Difficulties in the Learning and Teaching of Linear Algebra – A Personal Experience

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ABSTRACT

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Two different personal experiences of teaching Linear Algebra are analyzed – (1) teaching a college-level linear algebra course and (2) being the instructor in a pre-designed experiment that investigates a geometric approach to the concepts of vector and linear transformations using the dynamic geometry software *Cabri-géomètre II*. The analysis is conducted within a framework of three perspectives on students' difficulties in learning Linear Algebra: (a) the nature of Linear Algebra, (b) the didactic decisions made in teaching Linear Algebra, and (c) students' ways of thinking and their mathematical backgrounds. The historical look at the subject's development reveals that the content of an undergraduate linear algebra course is often the end product of a long process of intellectual struggle and research into deep mathematical problems with which students may never become acquainted in the course of their studies. The experiment tries to build up concepts using geometry instead of giving the final product, but fails to eliminate the structural approach. As a result, students still struggled with concepts. The detailed discussions of the situations result in an "interpretive understanding" of the situations. Recommendations on improving college-level Linear Algebra courses, such as the concentration on computations, and future research projects are given.

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I want to thank my mother Zahira, my brother Marwan and my sister Maha for believing in me.

I would like to dedicate this thesis to the memory of my father, Samir. I know he is proud.

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Introduction

My aim in this thesis is to analyze my two very different experiences as a Linear Algebra teacher. The first experience was teaching a college-level Linear Algebra course offered at Concordia University. The second experience was being the instructor in a tightly controlled experiment in which topics in Linear Algebra were introduced geometrically using the software *Cabri-géomètre II*. In this thesis I plan to analyze the two approaches to the teaching of Linear Algebra from the point of view of the difficulties that students were encountering in each. I am going to be looking at three sources of their difficulties: (1) the nature of Linear Algebra, (2) the way Linear Algebra is taught, and (3) how students learn and cope with Linear Algebra.

In Chapter I, I will discuss the above sources of difficulties in greater detail, thus setting up a framework for my analysis of the two teaching experiences. The discussion on the nature of linear algebra will be a historical look at the subject's development. A forceful conclusion gained by this perspective will be that the content of an undergraduate linear algebra course is often the end product of a long process of intellectual struggle and research into deep mathematical problems with which students may never become acquainted in the course of their studies. The concepts taught to students may appear simple but, outside of their proper context of application, they appear meaningless and incomprehensible. On top of this "objective" factor of difficulty of the linear algebra courses, there are other elements, such as certain teaching methods or approaches and students' study habits and past mathematical experience.

In Chapter II, I will analyze my experience teaching the college-level course MATH 204 *Vectors and Matrices*, using the "grid" of analysis developed in Chapter I. I will discuss the course format, the type of students that take this course, and the content of the course. Examples of student responses will illustrate my arguments, and I will give recommendations on improving the course.

In Chapter III, I will analyze my experience as the instructor in an experiment specially designed as a result of the concerns discussed above. The experiment was designed by a group of researchers at Concordia University, and my role in the project was being the instructor during the experimental sessions. The analysis will be very detailed account of what happened during the sessions and will involve many quotations from the transcripts. Recommendations for future projects will also be given.

In Chapter IV, I will discuss the overall teaching of Linear Algebra and whether the results of the experiment did really answer our concerns of the way this complex subject is presently being taught.

CHAPTER I

A Framework of Analysis

Introduction

The applications of Linear Algebra are many and varied. These days students of different disciplines --e.g., biology, economics, computer science, engineering, etc. -- are now required to take linear algebra courses. As a result, students usually enroll in a linear algebra class in their first year of undergraduate studies so as to complete the requirements for their core courses in their respective majors. This is in sharp contrast to the late 1950's when mathematics majors took their first linear algebra course in the first year of their graduate years. (Carlson, 1993) Since the course material is still as complex as it was in the 1950's, the students who study linear algebra today are, as Carlson (1994) points out, "generally not mathematically mature." From my own learning experience, the concepts did not start to make sense to me until the third Linear Algebra course, which I was taking concurrently with an Abstract Algebra class.

What makes the concepts of Linear Algebra so complex? Why do first-year undergraduate students have trouble comprehending them? Is Linear Algebra being taught to promote the understanding of these concepts? What are the consequences of trying to learn Linear Algebra at this early stage? These questions can be separated into three interconnected perspectives: (1) the nature of Linear Algebra, (2) the teaching of Linear Algebra, and (3) how students learn and cope with Linear Algebra.

"The nature of Linear Algebra" suggests that the complex concepts involved in Linear Algebra are a source of students' difficulties in learning the content. The historical development and its consequences on the students' learning of Linear Algebra will be discussed.

"The teaching of Linear Algebra" refers to the didactic decisions made and the teaching methods used that are blamed for causing students' difficulties in learning Linear Algebra.

"How students learn and cope with Linear Algebra" points to the students' ways of thinking and their mathematical backgrounds that are blamed for the difficulties in learningLinearAlgebra.

I am going to use the existing knowledge and opinions about the nature of Linear Algebra, the teaching of Linear Algebra, and students' thinking and coping with Linear Algebra as being sources of students' difficulties. I will analyze my personal experience as a teacher against this background.

The focus will be on mathematical rather than sociological or psychological factors. Students' difficulties will be explained not by psychological or sociological theories, but by (a) the difficult nature of the concepts involved, (b) the didactic decisions concerning its introduction, and (c) the history of students' understandings of prerequisite notions.

I shall be using what Carr and Kemmis (1986) would describe as an "interpretive" framework in my analysis of the two teaching approaches in which I participated as a teacher. In other words, I will try to understand why the students did the mistakes that they did and to find what is creating the stumbling blocks for them.

In order to achieve this goal, I have to become aware of the "subjective meaning" of the ideas and concepts that the students have. "Actions, unlike the behaviour of most objects, always embody the interpretations of the actor, and for this reason can only be understood by grasping the meanings that the actor assigns to them." (Carr and Kemmis, 1986, p. 88)

In reality, for an "interpretive understanding" of a situation to be valid, it must have the approval of the person who did the act. I can give reasons to what happened, justify them from videos, tapes, and transcripts, and compare them to similar cases, but unless the person who did and said these things confirms my reasons, my "interpretive understanding" is just an interpretation, and not some kind of "truth". After all, these actions and words are that person's, and he/she is the only one that knows the true meanings behind them. (Carr and Kemmis, 1986)

In other words, I am claiming that my analysis is only one possible interpretation of the students' understanding of concepts or lack thereof. The analysis is based on my interactions with the students during the experiment or class, reading the transcripts and reviewing the video and audio tapes of the sessions. There were few instances during the experiment where the students confirmed my interpretation of their knowledge, but most of the analysis is based on interpretations after the fact. For this reason, the experiment is open to different interpretations.

1.1. The Nature of Linear Algebra as a Source of Students' Difficulties

What makes the concepts of Linear Algebra so complex? To answer this question, one must look at the nature and historical development of Linear Algebra. Its development is different and unique compared to the types of mathematics encountered by students up to their first year of university. For example, Calculus was borne out of the necessity to solve specific problems such as: (1) find the slope of the tangent line to a curve (Differential Calculus), (2) find a rate of change (Differential Calculus), and (3) find the area under a curve (Riemann sums), or (4) find the velocity, given the acceleration (antiderivatives). On the other hand, Linear Algebra was developed mainly, as Dorier (1995a) would say, to unify and generalize concepts in mathematics. In other words, one of the main tasks of Linear Algebra was not to solve new problems (except in the Functional Analysis for non-countable infinite dimensions), but instead to "simplify" problems already solved. Common aspects of different methods, tools, and objects that already existed in varied mathematical settings were identified and abstracted to form a unified and generalized theory. This theory could then be adapted to other branches of mathematics and thus simplify and introduce new methods for problem solving. (Dorier, 1995a).

1.1.1 Historical Development of Linear Algebra

Methods of solving linear equations had existed since Ancient Chinese times, however, the history of Linear Algebra started with the publishing of René Descartes' *La Géometrie* in 1637. The introduction of analytical geometry and the Cartesian plane created an "algebraization of geometry" in which every geometrical curve can be expressed as an algebraic equation in two variables. These new ideas created a split in the mathematical thinking of the 1600s.

First, the concept of 'analysis' changed. Prior to Descartes, 'analyzing' a problem required two steps that were inseparable. First, one had to translate a geometrical problem into algebraic equations ('analysis'). Then, after algebraically manipulating and solving the equations, one had to interpret the results geometrically ('synthesis'). (Panza, 1996) But now, according to Newton,

> "... 'analytical' procedures are completely distinct from their possible geometrical (or mechanical) applications or interpretations. An equation, a finite symbolic expression, or a series is considered as an autonomous object. ...'analysis' is a self governing field and gains its mathematical meaning without the employing of any interpretation or construction." (Panza, 1996, p.243)

Second, up to that point in time, the line and the circle were considered as basic figures of Geometry, but analytical geometry separated the two concepts as first order and second order equations, respectively. With the line being the most basic entity of algebra (first order equation), questions of linearity began to be raised.

The Algebraization of Geometry and its Influences

From Descartes' *La Géometrie* till 1750, work was done to improve techniques for solving systems of linear equations. In 1750, Gabriel Cramer published *Introduction à l'analyse des courbes algébriques*, in which he introduced the use of determinants to solve systems of linear equations. Also in 1750, Cramer reformulated an inconsistency recognized since 1720 that became known as Cramer's paradox.

At that time, it had been known that two distinct algebraic curves of order n intersected in n^2 places. It had also been accepted that to determine a curve of order n, it was necessary and sufficient to know $\frac{n(n+3)}{2}$ points. The paradox occurred for $n > 2$. For example, if n=3, then nine (9) points are enough to define the curve. This contradicts the first proposition, which says that <u>two</u> distinct curves of order 3 can have nine (9) points in common. Therefore, nine (9) points are <u>not</u> sufficient to determine the curve since it could be one of two distinct curves.

Ideas of linear dependence began to emerge for the first time in Leonhard Euler's explanation of Cramer's paradox. The results were published in the treatise *Sur une contradiction apparente dans la doctrine des lignes courbes (1750).* Euler showed that n equations in n unknowns will not always have a unique solution since one equation could be "comprised" in one of the others. In other words, one equation could be obtained from the others by multiplying one equation by a scalar and/or adding it to a second equation. For example, in solving the following system of linear equations

$$\begin{aligned} x + y + z &= 1 && (1) \\ 3x + y + z &= 2 && (2) \\ -x - 4y - 2z &= -1, && (3) \end{aligned}$$

one determines that the solution is $x = 1 - 2z/3$ and $y = z/3$, where z is any real number. Although the system has three equations and three unknowns, there is more than one solution because equation (3) is equal to –3 times equation (1) added to equation (2). In other words, equation (3) "depends" on equations (1) and (2).

Despite his explanations, Euler's ideas were neglected and were not investigated further. Instead, Cramer's theory of determinants flourished and started its own branch of mathematics that included linear equations. The issue of undetermined and inconsistent linear systems was ignored for the next century. This meant that the concept of dependence was also disregarded since it arose only in the context of inconsistent and dependent systems.

In the meantime, other developments were occurring in different branches of mathematics that would eventually be integrated in our modern-day linear algebra. Analytical geometry generated the study of linear substitutions --i.e., change of coordinates. The study of linear substitutions was a precursor of what is called today linear transformations.

The calculations involved in the linear substitutions of variables resulted in long and complex equations that were difficult and tedious to manipulate. As problems became more complicated, the mathematicians looked for techniques that would help them in having better organization and more control over their work. One result was to represent expressions by arrays of their coefficients. An important development was the realization that one can create a calculus on these representations which gives results that would be extremely difficult to obtain through direct manipulations of the original expressions. Thus, the concept of matrix was developed as a result of the need to generalize and manipulate complicated algebraic expressions.

At first, the concept of matrix and the concept of determinant were not clearly distinguished from each other. One fundamental reason was that the multiplication of matrices was seen as a local process, not as an algebraic operation in itself. But by the mid-1800s, matrices began to be observed as a mathematical entity or generalized number. This was due to the acceptance of imaginary numbers by mathematicians and the discovery of quaternions (3-dimensional complex numbers) by Hamilton, which helped in expanding the field of algebra, in general.

Although generalizations to n-dimensional space had been done earlier – e.g., by Euler, in an algebraic context, most mathematicians did not go beyond 3-dimensional space because they saw geometry as the science of the physical space and more than three dimensions did not make physical sense to them. Cayley was one of the first mathematicians to break that barrier. In his paper *Sur quelques résultats de géométrie de position* (1846), he described how one can work in a space of an arbitrary dimension and get a result in 3-dimensional space. The use of n-dimensional spaces became justified in the 19th century as a result of the following events:

1. The discovery of non-Euclidean geometries and the advancement of projective and algebraic geometry started discussions on the foundations of geometry. If one can disregard one of Euclid's postulates to create a geometry that does not agree with our visual and physical world, why cannot one expand algebra into the nth-dimension?

2. The discovery of quaternions opened the way to many discoveries of new types of algebra.

The second half of the 19th century saw the development of n-dimensional geometry based on analytic geometry and the theories of determinants and matrices. (Dorier, 1995b)

Coordinate-Free Geometry and its Influences

Meanwhile, not everyone had embraced the idea of the "algebraization of geometry". Many mathematicians wanted to go back to the basics of "coordinate-free geometry". Gottfried Wilhelm Leibniz expressed his criticism in a letter to Christian Huygens written in 1679, but was not published until 1833:

> "I am still not satisfied with algebra because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that as far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation directly as algebra expresses magnitude directly..... Algebra is the characteristic for undetermined numbers or magnitudes only, but it does not express situation, angles and motion directly. Hence it is often difficult to analyze the properties of a figure by calculation, and still more difficult to find very convenient geometrical demonstrations and constructions even when the algebraic calculation is completed." (Cited in Crowe, 1967, p.3)

Although Leibniz failed to create a geometrical system in which the symbols representing geometrical entities can be added, subtracted and multiplied, he inspired many mathematicians in the 19th century to search for such a system.

According to Crowe (1967), there were three major ideas that influenced the creation and development of vectorial systems: one was related to viewing some physical magnitudes as vectors, another to geometric representations of complex numbers, and the third to a coordinate-free "geometric algebra" (Leibniz's idea mentioned above).

Representing physical entities such as velocity and force by a parallelogram was not uncommon in the 16th and 17th centuries (since it made physical sense), but the idea of the <u>addition</u> of the lines did not emerge until the creation of vectors. Therefore, the concept of the parallelogram had an important, but indirect influence on the development of vectorial systems, "for it was the first and most obvious case in which vectorial methods could be brought to the aid of physical science." (Crowe, 1967, p.2)

The acceptance of complex numbers was very slow. Between 1799 and 1828, in efforts to legitimize these numbers, five mathematicians (Wessel, Burée, Argand, Mourey, and Warren) worked independently of each other on setting up the principles of a geometric representation of complex numbers. But these principles did not get widely known and accepted by mathematicians until Gauss wrote about them in 1831. The acceptance of complex numbers opened the door for expanding the notion of "number" to also include vectors. Thus, the groundwork for vectorial geometry had been laid. (Crowe, 1967, p. 5; see also Dorier, 1995b)

Two early vectorial systems were created by August Ferdinand Möbius (1827) and Giusto Bellavitis (1833). The pre-occupying concern of these early systems was the notion of <u>equal</u> vectors. In his *Barycentrische Calcul* (1827), Möbius was one of the first mathematicians to give direction to a line segment by designating " a line segment from a point A to a point B by the notation AB and stated that AB = -BA ..." (Dorier, 1995b, p. 9) In *Calcolo delle Equipollenze* (1833), Bellavitis based his "vectors" on the behaviour of geometrically represented complex numbers. The difference was that he viewed his lines as <u>entities</u>, not as <u>representations</u>. (Bellavitis never accepted imaginary numbers as mathematical entities.) Although both Möbius and Bellavitis defined the addition and multiplication of their "vectors", their systems could not be extended to the 3-dimensional space. (Crowe, 1967)

The first vectorial system that could be generalized to *n*-dimensions was Hermann Grassmann's Calculus of Extension. His ideas and principles for this system were first developed as early 1832. By the time Grassmann wrote his *Theorie der Ebbe und Flut* in 1840 (this work was not published until 1911), he had already developed methods that are equivalent to today's vector addition, the two major kinds of vector products, and vector differentiation. When Grassmann extended his system to 3-dimensional space, it resulted in his book *Lineale Ausdehnungslehre* ("linear theory of extension") in 1844, and later modified in 1862.

Grassmann used a philosophical approach to present his work in *Lineale Ausdehnungslehre*. Grassmann claimed that his discovery is an independent field of mathematics, even though it could be applied to geometry, mechanics, and other scientific fields. He believed that geometry is a science independent of mathematics and that his "theory of extension" is the mathematical model to be applied to it. Grassmann's theory was self-contained; thus it included many preliminary definitions and introduced new notations, words, and concepts. (Dorier, 1995b)

Since Grassmann's work mixed mathematical results with philosophical considerations, mathematicians rejected it because they found it to be confusing and unclear. However, today, Grassmann's work is considered to be one of the first unifying and generalizing theories in mathematics. "Grassmann's theory contained the bases for a unified theory of linearity, as it introduced, with great accuracy and in a very general context, elementary concepts such as linear dependence, basis, and dimension." (Dorier, 1995b, p.18) His theory provided a framework for generating a rich model for linearity by defining the essential objects and proving most of the elementary properties of finite-dimensional vector spaces.

Although Grassmann's results correspond to the modern theory of vector spaces, it had no direct influence on its creation since most of these concepts were reestablished independently of his work. Nonetheless, Grassmann's work was the first theory to use the axiomatic approach. (Dorier, 1995b)

Giuseppe Peano wrote a condensed version of Grassmann's work called *Calcolo geometrico* (1888). At the end of his work, he gave an axiomatic definition of a "linear system", which is considered to be the first modern definition of a vector space. But again, at the time it appeared, his work was not followed by other studies.

Other Italian mathematicians, Cesare Burali-Forti (1897) and Roberto Marcolongo (1909), also published works with the axiomatic approach, but once again they were ignored. Their work is nevertheless significant because (1) unlike Peano, they opened their treatises with an axiomatic presentation of their linear system, and (2) it showed that axiomatic approach was beginning to be more widely used.

The Creation of Linear Algebra: A Unifying and Generalizing Theory

Since the 18th century, mathematicians have been interested in differential equations. Their study led to the creation of a branch of mathematics called functional analysis around the turn of the 20th century.

In 1822, while solving differential equations by power series, Joseph Fourier was led to the method of solving systems of countably infinite linear equations in countably infinite unknowns. Because of the lack of understanding of the convergence of power series, Fourier was not able to give a correct solution. As a matter of fact, the solution of infinite linear systems was not researched for about the next fifty years. It was not until 1886 that a text with consistent results was published. After that, many mathematicians worked on infinite linear systems. They used Fourier's ideas, but they changed the restrictive boundary conditions on the power series so that the convergence of the infinite determinant was guaranteed. In fact, up to the 1920s, mathematicians researched most of the known concepts and methods of the finite dimensional case --e.g., the theories of determinant, matrix, and quadratic and bilinear forms -- in the context of countably infinite dimension; thus creating the framework for a unified theory of linearity. Nonetheless, some of the methods became highly technical and difficult to manipulate, which led to inaccuracies. Slowly, mathematicians moved away from the use of determinants and began to consider more and more general vector spaces. Eventually, this led to the axiomatization of functional analysis.

Stefan Banach (1920-1922) and Hans Hahn (1922 and 1927) took the final decisive steps towards the axiomatization of functional analysis independently of each other. Banach wrote in the introduction of his book *Théorie des opérateurs linéaires* (1932):

> "The present book follows the goal of establishing a few theorems valid for various functional fields, which I will specify. Nevertheless, so that I do not have to prove them separately for each field, which would be painful, I have chosen a different method, that is: I will consider in a general sense the sets of elements of which I will postulate certain properties, I will deduce some theorems and then I will prove for each specific functional field that the chosen postulates are true." (Cited in Dorier, 1995b, p.29)

Although some of the axioms that Banach presented were redundant and some were missing, he demonstrated the effectiveness of the axiomatic approach. Since he was dealing with functional spaces of uncountably infinite dimension, an axiomatic approach was the better choice. Banach's book "gave the general framework and most of the results of axiomatic functional analysis and infinite-dimensional linear algebra; this book was an enormous success and rapidly opened a new era in these two fields of mathematics."(Dorier, 1995b, p.29)

1.1.2 Consequences of a Complex Historical Development for the Understanding of Students' Difficulties

As was shown above, it took nearly three centuries from Descartes' *La Géometrie* for linear algebra to come into existence. It was a long and winding road that raised many discussions, debates, and disagreements along the way. But finally, a unifying and generalizing theory emerged. This theory, as one can expect, is abstract, which is one of the fundamental reasons of why students have difficulty in learning linear algebra. First-year university students do not have a sufficient mathematical experience; thus they have not enough knowledge to abstract from. Also, as Harel (1989) points out, high school mathematics does not involve abstract systems and does not train students to generalize specific situations and problems. Yet, a review of the textbooks of a first course in linear algebra found the following implicit assumptions were made: beginning students

> "a) are capable of dealing with abstract structures without extensive preparation.
>
> b) can appreciate the economy of thought when particular concepts and systems are treated through an abstract representation." (Harel, 1989, p.140)

Another reason for students having difficulty in learning linear algebra is the axiomatic character of the theory. As was shown in the above historical review, the axiomatization of functional analysis resulted from a gradual movement away from the use of determinants (a mostly computational theory) to the consideration of vector spaces (a generalizing and unifying theory). An understanding of the axiomatic theory requires that the learner has access to multiple models which implies the ability to use different languages which in turn implies the ability to use different representations. The question, then, is how to make the students see that "two externally different representations indeed represent one and the same 'thing'". (Hillel & Sierpinska, 1994)

Finally, throughout the historical development of linear algebra, mathematicians had to overcome many obstacles and change their way of thinking about certain concepts --i.e., accepting imaginary numbers, quaternions, matrices, and vectors as mathematical entities and recognizing vector spaces. These were people with extensive mathematical knowledge. So, it is only natural to expect difficulties in student understanding of such complex topics, especially with students with a limited mathematical background. How should then linear algebra be taught to promote maximal student understanding?

1.2. Methods of Teaching and Contents of Linear Algebra Courses as Sources of Students' Difficulties

Finding ways to teach linear algebra to promote maximal student understanding is an ongoing investigation, but what is known is that some of the present didactic methods are not effective.

First, as mentioned above, abstract concepts are introduced quickly, not taking into account the students' previous mathematical experience and knowledge. In a first-year linear algebra course, most of the students' mathematical experience has been computational. Which is the reason why, according to Carlson (1993), students usually have no trouble with solving systems of linear equations and using matrix algebra. But when the topics of *subspaces*, *spanning*, and *linear independence* are studied, students become confused and disoriented. "The traditional reliance on the abstract, axiomatic approach in the presentation of the ideas of linear algebra often leaves beginning students with little understanding of the concepts and little idea of how they are related to their previous knowledge of mathematics." (Carlson, 1994, p.371)

When using the axiomatic approach, the final product is taught, while the process is ignored. Since the concepts are introduced in an abstract setting, students usually are not aware of the construction process, although some might understand the resulting abstraction. (Harel, 1989) This creates gaps in the students' mathematical knowledge and prevents them from making connections to their previous experiences, which gives rise to a lack of motivation. Carlson (1993) points out that mathematicians learn in a constructivist way: Starting with a stimulus (a talk, lecture, paper, or thought), they do examples, make conjectures, solve problems, prove theorems and communicate with colleagues. As was seen in the historical development of linear algebra, concepts took years to develop and become accepted in the mathematics community. During this time, mathematicians discussed, argued, and wrote papers about these notions. Most mathematicians had to change long-held beliefs to be able to understand and accept new ideas. So, how can we present to students in one 75-minute lecture the final products of a long process of development and expect them to understand it right away?

Second, linear algebra is presented with little or no examples of its applications, resulting in a loss of student motivation. Part of the problem is that the applications of linear algebra are many and varied (since it is a unifying and generalizing theory). Therefore, the teacher must find examples of applications that are relevant and familiar to the students. By the same token, for the students to appreciate the power of linear algebra, they must be well versed in many fields.

But one has to be careful not to assume that applications alone will increase the students' appreciation and understanding of the abstract nature because

> "a) students do not see the necessity of dealing with abstract concepts for treating a limited scope of situations, and
>
> b) applications may help to illustrate ideas, but do not help one in understanding the construction process of concepts." (Harel, 1989, p.141)

There are so many fields to which linear algebra can be applied that it is impossible to study all in a class. Thus, students are exposed to a few examples of application problems (if any) and then are expected to appreciate the abstract nature of linear algebra that allows them to solve problems that could have been easily answered using more familiar and less complex algorithms and procedures. At the same time, these algorithms and procedures are the final products of a long process of development. Using them to solve application problems will not provide a better understanding of the concepts involved, but instead supplies a motivation (a very important requirement for learning) for the need to use these methods.

Third, linear algebra is taught with no visual interpretation. Except for the unit on directed line segments, all topics are presented algebraically. Harel (1989) refers to results of research projects that indicate both high school and university students have trouble comprehending algebraic systems that cannot be easily represented visually or concretely. Carlson proposes "emphasizing a geometric interpretation of the properties of R^n, eigenvalues and eigenvectors, and orthogonality. This is important, both for its conceptual utility and as a tie to our students' prior experience with geometry." (Carlson, 1993, p.32)

Finally, very often teachers and textbooks of linear algebra ask the wrong questions. Many assigned problems are asked in such a way that students follow a certain procedure or algorithm, instead of testing the students' understanding of the concepts. A typical question, as can be seen later in the chapter, is to prove that a subset of a given linearly independent set is linearly independent. Students are told to solve this problem by showing that the trivial solution is a unique solution of the homogeneous system --i.e., showing that the coefficients can <u>only</u> be zero in the linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots\ldots + k_r\mathbf{v}_r = 0$. Thus, students are taught to follow a certain procedure so that their result is consistent with the definition of *linear independence*. A better question would be to ask to find, if possible, a linearly <u>dependent</u> subset of a linearly independent set. Leaving the students to work on this problem on their own and then discussing with them the results could give the students a better chance to see the difference between a dependent and independent set of vectors.

1.3. Cognitive Sources of Students' Difficulties

It is believed that when students do not understand the mathematical concepts for any of the above reasons, they create for themselves an *obstacle of formalism* . This means that the students are working on the level of the form of expressions, ignoring the "semantics" or the grammar of mathematics. For example, they cannot distinguish between "belongs to" a set ($\in$) and "contained in" a set ($\subset$). To them, $\mathbf{v} \in \{\mathbf{u},\mathbf{v},\mathbf{w}\}$ and $\mathbf{v} \subset \{\mathbf{u},\mathbf{v},\mathbf{w}\}$ mean the same thing. Usually this is witnessed when the students do not understand the concepts involved in a certain problem. In their solutions, they write a lot of mathematical symbols and notations because it looks "mathematical"' even though it does not make sense. Specific examples of this situation will be given in the next section. Student learning is also hindered when students bring into the classroom years of bad habits. First, they learn by memorizing algorithms; thus students learn concepts as being a result of a procedure, instead of understanding the concept itself. Second, students do not like reading mathematical textbooks (although admittedly, some are not well written). So, they just rely on looking at examples and learning the process. Third, students are concerned only if the final answer is correct, but not in why it is; thus gearing their studying towards passing the final exam, whether the concepts were understood or not.

CHAPTER II

The First Experience: Math 204

Introduction

This chapter will look at my experience in teaching a college-level Linear Algebra course, MATH 204 *Vectors and Matrices* course offered at Concordia University. MATH 204 is a college-level course that is a pre-requisite to the first-year university Linear Algebra course.

I am not the author of the course outline, which has been in use at Concordia for the last twenty years. I have taught this course twice; first during the Summer 1996 session and then in the Fall 1997 semester. The materials used in this chapter are from the latter experience.

The chapter will be divided into several sections, including the course content and format, the type of students who take the course and the consequences of teaching different topics. Each section will be analyzed separately, making frequent references to the framework of analysis discussed in the previous chapter. I will show how the three sources of students' difficulties in linear algebra , – i.e., the nature of linear algebra, the way it is taught, and how students learn – interact with each other and I will discuss the resulting problems. I shall conclude the chapter by making recommendations on how the course could be improved.

2.1 Course format and Course content

MATH 204 *Vectors and Matrices* runs for thirteen weeks, with a midterm test given in Week 7 and the last week reserved for review for the final exam. Some sections meet twice a week for a 75-minute lecture (total of 2 $\frac{1}{2}$ hours/week), while others meet once a week for a two-hour session. Depending on the course section, the actual time for teaching <u>new</u> topics is 11 or 11 $\frac{1}{2}$ weeks (22 or 28 $\frac{3}{4}$ hours).

In this short time, the instructor is required to teach the following major topics: systems of linear equations, matrices and matrix algebra, determinants, vector geometry in 2- and 3-space, vectors in R^n, vector spaces and subspaces, linear independence, eigenvalues and eigenvectors, and conic sections.

Comment

There is a problem in the course design and in the teaching of linear algebra if teachers realistically expect students to fully understand so many complex topics in such little time. Not only is the number of topics being taught ridiculously high, but the nature of some these notions is very complex --i.e., vector space, subspace, linear independence. These subjects, which took hundreds of years to develop, are covered in two weeks. These concepts require thinking in terms of algebraic structures, which has yet to develop in the type of students enrolled in MATH 204.

2.2 Type of student

Students who enroll in MATH 204 are majoring in either (a) mathematics, (b) the applied sciences (e.g., engineering, computer science, physics, etc.), or (c) non-math disciplines (e.g., psychology, sociology, history, leisure science, etc.). Students in the math and science oriented programs usually take this course because they have been out of school for a few years and/or they need it to be fully accepted into their university program, whereas the non-math majors take the course as a credit requirement in math for their respective discipline.

Comment

In other words, students enrolled in this course either lack the skills or have not had many recent experiences in mathematical thinking, reasoning, and generalizing. Bombard these students with new and complex topics and one will end up with a lot of confused and frustrated students who have build obstacles for themselves and a bad attitude towards linear algebra.

2.3 Consequences of teaching too many topics

As a result of teaching too many topics in a short period of time, important notions and ideas are covered superficially, while others are omitted completely. For example, the concept of linear independence is introduced with the following definition:

> "If $S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors, then the vector equation
>
> $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots\ldots + k_r\mathbf{v}_r = \mathbf{0}$
>
> has at least one solution, namely
>
> $k_1 = 0, k_2 = 0, \ldots, k_r = 0.$
>
> If this is the only solution, then S is called a ***linearly independent*** set. If there are other solutions, then S is called a ***linearly dependent*** set." (Anton, 1994, p.232)

Meanwhile, because of the shortage of time, the concept of basis is omitted. Although it is possible for students to learn the idea of linear independence/dependence without the notion of a basis, the relevance of the definition is not obvious when basis is not taught. A very common consequence of teaching this definition without the notion of basis is that the students learn (through no fault of their own) that linear independence means that the coefficients are equal to zero in the equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots\ldots + k_r\mathbf{v}_r = \mathbf{0}$, but they do not understand that this is the <u>only</u> solution to this equation. Students also have no idea as to what this result actually means in terms of the relationship between the vectors, and therefore make their own interpretations of the definition. This is illustrated in a student's response to a homework problem.

An example of a student's response

The following homework problem was assigned:

> "Show that if $S = \{ v_1, v_2, ..., v_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S." (Anton, 1994, p.240)

A student responded in this way:

> "if $S = \{ v_1, v_2, ..., v_r\}$[1] is linearly independent
>
> $k_1v_1 + k_2v_2 + ... + k_rv_r = \mathbf{0}$
>
> $k_1 = 0, k_2 = 0, ..., k_r = 0$
>
> Any subset will equal zero because all the k's are 0. This makes the subset linearly independent."

Analysis

From this answer, it was not clear whether the student does not have a clear understanding of the concepts of *sets* and *linear independence/dependence* or if the student understands the concept of *linear independence/dependence* but does not have the ability and training to express his thoughts and ideas in a correct mathematical language. What was meant by "any subset will equal zero"? Is the student able to distinguish between the concepts of *sets, vectors,* and *real numbers*? Is the student talking about the number of elements in the sets? Or is there a reference to the values of the constants k ("because all the k's are 0")? Does the student know that linear independence requires the uniqueness of the trivial solution? Or does it mean that since the coefficients k_n must

---

[1] Because the problem in the book gives a set of r vectors, the discussion of this solution will reverse the usual notation of v_r, where $0<r\leq n$ to v_n, where $0<n\leq r$

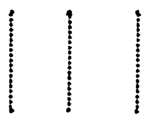
all be zero for the vector equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+.......+k_n\mathbf{v}_n = \mathbf{0}$ ($0<n\leq r$) to hold, then every nonempty subset of S is linearly independent but the student cannot express it correctly?

Follow-up session

To answer these questions, a meeting was arranged with the student. The student was not told about the situation with the homework problem. All that was known was that there will be a few questions asked about linear algebra. The meeting was held two days after the final exam. The student was asked the following question (a variation of the homework question):

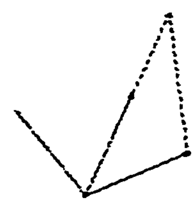
> "Can you give an example of two linearly dependent vectors from a set of three linearly independent vectors?"

At first, the student did not understand the question, which I must admit is rather unclear. I explained the question, but the student did not know how to proceed. So, I asked what linear independence means. The response was that it was vectors that "have nothing to do with each other. No common points," and drew the following diagram:



I, then, told him that it is possible to have linearly independent vectors that start from the same point. The student asked, "You mean like this?" and drew this diagram:

Then, the student proceeded to do the problem of finding two linearly dependent vectors from the three linearly independent ones and started to draw:



I warned the student that the problem involved the space R^3 or higher, but his diagram is in R^2.

I, then, tried a different approach. I asked the student to define linear independence in terms of an equation. The student did not remember and had to look it up in the textbook. During the discussion, I realized that the student had the misconception that linear independence/dependence depended on the value of the expression $k_1v_1 + k_2v_2 + \ldots\ldots + k_rv_r$. If the expression equaled zero, then the vectors are linearly independent, otherwise they are linearly dependent. This finally explained what was previously meant by "any subset will equal zero because all the k's are 0." His reasoning was that since it was given that $k_1=0$, $k_2=0$, ..., $k_r = 0$, then any nonempty subset will make the expression $k_1v_1 + k_2v_2 + \ldots\ldots + k_nv_n$ $(0<n\leq r)$ equal to zero (since the k's are all zero); thus making the subset "linearly independent".

Analysis

From the initial approach to the problem, it is evident that the student is not clear on the concepts of *vectors* and *linear independence*. The student is mistaking *vectors* for *linear equations* and treating *linearly independent vectors* as an *inconsistent linear system*.

Another oversight by the student that indicated a lack of understanding was that the student tried to solve the problem in $\mathbf{R}^2$ instead of $\mathbf{R}^3$. I believe that both stumbling blocks – i.e., (1) the belief that linearly independent vectors are parallel, and (2) not realizing which space one is in – probably could have been avoided if the student had known and seen examples that a basis is a set of linearly independent vectors that generates the whole space, and hence the maximum number of linearly independent vectors in $\mathbf{R}^n$ is n vectors.

Other observations and comments on skipping the notion of a basis

Another problem that could result, if instructors are not careful, from skipping the concept of basis, is encountered during the eigenvalue/eigenvector section. The students are assigned homework problems that ask them to "find the bases of the eigenspaces." Since they have not seen this term before, I have to show them how to find the bases without really explaining what a basis is. This problem could be avoided by carefully choosing homework problems that do not have the word "basis" in them. Unfortunately, the majority of the problems in most texts have this word in the eigenvector problems since this unit is introduced after the concept of basis had been studied in the text.

A consequence of telling the students how to solve the problem without explaining the concept of a basis is that the students will memorize the steps of a typical problem without comprehending what they are finding. I believe this will create problems for the students in future linear algebra courses when they are formally introduced to the concept of a basis. By then, the students will probably have the pre-conceived idea that a basis is the result of the trick of substituting "1" for the free variable of an eigenvector. To them, a basis becomes associated with the eigenvalue problem and is considered to be an answer to a school exercise rather than a general concept that is encountered in many contexts – e.g., eigenspaces, basis of $\mathbf{R}^n$, bases of general vector spaces, basis of the solution space of a homogeneous system of equations. This is analogous to the problem faced by high school students when finding the solution to a system of linear equations. They associate the solution as being the x and y values resulting from the elimination method learned in class, instead of the idea that the x and y values are a pair of numbers that satisfies both equations, and are represented by the point of intersection of the two lines in the graphical representation.

This situation does not create conceptual problems for the students, but instead exploits a problem in how students learn math. They believe that math is about processes and procedures, instead of ideas and concepts, and thus all they have to do is memorize the steps to a problem.

2.4 Consequences of the axiomatic approach

Another topic that is introduced quickly and covered in one week is the axiomatics of vector spaces. A *vector space* is defined as a structure composed of a set in which two operations are defined, called vector addition and scalar multiplication, and which satisfy certain given axioms.

Examples of students' responses

A typical exercise would give a certain set of vectors and define a set of operations and ask the students to determine whether the set is a vector space:

> "Determine whether the set of all pairs of real numbers (x,y) with operations (x,y) + (x', y') = (x+x',y+y') and k(x,y) = (kx,y) is a vector space with respect to those operations." (from the Final Exam)

A couple of students' responses:

1. "(x+y) + (x',y') = (x+x',y+y') & k(x,y) = (kx,y)
   k(x,y) = (kx,ky) fails this axiom"

2. "(x+y) + (x',y') = (x+x',y+y')
   if k(x,y) = (kx,y)
   (kx,y) + (x',y') = (kx+x',y+y')
   ∵ ku = k(u) not working
   ∴ k(x,y) = (kx,y) is not a vector space."

Analysis

These examples of student responses show that the students do not understand the concepts of *vector space*, *computation rules*, and *axioms*. The first student thinks that the property k(x,y) = (kx,ky) is one of the axioms that determine a vector space, while the second student thinks that the property k(x,y) = (kx, y) is the vector space being tested. It could be argued that the second student does not know the proper language in which to write his conclusion, but the previous steps show that the student does not know the axioms and does not know how to check them. Some might say that the student does not memorize well and has weak algebraic skills. But I say, how could one memorize a list of meaningless rules? Why is it important for vectors to follow these rules? Why is it important for something to be a vector space?

The concept of *vector space* is so abstract that the students cannot relate to it. They are still in the process of getting to grips with a vector being a directed line segment that the idea of a vector space other than R^n – e.g., nxm matrices – is very confusing. On top of that, after spending a few weeks learning how to add vectors and multiply by a scalar in R^n, they are now being told that they have to use new rules for each problem. To make things worse, none of the examples in the assigned problems turn out to be vector spaces --i.e., the students do not see an example of a vector space with non-standard operations. I do not know how students are expected to understand this difficult and abstract topic when they have nothing to draw on from all of their previous knowledge. This is definitely a flaw in the teaching of linear algebra, and I believe that vector spaces should not be taught at this early level.

2.5 Proofs

Another concern in MATH 204 is proofs and their role in the course. Since their secondary or college math courses, students have always disliked proofs because they have never seen or have ever understood their purpose. They believe that the role of a proof is to verify that a statement is true, and if they already believe that it is true (there is no need to doubt a theorem), then there is no need for a proof. Therefore, students already have a negative attitude towards proofs.

MATH 204 is the first course in which students experience different types of proof. Up to this point, geometric deductive proofs and verification of trigonometric identities are the only kinds of proof that they have substantially worked with, although they might have occasionally seen other types. In both cases, the objective is clear. In the geometric proofs, the 'given' and 'prove' statements are clearly defined and in the trigonometric identities case, the equation to be verified is given. But in linear algebra, students encounter new and different types of proofs. The different methods of proving is something new to them and making it even worse in their minds is that there is no preset algorithm for finding and writing the proofs. Consequently, they have to decide which method to use each time. This again highlights the problem in student learning in that they prefer to memorize steps and procedures instead of tackling a problem by using their knowledge of mathematical concepts and ideas.

Students have trouble deciding which method of proof to use because they do not know how to extract information from the question and they also cannot distinguish between (a) proving by satisfying certain definitions and axioms and (b) proving conditional statements – e.g., if... then... statements.

Proving by satisfying certain definitions and axioms

During the course, students encounter two different variations of this type of proof:

(1) proving that a given object belongs to a given category of objects by showing that it satisfies the defining property of this category: --e.g., *Prove that, if B is any matrix then* BB^T *is symmetric.*

To prove symmetry, one sets up the proof to satisfy the definition of symmetry – i.e., $(BB^T)^T = BB^T$. Most students have no trouble with this type of proof.

(2) proving a structural property of operations on a certain category of objects (e.g., that some two operations on matrices commute): – e.g., *prove* $(A^T)^{-1} =$

$(A^{-1})^T$ for any invertible matrix A .

The difficulty with this type of proof is that unlike trigonometric proofs, the equation presented to the students is not the one that they use in their proof. They have to set up a new equation –i.e., $A^T(A^{-1})^T = I$, – according to the definition of the inverse of a matrix. This task becomes even more difficult for the students if they do not realize that the statement $(A^T)^{-1} = (A^{-1})^T$ refers to the inverse of the transpose of A.

Proving conditional statements

When proving conditional statements, most students do not realize that what they have to prove depends on the given conditions, and if those conditions are changed, then the whole problem changes. This is evident from the final exam.

Examples of students' responses

The following problem was given:

> "Let **u**,**v**,**w** be vectors of a vector space V over R, show that **u-v**, **u-w**, **v+w** are linearly dependent."[2]

There were two misprints in the problem. It should have read "...., show that **u-v**, **w-u**, **v-w** are linearly dependent". In this case the three given vectors are indeed dependent:

---

[2] Teachers have to be consistent with the terminology being used. Note that the problem talks about 'linearly dependent vectors', whereas the textbook defines a 'linearly dependent **set of** vectors'

their sum is the zero vector. But many students did not question the fact that it is not possible to do the question as it was written and just assumed that **u**, **v**, **w** are linearly independent because all of their homework problems had this assumption. The students' responses showed many typical mistakes, misconceptions, and obstacles that are commonly encountered in linear algebra.

One such obstacle is the *obstacle of formalism*. The following examples of student answers illustrate this:

1. "since $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{0}$,
   $k_1, k_2, k_3 = (0,0,0)$
   so $k_1=0$, $k_2=0$ and $k_3=0$
   $\therefore k_1(\mathbf{u} - \mathbf{v}) + k_2(\mathbf{u} - \mathbf{w}) + k_3(\mathbf{v} + \mathbf{w})=0$ are linearly dependent."

2. "let's say $\quad$ u-v = u-w
   $\Leftrightarrow$ u-u -v = -w
   $\Leftrightarrow$ u-u = v+w
   But u-u = 0
   therefore they are linearly independent."

3. "For the vector to be linearly dependent
   $k_1 + k_2 \ldots\ldots\ldots + k_r \neq 0$

   $v_1 = k_1v_1 + k_2v_2 \ldots\ldots\ldots\ldots k_rv_r$
   .
   .
   .
   .
   $v_r = k_1v_1 + k_2v_2 \ldots\ldots\ldots\ldots\ldots k_rv_r$
   where k is not equal to 0

   let's take some scalar t
   so,
   $\Rightarrow t_1u + t_2v + t_3w = 0$
   $\Rightarrow t_1(u\text{-}v),\ t_2(u\text{-}w),\ t_3(v\text{+}w)$
   $\Rightarrow t_1u\text{-}t_1v,\ t_2u\text{-}t_2w,\ t_3v\text{+}t_3w$ are linearly dependent if
   t_1 & t_2 & $t_3 \neq 0$

   $$\Rightarrow R_1+R_2 \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow R_2+R_3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} .$$

Analysis

The first student assumes the linear independence of the vectors **u**,**v**,**w**, just like in the homework problems. The obstacle of formalism is seen in the statement $k_1, k_2, k_3 = (0,0,0)$. The student's conclusion satisfies the definition of linear independence, but the student declares the equation linearly dependent (another example of the obstacle of formalism) because that is what the question asks. The student probably believes that if one follows the procedure that is learned in class, then the right answer will be obtained.

The second student assumes the equality of the vectors. Algebraic manipulations are made and an answer of "0" is obtained. The student then declares the linear independence of the vectors because the student remembers "something" has to equal to "0" in the definition.

In trying to explain his reasoning, the third student also shows symptoms of the obstacle of formalism. He lists r equations in r unknowns. He writes that the <u>sum</u> of the k's cannot equal zero for linear dependence to be true. The 'implies' symbol ($\Rightarrow$) is used repeatedly, and equations are written that do not make sense. The student states what has to be shown – e.g., t_1 & t_2 & $t_3 \neq 0$ – and then sets up the matrix to be reduced. Using elementary row operations, the student correctly shows that the vectors are linearly independent (because of the misprint in the question). The student is probably taken aback by the result and stops without making a conclusion.

In all three examples, the students have a vague recollection that in linear independence/dependence proofs, it has to be shown that the constants are equal to zero, but they do not really understand the concept (although it could be argued that the third student knew what he/she was doing). So, they develop an obstacle of formalism by writing statements and symbols just to make the solution look mathematical.

Other concerns arising from proofs

Proofs also bring out another difficulty in student learning --i.e., *generalizing*. Many students have trouble distinguishing between proving a general statement and showing that the statement is true for some particular values of the variables. The following examples of students' answers show this fact (along with the obstacle of formalism and operation mistakes):

Examples of students' responses

The problem is the same as the erroneously formulated one discussed in the previous section:

"Let u,v,w be vectors of a vector space V over R, show that u-v, u-w, v+w are linearly dependent."

The following are samples of students' answers:

1. "Let u=(1,2,1), v=(2,1,1), w=(1,1,2)
   u-v = (1 • 2) – (2 • 1) – (1 • 1)
   = -1

   Now u-v = v-u
   So v-u = (2 • 1) - (1 • 2) – (1 • 1)
   = -1

   u-w = w-u
   (1 • 1) – (2 • 1) – (1 • 2) = (1 • 1) – (1 • 2) – (2 • 1)
   -3 = -3

   v+w = w+v
   (2 • 1) + (1 • 1) + (1 • 2) = (1 • 2) + (1 • 1) + (2 • 1)
   5 = 5

   If u-v, u-w and v+w were linearly independent the answers would of been =0, but it is not."

2. "if we said that u, v, w are not zeros
   ⇒ (u-v) and (u-w) and (v+w) are linearly dependent iff (if and only if) the result of these operation are zero

   For example

$$\underset{u}{\begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}} - \underset{v}{\begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

u-w

$$\begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Analysis

The first student assigns values to the vectors, but then uses the dot product instead of vector addition. On top of that, the student is showing examples of the commutative property. Since the results of the calculations were not "0", then the vectors are declared linearly dependent. Again the student vaguely remembers that "something" has to be equal to "0" for vectors to be linearly independent.

The second student does the opposite. The student also remembers that "something" has to equal to "0", but thinks that this proves linear dependence. One interesting surprise is that this student used 2x2 matrices for the vectors **u**, **v**, **w**. This is something that was rarely seen in class and homework assignments.

The above two examples and the previous three all support the argument that the introduction of the definition of the linear independence out of any context of its possible uses leads to major misconceptions by the students. The five students had five <u>different</u> interpretations of linear independence and what has to be equal to "0". Only one student was close to the correct meaning.

2.6 Conclusions and recommendations

All of the discussed examples demonstrate how the three sources of difficulty --i.e., the nature of linear algebra, the way it is taught, and how students learn -- are intertwined. It was shown that when complex and/or abstract topics are taught in a very short period of time, sometimes using poorly chosen and badly written questions, the concepts are lost on the students and obstacles of formalism are formed.

Coming into this course, most students already have weak study skills and habits in mathematics. These students do not attempt to read the text and if they do, it is only to look at solved examples, instead of reading about the concepts involved in solving a particular problem. Therefore, when students are required to learn very complex and abstract concepts (the nature of linear algebra) in a badly designed course, their study skills do not improve. On the contrary, this only supports their already poor habits by gearing their studying towards passing the Final Exam in lieu of understanding the concepts. They just study and memorize previous exams. Every semester, the Final Exam has had for the most part the same type of questions as the previous semester -- e.g., solve the system of linear equations, find the determinant, find the equation of the plane that passes through three given points, find the eigenvalues of a matrix, etc. This is clearly another flaw in the course design. Consequently, because of the shortage of time and the resulting lack of student understanding in such a setting, teachers tend to change their teaching philosophies and start to cater to the students' need to pass the Final Exam instead of promoting students' understanding of the mathematical concepts.

Therefore, in order to promote student understanding in college-level linear algebra courses, I am proposing a change in the course content and in the way this material is taught. I support the proposals in Carlson, Johnson, Lay & Porter (1993), but they are geared towards a first-year university course and MATH 204 is a college-level course.

I believe, just as Carlson et al. (1993), that there is too much emphasis on axiomatics at this early stage of linear algebra. Students have trouble understanding and visualizing these abstract concepts, and the questions being asked of them do not help their cause. Some people might argue that it is possible for students to abstract at this level if they are given the proper tools. But I think that it is not a question whether students can or cannot abstract, but it is a question of whether there is a need for them to abstract. By requiring the students to abstract, will they be better off in their understanding of math and have better tools to solve their problems with? Or is it just creating unnecessary confusion and frustration for most students and building a bad attitude towards linear algebra, and math in general? I believe in the latter.

I think that at this level the emphasis should be on the practical aspects of linear algebra, such as systems of linear equations, matrices and matrix algebra, determinants and vectors in R^2 and R^3. Half of the course time is already being spent on the first three topics, but a lot more time is needed to work with vectors. This extra time should be used to study vectors both analytically and geometrically and more importantly on the connection between the two representations. This will allow the students to get a better feel and understanding for vectors.

Understanding the notion of vector in different representations will help in learning the concept of linear independence. As was seen in the examples in this chapter, the concept of linear independence is practically non-existent in students when introduced as a definition with the notion of a basis. I suggest teaching the topics of basis and linear independence after the notion of vector has been properly introduced both analytically and geometrically. Students should be given a chance to slowly construct their understanding through a broad experience in meaningful contexts. This goal cannot be achieved through giving the students a formal definition followed by a series of exercises of verification if a given object satisfies this definition.

The way MATH 204 is now designed, the proofs required of the students are mathematically useless and meaningless: they are no more than another type of exercise, serving only the institutional purposes of evaluation and selection of students. Even though it was shown that part of the difficulty is the problem of how the students themselves learn math, using a "constructivist" approach could make students see the need for proofs. The activity of proving could be used in teaching so as to actually enhance the students' understanding, if proofs appeared as tools in solving investigative problems and trying to find out for oneself, (not for the mark or the teacher) if a given conjecture is true. Questions starting with "Is it possible that ?", leading to conjectures and their refutation or confirmation, could have a better effect on students' understanding than the usual "prove that..." questions.

CHAPTER III

The Second Experience: An Experimental Introduction to Linear Algebra with *Cabri*

Introduction

In view of the difficulties in the learning and teaching of Linear Algebra discussed in Chapter I and the availability of mathematical software, a research project was designed and implemented to test an alternative way to present certain topics. Anna Sierpinska, Tommy Dreyfus and Joel Hillel started work on the project in September 1996. I joined them in January 1997. The goal was to design about five to seven sessions that would be tested on two students in March 1997. I was to be the instructor in the experiment. This is the second teaching experience that I will analyze.

The design of the experiment took into account the previously discussed a priori sources of student difficulties in learning Linear Algebra. Reasons behind decisions made in the design have been discussed in Sierpinska, Dreyfus & Hillel (1999) and will not reviewed here.

The goal of the project was to expose the students to the notions of vectors, linear transformations and eigenvectors in the dynamic geometry environment of *Cabri-géomètre II*. In this environment the 2-dimensional vector space was represented by an arrow, the *Cabri* "vector", stemming from a distinguished point "O", called the origin. The arrow was meant to represent the arbitrary element of the 2-dimensional vector space. Thus, geometrically, a vector was given by a position with respect to the origin, but this position was not immediately described by a pair of numbers; there was no a priori system of coordinates. The two vector space operations, vector sum and scalar multiplication, were introduced in geometric terms as the diagonal of the parallelogram formed by the two vectors and a dilation of a vector by a factor k, respectively. Linear transformations were introduced as those that conserve the operations of vector addition and dilation. Eigenvectors appeared implicitly in questions about the existence of invariant lines of linear transformations. The introduction of the arithmetic representation of vectors as strings of numbers was to be done later in the sessions. in the context of the notion of "the coordinates of a vector with respect to a basis", where "basis" referred to a pair of non-collinear vectors on which axes could be constructed (using the command "New Axes" in Cabri). The notion of basis was not to be isolated as a separate concept during the sessions. Matrix representations of linear transformations were to be introduced after the students would have understood that, in the vector plane, a linear transformation is completely determined by its images on a pair of non-collinear vectors.

> "The idea was to separate the notions of vector, linear transformation and eigenvector from their representations in the form of arrays of numbers with the purpose of facilitating the students' understanding of these representations as relative to the choice of basis." (Sierpinska et al., 1999, p. 12)

In view of the complex nature of Linear Algebra, the project was trying a coordinate-free geometric approach to the introduction of the concepts, thus leading the students via a route favored by Leibniz. Also, with this approach and the aid of *Cabri*, it was hoped that the didactic situations would shift from the traditional Socratic method of teaching to a more discovery-oriented atmosphere, creating a learning environment in which the students would be less likely to develop the obstacle of formalism.

The approach used in this experiment is very different from the one used in MATH 204, however, both designs use a structural approach to the concepts of Linear Algebra. In particular, both use axiomatic definitions; the definition of *vector space* and *linear independence* in MATH 204, and the definition of *linear transformation* in the experiment.

In this chapter, first a summary of the contents of the sessions will be given, followed by a discussion of the students' backgrounds and a description of the physical conditions of the experiment. The main body of the chapter will be a detailed analysis of each session. Assessment of the experiment concludes the chapter along with some recommendations.

3.1 The general overview of the experiment

The experiment consisted of seven two-hour sessions. The first session introduced the students to a representation of R^2 in *Cabri*, labeled CR2. Session II presented linear transformations while the third session discussed the invariant lines and characteristic values of a linear transformation. Session IV began with the introduction of the coordinates of a vector. The rest of the session, along with Session V, was devoted to defining a linear transformation by its values on a basis. Arithmetization began in Session VI when linear transformations were represented by an array of numbers. In Session VII, the students were tested on the knowledge they had acquired.

In this thesis, I will focus on the students' understanding of vectors, operations on vectors and linear transformations only.

The aims of Session I were (1) to build a model, called CR2, for R^2 in *Cabri*, (2) to define the operations of CR2 (dilation and vector sum), (3) for students to conclude that any vector can be obtained by using the two operations of CR2, and (4) for students to develop some notion or intuition of coordinates of a vector in an arbitrary basis. The session ended with the problems of composition and decomposition of vectors. In the first problem, the students were expected to use both operations and to drag the vectors so that they can conclude that any vector in CR2 can be constructed from any two non-collinear vectors. In the second problem, the students were asked to decompose a given vector $\mathbf{v}$ into a linear combination of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. It was hoped that they would develop some preliminary notion of coordinates of vector in an arbitrary basis.

The aims of the second session were (1) to clarify certain concepts and ideas that might have been previously misunderstood, (2) to introduce the notion of linear transformation, and (3) for students to come up with an algebraic definition for linear transformation. A tree diagram was produced that showed the different ways in which a transformation can behave with respect to vector operations: conserve them or not. The students were then engaged in verifying a series of examples of transformations for their behavior with respect to the operations. The terminology of "conservation of operations" was not used; the language was guided by the paths on the diagram and led to informal phrases such as "the transformation goes from vectors v and w to v+w by vector sum / not by vector sum", and "the transformation goes from v to kv by dilation with the same factor k / not by dilation / by dilation but not with the same factor". A linear transformation was identified as a special case among these possible behaviors.

The aims of the third session were (1) to clarify any misunderstood concepts and ideas, and (2) for students to be able to verify if a certain transformation is linear. Using the testing procedure from the previous session, the students were to find out if a *reflection* and a *shear* were linear transformations.

The aims of the fourth session were (1) to review what and how to check if a given transformation is linear, (2) to define a system of axes in Cabri, (3) to define the coordinates of a vector in this system, (4) to calculate the coordinates of a vector sum and a dilated vector, and (5) to define a linear transformation by its values on a basis. The students redid the linearity test on the shear transformation because of a technical glitch during Session III. After being introduced to representing vectors in a coordinate system and applying the two operations in this environment, the students were given the problem of constructing vector T(v) after they had randomly placed the five vectors $\mathbf{v}_1$, $\mathbf{v}_2$,

$T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $\mathbf{v}$ on the screen, and asked to assume that T is a linear transformation. This problem will be labeled the "Linear Extension Problem" (or LEP), because, in fact, it consists in linearly extending a transformation of a basis to the whole vector plane.

The aims of the fifth session were (1) to clarify certain concepts and ideas that might have been previously misunderstood, (2) to continue laying the ground for an analytic-arithmetic representation of vectors and linear transformations, (3) for students to be able to find the linear transformation, given the images on a basis, and (4) for students to be able to define a specific linear transformation by configuring its images on a basis. The students went through a series of activities that cleared up misconceptions that were picked up along the way. They were then given problems of configuring vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, and $\mathbf{w}_2$ (where $\mathbf{w}_1$ and $\mathbf{w}_2$ are assumed to be the images of $\mathbf{v}_1$ and $\mathbf{v}_2$ under a linear transformation) so that a transformation of a certain kind (e.g. a projection, or a shear) or having certain properties (e.g. a single invariant line with a given characteristic value) would be obtained.

The aims of the sixth session were to clarify any misunderstood concepts and ideas through a set of activities. These involved the notion of what defines a specific linear transformation.

The seventh session was a test session and the instructor was not present. In the first part of the session the two students worked on a problem together. They had to verify a certain transformation for linearity and find its invariant lines. After that they were asked to write individual reports of their solution. In the second part of the session the students worked on a similar problem individually, in separate rooms. The aim of this arrangement was to find how much each of the students can do on his or her own without the support of the interactions with an instructor and another student.

3.2 Detailed analysis of the experimental sessions

The focus of my analysis will be on the concepts of *vector* and *linear transformation*. As mentioned in Chapter I, the analysis will be using an "interpretive" approach, making the experiment open to different interpretations. Therefore, I tried to present my analysis in such a way that it is possible for the Reader to make their own interpretation. The discussion of each worksheet, problem and activity is broken into three parts:

1. The Design section defines the problem and the researchers' expectations of how the students will solve it.

2. The What happened part describes the "action" from my point of view. A lot of detailed quotations and conversations are included. This is to try to put the Reader in the frame of mind of the situation. This way the reader gets a better understanding of my analysis and is put in a position to make his/her own interpretation and critique. I also wanted to show the difficulties, be they technical, conceptual, or physical, that are encountered during the implementation of research projects. This is a side of research that is usually not portrayed in the final reports.

3. The Analysis section has two perspectives: (a) at the time of the experiment and (b) after a detailed analysis of the transcripts and other material. When there is a discrepancy between the time frames, it is pointed out.

Before getting to the analysis, an introduction to the students' backgrounds and a description of the experimental setting are in order. The two students, whom I shall call Jack and Jill, had finished taking the MATH 204 course, previously described, in December 1996. Both had received a B grade (equivalent to 73% –77%). Jack was in an engineering program, and Jill was in liberal arts.

The experiment took place in a small room. The computer was on one side of the room, facing a wall, with three chairs in front of it. Directly behind the chairs, there was a small table that was used for discussions not involving the computer. When the students turned around and sat at the table, their backs were to the computer and they faced a small board, hung on the opposite wall. Next to the board, there was a desk, at which I sat when the students were required to work on their own.

The video camera was placed between the table and the board and off to the side (in front of the desk). There were two audio tape recorders placed next to the computer. When the students sat at the small table, one tape recorder was also transferred there.

The sessions took place in March 1997, every Monday and Wednesday. The sessions started on a Monday and also ended on a Monday, seven sessions later.

During the sessions, besides the two students and myself, there was at least one other person in the room. Tommy operated the camera every Monday, while Anna did it every Wednesday. There was one session when both were present at the same time. On two separate occasions, there was a different third observer each time. The students were introduced to every person in the room.

Most of the interaction was between the students and myself, while occasional interjections were made by Anna and Tommy. The third observer and the students did not communicate.

3.2.1 Session I

After months of discussions, preparations and speculations on how the students could possibly react to the planned activities and problems, the day finally arrived when all of our theories began to be tested. I was excited about this new approach to teaching Linear Algebra; after all it helped <u>me</u> during the design sessions to get a clearer understanding and a better feel for some of the concepts. So why shouldn't it do the same to others? But along with every exciting situation, there is also an element of nervousness. Because we were testing new teaching approaches to enhance the learning of Linear Algebra, I did not want to ruin the results by telling the students too much or not enough information during our interactions.

When I arrived at the testing room, the equipment (computer, video camera, audio recorder) was already set up. The two students, Jack and Jill, soon arrived. After the initial introductions and paper work (the reading and signing of the consent forms by the students), we were ready to get started.

Part of the intentions of the first session was to build a model called CR2 for R^2 with *Cabri* and then to define its operations. These aims were met by first explaining to the students on the board that CR2 consisted of a *Cabri* screen with a fixed point O called "the origin", and two types of objects. The first type, called *vectors*, are represented by arrows starting at the point O, while the second, called *numbers*, or *scalars*, are represented by the coordinate of a moveable point on a separate number line (see Fig. I-1). The students then learned how to draw vectors in *Cabri* and create the number line using a macro named SCALAR. When the students were asked to sit at the computer, it was decided that Jack would control the mouse and that Jill would read the instructions from the worksheets.

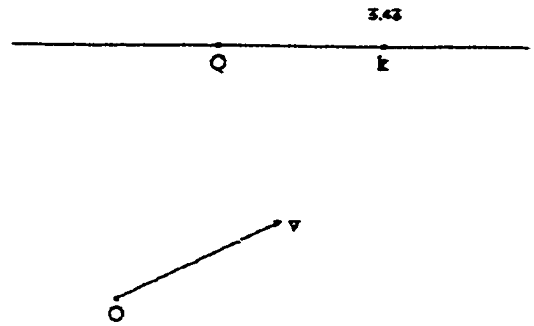


FIG. I-1: **Visual representations of a *vector v* and a *number* k in CR2**

Going back to the board, I introduced the two operations of CR2, *vector addition* and *dilation*. I defined the vector sum **v+w** as the vector that is represented by the diagonal of the parallelogram built on **v** and **w** and the dilated vector **kv** as the vector that lies on the same line as **v**, whose orientation depends on the sign of the scalar k, where the value |k| is the ratio of the lengths of kv to **v** (see Fig. I-2). The students seemed to have no trouble in completing the worksheet in which they learned how to use the *Cabri* commands VECTOR SUM and DILATION in two separate exercises and solve a problem on finding the value of the dilation factor k.

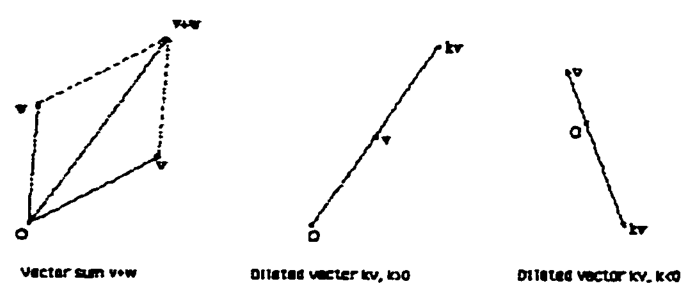


FIG. I-2: Operations in CR2

At this point, we were an hour into the session. I was thinking that everything is going quite smoothly. In fact, the students were mastering the *Cabri* commands quicker than I expected[1]. And then it happened – trouble – Worksheet #3!

Worksheet #3

The second set of aims of this session were for the students to conclude that any vector in CR2 can be obtained from any two given fixed non-collinear vectors by using the two operations and to start developing some kind of notion or intuition of coordinates of a vector in an arbitrary basis. In Worksheet #3, the students were required to solve two problems. The first involved the *composition* of vectors, while the second required the *decomposition* of a vector into a linear combination of two given ones.

Problem 1

Design

Problem 1 asked the students to name the vectors that can and cannot be obtained from (a) one given vector and (b) two given vectors "through the operations of vector sum and scalar multiplication".

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[1] One of my main concerns for introducing computers in a mathematics classroom has always been that a lot of time would be wasted while the students learned the commands and/or language of the software; until now.

The problem was not expected to be solved by using *Cabri* constructions. It was hoped that the students will be able to extrapolate from the experience they had had in visualizing each of the operations in *Cabri* and imagine what vectors could be obtained through these operations starting from one or two vectors. But the students were not told to not use the computer. In fact, they were free to play with some representations of, say, the vector sum of two vectors with the same direction (obtained by dilating the same vector), just to get some initial ideas from which to generalize to the case of dilating by all possible scalars and adding pairs of all possible resulting vectors. It was hoped that the students would eventually come to the conclusion that any vector in CR2 can be obtained from any two given non-collinear vectors through a sequence composed of the two operations.

Problem 1(a) – What happened

The students may have thought that this is an exercise in Cabri constructions and not a theoretical question, for they immediately turned to the computer and, starting to answer part (a), they proceeded to construct the origin O with vector **v** stemming from it. Then they drew a line through O and **v**. Another vector **w** was drawn with its endpoint on the line. Finally, they measured the lengths of vectors **v** and **w** (see Fig. I-3).

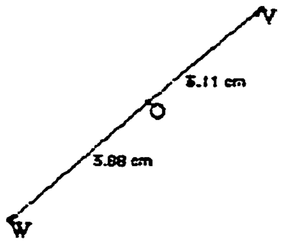

FIG. I-3: Students' diagram for the solution of Problem 1(a)

<u>Problem 1(a) – Analysis</u>

In a sense, the students' solution was correct: From one vector v, by dilation and vector sum any vector w with the same direction can be obtained. So it is, indeed, enough to draw a line through O and v and put a vector with its endpoint on this line. However, this reasoning introduces an object from outside CR2: the geometric line. One would rather expect the line in the direction of v to be defined as a result of operations in CR2, as the set of all vectors of the form kv, where k is any scalar. In the students' solution, the operations and the structure of CR2 are completely lost. Some of this structure could have appeared in the solution if the students had tried to represent w as a multiple of v. The students appeared to do one step in this direction: They measured the lengths of v and w. But they stopped there. Maybe they were just remembering measuring the lengths in a previous exercise (where the equality of vectors was discussed).

<u>Problem 1(b) – What happened</u>

In answering part (b) the students drew vectors **OV** and **WU** from different origins. They were not supposed to do that! Then I thought that it would be interesting to see what is going to happen next. They found the vector sum **U+V** relative to point **O**. Then, the number line was constructed, a point placed on **OV** (no idea why; it could have been a clicking mistake), and finally, vector **OV** was dilated by a scalar (see Fig. I-4).

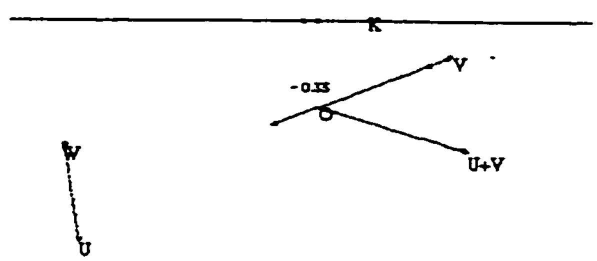

FIG I-4: Students' diagram for the solution of Problem 1(b)

Problem 1(b) – Analysis

In their procedure, the two operations were used separately. There was no attempt to combine the vector sum and dilation; thus indicating that the concept of a linear combination was non-existent. In retrospect, analyzing my own understanding during the linear algebra courses that I took, a linear combination was the expression $a_1\mathbf{v}_1+a_2\mathbf{v}_2+\ldots+a_n\mathbf{v}_n$. From the given, I was able to solve for the missing "values" (e.g., given the independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and vector $\mathbf{v}$, find $a_1, a_2, \ldots, a_n$ such that $v = a_1\mathbf{v}_1+a_2\mathbf{v}_2+\ldots+a_n\mathbf{v}_n$). But it had no meaning outside this type of exercises. It was not until I started working with vectors visually that I grasped the concept. Also to be fair to the students, we might have expected too much of them. We jumped from Worksheet #2, in which they learned about each operation separately, to Worksheet #3, in which they were expected to form linear combinations. Although linear combinations were never mentioned as a name for an object, it was assumed that that the students will discover that they could produce them as a sequence of the two operations.

I believe that we should have expected that the students will want to solve the problem with a substantial aid of *Cabri* and should have provided them with the possibility of creating linear combinations on the screen. For this purpose they would need two independent variable scalars, so putting two number lines on the screen with the SCALAR macro would have been helpful.

The other interesting point observed from their solution was the use of separate origins to draw vectors. I think that it was only natural for the students to be curious about vectors starting from different points. In a way, there was a contradiction in the experiment design. One of the main purposes of the experiment was to develop the synthetic thinking of the students by freeing them from the constraints of coordinates. But then, we turn right around and require that all vectors start from one given point.

Problem 2

Design

Problem 2 required the students to open a file with the configuration in Fig. I-5 and to decompose the vector **v** into a sum of the multiples of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ –i.e., find the numbers a_1 and a_2 such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$. The simplest possible configuration was given with **v** between $\mathbf{v}_1$ and $\mathbf{v}_2$ in the first quadrant.

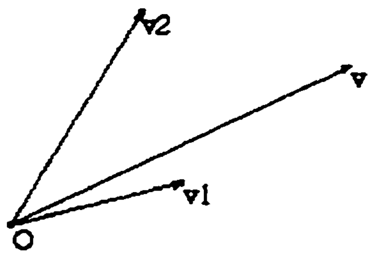

FIG. I-5: Given configuration for Problem 2

We expected them to draw the parallelogram with vector **v** as its diagonal and its sides along vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. After measuring the lengths of the sides, they would divide the results by the lengths of $\mathbf{v}_1$ and $\mathbf{v}_2$ to obtain the absolute values of a_1 and a_2. The signs of the coefficients would be decided to be positive in view of the position of v.

<u>What happened (part I)</u>

The students started by measuring the lengths of $\mathbf{v}_1$ (2.83 cm), $\mathbf{v}_2$ (4.19 cm), and **v** (6.13 cm). They did not know how to proceed using *Cabri*. During their discussion, Jill suggested several times to "put $\mathbf{v}_1$ on the v-line". So, when they resorted to pen and paper, Jack and Jill wrote:

$$v_1 = 2.83\ cm$$
$$v_2 = 4.19\ cm$$
$$v = 6.13\ cm$$

$$a_1\,(2.83) + a_2\,(4.19) = 6.13$$

$$a_1 \quad \frac{6.13}{4.19} = 1.46/2$$

$$a_2 \quad \frac{6.13}{2.83}$$

Analysis (part I)

The students' behavior brought to our attention another unanticipated problem. The students, especially Jill, considered vectors as lengths without direction. In other words, direction does not matter. One can move a vector around a circle with radius equal to its length and centre O and still have the same vector. So, her solution made perfect sense. And since both $a_1\mathbf{v}_1$ and $a_2\mathbf{v}_2$ had to add up to $\mathbf{v}$, she divided a_1 and a_2 by two.

What happened (part II)

When the values did not work in their equation, Jack realized that a combination of the operations is in order when he stated that "$\mathbf{v}$ is not $\mathbf{v}_1$ plus $\mathbf{v}_2$ but a factor times $\mathbf{v}_1$ plus a factor times $\mathbf{v}_1$". The students continued by drawing the lines through $\mathbf{v}$ and parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ and later the line through O and $\mathbf{v}_1$ to complete a parallelogram. They added point P at an intersection and measured the sides (see Fig. I-6). Jack calculated the values of a_1 and a_2 correctly, but then again resorted to the idea of vectors as lengths by writing "0.42(4.19) + 1.69(2.83) = 6.13". Since the equation did not work out, Jill decided to keep the value of a_1 and to solve for a_2 –i.e., "1.69(2.83) + a(4.19) = 6.13".

Analysis (part II)

As stated above, a major misconception that was discovered from these problems was the students' perception of vectors. To them, vectors have lengths only. Their direction is implicit. I believe that part of this problem is that in our everyday activities and language, people speak of vectorial quantities as magnitudes only. For example, velocities are treated as speeds. Also , people talk about the amount of force being applied, but not in which direction. If they happen to mention a direction, it is usually a general one such as up, down, forward, or backwards. Even then, I do not think that they realize that the direction and the amount of force are together as <u>one</u> entity.

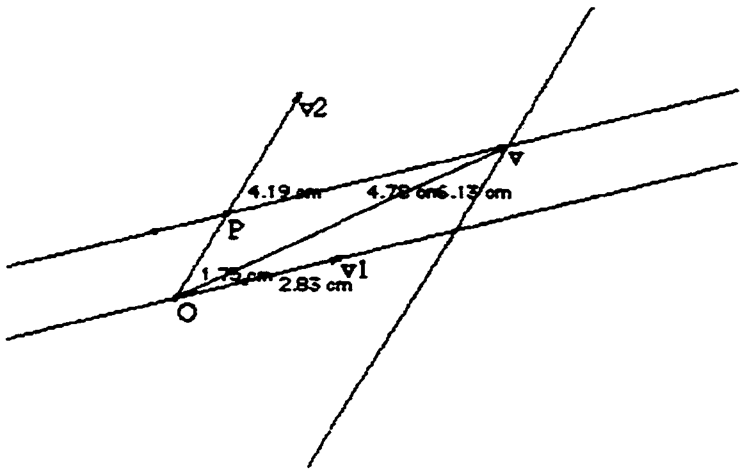

FIG. I-6: Students' diagram for the solution of Problem 2

But again, was it realistic for us to expect the students to understand these concepts? At the beginning of the session, they were introduced to two types of objects, *vectors* and *numbers*. It was shown that the macro SCALAR generates the *numbers*, with the value printed on the screen. Although the concept of the <u>direction</u> of a *vector* was talked about implicitly, the students were only shown how to measure the <u>length</u> of a *vector*. After going through several exercises of calculating the value of the dilation factor k on the screen, they were given a problem that asks to find the value of the scalars that make the linear combination $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ true for a given $\mathbf{v}$, $\mathbf{v}_1$, and $\mathbf{v}_2$. I believe that it is only natural for students to plug in numbers into an equation. Since Jack and Jill were not shown how to represent vectors "numerically" (one of the experiment's objectives was to introduce coordinate–free vectors), then it is reasonable for them to use the lengths of the vector in the equation. Two questions are raised: Are the students completely missing the concept of a vector or do they understand that a vector has both length and direction, but do not know how to represent them numerically?

3.2.2 Session II

Because of the misconceptions that arose in Session I, it was decided to add to the second session a few teacher-led exercises to clarify the notions of (1) a vector is not only determined by its length, but also by its direction, (2) a vector sum does not have a length equal to the sum of the lengths of its components, and (3) if two vectors are in-line with each other, then one is a multiple of the other. These exercises went smoothly and were effective in clarifying the misconceptions. At the end of the second exercise, Jill commented, "That was good. I see it better now."

It was time to introduce linear transformations. But first we had to acquaint the students with the notion of a *transformation*. This was not done by a formal definition, but rather by metaphorical language and visualization using two examples. I demonstrated to the students the results of two different transformations[2] of the plane

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[2] The transformations were not demonstrated in the same *Cabri* file. The students were not asked to decide whether S=T or not, given **v**, T(**v**), **w**, S(**w**) on the same *Cabri* screen.

when applied to any vector v. It was pointed out that under the first transformation the image R(v) is always a 60° rotation of v, while under the second transformation the vector v is projected and the image P(v) is always on a pre-drawn line.

The session then switched to the blackboard where a discussion was held on the possible behaviors of transformations with respect to the operations of dilation and vector sum. A tree diagram was produced to show the different possible ways of obtaining an image with respect to each operation: (1) T(kv) can be created from T(v) either by a dilation or not by a dilation; and if the former, by a dilation with either a factor k or not by k (see Fig. II-1), and (2) T(v+w) can be constructed from T(v) and T(w) either by vector sum or not by vector sum (see Fig. II-2).

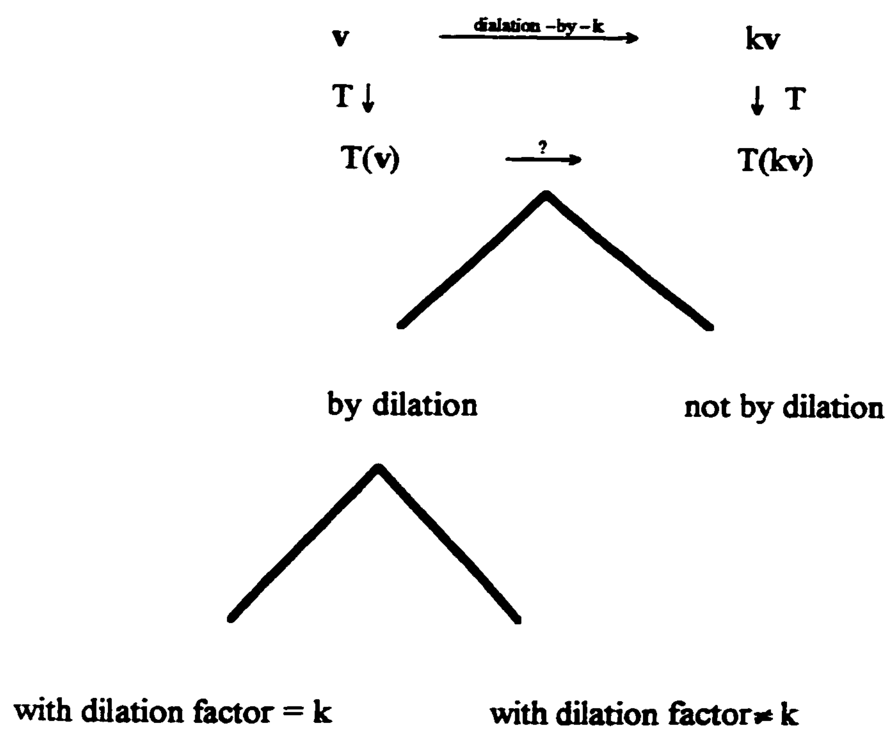

FIG. II-1 – The possibilities of going from T(v) to T(kv)

$$\begin{tikzcd}
\mathbf{v}, \mathbf{w} \arrow[r, "\text{vector-sum}"] \arrow[d, "T"'] & \mathbf{v}+\mathbf{w} \arrow[d, "T"] \\
T(\mathbf{v}), T(\mathbf{w}) \arrow[r, "?"] & T(\mathbf{v}+\mathbf{w})
\end{tikzcd}$$

by vector sum | not by vector sum

FIG. II-2 – The possibilities of going from T(v) and T(w) to T(v+w)

The students were then told that they are going to see several transformations that illustrate some of the above possibilities. Turning back to the computer, the intention was for me to demonstrate the first example so that the students can be shown how *Cabri* is used to test if T(**kv**) is the dilated vector T(**v**) by factor k and also if T(**v+w**) is indeed the vector sum of T(**v**) and T(**w**). The method for testing for the preservation of dilation was: (1) apply the transformation to a vector **v** to produce T(**v**), (2) dilate both vectors **v** and T(v) by factor k, thus creating vectors k**v** and kT(**v**), and (3) apply the transformation to vector k**v** to see if the resulting vector T(k**v**) always coincides with vector kT(**v**), as k and v vary. Similarly, the vector sum test was: (1) apply the transformation to the given vectors **v** and **w** to obtain vectors T(**v**) and T(**w**), (2) construct the vector sums **v+w** and T(v)+T(**w**), and (3) apply the transformation to vector **v+w** to see if the resulting vector T(**v+w**) always coincides with vector T(**v**)+T(**w**), as v and w vary.

Example 1

Design

This example required the use of the macro LINE-3, which rotates any vector **v** by a 45° counter-clockwise angle and then dilates it by a factor of 1.5. This, of course, is a linear transformation. One goal was to show the students that T(k**v**) and kT(**v**) are the same vector by constructing both vectors and observing that they coincide, no matter what the position of **v** and the value of k were. Similarly, the procedure was to be repeated to show that T(**v**+**w**) and T(**v**)+T(**w**) are also the same vector, for any **v** and **w**.

To help the students visualize the preservation of dilation by a linear transformation, segments **v**T(**v**) and k**v**T(k**v**) were to be drawn to create a pair of similar triangles (see Fig. II-3). By altering the value of k, the students would observe that triangle k**v**OT(k**v**) always moved in proportion to triangle **v**OT(**v**); thus creating a nice dynamic picture. In other words, vectors **v** and T(**v**) were always stretched by the same factor.

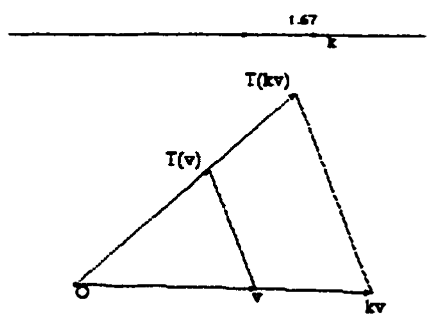

FIG. II-3 - Similar triangles are created to help visualize the preservation of dilation in a linear transformation.

What happened

The example began with me setting up the *Cabri* screen:

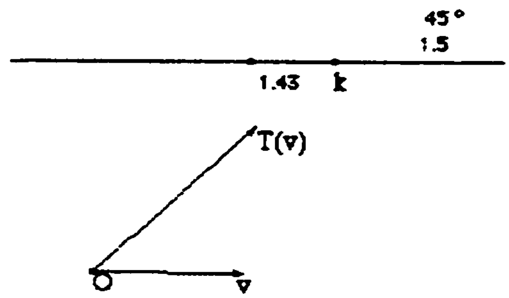

FIG II-4 – Configuration at Line 77

75. M[3]: So we'll put two numbers here. And then I'm going to put our vector... Label this point v and this point O. Now transform this vector. (applies LINE-3 to v). So now this red vector is the image of v which I will call T(v). Now I am not going to tell you what this transformation does. I'll move this around [vector v] and ask you if you have any idea about what this transformation does.

76. Jill: What do you mean what it does?

77. M: What is the relation between v and its image T(v)? (see Fig. II-4)

After a few suggestions,

81. M: At this point we are not interested in what it does. It is just to show you that we have a certain transformation, then we always have the vector and its image. What we are really interested in is the relationship between T(v) and T(kv)...

[3] M is Me (Michael); AS is Anna Sierpinska; T is Tommy Dreyfus

Through a discussion, the students determined that kT(v) and T(kv) are equal and they learned a method to verify this conclusion using *Cabri*.

107. M: O.K. So the relation between these two is a dilation. Let's check something else though. I am going to draw a line segment between this and this [endpoint of vector v and endpoint of vector T(v) and endpoint of vector kv and endpoint of vector T(kv)]. So we have a line between v and its image. And a line between kv and its image. I am going to drag this point (k). What is happening? (see Fig. II-5)

108. Jack: If k is moving, the two triangles are not equal but they have, uhm, they remain at the same angle. And the relationship of the sides of the triangles are the same.

109. M.: What do you call that relationship?

110. Jack: Congruent?

111. Jill: They are depending on each other? When the factor changes they are changing but still they are equal in proportion?

112. M.: O.K. So what is changing in proportion?

113. Jill: k... No. T(kv), the transformation of kv and the vector kv.

114. M.: So the two images in other words?

115. Jill: Yes.

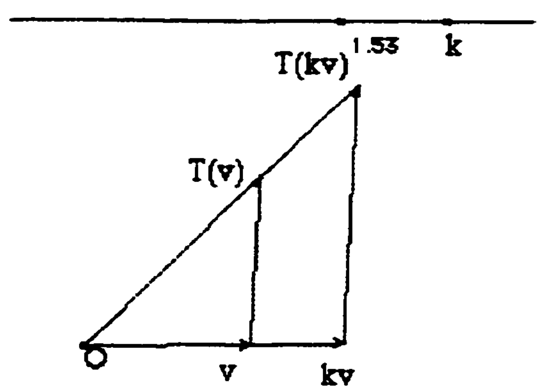

FIG. II-5 – Configuration used after Line 107

After dragging point k some more and reminding them that the triangles are called similar triangles,

121. M.: So what can we conclude about the behaviour with respect to dilation? The dilation factor of k and seeing these images, can you make any conclusion ... a statement about this...

122. Jill: When the value of k changes (silence); basically what I said before. The value of k changes then kv changes in proportion with the transformation of kv. Is that what you asked for?

123.M.: Are they changing in proportion? Is kv in proportion to this one [(T(kv)]? Or to what?

124. Jack: T(kv) is proportional to T(v) by k factor.

125. M.: Good. And what else? This [T(v)] is proportional to this one [T(kv)]. And...

126. Jack: There is the same relationship between v and kv and T(v) and T(kv).

127. M.: And that same relationship is ...

128. Jack: The dilation factor.

It was time to look at the behaviour of the transformation under the vector sum operation. A smooth discussion ensued. The students confirmed that T(v+**w**) is equal to T(v)+T(**w**) and learned how to verify it using *Cabri*.

Analysis

As was intended, we started Example 1 with the emphasis on the notion of applying the transformation to the whole plane (Lines 75 – 81 above). Unfortunately, the language used may not have been very clear –e.g., "it is just to show you that we have a certain transformation, then we always have the vector and its image."

Our intention in drawing the two similar triangles was not only to create a nice, dynamic picture when k is varied, but also to show that both v and T(v) are being stretched by the same factor; thus creating the proportion $\frac{|kv|}{|v|} = \frac{|T(kv)|}{|T(v)|} = |k|$. In other words, the ratios were a comparison of the corresponding sides of the two triangles (see Fig. II-6).

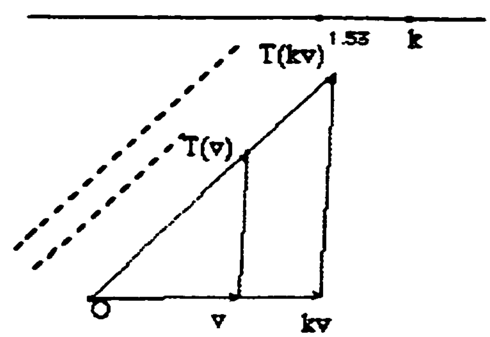

FIG II-6 – The dotted lines show the sides being compared in the similarity proportion

Jack had no trouble visualizing this, but Jill saw it differently. She focused on the relationship of the sides within the <u>same</u> triangle, thus creating the proportion $\frac{|T(v)|}{|v|} = \frac{|T(kv)|}{|kv|} =$ constant. In this case, the constant is equal to 1.5 (the dilation factor of the transformation). In other words, the ratios were a comparison of "side a" and "side b" of the same triangle (see Fig. II-7).

Jill focused on the movement of the triangle changing size–i.e., the vectors kv and T(kv) – that when she created her ratios, she compared the sides of the same triangle. Not that it is wrong to see it that way, but throughout the whole discussion she was not able to see it the other way, even though I tried to steer her in that direction (Lines 112- 123 above). Jack was able to snap out of the hypnotic state created by the triangle's movement and focus on the relationship of the corresponding sides of the triangles. Maybe his stronger geometric background helped him visualize both ways (Line 124).

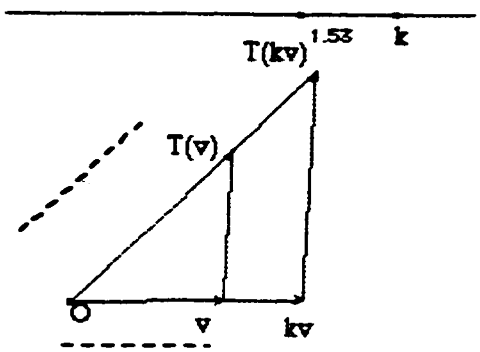

FIG II-7 – Dotted lines show the sides being compared in the similarity proportion

My mistake was that once Jack said the intended answer, I shifted my attention to him and ended the discussion (Lines 125-128). Although I was probing Jill all along, I left her hanging without any confirmation on her misunderstanding.

Examples 2 & 3

Design

The aim of these examples (including Example 4) was for the students to practice the process of testing the linearity of a transformation, although they were not told that that was what they were doing. They were trying to find the appropriate cases on the tree diagrams.

Example 2 used the macro QUADRATIC-7. The macro is designed to rotate any vector **v** by a 45° counter-clockwise angle and then dilate it by a factor of $\|v\|^2$. In this transformation, T(kv) is produced from T(v) by a dilation by a factor not equal to k. It also does not conserve the operation of vector sum.

Example 3 employed the macro Translate-tip, which translates the tip of **v** by a given vector **a** --i.e., T(v) = **v** + **a**. This transformation follows the "not by dilation" and "not by vector sum" branches in Figs. II-1 and II-2 above.

<u>What happened</u>

With my instructions, Jack set up the configuration for testing the behaviour of the transformation in Example 2 with respect to dilation.

> 119. M.: ...Let us vary k. (Jack moves 'k' along the number line) It would probably be easier if you draw the line segments between v and T(v), and kv and T(kv). (see Fig. II-8)
>
> 120. Jill: So they are not...
>
> 121. Jack: Yes. They are changing. (Moves 'k' very slowly, stops at the point where k=1) It's the same now.
>
> 122. Jill: Dilation by a factor not equal to k.
> (...)
> 133. Jack: If the dilation factor were equal to k, we would always have similar triangles. But we don't.

134. M.: O.K. Good. Let us check for vector sums. (AS suggests skipping this part) We'll try another one.

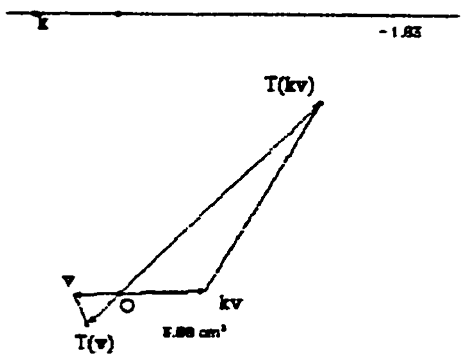

FIG. II-8 – Configuration after Line 119 (Note: a ghost number appears on the screen upon activating macro Quadratic, we had no way of getting rid of it)

Again I instructed Jack in setting up the opening configuration for testing the behaviour of the transformation in Example 3 with respect to dilation (see Fig. II-9).

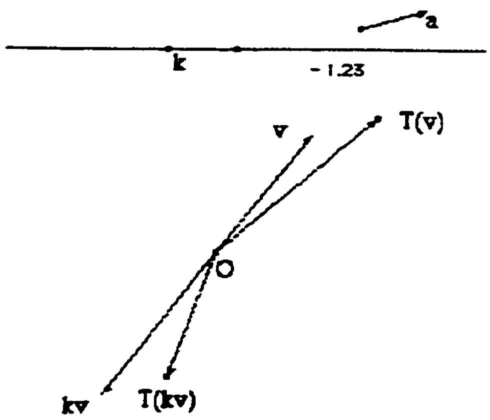

FIG. II-9 – Configuration for Example 3

135. M: ...Now drag 'k'.

136. Jack: It is just like the other one. I don't think it is dilated by factor k.

137. M.: So what is this relationship? What case do we have? How does the transformation behave with respect to dilation?

138. Jack: Dilation factor different from k.

139. M.: Do you agree, Jill?

140. Jill: Yes.

141. Jack: Because I think if it were dilated by a dilation factor k, T(v) would be on the same line as T(kv). But it's not.

142. Jack: If it were the dilation factor k, then T(v) would be on the same line as T(kv). But as I was saying, it is not.

143. Jill: It will keep the same angle.

144. Jack: kv is always on the same line as v. But T(kv) is not on the same line as T(v).

145. M.: So you are saying that is a dilation by a factor not equal to k, or ...?

146. Jack: Yes. Maybe it is not even a dilation. No, I mean, it is. No. What I am sure of is that it is not by factor k.

147. M.: Why don't we click (uncovers the figure from Example 2) Refer to that figure here. This was the case, we said, of what?

148. Jack: Dilation with a dilation factor different than k.

149. M.: So if you move k around, observe T(v) and T(kv).

150. Jack: They are on the same line.

151. M.: And the other one, if you go back (back to figure in Example 3) and move k around, are they the same case?

152. Jack: No. And...

153. Jill: But they are both not...

154. Jack: And I think T(kv) is not obtained by a dilation of T(v). Because they are not on the same line. So we don't always get T(kv) by a dilation.

155. M.: Do you agree, Jill?

156. Jill: I think they are both a dilation by a factor not equal to k.

157. M.: So you are saying it is this case here. (points to the 'dilation not by factor k' branch on the tree diagram) but is it a dilation? To get this case, it has to be a dilation. Do you agree that it is a dilation but not by a factor k?

158. Jill: It can't be a dilation.

159. M.: So which case is it in this diagram?

160. Jill: Not by dilation.

161. M.: What do you think would happen to the vector sum?

(...)

166. Jack: I think it would be 'not by vector sum'.

167. AS: Yes, it's true and let's skip that part, let's go to the next example and do the vector sum there. We've only got half an hour.

Analysis

Our intention in introducing the dynamic similar triangles was to create a nice, visual image to help in the understanding of the concept of linearity of transformations with respect to dilation. Example 2 demonstrated that that the moving image was useful in distinguishing between a dilation by factor k and a dilation by a factor not equal to k (Lines 119-133 above).

What we did not anticipate is that the dynamic picture would detract the students from the idea of dilation. Since the triangles could be drawn in the configuration of Example 3, the students attention was so focused on the movement of these triangles that they did not notice that T(kv) was not on the same line as T(v). Their first reaction was that it was the case of a dilation by a factor not by k (Lines 135-140). Since Jack has the ability to visualize both ways of comparing "similar" triangles as mentioned in the analysis of Example 1, he felt that something was wrong (Lines 141-146). He became confused with the definition of dilation, although in past examples he always confirmed that a dilated vector is on the same line as the original vector. After comparing Examples 2 and 3, he was able to see the difference (Line 154).

The same could not be said of Jill. Her main concern was the triangles themselves. She focused on the shapes of the triangles –i.e., the angles—instead of the sides. When the students were explaining how the case of dilation by factor k would look like, Jack talked about the vectors being on the same line, while Jill talked about the triangles having the same angles (Lines 142-143). When Example 2 and Example 3 were compared, she still had doubts that they are not the same case since both of the transformations did not result in similar triangles. For her, both examples were the case of a dilation by a factor other than k (Lines 151-156). It was not until I explained to her that for a transformation to be that case, it has to be a dilation first that she agreed (probably reluctantly) that it was the case of "not by dilation". It seemed that Jill's main concern is the movement of the triangles, not the position of the vectors.

In both examples, the check for the behaviour of the transformation under vector sum was skipped because we were running short of time. Unfortunately, this led the students to believe that it is only necessary to check for one of the operations. This problem was to surface in a later session.

Example 4

This example involved the macro SEMILINEAR, which sends the zero vector onto the zero vector and rotates any non-zero vector **v** by a 90° counter-clockwise angle and then dilates it by a variable factor c, where $c = \frac{|x_1|}{\|v\|}$ and $v = (x_1, x_2)$. This transformation conserves dilation but not vector sum.

<u>What happened</u>

After setting up the *Cabri* screen, I asked the students about the behaviour of the transformation with respect to dilation. Jack, without hesitation, constructed the similar triangles and noted that it is the dilation by factor k case. He added, "But if we want to be really sure, we can dilate T(**v**)." Jill did not say a word the whole time.

To test the vector sum case, some of the vectors were hidden, leaving only three vectors, **v**, T(**v**), and **w**. Jack produced the vectors T(**w**) and T(**v**+**w**).

> 188. Jack: ...If we move v, we see that T(v) changes and T(v+w) changes also. So it means that they are always vector sum. Because they keep changing at the same time (Jack drags v so that it overlaps with, probably, w and then stops for a while, contemplating).
>
> 189. Jill: They are in proportion.
>
> 190: M.: So do you agree with him?
>
> 191. Jill: Yes.
>
> 192. M.: So what were we checking? What did you want to check?

193. Jill: We wanted to check if the transformation of v and vector v are uhm .. If they are proportional to the transformation of vector sum v+w. Well... the sum of the transformations and the sum of the vectors...

194. M.: Why don't you look at the diagram and see what happened.

195. Jill: The vector sum of the transformations and the vector sum of the vectors...

196 AS: I don't think you have the sum of the transformations. Did you get...

197. Jack: Yeah we did not....So let's find this vector.

He produced T(**v**)+T(**w**). It was almost overlapping with T(**v**+**w**), but it was obviously shorter and slightly to the right of T(**v**) (see Fig. II-10).

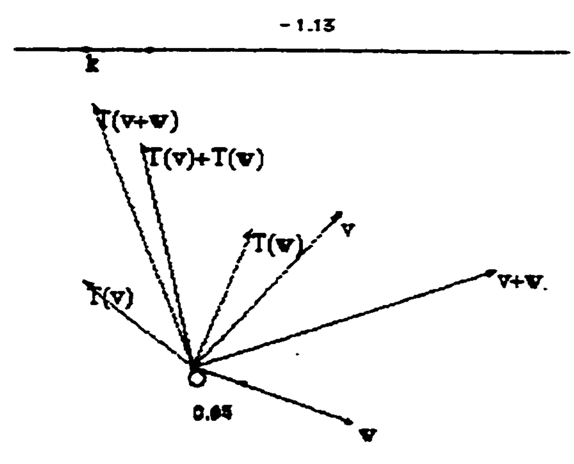

FIG. II-10 – Configuration after Line 197

199. M.: So what can you conclude? What case is this?

200. Jack: That the vector sum of T(w) and T(v) is not equal to the transformation of the vector sum. We don't get T(v+w) by vector sum.

201. M.: Right. O.K. Questions?

We then turned to the board, where I defined a linear transformation by highlighting the appropriate branches on the tree diagrams (see Fig. II-11).

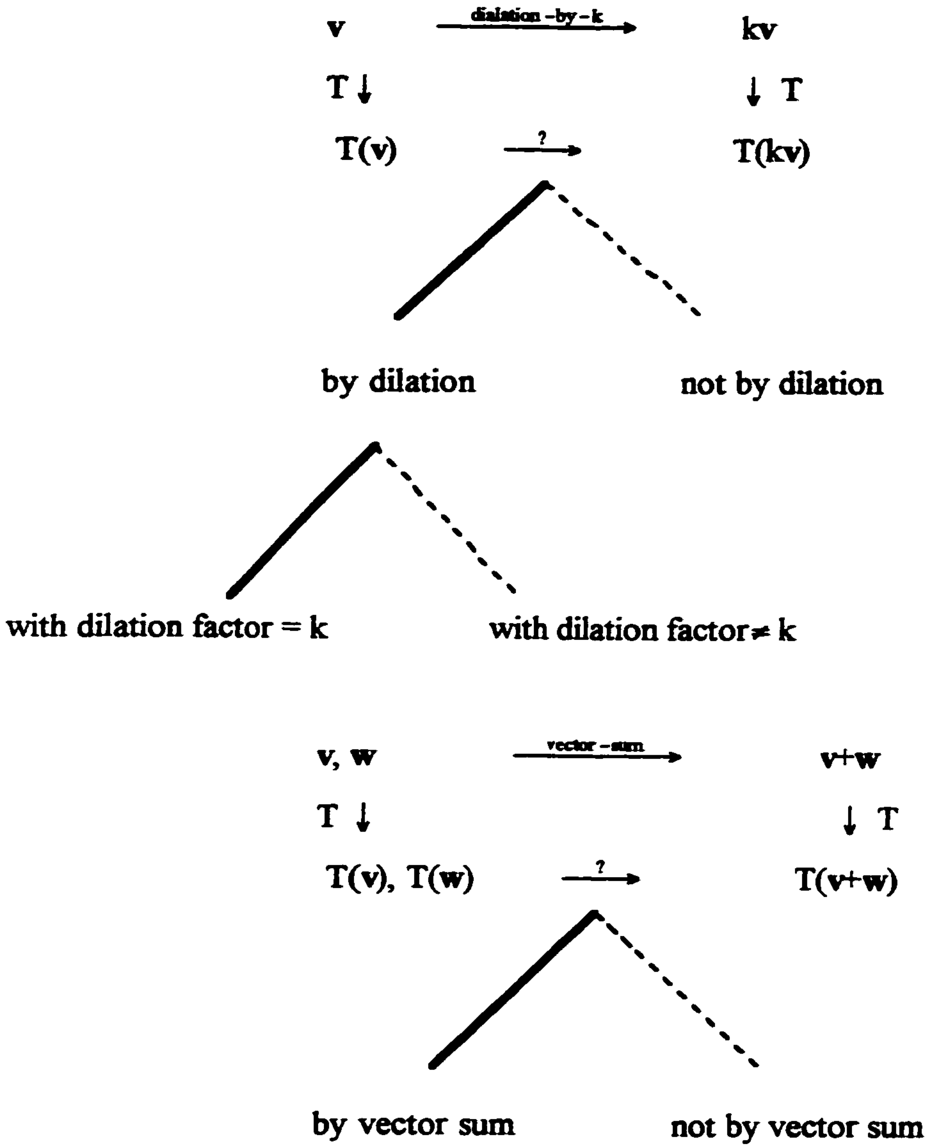

FIG. II-11 – A linear transformation.

Analysis

Jack seemed to understand the concept of the linearity of transformations (although we have not used the term *linear* yet). He was quick in making the conclusions, although he had to be reminded to produce T(**v**)+T(**w**).

Jill was still not clear on the concept, as witnessed by her silence in the first part of the example and her confusion in the second. She was so mesmerized by the moving vectors that she called any relationship "proportional". In Line 189 above, she said that **v**, T(**v**), and T(**v**+**w**) are "in proportion" because she saw all three move when vector **v** was dragged. Of course, T(**v**) and T(**v**+**w**) moved because they are dependent on **v**.

Again I made the mistake of not confirming Jill's understanding at the end of the example. In Lines 189 – 196, I was leading her through the discussion, but once again as soon as Jack came up with the intended response, I concluded the example.

We later felt that defining linear transformations on the board using the tree diagrams was meaningless to the students.

Worksheet 1

Design

After defining linear transformations using the tree diagrams, the students were given Worksheet #1:

"Write the definition of a linear transformation in an algebraic language. The definition should start with the words; 'A transformation T is linear if ......'. Use symbols such as: T v **w** k ( ) + = ."

By comparing the results of the above examples with the tree diagram, the students were expected to write the equations $T(kv) = kT(v)$ and $T(v+\mathbf{w}) = T(v) + T(\mathbf{w})$.

What happened

Jill was still reading the question when Jack wrote $T(v) + T(\mathbf{w}) = T(v+\mathbf{w})$. He said, "So that's all about the vector sum. Now you do the second." Jill was thinking out loud while writing an equation part by part. As she was saying her thoughts slowly, Jack was completing her sentences. She finally wrote the equation $T(v) = T(kv)$. Jack said, "But we have to put... But this is not equal. It's $T(kv)$ that's equal to $kT(v)$." Jill then asked if she can look at the sketches. She wanted to see the diagram of the similar triangles. She could not find the diagram, so she drew it on paper.

229. Jill: I see it like this. When this (moves her pen along v and kv as if imitating the effects of dragging 'k' on the screen) then this (makes rhythmic movements along v, kv and T(v), Tk(v) (i.e., T(kv)))

230. Jack: (Fixes Jill's equation to look like k(T(v)) = T(kv))

231. Jill: I wouldn't write it like this... (reads Jack's formula and then comes back to her diagram, trying to translate from the formula to the diagram) kv is here...

232. Jack: But you see kv over v is equal to kT(v) over (crosses out 'kT(v)) T(kv) over kT(v)... No, (crosses out 'k' in the denominator) That's the same thing.

Jack had written the equation $\frac{\mathbf{kv}}{\mathbf{v}} = \frac{T(\mathbf{kv})}{T(\mathbf{v})}$. Jill agreed with him. They, then, called me over to discuss their results. Jill explained, "I knew what I was trying to say but it is the writing of it in a formula type of thing that I had trouble with."

Analysis

It is clear that Jack was grasping the definition of linear transformation, but Jill was still confused and not confident in her understanding. Jack showed that he was able to switch easily between a visual representation and an algebraic one. On the other hand, Jill was so mesmerized by the moving pictures that formulas did not mean much to her. She was able to understand the equation when it was written as a proportion since it was derived directly from the diagram. It would have been interesting if she had tried to write the formula for the vector sum. Would she have written proportions?

When Jack wrote the ratio $\frac{\mathbf{kv}}{\mathbf{v}} = \frac{T(\mathbf{kv})}{T(\mathbf{v})}$, we accepted it. We did not point out that the ratio should be of the lengths of the vectors – i.e. $\frac{|\mathbf{kv}|}{|\mathbf{v}|} = \frac{|T(\mathbf{kv})|}{|T(\mathbf{v})|}$ – thus compounding to the already misunderstood concept of a *vector*. First, we show them that a vector has both length and direction, but we do not show them how to write it. That is because to represent a vector algebraically requires the introduction of a coordinate-system, which contradicts the aims of the experiment.

3.3.3 Session III

I began Session III by interviewing Jack and Jill about their understanding of linear transformations. They sat at the small table and were given a reproduction of the tree diagrams that were developed in Session II. I was to correct the students if they had made a mistake concerning the meaning. Otherwise, they were allowed to use any form of expression they liked.

After clarifying and/or confirming their understanding, the students were to continue with Worksheet #1 left from the previous session.

Parts II and III of this session involved the introduction of invariant lines. This topic will not be analyzed in this paper.

Episode 1

<u>What happened</u>

Sitting at the table, facing the board with our backs to the computer, I asked the students if they could give a definition of a linear transformation.

3. Jack: Linear transformation is when... the transformation... the vector sum of two vectors that are transformed... is equal to the transform of the vector sum of the vectors... T of u plus T of v equals T of (makes gestures imitating brackets with his hands) u plus v.

4. M.: OK... So that, do you agree with that?

5. Jill: (looks surprised; slight pause, then seems to recite what Jack has just said) um... OK... transformation

of say vectors v and w are equal to the transformation of v+w (she looks up to me for approval).

6. M.: (sensing that she is reciting) One more try?

7. Jill: Then what I am saying is not right?

8. M.: Yeah, that's okay...

9. Jill: OK... I can... say only if I see it...

10. M.: If you want to draw something... there's paper... it's right here.

11. Jill: OK... (Makes gestures: palms of the hands straightened and close to each other and then moving forward and away from each other, as if simulating two vectors with common origin moving along straight lines. But decides not to draw and instead gives the condition for vector sum; this time it sounds less like reciting) The translation... the transformation of v plus the transformation of w equals to transformation of v plus w.

I got confused with the wording. Thinking she said that the sum of **v** and **w** is equal to the transformation of **v** plus **w** –i.e., **v**+**w** = T(**v**+**w**) --, I asked her to repeat herself. She decided to draw the situation. After thinking for a while, she asked Jack to help her remember the situation. The following short discussion ensued:

20. Jack: Yeah, and we did the vector sum of the transform of v and transform of w, and if it's equal the transform of v+w, it means that it is linear...

21. M.: OK, would you like..., so that she sees it better... Can you draw the situation for her?

22. Jack: (draws a diagram) OK this is u, no v, this is w...OK let's say this is the vector sum of v+w.. OK.. if we transform v, OK... let's, I don't know, rotate it, this is T(v) and let's rotate w again. (see Fig. III-1)

They both continued to explain that if T(**v**+**w**) equaled T(**v**) + T(**w**), then it would be linear.

31. M.: OK so is this the only thing that would make a linear transformation?

32. Jack: No, there is also by dilation.

33. M.: OK... what about dilation?

34. Jack: OK if we dilate the vector by k... and we transform vector T(kv), it's supposed to be equal to the transform of v by k.

35. M.: (to Jill) OK... do you agree with that?

36. Jill: Yeah.. we have v and kv (draws v and kv), I don't remember the transformation...

37. M.: Sorry, you don't remember? Remember what...

38. Jill: The transformation... like T(v)

39. M.: You mean... where the image would be...

40. Jill: Right.

After I told her to use any transformation, she drew the diagram in Fig. III-2.

44. Jill: OK, so transformation of v by factor k would be transformation of kv.

45. M.: OK... so?

46. Jill: So they are equal and they are proportional...

47. M: OK, so what is proportional?

48. Jill: Um... the transformations of... um... They are by the same factor k.

49. M.: OK...

50. Jill: Umm, I have trouble wording it.

51. M.: So if you extended this by k that you showed here .. (pointing to v and kv line)

52. Jill: OK.. right.. and so this will extend by the same factor.

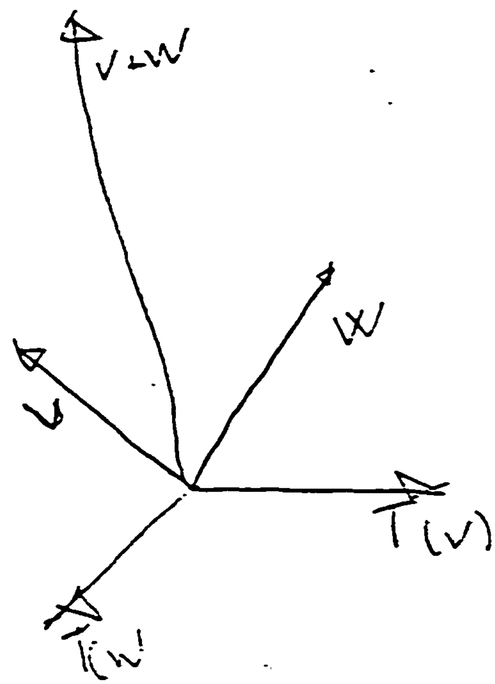

FIG. III-1 - Diagram drawn on paper by Jack

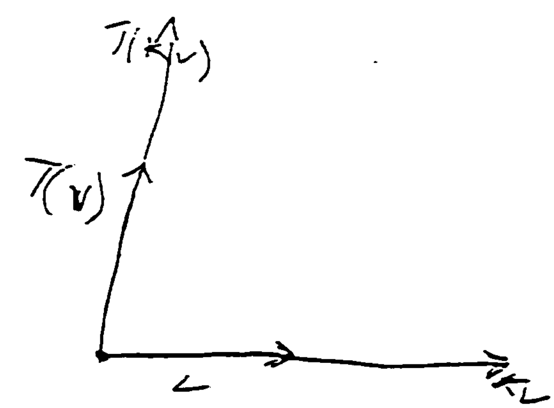

FIG. III-2 - Diagram drawn on paper by Jill

Analysis

The above discussion confirms the conclusions about the students' understanding of linear transformations after completing Session II. Jill understands visually as seen in her need to draw diagrams to explain herself (Line 9 and Lines 35-52). Her understanding of linear transformations is based on the image of the moving similar triangles. This is seen in Line 11 above when she imitates the similar triangles image with her gestures even though we were discussing the vector sum case. When asked to explain the dilation case, she immediately drew the diagram and said that kT(v) and T(kv) are equal and proportional (the "buzz" word). When asked what is proportional, she could not word her explanation. I think that is because she is visualizing in her mind the two sides extending, but at the same time she sees the proportion formula, which compares the large triangle to the small triangle. She sees the comparison of the sides of the big (moving) triangle. Therefore, she cannot put it into words.

On the other hand, Jack can explain himself verbally, with formulas, and graphically. As for his understanding of linear transformations, I believe that he understands both conditions individually, but he might think that only one needs to be tested. This could be the reason why he gave only one condition when defining a linear transformation (Line 3).

At the time when the experiment was being conducted, we thought that the students did not understand the idea of a transformation being applied to the whole plane.

The language they spoke indicated that the transformation is the same as the image. Jack talked about the "transform of v" (Lines 20,22,34) and Jill explained, "The transformation… like T(v)." (Line 38) But now I believe that Jack (I'm not so sure about Jill) understood that a transformation is applied to the whole plane, but did not have the proper vocabulary to express himself correctly. When I asked Jack to draw the vector sum case for Jill, he used a specific transformation (rotation) to draw the images of vectors **v** and **w**. He could have easily drawn random vectors T(**v**) and T(**w**), but instead he showed that the relation between any vector and its image is preserved (Line 22).

Worksheet 1

Design

In this worksheet, the students were given the descriptions of two transformations and were asked to find out if they were linear. The students were expected to use the testing method of Session II.

In problem 1(a) the transformation was a reflection. It required the use of the 'reflection' command under the TRANSFORM menu. It constructs an image vector that is a reflection of a given vector v through a given line L.

In problem 1(b) the transformation was a shear. In linear algebra, the shear transformation can be defined as follows: Let **u** be a non-zero vector and q a scalar.

The shear with factor q in the direction of $\mathbf{u}$ is the linear transformation S of the vector plane for which $S(\mathbf{u}) = \mathbf{u}$ and $S(\mathbf{u}_{\perp}) = q\mathbf{u} + \mathbf{u}_{\perp}$, where $\mathbf{u}_{\perp}$ is the rotation of $\mathbf{u}$ by 90°. But this definition could not be given to the students in the experiment because they did not have the necessary background to understand it. So a geometric description of the transformation was given: a shear with factor q (q>0) in the direction of a line L is a transformation such that for any vector **v** in the plane, the line through the tips of **v** and its image S(**v**) is parallel to L, the ratio of the distance from **v** to S(**v**) to the distance from the tip of **v** to L is constant and equal q, and the vector S(**v**) is to the right of **v** when one's eyes are positioned in line with L and so that **v** is above the line.

For this session, the shear transformation was programmed in Cabri by a macro construction labeled SHEAR-4. The parameters for this macro were a line L through the origin and a positive number q. The output was supposed to be the image of v under the shear with factor q in the direction of L. Unfortunately, there was an error in the design of the macro[4], and the vector S(v) was not always appearing to the right of v (see Figs. III-3,4). This made the transformation not linear: it failed to conserve scalar

---

[4] SHEAR-4 was designed to produce the image using the following steps: (1) v is orthogonally projected onto a line K perpendicular to L through O; obtaining vector $\mathbf{w}_1$, (2) Vector $\mathbf{w}_1$ is reflected in the angle bisector of lines L and K and dilated by q; producing vector $\mathbf{w}_2$, and (3) the vector S(v) is the vector sum of v and $\mathbf{w}_2$. The error occurred because for any pair of intersecting lines, there are two angle bisectors and *Cabri* was randomly making the choice of the angle, depending on the position of v, the value of q, and the inclination of line L. In SHEAR-11, the revised macro used in Session IV, a reflection by –90° replaced the reflection in the angle bisector. This took care of the problem but required an extra parameter (-90 °) to activate the macro.

multiplication. We were not aware of this flaw, and it caught us by surprise during the session.

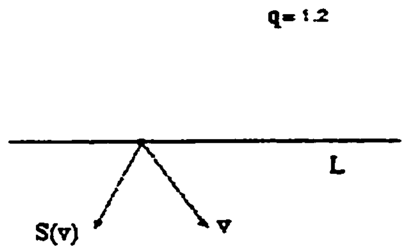	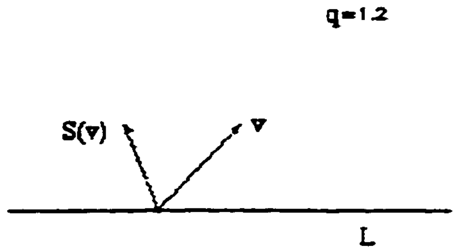
FIG. III-3 - For the vector v below the horizontal line L, the SHEAR-4 macro produces S(v) to the right of v for eyes in line with L and seeing v above L.	FIG. III-4 - For the vector v above the horizontal line L, the SHEAR-4 macro produces S(v) to the left of v for eyes in line with L and seeing v above L.

<u>What happened</u>

The students started by reading Problem 1(a) and the description of the reflection transformation. Following the instructions, they created the diagram in Fig. III-5. Jack suggested that they "have to check if it's [the case of] 'by vector sum' and 'by dilation' when it's a reflection." Jill agreed and started to read the definition of reflection again. In the meantime, Jack was moving the endpoint of v and T(v) moved accordingly.

72. Jack: Hey watch on.. like in a mirror...
73. Jill: Yeah, that goes through...
74. Jack: Yeah...
75. Jill: But you know what...
76. Jack: They will always be equal, there will always be the same angle between the two.
77. Jill: This is probably linear.

78. Jack: So, I think it will be linear because reflection doesn't change length or anything. I think this is linear.

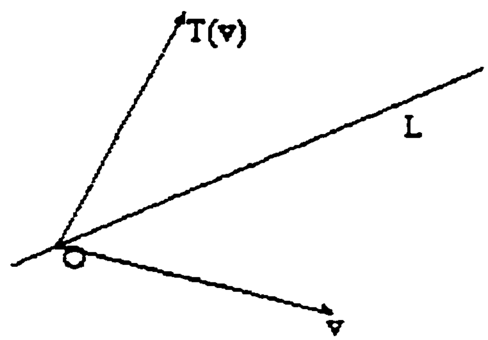

FIG. III-5 – Configuration used in Problem 1(a) (Reflection)

They started Problem 1(b) by reading the instructions on how to apply the shear transformation. They produced the figure in Fig. III-6. Jill read the description of the shear transformation and a discussion between the students ensued on how it applies to their figure on the screen. Jack understood the fact that the tips of vectors **v** and S(**v**) will always lie on a line parallel to line L. In discussing the role of q, Jill suggested to "put a new number q." Jack put the number 0.5 on the screen and used it as the parameter q to produce the unlabeled vector in Fig. III-7.

111. Jack: ... You see here... It's always equal. They're always on a parallel line but for it to be linear it means that it's supposed to be dilated by a factor k. But we see that this is factor k and it has no influence on the vectors (moves k along the number line and nothing changes with v and S(v)). It's only by the number q, but number q it only means that the distance between the points...

112. Jill: The distance between the endpoints of S(v) and v is equal to q times the distance from the endpoint of v to the line L.

113. Jack: Yeah. To get this we have to multiply by this number by the distance of v (moves the cursor along vector v as if indicating that 'the distance of v' is the length of vector v), like the distance between this and this [vector v and unlabeled vector] is half of the length of vector v, and between this and this [vectors v and S(v)] is 1.3 of vector v... But this does not mean that it's linear because if it were by dilation, the vector S(v) was supposed to be on the same line (points to the unlabeled vector) and it's not...

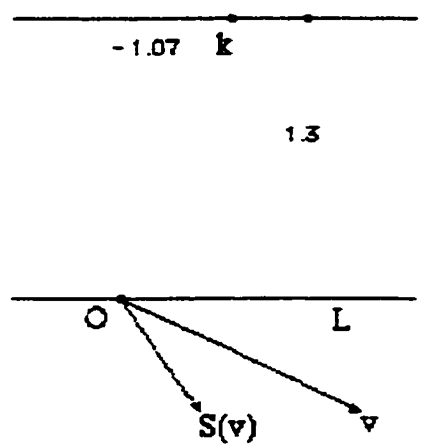

FIG. III-6 – Opening configuration of Problem 1(b) (Shear, where q = 1.3)

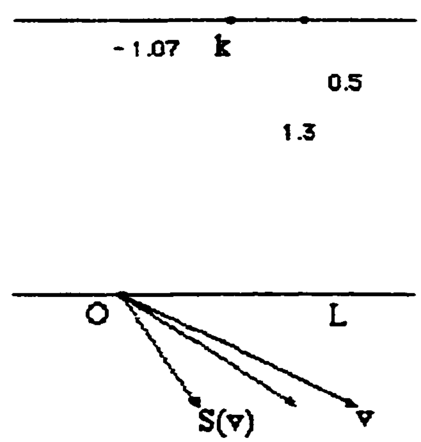

FIG. III-7 – Unlabeled vector is produced using SHEAR-4 with q = 0.5

Jill did not understand. Jack explained how the distance between the tips of an image vector and vector **v** is equal to the distance from the endpoint of **v** to line L times the corresponding q. This time Jack correctly pointed to the perpendicular distance between endpoint **v** and line L. He went on to explain how the tips of the images will always be on a line parallel to line L. But Jack was still not sure about the linearity of the transformation. They decided to try a negative value for q. They typed the value -0.8 and produced another image whose tip was on the line parallel to line L (see Fig. III-8).

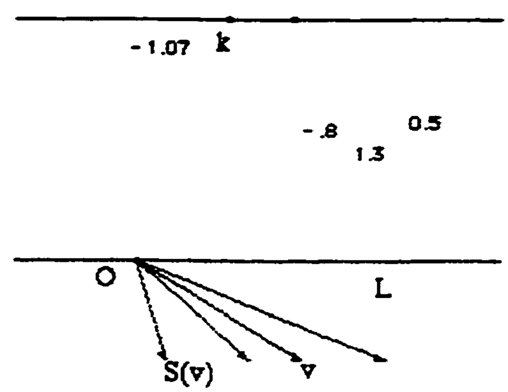

FIG. III-8 - Unlabeled vector to the far right is produced using SHEAR-4 with q = -0.8

138. Jack: Two different vectors we will get [when using different values for q], but I don't think it's linear, because it's not 'by dilation', these vectors (pointing to the three image vectors) are not in line, and you cannot get (Silence. Jack starts moving k on the number line.)

139. Jill: No.. no..

140. Jack: But there is a number that I think its proportional or ... it's not v by that number... No, I think there is something wrong with this figure...

(...)

143. Jack: (Silence, moves the pointer from one vector to another) This follows some other rule... (Long silence).

At this point, I decided to start the discussion. I joined the students in front of the computer and asked them about what they have discovered. They explained how an image is formed under a shear transformation. Jack added that he was not sure if it was linear because

152. Jack: When a transformation is linear it means that it has to be dilated by k and by vector sum.

153. M.: Yes, that's what we said before.

154. Jack: But as v and the others are not the same dilation. Because it depends on q, basically. What we know is that they'll always be on the same line. The endpoints of these will always be on a line parallel to L. We get any vector we want, but um... it doesn't mean it's linear.

(...)

157. AS: They are confusing q with k.

158. M.: Ok... q is not.. Ok.. q is just a parameter to give you the image... Ok.. so if you do change q you are getting a different transformation..

159. Jack: It's not.. ah.. it's not k

160. M.: It's not k..

161. Jack: I tried to move k and it has no effect whatsoever on the transformation...

162. M.: Well... why do you think that?

163. Jack: Well... I ah... it [k] can be anything but it [the vectors] won't change if it [k] does... we don't need k for getting something that...

164. M.: Why did you use k before... in the previous examples?

165. Jack: Um... kS(v) must equal S(kv).. (Jack & Jill giggle, realizing their mistake).

They went ahead and produced vectors **kv** and S(**kv**). But because of the design error discussed above, S(**kv**) was not in line with S(**v**) (see Fig. III-9). Jack correctly concluded that the transformation was not linear.

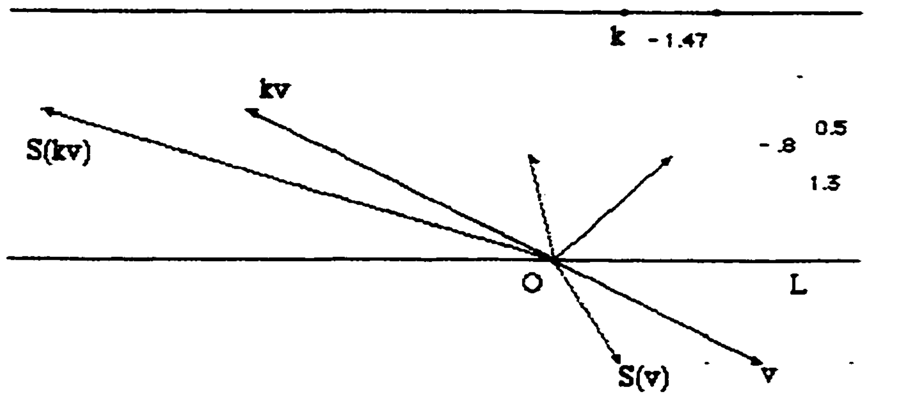

FIG. III-9 – Final configuration of Problem 1(b)

I then turned the discussion back towards Problem 1(a). Jack retrieved the figure with the reflection and proceeded to add the number line and produce vectors **kv** and T(**kv**). By moving k along the number line, they both concluded that it is the 'dilation by factor k' case. When asked about the vector sum case, Jill said that a new vector **w** was needed and that "the sum of the transformation of **w** plus the transformation of **v** was equal to transformation of **v**+**w**". I then asked them if they are sure that this would happen.

> 221. Jack: Oh.. its um.. if we use dilation to see if it was ah... linear so if we use vector sum we are supposed to get... to draw the same conclusion... we are supposed to find that T(**v**)+T(**w**) is equal to T(**v**+**w**). I don't think... they both have to be true.
>
> 222. M.: OK. So is the other one true. The sum..

Jack constructed vectors **w**, T(**w**), **v**+**w**, T(**v**+**w**) and T(**v**)+T(**w**) and concluded that the vector sum case is satisfied.

227. Jack: ...So it is linear... We proved it by dilation and vector sum. That's all! So dilation and vector sum are ways to see it's linear.

228. T.: Ok, so can I interject one question on the activity just done? You just said you did it by dilation and you did it by vector sum.

229. Jack: Yes...

230. T.: Ok if we do one of these [linearity tests], would that satisfy you?

231. Jill: I think so...

232. T.: Either one...

233. Jill: Yes...

234. Jack: Yes... Ok because I don't think...

235. Jill: I don't think they could be linear in one way and not the other.

236. Jack: Exactly.

237. M.: Ok.

Analysis

At the beginning of Problem 1(a), Jack suggested doing the dilation and vector sum tests for the reflection transformation but did not act on it. Instead, he (and Jill) concluded that the transformation is linear because the vectors **v** and T(**v**) will always have the same length (Line 78). He compared this transformation to a rotation transformation which is linear and always produces an image vector T(**v**) equal in length to vector **v**. In all the examples of non-linear transformations that they have seen, the length of T(**v**) was variable according to the position of vector **v**. Therefore, the present situation fit the criteria for it to be a linear transformation. According to Rumelhart (1980), the students have developed their own schemata of what a linear transformation represents, and the reflection transformation fit that prototype.

Confusing parameter q for factor k could represent an obstacle that we had not seen before, but it does confirm Jack's understanding of linear transformations. In Line 111, he states that for the transformation to be linear, it had to be dilated by a factor k. But when he moved k along the number line, nothing happened. He repeats several times his concerns that the transformation is "not by dilation" since the image vectors are not in line (Lines 113 and 138). He knows that there should be a variable number that will make the picture move proportionally, but this is not happening (Line 140). He is so puzzled that he concludes that "this follows some other rule." (Line 143) Refering to Rumelhart's (1980) prototype theory, this time the present situation did not fit the criteria of a linear transformation. "If a promising schema fails to account for some aspect of a situation, one has the options of accepting the schema as adequate in spite of its flawed account or of rejecting the schema as inadequate and looking for another possibility." (Rumelhart, 1980, p. 38) Jack decided the latter.

Jack and Jill did not realize that they had not created the vectors kv and S(kv) since they thought that parameter q was the factor k. Was this just an innocent slip up because of the lack of practice that they have had and the long time in between Sessions II and III (from Wednesday to Monday)? Or was it deeper than that? Is there no connection between what they do on the computer and the formulas -- i.e., between performing the linearity tests and the equations of the conditions of a linear transformation? Is any scalar value a dilation factor? What is a parameter? At this point I cannot tell what is the reason(s) behind the students' confusion, but we have to remember that a dilation is in itself a linear transformation; thus making the factor k a parameter of the transformation.

In discussing Problem 1(a) we realized that the students had the misconception that it was sufficient to do only one operational test when checking for linearity. Jack even said at one point that he does not think that "both [tests] have to be true" (Line 221). After performing both tests, Jack said, " So dilation and vector sum are ways to see if it's linear." (Line 227) It seems that, for the students, these procedures are individual tests that are done in *Cabri* to see if a transformation is linear and not the conditions that a transformation must follow to be called linear[5].

---

[5] There is no way to explain to the students at this stage of their mathematical studies that both conditions are necessary in the definition of linear transformations. It was possible to give an example of a transformation satisfying the dilation property but not the vector sum condition ('Semilinear'). But, since every transformation which satisfies the vector sum condition satisfies the dilation condition for rational scalars, things may go wrong only with irrational scalars. This implies that an example of a transformation satisfying vector addition but not the dilation condition cannot be constructed (and certainly cannot be represented in Cabri which has no representation of irrational numbers) (see Sierpinska, Dreyfus and Hillel, 1999).

When Tommy asked them if they would be satisfied with the results of only one test, Jill said that she would be since a transformation cannot be "linear in one way and not the other" (Line 235) Obviously, she had forgotten the Semi-linear example (Example #4) in Session II. But when nobody corrected her mistake and I even said okay (Line 238), the students thought that their reasoning was correct. This problem will come up in the next session.

This little incident shows the delicate line between a classroom situation and an educational experiment. A classroom teacher would have quickly corrected Jill's mistake, whereas a researcher is more interested in how students acquire knowledge and therefore he/she sometimes decides not to correct the students. The problem is the students do not distinguish between a classroom and a research setting. To them, whatever the authority figure says is true and their silence is a sign of confirmation.

By the end of the session, we felt that Jill was being left out during the worksheets. Jack was much quicker in making a plan to solve the problem. Since he was also in control of the mouse, he was going ahead with his plan without waiting for Jill. He also was not listening to her suggestions. We decided that Jill would take control of the mouse in the next session to slow the pace down to her level.

3.2.4 Session IV

As promised, the Shear transformation was re-looked at since the macro SHEAR-4 in Session III was giving a non-linear transformation. In this session the revised macro SHEAR-11 was used to investigate the linearity of the shear transformation and its invariant lines and characteristic values. The invariant lines discussion will not be analyzed here.

What happened

We sat at the computer, but this time Jill was in control of the mouse. Since it was her first time using *Cabri*, she had trouble drawing a vector. Everybody in the room gave advice. It got so noisy that Jill, sighing heavily, said, " everybody talking at once." After some time, the investigation finally began.

92. M.: OK. So now you've constructed a vector and you got its image under the shear transformation.(see Fig. IV-1) How would we know if this shear transformation is a linear transformation?

93. Jack: If... what we can do is that we can use the dilation... we dilate... the v and then we'll dilate S(v) if... if... they're on the same line it means that S(v) is a linear transformation.

94. Jill: If the dilation stays each on its own line.

(...)

97. M.: It's not only a dilation but it has to be in proportion.

98. Jack: The same factor.

99. M.: Exactly.

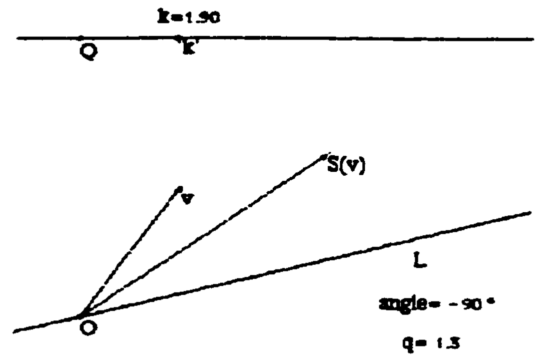

FIG. IV-1 – Opening configuration: the SHEAR problem

With Jack's instructions, Jill constructed the configuration in Fig. IV-2.

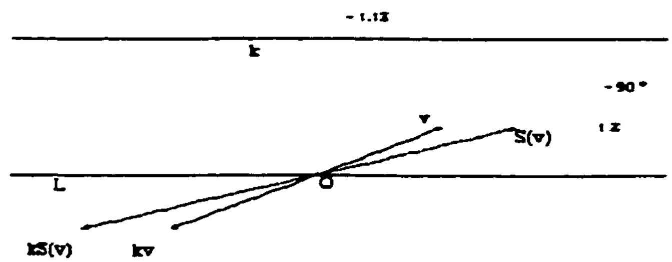

FIG. IV-2 – Students' setup for checking for preservation of dilation.

109. Jack: ... Now we'll move k to see if they are proportional.

110. Jill: Yeah.

111. Jack: ... They're on the same line. So it's proportional, so it means that... S(v) is a linear transformation.

112. M.: So why do you think that they are moving at the same rate?

113. Jack: Because... like if we... drew a line between the endpoint of kS(v) and kv, and the v and S(v), the two lines would be parallel. So it means that every time they are getting bigger it always has the same rate.

114. M.: Right. But... [silence]

Then I proposed to them that their construction would work for all transformations, linear or not. A discussion ensued. At the end, Jack said:

> 122. Jack: [When the transformation was not linear] it was not on the same line so... and it was... and sometimes it was on the same line but it was much bigger than the other things which was not... maybe it was dilated but not by a factor k which meant that it was not linear. I think if it's on the same line and it's proportional, it means that it's always that it's always... linear.
>
> 123. M.: OK. I agree with what you said because this is what the definition of a linear transformation, but what I'm saying is the steps you took... I'll always get this proportionality and can you.. can you find out why? Maybe you can use that diagram [the tree diagram from Session II] to see what you did?

With the aid of the diagram, Jack noticed the mistake. He explained it to Jill. Although she repeated the explanation in her own words, she still dilated S(v).

> 143. M.: No. See what happened? You used the wrong... you used the dilation. You dilated this way. Transform kv...
>
> 144. Jack: What we have to do now...
>
> 145. Jill: ... to see if it's on the line.
>
> 146. Jack: Yeah. We have to do this again. Select SHEAR-11 from the macros menu and click on L. angle, q, v...S.
>
> 147. Jill: Like can't we just...
>
> 148. Jack: No. We have to use... because we have... we must use... shear...

After constructing S(kv), they concluded that dilation was preserved under the shear transformation.

> 169. M.: ... OK. And so now, is this a linear transformation?
>
> 170. J&J: Yes.
>
> 171. M.: OK. And so you don't have to check for the sums?

172. Jack: No, because... We found if it... we find if it's linear by dilation or by the vector sum. Because... we cannot find that the transformation is linear by dilation and it won't be linear by vector sum.

173. Jill: That's what Dr. Dreyfus asked us last time. It could be linear with one and not linear when you try to prove it the other way.

At this point, I reminded them of the semi-linear transformation (Example 4) from Session II. I opened a new *Cabri* file and I demonstrated how the macro is used. We proceeded to check for the preservation of the operations. There was no problem with the dilation.

To check for the vector sum, the following configuration was produced (see Fig. IV-3a). After the vector sum T(v)+T(w) was constructed, they concluded that the vector sum was not preserved (see Fig. IV-3b). I asked them if the whole transformation was linear. They said no. Jack concluded that both tests have to be done and that since only the dilation test worked, that was probably why the transformation was called semi-linear.

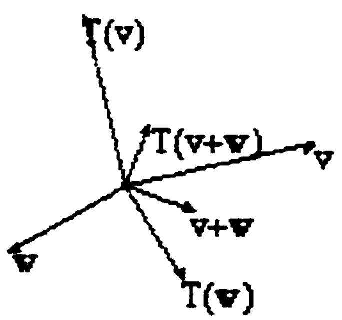

FIG. IV- 3a – Configuration to be checked for preservation of vector sum.

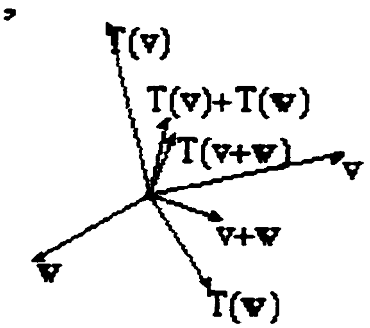

FIG. IV-3b – Vector sum is not preserved

They proceeded to test the vector sum of the shear transformation. They had no trouble concluding that shear is a linear transformation. Jack said, "This transformation is completely linear", as opposed to "semi-linear".

Analysis

At the time of the experiment, we thought that the students did not understand the concept of applying a transformation to the whole plane because of the language that they were using. They were interchanging the words "transformation" and "S(v)" as if they were the same word. In this session, Jack twice said, "S(v) is a linear transformation"[6] (Lines 93 &111), indicating that the image vector represents the transformation. But now I believe that Jack (I am not sure about Jill) understood that a transformation is applied to every vector in the plane but did not have the proper vocabulary or symbols to express

[6] Throughout the analysis, I will talk about the students' misconception that an image vector represents the linear transformation -- i.e., T = T(v). This is correct if one considers v to be a general or arbitrary vector; however, I do not believe that the students can distinguish between a specific vector and a general vector, which is why I consider their interpretation of T = T(v) to be a misconception.

himself correctly. In my discussions with them, I never stressed the fact that T represents the transformation and T(v) represents a vector. The only place that they saw the distinction was in the instructions of the worksheet. I am a firm believer that students do not learn proper terms and symbols on their own. They have to be told of the correct way of saying things (verbal interpretations) and how to write them symbolically, otherwise they will write them in any way they want. I think that is because students believe that math is about correct answers, and as long as the final answer is correct, then that is what matters.

But as important as the idea of applying the transformation to the whole plane is the concept of linearity. I do not think that Jack has fully internalized the concept of a linear transformation yet. It seems that he can explain definitions (Line 122) and concepts (Line 113) correctly, but has to go through a process to retrieve them. But even then, it seems that he has trouble applying these definitions and concepts to his work. This is seen from the beginning when he tried to describe the preservation of dilation under the shear transformation (Line 93). He said that if kv and kS(v) are on the same line, then shear is a linear transformation. He also only mentioned the ON-LINE property of a dilation, but not the RATIO property. I had to remind the students of the dilation factor k (Line 97).

When the students proceeded to draw the figure for the dilation test, they created vector kS(v) instead of S(kv) to do the "similar triangle" test. This is consistent with

Jack's explanation of the dilation test discussed above (Line 93). When Jill moved k along the number, obviously the vectors moved in proportion. Since the transformation passed the visual test of their schema for a linear transformation, they declared that shear is a linear transformation.

When confronted with the fact that their construction will always give the proportional image, whether the transformation is linear or not, Jack defended his conclusions very well with a good explanation of linearity visually (Line 122). He gave counter-examples when he described the Translate-tip and Quadratic examples from Session II (Examples 2 &3). But he still did not realize that kS(v) had been constructed instead of S(kv). It seems that he understood the final results of the dilation test, but does not know how to get there by himself. It was not until I referred him to the tree diagram that he realized the mistake.

The difference between Jack's and Jill's comprehension of linear transformations is that Jack understood some of the ideas and concepts, but had not fully internalized them. In other words, he could not recall information immediately and could not apply concepts and definitions to procedures because he did not see the connection. They were separate entities in his mind, but with some help, he was able to make the connections. On the other hand, Jill's base knowledge on linear transformations was far smaller than Jack's that it was very difficult for her to make connections between different concepts and procedures. She still made the same mistake of producing kS(v) instead of S(kv), even though a few minutes before she was able to explain linearity tests and equations (Lines 143-148).

After proving that the shear transformation preserved dilation, the students were asked if it was necessary for them to check the vector sum property. Jack said that it was not needed because a transformation that is linear under dilation but not linear under vector sum did not exist (Line 172). Obviously, he had forgotten about the Semi-linear example from Session II (Example 4). I could think of two possible reasons for this: (1) When first introduced in Session II, the semi-linear example had no meaning to the students. It was just another example of a possibility on a tree diagram. They did not know what a linear transformation was or that we were leading them to it. Thus, it did not leave a lasting impression on them. On the other hand, the translate-tip and quadratic examples (Examples 2&3) had left an impression because the similar triangles diagram did not work. That is probably why Jack was able to recall these two examples earlier. (2) The students did not realize that the two tests went together and thought of them as two independent tests.

Jill's response in Line 173 brought back ghosts from the previous session. She was referring to the episode in Session III in which Tommy had asked them if it was necessary to check the conservation of both operations to determine the linearity of transformation. When they had said that only one check was needed, he did not correct them; thus they took their answer to be right. But here Jill took it a step further and said that the second operation did not even have to be conserved.

Worksheets #1 & #2

After completing the activities for the shear transformation, the students were handed Worksheet #1. They constructed a coordinate system based on two non-collinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Through a construction, the students were led to the conclusion that any vector $\mathbf{v}$ is equal to $a\mathbf{v}_1+b\mathbf{v}_2$, where (a, b) is an ordered pair of numbers, representing the position of $\mathbf{v}$ with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$. In the discussion, I formally defined for them that the ordered pair (a, b) is called the "coordinates of vector $\mathbf{v}$ in the basis $<\mathbf{v}_1, \mathbf{v}_2>$".

In Worksheet #2, the students discovered that the coordinates of $\mathbf{v} + \mathbf{w}$ are (a + c, b+ d) and the coordinates of $k\mathbf{v}$ are (ka, kb), where (a, b) and (c, d) are the coordinates of vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, and k is any scalar.

Both worksheets were completed successfully that an analysis is not warranted.

Worksheet #3

Design

The theme of Worksheet #3 was the Linear Extension Problem (see Section 3.1 of the present chapter). The problem for the students was: Given the images of a pair of non-collinear vectors under a linear transformation, to find the image, under this transformation, of an arbitrary vector. More precisely, the students were told to construct five different vectors from the origin in a new *Cabri* figure. The vectors were labeled $\mathbf{v}_1$, $\mathbf{v}_2$, $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$ and $\mathbf{v}$. The students were to assume that $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are the images of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, under a certain linear transformation. The question was: "From the information given, would you know where should the vector $T(\mathbf{v})$ be? Can you construct it?"

The students were expected to construct a coordinate system on basis $<\mathbf{v}_1, \mathbf{v}_2>$ and to find the coordinates (a, b) of $\mathbf{v}$. Since $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and the transformation is linear, then

$$\begin{aligned} T(\mathbf{v}) &= T(a\mathbf{v}_1 + b\mathbf{v}_2) \\ &= T(a\mathbf{v}_1) + T(b\mathbf{v}_2) \\ &= aT(\mathbf{v}_1) + bT(\mathbf{v}_2) \end{aligned}$$

The students would then dilate $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ by factors a and b, respectively. Vector $T(\mathbf{v})$ would be the vector sum of the resulting vectors.

We wanted the students to work completely on their own. I was not supposed to give any hints towards the solution of the problem. A discussion of their results was to follow only when they had finished the problem. No discussion was to be held if the students did not finish by the end of the session.

What happened

Following the instructions, the students created the configuration in Fig. IV-4. Jill started to read the question,

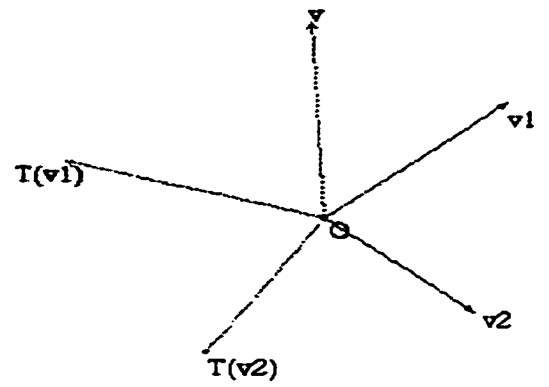

FIG. IV-4 – Configuration drawn by students for Worksheet #3

272. Jill: (reading) "... The question is: From the information given, would you know where should the vector T(v) be? Can you construct it?"

273. Jack: (after some thought) I don't think we could construct it because we've only... we put $T(v_1)$ and $T(v_2)$ ... I think that's supposed to be the same transformation but we put it at random. You know we don't really know where to put the four vectors. So $T(v_1)$...

274. Jill: (reading) " Would you know where should the vector T(v) be?"

275. Jack: Put... like $T(v_1)$ and $T(v_2)$ are supposed to be the images of v_1 and v_2 by the one linear transformation but $T(v_1)$ could be this, could be that, you know it's not... it's not like it's um... um... it's not like it's it's real... it's a real transformation. So v_1 could be any...T(v) could be anywhere... since... there's not a real transformation... well not a real transformation but...

276. Jill: You mean we could've put v_2 there?

277. Jack: Yeah... If we knew where to put $T(v_1)$, where to put v_1 and... and the coordinates, that would make more sense but since it's... we put it wherever we wanted. We don't know if T is linear or... I don't know if... I don't think we could construct T(v).

At this point, Anna reminded them that they were assuming that the transformation was linear[7]. After a short discussion between the students, Jill asked if they could move the vectors around. Anna told her that that would change the transformation. Jack agreed.

290. Jack: Exactly. So if we move them I think that... they're supposed to be the same transformation but um... I don't know, I don't... I don't think we have enough information to get T(v). (a long silence, kept looking at the tree diagram) Since... the only thing we know is that since it's linear kT(v) is equal to T(kv), like $kT(v_1)$ is equal to $T(kv_1)$ and if we had v_1 and v_2 [v_1+v_2]... the transformation of v_1 and v_2 [$T(v_1+v_2)$] is equal...

291. Jill: I think... both...

292. Jack: ... to the sum of $T(v_1)$ and $T(v_2)$. But...

293. Jill: But you have to have... think that they are direct images.

294. Jack: But we don't know if it's by rotation or... either as I told you, do you remember when you ... we could've put $T(v_1)$ here, $T(v_2)$ there, I don't know... we could've... we put the vectors wherever we wanted.

This discussion between them went on for a quite while, in which Jill kept insisting that "assuming you have done the transformation" and Jack continued to argue that "it's not like it [the transformation] really means something [e.g., a rotation, a reflection, etc.]". Jill redrew the configuration on the worksheet to explain the

[7] From this point on, all our discussions will involve linear transformations. Therefore, any talk about transformations implies linear transformations.

assumption of a linear configuration. To explain his point, he drew a different configuration and asked her to find T(**v**) (see Fig. IV-5). She saw his point and agreed with him. That was when we decided to break the "no hints" rule. I asked them what the information can they get from the problem. Jack said that since it is a linear transformation, they can get $T(\mathbf{v}_1) + T(\mathbf{v}_2)$ and $T(\mathbf{v}_1+\mathbf{v}_2)$. At this point Anna told them that they could find the coordinates of **v** in the basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. Ralph sounded surprised when he said, "Oh, we have to use coordinates."

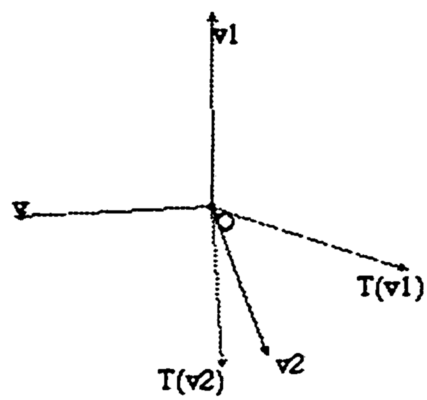

FIG. IV-5 – Configuration drawn by Jack on paper

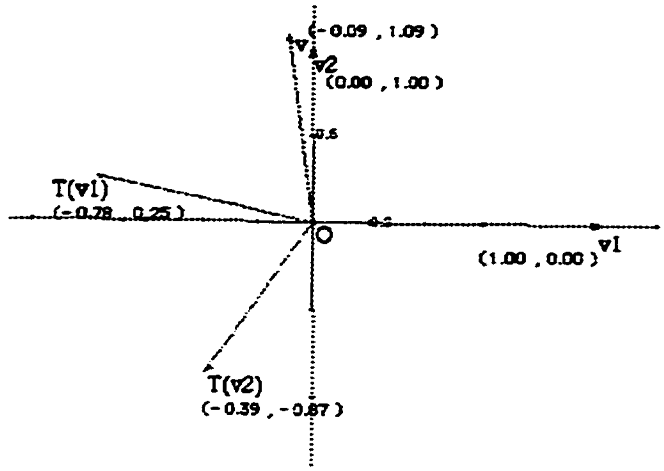

FIG. IV-6 – Configuration of the problem after axes were moved.

Jill drew the axes on $\mathbf{v}_1$ and $\mathbf{v}_2$. After finding the coordinates of all the vectors, she moved the axes so that they looked like the ordinary, orthogonal axes (see Fig. IV-6). Using the coordinates, Jack wrote down the following information on the worksheet:

$$T(v_1) = -0.78v_1 + 0.25\, v_2$$
$$T(v_2) = -0.39v_1 - 0.87v_2$$

$$v_1 = (1, 0)$$
$$v_2 = (0, 1)$$

$$v = -0.09v_1 + 1.09v_2$$

Looking at his equations, he said:

> 347: Jack: ...So it means that... what happened to $\mathbf{v}_1$ is that... that, well, minus... by .78... but initially $\mathbf{v}_1$ was (1,0)... right? And by the transformation it became -.78, so it means that... it was... substracted by .78. And they add .25 to $\mathbf{v}_1$... to $\mathbf{v}_2$.[8]

---

[8] These calculations and the ones following will be shown clearly in the analysis section.

The students stared at the screen, mumbled to each other, and sat in silence. After a while, Jack proposed

> 355. Jack: ... v_1 was one, right? And it turned into -.78... which got subtracted by -.87 and added by .25. If we do the same to v maybe we'll get T(v). And if we do that like... v_2 is zero minus .39... one minus well... 1.87. If we do that to v we'll get T(v). That means $T(v_1)$ is equal to...they don't have the same... but I think that's supposed to be the same... it's supposed to be the same transformation.

The students reread the problem. Jack said that he could find the coordinates of T(v). He started to do calculations on the screen. Meanwhile, Jill was looking at Worksheet #1. She wrote the equation $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Then she applied the transformation T to the equation and wrote "$Tv = Tav_1 + Tbv_2$". She showed the equation to Jack. He told her that it made sense, but explained again what he was trying to do:

> 371. Jack: What I did is... Well I moved v_1... from v_1 to... was transformed into $T(v_1)$, like their x-coordinate... equals one, then it turned to -.78. And we say it was −1.87. Because (looking at Worksheet #2)... I think they give the addition...
>
> (...)
>
> 377. Jack: (Doing calculations out loud)...So, if what I'm doing is correct, T(v) would be... would have these coordinates. So T(v)... -1.87... and 1.34. So it would be... (a long silence)... if we use $T(v_1)$ plus $T(v_2)$...no. If we do $T(v_1)$ plus $T(v_2)$, the vector sum, we have $T(v_1)$ plus $T(v_2)$ and... well... no. (a shorter silence) I don't know but I think what I am doing is correct. Well, I'm not sure. But maybe...

At this point I told them that they had a few more minutes left and if they could wrap things up. Jill said that they had an idea. Anna asked her about it.

386. Jill: Because from our other formula $av_1 + bv_2$, we get v. 'Cause we have our transformation by a factor a of v_1 and a transformation of v_2 by a factor b, we get out transformation of v.

387. Jack: v_1... well when it's transformed... the coordinates will be changed. If we apply the same... operations on v, you should get T(v).

Anna reminded them that T is assumed linear and asked them about the factor in Jill's equation.

396. AS: ... You have T of av_1 on the screen. You have $T(v_1)$, how can you get T of av_1 on the screen?

397. Jill: By putting in our av and then transforming our av.

398. AS: You can use the fact that T is linear.

After a few answers that were going nowhere, I asked them how they got the equation. Jill explained:

410. Jill: Basically from this one... the one we did beforehand a factor a and a vector... when we did the dilation...plus the factor b on v_2 would give us v. I just... so I just thought of our transformation of av_1 plus our transformation of bv_2... transformation of v.

411. Jack: Can we have like -.09 multiplied by -.78 plus 1.09 multiplied by .25... no by $T(v_2)$, for v.

Because it was difficult to follow the numbers he was trying to say, I asked him to write it down. He wrote "Tv = -0.09(-0.78, 0.25) + 1.09 (-0.39, -0.87)". Anna said that they had the coordinates and that they will draw it on the screen next time.

Analysis

At the time of the experiment we were surprised that Jack was stuck on the fact that he had to assume that the transformation was linear and that it was Jill who was more open-minded about the random placement of the vectors. But after careful analysis of this worksheet, I believe that it was Jack's understanding of concepts that created this stumbling block. In the discussion between them on this subject, Jack was very adamant about the fact that the transformation was not 'real' –i.e., not concrete (Line 275). He insisted that by randomly placing the vectors, he cannot tell the type of transformation. Was it a rotation? A reflection, perhaps? (Line 294) I believe this shows that Jack understood the fact that a transformation is applied to the whole plane and that there is a preservation of a relationship between vectors and their images. The only problem was that randomly placed vectors do not fit his schema of a linear transformation. He believed that a linear transformation had to make visual sense, where he can see how each image was similarly created from its corresponding vector.

On the other hand, I think that it was Jill's lack of comprehension of these ideas that made her accept the assumption that a linear transformation had been applied to their configuration (Line 293). She took the problem at face value without questioning it. This is a perfect example that shows how Jack's thinking and Jill's thinking are at different levels. She thinks in very specific terms. She takes one problem and/or situation at a time. She does not generalize. For example, when she wanted to explain her point of view to Jack, she drew the exact configuration from the screen on a piece of paper. When he drew for her a different configuration, she got convinced of his concern and said, "That's why you're saying [it is not linear]... I'm saying it [that it is linear] looking at this one [the configuration on the screen]." Consistent with previous cases, a concrete, visual image was again needed for Jill to understand a point.

Further proof that Jack understood the conservation of the relationships between vectors was when he agreed with Anna that moving the vectors around changes the transformation (Line 290). Surprisingly, he did not object when Jill, after constructing the axes on $\mathbf{v}_1$ and $\mathbf{v}_2$, moved them to create a perpendicular pair. Maybe for him, the axes and vectors are not connected, even though one was defined on the other. This could be explained from an incident that occurred in Worksheet #1. After learning about the coordinate axes in *Cabri*, the students moved them so that they were perpendicular to each other. When asked about why they did it and if it made a difference to the problem, Jack replied that the solution was not affected and that ever since he was a kid, they had always used perpendicular axes. Unfortunately, that problem was introducing the notion of coordinates in a basis. Therefore, changing the axes around did not matter since just the basis was changing. He must have thought the same way in Worksheet #3, not realizing that he was contradicting his beliefs about transformations. This highlights the fact that he lacked connections between concepts.

When Anna told them that they could use coordinates, he sounded surprised. He did mention coordinates before (Line 277). I believe at the time he was so bothered by the idea of being able to create a linear transformation by randomly placing vectors that the idea of the coordinates was secondary and was later lost when he was trying to justify his objections to Jill.

After finding the coordinates of all the vectors, Jack showed that he understood Worksheet #1 by expressing them as linear combinations. The following is a detailed and organized description of his analysis in Lines 347, 355, 371 and 377:

Given the following vectors:

$$T(\mathbf{v}_1) = -0.78\mathbf{v}_1 + 0.25\,\mathbf{v}_2$$
$$T(\mathbf{v}_2) = -0.39\mathbf{v}_1 - 0.87\mathbf{v}_2$$

$$\mathbf{v}_1 = (1,\ 0)$$
$$\mathbf{v}_2 = (0,\ 1)$$

$$\mathbf{v} = -0.09\mathbf{v}_1 + 1.09\mathbf{v}_2\text{"}$$

<u>What happened to $\mathbf{v}_1$</u>

The coordinates of $\mathbf{v}_1$ (1,0) became the coordinates of $T(\mathbf{v}_1)$ (-0.78, 0.25). The 'operation' was a subtraction of $T(\mathbf{v}_1)$ and $\mathbf{v}_1$:

$$-0.78 - 1 = -1.78$$
$$0.25 - 0 = 0.25$$

Therefore, (-1.78, 0.25) represents the transformation.

Since $T{:}\mathbf{v}_1 \longrightarrow T(\mathbf{v}_1)$

$$(1,0) + (-1.78,\ 0.25) = (-0.78,\ 0.25)$$

then $T{:}\mathbf{v} \longrightarrow T(\mathbf{v})$

$$(-0.09,\ 1.09) + (-1.78,\ 0.25) = (-1.87,\ 1.34)$$

Therefore, $T(\mathbf{v}) = (-1.87,\ 1.34)$

<u>**What happened to $\mathbf{v}_2$**</u>

The coordinates of $\mathbf{v}_2$ (0,1) became the coordinates of $T(\mathbf{v}_2)$ (-0.39, -0.87). The 'operation' was a subtraction of $T(\mathbf{v}_2)$ and $\mathbf{v}_2$:

$$-0.39 - 0 = -0.39$$
$$-0.87 - 1 = -1.87$$

Therefore, (-0.39, -1.87) represents the transformation.

Since $T: \mathbf{v}_2 \longrightarrow T(\mathbf{v}_2)$
$$(0,1) + (-0.39, -1.87) = (-0.39, -0.87)$$

then $T: \mathbf{v} \longrightarrow T(\mathbf{v})$
$$(-0.09, 1.09) + (-0.39, -1.87)) = (-0.48, -0.78)$$

Therefore, $T(\mathbf{v}) = (-0.48,\ -0.78)$

His analysis of the problem clearly showed that he understood that a transformation is applied to every vector, but created his own "rules". In matrix algebra, the transformation is represented by a matrix A, whose columns are the coordinates of the images of the basis. The image of any vector can then be obtained by multiplying matrix A and the vector. Obviously, Jack did not know this. But he knew that when a transformation is applied to a vector, an image, with different coordinates, is obtained. He also knew that a transformation is applied to the whole plane; thus every vector is affected in the same way. So, to find out what happened to each vector, he subtracted the coordinates. That is probably because students usually associate "change" with subtraction. Clearly, Jack lacked the proper algebraic interpretation of vectors and linear transformations, but the way he manipulated the coordinates in his calculations was consistent with what he had just learned in Worksheet #2, --i.e., (a, b) + (c, d) = (a+c, b+d). His systematic approach of applying the same "change" to the vectors shows, I think, that Jack is starting to get an intuitive feeling for a transformation matrix.

At first, he thought that if you did the above calculations separately, first using $T(v_1)$ and then $T(v_2)$, then the two resulting T(v)'s would be equal (Line 355). This was just like the linearity tests, where two separate operations were done and ended up with the vectors overlapping. But when he actually did the calculations, he realized something was wrong but could not see what it was (Line 377).

Unfortunately, other misconceptions and lack of knowledge prevented him from solving the problem correctly. Just as before, he still worked with one vector operation at a time. He did not combine dilation and vector sum together in the same problem. In fact, in this problem, he did not even mention dilation. He referred back to Worksheet #2, in which they had found the rules of both operations in coordinate notation, but applied only the vector sum (Lines 371 – 377). Later on, while the students were discussing their equation "$Tv = Tav_1 + Tbv_1$", Anna asked, "What can you do with this factor [a or b]?" She reminded them that the transformation was linear and referred them to the configuration on the screen, since they had a better feeling for linear combinations visually (Line 396 - 397). Still, they did not realize that they could pull the factors out. This could be because (1) previously, they had seen dilation in the context of the equation

$T(kv) = kT(v)$ only and not as a component of a linear combination, (2) the context of the factor had changed –i.e., k was a scalar on a number line, a and b are coordinates, and/or (3) the way the equation was written, no parentheses were used.

The difference between Jack's and Jill's problem solving techniques is that Jack tries to back up what he is doing --i.e., it has to make sense to him--, whereas Jill works more on intuition. Even though she was the one who came up with the formula "$Tv = Tav_1 + Tbv_1$", she did it by placing a T in front of each term of the linear combination $v = av_1 + bv_2$ because she was transforming the vectors (Line 410). If she had used the proper way of applying a transformation –i.e., $T(v) = T(av_1 + bv_2)$ --, then maybe they would have realized sooner about pulling out the factors.

It was only when I focused their attention to the formula that Jack proposed substituting the values into the equation (Lines 410 – 411). Up to that point, he did not think that the equation was important. He was sticking to the idea of changing the coordinates. When Anna asked them about the idea that they had had, Jill talked about the equation, while Jack mentioned the coordinates (Lines 386 – 387). My teaching intervention (because I was acting as a teacher more than as a research at this moment) of turning their attention to the equation provoked a type of interaction that has been labeled by Wood (1998) as a "focusing pattern". In this form of communication, students are not led to the solution by a series of questions, but instead their attention is focused on an idea/process that they have to re-explain themselves. This promotes revaluation of the students' own thinking and understanding.

The session ended after Jack wrote the equation "Tv = -0.09(-0.78, 0.25) + 1.09 (-0.39, -0.87)" and Anna telling the students that they now have the coordinates of vector **v**. I think that she assumed that they would know how to do the algebra from there. From what we have seen so far, I would not count on it.

Worksheet #1 (from Session V)

Design

In Session IV, we saw that the students still did not think in terms of linear combinations, which translated into not using both operations at the same time. To begin Session V, we designed a worksheet to help the students to overcome these difficulties and to guide them towards a geometric solution of the problem in Worksheet #3, as promised.

Worksheet #1 consisted of four activities. The first three activities conveyed notions that were needed to solve Activity 4, which was the exact problem from Worksheet #3 (Session IV). The four activities were:

"ACTIVITY 1

Open a new CABRI figure.
Put the origin O and draw two vectors, **v** and **w**.
Construct the vector $2.1\mathbf{v} - 3.7\mathbf{w}$ on the screen.

ACTIVITY 2

Open Figure V.1 from the desktop.
You are given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
Construct a vector $\mathbf{v}$ whose coordinates in the basis $<\mathbf{v}_1, \mathbf{v}_2>$ are (-1.2, 2.3).

ACTIVITY 3

Open Figure V.2 from the desktop.
Under a certain linear transformation T, the vector $\mathbf{v}_1$ got transformed into $T(\mathbf{v}_1)$ and $\mathbf{v}_2$ got transformed into $T(\mathbf{v}_2)$.
Construct the vector $T(2.4\mathbf{v}_1 - 3.1\mathbf{v}_2)$.

ACTIVITY 4

Open Figure V.3 from the desktop.
Under a certain linear transformation T, the vector $\mathbf{v}_1$ got transformed into $T(\mathbf{v}_1)$ and $\mathbf{v}_2$ got transformed into $T(\mathbf{v}_2)$.
Given the vector $\mathbf{v}$ construct the vector $T(\mathbf{v})$."

Upon completion of the worksheet, a discussion was to follow, in which I had to make sure that the students understood that a linear transformation can be completely determined by giving the images of two non-collinear vectors.

What happened

The students completed the worksheet without any major incidents, but a few interesting observations were made.

After reading Activity 1, Jack remarked, "So it's a vector sum of dilated vectors." In Activity 2, they put the axes, but did not use them. After constructing the required vector **v**, Jack wanted to move the axes to make them perpendicular. He could not because we had fixed the given vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that they will not be moved. Jack recognized right away that Activity 4 was the same problem as the one in Session IV.

In all of these activities, it was Jack who came up with the plan. Then, he would explain to Jill what to do. Sometimes, they had to go over the plan a couple of times for her to understand it, but then, as seen before, she would do the same mistake in the construction. This could be attributed to her lack of understanding, but at the same time she had not yet mastered the *Cabri* commands. On several occasions, her reactions reflected her uneasiness with the software. When her dilated vector went off the screen and Jack told her that is because vector **w** is too big, she sighed, "Oh, boy!" And when he told her that they had to do a vector sum, she exclaimed, "Oh, my god!"

Upon completing the activities, I sat with them at the computer. I asked them about the first activity. Both of them had no trouble explaining what they did.

> 229. Jack: Well, we found that...
>
> 230. Jill: ... the sum of the dilations of vector v and vector w.
>
> 231. Jack: You mean that we have to dilate v and w and then... do the sum of those two vectors.

I concluded by telling them, "The point of this is that you can do both operations... you can work them together".

I then asked them about what they learned from Activity 2. Jack had trouble with wording his explanation, but essentially he said that the coordinates of a vector **v** are the factors in the linear combination of **v** in basis <$\mathbf{v}_1$, $\mathbf{v}_2$>. He realized that both Activities 1 and 2 gave the same result. When I asked him about the methods used in each activity, he said:

> 250. Jack: The only thing is that we have to... draw the axes... because they gave us the numbers in coordinates. That's the only thing.
> (...)
> 254. Jack: ...Like the first activity, if they give us "Draw the vector that is 2.1", that's the coordinates in the basis <v, w>. Or (2.1, -3.7)... it's the same thing.

But when Tommy asked him about what he did after drawing the axes, Jack realized that they were not used and said, "In fact we don't... the axes is just to see where it will go, but it's not... it's not that important."

Next, I asked them about their construction in Activity 3. Jill explained that they did the dilations of the transformations, and then they took their vector sum. Looking at their diagram, I asked them about the factor being on the outside of the label. Jack explained that $-3.1T(\mathbf{v}_2)$ was the same as $T(-3.1\mathbf{v}_2)$ since the transformation was linear.

After explaining their procedure for Activity 4, I wanted them to make a conclusion about what determines a linear transformation. I started to ask that if they were given $\mathbf{v}_1$, $\mathbf{v}_2$ and their images, when Jack interrupted me and explained the whole procedure again, concluding, "... since we have the coordinates of **v**, we will be able to get

$T(\mathbf{v})$ just by dilating $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ by those factors." I asked Jill if she agreed, and she said that she did not understand my question. So, we turned to the green table, and I wrote the following and asked her to complete the equation:

Given $\mathbf{v}_1$, $\mathbf{v}_2$, $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$
$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ in basis $<\mathbf{v}_1, \mathbf{v}_2>$
$T(\mathbf{v}) =$

She wrote:

$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$
$T(\mathbf{v}) = aT\mathbf{v}_1 + bT\mathbf{v}_2$
$T(\mathbf{v}) = T(a\mathbf{v}_1) + T(b\mathbf{v}_2)$

When I asked her about the difference between the two equations that she wrote, she explained that they were the same. She went on to explain that $T(\mathbf{v})$ can be obtained by either (1) finding the vector sum of the transformations of the dilated vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, or (2) dilating the transformations of $\mathbf{v}_1$ and $\mathbf{v}_2$, and then adding the resulting vectors. Then I asked them about which method was used in their solution. She said that they had used the first method. Jack agreed, but after some discussion, he said that the first method could be used only if they had a macro of the transformation.

Wrapping things up, I asked them:

317: M.: So what we can conclude from here is that we can get an image of any vector as long as we know what?

318. Jack: The coordinates on its own basis.

319. M.: So all we need is the coordinates? All I need is (a,b)?

320. Jack: No, but you need also (looks through some papers)...

321. Jill: It's linear, too.

322. M.: Yeah, right. Yeah... To determine any transformation... the image of a transformation... you need the images of the basis vectors and the coordinates of the vector. And you can find the image.

Analysis

At the end of Session IV, we thought that Jill began holding her own during their discussions. From these activities, things went back to what they were before, with Jack coming up with the plan first and telling Jill what to do on the screen. One could sense that she was still not comfortable with the software and the concepts.

From Activity 1, I think that they finally understood that both operations could be used at the same time. (Lines 230 –231).

After discussing Activity 2, Jack seemed to understand the connection between the first two activities (Line 254). But, at the beginning, he did two things typical of students: (1) assumed that if coordinates were given, then axes are needed (Line 250), and (2) tried to make the axes perpendicular to each other, even though different axes were discussed when coordinates were introduced in Worksheet #1 of Session IV.

Activity 4 showed us that the students were now comfortable using the linearity equations and could switch from one to the other. But they still had trouble relating their written work to *Cabri*. Jill had done this mistake several times before. Although Jack also made the same mistake, he was able to correct himself.

From my concluding remarks, it was agreed that given the images of the basis vectors, the image T(**v**) of any vector **v** can be constructed (Line 322). Even though I did mention "to determine the transformation", my emphasis was on constructing the image of a vector. This could be part of the reason why the students had trouble with Worksheet #2 (see next section).

3.2.5 Session V

Worksheet #2

Design

In this worksheet, the previous problem was reversed. Instead of finding the transformation given the images of the basis vectors, they now had to define a given linear transformation by its images on a basis. In other words, they had to find a configuration of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, and $\mathbf{w}_2$ so that a specific transformation is obtained. The question of the uniqueness of the configuration was to be brought up in the discussion after the completion of the worksheet.

The students were given the following problem:

•Open Figure V.4.1 from the desktop.

In this figure you have six vectors and a line. You can move these vectors around, but the line is fixed.

We assume that these vectors are: v_1, v_2 – two non-collinear vectors which are regarded as a basis; w_1, w_2 – regarded as images of v_1 and v_2 in some linear transformation T. Check this: move the endpoint of v so that v overlaps with v_1: observe that T(v) overlaps with w_1; move the endpoint of v so that v overlaps with v_2: observe that T(v) overlaps with w_2.

Different configurations of the vectors v_1, v_2, w_1 and w_2 usually give different linear transformations. Check this: move the endpoints of vectors v_1, v_2, w_1, w_2 and observe that T(v) changes.

The question is: Can you find a configuration of the vectors v_1, v_2, w_1, w_2 that would make the transformation into
(a) a shear along the given line L?
(b) a projection[9] onto the given line L? (Open Figure V.4.2 from the desktop to work on this question b).

Call the teacher when you are done."

To solve Problem (a), we expected the students to build the shear transformation using one of these two methods:

1. Apply the SHEAR-11 macro to v_1 using a number q that must be numerically edited on the screen, and then redefine the endpoint of w_1 to be the endpoint of the image of v_1 under shear. Repeat the procedure for v_2 and w_2.

2. Put vectors v_1 and w_1 on the line L and making them equal. Draw a parallel through the endpoint of v_2 and redefine the endpoint of w_2 to lie on this line.

To answer Problem (b), the students were expected to create a projection by drawing a line, perpendicular to line L, through the endpoint of v_1, and then to redefine the

[9] All projections are assumed to be orthogonal.

endpoint of $\mathbf{w}_1$ to be the intersection point of line L and the perpendicular line. They had to repeat the procedure for $\mathbf{v}_2$ and $\mathbf{w}_2$.

Problem (a)

What happened

Jill opened the required file with the configuration in Fig. V-1. Both students started to read the problem. As instructed, Jill moved vector **v** to overlap with $\mathbf{v}_1$.

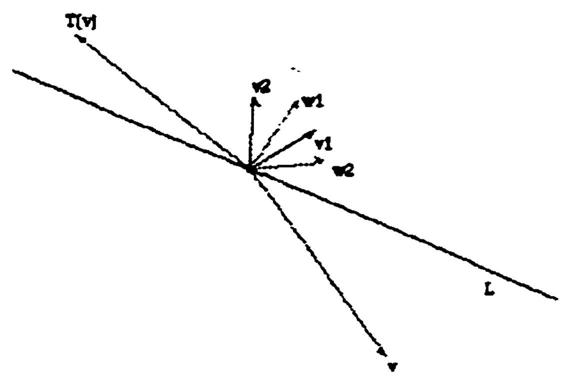

FIG. V-1 – Opening configuration of Problem (a)

10[10]. Jack: Like the... so T(v) would be equal to w_1, right?

11. Jill: Well we can see by...

12. Jack: Yeah. We have observed that T(v) overlaps with w_1. Yeah. (reading) "Moving the endpoint of v so that v overlaps with v_2:"

13. Jill: (moves v to overlap with v_2)

14. Jack: We see that T(v) overlaps with w_2. (reading) "Different configurations of the vectors v_1, v_2, w_1, w_2 give different linear transformations. Check this: move the endpoints of v_1, v_2, w_1, w_2 and observe that T(v) changes."

---

[10] In the transcript for this session, the line numbers were reset to start at 1 at the beginning of Worksheet #2.

Jill moved the vectors around the screen. The students discussed the problem represented on the screen.

> 26. Jack: ... But you see v is never moved. v is not supposed to move, right? v is never moving. It's only T(v) that changes. Here the question is ...
>
> 27. Jill: (reading) " Can you find a configuration of the vectors v_1, v_2, w_1, w_2 that would make the transformation into – a shear along the given line L."

The students did not remember the details of the shear transformation and retrieved the handout with its description. A discussion ensued on what a shear was and how to proceed to solve the problem. Although the factor q was mentioned briefly, most of the discussion involved the parallel line condition of the shear transformation. Jack came up with the plan of drawing a line parallel to line L, redefining the endpoint of **v** on the parallel line, and moving $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ so that the tip of T(**v**) would also be on the parallel line. But it took Jill some time to understand what Jack wanted to do. At some point during the discussion, Jill asked:

> 61. Jill: What are we looking for?
>
> 62. Jack: We have to find a configuration of vectors v_1, v_2, w_1, w_2 that gets us a shear along the given line L.
>
> 63. Jill: So we can put um...
>
> 64. Jack: So once we get the parallel line that is parallel to L. So we should put... we could draw a parallel line and then move v and T(v) so that T(v) and v is on the same line.

The students had trouble remembering how to draw a parallel line through **v**. When they finally did it, it was not through **v**, so Jack suggested putting **v** on the line and moving "w_1, w_2 so that T(v) is on the line too." After some more difficulty with remembering the *Cabri* commands, Jill redefined the endpoint of v on the parallel line.

> 100. Jill: There!
>
> 101. Jack: And then you ... and then let's move v_1, v_2,...
>
> 102. Jill: There! (moves v along the parallel line and watches T(v) move along with it around the screen but not on the line) (whispers something to Jack)
>
> 103. Jack: Ok. So now... you want to do um... you're going to redefine this again... redefine T(v)? Yes. You could do that. Just like... just like... do what, like you did. Maybe it'll work.
>
> 104. Jill: Um... T(v) is on the line.

After redefining T(v), Jill moved v_1, but T(v) did not move like before. They both giggled, and Jack said, "Maybe, I think we shouldn't have done that." At this point, Tommy interrupted them and told them that they were allowed to move any vector except T(v). He explained that it was a flaw in the design that enabled them to redefine T(v).

They restarted the problem in a new file and quickly redefined the endpoint of vector v on a line parallel to line L. They started moving the other four vectors to get T(v) on the parallel line (see Fig. V-2). During their discussion on whether the tip of T(v) was actually on the parallel line, the factor q was mentioned. Jack said, "We don't need q there." Later in the discussion, his reasoning was explained:

> 152. Jack: ... Because it's any number q, you know? That's what we're looking for basically. It has to be on the line. So we know. But now it's not. (Jill had moved the vectors). Put it...
>
> 153. Jill: So make it move...

154. **Jack:** ... such that the line through the endpoint v and T(v) is parallel to L. We've got that already. So it's...

155. **Jill:** The line through... like here you mean?

156. **Jack:** Yeah. It's parallel to L so it's um... it's shear along the given line L. That's all they asked for. They don't ask for q, vq, whatever... Now we have to do for projection onto the given line L.

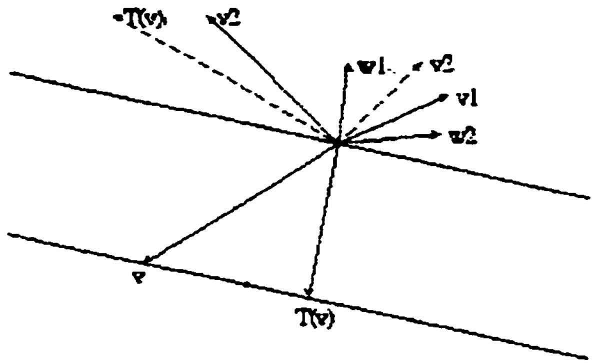

FIG. V-2 – Before (dotted) and after (solid) positions of vectors v_2 and T(v) after students moved v_2, to get T(v) on the parallel line.

Problem (b)

What happened

Jill opened the appropriate file to reveal the same starting configuration as before (see Fig. V-1). The students did not remember what a projection was and there was no previous handout for it since it was introduced as a demonstration. After a brief interruption caused by Jill's nosebleed, I explained a projection to the students. Using v_1 as an example I told them that if you take the line perpendicular to line L through the endpoint of v_1, then the projection would be on line L, from the origin to the intersection with the perpendicular line. I used my finger to point at the vectors on the screen while I talked. They proceeded to solve the problem.

During their discussion, it became clear that Jill still did not understand what a projection was. At one point, she thought that the only condition for a projection was T(v) being on line L. Jack, gesturing and pointing to the screen, explained the transformation to her again. This time she had trouble drawing the appropriate perpendicular line (see Fig. V-3). She finally said, " I'm not sure what we're doing here. I don't know... I don't understand what we're trying... Projection onto the given line L."

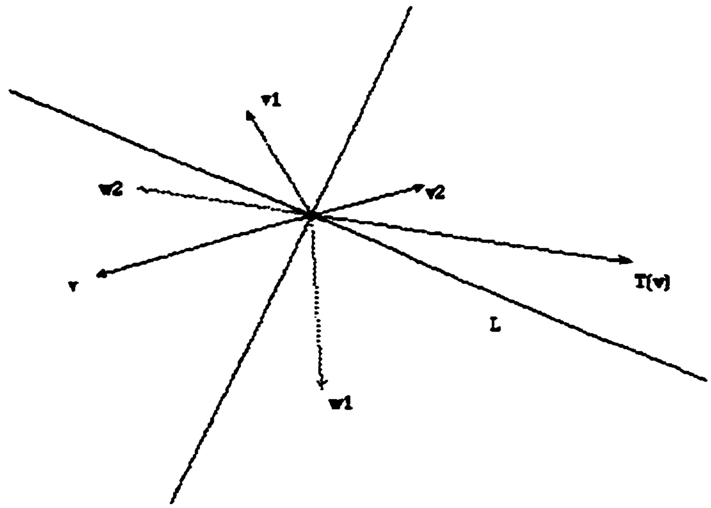

FIG. V-3 – Wrong perpendicular line is drawn.

I sat with them at the computer and took control of the mouse. I erased all the lines they had drawn to get back the original configuration:

> 223. M.: ... what a projection is... Ok. Let's do it for v_1...
> Now when you.... Let's project this vector on this
> (applies macro PROJECTION-1 to v_1). Ok. So this

is your image, this should be $T(v_1)$. Now the way this was constructed is if you take a perpendicular line from this point to this line. (see Fig. V-4)

224. Jill: That's with that.

225. M.: So when you do this, then you... the vector that is on line L from the origin to there is the projection. So you see if I move v_1, now I'll see that the image always is perpendicular to that line. So you are clear what a projection is?

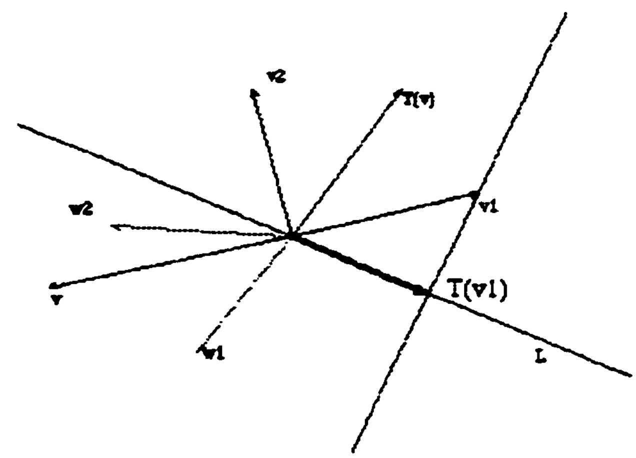

FIG. V-4 – Demonstration of what a projection is.

After answering yes, I explained what the question was asking:

231. M.: This is the projection of v_1 (points to the vector on the screen). Ok. Now we want you to configure v_1, v_2, w_1, w_2, this line, place them on the screen, arrange them so that you have a projection, a linear transformation that is a projection.

232. Jack: So that the line that pass through T(v) and v is para... is perpendicular to L.

233. M.: Are you clear or still...?

234. Jack: Clear.

235. Jill: I'm not sure what we're being asked to do.

236. Jack: Like we... well... like let's say this is v, T(v)'s supposed to be like that. Like if we draw a ... draw the line between the endpoint of T(v) and v it's supposed to be perpendicular to L. You know like it's supposed to be a right angle.

237. Jill: Ok.

238. Jack: A right triangle. So that we have to move like those four points so that we get the same thing for v and T(v). T(v)'s supposed to be on the line and per... it's supposed to be perpendicular to L.

239. M.: Do you...?

240. Jill: So if we stick our T(v) on this line...

241. Jack: Yeah. But you're not supposed to move your T(v), you're supposed to move your v_i's.. And T(v)'s supposed to be on the line and it's... it's parallel... it's perpendicular.

242. Jill: Just put a line perpendicular to L with T(v) to the line.

243. Jack: Yeah.

Jill drew the line perpendicular to line L, passing through **v**. They moved vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ so that T(**v**) became the projection of **v**.

255. Jack: ... There it is. Voilà!

256. Jill: But as soon as we move all these...

257. Jack: Yeah. It will move, of course. But that's what they...

258. Jill: Ok. So we positioned it.

259. Jack: Yeah, That's what they asked you for, Since...

260. Jill: So couldn't we have...

261. Jack: Yeah. You can move any, any...

262. Jill: We didn't have to put it in relation to v. We could have put it with any other point.

263. Jack: Exactly. No. Well...

While they were talking, Jill was moving the vectors around. Then the students reconfigured $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ so that T(**v**) is once again the projection of **v**.

273. Jack: Well, you see if you move the vectors, T(v) won't be the projection anymore.

274. Jill: Which ones did we put...

275. Jack: Cause that's all they asked us for. We don't have to do anything else. We have to find a way that T(v) is a projection. And there are many ways we can do that.

Jill was moving w_1 slowly around the screen and was thinking out loud. She then asked Jack,

282. Jill: Projection of w_1 on...?

283. Jack: No, it's not w we're looking for... it's the projection of v we're looking for.

284. Jill: So if we were to put w_1 on our line (moves the cursor from the tip of w_1 to the perpendicular line), it would be the projection of w_1? If we did our line...

285. Jack: Yeah. But it's not... yeah but exactly. I guess we're done.

Discussion of Problem (a)

<u>What happened</u>

I sat with the students at the computer and retrieved their configuration for the shear transformation (see Fig. V-2). I asked them to explain what they did. Jack said that since in a shear transformation the line through the tips of v and T(v) was parallel to line L, they drew a line parallel to L, through v, and moved the other vectors to get T(v) on the line. I then asked them if they would still get a shear if v was moved. Jack answered:

297. Jack: Well um... like you can try but for this specific... no, like it's very specific. It's only for this v that it works because of where v_1 and v_2 are.

298. M.: Ok. So you're saying that...

299. Jack: Like for this one it works but if v's somewhere else it won't work.

We both agreed that what they had created was a transformation that works for only one vector. When I reminded them that a transformation should work for all vectors, Jack agreed. So I asked them, "What would you have to do to make it a transformation that will work for any v?"

310. Jack: Well like if that... like if it was supposed to be a transformation on any v there should... there should be this... there should be the same relationship between like v...

(...)

314. Jack: Ok. Like v is ... like in function um... is in relation with v_1 and v_2. Ok? So ... T(v) would have to be like ... the same relation between w_1 and w_2. I think w_1 is $T(v_1)$ and w_2 is $T(v_2)$.

315. M.: Ok.

316. Jack.: So there should be the same factors or the same vectors on...

317. M.: Do you agree with what he said?

318. Jill: Yeah. They would have to be ... I think like... The same kind of proportion with v to be... to be the same... to be the same... relation to T(v)

319. M,: So yeah, w_1 represents what?

320. Jack: The transformation of v_1.

321. M.: Ok. And w_2 represents...?

322. Jack: The transformation of v_2.

323. M.: Ok. Now, looking at the configuration here, is w_1 the same transformation as w_2? The same transformation as T(v)? In other words, was w_1 obtained by the same transformation as w_2 was obtained?

324. Jack: It doesn't look like it, like it was. Like everything like this is, like what we did like it's not linear or something but not... I doubt it.

325. M.: Ok. So now if you wanted to make this a shear...

326. Jack: Yeah.

327. M.: ... a transformation that is... that works for <u>all</u> vectors v, what should be done?

328. Jack: There should be the same relationship between... like v is equal to av_1 plus bv_2 so that T(v) would be equal to... would be equal to aw_1 plus bw_2.

329. M.: Ok.

330. Jack: Like the same coordinates... not the same coordinates but the same factor.

331. M.: Right. Ok. And w_1 should be...?

332. Jack: Transformation of v_1.

333. M.: Ok. So in this particular case you should have... how should you configure w_1?

After a long silence, Jack said something to the effect that there should be the same relationship between $\mathbf{v}_1$ and $\mathbf{w}_1$, and $\mathbf{v}_2$ and $\mathbf{w}_2$. He came up with the following plan:

343. Jack:...v_1 and w_1 are supposed to be on the same line parallel to L and w_2 and v_2 are supposed to be on the same line parallel to L, so that if we move v, we'll always get T(v) as a shear along ... L of v.

They followed the plan by placing the endpoints of vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ onto their respective parallel lines (see Fig. V-5). I wanted to show them that what they had created was not a shear because $\mathbf{w}_1$ and $\mathbf{w}_2$ were not redefined on the parallel lines; thus will not follow the movements of $\mathbf{v}_1$ and $\mathbf{v}_2$. So I asked them to move vector **v** around , thinking that T(**v**) will move off the line. I was surprised when it did not (at that moment, I did not realize that I should have asked them to move $\mathbf{v}_1$ or $\mathbf{v}_2$ instead). I did not pursue the matter.

I decided instead to go on to the projection problem. I asked them if they knew what their mistake was. Jack explained that since **v** is equal to $a\mathbf{v}_1 + b\mathbf{v}_2$, they should have also made the transformations of $\mathbf{v}_1$ and $\mathbf{v}_2$ into projections, whereas they only made T(**v**) into a projection. I then asked them, "What do you think will determine a certain transformation?"

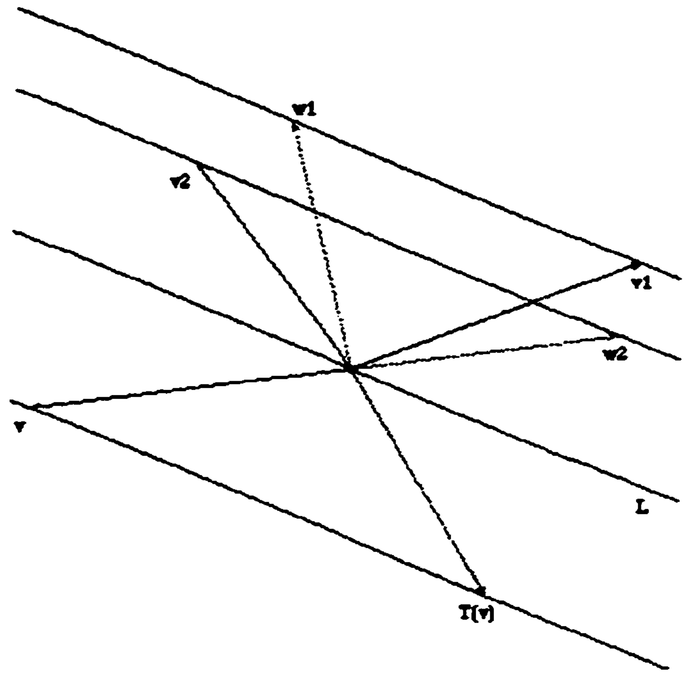

FIG. V-5 – Students' solution to Problem (a)

387. Jill: What do you mean by 'determine'?

388. M.: Yeah. Ok. In other words. Let's see. Determine means, how will you obtain a certain transformation? What do you have to do?

389. Jill: Oh! The two points are on the lines parallel to your ...

390. Jack: No but...

391:. M.: But in any situation, any linear transformation, if I ask you...

392. Jack: By doing the same transformation on v_1 and v_2. And that way we'll get v.

393. Jill: Ok.

394. M.: So the positioning... what determines a certain transformation is the positioning.

395. Jack: Yeah. Of...

396. M.: And what's important is the positioning of...? What's the real important...

397. Jack: What do you mean?

398. M.: Could v_1 and v_2 be anywhere and you still get the same transformation?

399. Jack: Well the... they could be ... well they can be anywhere but the transformation of v_2 is... the transformation has to be the same. Like it's not... it doesn't really matter where v_1 and v_2 are but it's like the transformation of v_1 and v_2 are supposed... are supposed to be the same transformation. It's supposed to be constant

Asking him to clarify what he means by "constant",

405. Jack: v_2 and w_2 are supposed to be... like there's supposed to be the same relationship between v_1 and w_1, and v_2 and w_2.

406. M.: Ok. I think what you're trying to say...

407. Jill: I don't understand...

408. M.: So it's really the positioning of...

409. Jack: The image.

410. M.: Yeah. The image. Ok, in relation to v_1 and v_2... and they have to be the same transformation.

At this point, Tommy and I discussed if they should start Worksheet #3. But we decided that with fifteen minutes left, it was not worth it, and instead, the students (at their own request) should fix the projection problem.

Jill opened a new starting configuration. She drew three perpendicular lines to line L, through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ (see Fig. V-6). Jill, like before, started to move the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ to get T(v) as a projection. Jack explained to her that if they made $\mathbf{w}_1$ and $\mathbf{w}_2$ the projections of $\mathbf{v}_1$ and $\mathbf{v}_2$, then T(v) will automatically become the projection of v. Just like the shear, they placed $\mathbf{w}_1$ and $\mathbf{w}_2$ as projections instead of redefining them. In the ensuing discussion, I asked them about their method:

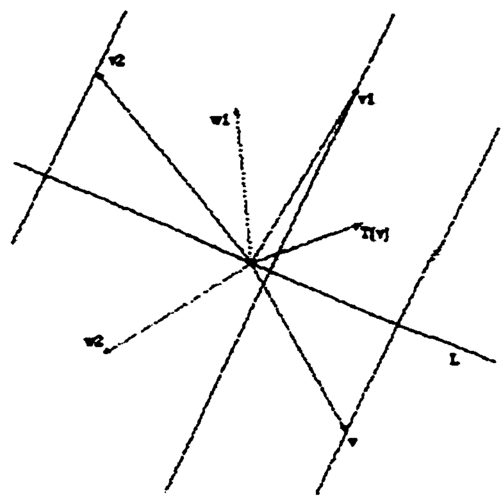

FIG. V-6 – Starting configuration for Problem (b) after discussion.

438. M.: Why do think you got it [T(v)] automatically there [as the projection of v]?

439. Jack: Because v is equal to, well, $av_1 + bv_2$ so that... so we know that T(v) is equal to... vector sum of $aT(v_1) + bT(v_2)$ so if we get $T(v_2)$ and $T(v_1)$, like the image of v_1 and the image of v_2, we'll get T(v).

I then asked them that if $\mathbf{v}_1$ or $\mathbf{v}_2$ were to be moved, would the transformation still be a projection. They tried it and noticed that it did not. The students decided that $\mathbf{w}_1$ and $\mathbf{w}_2$ had to be redefined as the projections of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The students proceeded to redefine the vectors. They ran into some technical difficulties while applying the commands. We all tried to fix it, but time ran out. So I concluded by saying, "As long as you have $\mathbf{w}_1$ and $\mathbf{w}_2$ as the same transformation, you've created a particular transformation."

As the students were putting on their coats, Tommy remarked that there was no problem with the projection, but what they did in the shear problem was not a shear. Jack quickly said that he knows because "it was not the same q."

Analysis

Problem (a) clearly shows that both students still did not have a good grasp of the concept of a linear transformation, although Jack had a much better understanding than Jill. I think they both understood that when T(**v**) moves and **v** stays in the same position, the transformation is changing. But the fact that they redefined T(**v**) onto the parallel line shows that they still did not understand that the relationship between **v** and T(**v**) resulted from the relation between $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$, and the only way to move T(**v**) was by moving $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$. This makes me wonder whether the little "exercise" in the introduction of the problem had any effect on the students. When Jack stated that they observed that T(**v**) overlapped with $\mathbf{w}_1$ when **v** was placed on $\mathbf{v}_1$ and the same happened with $\mathbf{v}_2$ and $\mathbf{w}_2$ (Lines 12 and 14), was he merely repeating what was written in the problem or did he understand the relationships between the vectors (as his statements in Line 10 and 26 indicated)?

Whereas I am sure that Jill did not understand the idea behind this demonstration, I am not so sure about Jack. Throughout the whole problem, he would say things that showed that he did not understand the concepts, but in his next statement he would demonstrate an understanding of the ideas, or vice versa. One example is when at first he correctly planned to move $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ to make T(**v**) a shear (Line 62), and then suggested moving both **v** and T(**v**) onto the parallel line (Line 64). Even though Jack kept suggesting to move $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, and $\mathbf{w}_2$, but by quickly agreeing with Jill's idea of redefining T(**v**) to the parallel line (Line 103) makes me wonder again about how strong was his understanding and whether he was just reading the instructions.

When the students redid the shear problem by moving $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ so that the tip of T(**v**) is on the parallel line, they solved the problem for only one vector –i.e., the current position of vector **v**. Jack was fully aware that they created an 'one-vector' transformation, as seen in their discussion with me of the shear problem (Lines 297 – 299) and when they were solving the projection problem (Lines 255 – 259, 273 - 275). His reasoning was that "that is what they ask for." After a careful analysis of these sessions, I believe that Jack had every right to have this misconception. There are at least two factors that contribute to this misunderstanding:

First, the language being used is not very clear in nature. For example, under a projection, the resulting image is the projection of vector **v** on line L. Under a reflection, the image is the reflection of vector **v** about L. T(**v**) is the shear of vector **v** under a shear along line L with some factor q. When I was explaining what a projection was to the students, I told them that the resulting image was called the projection (Line 225) and that they had to arrange vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ "so that you have a projection, a linear transformation that is a projection." (Line 231) Seeing how the words projection, reflection, and shear refer to both the transformation itself and to the image, I believe that it was natural for the students to assume that finding "a configuration of the vectors $\mathbf{v}\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ that would make the transformation into a ..." meant making only T(**v**) into the desired transformation. To the students the equation $T = T(\mathbf{v})$ is true. Therefore, under their definition of a transformation, what they did was correct, and Jack was right in saying that the factor q is not needed since they were applying the shear to one vector only. As long as the tip of T(**v**) was on the parallel line, a factor q existed (Lines 152 – 156).

Second, in previous problems the students were asked to find only one vector. For example in the Worksheet #3 of Session IV, they had to find the position of T(**v**) for a given **v**, given the positions of the basis vectors and their images. The exact wording was: "From the information given [a configuration of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$, and **v**], would you know where should the vector T(**v**) be?" So, in this problem, they did the same thing: they constructed <u>vector</u> T(**v**) for the given vector **v**.

During the projection problem, it became clear that Jill was very lost. Maybe her nosebleed had been an omen of the things to come. Not only did she not understand what was required of them, she did know what a projection was. After my initial explanation to remind them of the projection, the students worked for several minutes before Jill declared that she did not understand what they were trying to do. She got a clearer picture of a projection when I sat down with them and showed them a concrete example (Lines 223 – 225), which supports our observations from previous sessions that Jill needs visual images to understand.

The conversation in Lines 231 – 243 shows the widening gap between Jack's and Jill's understanding. Jack understood what a projection was (Line 236) and except for the misconception of T = T(v), knew what to do to solve the problem (Lines 238 and 241). Both Jack and I took turns describing what was required of them to Jill. In Lines 240 and 242 when she asked that if T(v) should be on line L, reaching the perpendicular, I think she finally understood what a projection does, but her other misconceptions about the relationships between the vectors prevented her from fully understanding the problem.

In their conversation in Lines 255- 285, they were talking about two different things. They both agreed that there was more than one answer to the problem. Jack said, "And there are many ways we can do that [a projection]." (Line 275), meaning that there are many configurations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ that will give a projection. (Is this his idea of different bases?) Because Jill did not understand that the relationship between **v** and T(**v**) was dependent on the positions of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$, she thought that T(**v**) can be the projection of any vector as long as it is on line L and reaches the perpendicular line through the desired vector (see Fig. V–7a,b). I think that is what she meant when she said that they did not have to do this problem with respect to **v**, but could have done it instead with respect to any other vector (Lines 262 and 284).

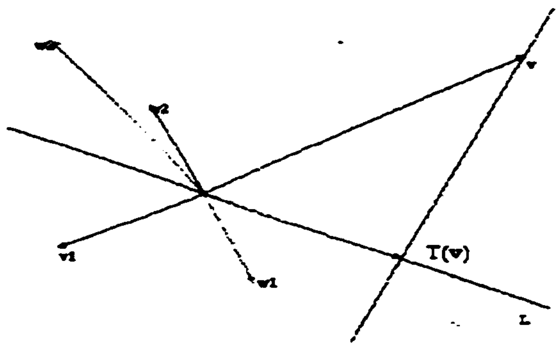

FIG. V-7a – T(v) is the projection of v

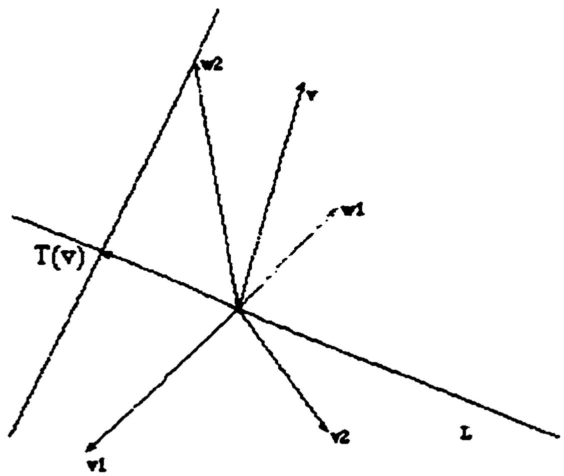

FIG. V-7b – T(v) is the projection of $\mathbf{w}_2$

I believe that the wording of the problem could have contributed to and enhanced this misunderstanding. At the beginning, the students were asked to move the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ around the screen and to observe that T(**v**) changes. When $\mathbf{v}_1$ was moved, T(**v**) was the only other vector that moved. When $\mathbf{v}_2$ was moved, again T(**v**) was the only other vector that moved. The results were the same when $\mathbf{w}_1$ and $\mathbf{w}_2$ were moved. For someone having trouble with the concepts of linear combinations and relationships between vectors could easily conclude that T(v) was also the image of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$. Maybe the explanation that the transformation changed *because the relationship between the basis vectors and their images had been changed* could be added to the problem. I am not sure how effective this addition would to someone at Jill's level of understanding, but I think it would definitely help someone who is at Jack's level.

Our discussion on the shear problem shed a lot of light on Jack's understanding and misconceptions. It seems that he understood that a transformation is applied to all vectors all along, but solved the problems for one vector only because he thought (rightly so, as shown above) that is what we had asked for. When I asked him about what should be done to make the transformation work for any vector v, he replied that if it was *supposed* to be on any v, then there should be the same relationship (Line 310). Not only he knew that a transformation applies to any v, but he also knew about the preservation of relationships.. Unfortunately, he thought that the relationships were the equations $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$ (Line 314). When I asked Jill if she agreed with him, she said something about "the same proportion", indicating that she is completely lost (Line 318).

On the other hand, I believe that Jack's thinking is progressing along a natural path. Jack's algebraic thinking was replacing his geometric thinking. At first, he thought that a transformation had to be something concrete –i.e., rotation, reflection. He resisted the idea of randomly placing vectors to define a transformation because it did not make sense visually. He was able to overcome this obstacle by justifying the transformation through equations. So now, the visual relationships between $\mathbf{v}_1$ and $\mathbf{w}_1$, and $\mathbf{v}_2$ and $\mathbf{w}_2$ were being replaced by the algebraic relationships $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$.

Jack's misconception about the relationship between the vectors led him to another misunderstanding. He thought that there were two bases –i.e., $\mathbf{v}$ was in basis $<\mathbf{v}_1, \mathbf{v}_2>$, and $T(\mathbf{v})$ was in basis $<\mathbf{w}_1, \mathbf{w}_2>$[11]. Even though there were signs in his conversations that alluded to this idea, we did not realize this misconception at the time (it became apparent in Session VI). Throughout the entire session, he referred to the relationships between $\mathbf{v}$, $\mathbf{v}_i$ and $\mathbf{v}_2$, and $T(\mathbf{v})$, $\mathbf{w}_1$ and $\mathbf{w}_2$ (Lines 314, 328 and 439), but he never once mentioned that $\mathbf{w}_1$ can be written in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, in Line 314 he said that $\mathbf{w}_1$ is $T(\mathbf{v}_1)$ and $\mathbf{w}_2$ is $T(\mathbf{v}_2)$, showing that he viewed $\mathbf{w}_1$ and $\mathbf{w}_2$ as, in a sense, "independent" of $\mathbf{v}_1$ and $\mathbf{v}_2$. I believe that he used the following logic in his reasoning: He knew that the factors in the equation $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ represent the coordinates (a, b) of vector $\mathbf{v}$ in basis $<\mathbf{v}_1, \mathbf{v}_2>$. Using the same logic for the equation $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$, vector $T(\mathbf{v})$ must have coordinates (a, b) in basis $<\mathbf{w}_1, \mathbf{w}_2>$. If it was in basis $<\mathbf{v}_1, \mathbf{v}_2>$, then $T(\mathbf{v})$ would have the same coordinates as $\mathbf{v}$.

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[11] The students were not explicitly taught the notion of basis. The terminology of "bases" was introduced only in the context of "coordinates of a vector in a basis", and the non-collinearity condition of the vectors of the basis, although mentioned, was not discussed and appeared natural: collinear vectors would not

Jack still did not have a *complete* understanding of transformations. He has bits and pieces of information about them, but he has not made the connections to bridge these ideas. When he faced a contradiction or a difficulty, he simply picked the simplest and most convenient explanation. This is illustrated when he was explaining that since $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$, then the relationship is preserved by having "the same coordinates… not the same coordinates but the same factor" (Lines 328 – 330). Since the idea of coordinates was confusing him (because he was, naturally (see Note 13), not sure about what a "basis" is), he decided instead to call a and b factors.

Sensing that Jack was really close to the correct answer, unconsciously my interaction with the students changed (From Line 319 and on). I ceased to be a researcher and became a teacher. I began asking direct questions, hoping to, as Wood (1998) would say, "funnel" the students to the desired solution. After establishing that $\mathbf{w}_1$ and $\mathbf{w}_2$ are the images of $\mathbf{v}_1$ and $\mathbf{v}_2$, I tried to guide them to the proper relationships by asking them whether all of the images were obtained by the same transformation (Lines 319 – 323). Despite answering that they were not, Jack still referred to the wrong relationships (Line 328). As can be seen and as Wood (1980) points out, funneling does not promote understanding. It requires students to answer a series of questions that lead to a desired

---

define two separate axes. It is therefore quite justified that, for the students, a "basis" was just a pair of vector used as a reference for another vector.

solution, but students do not see the overall picture. They are only concerned with answering one question at a time.

I tried leading them again, but this time I only asked them about $\mathbf{w}_1$. After thinking for a few minutes, Jack noticed that each vector and its image must be on the same parallel line (Line 343). Even though they did not take into account the factor q, through my own confusion, I was not able to disprove their configuration.

In our summing up discussion, I asked the students about what determined a transformation. By answering that **v** and T(**v**) had to be on the same parallel line (Line 389), Jill showed that she was still applying a transformation to one vector. On the other hand, Jack understood that the basis vectors and their images must have the same relationship between them (Lines 392, 399 and 405). My "funneling" approach continued. I kept asking the same question –i.e., "what is important is the positioning of …" -- until I got the desired answer –i.e., "the positioning of the images" (Lines 394 – 410). Jill was completely lost during this questioning period. At one point, she even said that she did not understand (Line 407). But since I was in my "teacher" mode, I ignored her because I was focusing on Jack, who was so close to the answer. Teachers usually say that they are happy if they reach at least one student. Well, I was going for the minimum.

One conclusion that can be made from this worksheet is that the students were not able to reverse the given problem. As stated before, this problem is the opposite of the problem in Session IV. It is also a more complicated problem that needs a higher level of thinking since the students need to think in a more general way. In Session IV the students were given the basis vectors and their images, and the students had to find the position of T(**v**) for a given vector **v**. In other words, they had to position one specific vector. Although T(**v**) represents the image of every vector in the plane, the method employed required the use of the coordinates of a particular vector **v**. This process could easily be interpreted as applying the transformation to only one vector and gives the impression that T(**v**) = T. But now in Session V, the students were given a specific transformation, and they had to configure the basis vectors and their images so that the given transformation resulted. This was a much harder problem since they were required to move vectors around (in the previous problem, all vectors were fixed). There were also infinitely many correct configurations, as long as the relationships between $\mathbf{v}_1$ and $\mathbf{w}_1$, and $\mathbf{v}_2$ and $\mathbf{w}_2$ were the desired transformation. Therefore, for people that have yet to master the concepts of linear combinations and basis vectors, it is very difficult, if not impossible, for them to do this problem globally.

One could also describe their difficulties by stating that the students have trouble thinking in terms of sets. This explains why they had trouble with linear combinations and the idea of a basis. But the difficulty of thinking in terms of sets could also explain why Jack was able to see the correct relationship only after I asked him about the configuration of $\mathbf{v}_1$ and $\mathbf{w}_1$ alone (Line 331). When I asked them about the relationships off all vectors (Line 323), he still thought that the equations $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ and $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$ had to preserved (Line 328).

Finally, this worksheet showed that Jack and Jill were working at two different levels. He was working at a conceptual level, whereas she was working on an intuitive level and following procedures. Luckily for her, she was in control of the mouse. Otherwise, Jack would have left her completely behind. Throughout the session, Jack would quickly come up with an idea, but he would have to say it several times before she would act on it because she would be trying something on her own. An example of this occurred after redefining **v** onto the parallel line during the shear problem. He told her to move vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$, but she was preoccupied with moving **v** along the parallel line and watching $T(\mathbf{v})$ move around the screen (Lines 100 – 102).

Some people might argue that maybe it would have been better for Jill *not* to be in control of the mouse. They might say that by being in control of the mouse, she became mesmerized by the moving vectors and became preoccupied with working the *Cabri* commands correctly, leaving her no time to think about the concepts. Although it is true that both these things did happen to Jill, I believe that with Jack in control of the mouse, he would not have waited for her to catch up. That is precisely why we put her in charge of the mouse at the beginning of Session IV.

3.2.6 Session VI

Worksheet #1

<u>Design</u>

From the previous sessions, it became apparent that that the students did not have a clear understanding of (1) a transformation was applied to every vector in the plane, and not just to one or two vectors., and (2) a transformation could be defined in different ways, but if that definition changed, then so did the transformation.

Four activities were designed to clarify these concepts for the students. I was supposed to lead them through the activities so that we could discuss the different notions while doing the problems.

In Activity 1, the students were to determine whether a given vector **v** and its image represent a shear transformation. The students were expected to test for both the parallel line and the factor q conditions. We wanted the students to move **v** around the screen so that they understand that a transformation worked for all vectors.

In Activities 2 – 4, the students were given configurations of six vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$, v and T(v), in which $\mathbf{w}_1$ and $\mathbf{w}_2$ were the images of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. In Activity 2, I was to move vector **v** around the screen, and the students were to determine whether the transformation T was also changed since T(**v**) also changed positions. In Activity 3, I was to move $\mathbf{w}_1$ and then $\mathbf{v}_2$, and again they had to determine if T changed. In both these activities, the students were expected to find out that by changing the relationship between the basis vectors and their images was the only way to change a transformation, whereas moving $\mathbf{v}$ merely indicated that the coordinates had changed in the same basis, but not the transformation.

Activity 4 combined the previous activities. They again had to determine whether the given configuration was a shear. The configuration was built such that $\mathbf{v}_1$ and $\mathbf{w}_1$ overlapped and the tips of $\mathbf{v}_2$ and $\mathbf{w}_2$ were on the same parallel line to $\mathbf{v}_1$. The design of this configuration was made such that $\mathbf{w}_2$ could move freely on the line it was on, but when $\mathbf{v}_2$ was moved along the same line, $\mathbf{w}_2$ also moved, keeping the same distance between the two endpoints. Thus, the students had to determine the status of the transformation when I was to move vector $\mathbf{w}_2$, which would still give a shear but with a different factor q. Finally, I was to move $\mathbf{v}_2$. This time $\mathbf{w}_2$ would move with $\mathbf{v}_2$, which meant that the transformation remained unchanged. The students were then expected to conclude that a transformation remained the same as long as the relationship between its basis vectors and their images was preserved.

<u>What happened</u>

To begin Activity 1, I opened the file with the configuration in Fig. VI-1. I told them that these vectors are linked with some kind of transformation and if they thought that this transformation is a shear. At first they said that there was no line for it to be parallel to.

I told them that they could add things to the diagram. It was decided that a line should be drawn, but Jill was surprised that segment between v and T(v) should be drawn first. She said that then it would not be parallel to the horizontal line. Jack explained that the lines had to be parallel, not necessarily horizontal. He went on to instruct her on how to draw the parallel lines. As soon as the lines were drawn, he said that they had to find the value of q. I told him that we will get to that later, but first I wanted to clarify with Jill if she thought that in a shear transformation, the parallel lines had to be horizontal. Jack said that they only had to be parallel, and I confirmed.

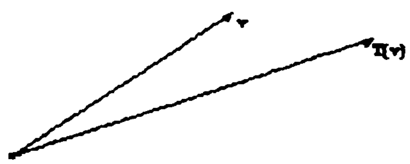

FIG. VI-1 – Opening configuration for Activity 1

29. M.: Ok. Yup. As long as these two lines are parallel, then it could be a shear. Now, are they always parallel?

30. Jack: What do you mean by "always parallel"?

31. M.: I mean, it looks parallel now, but will it always be parallel?

32. Jill: Do you mean, will they ever cross?

33. M.: Ah, no.

34. Jack.: Oh, you mean if we move v?

35. M.: That's right.

36. Jack: They won't.

Jill moved v around the screen, and Jack remarked, "It seems like they're always parallel." Then I asked them:

41. M.: Now, why is it important to know if it's always parallel?

42. Jack: So it means that it's always a shear whenever... I think, since it's always parallel it means that... like v is equal to av_1, av_2, so that, like v_1 and v_2 that makes up v... well that which we don't see now... are a shear also.

(...)

45. M: ... I don't know if you answered what I wanted, besides mentioning v_1 and v_2. Now, my question, why did you have to check it for different v's?

46. Jack: Because we wanted to know if it ... only because v is like... in position of shear. So we checked it and realized it was always a shear so it means that it will always be parallel. Because, just like on Monday [Session V], we did... we positioned v a certain way and it was... the transformation was right, but if you moved v, it wasn't working anymore. So it means it was not.... Since we moved v, we realized that it's always constant... the transformation is always constant.

I told them that the point was that a transformation has to act on all vectors. Since by leaving vector v in one position, you would not know if the transformation was the same on another vector. That was why you had to move v around.

We decided that it was time to check for the factor q by calculating it. Again, they had trouble remembering what the factor q represents. But once they figured it out, they came up with a plan of how to calculate it. I helped them with the *Cabri* commands, and we printed the value on the screen. Then Jack said:

94. Jack: Yeah. We have to use another vector to see if q is constant.

95. Jill: (working on the computer)

96. Jack: Move it. Change v.

97. Jill: Move it?

98. Jack: Yeah. To check if it's going to be the same. We see that our q is changing. I don't think it's supposed to change if it was a real shear.

99. M.: Do you agree?

100. Jill: Yes, because that line should be the same distance, although these will move, that's how q stays the same.

101. M.: What line should be the same?

102. Jill: Like one line will be the same distance from the other parallel one.

103. Jack: The ratio of these lines.

104. Jill: Yeah. Yeah.

They both agreed that this transformation was not a shear. I concluded by saying:

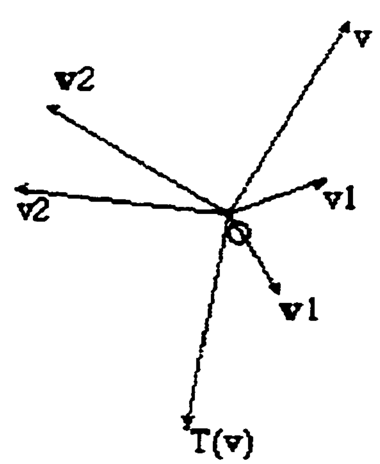

FIG. VI-2 – Opening configuration for Activity 2

107. M.: So it satisfies <u>one</u> of the conditions of a shear, but not both. Ok. And another thing you should always remember is that a transformation should work on <u>all</u> vectors. Ok. 'Cause now you have a

certain q, but you're not sure this q is always the same, so you test it for all vectors. Ok?

I opened the configuration for Activity 2 (see Fig. VI-2), where $\mathbf{w}_1$, $\mathbf{w}_2$, and T(**v**) are the images of $\mathbf{v}_1$, $\mathbf{v}_2$ and **v**, respectively, under the same transformation. I moved vector **v** around the screen and told them to notice that T(**v**) was also changing. I asked them if the transformation was changing, too. Jill said that she did not understand what I meant by "changing the transformation". I asked Jack to explain it to her, but he could not. So I tried to explain using a concrete example, asking them to pretend that a rotation had occurred between every vector on the screen and its image. When vector **v** was moved, was T(**v**) still a rotation of **v**, i.e. "is T(**v**) still the image of **v** under the same transformation as it was before?" Jack replied:

> 119. Jack: Well for me, I don't think it's the same transformation, 'cause if it were, the image of $\mathbf{v}_1$ and $\mathbf{v}_2$ would move too.
>
> 120. M.: Why would you say that?
>
> 121. Jack: Because we know that **v** is... is the vector sum of a$\mathbf{v}_1$ plus b$\mathbf{v}_2$ so... it implies that if **v** changes $\mathbf{v}_1$ and $\mathbf{v}_2$ are supposed to change if **v** is changing.

Jill was also not sure if $\mathbf{v}_1$ and $\mathbf{v}_2$ should change or not. So, I wrote the equation $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ on a piece of paper, and I said that **v** was defined according to the $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. As I moved **v** around on the screen, I asked them about what was changing in the equation. Jack quickly replied that the coordinates a and b were changing. I continued:

> 128. M.: .. Now if these change (circling a and b in the equation)... can you explain why you're saying that $\mathbf{v}_1$ and $\mathbf{v}_2$ should also change?
>
> 129. Jack: (after some thought) Yeah, it's true. Well maybe $\mathbf{v}_1$ and $\mathbf{v}_2$ are not supposed to change. Maybe it's um... (silence)...

I then asked them to explain how to find T(v), given v_1, v_2 and their images. Jill wrote the equations "$T(v) = Tav_1 + Tbv_2$ "and "$T(v) = aT(v_1) + bT(v_2)$".

136. M.: ... Now looking at these formulas and looking at what I am doing here (moving v), do you think it's a different transformation?... and this is one case, my first v (leaves v in one position), and I get a certain transformation, on this basis. And I find another v (moves v to another position and leaves it there), giving a new image. Is this the same transformation?

137. Jack: It can be the same transformation if T(v) and the v are the same... like for T(v) is $aw_1 + bw_2$.

138. AS: You mean if a and b are the same?

139. Jack: Yes... like v has a and b in the basis v_1, v_2, but I think T(v) is supposed to have the same coordinates in the basis w_1, w_2.

140. Jill: Here it's the factor that change a and b, and it's Jack: Yes... like v has a and b in the basis v_1, v_2, but I think T(v) is supposed to have the same coordinates in the basis w_1, w_2. not the vector.

141. M.: What I understood from Jack, correct me if I'm wrong, you're saying that v has the basis v_1, v_2...

142. Jack: Yes.

143. M.: ... but T(v) has the basis... What's the basis of T(v)?

144. Jack: w_1 and w_2.

I thought for a while on how to respond. I then told them directly that the basis of T(v) was $\langle v_1, v_2\rangle$. I explained that since we were finding our coordinates of every vector with respect to the axes, then the coordinates were respect to v_1 and v_2, which meant that the basis was always $\langle v_1, v_2\rangle$. Then, Anna added:

150. AS: We have to make a distinction between image and transformation because we call T(v) the transformation of v. We should be saying image of v under the transformation. So when we ask is the transformation the same, we're talking about the whole... what binds the v and T(v) and not about the concrete T(v) connected to the concrete v

which has coordinates a and b, concrete numbers. And now if we re-ask the question...

Jack was still not sure whether the same transformation was used to get $\mathbf{w}_2$ and $T(\mathbf{v})$. So, I moved vector on top of $\mathbf{v}_2$, and he observed that $T(\mathbf{v})$ and $\mathbf{w}_2$ overlapped. I did the same thing for $\mathbf{v}_1$ and $\mathbf{w}_1$, and the students agreed that it was the same transformation. Then I asked them if they knew what the transformation between $\mathbf{v}_1$ and $\mathbf{w}_1$ was.

169. Jack: Well it can be either a rotation or... it looks like a rotation... by 90 degrees.

170. M.: Ok. And between v_2 and w_2? Now remember we said that w_1 and w_2 are the images of v_1 and v_2 under the same transformation.

171. Jack: Yeah. But it's... something else then.

172. M.: So if you're saying this is a rotation by 90...

173. Jack: ... this should be rotated by 90 also. It's not a rotation by 90 degrees. It's another transformation.

They also wondered if the transformation was a projection or a shear. I asked them about other transformations that they have learned. Looking through the notes, hesitantly Jack said, that it was a reflection. After reading the description, he realized that an angle bisector could be drawn. I constructed the angle bisector, and Jack declared, "So this is the symmetry axis and it's... so it's always the same transformation." To probe their understanding, I asked them about the need to move v. Jack replied that it was to see if the transformation was working for any vector, but Jill was still not sure. I explained again:

194. M.: So when I move v (moves v), it could be this vector (stops moving v), this vector (moves v to another position), and so on and the image is always the reflection. Ok? So by moving... when we have a case like this when we have... when we've defined

a basis and we have the images of the basis... when a transformation's defined on these, then when moving **v** we're not changing the transformation. We're just checking it for any vector in the plane.

195. Jack: But if when we're moving **v**, v_1 and v_2 are moving, does it mean that it's not a trans... not the same transformation.

196. M.: When you move **v**...?

197. Jack: Yeah, and like v_1 and v_2... the images are moving.

198. M.: Well, you see **v** is defined on these two...

199. Jack: Ok.

200. M.: ... so when you move **v**, it will not...

201. AS: What depends on what? If v_1 was dependent on **v**, then it would move. But it's **v** that depends on v_1.

202. Jack: But without a line, it was quite hard to see that it was a reflection.

From the way he abruptly changed the topic, I assumed that he understood. So, I began Activity 3 by opening the configuration in Fig. VI-3, in which the vectors had the usual relationships between them. Then I moved $\mathbf{w}_1$ around the screen.

204. M.: ... Now, if I move, say w_1, T(v) moves. Am I changing the transformation?

205. Jack: Well I think... now you are because it's not the same... like w_1 does not have the same relationship with v_1 as v_2 with w_2. So now it's not the same transformation.

Jill agreed. They also said the same thing when I moved $\mathbf{v}_2$.

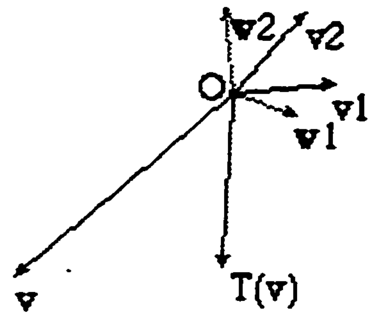

FIG. VI-3 – Opening configuration for Activity 3

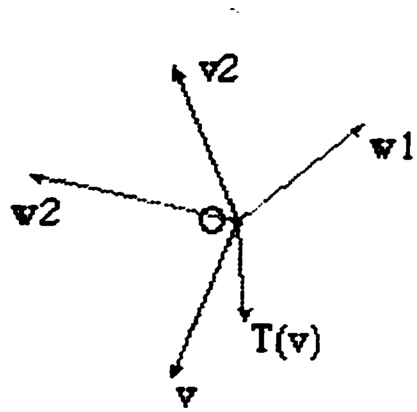

FIG. VI-4 – Opening configuration for Activity 4

Activity 4 started with the configuration in Fig. VI-4. I had to explain that $\mathbf{v}_1$ and its image $\mathbf{w}_1$ were the same vector, and thus were overlapping in the diagram. The students had to determine if the transformation was a shear. After a short silence,

> 221. Jack: I don't know, but if we move v... on v_2... and T(v) is equal to w_2, we can see if it's a shear... oh no... I don't know.
>
> 222. M.: You can control the mouse.
>
> 223. Jill: Ok.... I'd like to do what we did last time, putting in our line.

But, as before, she wanted to draw the line through the origin first, instead of the line through **v** and T(**v**). Jack corrected her, and helped her with the commands. They found out that the lines would always be parallel. Next, they calculated the factor q by measuring the distance between the endpoints of $\mathbf{v}_2$ and $\mathbf{w}_2$ and dividing it by the measured distance between $\mathbf{v}_2$ and the line containing $\mathbf{w}_1$. They determined that it remained constant. I continued:

256. M.: Ok. So now we've determined that this is a shear. Now what happens if I move w_2 a little bit to the right?

257. Jack: It's not a shear anymore.

258. M.: Why?

259. Jack: Because now w_2 is not a shear along L of v_2. It's something... it's not a transformation... it's not the same transformation and it's not the same constant. So therefore T(v) is not the same transformation... of v.

Jill disagreed, and said that the factor would stay the same. Jack suggested to change $\mathbf{w}_2$, then move **v** and to check if the value of q changes. They both concluded that they are both shears, but with different q's, and thus they were different transformations. I, then, asked:

278. M.: So when you moved your w_2, you changed the transformation by changing the q. Let's move v_2 now. (moves v_2) Reactions? What's happening when you're moving v_2?

279. Jack: Actually nothing happens.

280. M.: Why?

281. Jack: Because when we're moving v_2, the image is moving too. Because it's... w_2 becomes the shear of v_2. So whenever we move v_2, w_2 it's still the same image of the same transformation.

282. M.: Ok. And this goes back to what you said before. w_2 is dependent on v_2, so when you move v_2 anything that's dependent on it will move. But when we move w_2... So the transformation doesn't change as long as... what happens?

Jack explained that the transformation does not change as long as w_2 is found by the same transformation as before. I concluded the worksheet by saying, "So as long as you preserve the relationship, the transformation doesn't change."

Analysis

From the beginning, the students confirmed our suspicions that they still did not understand that "finding a transformation" meant finding a transformation that works on all vectors. As soon as they drew the parallel lines in Activity 1 to check for a shear, Jack wanted to find the factor q. When asked whether the lines were always parallel, he answered, with surprise, "Oh, you mean if we move v?" (Line 34) Then he guessed that they will not stay parallel, but again seemed surprised that they did (Line 36). I think this shows that Jack knows that a transformation is applied to every vector, but does not understand that "giving a transformation" means giving the relationship of the transformation, and not just one image. By saying that the lines will not stay parallel showed that he was comparing it to the Activities in Session V. In those examples, he had found a configuration for only one vector, and when he had moved v, the transformation did not work. Further proof that he understood the conservation of relationships is in Line 42. Although in this statement he did not answer the question that was being asked, he knew that the basis vectors and their images (not shown in the figure) had to be related by a shear also. He finally answered my question, in his own way (Line 46), about the importance of knowing that the lines are always parallel.

I think the idea of checking a transformation for all vectors was finally understood by Jack, but not so much by Jill. Jack suggested using another vector by moving **v** to see if q changes, but Jill did not understand When Jack re-explained why they had to do it, she agreed, saying that the distances have to be equal. Jack corrected her by saying that the ratios of the distances had to be constant (Lines 94 – 104). Jill knew something had to stay constant when vector **v** was moved. I think that it is ironic that the one time ratios were part of the explanation, Jill, who seemed to explain everything by things "being proportional", did not mention the subject. I believe that is because the configuration for Activity 1 did not have the usual similar triangles picture.

Although Jack understood that a transformation applies to all vectors, when I moved vector **v** in Activity 2, it showed that both students did not know what it means to "change a transformation". I think that at this point they still thought that if T(**v**)'s position on the screen changed, then the transformation had also been changed, regardless of its relationship with vector **v**. This is due to the misconception that the transformation and the image represent the same thing and are interchangeable.

To explain the meaning of "changing a transformation", I had to use the concrete example of a rotation to tell them that if the relationship between every vector and its image, which must be the same for each pair, was not preserved, then the transformation had been changed. Jack said that in that case, the transformation was not the same because when **v** changes, then $\mathbf{v}_1$ and $\mathbf{v}_2$ also change (Lines 119 – 121). This is something that he had mentioned before, but I either did not pick up on it at the time or I dismissed it as another case of Jack not being able to express himself properly and correcting himself in his next statement. But this time I questioned him about it. Did he understand which vectors depend on which? Did he think that coordinates (a, b) stay constant and the basis vectors move?

Jack realized his mistake when I made them compare the linear combination of $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ to the movements on the screen. He could not say why he thought that it was the vectors that changed and not the coordinates. To show the students that the reason why $T(\mathbf{v})$ also moved was because a and b changed, I asked them to compare the movements on the screen with the equations that Jill had written –i.e., $T(\mathbf{v}) = Ta\mathbf{v}_1 + Tb\mathbf{v}_2 = aT(\mathbf{v}_1) + bT(\mathbf{v}_2)$. But, instead, another unanticipated misconception popped up. Jack had thought that **v** was in basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $T(\mathbf{v})$ was in basis $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$ (Lines 136 – 144). As discussed earlier, there were signs of this misconception in previous sessions, but we had not caught on to them at the time, and so this came to me by surprise. After thinking for a while, I decided to tell them straight out that the basis for all the vectors on the screen was $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. At this point, Anna also decided to clarify another misconception by explaining the difference between a transformation and an image (Line 150).

After some more explanations, the students were asked to find a name for the transformation. Jack was either still shaky on the idea that the transformation had to be the same between all vectors and their images or still did not think in terms of sets, and only looked at each pair of vectors as individual cases. I think if it was not for my major hint about the "same transformation" (Line 170), Jack was about to guess a different transformation for $\mathbf{v}_2$ and $\mathbf{w}_2$ than the one he chose for $\mathbf{v}_1$ and $\mathbf{w}_2$ (Lines 169 – 173).

It took them a while to guess that the transformation was a reflection. One reason was that not only was the reflection line was not shown, but it was not horizontal. As seen in many instances before, students prefer to see things horizontally and vertically. We saw it with the students changing their axes to the preferred "right-side-up" orthogonal axes, Jill not being sure if a transformation was a shear if the parallel lines not being horizontal, and now the students not being able to recognize a reflection because the mirror line was not horizontal. I think that this preference of orientations being horizontal and/or vertical is a natural occurrence in humans that cannot be avoided. That is why a geometric approach to vectors with a software like *Cabri* is useful in making students see that other coordinate systems do exist, and thus help in the understanding of the notion of different bases.

After concluding that the transformation was always a reflection, I emphasized that every vector had to be checked by moving **v** around the screen (Line 194). I think an effective method of showing why **v** has to be moved is to place the vector in one position and say that this one vector. Move **v** to another position and declare that this is another vector, and so on. I believe that the idea of stopping the movement in several different positions is understood better than the continuous movement of **v** around the screen. Stopping the movement gives the impression of different cases, whereas continuous movements give the impression of one case only.

Jack still had trouble accepting that $\mathbf{v}_1$ and $\mathbf{v}_2$ did not change when **v** was moved. He went back to the incorrect notion that $\mathbf{v}_1$ and $\mathbf{v}_2$ must move with vector **v**. Anna and I explained to him that it was a matter of what vectors depend on what (Lines 195 – 201). This is something that Jack had done throughout the whole experiment. After we thought that he understood a certain concept, he would say something that would show a lack of comprehension. I think that this indicates that Jack's understanding had not been fully interconnected. In his mind, he has several notions about transformations, of which some were correct while others were wrong. He could tell when there was a contradiction between his concepts, but he could not tell which ones were right and which one were wrong because he still did not have the necessary connections between all of his ideas.

Activity 3 showed that the students were beginning to realize that preserving the relationship between all vectors and their images was necessary to keep the same transformation (Line 205).

It was Jill who came up with the correct idea of checking for a shear in Activity 4 (Line 223), but again she did not know how to proceed. On the other hand, although Jack came up with the wrong plan (Line 221), he was able to adapt his thinking according to the plan. In other words, he could jump from one idea to another in a split second. This again shows that concepts were clear in his mind, but he could not go from one to the other on his own. But once he was guided to the required notion, he had no problems adjusting his thinking.

Lines 256 – 259 show that Jack really understood the need to preserve the relationships between vectors in order to maintain the same transformation. When Jill disagreed with him, he suggested moving vector $\mathbf{v}$ to see if the transformation applied to every vector. When vector $\mathbf{v}_2$ was moved and $\mathbf{w}_2$ changed with it, he realized that it did not matter if the vectors were moved, but as long as the relationship between the vectors and their images remained unchanged then the transformation was still the same (Lines 278 – 281),. I brought their attention to the notion of a vector being dependent on another in *Cabri* (Line 282).

It seems that these activities revealed many unintended conceptions in the students, which in turn made us do extensive explanations and design special "remedial" activities. I believe that this was both necessary and beneficial to the students. Today's educationists find many arguments to support the claim that students learn better when they find out the facts themselves. I totally agree, but at some point there should also be some guidance from teachers. Since students, given the time, could discover a "million" things from an activity, the teacher is needed to help them decipher between the relevant and the irrelevant information and draw consistent conclusions that would also be pertinent from the point of view of the target knowledge which the teacher knows and the students do not. Students have all the right and reasons for initially missing the point. In other words, a teacher is needed to summarize concepts and to lay to rest the unintended conceptions. After five sessions of working for the most part on their own, I believe that these activities showed us that it was the right time for us to intervene and clarify certain notions. This occurred several times during the activities. The notion of having two conditions satisfied to obtain a shear was explained in Line 107. I emphasized on a couple of occasions that a transformation was applied to all vectors (Lines 107 and 194). I explained that all vectors are in basis $<v_1, v_2>$, while Anna explained the difference between a transformation and an image of a vector (Line 150). The idea that some vectors depend on other vectors was first brought up by Anna (Line 201), and then reiterated by me at the end (Line 282). I believe that this session brought all the loose ends from previous sessions together.

3.3 Conclusions and recommendations

The aim of this research project was to test an alternative way, namely a geometric approach, to introduce topics in Linear Algebra. The motivation was the disenchantment with the current approach to the teaching of this subject. Thus, the design team wanted to explore a coordinate-free introduction, with the aid of the dynamic geometry software *Cabri*, to the concepts of vectors and linear transformations, with the intention of introducing a coordinate system later on.

As it turned out, this approach was not as successful as was hoped. One positive aspect is the use of *Cabri*, or any good software. The motivation and interest of the students was definitely enhanced with the use of *Cabri*. I believe that it was also very beneficial for them to see a different representation of the concepts of Linear Algebra. Unfortunately, as it turned out, one interpretation was not enough for them to grasp the notions of vector, linear combination, and linear transformation.

My analysis was done using an "interpretive understanding" approach. Therefore, the following conclusions and recommendations are based on my best interpretation of why things happened the way they did.

3.3.1 Problems with the experiment

One of the aims of this project was to introduce the notion of vector using a visual image instead of coordinates. From the different ways of representing a vector, it was decided to choose the representation by arrows that start from the same origin O and go off in a direction a certain length. That was the whole introduction. Equality of vectors was not mentioned. This is in sharp contrast to the historical development of vectors. The primary concern of the first vectorial systems of both Möbius (in 1827) and Bellavitis (in 1833) was the notion of equal vectors. On top of that, while the students were instructed on how to measure the length of a vector **v** using *Cabri*, the question of "measuring" the direction was not discussed. It was assumed that the students knew the difference between a scalar and a vector. So, it was only natural that Jack and Jill thought of vectors as lengths only.

Some might recommend to show vectors as representations of translations. It is definitely a better idea, however, I still believe that our approach to vectors is a good way to represent them, but a better introduction is needed. We cannot make any assumptions about the students' background knowledge since they all have had different experiences, if any, with vectors. I think that no matter which representation is used, we still need to talk about the topic and have students do activities (similar to the ones used in Session II) that bring out the question of vector equality. After all, this notion was the pre-occupying concern for Möbius and Bellavitis.

Another topic with a bad introduction, if it can be called that, was linear combinations. We did not want to give it a name, but instead we wanted the notion to develop in the students as part of a natural progression of applying one operation after the other. As a result, during the first five sessions, the idea of a linear combination was non-existent. They used each operation separately. The notion of linear combinations is totally new to the students in terms of a visual representation. To promote its understanding, I suggest that one should either (1) introduce activities like the ones from

Session V at the beginning to help students combine the operations in the same problem or (2) introduce linear combination as one more concept.

A major misconception of students was that a linear transformation is applied only to one or two vectors in the plane. This occurred despite promoting the idea of a global transformation right from the beginning when transformations were first introduced. One reason was that the students did not realize that vector **v** on their screen, by being created independently of all other constructions on the screen except for starting at the "origin", and variable, represented all vectors in the plane. To them, the Cabri arrow represented one vector that could be stretched and draged at will around the screen. To avoid this kind of interpretation, during an activity, we should move **v** to one place and stop, then move **v** to a different place and stop, and so on. To be effective, the instructor should point out that each placement of vector **v** is a different vector. I am not sure if students have other stumbling blocks that will prevent them from seeing vector **v** as a general vector, but it seemed to work when I finally did it in Session VI.

A second reason for not globalizing transformations is the students' notion that the transformation and the image are the same –i.e., $T=T(\mathbf{v})$. This could be seen in the language that the students used. To be fair, I was just as guilty as they in promoting that idea through my word choices. This is something hard to avoid because of the differences between mathematical and everyday usage of words. The only thing to be done to reduce this problem is for teachers to be extra careful with their words and to correct students when words are misused. Hopefully, that will make students aware of the differences and make them correct the teachers as well.

A third reason for not visualizing a global transformation comes from the students' mathematical background. In high school mathematics, all transformations are local and are seen in the context of geometric figures. This is usually illustrated as a triangle in a Cartesian plane that has been translated, reflected, rotated or dilated to another position in the plane. The coordinate axes remain unchanged.

An unintended concept that arose was the idea that a linear transformation has two bases – i.e., $\mathbf{v}$ is in basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $T(\mathbf{v})$ is in basis $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the images of $\mathbf{v}_1$ and $\mathbf{v}_2$. This misunderstanding comes from the fact that the notion of basis was not separately introduced to the students, but only in the context of "coordinates of a vector in a basis". Geometrically, the coordinates were obtained by decomposing a vector $\mathbf{v}$ into a linear combination $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$, and then (a,b) were the "coordinates of $\mathbf{v}$ in the basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. So naturally, when $T(\mathbf{v})$ was generated from $T(\mathbf{v}) = a\mathbf{w}_1 + b\mathbf{w}_2$, the students felt free to use the same language and say that "$T(\mathbf{v})$ has coordinates (a, b) in basis $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle$".

I also believe that the introduction of the similar triangles backfired on us. We introduced it to show the students a nice, dynamic picture of proportional change. I think we overused the geometric approach in this instance. Jill, the weaker student, was so mesmerized by the movements that for the rest of the experiment, everything was "proportional". Alternatively, Jack used the visual image as a shortcut to check for linearity. He was so preoccupied with looking at their movement that at one point he did not realize that a vector and its dilation were not even on the same line. I do not think that the similar triangle image should be used in the future.

Another change that I recommend is the name of the "dilation" operation. I think that it is confusing to have an operation named after a transformation. In *Cabri*, the "vector sum" command is in the CONSTRUCTION menu, whereas "dilation" is under the TRANSFORMATIONS heading. The usual term of "scalar multiplication" should be used and a special macro designed to perform this operation. The macro should then be put into the same menu as 'vector sum'. This way the two operations are a "sum" and a "multiplication" and both are algebraic terms, as opposed to one being algebraic and the other geometric.

Despite the misconceptions created by the geometric approach (which approach does not lead to misconceptions?), I still support this new way of introducing topics. What has to change, I think, is that both geometric and algebraic ways should be employed in a Linear Algebra course. Introducing a topic in one interpretation limits the understanding of certain concepts. For example, we expected the students to use equations for linearity structurally to be able to construct the image vector. But students are used to putting in numbers into equations. Coupled with the fact that coordinates were not introduced, the students resorted to substituting lengths for vectors in the equations; thus helping in creating the misconception that vectors are lengths only.

Aside from the "common" algebraic and geometric representations, it is also very important to include a verbal interpretation. It is very important to be able to describe the concepts clearly and to express oneself correctly. Language can lead thought towards a more precise and consistent knowledge.

3.3.2 The Role of the instructor

In a research experiment, the instructor walks a thin line between the role of a researcher and the role of a teacher. A researcher's role is to find out how students learn mathematics and what is the nature of the difficulties they encounter, but a teacher has the responsibility of bringing the students to some pre-determined knowledge and understanding. Therefore, the instructor in an experiment has to decide when to correct the unintended students' conceptions and when to keep silent. But the students do not know of these distinctions between the roles, and may take silence as a confirmation of their answers. This occurred a couple of times during our experiment.

Instructors in a research have to be careful not to lead students through question so that they can be "funneled" to the solution. They should try to focus the students' attention on the important parts of a solution. This is true for a classroom teacher as well.

3.3.3 Students' understanding

As one could tell from the analysis of the experiment that Jack's and Jill's understanding of concepts were totally different. Jill, I believe, for most of the experiment was lost in terms of understanding the concepts. She worked on intuition (an important aspect in problem solving) but had a lot of misconceptions that prevented her from understanding the concepts. Some of her misconceptions were preconceived before the experiment, while others were borne out of the activities. This is partly due to her weak background and also to the design problems discussed above.

Even though Jack had some of his own misconceptions –e.g., the transformation is the image (T=T(v)) idea –, I believe that Jack's learning was following the path presented to him. Jack had many concepts about transformations in his mind. Some of these ideas about linear transformations were true, while others were misconceived. But like any other novice learning something new, he had no connections between these ideas. The lack of bridges between notions prevented him from seeing contradictions in his responses. And when he actually did notice a contradiction, he took the easier or more recent explanation. This explains why throughout the whole experiment, Jack would give correct responses, but in the next sentence, he would contradict himself. He was jumping from one concept to another, instead of taking a leisurely walk across a bridge. I guess the same could be applied to Jill, except her little conceptual islands were too few and farther apart.

No obstacle of formalism was detected in the experiment. This is due to two factors. First, there was not a lot of written problem solving in the experiment to indicate if such an obstacle existed. Second, the obstacle of formalism usually occurs in a loose classroom situation where students work more independently rather than in a closely monitored atmosphere where an instructor checks every move that is made.

As an overall assessment on their understanding, I would say that Jack and Jill went up the hill to fetch some knowledge. Jill, with a little push from us, fell down, but Jack did not come tumbling after.

CHAPTER IV

Conclusions and Recommendations

I have discussed students' difficulties in learning Linear Algebra from three perspectives:

1. The nature of Linear Algebra – Linear algebra is a product of a long process of development and intellectual struggle of several generations of outstanding mathematical minds. The result, for now, is an abstract "unifying and generalizing" theory with many ramifications and a wide range of applications.

2. The teaching of Linear Algebra – Courses in linear algebra are often aimed at presenting the concise and apparently simple general theory without giving the students a chance of experiencing the above mentioned process of genralization and unification and appreciating its applications. The first course in structural theory of linear algebra is given too early for the students to have anything to unify or generalize and apply the theory.

3. How students learn and cope with Linear Algebra – Besides bringing with them bad study habits, as a result of the two factors above, students develop the *obstacle of formalism*. The theory appears to them as a formal play on meaningless strings of symbols.

Using these sources of students' difficulties as a framework, I analyzed two of my own different experiences in teaching Linear Algebra: (a) teaching MATH 204 - a college-level course, and (b) conducting experimental sessions on a geometric approach to vectors and linear transformations.

My analysis concludes that MATH 204 is in need of a curriculum change. There are too many topics taught in too little time. Some of these topics are complex, abstract, and unnecessary at that stage of most students' educational careers, or *any* stage for that matter. Linear algebra is being taught through an axiomatic approach and concepts are introduced by definitions. Aside from the one week in which vector geometry is taught, the rest of the course employs algebraic representations of the concepts. The course is usually geared towards a final exam, almost the same from year to year. This, in turn, promotes memorization of procedures and solutions, instead of understanding the concepts behind the solutions.

In terms of the content of Linear Algebra classes, I believe a college-level course, like MATH 204, should concentrate only on systems of linear equations, matrices and matrix algebra, determinants, and vectorial geometry. The first two topics are strictly computational, thus agree with the students' mathematical background knowledge. On top of that, many application problems can be presented using these two topics. As for the vectorial geometry, most students at this level have seen little or no geometrical representations of vectors. In a college-level course, I would concentrate on the notions of vectors and linear combinations, geometrically, algebraically, and verbally. I would leave the concept of linear transformations for a first-year university course, and at that level, the notion should be studied only in $\mathbf{R}^n$, practically as a multiplication by a matrix. There is no sense in introducing linear transformations if the concepts of vectors and linear combinations are not fully understood. As for the concepts of vector spaces and subspaces, and the structural notion of linear transformation of vector spaces, I would recommend that they be left for a second-year university course or higher. These ideas are only useful for very high level mathematics. In other words, I recommend that Linear Algebra courses follow a similar path to its development. After all, linear algebra took a long time to become accepted for a reason -- it involved very complex and abstract topics that need time to be accepted and understood.

In view of these concerns, a research project was launched to test an alternative way, namely a geometric approach, to introduce the concepts of Linear Algebra. Two students, who had successfully completed MATH 204 (the final grade of both was B) two months before, were involved in six sessions in which the concepts of vectors and linear transformations were introduced in a coordinate-free environment with *Cabri*. This approach did not fare any better than the algebraic approach. The students never grasped the concept of vector, which in turn prevented them from understanding linear combinations, which then made the notion of linear transformation difficult to comprehend. These difficulties occurred partly because the experiment did not properly introduce the notions of vectors and linear combinations. The designers assumed that the students understood what vectors were. They also hoped to build the notion of linear combination as a natural progression of the two vector operations, but this is difficult without a solid understanding of vectors. Another reason for the misconceptions is the lack of an algebraic interpretation. I believe that more than one representation is necessary to promote a better understanding of the concepts. It was seen in MATH 204 with a strictly algebraic approach and with the experiment using geometric interpretations. Moreover, the notion of linear transformation was introduced by its formal definition, for which the students were not conceptually prepared.

Seeing that students struggled in these two very different environments, one has to look at what they had in common to find out the essence of these difficulties. In both teaching situations a structural approach was used. Axiomatic definitions of vector space and linear independence were given in MATH 204 and of linear transformations in the experiment. It seems that the main problem is that students have trouble thinking in terms of structures. They think at the level of elements of structures: individual vectors, operations on concrete vectors. They see properties of operations as properties of actions, not as axioms that could define an abstract structure such as a vector space. This way of thinking is usually seen in the "proofs" in MATH 204 where students find it difficult to know what they are supposed to prove and how to prove it. In the experiment, Jack and Jill had the same trouble in showing that a transformation was linear.

What is the solution? Do we give up the structural approach? Would that be the same as giving up on Linear Algebra altogether? But one has to wonder whether the general vector space theory really is the essence of Linear Algebra. Could a kind of "calculus" approach to this extremely complex subject be possible? Until an answer is found, I would stick as much as possible with computational problems, especially in a college-level course, and leave the structural approach to the higher level courses, whose students are ready for and need these complex and abstract concepts.

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