alpha < 1$, there exists $E \in \mathcal{B}$ and there exists a positive integer n_0 such that

$$\lambda(E) < \delta \Rightarrow \Lambda^{\lambda}_{n_0}(E) > \alpha, \tag{2.6}$$

where $E \in \mathcal{B}$.

Choose $\delta > 0$ such that $M\delta + \frac{1}{10} < \frac{1}{4}$ where M is the constant of Lemma 2.3. Let n_0 be the index corresponding to δ in 2.6. Then by Lemma 2.3, we have for $A \in \mathcal{B}$

$$\begin{aligned} \lambda(A) < \delta &\Rightarrow \mu(A) < \frac{1}{4}; \\ \Lambda^{\lambda}_{n_0}(A) < \delta &\Rightarrow \Lambda^{\mu}_{n_0}(A) < \frac{1}{4}. \end{aligned} \tag{2.7}$$

Let $\alpha = 1 - \frac{\delta}{2}$. Then,

$$\Lambda^{\lambda}_{n_0}(X \setminus E) = 1 - \Lambda^{\lambda}_{n_0}(E) < 1 - 1 + \delta = \delta.$$

By our choice of δ, we get

$$\Lambda^{\mu}_{n_0}(X \setminus E) < \frac{1}{4}.$$

Since μ is invariant, we have

$$\mu(X \setminus E) = \Lambda^{\mu}_{n_0}(X \setminus E) < \frac{1}{4}.$$

Thus,

$$1 = \mu(X) = \mu(E) + \mu(X \setminus E) < \frac{1}{4} + \frac{1}{4},$$

a contradiction.

Conversely, suppose that there exists $\delta > 0$ and $0 < \alpha < 1$ such that for any measurable set E and any positive integer k, $\lambda(E) < \delta$ implies

$$p_1\lambda(\tau_1^{-1}(E)) + p_2\lambda\left(\tau_2^{-1}(E)\right) < \alpha;$$

$$p_1^2\lambda\left(\tau_1^{-2}(E)\right) + p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + p_1p_2\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2^2\lambda\left(\tau_2^{-2}(E)\right) < \alpha;$$

$$\vdots$$

$$\sum_{(i_1,i_2,i_3,\dots,i_k)} p_{i_1}p_{i_2}\dots p_{i_k}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_k}^{-1}(E)\right) < \alpha.$$

We want to show that there exists a measure μ such that $\mu(E) = \sum_{i=1}^{2} p_i\mu\left(\tau_i^{-1}(E)\right)$, $E \in \mathcal{B}$, $\mu(X) = 1$ and $\mu << \lambda$.

Consider the measures λ_n defined by

$$\lambda_n(E) := \frac{1}{n}\sum_{k=0}^{n-1}\Lambda_k^{\lambda}(E),\ E \in \mathcal{B}. \tag{2.8}$$

It can be shown that, for all n, λ_n are normalized measures. Moreover, if $\lambda(E) = 0$, then

$$\begin{aligned}
\lambda_n(E) &= \lambda(E) + p_1\lambda\left(\tau_1^{-1}(E)\right) + p_2\lambda\left(\tau_2^{-1}(E)\right) \\
&\quad + p_1^2\lambda\left(\tau_1^{-2}(E)\right) + p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + p_1p_2\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2^2\lambda\left(\tau_2^{-2}(E)\right) \\
&\quad + \dots + \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1}p_{i_2}\dots p_{i_n}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_n}^{-1}(E)\right) \\
&= 0,
\end{aligned}$$

by non-singularity of τ_1 and τ_2. Hence $\lambda_n << \lambda$. We imbed λ_n in the dual space $L_\infty(\lambda)^*$ of $L_\infty(\lambda)$ in the following way:

$$g_n(f) = \int_X f d\lambda_n, f \in L_\infty(\lambda).$$

For every n,

$$|g_n(f)| = |\int_X f d\lambda_n| \leq ||f||_\infty \int_X d\lambda_n = ||f||_\infty.$$

Hence, for each n, $||g_n|| \leq 1$. Thus, the λ_n can be thought of as elements of the unit ball of $L_\infty(\lambda)^*$. This unit ball is weak$^*-$ compact by Alaoglu's Theorem [9]. Let ν be a cluster point in the weak$^*-$ topology of $L_\infty(\lambda)^*$ of the sequence $\{\lambda_n\}_{n\geq 1}$.

Define a set function μ on $\mathcal{B}$ by

$$\mu(E) = \nu(\chi_E). \tag{2.9}$$

We claim that μ is finitely additive, bounded, and that it vanishes on sets of $\lambda-$ measure zero: $\mu(\emptyset) = \nu(\chi_\emptyset) = \nu(0) = 0$, since ν is a linear functional. For any $E \in \mathcal{B}$,

$$\begin{aligned} \mu(E) &= \nu(\chi_E) = \lim_{s\to\infty} g_{n_s}(\chi_E) = \lim_{s\to\infty} \int_E d\lambda_{n_s} = \lim_{s\to\infty} \lambda_{n_s}(E) \\ &= \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=0}^{n_s-1} \Lambda_k^\lambda(E) \geq 0, \end{aligned}$$

since Λ_k^λ is a measure. Thus,

$$0 \leq \mu(E) \leq \mu(X) = \lim_{s\to\infty} \lambda_{n_s}(X) = 1.$$

Now,

$$\begin{aligned} \mu(\cup_{i=1}^m E_i) &= \lim_{s\to\infty} \lambda_{n_s}(\cup_{i=1}^m E_i) = \lim_{s\to\infty} \sum_{i=1}^m \lambda_{n_s}(E_i) \\ &= \sum_{i=1}^m \lim_{s\to\infty} \lambda_{n_s}(E_i) = \sum_{i=1}^m \mu(E_i). \end{aligned}$$

Let $\lambda(E) = 0$. Then $\mu(E) = \lim_{s\to\infty} \lambda_{n_s}(E) = 0$, because $\lambda_{n_s} << \lambda$. Hence, μ is finitely additive, bounded, and it vanishes on sets of $\lambda-$ measure zero.

μ **is** $T-$**invariant:**

$$\begin{aligned}
\mu(E) &= \lim_{s\to\infty} \lambda_{n_s}(E) = \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=0}^{n_s-1} \Lambda_k^\lambda(E) \\
&= \lim_{s\to\infty} \frac{1}{n_s}[\Lambda_0^\lambda(E) + \Lambda_1^\lambda(E) \ldots + \Lambda_{n_s-1}^\lambda(E)] \\
&= \lim_{s\to\infty} \frac{1}{n_s}[\lambda(E) + p_1\lambda\left(\tau_1^{-1}(E)\right) + p_2\lambda\left(\tau_2^{-1}(E)\right)] \\
&\quad + \ldots + \sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})} p_{i_1}p_{i_2}\ldots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\ldots\tau_{i_{n_s-1}}^{-1}(E)\right)].
\end{aligned}$$

On the other hand,

$$\sum_{i=1}^{2} p_i\mu\left(\tau_i^{-1}(E)\right) = p_1\mu\left(\tau_1^{-1}(E)\right) + p_2\mu\left(\tau_2^{-1}(E)\right)$$

Using definition of Λ_k^λ we get,

$$= p_1 \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=0}^{n_s-1} \Lambda_k^\lambda\left(\tau_1^{-1}(E)\right) + p_2 \lim_{s\to\infty} \frac{1}{n_s} \sum_{k=0}^{n_s-1} \Lambda_k^\lambda\left(\tau_2^{-1}(E)\right)$$

Spliting the sum we get,

$$= \lim_{s\to\infty} \frac{1}{n_s}[p_1\{\lambda\left(\tau_1^{-1}(E)\right) + p_1\lambda\left(\tau_1^{-2}(E)\right) + p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + \ldots$$

By rearranging we get

$$
\begin{aligned}
& + \sum_{(i_1,i_2,i_3,\dots,i_{n_s-1})} p_{i_1}p_{i_2}\dots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_{n_s-1}}^{-1}(\tau_1^{-1}(E))\right)\} \\
& + p_2\{\lambda(\tau_2^{-1}(E)) + p_1\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2\lambda\left(\tau_2^{-2}(E)\right) + \dots \\
& + \sum_{(i_1,i_2,i_3,\dots,i_{n_s-1})} p_{i_1}p_{i_2}\dots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_{n_s-1}}^{-1}(\tau_2^{-1}(E))\right)\}] \\
= & \lim_{s\to\infty}\frac{1}{n_s}[p_1\lambda(\tau_1^{-1}(E)) + p_1^2\lambda\left(\tau_1^{-2}(E)\right) + p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + \dots \\
& + p_1 \sum_{(i_1,i_2,i_3,\dots,i_{n_s-1})} p_{i_1}p_{i_2}\dots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_{n_s-1}}^{-1}(\tau_1^{-1}(E))\right) \\
& + p_2\lambda\left(\tau_2^{-1}(E)\right) + p_2p_1\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2^2\lambda\left(\tau_2^{-2}(E)\right) + \dots \\
& + p_2 \sum_{(i_1,i_2,i_3,\dots,i_{n_s-1})} p_{i_1}p_{i_2}\dots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_{n_s-1}}^{-1}(\tau_2^{-1}(E))\right)].
\end{aligned}
$$

Clearly,

$$\mu(E) = \sum_{i=1}^{2} p_i\mu\left(\tau_i^{-1}(E)\right).$$

Thus, we have shown that μ is a finitely additive $T-$invariant measure. By Theorem 2.1, μ has a unique representation

$$\mu = \mu_c + \mu_p,$$

where μ_c is countably additive and $\mu_c \geq 0$ and μ_p is purely additive and $\mu_p \geq 0$. We claim that $\mu_c \neq 0$. Suppose $\mu_c = 0$. Then by Theorem 2.2, there exists a decreasing sequence $\{E_n\}_{n\geq 1}$ of elements of $\mathcal{B}$ such that $\lim_{n\to\infty}\lambda(E_n) = 0$ and $\mu(E_n) = \mu(X) = 1$. Thus, there exists an integer n_0 such that for all $n \geq n_0$, $\lambda(E_n) < \delta$ and, as a consequence of our hypothesis, we have for all k,

$$\Lambda_k^{\lambda}(E_n) < \alpha.$$

Hence,

$$\lambda_k(E_n) < \alpha, k = 1, 2, 3, \ldots .$$

Thus, $\mu(E_n) = \lim_{s\to\infty} g_{n_s}(E_n) < \alpha < 1$, a contradiction. Now,

$$\begin{aligned}
\mu(E) &= p_1\mu\left(\tau_1^{-1}(E)\right) + p_2\mu\left(\tau_2^{-1}(E)\right) \\
&= p_1\{\mu_c\left(\tau_1^{-1}(E)\right) + \mu_p\left(\tau_1^{-1}(E)\right)\} + p_2\{\mu_c\left(\tau_2^{-1}(E)\right) + \mu_p\left(\tau_2^{-1}(E)\right)\} \\
&= \{p_1\mu_c\left(\tau_1^{-1}(E)\right) + p_2\mu_c\left(\tau_2^{-1}(E)\right)\} + \{p_1\mu_p\left(\tau_1^{-1}(E)\right) + p_2\mu_p\left(\tau_2^{-1}(E)\right)\}.
\end{aligned}$$

Clearly $m : \mathcal{B} \to R$, defined by

$$m(E) = p_1\mu_c(\tau_1^{-1}(E)) + p_2\mu_c(\tau_2^{-1}(E)),$$

is a countably additive measure, and $m \leq \mu$. Thus, by Lemma 2.1, we have $m \leq \mu_c$ and hence

$$E \mapsto \mu_c(E) - m(E) = \mu_c(E) - \{p_1\mu_c\left(\tau_1^{-1}(E)\right) + p_2\mu_c\left(\tau_2^{-1}(E)\right)\}$$

is a positive measure. But this measure has total mass zero. Hence, it is a zero measure. Thus, μ_c is $T-$ invariant. Because μ vanishes on sets of $\lambda-$ measure zero and $0 \leq \mu_c \leq \mu$, we have $\mu_c << \lambda$. Finally, $\gamma(E) = \frac{\mu_c(E)}{\mu_c(X)}$ is a normalized, $T-$ invariant and absolutely continuous with respect to λ.

We now state the analogous result for position dependent random maps.

Theorem 2.5 *Let (X, B, λ) be a measure space with normalized measure λ and $\tau_i : X \to X$, $i = 1, 2$ be nonsingular transformations. Consider the random map*

$T = \{\tau_1, \tau_2; p_1, p_2\}$ with position dependent probabilities p_1, p_2. Then there exists a normalized absolutely continuous (w.r.t. λ) $T-$invariant measure μ if and only if there exists $\delta > 0$ and $0 < \alpha < 1$ such that for any measurable set E and any positive integer k, $\lambda(E) < \delta$ implies

$$\int_{\tau_1^{-1}(E)} p_1(x)d\lambda + \int_{\tau_2^{-1}(E)} p_2(x)d\lambda < \alpha;$$

$$\int_{\tau_1^{-2}(E)} p_1(x)p_1(\tau_1(x))d\lambda + \int_{\tau_2^{-1}\tau_1^{-1}(E)} p_1(x)p_2(\tau_1(x))d\lambda$$
$$+ \int_{\tau_1^{-1}\tau_2^{-1}(E)} p_2(x)p_1(\tau_2(x))d\lambda + \int_{\tau_2^{-2}(E)} p_2(x)p_2(\tau_2(x))d\lambda < \alpha;$$

$\vdots$

$$\sum_{(i_1, i_2, i_3, \dots, i_k)} \int_{\tau_{i_1}^{-1}\tau_{i_2}^{-1}\dots\tau_{i_k}^{-1}(E)} p_{i_1}(x)p_{i_2}(\tau_{i_1}(x))\dots p_{i_k}(\tau_{i_1}\tau_{i_2}\dots\tau_{i_{k-1}}(x))d\lambda < \alpha.$$

Proof. The proof is analogous to the proof of Theorem 2.4. ∎

Chapter 3

Approximation of Invariant Measures for Markov Switching Position Dependent Random Maps

3.1 Introduction

Approximation of invariant measures for dynamical systems was suggested by Ulam [37]. For a single transformation, Li [25] first proved convergence of Ulam's approximation. In [10] Froyland extended Ulam's method for a single transformation to random maps with constant probabilities [30]. Góra and Boyarsky in [13] proved the convergence of Ulam's approximation for position dependent random maps. The existence of absolutely continuous invariant measure for Markov switching position dependent random maps was proved by Bahsoun, Góra and Boyarsky [2] using spectral properties of the Frobenius-Perron operator. In this chapter we present a bounded variation proof for the existence of acims and we describe Ulam's method of approximating the acims for Markov switching position dependent random maps. In Section 2 we present the proof of existence of acims. In Section 3 we prove the main result, the convergence of Ulams's approximation. In Section 4 we describe the error bounds for approximation of invariant densities. In Section 5 we present a numerical example.

Let $([a,b],\mathcal{B},\lambda)$ be a measure space and $\tau_k : X \to X, k = 1,2,\ldots,K$, be piecewise one-to-one monotonic non-singular transformations on a common partition $\mathcal{P}$ of $[a,b]$: $\mathcal{P} = \{J_1, J_2, \ldots, J_q\}$ and $\tau_{k,i} = \tau_k | J_i,\ i = 1,2,\ldots,q,\ k = 1,2,\ldots,K$. Let W be a K by K position dependent stochastic matrix whose elements are piecewise continuous on the same partition $\mathcal{P}$. Now consider the Markov switching position dependent random map $T = \{\tau_1,\tau_2,\ldots,\tau_K;\ p_1,p_2,\ldots,p_K;\ W\}$. In Chapter 2 we defined the Frobenius-Perron operator $\widehat{P_T}$ and invariant density as follows. $\widehat{P_T} : \widehat{BV} \to \widehat{BV}$ by

$$\widehat{P_T}(f_1, f_2, \ldots, f_K) = \left(\sum_{k=1}^{K} \mathcal{P}_{\tau_k}(W_{k,1}f_k), \sum_{k=1}^{K} \mathcal{P}_{\tau_k}(W_{k,2}f_k), \ldots, \sum_{k=1}^{K} \mathcal{P}_{\tau_k}(W_{k,K}f_k)\right), \tag{3.1}$$

where $\widehat{BV} = \prod_{k=1}^{K} BV$ denote the $k-$fold product of the space BV of functions of bounded variation. If $(f_1^*, f_2^*, \ldots, f_K^*)$ is fixed point of $\widehat{P_T}$, we call

$$f^* = \sum_{k=1}^{K} f_k^*$$

an invariant density of the Markov switching position dependent random map T. A norm on $\widehat{BV}$ as $\| (f_1, f_2, \dots, f_K) \|_{\widehat{BV}} = \sum_{k=1}^{K} \| f_k \|_{BV}$. We also define L^1 norm on $\widehat{BV}$ $: \| (f_1, f_2, \dots, f_K) \|_1 = \sum_{k=1}^{K} \| f_k \|_1$.

3.2 Existence of acim using bounded variation

Definition 3.1 *We say that $\tau : [a, b] \to [a, b]$ is a Lasota-Yorke map if τ is piecewise monotone and C^2 and τ is non-singular, i.e., τ is nonsingular and there exists a partition of $[a, b], a = x_0 < x_1 < \dots < x_n = b$ such that for each $i = 0, 1, \dots, n-1, \tau|_{(x_i, x_{i+1})}$ is monotonic and can be extended to a C^2 function on $[x_i, x_{i+1}]$.*

Lemma 3.1 *Let τ_k be a Lasota-Yorke map on $I = [0, 1]$ and $W_{k,l}$ be piecewise of class C^1, for $k = 1, 2, \dots, K$ and $l = 1, 2, \dots, K,$. Let*

$$\alpha_l = \max_k (\sup_x \frac{2 \cdot W_{k,l}(x)}{|\tau_k'(x)|}), \ l = 1, 2, \dots, K.$$

Then,

$$V_I(\widehat{P}_T f)_l \leq \alpha_l \sum_{k=1}^{K} V_I f_k + B_l \sum_{k=1}^{K} \|f_k\|_1, \tag{3.2}$$

where, $h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}, \delta = \min_i \lambda(J_i)$ and $B_l = \frac{2}{\delta}(\max_k \sup_x h_k(x)) + (\max_k \sup_x |h_k'(x)|)$.

Proof. Since f_k is Riemann integrable, for arbitrary $\epsilon > 0$, we can find a number θ such that for any $J_i \in \mathcal{P}$ and any partition finer than : $J_i = \cup_{p=1}^{L_i} [s_{p-1}, s_p]$ with $|s_p - s_{p-1}| < \theta$, we have

$$\sum_{p=1}^{L_i} |f_k(s_{p-1})||s_p - s_{p-1}| \leq \int_{J_i} |f_k| d\lambda + \epsilon. \tag{3.3}$$

Let $0 = x_0 < x_1 < \ldots \leq x_r = 1$ be such a fine partition of $I = [0,1]$. Define $\phi_{k,i} = \tau_{k,i}^{-1}$. Let $h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}$. We have,

$$V_I(\widehat{P}_T f)_l \leq \sum_{k=1}^{K} V_I P_{\tau_k}\left(W_{k,l} f_k\right). \tag{3.4}$$

We estimate $V_I P_{\tau_k}\left(W_{k,l} f_k\right)$:

$$\begin{aligned}
&\sum_{j=1}^{r} |P_{\tau_k}\left(W_{k,l} f_k\right)(x_j) - P_{\tau_k}\left(W_{k,l} f_k\right)(x_{j-1})| \\
&= \sum_{j=1}^{r} |(\sum_{i=1}^{q} h_k(\phi_{k,i}(x_j)) f_k(\phi_{k,i}(x_j)) \chi_{\tau_k(J_i)}(x_j) - \sum_{i=1}^{q} h_k(\phi_{k,i}(x_{j-1})) f_k(\phi_{k,i}(x_{j-1})) \chi_{\tau_k(J_i)}(x_{j-1}))| \\
&\leq \sum_{j=1}^{r} \sum_{i=1}^{q} |h_k(\phi_{k,i}(x_j)) f_k(\phi_{k,i}(x_j)) \chi_{\tau_k(J_i)}(x_j) - h_k(\phi_{k,i}(x_{j-1})) f_k(\phi_{k,i}(x_{j-1})) \chi_{\tau_k(J_i)}(x_{j-1}))|
\end{aligned} \tag{3.5}$$

We divide the sum on the right hand side into three parts:

(I) the summands for which $\chi_{\tau_k(J_i)}(x_j) = \chi_{\tau_k(J_i)}(x_{j-1}) = 1$;

(II) the summands for which $\chi_{\tau_k(J_i)}(x_j) = 1$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 0$;

(III) the summands for which $\chi_{\tau_k(J_i)}(x_j) = 0$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 1$.

First, we will estimate (I).

$$\begin{aligned}
&\sum_{j=1}^{r} \sum_{i=1}^{q} |h_k(\phi_{k,i}(x_j)) f_k(\phi_{k,i}(x_j)) - h_k(\phi_{k,i}(x_{j-1})) f_k(\phi_{k,i}(x_{j-1}))| \\
&\quad \leq \sum_{i=1}^{q} \sum_{j=1}^{r} |f_k(\phi_{k,i}(x_j))[h_k(\phi_{k,i}(x_j)) - h_k(\phi_{k,i}(x_{j-1}))]| \\
&\quad + \sum_{i=1}^{q} \sum_{j=1}^{r} |h_k(\phi_{k,i}(x_{j-1}))[f_k(\phi_{k,i}(x_j)) - f_k(\phi_{k,i}(x_{j-1}))]|
\end{aligned}$$

$$\leq \sup_x |h_k'(x)| \sum_{i=1}^{q} \sum_{j=1}^{r} |f_k(\phi_{k,i}(x_j))[\phi_{k,i}(x_j) - \phi_{k,i}(x_{j-1})]| + (\sup_x h_k(x)) \sum_{i=1}^{q} V_{J_i} f_k$$

$$\leq \sup_x |h_k'(x)| \sum_{i=1}^{q} \left(\int_{J_i} |f_k| d\lambda(x) + \epsilon \right) + (\sup_x h_k(x)) \sum_{i=1}^{q} V_{J_i} f_k, \text{ using } (3.3)$$

$$\leq \sup_x |h_k'(x)| \int_I |f_k| d\lambda(x) + (\sup_x h_k(x)) V_I f_k + q(\sup_x |h_k(x)|)\epsilon.$$

We now consider (II) and (III) together. Notice that $\chi_{\tau_k(J_i)}(x_j) = 1$ and $\chi_{\tau_k(J_i)}(x_{j-1}) = 0$ occurs only if $x_j \in \tau_l(J_i)$ and $x_{j-1} \notin \tau_l(J_i)$, i.e., if x_j and x_{j-1} are on opposite sides of an end point of $\tau_l(J_i)$, we can have at most one pair x_j, x_{j-1} like this and another pair $x_{j'} \notin \tau_l(J_i)$ and $x_{j'-1} \in \tau_l(J_i)$. Thus,

$$\begin{aligned} &\sum_{i=1}^{q} (|h_k(\phi_{k,i}(x_j)) f_k(\phi_{k,i}(x_j))| + |h_k(\phi_{k,i}(x_{j'-1})) f_k(\phi_{k,i}(x_{j'-1}))|) \\ &\leq \sup_x h_k(x) \sum_{i=1}^{q} (|f_k(\phi_{k,i}(x_j))| + |f_k(\phi_{k,i}(x_{j'-1}))|). \end{aligned} \tag{3.6}$$

Since $s_i = \phi_{k,i}(x_j)$ and $r_i = \phi_{k,i}(x_{j'-1})$ are both points in J_i, we can write

$$\sum_{i=1} (|f_k(s_i)| + |f_k(r_i)|) \leq \sum_{i=1}^{q} (2|f_k(v_i)| + |f_k(v_i) - f_k(r_i)| + |f_k(v_i) - f_k(s_i)|),$$

where $v_i \in J_i$ is such that $|f_k(v_i)| \leq \frac{1}{\lambda(J_i)} \int_{I_i} |f_k| d\lambda(x)$. Thus,

$$\begin{aligned} &\sup_x h_k(x) \sum_{i=1} (|f_k(\phi_{k,i}(x_j))| + |f_k(\phi_{k,i}(x'_{j-1}))|) \\ \leq\ & \sup_x h_k(x) \sum_{i=1} \left(V_{I_i} f_k + \frac{2}{\lambda(J_i)} \int_{I_i} |f_k| d\lambda(x) \right) \\ \leq\ & \sup_x |h_k(x)| V_I f_k + \frac{2 \sup_x h_k(x)}{\delta} \int_I |f_k| d\lambda(x). \end{aligned}$$

Therefore,

$$\begin{aligned} V_I P_{\tau_k}(W_{k,l} f_k) \leq 2 \sup_x |h_k(x)| V_I f_k + (\frac{2}{\delta}(\sup_x h_k) \\ +(\sup |h_k'(x)|)) \|f_k\|_1 + q(\sup_x h_k(x))\epsilon. \end{aligned}$$

Thus,

$$V_I(\widehat{P}_T f)_l \leq \sum_{k=1}^{K} (2 \max_k \sup_x |h_k(x)| V_I f_k + (\frac{2}{\delta} (\max_k \sup_x h_k)$$

$$+ (\max_k \sup |h'_k(x)|)) \|f_k\|_1 + q(\sup_x h_k(x))\epsilon).$$

Since ϵ is arbitrarily small this proves the lemma. ■

Theorem 3.1 *Let τ_k be Lasota-Yorke maps and let $W_{k,l}$ be piecewise of class C^1, for $k, l = 1, 2, \ldots, K$, and*

$$\alpha_l = \max_k \left(\sup_x \frac{2W_{k,l}(x)}{|\tau'_k(x)|} \right), \quad l = 1, 2, \ldots, K,$$

and $\sum_{l=1}^{K} \alpha_l < 1$. Then the operator $\widehat{P}_T$ is quasi-compact and admits a fixed point in $\widehat{BV}$, i.e., the Markov switching random map T admits an absolutely continuous invariant measure.

Proof. The space $\widehat{BV}$ is a Banach space with norm $\|\cdot\|_{\widehat{BV}} = \sum_{k=1}^{K} \|\cdot\|_{BV}$. First, if $f = (f_1, f_2, \ldots, f_K)$ with $f_k \geq 0$, then we have

$$\begin{aligned} \|\widehat{P}_T f\|_1 &= \sum_{l=1}^{K} \|(\widehat{P}_T f)_l\|_1 = \sum_{l=1}^{K} \int_I \sum_{k=1}^{K} P_{\tau_k}(W_{k,l} f_k) d\lambda \\ &= \sum_{k=1}^{K} \int_I \sum_{l=1}^{K} P_{\tau_k}(W_{k,l} f_k) d\lambda = \sum_{k=1}^{K} \int_I \sum_{l=1}^{K} P_{\tau_k}(f_k) d\lambda = \|f\|_1. \end{aligned}$$

For a general f it is easy to show that $\|\widehat{P}_T f\|_1 \leq \|f\|_1$. For $f \in \widehat{BV}$, by the above lemma, we obtain

$$\begin{aligned}
\| \widehat{P_T f} \|_{\widehat{BV}} &= \sum_{l=1}^{K} \| (\widehat{P}_T f)_l \|_{BV} = \sum_{l=1}^{K} V_I(\widehat{P}_T f)_l + \| \widehat{P}_T f \|_1 \\
&\leq \sum_{l=1}^{K} V_I(\widehat{P}_T f)_l + \| \widehat{P}_T f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \left(\sum_{k=1}^{K} V_I f_k \right) + \sum_{l=1}^{K} \left(B_l \sum_{k=1}^{K} \| f_k \|_1 \right) + \| f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \| f \|_1 .
\end{aligned}$$

Thus, by Ionescu-Tulcea and Marinescu Theorem [12], $\widehat{P}_T$ is quasi-compact on $\widehat{BV}$ and admits a fixed point f in $\widehat{BV}$. ■

Example 3.2

Consider the Markov switching position dependent random map $T = \{\tau_1, \tau_2; p_1, p_2; W\}$, where τ_1, τ_2 are maps on $I = [0, 1]$ defined by

$$\tau_1(x) = \begin{cases} 4x, & 0 \leq x \leq \frac{1}{4}, \\ 4x - 1, & \frac{1}{4} < x \leq \frac{1}{2}, \\ 4x - 2, & \frac{1}{2} < x \leq \frac{3}{4} \\ 4x - 3, & \frac{3}{4} < x \leq 1 \end{cases} \tag{3.7}$$

and

$$\tau_2(x) = \begin{cases} \frac{8}{3}x & , 0 \leq x \leq \frac{1}{3}, \\ \frac{8}{3}x - \frac{8}{9} & , \frac{1}{3} < x \leq \frac{2}{3}, \\ \frac{8}{3}x - \frac{16}{9} & , \frac{2}{3} < x \leq 1. \end{cases} \tag{3.8}$$

and W ia a stochastic switching matrix defined by

$$W = \begin{bmatrix} \frac{1}{2}x + \frac{1}{10} & \frac{9}{10} - \frac{1}{2}x \\ \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix},$$

and p_1, p_2 are initial probabilities. It is easy to show that $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{9}{20}$. Hence

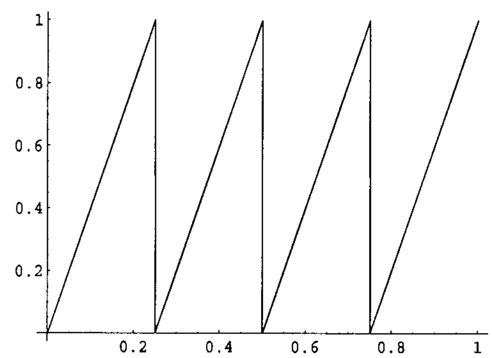


Figure 3.1: The graph of τ_1.

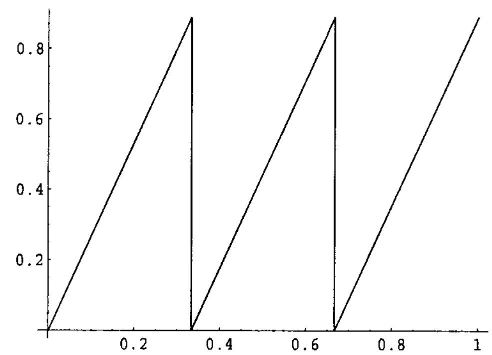


Figure 3.2: The graph of τ_2.

the Markov switching random map $T = \{\tau_1, \tau_2\}$ with switching matrix W satisfies the condition of Theorem 3.1 and T has an acim.

3.3 Approximation of invariant measure for Markov switching position dependent random maps

In this section we consider Markov switching position dependent random maps T with position dependent switching matrix W and describe a method of approximating the fixed point of the operator $\widehat{P_T}$ by the fixed points of a matrix operators. Let $\tau_k : [0,1] \to [0,1], k = 1, 2, \ldots, K$, be Lasota-Yorke maps on a common partition $\mathcal{P}$

of $[0,1] : \mathcal{P} = \{J_1, J_2, \dots, J_q\}$ and

$$W = \begin{bmatrix} W_{1,1}(x) & W_{1,2}(x) & \dots & W_{1,K}(x) \\ W_{2,1}(x) & W_{2,2}(x) & \dots & W_{2,K}(x) \\ \vdots & \vdots & \dots & \vdots \\ W_{K,1}(x) & W_{K,2}(x) & \dots & W_{K,K}(x) \end{bmatrix}$$

be a switching matrix piecewise continuous on the same partition $\{J_1, J_2, \dots, J_K\}$ of $I = [0,1]$, satisfying the condition of Theorem 3.1. Hence, the Markov switching random map $T = \{\tau_1, \tau_2, \dots, \tau_K\}$ map has an acim μ. We want to approximate the invariant density f of μ for the random map T. Let $\mathcal{P}^{(n)} = \{I_1, I_2, \dots, I_n\}$ be a partition of $[0,1]$ into subintervals of equal length and let $M_n(k)$ be the matrix of transition probabilities between the elements of $\mathcal{P}^{(n)}$ for the map $\tau_k, k = 1, 2, \dots, K$:

$$M_n(k) = \left(\frac{\lambda(I_i \cap \tau_k^{-1}(I_j))}{\lambda(I_i)} \right)_{1 \le i,j \le n}.$$

Let $L^{(n)} = \{f \in BV(I) : f = \sum_{i=1}^{n} f_n^i \chi_{I_i} = (f_n^1, f_n^2, \dots, f_n^n)\}$. Define an operator $Q^{(n)} : BV(I) \to L^{(n)}$ by

$$Q^{(n)}(f) = \sum_{i=1}^{n} n \left(\int_{I_i} f d\lambda \right) \chi_{I_i} = \left(n \int_{I_1} f d\lambda, n \int_{I_2} f d\lambda, \dots, n \int_{I_n} f d\lambda \right).$$

Let P_{τ_k} be the Frobenius-Perron operator of τ_k and $P_{\tau}^{(n)} : L^{(n)} \to L^{(n)}$ be a finite approximation of P_{τ} defined by

$$P_{\tau_k}^{(n)} f = (M_n(k))^t f.$$

Li [25] proved the following results:

1. For $f \in L_1$, $Q^{(n)} f \longrightarrow f$ in L_1 as $n \to \infty$;

2. For $f \in L^{(n)}, P^{(n)}_{\tau_k} f = Q^{(n)} P_{\tau_k} f$;

3. For $f \in BV(I), V_I Q^{(n)} f \leq V_I f$;

4. For $f \in L_1, P^{(n)}_{\tau_k} f \longrightarrow P_{\tau_k} f$ in L_1 as $n \to \infty$;

Let $\widehat{L^{(n)}} = \prod_{k=1}^{K} L^{(n)}$ and define $\widehat{Q^{(n)}} : \widehat{BV} \to \widehat{L^{(n)}}$ by

$$\widehat{Q^{(n)}}(f_1, f_2, \dots, f_K) = (Q^{(n)}(f_1), Q^{(n)}(f_2), \dots, Q^{(n)}(f_n)).$$

We define an operator $\widehat{P^{(n)}_T} : \widehat{L^{(n)}} \to \widehat{L^{(n)}}$ by

$$\begin{aligned} \widehat{P^{(n)}_T}(f_1, f_2, \dots, f_K) &= (\sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,2}) f_k, \dots, \\ &\quad \sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,K}) f_k). \end{aligned} \tag{3.9}$$

For a fixed n, let

$$f_k = (f^1_{k,n}, f^2_{k,n}, \dots, f^n_{k,n}), \; k = 1, 2, \dots, K,$$

and

$$Q^{(n)}(W_{k,l}) = (W^1_{k,l,n}, W^2_{k,l,n}, \dots, W^n_{k,l,n}), \; k, l = 1, 2, \dots, K.$$

Then, for any $1 \le i \le n$, $\sum_{l=1}^{K} W_{k,l,n}^{i} = \sum_{l=1}^{K} n \int_{I_i} W_{k,l} d\lambda = 1$ and hence

$$\begin{aligned}
\widehat{P_T^{(n)}}(f_1, f_2, \dots, f_K) = \Bigg(& \sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,2}) f_k, \dots \\
& , \sum_{k=1}^{K} M_n(k) Q^{(n)}(W_{k,K}) f_k \Bigg) \\
= (& \sum_{k=1}^{K} M_n(k) \left[W_{k,1,n}^{1} f_{k,n}^{1}, W_{k,1,n}^{2} f_{k,n}^{2}, \dots, W_{k,1,n}^{n} f_{k,n}^{n} \right], \qquad (3.10) \\
& \sum_{k=1}^{K} M_n(k) \left[W_{k,2,n}^{1} f_{k,n}^{1}, W_{k,2,n}^{2} f_{k,n}^{2}, \dots, W_{k,2,n}^{n} f_{k,n}^{n} \right], \\
& \dots, \sum_{k=1}^{K} M_n(k) \left[W_{k,K,n}^{1} f_{k,n}^{1}, W_{k,2,n}^{2} f_{k,n}^{2}, \dots, W_{k,K,n}^{n} f_{k,n}^{n} \right]).
\end{aligned}$$

Lemma 3.2 *For $f \in \widehat{L^{(n)}}$, we have $\widehat{P_T^{(n)}} f = \widehat{Q^{(n)}} \widehat{P_T} f$.*

Proof. Let $f = (f_1, f_2, \dots, f_K) \in \widehat{L}^{(n)}$, where $f_k = (f_{k,n}^{1}, f_{k,n}^{2}, \dots, f_{k,n}^{n})$, $k = 1, 2, \dots, K$. Then,

$$\widehat{Q^{(n)}} \widehat{P_T} f = \widehat{Q^{(n)}} \left(\sum_{k=1}^{K} \mathcal{P}_{\tau_k}(Q^{(n)}(W_{k,1}) f_k), \sum_{k=1}^{K} \mathcal{P}_{\tau_k} Q^{(n)}(W_{k,2}) f_k) f_k, \dots, \sum_{k=1}^{K} \mathcal{P}_{\tau_k} Q^{(n)}(W_{k,K}) f_k) f_k \right)$$

$$\begin{aligned}
&= \left(Q^{(n)} \sum_{k=1}^{K} \mathcal{P}_{\tau_k} \left(Q^{(n)}(W_{k,1}) f_k \right), Q^{(n)} \sum_{k=1}^{K} \mathcal{P}_{\tau_k} \left(Q^{(n)}(W_{k,2}) f_k \right), \dots, Q^{(n)} \sum_{k=1}^{K} \mathcal{P}_{\tau_k} \left(Q^{(n)}(W_{k,K}) f_k \right) \right) \\
&= \left(\sum_{k=1}^{K} Q^{(n)} \left(\mathcal{P}_{\tau_k} Q^{(n)}(W_{k,1}) f_k \right), \sum_{k=1}^{K} Q^{(n)} \left(\mathcal{P}_{\tau_k} Q^{(n)}(W_{k,2}) f_k \right), \dots, \sum_{k=1}^{K} Q^{(n)} \left(\mathcal{P}_{\tau_k} Q^{(n)}(W_{k,K}) f_k \right) \right) \\
&= \left(\sum_{k=1}^{2} (M_k^n)^t Q^{(n)}(W_{k,1}) f_k, \sum_{k=1}^{2} (M_k^n)^t Q^{(n)}(W_{k,2}) f_k), \dots, \sum_{k=1}^{K} (M_k^n)^t Q^{(n)}(W_{k,2}) f_k) \right) \\
&= \widehat{P_T^{(n)}} f.
\end{aligned}$$

■

Using properties of $Q^{(n)}$and $\widehat{Q}^{(n)}$, we can prove the following lemmas.

Lemma 3.3 *For $f \in \widehat{BV}$, $\widehat{Q}^{(n)}f$ converges to f in $\widehat{BV}$.*

Lemma 3.4 *For $f \in \widehat{L^{(n)}}$, $\widehat{P}_T^{(n)}f$ converges to $\widehat{P}_T f$ in $\widehat{BV}$.*

Proof. By Lemma 3.2, for $f \in \widehat{L^{(n)}}$ we have $\widehat{P_T^{(n)}}f = \widehat{Q^{(n)}}\widehat{P_T}f$. By Lemma 3.3 we have $\widehat{Q^{(n)}}\widehat{P_T}f$ converges to $\widehat{P_T}f$. ■

Now we prove a theorem which will be useful later.

Theorem 3.3 *Let τ_k be of class C^2 and $W_{k,l}$ be of class C^1, for $k = 1, 2$ satisfying the conditions of Theorem [3.1], i.e.,*

$$\alpha_l = \max_k(\sup_x \frac{2W_{k,l}(x)}{|\tau_k'(x)|}),\ l = 1, 2, \ldots, K.$$

and $\sum_{l=1}^K \alpha_l < 1$. Then for any positive integer n, $\widehat{P_T^{(n)}}$ has a fixed point f_n in $\widehat{L^{(n)}}$.

Proof.

$$\begin{aligned} V_I(\widehat{P_T^{(n)}}f)_l &= V_I(\widehat{Q^n}\widehat{P_T}f)_l \\ &\leq V_I(\widehat{P_T}f)_l \\ &\leq \alpha_l \sum_{k=1}^K V_I f_k + B_l \sum_{k=1}^K \| f_k \|_1, \end{aligned}$$

where $\alpha_l = \max_k(\sup_x \frac{2 \cdot W_{k,l}(x)}{|\tau_k'(x)|})$, $h_k(x) = \frac{W_{k,l}(x)}{|\tau_k'(x)|}$, $\delta = \min_i \lambda(I_i)$ and $B_l = \frac{2}{\delta}(\max_k \sup_x h_k(x)) + (\max_k \sup_x |h_k'(x)|)$.

Now,

$$\begin{aligned}
\| \widehat{P_T^{(n)}} f \|_{\widehat{BV}} &= \sum_{l=1}^{K} \| (\widehat{P_T^{(n)}} f)_l \|_{BV} \\
&= \sum_{l=1}^{K} V_I(\widehat{P_T^{(n)}} f)_l + \| \widehat{P_T^{(n)}} f \|_1 \\
&\leq \sum_{l=1}^{K} V_I(\widehat{P_T} f)_l + \| \widehat{P_T^{(n)}} f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \left(\sum_{k=1}^{K} V_I f_k \right) + \sum_{l=1}^{K} \left(B_l \sum_{k=1}^{K} \| f_k \|_1 \right) + \| f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \| f \|_1 .
\end{aligned}$$

Thus, by the Ionescu-Tulcea and Marinescu Theorem [12], $\widehat{P_T^{(n)}}$ is quasi-compact on $\widehat{L^{(n)}}$ and admits a fixed point f_n in $\widehat{L^{(n)}}$. ■

This theorem can be also proved using simple matrix theorem. It can be shown that the operator $\widehat{P_T^{(n)}}$ can be represented by the following $(K \times n)$ by $(K \times n)$ stochastic matrix

$$S_n = \begin{bmatrix}
M_n(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,1})] & M_n(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,2})] & \dots & M_n(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,\mathrm{K}})] \\
M_n(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,1})] & M_n(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,2})] & \dots & M_n(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,\mathrm{K}})] \\
\vdots & \vdots & \dots & \vdots \\
M_n(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},1})] & M_n(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},2})] & \dots & M_n(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},\mathrm{K}})]
\end{bmatrix}.$$

Since 1 is a left eigenvalue of any stochastic matix [8, 29], the matrix S_n has 1 as a left eigenvalue. Let

$$s_n = (s_{n,1}, s_{n,2}, \dots , s_{n,K})$$

be a left eigenvector of S_n associated with the eigenvalue 1 and

$$s_{n,k} = \left(s^1_{n,k}, s^2_{n,k}, \dots, s^n_{n,k}\right), \sum_{i=1}^{n} s^i_{n,k} = 1, \; k = 1, 2, \dots, K.$$

Define the approximating invariant density

$$d_n = \sum_{i=1}^{n} \left(\frac{\sum_{k=1}^{K} s^i_{n,k}}{\lambda(I_i)} \right) \chi_{I_i}. \tag{3.11}$$

Theorem 3.4 *Suppose that the Markov switching random map T satisfies the hypothesis of Theorem 3.1 and the operator $\widehat{P}_T$ has a unique invariant density d. Then $\| d - d_n \|_1 \to 0$ as $n \to 0$, for the approximate density d_n in (3.9).*

Proof. Let $f = (f_{n,1}, f_{n,2}, \dots, f_{n,K}) \in \widehat{L}^{(n)}$ be a fixed point of $\widehat{P}_T^{(n)}$. Then

$$\begin{aligned}
\max_{1\le l\le K} V_I(\widehat{P}_T^{(n)} f)_l &= \max_{1\le l\le K} V_I(\widehat{Q}^{(n)} \widehat{P}_T (f_{n,1}, f_{n,2}, \dots, f_{n,K}))_l \\
&= \max_{1\le l\le K} V_I(Q^{(n)}(\widehat{P}_T (f_{n,1}, f_{n,2}, \dots, f_{n,K})))_l \\
&\le \max_{1\le l\le K} V_I(\widehat{P}_T (f_{n,1}, f_{n,2}, \dots, f_{n,K}))_l \\
&\le \max_{1\le l\le K} \alpha_l \sum_{k=1}^{K} V_I f_{n,k} + \max_{1\le l\le 2} B_l \sum_{k=1}^{K} \| f_{n,k} \|_1 . \qquad (3.12)
\end{aligned}$$

Thus the sequence $\{V_I(P_T^n f)_l)\}_{n\ge1}$ is uniformly bounded . So by Helly's theorem, the set $C = \{(f_{n,1}, f_{n,2}, \cdots, f_{n,K}); n = 1, 2, \cdots\}$ is sequentially compact in $\prod_{k=1}^{r} L^1$. Let $\{(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\}_{j\ge1}$ be any subsequence of C and assume that

$\{(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})\}_{j\geq 1}$ converges to $(f_1, f_2, \dots, f_K)$ as $j \to \infty$. Then

$$
\begin{aligned}
& \| (f_1, f_2, \dots, f_K) - \widehat{P}_T (f_1, f_2, \dots, f_K) \|_1 \\
\leq\ & \| (f_1, f_2, \dots, f_K) - (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) \|_1 \\
& + \| (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) \|_1 \\
+\ & \| \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_1, f_2, \cdots, f_K) \|_1 \\
& + \| \widehat{Q}^{(n_j)} \widehat{P}_T (f_1, f_2, \cdots, f_K) - \widehat{P_T} (f_1, f_2, \dots, f_K) \|_1 .
\end{aligned}
$$

Note that

$$
\widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) = \widehat{P}_T^{n_j} (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})
$$

and $(f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K})$ is a fixed point of $\widehat{P}_T^{n_j}$. Thus

$$
\| (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) \|_1 = 0.
$$

Moreover,

$$
\begin{aligned}
& \| \widehat{Q}^{(n_j)} \widehat{P}_T (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - \widehat{Q}^{(n_j)} \widehat{P_T} (f_{n,1}, f_{n,2}, \cdots, f_{n,K}) \|_1 \\
& \leq \| \widehat{Q}^{(n_j)} \widehat{P}_T \|_1 \| (f_{n_j,1}, f_{n_j,2}, \cdots, f_{n_j,K}) - (f_{n,1}, f_{n,2}, \cdots, f_{n,K}) \|_1
\end{aligned}
$$

and

$$
\widehat{Q}^{n_j} (\widehat{P}_T h) \to \widehat{P}_T h.
$$

Hence $\widehat{P_T} (f_1, f_2, \dots, f_K) = (f_1, f_2, \dots, f_K)$

Therefore, any convergent subsequence of C converges to a fixed point of $\widehat{P_T}$. By assumption, $\widehat{P_T}$ has a unique fixed point, that is, $\|d - d_n\|_1 \to 0$ as $n \to \infty$. ■

Now, we restrict our attention to the space $\widehat{BV}_0 = \{\hat{f} \in \widehat{BV} : \int f_k = 0$ for all $k = 1, 2, \ldots, K\} \subset \widehat{BV}$. In this situation we have $\| f_k \|_1 \leq \frac{1}{2} V_I f_k, k = 1, 2, \ldots, K$ [18]. Moreover, the operator $\widehat{P}_T$ is a contraction on $\widehat{BV}_0$. We have the following lemma.

Lemma 3.5 *Let* $f = (f_1, f_2, \ldots, f_K) \in \widehat{BV}_0$. *Then*

$$\| (\widehat{P}_T f) \|_{\widehat{BV}} \leq \frac{1}{2} \left(\sum_{l=1}^{K} \alpha_l + \sum_{l=1}^{K} B_l + 1 \right) \| f \|_{\widehat{BV}} . \tag{3.13}$$

Proof.

$$\begin{aligned}
\| \widehat{P_T f} \|_{\widehat{BV}} &= \sum_{l=1}^{K} \| (\widehat{P_T f})_l \|_{BV} \\
&= \sum_{l=1}^{K} V_I (\widehat{P_T f})_l + \| \widehat{P_T f} \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \left(\sum_{k=1}^{K} V_I f_k \right) + \sum_{l=1}^{K} B_l \left(\sum_{k=1}^{K} \| f_k \|_1 \right) + \| f \|_1 \\
&= \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \| f \|_1 \\
&\leq \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \cdot \sum_{l=1}^{K} \frac{1}{2} V_I (f_l) \\
&= \sum_{l=1}^{K} \alpha_l \| f \|_{\widehat{BV}} + \frac{1}{2} \left(\sum_{l=1}^{K} B_l + 1 - \sum_{l=1}^{K} \alpha_l \right) \| f \|_{\widehat{BV}} \\
&= \frac{1}{2} \left(\sum_{l=1}^{K} \alpha_l + \sum_{l=1}^{K} B_l + 1 \right) \| f \|_{\widehat{BV}} .
\end{aligned}$$

■

3.4 Error bound for density of Markov switching position dependent random map

In this section we restrict our attention to the space $\widehat{BV}_0 = \{\hat{f} \in \widehat{BV} : \int f_k = 0$ for all $k = 1, 2, \dots, K\} \subset \widehat{BV}$. Let $d = \sum_{k=1}^{K} d_k$, where $\widehat{P}_T(d_1, d_2, \dots, d_K) = (d_1, d_2, \dots, d_K)$ is the unique invariant density of the invariant measure μ for a Markov switching position dependent random map T, and for a fixed n let d_n be the approximate invariant density of T. In this section we want to find a bound for $\|d - d_n\|_1$. The measure μ can be decomposed as $\mu = \sum_{k=1}^{K} \mu_k$, where $(\mu_1, \mu_2, \dots, \mu_K)$ is fixed by the operator

$$(\mu_1, \mu_2, \dots, \mu_K) \mapsto \left(\sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,1} d\mu_k, \sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,2} d\mu_k, \dots, \sum_{k=1}^{K} \int_{\tau_k^{-1}} W_{k,K} d\mu_k \right).$$

Let $\widehat{M_n}(k)$ be the matrix of transition probabilities between the elements of $\mathcal{P}^{(n)}$ for the map τ_k with respect to measure μ_k, $k = 1, 2, \dots, K$:

$$\widehat{M_n}(k) = \left(\frac{\mu_k(I_i \cap \tau_k^{-1}(I_j))}{\mu_k(I_i)} \right)_{1 \le i,j \le n}$$

Now, consider the following $(K \times n)$ by $(K \times n)$ matrix

$$\hat{S}_n = \begin{bmatrix} \widehat{M_n}(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,1})] & \widehat{M_n}(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,2})] & \dots & \widehat{M_n}(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,\mathrm{K}})] \\ \widehat{M_n}(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,1})] & \widehat{M_n}(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,2})] & \dots & \widehat{M_n}(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,\mathrm{K}})] \\ \vdots & \vdots & \dots & \vdots \\ \widehat{M_n}(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},1})] & \widehat{M_n}(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},2})] & \dots & \widehat{M_n}(K)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{\mathrm{K},\mathrm{K}})] \end{bmatrix}.$$

Let $\hat{s}_n = (\hat{s}_{n,1}, \hat{s}_{n,2}, \dots, \hat{s}_{n,K})$ be a left eigenvector of $\hat{S}_n$ associated with eigenvalue 1 and $\hat{s}_{n,k} = \left(\hat{s}^1_{n,k}, \hat{s}^2_{n,k}, \dots, \hat{s}^n_{n,k}\right), \sum_{i=1}^n \hat{s}^i_{n,k} = 1, k = 1, 2, \dots, K$. It can be shown that

$$\hat{s}_{n,k} = \left(\hat{s}^1_{n,k}, \hat{s}^2_{n,k}, \dots, \hat{s}^n_{n,k}\right) = (\mu_k(I_1), \mu_k(I_2), \dots, \mu_k(I_n)).$$

Lemma 3.6 *For $f_n \in L^{(n)}$, $\|f_n\|_m \leq n\|f_n\|_{BV}$ and $\|f_n\|_{BV} \leq 3\|f_n\|_m$, where $\|f_n\|_{BV} = V(f_n) + \|f_n\|_1$ and $\|f_n\|_m = \sum_{i=1}^n |f_n^i|$.*

Proof. $\|f_n\|_m = \sum_{i=1}^n |f_n^i| = n\left(\sum_{i=1}^n |f_n^i| \frac{1}{n}\right) = n\|f_n\|_1 \leq nV(f_n) + n\|f_n\|_1 = n\|f_n\|_{BV}$. Notice that $V(f_n) \leq 2\|f_n\|_m$. Thus,

$$\|f_n\|_{BV} = V(f_n) + \|f_n\|_1 \leq 2\|f_n\|_m + \frac{1}{n}\|f_n\|_m \leq 2\|f_n\|_m + \|f_n\|_m = 3\|f_n\|_m.$$

■

Define $\|s_n\|_m = \sum_{k=1}^K \sum_{i=1}^n |s^i_{n,k}|$.

Lemma 3.7 *For $\hat{f}_n = (f_{n,1}, f_{n,2}, \dots, f_{n,K}) \in \widehat{L}^{(n)}$, $\|\hat{f}_n\|_m \leq n\|\hat{f}_n\|_{\widehat{BV}}$ and $\|\hat{f}_n\|_{\widehat{BV}} \leq 3\|\hat{f}_n\|_m$.*

Proof. $\|\hat{f}_n\|_m = \sum_{k=1}^K \|f_{n,k}\|_m \leq n\sum_{k=1}^K \|f_{n,k}\|_{BV} \leq n\|\hat{f}_n\|_{\widehat{BV}}$. On the other hand, $\|\hat{f}_n\|_{\widehat{BV}} = \sum_{k=1}^K \|f_{n,k}\|_{BV} \leq 3\sum_{k=1}^K \|f_{n,k}\|_m = 3\|\hat{f}_n\|_m$. ■

Theorem 3.5 $\|d - d_n\|_1 \leq \frac{1}{n}\sum_{k=1}^K Lip(d_k) + \inf_{0<\delta<1}\left(2 + \frac{\delta}{1-\delta}\right)\left(\left[\frac{\log(6K^2n/\delta)}{-\log\gamma}\right] + 1\right) - 1\}$.

Proof. $\|d - d_n\|_1 \leq \|d - Q^{(n)}(d)\|_1 + \|Q^{(n)}(d) - d_n\|_1$. First we find a bound for $\|d - Q^{(n)}(d)\|_1$.

$$\begin{aligned}
\|d - Q^{(n)}(d)\|_1 &= \|\sum_{k=1}^{K} d_k - Q^{(n)}(\sum_{k=1}^{K} d_k)\|_1 \leq \sum_{k=1}^{K} \|d_k - Q^{(n)}(d_k)\|_1 \\
&\leq \sum_{k=1}^{K} \left(\int_I |d_k - Q^{(n)}(d_k)| d\lambda \right) \leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \int_{I_i} |d_k - n \int_{I_i} d_k d\lambda| d\lambda \right) \\
&\leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \frac{1}{2n} \left(\sup_{I_i} d_k - \inf_{I_i} d_k \right) \right) \leq \sum_{k=1}^{K} \left(\sum_{i=1}^{n} \frac{1}{n} \cdot Lip(d_k) \cdot \frac{1}{2n} \right) \\
&= \sum_{k=1}^{K} Lip(d_k) \cdot \frac{1}{2n},
\end{aligned}$$

where $Lip(d_k)$ is the maximum Lipschitz constant calculated over each of the Lipschitz pieces of d_k separately.

Now, we want to bound $\|Q^{(n)}(d) - d_n\|_1$:

$$\begin{aligned}
\|Q^{(n)}(d) - d_n\|_1 &= \|\sum_{i=1}^{n} n \int_{I_i} d - \sum_{i=1}^{n} n \left(\sum_{k=1}^{K} s_{n,k}^i \right) \chi_{I_i} \|_1 \\
&= \|\sum_{i=1}^{n} n \left(\sum_{k=1}^{K} \hat{s}_{n,k}^i \right) \chi_{I_i} - \sum_{i=1}^{n} n \left(\sum_{k=1}^{K} s_{n,k}^i \right) \chi_{I_i} \|_1 \\
&= \sum_{i=1}^{n} |\sum_{k=1}^{K} (\hat{s}_{n,k}^i - s_{n,k}^i)| \leq \sum_{i=1}^{n} \sum_{k=1}^{K} |\hat{s}_{n,k}^i - s_{n,k}^i| \\
&= \sum_{k=1}^{K} \sum_{i=1}^{n} |\hat{s}_{n,k}^i - s_{n,k}^i| = \|\hat{s_n} - s_n\|_m,
\end{aligned}$$

where $\|.\|_m$ denotes the L^1 vector norm. Now the stochastic matrix S_n has a unique left eigenvector and we have by [32],

$$\|\hat{s_n} - s_n\|_m \leq \|\hat{S_n} - S_n\|_m \| \left(I_{Kn} - S_n + {S_n}^{\infty} \right)^{-1} \|_m,$$

where each row of ${S_n}^{\infty}$ is the unique left eigenvector of S_n.

Now, we want to bound $\|\hat{S}_n - S_n\|_m$:

$$\begin{aligned}
|M_{n,ij}(k) - \hat{M}_{n,ij}(k)| &= M_{n,ij}(k)|1 - \frac{\mu_k(I_i \cap \tau_k^{-1}(I_j))}{\mu_k(I_i)} \frac{\lambda(I_i)}{\lambda(I_i \cap \tau_k^{-1}(I_j))}| \\
&= M_{n,ij}(k)|1 - \frac{\int_{I_i \cap \tau_k^{-1}(I_j)} d_k}{\lambda(I_i \cap \tau_k^{-1}(I_j))} \frac{\lambda(I_i)}{\int_{I_i} d_k}| \\
&\leq M_{n,ij}(k)|1 - \frac{\sup_{I_i \cap \tau_k^{-1}} d_k}{\inf_{I_i} d_k}| \\
&\leq \frac{M_{n,ij}(k)}{\inf_{I_i} d_k} | \sup_{I_i \cap \tau_k^{-1}} d_k - \inf_{I_i} d_k| \leq \frac{M_{n,ij}(k)}{\inf_{I_i} d_k} |\sup_{I_i} d_k - \inf_{I_i} d_k| \\
&\leq \frac{M_{n,ij}(k)}{\inf_I d_k} \mathrm{Lip(d_k)}. \frac{1}{2\mathrm{n}}.
\end{aligned}$$

where $\mathrm{Lip}(d_k)$ is as before.

Now,

$$\begin{aligned}
\|\hat{M}_n(k) - M_n(k)\|_m &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |M_{n,ij}(k) - \hat{M}_{n,ij}(k)| \\
&\leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \frac{M_{n,ij}(k)}{\inf_I d_k} \mathrm{Lip(d_k)}. \frac{1}{2\mathrm{n}} = \frac{\mathrm{Lip(d_k)}}{\mathrm{inf_I\, d_k}}. \frac{1}{2\mathrm{n}}.
\end{aligned}$$

$$\begin{aligned}
\|\hat{S}_n - S_n\|_m &= \max_{1 \leq k \leq K} \sum_{l=1}^{K} \|\hat{M}_n(k) \mathrm{diag}[\mathrm{Q^n(W_{k,l}]} - \mathrm{M_n(k)diag[Q^n(W_{k,l}]}\|_\mathrm{m} \\
&= \max_{1 \leq k \leq K} \sum_{l=1}^{K} \left(\max_{1 \leq i \leq n} \sum_{j=1}^{n} | \left(\hat{M}_{n,ij}(k) \mathrm{diag}[\mathrm{Q^n(W_{k,l}]} \right)_{ij} - (M_{n,ij}(k) \mathrm{diag}[\mathrm{Q^n(W_{k,l}]})_{ij}| \right) \\
&\leq \max_{1 \leq k \leq K} \sum_{l=1}^{K} \left(\left(\max_{I_i \in \mathcal{P}^n} n \int_{I_i} W_{k,l} \right) \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\hat{M}_{n,ij}(k) - M_{n,ij}(k)| \right) \\
&\leq \max_{1 \leq k \leq K} \sum_{l=1}^{K} \left(\left(\max_{I_i \in \mathcal{P}^n} n \int_{I_i} W_{k,l} \right) \frac{\mathrm{Lip(d_k)}}{\mathrm{inf_I\, d_k}}. \frac{1}{2n} \right).
\end{aligned}$$

Now,

$$\| \left(I_{Kn} - S_n + S_n^\infty \right)^{-1} \|_m \leq 1 + \sum_{N=1}^{\infty} \|S_n^N - S_n^\infty\|_m.$$

Define $\widehat{L^{(n)}}_0 = \{(f_{n,1}, f_{n,2}, \dots, f_{n,K}) \in \widehat{L^{(n)}} : \sum_{i=1}^{n} f_{n,k}^i = 0 \text{ for all } k = 1, 2, \dots, K\}$.

$$\begin{aligned}
\|S_n^N - S_n^\infty\|_m &= \sup_{\hat{f}_{n,0} \in \widehat{L^{(n)}}_0} \frac{\|(S_n^N - S_n^\infty)\hat{f}_{n,0}\|_m}{\|\hat{f}_{n,0}\|_m} = \sup_{\hat{f}_{n,0} \in \widehat{L^{(n)}}_0} \frac{\|S_n^N(\hat{f}_{n,0} - S_n^\infty \hat{f}_{n,0})\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\leq \|S_n^N|_{\widehat{L^{(n)}}_0}\|_m \sup_{\hat{f}_{n,0} \in \widehat{L^{(n)}}_0} \frac{\|(\hat{f}_{n,0} - S_n^\infty \hat{f}_{n,0})\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\leq 2\|S_n^N|_{\widehat{L^{(n)}}_0}\|_m = 2 \sup_{\hat{f}_{n,0} \in \widehat{L^{(n)}}_0} \frac{\|[\hat{Q}^n \widehat{P}_T]^N \hat{f}_{n,0}\|_m}{\|\hat{f}_{n,0}\|_m} \\
&\leq 2 \sup_{\hat{f}_{n,0} \in \widehat{L^{(n)}}_0} \frac{n\|[\hat{Q}^n \widehat{P}_T]^N \hat{f}_{n,0}\|_{\widehat{BV}}}{\frac{\|\hat{f}_{n,0}\|_{\widehat{BV}}}{3}} \\
&\leq 6n\|\widehat{P}_T|_{\widehat{BV}_0}\|_{\widehat{BV}}^N \leq 6n\gamma^N,
\end{aligned}$$

where $\gamma = \frac{1}{2}\left(\sum_{l=1}^{K} \alpha_l + \sum_{l=1}^{K} B_l + 1\right)$. Using Lemma 6.11 and Lemma 6.15 of [10], we get,

$$\begin{aligned}
\|\,(I_{Kn} - S_n + S_n^\infty)^{-1}\,\|_m &\leq 1 + \sum_{N=1}^{\infty} \|S_n^N - S_n^\infty\|_m \\
&= 1 + \sum_{N=1}^{m_n} \|S_n^N - S_n^\infty\|_m + \sum_{N=m_n}^{\infty} \|S_n^N - S_n^\infty\|_m \\
\leq 1 + \sum_{N=1}^{m_n} 2 + \sum_{N=m_n}^{\infty} \beta^{[n/m_n]} &= \left(2 + \frac{\beta}{1-\beta}\right) m_n - 1 \\
&\leq \inf_{0<\beta<1} \left(2 + \frac{\beta}{1-\beta}\right)\left(\left[\frac{\log(6n/\beta)}{-\log\gamma}\right] + 1\right) - 1\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|d - d_n\|_1 \leq \|d - Q^{(n)}(d)\|_1 + \|Q^{(n)}(d) - d_n\|_1 &\leq \sum_{k=1}^{K} Lip(d_k).\frac{1}{2n} \\
&+ \inf_{0<\beta<1} \left(2 + \frac{\beta}{1-\beta}\right)\left(\left[\frac{\log(6n/\beta)}{-\log\gamma}\right] + 1\right) - 1\}.
\end{aligned}$$

■

3.5 Numerical example

In this section we present a numerical example. We use Maple (version 9.5) to find the invariant density of the acim for a Markov switching position dependent random map.

Example 3.6

Consider the Markov switching position dependent random map $T = \{\tau_1, \tau_2; p_1, p_2; W\}$, where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by

$$\tau_1(x) = 6x^3 - 9x^2 + 8x \pmod 1,$$

$$\tau_2(x) = \begin{cases} x^2 + 3x & , 0 \le x < \frac{-3}{2} + \frac{1}{2}\sqrt{13}, \\ \frac{x - \frac{3}{4}}{\frac{9}{4} - \frac{1}{2}\sqrt{13}} + 1 & , \frac{-3}{2} + \frac{1}{2}\sqrt{13} \le x < \frac{3}{4}, \\ 4x - 3 & , \frac{3}{4} \le x \le 1 \end{cases}$$

and the position dependent switching matrix W,

$$W = \begin{bmatrix} W_{1,1}(x) & W_{1,2}(x) \\ W_{2,1}(x) & W_{2,2}(x) \end{bmatrix}$$

is defined by

$$W_{1,1}(x) = \begin{cases} .8 & , 0 \le x < \frac{1}{2}, \\ .2 & , \frac{1}{2} \le x \le 1. \end{cases}$$

$$W_{1,2}(x) = \begin{cases} .2 & , 0 \le x < \frac{1}{2}, \\ .8 & , \frac{1}{2} \le x \le 1. \end{cases}$$

$$W_{2,1}(x) = \begin{cases} .5 & , 0 \le x < \frac{1}{2}, \\ .2 & , \frac{1}{2} \le x \le 1. \end{cases}$$

$$W_{2,2}(x) = \begin{cases} .5 & , 0 \le x < \frac{1}{2}, \\ .8 & , \frac{1}{2} \le x \le 1. \end{cases}$$

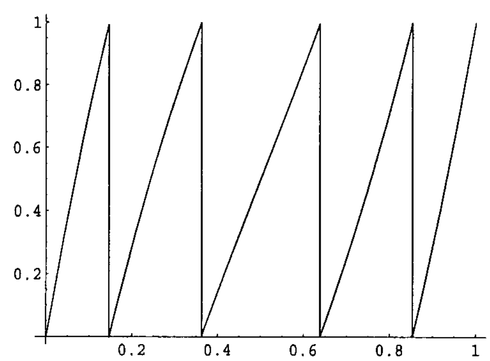


Figure 3.3: The graph of τ_1.

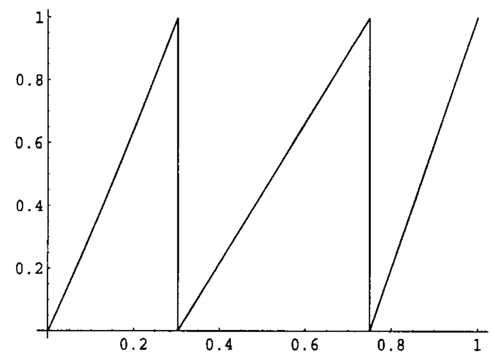


Figure 3.4: The graph of τ_2.

Notice that $\inf_x|\tau_1'(x)| = 3.5$ and $\inf_x|\tau_2'(x)| = \frac{1}{\frac{9}{4}-\frac{1}{2}\sqrt{13}} = 2.236014146$. We have, for $x \in [0, \frac{1}{2}), \alpha_1 + \alpha_2 = .45714 + .44723 = .90437 < 1$ and for $x \in [\frac{1}{2}, 1], \alpha_1 + \alpha_2 = .71556 + .17889 = .89445 < 1$. Thus, by Theorem 3.1, the random map T has an acim. Now we want to approximate the invariant density of the acim using our method described in Section 3.3. We have a Maple program (Maple 9.5) that gives, for any positive integer n, the transition matrices

$$\widehat{M_n}(k) = \left(\frac{\lambda(I_i \cap \tau_k^{-1}(I_j))}{\lambda(I_i)}\right)_{1\leq i,j\leq n}, k = 1, 2\ ,$$

and the $2 \times n$ by $2 \times n$ stochastic matrix

$$\hat{S_n} = \begin{bmatrix} \widehat{M_n}(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,1})] & \widehat{M_n}(1)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{1,2})] \\ \widehat{M_n}(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,1})] & \widehat{M_n}(2)\mathrm{diag}[\mathrm{Q}^{(\mathrm{n})}(\mathrm{W}_{2,2})] \end{bmatrix},$$

and finally the left eigenvector of the matrix $\widehat{S_n}$. Here is a typical example for $n = 8$:

$$\widehat{M_n}(1) = \begin{bmatrix} .12726 & .13199 & .13711 & .14266 & .14867 & .15518 & .15712 & 0 \\ .17840 & .18752 & .19728 & .20792 & .05368 & 0 & .00520 & .17000 \\ .10656 & 0 & 0 & 0 & .16536 & .23088 & .24272 & .25448 \\ .15880 & .27464 & .28144 & .28512 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .28512 & .28144 & .27464 & .15880 \\ .25448 & .24272 & .23088 & .16536 & 0 & 0 & 0 & .10656 \\ .17000 & .00520 & 0 & .05368 & .20792 & .19728 & .18752 & .17840 \\ 0 & .15712 & .15520 & .14864 & .14264 & .13712 & .13200 & .12728 \end{bmatrix},$$

$$\widehat{M_n}(2) = \begin{bmatrix} .32883 & .32028 & .31241 & .03848 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .26648 & .29816 & .29176 & .14360 & 0 \\ .44720 & .13040 & 0 & 0 & 0 & 0 & .14216 & .28008 \\ 0 & .31688 & .44720 & .23592 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .21128 & .44720 & .34152 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .10568 & .44720 & .44712 \\ .25000 & .25000 & .25000 & .25000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .25000 & .25000 & .25000 & .25000 \end{bmatrix}.$$

Solving $x \cdot S_n = x$ we get,

$$x = [1.1698, 1.2088, 1.2280, 1.3348, .41270, .41454, .41760, .42095, .64043,$$
$$.65901, .66676, .76380, 1.6495, 1.6587, 1.6701, 1.6844]$$

and hence the approximate normalized density

$$d_n = [.90544, .93413, .94738, 1.0493, 1.0309, 1.0364, 1.0439, 1.0525].$$

For $n = 16$,

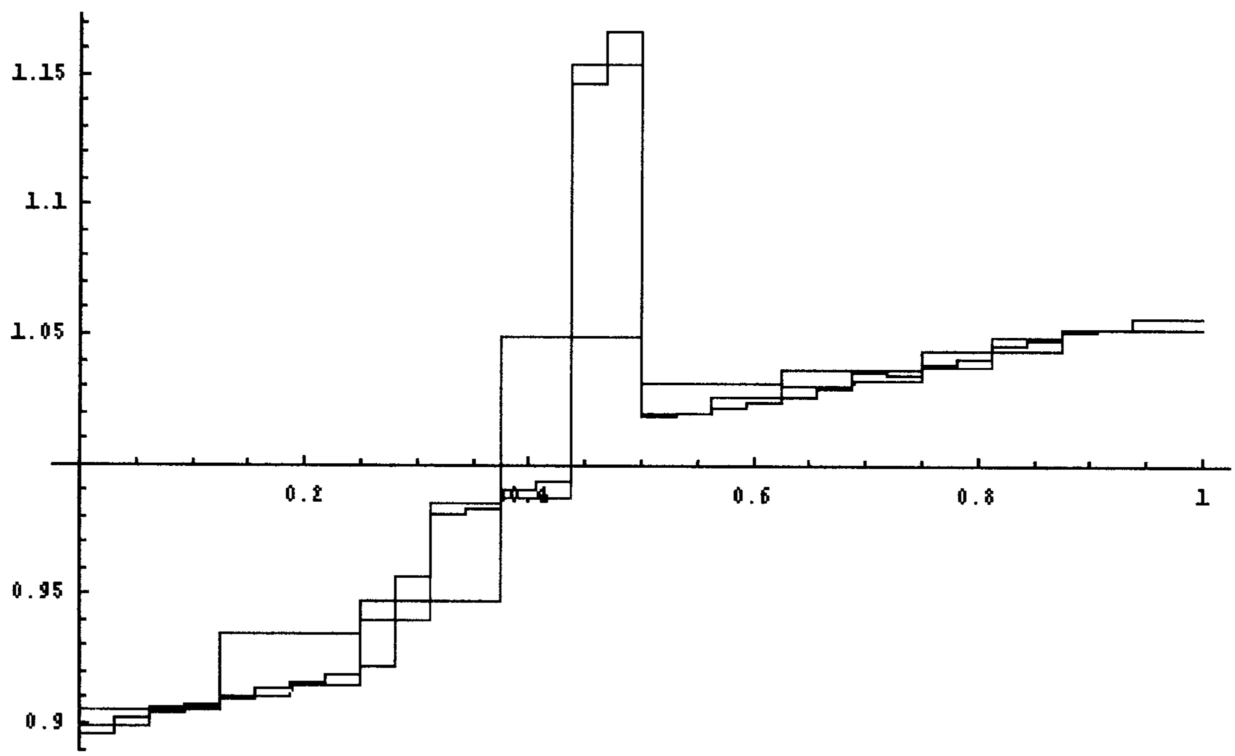
$$\begin{aligned} d_n = [&.89878, .90659, .91053, .91503, .94037, .98519, .98728, 1.1539, \\ &1.0198, 1.0257, 1.0296, 1.0325, 1.0373, 1.0491, 1.0526, 1.0560]. \end{aligned} \tag{3.14}$$

For $n = 32$,

$$\begin{aligned} d_n = [&.89625, .90175, .90429, .90745, .90950, .91370, .91559, .91859, \\ &.92179, .95654, .98114, .98259, .99025, .99300, 1.1465, 1.1661, \\ &1.0189, 1.0201, 1.0220, 1.0233, 1.0255, 1.0286, 1.0351, 1.0348, \\ &1.0383, 1.0405, 1.0456, 1.0478, 1.0506, 1.0525, 1.0566, 1.0565]. \end{aligned} \tag{3.15}$$

For $n = 64$,

$$\begin{aligned} d_n = [&.89415, .89704, .89923, .90345, .90524, .90860, .90712, .90985, \\ &.91032, .91079, .91305, .91524, .91688, .91813, .91844, .92024, \\ &.92227, .92469, .94149, .96039, .96891, .98172, .98906, .98813, \\ &.98891, .99078, .99344, .99414, 1.1225, 1.1633, 1.1666, 1.1657, \\ &1.0123, 1.0182, 1.0173, 1.0208, 1.0226, 1.0253, 1.0238, 1.0258, \\ &1.0241, 1.0280, 1.0272, 1.0302, 1.0332, 1.0340, 1.0329, 1.0337, \\ &1.0366, 1.0428, 1.0389, 1.0422, 1.0440, 1.0471, 1.0449, 1.0472, \\ &1.0494, 1.0532, 1.0515, 1.0523, 1.0544, 1.0584, 1.0557, 1.0585]. \end{aligned} \tag{3.16}$$


1.15
1.1
1.05
0.95
0.9
0.2
0.4
0.6
0.8
1

In the graph of the previous page, the red graph is the approximate density corresponding to partition points 8, the green graph is the approximate density corresponding to partition points 16 and the blue one is the approximate density corresponding to partition points 64.

Now we plot errors to see the convergence rate of our method. In the x direction we consider number of partition points n and in the y direction we consider the difference of the L^1 norm with partition points n and L^1 norm with partition points $2n$.

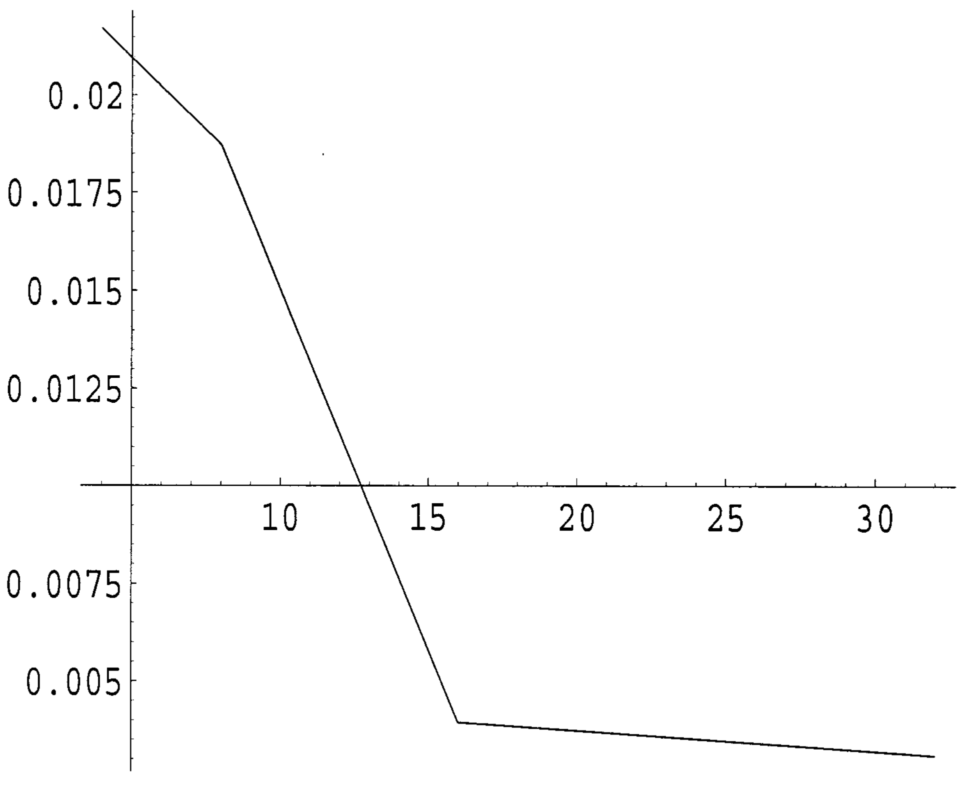


Figure 3.5: The graph of errors versus number of partitions.

Chapter 4

Random Maps with an Ergodic Infinite Absolutely Continuous Invariant Measures

4.1 Introduction

Piecewise expanding is established as a sufficient condition for the existence of a finite absolutely continuous invariant measures (acim) for a one dimensional map by Lasota-Yorke in [26]. In [26] there is an interesting counter example, namely, if the map fails to be expanding at even one point (it has slope 1 at a fixed point), then there is no finite acim. The only acim is infinite, with a singularity at the fixed point.

In this chapter we consider the question of whether analogous behavior occurs for more general dynamical systems, namely random maps. We consider random maps constructed from maps which admit finite acim, but which do not have finite acim themselves.

In Section 4.2 we present three random maps, and prove the existence of an infinite acim for each of them. In Section 4.3, we show that with position dependent probabilities a random map of Section 4.2 admits a finite acim.

4.2 A random map which admits an infinite acim

Consider the random map $T = \{\tau_1, \tau_2; p_1, p_2\}$, where $\tau_1, \tau_2 : [0,1] \to [0,1]$ (see Fig. 4.1 and 4.2) are defined by

$$\tau_1(x) = \begin{cases} 1 + \frac{(x-\frac{1}{2})}{4} & , \text{ for } 0 \le x < \frac{1}{2}; \\ \frac{1}{2} + 2|x - \frac{3}{4}| & , \text{ for } \frac{1}{2} \le x \le 1. \end{cases} \tag{4.1}$$

$$\tau_2(x) = \begin{cases} 2|x - \frac{1}{4}| & , \text{ for } 0 \le x < \frac{1}{2}; \\ \frac{1}{2} + \frac{x-1}{4} & , \text{ for } \frac{1}{2} \le x \le 1. \end{cases} \tag{4.2}$$

It is easy to see that each of these maps admits a finite acim.

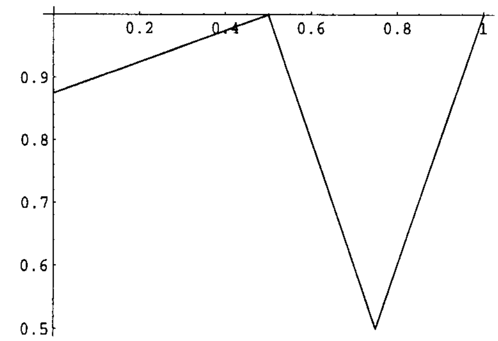


Figure 4.1: The graph of τ_1

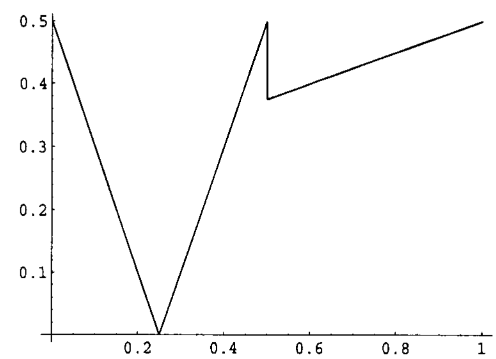


Figure 4.2: The graph of τ_2

4.2.1 Existence of an infinite acim

Theorem 4.1 *The random map* $T = \{\tau_1, \tau_2; \frac{1}{2}, \frac{1}{2}\}$ *admits an ergodic infinite absolutely continuous invariant measure and it has no finite absolutely continuous invariant measure.*

Proof. Consider the infinite Markov partition: $P = \{I_k^{(1)}, I_k^{(2)}\}_{k=2}^{\infty}$, where

$$I_k^{(1)} = [\frac{1}{2} - \frac{1}{2^{k-1}}, \frac{1}{2} - \frac{1}{2^k}], \ k = 2, 3, 4, \dots \tag{4.3}$$

and

$$I_k^{(2)} = [1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}], \; k = 2, 3, 4, \ldots \tag{4.4}$$

The Perron-Frobenius operators of τ_1 and τ_2 can be represented as matrices:

$$M_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 4 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 4 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 4 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots \\ 0 & 0 & \ldots & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \end{bmatrix}, \tag{4.5}$$

$$M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 & 0 & 0\dots & \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{4.6}$$

Now, let us consider $p_1 = p_2 = \frac{1}{2}$. An invariant density f of the random map $T = \{\tau_1, \tau_2; \frac{1}{2}, \frac{1}{2}\}$ satisfies the following Perron-Frobenius equation:

$$f = \frac{1}{2}\{fM_1 + fM_2\}. \tag{4.7}$$

Let

$$M = M_1 + M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 & 4 & 0\dots & \\ \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 4 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 4 & 0 & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ 0 & 0 & 0 & 4 & \dots & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \end{bmatrix}$$

Then (4.7) reduces to

$$2f = fM. \tag{4.8}$$

If the value of the invariant density f on $I_k^{(i)}$ is constant $x_k^{(i)}$ and we assume that $x_k^{(1)} = x_k^{(2)} = x_k$, for all $k = 2, 3, \dots$, then the density f can be represented as an infinite vector $(x_2, x_3, \dots)$ and equation (4.8) can be written as:

$$2\,(x_2, x_3, \dots) = (x_2, x_3, \dots)\,M. \tag{4.9}$$

From equation (4.9), we get:

$$\begin{aligned} x_3 &= 3x_2; \\ x_4 &= 11x_2; \\ x_i &= 4x_{i-1} - 8x_{i-3} - x_2, \quad i = 5, 6, \dots . \end{aligned} \tag{4.10}$$

From the above system we get

$$x_3 = 3 \cdot x_2,\ x_4 = 11 \cdot x_2,\ x_5 = 35 \cdot x_2,\ x_6 = 115 \cdot x_2,\ x_7 = 371 \cdot x_2,\ \dots .$$

Now, we will find a general solution of the system (4.10) of discrete diffrence equations. Consider the homogeneous equation $x_i = 4x_{i-1} - 8x_{i-3}$. Let $x_i = \lambda^i x_2$. Then the characteristic equation of this homogenious equation is

$$\lambda^3 - 4\lambda^2 + 8 = 0$$

and the roots are $\lambda = 2, \lambda = 1 + \sqrt{5}, \lambda = 1 - \sqrt{5}$. Thus, the general solution of the system (4.10) is

$$x_i = c_1 2^i x_2 + c_2(1+\sqrt{5})^i x_2 + c_3(1-\sqrt{5})^i x_2 + k,$$

where c_1, c_2, c_3 and k are constants. By substituting back into (4.10), we obtain $k = -\frac{x_2}{5}$. Moreover, using the initial conditions $x_5 = 35x_2, x_6 = 115x_2, x_7 = 371x_2$, we obtain $c_1 = -.0000000004632812506, c_2 = .1000000001, c_3 = .1000000009$. Hence, the solution of (4.10) is

$$x_3 = 3 \cdot x_2;$$

$$x_4 = 11 \cdot x_2;$$

$$x_i = a_i \cdot x_2, \quad i = 5, 6, \dots .$$

where

$$a_i = -0.0000000004632812506 \cdot 2^i \qquad + 0.1000000001 \cdot (1 + \sqrt{5})^i$$
$$+ 0.1000000009 \cdot (1 - \sqrt{5})^i - \tfrac{1}{5}.$$

Now,

$$\begin{aligned}\sum_{i=2}^{\infty} \sum_{k=1,2} x_i^{(k)} \lambda(I_i^{(k)}) &= 2\left[x_2 \cdot \frac{1}{2^2} + x_3 \cdot \frac{1}{2^3} + x_4 \cdot \frac{1}{2^4} + x_5 \cdot \frac{1}{2^5} \cdots\right] \\ &= \frac{1}{2}\left[1 + \frac{3}{2} + \frac{11}{2^2} + \frac{a_5}{2^3} + \frac{a_6}{2^4} \cdots\right] x_2.\end{aligned}$$

Note that $1 + \sqrt{5} > 3$. Thus, the above series diverges, because for large i, $a_i \approx \text{const.} \cdot \left(1 + \sqrt{5}\right)^i$. This means that the random map has an infinite acim $m = f \cdot \lambda$. The measure m is supported on the whole interval $[0, 1]$.

Now, we prove that the measure m is ergodic. It is enough to show that the matrix M is irreducible [11]. The transition graph of matrix M is shown in Fig 4.3. It is easy to see that the graph is strongly connected, i.e., every state communicates with

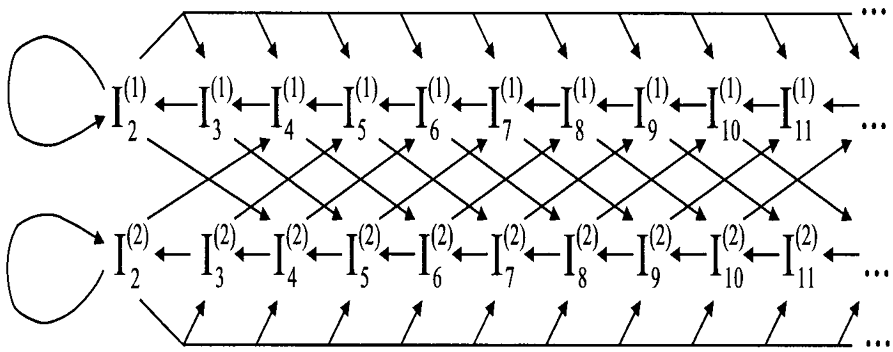


Figure 4.3: The transition graph of the matrix M.

every other. Thus, the matrix M is irreducible and the measure is ergodic. Since m is supported on all of [0,1], there is no other acim. In particular, this random map does not admit a finite acim. ■

Now we consider the same map but more general probabilities.

Theorem 4.2 *For any probability p, the random map $T = \{\tau_1, \tau_2; p, 1-p\}$ admits an ergodic infinite absolutely continuous invariant measure and it has no finite absolutely continuous invariant measure.*

Proof. Let the invariant density of the random map $T = \{\tau_1, \tau_2; p, 1-p\}$ be

$$f = [x_2, x_3, x_4, \dots, y_2, y_3, y_4, \dots,].$$

Then,

$$f = (pf) \cdot M_1 + ((1-p)f) \cdot M_2. \tag{4.11}$$

From equation (4.11), we get:

$$
\begin{aligned}
x_3 &= \frac{1+p}{1-p}x_2; \\
x_4 &= \frac{2}{1-p}\left(\frac{2(1+p)}{(1-p)^2} - 1\right)x_2; \\
x_i &= \frac{2}{1-p}x_{i-1} - 8y_{i-3} - x_2, \quad i = 5, 6, \dots ; \\
y_3 &= \frac{2-p}{p}y_2; \\
y_4 &= \left(\frac{(2(2-p)}{(p)^2} - 1\right)y_2; \\
y_i &= \frac{2}{p}y_{i-1} - 8x_{i-3} - y_2, \quad i = 5, 6, \dots .
\end{aligned}
\tag{4.12}
$$

Let us make the following change of variables:

$$u_i = x_{i-1}, v_i = y_{i-1}, s_i = u_{i-1}, t_i = v_{i-1}, i = 5, 6, \dots, .$$

Then (4.12) reduces to

$$
\begin{aligned}
x_i &= \frac{2}{1-p}x_{i-1} - 8t_{i-1} - x_2; \\
y_i &= \frac{2}{p}y_{i-1} - 8s_{i-1} - y_2; \\
u_i &= x_{i-1}; \\
v_i &= y_{i-1}; \\
s_i &= u_{i-1}; \\
t_i &= v_{i-1}.
\end{aligned}
\tag{4.13}
$$

From equation (4.11), we get:

$$
\begin{aligned}
x_3 &= \frac{1+p}{1-p}x_2; \\
x_4 &= \frac{2}{1-p}\left(\frac{2(1+p)}{(1-p)^2}-1\right)x_2; \\
x_i &= \frac{2}{1-p}x_{i-1} - 8y_{i-3} - x_2, \quad i = 5, 6, \dots ; \\
y_3 &= \frac{2-p}{p}y_2; \\
y_4 &= \left(\frac{(2(2-p)}{(p)^2}-1\right)y_2; \\
y_i &= \frac{2}{p}y_{i-1} - 8x_{i-3} - y_2, \quad i = 5, 6, \dots .
\end{aligned}
\tag{4.12}
$$

Let us make the following change of variables:

$$
u_i = x_{i-1}, v_i = y_{i-1}, s_i = u_{i-1}, t_i = v_{i-1}, i = 5, 6, \dots , .
$$

Then (4.12) reduces to

$$
\begin{aligned}
x_i &= \frac{2}{1-p}x_{i-1} - 8t_{i-1} - x_2; \\
y_i &= \frac{2}{p}y_{i-1} - 8s_{i-1} - y_2; \\
u_i &= x_{i-1}; \\
v_i &= y_{i-1}; \\
s_i &= u_{i-1}; \\
t_i &= v_{i-1}.
\end{aligned}
\tag{4.13}
$$

Let $X_i = (x_i, y_i, u_i, v_i, s_i, t_i)$, $A = \begin{bmatrix} \frac{2}{1-p} & 0 & 0 & 0 & 0 & -8 \\ 0 & \frac{2}{p} & 0 & 0 & -8 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and $b = (x_2, y_2, 0, 0, 0, 0)$.

Then, the system (4.24) reduces to

$$X_i = AX_{i-1} + b. \tag{4.14}$$

In general, a system of the form

$$Z_{i+1} = BZ_i + C_i, \quad i = 0, 1, 2, \dots,$$

has the following form of solution [8]:

$$Z_i = B^i Z_0 + \sum_{k=0}^{i} B^{i-k} C_i, \quad i = 0, 1, 2, \dots.$$

Note that one of the eigenvalues of the matrix A in the system (4.14) is exactly 2. Thus,

$$\begin{aligned} \sum_{i=2}^{\infty} \sum_{k=1,2} x_i^{(k)} \lambda(I_i^{(k)}) &= \left[x_2 \cdot \frac{1}{2^2} + x_3 \cdot \frac{1}{2^3} + x_4 \cdot \frac{1}{2^4} + x_5 \cdot \frac{1}{2^5} \cdots \right] \\ &\quad + \left[y_2 \cdot \frac{1}{2^2} + y_3 \cdot \frac{1}{2^3} + y_4 \cdot \frac{1}{2^4} + y_5 \cdot \frac{1}{2^5} \cdots \right] \\ &= \infty. \end{aligned}$$

This means that the random map has an infinite acim $m = f \cdot \lambda$. The same argument as in the proof of Theorem 4.1 proves the ergodicity of m.

■

4.2.2 Another random map

Now we consider a random map $T = \{\tau_1, \tau_2; p_1, p_2\}$, where $\tau_1, \tau_2 : [0,1] \to [0,1]$ (see Fig. 4.4 and 4.5) are defined by

$$\tau_1(x) = \begin{cases} 1 + \frac{(x-\frac{1}{2})}{2} & , \text{ for } 0 \leq x < \frac{1}{2}; \\ \\ \frac{1}{2} + 2|x - \frac{3}{4}| & , \text{ for } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{4.15}$$

The slopes of τ_1 and τ_2 are different than the slopes of the maps we have considered in the previous example. We show that this random map also admits an infinite acim.

$$\tau_2(x) = \begin{cases} 2|x - \frac{1}{4}|, & \text{for } 0 \leq x < \frac{1}{2}; \\ \\ \frac{1}{2} + \frac{x-1}{2}, & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{4.16}$$

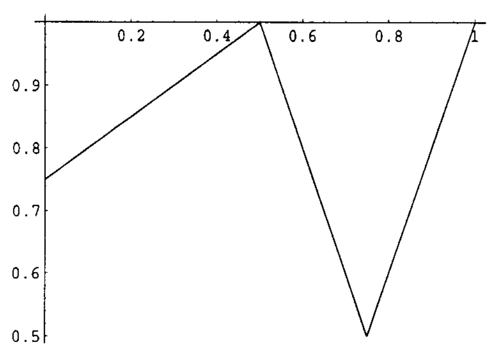


Figure 4.4: The graph of τ_1

Theorem 4.3 *The random map $T = \{\tau_1, \tau_2; \frac{1}{2}, \frac{1}{2}\}$ admits an ergodic infinite absolutely continuous invariant measure and it has no finite absolutely continuous invariant measure.*

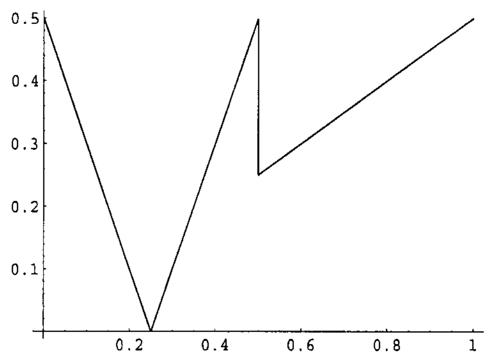


Figure 4.5: The graph of τ_2

Proof. Consider the infinite Markov partition: $P = \{I_k^{(1)}, I_k^{(2)}\}_{k=2}^{\infty}$ where

$$I_k^{(1)} = [\frac{1}{2} - \frac{1}{2^{k-1}}, \frac{1}{2} - \frac{1}{2^k}],\ k = 2, 3, 4, \ldots \tag{4.17}$$

and

$$I_k^{(2)} = [1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}],\ k = 2, 3, 4, \ldots \tag{4.18}$$

The Perron-Frobenius operators of τ_1 and τ_2 can be represented by the matrices:

$$M_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & 2 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ 0 & 0 & \dots & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \end{bmatrix} \tag{4.19}$$

and

$$M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \dots & \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 2 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{4.20}$$

Now, let us consider $p_1 = p_2 = \frac{1}{2}$. An invariant density f of the random map $T = \{\tau_1, \tau_2; \frac{1}{2}, \frac{1}{2}\}$ satisfies the following Frobenius-Perron equation:

$$f = \frac{1}{2}\{fM_1 + fM_2\}. \tag{4.21}$$

Let

$$M = M_1 + M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 2 & 0 & 0\ldots & \\ \frac{1}{2} & 0 & 0 & 0 & \ldots & 0 & 0 & 2 & 0 & \ldots \\ 0 & \frac{1}{2} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 2 & 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots \\ 0 & 0 & 2 & 0 & \ldots & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \end{bmatrix}$$

Then (4.21) reduces to

$$2f = fM. \tag{4.22}$$

If the value of the invariant density f on $I_k^{(i)}$ is constant $x_k^{(i)}$ and we assume that $x_k^{(1)} = x_k^{(2)} = x_k$, for all $k = 2, 3, \ldots$, then the density f can be represented as an infinite vector $(x_2, x_3, \ldots)$ and equation (4.22) can be written as:

$$2\,(x_2, x_3, \ldots) = (x_2, x_3, \ldots)\, M. \tag{4.23}$$

From equation (4.23), we get:

$$\begin{aligned} x_3 &= 3x_2; \\ x_i &= 4x_{i-1} - 4x_{i-2} - x_2, \quad i = 4, 6, \ldots . \end{aligned} \tag{4.24}$$

From the above system we get

$$\begin{aligned} x_3 &= 3 \cdot x_2, \\ x_4 = (2 \cdot 3 + 1) \cdot x_2 &= 7 \cdot x_2, \\ x_5 = (2 \cdot 7 + 1) \cdot x_2 &= 15 \cdot x_2, \\ x_6 = (2 \cdot 15 + 1) \cdot x_2 &= 31 \cdot x_2, \\ x_7 = (2 \cdot 31 + 1) \cdot x_2 &= 63 \cdot x_2, \\ &\vdots \end{aligned} \tag{4.25}$$

In general, $x_i = (2\cdot \text{ constant in } x_{i-1} + 1) \cdot x_2$.

Now,

$$\begin{aligned} \sum_{i=2}^{\infty} \sum_{k=1,2} x_i^{(k)} \lambda(I_i^{(k)}) &= 2\left[x_2 \cdot \frac{1}{2^2} + x_3 \cdot \frac{1}{2^3} + x_4 \cdot \frac{1}{2^4} + x_5 \cdot \frac{1}{2^5} \cdots\right] \\ &= \frac{1}{2}\left[1 + \frac{3}{2} + \frac{7}{2^2} + \frac{15}{2^3} + \frac{31}{2^4} + \frac{63}{2^5} \cdots\right] x_2. \end{aligned}$$

Note that each of the terms of the above series is bigger or equal to 1. Thus, the above series diverges. This means that the random map has an infinite acim $m = f \cdot \lambda$. The measure m is supported on the whole interval $[0, 1]$. The proof of ergodicity of m is similar to that in the previous example.

■

4.3 A Position dependent random map with finite acim

In this section we show that using the maps in equation 4.1 and equation 4.2 we can construct a random map with position dependent probabilities which admits a finite acim. Consider the position dependent random map $T = \{\tau_1, \tau_2; p_1(x), p_2(x)\}$, where τ_1, τ_2 are the same as in the previous section and the position dependent probabilities $p_1(x), p_2(x)$ are as follows:

$$p_1(x) = \begin{cases} \frac{1}{9} & , 0 \le x < \frac{1}{2}, \\ \frac{8}{9} & , \frac{1}{2} \le x \le 1. \end{cases}$$

$$p_2(x) = \begin{cases} \frac{8}{9} & , 0 \le x < \frac{1}{2}, \\ \frac{1}{9} & , \frac{1}{2} \le x \le 1. \end{cases}$$

We will show that:

$$\sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} \le \alpha < 1.$$

For $x \in [0, \frac{1}{2})$, we have

$$\sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{1}{9}}{\frac{1}{4}} + \frac{\frac{8}{9}}{2} = \frac{8}{9} < 1.$$

For $x \in [\frac{1}{2}, 1]$, we have

$$\sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{8}{9}}{2} + \frac{\frac{1}{9}}{\frac{1}{4}} = \frac{8}{9} < 1.$$

Hence, T admits a finite acim.

Chapter 5

Invariant Densities of Random Maps have Lower Bounds on their Support

5.1 Introduction

For a single transformation, much is known about the densities of absolutely continuous invariant measures (acim). For example it is known that the densities inherit smoothness properites from the map itself (Halfant[17]), that the supports consists of a finite union of intervals and that the densities are bounded below on their supports (Keller [23] and Kowalski [24]). In this chapter we generalize to random maps results of Keller [23] and Kowalski [24], which prove that the density of an absolutely continuous invariant measure (acim) is strictly positive on the support of the acim of a nonsingular map.

In Section 5.2 we present the notation and summarize results we shall need in the sequel. In Section 5.3 we prove the main result.

5.2 Preliminaries

Let $(X, \mathcal{B}, \lambda)$ be a measure space, where λ is an underlying measure and $\tau_k : X \to X$, $k = 1, 2, \ldots, K$ are nonsingular transformations. A random map T with constant probabilities is defined as

$$T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\},$$

where $\{p_1, p_2, \ldots, p_K\}$ is a set of constant probabilities. For any $x \in X$, $T(x) = \tau_k(x)$ with probability p_k and, for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^{N} p_{k_j}$. A measure μ is $T-$invariant if and only if it satisfies the following condition [30]:

$$\mu(E) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{5.1}$$

for any $E \in \mathcal{B}$.

We now recall some definitions and results which will be needed to prove the main results in Section 3.

Definition 5.1 *Let $\tau : (X, \mathcal{B}, \lambda) \rightarrow (X, \mathcal{B}, \lambda)$ be a non-singular transformation and μ an acim with respect to Lebesgue measure λ possessing density function f. We define the support of μ as follows:*

$$\text{supp}(\mu) = \text{supp}(f) = \{x \in X : f(x) > 0\}$$

Definition 5.2 *A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a lower semicontinuous function if and only if $f(y) \leq \liminf_{x \rightarrow y} f(x)$ for any $y \in \mathbb{R}$.*

Theorem 5.1 *[12] If f is lower semi continuous on $I = [a, b] \subset \mathbb{R}$, then it is bounded below and assumes its minimum value. For any $c \in \mathbb{R}$, the set $\{x : f(x) > c\}$ is open.*

Lemma 5.1 *[12] If f is of bounded variation on I, then it can be redefined on a countable set to become a lower semicontinuous function.*

Let $\mathcal{T}_0(I)$ denote the class of transformations $\tau : I \rightarrow I$ that satisfy the following conditions:

(i) τ is piecewise monotonic, i.e., there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I such that $\tau_i = \tau|I_i$ is C^1, and

$$|\tau_i'(x)| \geq \alpha > 0, \tag{5.2}$$

for any i and for all $x \in (a_{i-1}, a_i)$;

(ii) $g(x) = \frac{1}{|\tau_i'(x)|}$ is a function of bounded variation, where $\tau_i'(x)$ is the appropriate one-sided derivative at the end points of $\mathcal{P}$.

We say that $\tau \in \mathcal{T}_1(I)$ if $\tau \in \mathcal{T}_0(I)$ and $\alpha > 1$ in condition (5.2), i.e., τ is piecewise expanding.

Theorem 5.2 *[23, 24] Let $\tau \in \mathcal{T}_1(I)$ and f be a $\tau-$invariant density which can be assumed to be lower semicontinuous. Then there exists a constant $\beta > 0$ such that $f|_{\text{supp}(f)} \geq \beta$.*

Theorem 5.3 *[?, 30] Let $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ be a random map, where $\tau_k \in \mathcal{T}_0(I)$, with the common partition $\mathcal{P} = \{J_1, J_2, \dots, J_q\}$, $k = 1, 2, \dots, K$. If, for all $x \in [0,1]$, the following Pelikan's condition*

$$\sum_{k=1}^{K} \frac{p_k}{|\tau_k'(x)|} \leq \gamma < 1, \tag{5.3}$$

is satisfied, then for all $f \in L^1 = L^1([0,1], \lambda)$:

(i) *The limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} P_T^i(f) = f^* \quad \text{exists in} \quad \mathrm{L}^1;$$

(ii) $P_T(f^*) = f^*$;

(iii) $V_{[0,1]}(f^*) \leq C \cdot \|f\|_1$, for some constant $C > 0$, which is independent of $f \in L^1$.

5.3 Support of invariant density of random maps

In this section we prove that the invariant density of an acim of the random map $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$, $\tau_1, \tau_2, \dots, \tau_K \in \mathcal{T}_1$, is strictly positive on its support. For notational convenience, we consider $K = 2$, that is, we consider only two transformations τ_1, τ_2. The proofs for larger number of maps are analogous. We consider random maps with constant probabilities.

Let $\mathcal{Q}$ denote the set of endpoints of intervals of partition $\mathcal{P}$ except the point 0 and 1.

The main result of this note applies to random maps, where each component map is in $\mathcal{T}_1(I)$, but the first two lemmas are proved under more the general assumptions of Theorem 5.3:

Lemma 5.2 *Let the random map $T = \{\tau_1, \tau_2; p_1, p_2\}$ satisfy the assumptions of Theorem 5.3. In particular,*

$$\frac{p_1}{|\tau_1'(x)|} + \frac{p_2}{|\tau_2'(x)|} \leq \gamma < 1, \tag{5.4}$$

for all $x \in I \setminus \mathcal{Q}$. Then, for any interval J disjoint with $\mathcal{Q}$, at least one of the images $\tau_1(J)$, $\tau_2(J)$ is longer than J.

Proof. First, let us note that if ν is the normalized Lebesgue measure on J, then

$$1 = \left(\int_J 1 d\nu\right)^2 = \left(\int_J \sqrt{|\tau'(x)|} \cdot \frac{1}{\sqrt{|\tau'(x)|}} d\nu(x)\right)^2 \leq \int_J |\tau'(x)| d\nu(x) \cdot \int_J \frac{1}{|\tau'(x)|} d\nu(x), \tag{5.5}$$

or

$$\frac{1}{\int_J |\tau'(x)| d\nu(x)} \leq \int_J \frac{1}{|\tau'(x)|} d\nu(x), \tag{5.6}$$

or

$$\frac{1}{\frac{1}{\lambda(J)} \int_J |\tau'(x)| dx} \leq \frac{1}{\lambda(J)} \int_J \frac{1}{|\tau'(x)|} dx. \tag{5.7}$$

Integrating (5.4) over J, we obtain

$$p_1 \int_J \frac{1}{|\tau_1'(x)|} dx + p_2 \int_J \frac{1}{|\tau_2'(x)|} dx \le \gamma \cdot \lambda(J), \tag{5.8}$$

and, using (5.7), we obtain

$$\frac{p_1 \lambda(J)}{\lambda(\tau_1(J))} + \frac{p_2 \lambda(J)}{\lambda(\tau_2(J))} \le \gamma.$$

Thus, at least one of the numbers $\lambda(\tau_1(J))$, $\lambda(\tau_2(J))$ is larger than $\lambda(J)$. ■

Lemma 5.3 *Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be a random map on $[0, 1]$ satisfying the conditions of Theorem 5.3. Then, the support of the invariant density of T contains an interval which is not disjoint with $\mathcal{Q}$.*

Proof. Let $\text{supp}(f) = \{x \in [0,1] : f(x) > 0\}$. The density function f of the acim μ is a function of bounded variation by Theorem 5.3 and thus, by Lemma 5.1, f can be redefined on a countable set to become a lower semicontinuous function $\bar{f}$ and $\bar{f} = f$ a.e.. Thus $\text{supp}(f) = \text{supp}(\bar{f}) = \{x : \bar{f} > 0\}$ is an open set by Theorem 5.1. Thus, $\text{supp}(f) = \cup_{i=1}^{\infty} I_i$, where I_i's are intervals separated by sets of positive measure. Without loss of generality, let us assume that $\lambda(I_i) \ge \lambda(I_{i+1})$ for $i = 1, 2, \ldots$. We will prove that $\mathcal{Q} \cap I_1 \ne \emptyset$. Suppose $\mathcal{Q} \cap I_1 = \emptyset$. Then I_1 is contained in one of the subintervals, J_*, of the partition $\mathcal{P}$ and $\tau_1(I_1)$ and $\tau_2(I_1)$ are both open intervals. Since f is an invariant density of the random map T, we have,

$$f(x) = p_1 \cdot \sum_{i=1}^{q} \frac{f(\tau_{1,i}^{-1}(x))}{|\tau_1'(\tau_{1,i}^{-1}(x))|} \chi_{\tau_1(I_i)}(x) + p_2 \cdot \sum_{i=1}^{q} \frac{f(\tau_{2,i}^{-1}(x))}{|\tau_2'(\tau_{2,i}^{-1}(x))|} \chi_{\tau_2(I_i)}(x).$$

Let $x \in \tau_1(I_1)$. It is clear that at least one element of $\{\tau_{1,*}^{-1}(x)\}$ is in I_1 and since $I_1 \subset \text{supp}(f)$ we have $f(x) > 0$. Thus, $\tau_1(I_1)$ is a subset of supp(f). Similarly, $\tau_2(I_1)$ is a subset of supp(f). By Lemma 5.2, at least one of the intervals $\tau_1(I_1)$, $\tau_2(I_1)$ has larger length than the length of I_1. This is a contradiction because supp(f) does not contain an interval of length greater than $\lambda(I_1)$. This proves that $\mathcal{Q} \cap I_1 \neq \emptyset$. ■

Corollary 5.1 *The number of different ergodic acim for the random map T satisfying the assumptions of Theorem 5.3 is at most equal to the cardinality of the partition $\mathcal{P}$ minus one.*

We now assume that $\tau_1, \tau_2 \in \mathcal{T}_1$.

Theorem 5.4 *Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be a random map on $[0, 1]$, where $\tau_1, \tau_2 \in \mathcal{T}_1$ and have a common partition $\mathcal{P} = \{J_1, J_2, \dots, J_q\}$. Then the support of the invariant density f of T,* supp(f), *is a finite union of open intervals almost everywhere.*

Proof. Again we can assume that supp$(f) = \cup_{i=1}^{\infty} I_i$, where I_i's are intervals separated by sets of positive measure. Let $\mathcal{D} = \{j \geq 1 : I_j$ contains a discontinuity of τ_1 or τ_2 or both$\}$. By Lemma 5.3, $\mathcal{D}$ is not empty. Also, $\mathcal{D}$ is a finite set. If $j \in \mathcal{D}$, then $\tau_i(I_j), i = 1, 2$, is a finite union of intervals. Let J be the shortest interval of the family

$$\{I_j\}_{j \in \mathcal{D}} \cup \{I : I \text{ is a connected component of } \tau_{\rm i}({\rm I_j}), {\rm i} = 1, 2, {\rm j} \in \mathcal{D}\}.$$

Let $\mathcal{F} = \{i \geq 1 : \lambda(I_i) \geq \lambda(J)\}$, where i is not necessarily in $\mathcal{D}$ and let

$$S = \cup_{i \in \mathcal{F}} I_i \subset \cup_i I_i = \text{supp}(f).$$

S is a finite union of open separated intervals since it is a family of disjoint intervals with length $\geq \lambda(J) > 0$. For any $j \in \mathcal{D}, I_j \subseteq S$.

We will prove that $\tau_i(S) \subseteq S$, $i = 1, 2$. Let $I_k \subset S$. If $k \notin \mathcal{D}$, then $\tau_i(I_k)$ is contained in an interval I_{k_i}, $i = 1, 2$ and

$$\lambda(I_{k_i}) \geq \lambda(\tau_i(I_k)) = \int_{I_k} |\tau_i'(x)| d\lambda > \inf_{x \in [0,1]} |\tau_i'(x)| \cdot \lambda(I_k), \ i = 1, 2.$$

Since $\inf_{x \in [0,1]} |\tau_i'(x)| > 1$ for $i = 1, 2$, we have

$$\lambda(I_{k_i}) \geq \lambda(I_k) > \lambda(J).$$

Thus, by the definition of S, we get $I_{k_i} \subseteq S$, $i = 1, 2$ and hence $\tau_i(I_k) \subset I_{k_i} \subseteq S$, $i = 1, 2$. If $k \in \mathcal{D}$, then by the definition of S, $\tau_i(I_k) \subset S$, $i = 1, 2$. Thus, $\tau_i(S) \subseteq S$, $i = 1, 2$.

Now we will prove that $\text{supp}(f) \subseteq S$. Suppose not. Let I_s be the largest interval of $\text{supp}(f) \setminus S$. Thus, $s \notin \mathcal{D}$ and

$$\lambda(\tau_i(I_s)) = \int_{I_s} |\tau_i'(x)| d\lambda > \inf_{x \in [0,1]} |\tau_i'(x)| \cdot \lambda(I_s) > \lambda(I_s), \ i = 1, 2.$$

Then $\tau_i(I_s) \subset S$, $i = 1, 2$. Thus, $I_s \subset \tau_i^{-1}(S)$, $i = 1, 2$. But $I_s \not\subset S$, so $I_s \subset \tau_i^{-1}(S) \setminus S$, $i = 1, 2$. Let $\mu = f \cdot \lambda$ be the T-invariant absolutely continuous measure. We will show that $\mu(\tau_i^{-1}(S) \setminus S) = 0$, $i = 1, 2$. Since $\tau_i(S) \subseteq S$, $i = 1, 2$, we have

$$S \subseteq \tau_i^{-1}(\tau_i(S)) \subseteq \tau_i^{-1}(S), \ i = 1, 2.$$

Thus,

$$\begin{aligned} 0 = \mu(S) - \mu(S) &= [p_1\mu(\tau_1^{-1}S) + p_2\mu(\tau_2^{-1}S)] - [p_1\mu(S) + p_2\mu(S)] \\ &= p_1[\mu(\tau_1^{-1}S) - \mu(S)] + p_2[\mu(\tau_2^{-1}S) - \mu(S)] \\ &= p_1\mu(\tau_1^{-1}S \setminus S) + p_2\mu(\tau_2^{-1}S \setminus S). \end{aligned} \tag{5.9}$$

Thus, both $\mu(\tau_1^{-1}S \setminus S) = 0$ and $\mu(\tau_2^{-1}S \setminus S) = 0$ if $p_1, p_2 > 0$. Thus, $\mu(I_s) = 0$, which contradicts the fact that $I_s \subset \text{supp}(f)$. ■

Lemma 5.4 *Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be a random map on $[0,1]$, where $\tau_1, \tau_2 \in \mathcal{T}_1$ and have a common partition $\mathcal{P} = \{J_1, J_2, \ldots, J_q\}$. Let f be the invariant density of an acim μ of the random map T and $S = \text{supp}(f) = \{x : f(x) > 0\}$. Then*

(i) $\tau_i(S \setminus \{a_0, a_1, \ldots, a_q\}) \subseteq S,\ i = 1, 2$;

(ii) $\lambda(S \setminus \tau_i(S \setminus \{a_0, a_1, \ldots, a_q\})) = 0,\ i = 1, 2$;

where $\{a_0, a_1, \ldots, a_q\}$ are the endpoints of the intervals in the partition $\mathcal{P}$.

Proof. We assume that f is lower semicontinuous. If it is not, we modify it on at most a countable set. By Theorem 5.4, $S = \cup_{i=1}^{r} I_i$. Let $x \in S \setminus \{a_0, a_1, \ldots, a_q\}$. Then $x \in \mathrm{IntI_k}$, for some $k \in \{1, 2, \ldots, r\}$ and there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset I_k$ and $f(y) > \frac{1}{2}f(x) > 0$ for all $y \in B(x, \epsilon)$ since f is lower semi-continuous. We may assume that $\tau_i|I_k, i = 1, 2$ is increasing and that $f(\tau_i(x)) = \lim_{y \to \tau_i(x)^+} f(y)$, $i = 1, 2$. Now, for any $\delta > 0$, $\tau_i([x, x+\delta)) = [\tau_i(x), \tau_i(x) + \delta')$, $i = 1, 2$ and $\delta' \to 0$ as $\delta \to 0$.

Then, for $i = 1, 2$, we have

$$\begin{aligned}
\int_{[\tau_i(x), \tau_i(x)+\delta')} f d\lambda = \mu(\tau_i([x, x+\delta))) &\geq \mu([x, x+\delta)) \\
&= \int_{[x,x+\delta)} f d\lambda \geq \frac{1}{2} f(x) \lambda([x, x+\delta)) \\
&\geq \frac{1}{2} f(x) \frac{1}{\max|\tau_i'|} \lambda([\tau_i(x), \tau_i(x)+\delta')).
\end{aligned}$$

Since f is lower semicontinuous,

$$\begin{aligned}
f(\tau_i(x)) &= \lim_{\delta' \to 0} \frac{1}{\lambda([\tau_i(x), \tau_i(x)+\delta'))} \int_{[\tau_i(x), \tau_i(x)+\delta')} f d\lambda \\
&\geq \frac{1}{2} f(x) \frac{1}{\max|\tau_i'|} > 0, i = 1, 2.
\end{aligned}$$

Hence, $\tau_i(x) \in S$, $i = 1, 2$, and part (i) is proved.

Part (ii) is proved using reasoning similar to the end of Theorem 5.4 (equation (5.9)). ■

Theorem 5.5 *Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be a random map on $[0, 1]$, where $\tau_1, \tau_2 \in \mathcal{T}_1$ and have a common partition $\mathcal{P} = \{J_1, J_2, \ldots, J_q\}$. Let f be the invariant density of an acim μ of the random map T and let $S = \text{supp} f = \{x : f(x) > 0\}$. Then there exists a constant $a > 0$ such that $f_{|S} \geq a$.*

Proof. Since $S = \{x : f(x) > 0\}$ is a finite union of open intervals, $S = \cup_{i=1}^{r} I_i$, we can assume they are separated by intervals of positive measure. Then $\widehat{S} = S \backslash \{a_0, a_1, \ldots, a_q\}$ is also a finite union of intervals: $\widehat{S} = \cup_{i=1}^{s} J_i$. Let $\mathcal{F} = \{I_i\}_{i=1}^{r}$, and $\mathcal{C} = \{J_j\}_{j=1}^{s}$. For any $J_k \in \mathcal{C}, \tau_j|_{J_k}$, $j = 1, 2$ is of class C^1. Therefore, there exist $I_{i_j} \in \mathcal{F}$, $j = 1, 2$ such that $\tau_j(J_k) \subseteq I_{i_j}$, $j = 1, 2$.

Let (c, d) be any interval in $\mathcal{F}$ or $\mathcal{C}$. We associate with its endpoints two classes of standard intervals:

$$\eta_c = \{(c, c+\epsilon) : \epsilon > 0\} \text{ and } \eta_d = \{(d-\epsilon, d) : \epsilon > 0\}.$$

The points c and d are referred to as the endpoints of the classes η_c and η_d respectively. Let

$$\mathcal{K} = \{\eta_c, \eta_d : c, d \text{ are endpoints of intervals of } \mathcal{F} \text{ and } \mathcal{C}\}.$$

We now define a relation $\mapsto$ between elements of $\mathcal{K}$. For $\eta, \eta' \in \mathcal{K} : \eta \mapsto \eta'$ if and only if $\tau_j(U) \in \eta'$ for at least one of $j = 1, 2$ and for sufficiently small $U \in \eta$. The relation has the following two properties:

(1) If η' is associated with an end point of $I_i \in \mathcal{F}$, then there exists at least one η such that $\eta \mapsto \eta'$. To prove this, let us fix an I_i and η' associated with one of its endpoints. We claim that for each $J_k \in \mathcal{C}$, either $\tau_j(J_k) \subseteq I_i$, $j = 1, 2$ or $\tau_j(J_k) \cap I_i = \emptyset$, $j = 1, 2$. To show this, let us note that since $\tau_j(J_k)$, $j = 1, 2$ is contained in $S = \cup_{i=1}^{r} I_i$ and $\{I_i\}$ are separated, $\tau_j(J_k)$, $j = 1, 2$ is contained in one of I_{i_j}, $j = 1, 2$. Now, since

$$\lambda(I_{i_j} \setminus \tau_j(\widehat{S})) \leq \lambda(S \setminus \tau_j(\widehat{S})) = 0, \;\; j = 1, 2,$$

there must exists a J_k with $\tau_j(J_k) \in \eta'$, $j = 1, 2$ and this implies (1).

(2) If c' is an endpoint of I_i such that $\lim_{x \to c'} f(x) = 0, \eta'$ is associated with $c', \eta \mapsto \eta'$, and c is an endpoint of η, then for any $U \in \eta$,

$$\lim_{\substack{x \to c \\ x \in U}} f(x) = 0.$$

To prove this let us suppose that $\lim_{\substack{x \to c \\ x \in U}} f(x) = a > 0$, for some $U \in \eta$. By the definition of the relation $\mapsto$, for at least one of $j = 1, 2$, say $j = 1$, if $\eta = \{(c, c + \varepsilon)\}$, then we have $\tau_1(c, c + \varepsilon) = (c', c + \varepsilon')$, for ε small enough. Then,

$$\begin{aligned}\lim_{\substack{x \to c' \\ x \in U'}} f(x) &= \lim_{\varepsilon \to 0} \frac{1}{\lambda((c', c' + \varepsilon'))} \int_{(c', c' + \varepsilon')} f(t)dt = \lim_{\varepsilon \to 0} \frac{\mu((c', c' + \varepsilon'))}{\lambda((c', c' + \varepsilon'))} \\ &\geq \lim_{\varepsilon \to 0} p_1 \cdot \frac{\mu((c, c + \varepsilon))}{\max|\tau_1'| \cdot \lambda((c, c + \varepsilon))} = \frac{p_1}{\max|\tau_1'|} a > 0,\end{aligned} \tag{5.10}$$

which is a contradiction.

We make the following observations:

(3) In the setting of (2) above, $c \notin S$. Therefore, c is an end point of an interval $I_i \in \mathcal{F}$.

Now we define

$\mathcal{K}_0 = \{\eta : \eta$ is associated with an endpoint c of an $I_i \in \mathcal{F}$ and $\lim_{\substack{x \to c \\ x \in I_i}} f(x) = 0\}$.

From (2) and (3) we obtain:

(4) If $\eta' \in \mathcal{K}_0$ and $\eta \mapsto \eta'$, then $\eta \in \mathcal{K}_0$.

We note that by (1) and (4), for each $\eta' \in \mathcal{K}_0$ there exists at least one $\eta \in \mathcal{K}_0$ such that $\eta \mapsto \eta'$.

Now let $\eta \in \mathcal{K}_0$. For any $n \geq 1$, any η' such that $\eta' \mapsto \eta$ in n-steps also belongs to $\mathcal{K}_0$. Choose $U \in \eta$ to be sufficiently small. Then, all the preimages $\tau_{j_n}^{-1} \circ \tau_{j_{n-1}}^{-1} \circ \ldots \circ \tau_{j_2}^{-1} \circ (\tau_{j_1}^{-1}(U))$, $j_i \in \{1, 2\}$, touch an endpoint of some η' in $\mathcal{K}_0$. If M is the number of elements in $\mathcal{K}_0$ and $\alpha = \min\{\inf_{x \in I}|\tau_1'(x)|, \inf_{x \in I}|\tau_2'(x)|\}$, then

$$\begin{aligned}
\mu(U) &= \sum_{(j_n, j_{n-1}, \cdots, j_1)} p_{j_n} p_{j_{n-1}} \dots p_{j_1} \mu(\tau_{j_n}^{-1} \circ \tau_{j_{n-1}}^{-1} \circ \dots \circ \tau_{j_1}^{-1}(U))) \\
&\leq \sup f \cdot \sum_{(j_n, j_{n-1}, \cdots, j_1)} p_{j_n} p_{j_{n-1}} \dots p_{j_1} \lambda(\tau_{j_n}^{-1} \circ \tau_{j_{n-1}}^{-1} \circ \dots \circ \tau_{j_1}^{-1}(U))) \\
&\leq \sup f \cdot \sum_{(j_n, j_{n-1}, \cdots, j_1)} p_{j_n} p_{j_{n-1}} \dots p_{j_1} M \alpha^{-n} \lambda(U).
\end{aligned}$$

Thus, $\mu(U) = 0$ which implies that $\lambda(U) = 0$ since $U \subseteq S$. This contradicts the fact that $U \in \eta$ is an open, nonempty interval. Hence, $\mathcal{K}_0 = \emptyset$ and $\lim_{\substack{x \to c \\ x \in U}} f(x) > 0$ for each of finitely many endpoints of intervals $I_i \in \mathcal{F}$. On the other hand, since f is lower semicontinuous, it assumes its infimum on any closed interval. Hence, there exists $a > 0$ such that $f(x) \geq a$ for all $x \in S$. ∎

Appendix

In Chapter 2 we presented an example of Markov switching position dependent random map and we obtained piecewise constant density function . In this section we present Maple codes [27] for density calculations. This program is also available upon request.

```
> restart;

> Digits:=5:

> tau1:=x->6*x^3-9*x^2+8*x:

> v1:=fsolve(tau1(x)=1,x,0..1);

> tau2:=x->tau1(x)-1:

> v2:=fsolve(tau2(x)=1,x,0..1);

> tau3:=x->tau1(x)-2:

> v3:=fsolve(tau3(x)=1,x,0..1);

> tau4:=x->tau1(x)-3:

> v4:=fsolve(tau4(x)=1,x,0..1);
```

```
> tau5:=x->tau1(x)-4:
> v5:=fsolve(tau5(x)=1,x,0..1);
> f:=x->piecewise(x>=0 and x<v1,
> tau1(x),x>=v1 and x<v2,tau2(x),x>=v2
>           and x<v3, tau3(x),x>=v3 and x<v4,tau4(x),tau5(x));
> plot(f(x),x=0..1, discont=true);
> tau11:=x->3*x +x^2:
> v11:=evalf((-3/2+1/2*sqrt(13)));
> tau22:=x->1/(9/4-1/2*sqrt(13))*(x-3/4)+1:
> v22:=3/4;
> tau33:=x->4*x-3:
> v33:=1;
> w11:=x->piecewise(x>=0 and x<1/2,.8,.2):
> w12:=x->piecewise(x>=0 and x<1/2,.2,.8):
> w21:=x->piecewise(x>=0 and x<1/2,.5,.2):
> w22:=x->piecewise(x>=0 and x<1/2,.5,.8):
> g:=x->piecewise(x>=0 and
> x<(-3/2+1/2*sqrt(13)), 3*x +x^2, x>=(-3/2+1/2*sqrt(13))and
> x<3/4,1/(9/4-1/2*sqrt(13))*(x-3/4)+1,4*x-3);
> plot(g(x),x=0..1, discont=true);
> a:=0:
```

```
> b:=1:
> n:=8:
> h:=(b-a)/n:
> for i from 0  to n do
>   y[i]:=a+i*h:
> od:
> p:=y:
> z1[0]:=y[0]:
> z2[0]:=v1:
> z3[0]:=v2:
> z4[0]:=v3:
> z5[0]:=v4:
> z11[0]:=y[0]:
> z22[0]:=v11:
> z33[0]:=v22:
> z44[0]:=v33:
> z55[0]:=v44:
> z66[0]:=v55:
> p:=y:
> B:=array(1..n,1..n):
> for i from 1 to n do
```

```
> for j from 1 to n do
> if y[i]<v11 and y[i-1]>=0 then
>    z11[j]:=fsolve(tau11(x)=p[j],x,0..1):
>     if z11[j]<=y[i] and z11[j]>=y[i-1] and
>        z11[j-1]>=y[i-1] and z11[j-1]<=y[i] then
>       contr1:=(z11[j]-z11[j-1]):
>          elif z11[j-1]<=y[i] and
>           z11[j-1]>=y[i-1]and z11[j]>y[i]  then
>           contr1:=(y[i]-z11[j-1]):
>           elif z11[j]<=y[i] and
>            z11[j]>=y[i-1]and z11[j-1]<y[i-1] then
>            contr1:=(z11[j]-y[i-1]):
>            else
>     contr1:=0:  fi:
> B[i,j]:=contr1/(1/n):
> elif y[i]<v22 and y[i-1]>=v11 then
>     z22[j]:=fsolve(tau22(x)=p[j],x,0..1):
>     if z22[j]<=y[i] and z22[j]>=y[i-1] and
>       z22[j-1]>=y[i-1] and z22[j-1]<=y[i] then
>       contr2:=(z22[j]-z22[j-1]);
>         elif z22[j-1]<=y[i] and
```

```
>          z22[j-1]>=y[i-1]and z22[j]>y[i]  then
>          contr2:=(y[i]-z22[j-1]);
>        elif z22[j]<=y[i] and z22[j]>=y[i-1]
>             and z22[j-1]<y[i-1] then
>           contr2:=(z22[j]-y[i-1]);
>    contr2:=0;
>  fi;
> B[i,j]:=contr2/(1/n);
>  elif  y[i]<=1 and y[i-1]>=v22  then
>   z33[j]:=fsolve(tau33(x)=p[j],x,0..1):
>    if z33[j]<=y[i] and z33[j]>=y[i-1] and
>        z33[j-1]>=y[i-1] and z33[j-1]<=y[i] then
>      contr3:=(z33[j]-z33[j-1]);
>        elif z33[j-1]<=y[i] and z33[j-1]>=y[i-1]
>          and z33[j]>y[i]  then
>          contr3:=(y[i]-z33[j-1]);
>         elif z33[j]<=y[i] and
>          z33[j]>=y[i-1]and z33[j-1]<y[i-1] then
>           contr3:=(z33[j]-y[i-1]);
>        else
>    contr3:=0;
```

```
>    fi;
> B[i,j]:=contr3/(1/n):
> else
>  z11[j]:=fsolve(tau11(x)=p[j],x,0..1):
>     if z11[j]<=y[i] and z11[j]>=y[i-1] and
>      z11[j-1]>=y[i-1] and z11[j-1]<=y[i] then
>       contr1:=(z11[j]-z11[j-1]):
>         elif z11[j-1]<=y[i] and
>           z11[j-1]>=y[i-1]and z11[j]>y[i]  then
>           contr1:=(y[i]-z11[j-1]):
>         elif z11[j]<=y[i] and z11[j]>=y[i-1]and
>            z11[j-1]<y[i-1] then
>            contr1:=(z11[j]-y[i-1]):
>       else
>     contr1:=0:
>      fi:
>     contr11:=contr1:
> z22[j]:=fsolve(tau22(x)=p[j],x,0..1):
>     if z22[j]<=y[i] and z22[j]>=y[i-1] and
>        z22[j-1]>=y[i-1] and z22[j-1]<=y[i] then
>       contr2:=(z22[j]-z22[j-1]);
```

```
>             elif z22[j-1]<=y[i] and
>               z22[j-1]>=y[i-1]and z22[j]>y[i]  then
>          contr2:=(y[i]-z22[j-1]);
>                elif z22[j]<=y[i] and
>                z22[j]>=y[i-1]and z22[j-1]<y[i-1] then
>           contr2:=(z22[j]-y[i-1]);
>  else
>    contr2:=0;
>  fi;
> contr22:=contr2:
>  z33[j]:=fsolve(tau33(x)=p[j],x,0..1):
>    if z33[j]<=y[i] and z33[j]>=y[i-1] and
>       z33[j-1]>=y[i-1] and z33[j-1]<=y[i] then
>      contr3:=(z33[j]-z33[j-1]);
>        elif z33[j-1]<=y[i] and z33[j-1]>=y[i-1]and
>               z33[j]>y[i]  then
>          contr3:=(y[i]-z33[j-1]);
>        elif z33[j]<=y[i] and z33[j]>=y[i-1]and
>          z33[j-1]<y[i-1] then
>           contr3:=(z33[j]-y[i-1]);
> else
```

```
>     contr3:=0;
>   fi;
> contr33:=contr3: B[i,j]:=(contr11+contr22+contr33)/(1/n):
> fi:
> od:
> od:
> t:=array(1..n):
> for i from 1 to n do
>
> su:=0:
> for j from 1 to n do
>   su:=su+B[i,j]:
> od:
> t[i]:=su:
> od:
> print(t);
> A:=array(1..n,1..n):
> for i from 1 to n do
> for j from 1 to n do
> if y[i]<v1 and y[i-1]>=0 then
>     z1[j]:=fsolve(tau1(x)=p[j],x,0..1):
```

```
>     if z1[j]<=y[i] and z1[j]>=y[i-1] and
>         z1[j-1]>=y[i-1] and z1[j-1]<=y[i] then
>       cont1:=(z1[j]-z1[j-1]):
>         elif z1[j-1]<=y[i] and z1[j-1]>=y[i-1]and
>            z1[j]>y[i]  then
>           cont1:=(y[i]-z1[j-1]):
>         elif z1[j]<=y[i] and z1[j]>=y[i-1]and
>           z1[j-1]<y[i-1] then
>            cont1:=(z1[j]-y[i-1]):
> else
>     cont1:=0:
>   fi:
> A[i,j]:=cont1/(1/n):
> elif y[i]<v2 and y[i-1]>=v1 then
>     z2[j]:=fsolve(tau2(x)=p[j],x,0..1):
>     if z2[j]<=y[i] and z2[j]>=y[i-1] and
>         z2[j-1]>=y[i-1] and z2[j-1]<=y[i] then
>       cont2:=(z2[j]-z2[j-1]);
>         elif z2[j-1]<=y[i] and
>           z2[j-1]>=y[i-1]and z2[j]>y[i]  then
>           cont2:=(y[i]-z2[j-1]);
```

```
>         elif z2[j]<=y[i] and z2[j]>=y[i-1]and
>               z2[j-1]<y[i-1] then
>            cont2:=(z2[j]-y[i-1]);
> else
>     cont2:=0;
>   fi;
> A[i,j]:=cont2/(1/n);
>  elif  y[i]<v3 and y[i-1]>=v2  then
>    z3[j]:=fsolve(tau3(x)=p[j],x,0..1):
>     if z3[j]<=y[i] and z3[j]>=y[i-1] and
>          z3[j-1]>=y[i-1] and z3[j-1]<=y[i] then
>       cont3:=(z3[j]-z3[j-1]);
>          elif z3[j-1]<=y[i] and z3[j-1]>=y[i-1]and
>               z3[j]>y[i]  then
>            cont3:=(y[i]-z3[j-1]);
>          elif z3[j]<=y[i] and z3[j]>=y[i-1]and
>             z3[j-1]<y[i-1] then
>             cont3:=(z3[j]-y[i-1]);
>  else
>     cont3:=0;
>   fi;
```

```
> A[i,j]:=cont3/(1/n):
> elif y[i]<v4 and y[i-1]>=v3  then
>     z4[j]:=fsolve(tau4(x)=p[j],x,0..1):
>     if z4[j]<=y[i] and z4[j]>=y[i-1] and
>        z4[j-1]>=y[i-1] and z4[j-1]<=y[i] then
>       cont4:=(z4[j]-z4[j-1]);
>         elif z4[j-1]<=y[i] and z4[j-1]>=y[i-1]and
>              z4[j]>y[i]  then
>           cont4:=(y[i]-z4[j-1]);
>         elif z4[j]<=y[i] and z4[j]>=y[i-1]and
>           z4[j-1]<y[i-1] then
>            cont4:=(z4[j]-y[i-1]);
> else
>      cont4:=0;
>   fi;
> A[i,j]:=cont4/(1/n):
> elif  y[i]<v5 and y[i-1]>= v4 then
>        z5[j]:=fsolve(tau5(x)=p[j],x,0..1):
>     if z5[j]<=y[i] and z5[j]>=y[i-1] and
>        z5[j-1]>=y[i-1] and z5[j-1]<=y[i] then
>       cont5:=(z5[j]-z5[j-1]);
```

```
>         elif z5[j-1]<=y[i] and z5[j-1]>=y[i-1]and
>                z5[j]>y[i]  then
>           cont5:=(y[i]-z5[j-1]);
>                   elif z5[j]<=y[i] and z5[j]>=y[i-1]and
>             z5[j-1]<y[i-1] then
>            cont5:=(z5[j]-y[i-1]);
> else
>      cont5:=0;
> fi;
> A[i,j]:=cont5/(1/n):
> else
>  z1[j]:=fsolve(tau1(x)=p[j],x,0..1):
>     if z1[j]<=y[i] and z1[j]>=y[i-1] and
> z1[j-1]>=y[i-1] and z1[j-1]<=y[i] then
>       cont1:=(z1[j]-z1[j-1]):
>         elif z1[j-1]<=y[i] and
> z1[j-1]>=y[i-1]and z1[j]>y[i]  then
>           cont1:=(y[i]-z1[j-1]):
> elif z1[j]<=y[i] and z1[j]>=y[i-1]and
>          z1[j-1]<y[i-1] then
>            cont1:=(z1[j]-y[i-1]):
```

```
>           else
>     cont1:=0:
> fi:
>
>     cont11:=cont1:
>     z2[j]:=fsolve(tau2(x)=p[j],x,0..1):
>     if z2[j]<=y[i] and z2[j]>=y[i-1] and
>  z2[j-1]>=y[i-1] and z2[j-1]<=y[i] then
>       cont2:=(z2[j]-z2[j-1]);
>         elif z2[j-1]<=y[i] and
>         z2[j-1]>=y[i-1]and z2[j]>y[i]  then
>           cont2:=(y[i]-z2[j-1]);
>         elif z2[j]<=y[i] and z2[j]>=y[i-1]and
>              z2[j-1]<y[i-1] then
>            cont2:=(z2[j]-y[i-1]);
> else
>     cont2:=0;
>   fi;
> cont22:=cont2:
> z3[j]:=fsolve(tau3(x)=p[j],x,0..1):
>     if z3[j]<=y[i] and z3[j]>=y[i-1] and
```

```
> z3[j-1]>=y[i-1] and z3[j-1]<=y[i] then
>       cont3:=(z3[j]-z3[j-1]);
>         elif z3[j-1]<=y[i] and z3[j-1]>=y[i-1]and
>  z3[j]>y[i]  then
>           cont3:=(y[i]-z3[j-1]);
>          elif z3[j]<=y[i] and z3[j]>=y[i-1]and
>             z3[j-1]<y[i-1] then
>            cont3:=(z3[j]-y[i-1]);
>         else
>     cont3:=0;
>     # print(cont3);
>   fi;
> cont33:=cont3:
> z4[j]:=fsolve(tau4(x)=p[j],x,0..1):
>     if z4[j]<=y[i] and z4[j]>=y[i-1] and
>        z4[j-1]>=y[i-1] and z4[j-1]<=y[i] then
>       cont4:=(z4[j]-z4[j-1]);
>         elif z4[j-1]<=y[i] and z4[j-1]>=y[i-1]and
>  z4[j]>y[i]  then
>           cont4:=(y[i]-z4[j-1]);
>         elif z4[j]<=y[i] and z4[j]>=y[i-1]and
```

```
>              z4[j-1]<y[i-1] then
>            cont4:=(z4[j]-y[i-1]);
> else
>      cont4:=0;
>  fi;
> cont44:=cont4:
> z5[j]:=fsolve(tau5(x)=p[j],x,0..1):
>     if z5[j]<=y[i] and z5[j]>=y[i-1] and
>  z5[j-1]>=y[i-1] and z5[j-1]<=y[i] then
>       cont5:=(z5[j]-z5[j-1]);
>         elif z5[j-1]<=y[i] and
>  z5[j-1]>=y[i-1]and z5[j]>y[i]  then
>           cont5:=(y[i]-z5[j-1]);
>  elif z5[j]<=y[i] and z5[j]>=y[i-1]and
>              z5[j-1]<y[i-1] then
>            cont5:=(z5[j]-y[i-1]);
>  else
>      cont5:=0;
>         fi;
> cont55:=cont5:
> A[i,j]:=(cont11+cont22+cont33+cont44+cont55)/(1/n):
```

```
> fi:
> od:
> od:
> print(A):
> tt:=array(1..n):
> for i from 1 to n do
> suu:=0:
> for j from 1 to n do
>   suu:=suu+A[i,j]:
> od:
> tt[i]:=suu:
> od:
> print(tt);
> #************************************
> for i from 1 to n do
>   intw11[i]:=n*int(w11(x),x=(i-1)*h..(h*i)):
> od:
> for i from 1 to n do
>   intw12[i]:=n*int(w12(x),x=(i-1)*h..(h*i)):
> od:
> for i from 1 to n do
```

```
>   intw21[i]:=n*int(w21(x),x=(i-1)*h..(h*i)):
> od:
> for i from 1 to n do
>   intw22[i]:=n*int(w22(x),x=(i-1)*h..(h*i)):
> od:
> for i from 1 to 4 do
>   s[i]:=intw11[i]+intw12[i]:
> od:
> C:= array(1..n,1..n):
> for i from 1 to n do
>    for j from 1 to n do
>      if abs(i-j)=0 then
>         C[i,j]:=intw11[i];
>      else
>        C[i,j]:=0;
>             fi:
>   od;
> od;
> print(C):
> D1:= array(1..n,1..n):
> for i from 1 to n do
```

```
>    for j from 1 to n do
>      if abs(i-j)=0 then
>         D1[i,j]:=intw12[i]:
>     else
>        D1[i,j]:=0:
>             fi:
>   od;
> od;
> print(D1):
> E:= array(1..n,1..n):
> for i from 1 to n do
>    for j from 1 to n do
>      if abs(i-j)=0 then
>         E[i,j]:=intw21[i]:
>      else
>        E[i,j]:=0:
>             fi:
>   od:
> od:
> F:= array(1..n,1..n):
> for i from 1 to n do
```

```
>    for j from 1 to n do
>      if abs(i-j)=0 then
>         F[i,j]:=intw22[i]:
>       else
>        F[i,j]:=0:
>             fi:
>   od;
> od;
> with(linalg):
> S11:=multiply(A,C):
> S12:=multiply(A,D1):
> S21:=multiply(B,E):
> S22:=multiply(B,F):
> S:=blockmatrix(2,2,[S11,S12,S21,S22]);
> ttt:=array(1..2*n):
> for i from 1 to 2*n do
> suuu:=0:
> for j from 1 to 2*n do
>   suuu:=suuu+S[i,j]:
> od:
> ttt[i]:=suuu:
```

```
> od:
> print(ttt);
> with(linalg):
> S := transpose(S):
> H:= array(1..2*n,1..2*n):
> for i from 1 to 2*n do
>    for j from 1 to 2*n do
>      if abs(i-j)=0 then
>         H[i,j]:=1:
>      else
>        H[i,j]:=0:
>             fi:
>   od:
> od:
> T:= array(1..2*n+1,1..2*n):
> T1:=evalm(S-H):
> for i from 1 to 2*n do
>  for j from 1 to 2*n do
>    T[i,j]:=T1[i,j]:
>  od:
> od:
```

```
>   for j from 1 to 2*n do
> T[2*n+1,j]:=1:
> od:
> print(T):
> b := array(1..2*n+1):
> for i from 1 to 2*n do
> b[i]:=0:
> od:
> b[2*n+1]:=2*n:
> print(b):
> sol:=array(1..2*n):
> sol:=evalf(leastsqrs(T, b));
> summ:=0:
> for i from 1 to 2*n do
> summ:=summ+sol[i]:
> od:
> print(summ):
> density:=array(1..n):
> for i from 1 to n do
>  density[i]:=n*(sol[i]+sol[n+i]):
> od:
```

```
> print(density):
>
> summm:=0:
> for i from 1 to 2*n do
> summm:=summm+density[i]:
> od:
> summm:=1/n*summm:
> for i from 1 to n do
>  density[i]:= density[i]/summm:
> od:
> print(density):
> ##------------------density output---------------
> u2:=vector([.94663, 1.0534]);
> u4:=vector([.91665, .98045, 1.0431, 1.0597]);
> u6:=vector([.91692, .93533, 1.0199, 1.0325,
> 1.0389, 1.0563]);
> u8:=vector([.90513, .93438, .94763, 1.0498,
> 1.0308, 1.0362, 1.0438, 1.0525]);
> u10:=vector([.89651, .91135, .94740, .96240,
> 1.0785, 1.0236, 1.0318, 1.0422, 1.0488, 1.0567]);
> u12:=vector([.89700, .90396, .92004, .96100, .96921,
```

```
>  1.1037, 1.0245, 1.0322, 1.0375, 1.0425, 1.0522, 1.0563]);
> u16:=vector([.89997, .90666, .91107, .91457, .93972,
> .98422, .98753, 1.1537, 1.0203, 1.0262, 1.0297, 1.0324,
>  1.0374, 1.0487, 1.0528, 1.0545]);
> u32:=vector([.89550, .90166, .90441, .90841, .90991,
> .91270, .91505, .91780, .92270, .95750, .98034, .98209,
>  .98950, .99195, 1.1449, 1.1645, 1.0171, 1.0194, 1.0209,
>  1.0223, 1.0258, 1.0283, 1.0353, 1.0368, 1.0393, 1.0402,
>  1.0466, 1.0487, 1.0513, 1.0527, 1.0593, 1.0583]);
> u64:=vector([.89508, .89766, .89906, .90195, .90727,
> .90688, .90852, .91063, .91125, .91195, .91375, .91438,
>  .91508, .91617, .91852, .91969, .91797, .92148, .94016,
>  .96266, .97086, .98438, .98875, .98914, .99070, .99297,
>  .99234, .99172, 1.1263, 1.1678, 1.1691, 1.1711, 1.0171,
>  1.0207, 1.0175, 1.0202, 1.0205, 1.0228, 1.0199, 1.0239,
>  1.0251, 1.0259, 1.0223, 1.0288, 1.0326, 1.0351, 1.0333,
>  1.0341, 1.0376, 1.0422, 1.0398, 1.0402, 1.0409, 1.0467,
> 1.0420, 1.0473, 1.0484, 1.0519, 1.0514, 1.0543, 1.0547,
> 1.0583, 1.0562, 1.0585]);
> u128:=vector([.88614, .89122, .89169, .89493, .89833,
> .89848, .90005, .90548, .91641, .90223, .90708, .90801,
```

```
> .90704, .90962, .90817, .90813, .91219, .91134, .91122,
> .91184, .91641, .91016, .91044, .91497, .91423, .91348,
> .90887, .91606, .91762, .92102, .92243, .92544, .91801,
> .91782, .92258, .92239, .93504, .93778, .94356, .96274,
> .97082, .98453, .98996, .99398, .99227, .99121, .99348,
> .99320, .99418, .99633, .99539, .99844, .99570, .99352,
> .99922, .99844, 1.0931, 1.1702, 1.1839, 1.1878, 1.1801,
> 1.1822, 1.1854, 1.1864, 1.0348, 1.0328, 1.0300, 1.0330,
> 1.0248, 1.0214, 1.0133, 1.0151, 1.0119, 1.0132, 1.0143,
> 1.0221, 1.0175, 1.0197, 1.0219, 1.0182, 1.0189, 1.0203,
> 1.0227, 1.0297, 1.0202, 1.0233, 1.0229, 1.0231, 1.0311,
> 1.0314, 1.0265, 1.0334, 1.0237, 1.0262, 1.0282, 1.0301,
> 1.0304, 1.0352, 1.0311, 1.0406, 1.0385, 1.0343, 1.0338,
> 1.0368, 1.0375, 1.0365, 1.0386, 1.0454, 1.0424, 1.0399,
> 1.0506, 1.0550, 1.0572, 1.0513, 1.0519, 1.0517, 1.0474,
> 1.0478, 1.0496, 1.0533, 1.0471, 1.0500, 1.0529, 1.0572,
> 1.0518, 1.0519, 1.0538, 1.0499]);
> ##-----------Error calculation---------------------------
> u:=u4:v:=u8:summ[0]:=0:for i from 1 to 4 do
> summ[i]:=summ[i-1]+abs(u[i]-v[2*i-1])+abs(u[i]-v[2*i]):
> od:
```

```
> summ[4]/8;

> u:=u8:v:=u16:summ[0]:=0:for i from 1 to 8 do

> summ[i]:=summ[i-1]+abs(u[i]-v[2*i-1])+abs(u[i]-v[2*i]):

> od:

> summ[8]/16;

> u:=u16:v:=u32:summ[0]:=0:for i from 1 to 16 do

> summ[i]:=summ[i-1]+abs(u[i]-v[2*i-1])+abs(u[i]-v[2*i]):

> od:

> summ[16]/32;

> u:=u32:v:=u64:summ[0]:=0:for i from 1 to 32 do

> summ[i]:=summ[i-1]+abs(u[i]-v[2*i-1])+abs(u[i]-v[2*i]):

> od:

> summ[32]/64;

> u:=u64:v:=u128:summ[0]:=0:for i from 1 to 64 do

> summ[i]:=summ[i-1]+abs(u[i]-v[2*i-1])+abs(u[i]-v[2*i]):

> od:

> summ[64]/128;

> plot([[4,.21715e-1],[8,.18718e-1],[16,.39488e-2],[32,.31073e-2]]);

>

>
```

Bibliography

[1] Adler, R., and Flatto, L., *Geodesic flows, interval maps and symbolic dynamics*, Bull. Amer. Math. Soc. 25 (1991), 229-334.

[2] Bahsoun, W., Góra, P. and Boyarsky, A., *Markov switching for position dependent random maps with application to forecasting in financial markets*, to appear in SIAM Journal on Applied Dynamical Systems.

[3] Bahsoun, W., Góra, P. and Boyarsky, A., *Stochastic perturbations for position dependent random maps*, Stoch. Dynam. **3**, no. 4 (2003), 545-557.

[4] Bahsoun, W. and Góra, P., *Weakly Convex and Concave Random Maps with Position Dependent Probabilities*, Stoch. Anal. App. **21**, no. 5 (2003), 983-994.

[5] Bahsoun, W. and Góra, P., *Position Dependent Random Maps in One and Higher Dimensions*, to appear in Studia Math.

[6] Barnsley, M., *Fractals everywhere*, Academic Press, London, 1998.

[7] Boyarsky, A. and Góra, P., *A dynamical model for interference effects and two slit experiment of quantum physics*, Phys, Lett. A 168 (1992), 103-112.

[8] Chatelin, Francoise, *Eigenvalues of Matrices*, John Wiley and Sons(1993), 114-117.

[9] Dunford, N. and Schawartz, J.T., *Linear operators, Part I*, Wiley Interscience (Wiley Classics Library): Chichester,(1988).

[10] Froyland, G., *Ulam's method for random interval maps*, Nonlinearity, 12(1999) 1029-1052.

[11] Góra, Pawel, *Countably piecewise expanding transformations without absolutely continuous invariant measure*, Proceedings of Stefan Banach Center - Semester on Dynamical Systems and Ergodic Theory, Warsaw 1986 , PWN - Polish Scientific Publishers, 1989, pp.113-117.

[12] Góra, Pawel and Boyarsky, Abraham, *Laws of Chaos*, Birkhaüser, 1997.

[13] Góra, Pawel and Boyarsky, Abraham, *Absolutely continuous invariant measures for random maps with position dependent probabilities*, Math. Anal. and Appl. 278 (2003), 225-242.

[14] Góra, Pawel and Boyarsky, Abraham, *Attainable densities for random maps*, Preprint.

[15] Góra, P., and Boyarsky, A., *Why computers like Lebesgue measure*, Comput. Math. Applic., vol. 16, (1988), 321-329.

[16] Góra, P., Boyarsky, A., Islam, M.S., and Bahsoun,W., *Absolutely Continuous Invariant Measures that cannot be observed experimentally*, Submitted to SIAM .

[17] Halfant, M., *Analytic properties of Rényi's invariant density*, Isreal J. Math. 27(1977), 1-20.

[18] Hunt, B.R., *Estimating invariant measures and Lyapunov exponents*, Ergod. Theor. Dynam. Syst. 16 (1996)735-749.

[19] Islam, M.S. *Existence, approximation and properties of absolutely continuous invariant measures for random maps*, Ph.D. Thesis, Concordia University, Canada, 2004.

[20] Islam, M.S.,Góra, P., and Boyarsky, A. *A Generalization of Straube's Theorem: Existence of Absolutely Continuous Invariant Measures for Random Maps*, Journal of Applied Mathematics and Stochastic Analysis (accepted).

[21] Góra, P., Boyarsky, A. and Islam, M.S. *Invariant densities of random maps have lower bounds on their supports*, submitted to the proceedings of the American Mathematical Society.

[22] Jabłoński, M., Góra, P. and Boyarsky, A., *A general existence theorem for absolutely continuous invariant measures on bounded and unbounded intervals,* Nonlinear World 3(1996), 183-200.

[23] Keller, G *Piecewise monotonic transformations and exactness* Collection: Seminar on probability, Rennes 1978 (French);Exp. No. 6, 32 pp., Univ. Rennes, Rennes, 1978.

[24] Kowalski, Z. S., *Invariant measure for piecewise monotonic transformation has a lower bound on its support,* Bull. Acad. Aolon. Sci. Ser. Sci. Math. 27 (1979b), 53-57.

[25] Li, T-Y., *Finite approximation for the Frobenius-Perron operator: A solution to Ulam's conjecture,* J. Approx. Theory. 17 (1976), 177-186.

[26] Lasota, A. and Yorke, J.A., *On the existence of invariant measures for piecewise monotonic transformations* , Trans. Amer. Math. Soc 186 (1973) ,481-488.

[27] Maple 9.5, *Waterloo Maple Inc.*, 1997-2005.

[28] Morita , T., *Random iteration of one-dimensional transformations,* Osaka J. Math (1985), 22, 489-518.

[29] Motwani, R and Raghavan, P., *Randomized Algorithms,* Cambridge University Press, (1995).

[30] Pelikan, S., *Invariant densities for random maps of the interval*, Proc. Amer. Math. Soc. 281 (1984), 813-825.

[31] Rényi, A., *Representation for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. 8 (1957), 477-493, 27-30.

[32] Schweitzer, P. J., *Perturbation theory and Finite Markov chains*, J. Appl. probab 5 , 401-404.

[33] Schenk-Hoppe, K. R., *Random Dynamical Systems in Economics*, working paper series, ISSN 1424-0459, Institute of empirical research in economics, University of Zurich, Dec 2000.

[34] Slomczynski, W., Kwapien, J. and Zyczkowski, K., *Entropy computing via integration over fractal measures*, Chaos 10 (2000), 180-188.

[35] Straube, E., *On the existence of Invariant, Absolutely Continuous Measures.*, Comm. Math. Phys. 81 (1981), 27-30.

[36] Yosida, K and Hewitt, E., *Finitely additive measures*, Trans. Amer. Math. Soc. 72 (1952), 46-66.

[37] Ulam, S. M., *A collection of Mathematical problems* Interscience Tracts in pure and Applied Math.,vol 8, Interscience, new York, 1960.

[38] Ulam and von Neumann, *Random Ergodic theorem*, Bul Amer Math Soc. 72 (1952), 46-66.