Embedding the New York Stock Exchange

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This paper is dedicated to my loving mother Lise Best and to the memory of my father Neville Best, "I did it Dad!!".

ABSTRACT

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Given data in a time series we will create a phase space using methods based upon the work of Takens and Whitney. Our phase space will be approximated using a single record observed $s(n)$ of the New York Stock Exchange. This procedure of creating a phase space will create a complete vector space by defining $s(n)$ to be the first coordinate, $s(n+T)$ the second and $s(n+(D_E-1)T)$ the last coordinate, where T is a suitable delay and D_E is the embedding dimension. The observed phase space will be shown to be chaotic in its behavior and a reconstructed attractor in the phase space will provide us with predictions of future the stock market prices.

All algorithms for computation are written in Borland C++ version 5.

Contents

1 Theory of Abstract Spaces

In the first chapter we will review all the general concepts of topological and metric spaces that will be needed in the following chapters. This will encompass all proper definitions and theorems as well as any proposition or lemmas that will have any bearing in further chapters.

1.1 Topological Spaces

In the following section we will discuss the notion of a topological space and the general properties of open and closed sets. It will then be followed by the definition of metric spaces in the following section.

Definition 1.1.1 : *A **topological space** $\langle X, \mathcal{F} \rangle$ is a nonempty set X of points together with a family $\mathcal{F}$ of subsets (which we shall call open) possessing the following properties:*

1) $\mathbf{X} \in \mathcal{F}$, $\varnothing \in \mathcal{F}$
2) $\mathbf{X}_1 \in \mathcal{F}$ *and* $\mathbf{X}_2 \in \mathcal{F}$ *imply* $\mathbf{X}_1 \cap \mathbf{X}_2 \in \mathcal{F}$
3) $\mathbf{X}_i \in \mathcal{F}$ *implies* $\bigcup_i \mathbf{X}_i \in \mathcal{F}$, *where* $i \in I$ *and I is any set of indices*

The family $\mathcal{F}$ is called a topology for the set **X**.

Now that we have the definition of a topology we can see that there are automatically two types of topologies. The first is the trivial topology that contains only two open sets, the empty set Ø and the set **X** itself. The other is the discrete topology where any point **X** forms an open set.

Definition 1.1.2 : *A point $x \in X$ is called a **point of closure** of the set E if every open set O containing x meets E, i.e., has a nonempty intersection with E.*

The definition of a closed set in a topology **X** is a subset O such that $\bar{O} = O$ where $\bar{O}$ is the set that contains all the points of closure for O. Now that we defined what a open and closed set are we can look at some of the properties of sets in a topology.

Proposition 1.1.1 : *The complement of an open set is closed and the complement of a closed set is open.*

In topological spaces we can consider the idea of a **limit**. A sequence $\langle x_n \rangle$ is said to have a limit x if there exist an integer N such that for any open set O that contains x we have $x_n \in O$ for all $n \geq N$. Now, we can define continuity for functions that map one topological space into another.

Definition 1.1.3 : *A mapping f of a topological space $\langle X, \mathcal{F} \rangle$ into a topological space $\langle Y, \mathcal{S} \rangle$ is said to be* ***continuous*** *if and only if the inverse image of every open set is open, that is, if $Y \in \mathcal{S} \Rightarrow f^{-1}(Y) \in \mathcal{F}$*

Continuing along this line we can define the idea of a homeomorphism.

Definition 1.1.4 : *A* ***homeomorphism*** *between two topological spaces is a one-to-one continuous mapping of **X** onto **Y** for which f^{-1} is also continuous. The spaces **X** and **Y** are said to be* ***homeomorphic*** *if there is a homeomorphism between them.*

We will now move to the foundations of metric spaces.

1.2 Metric Spaces

The following section will define and discuss all main concepts of metric spaces that will apply in our discussion. We will first define the idea of metric spaces and then redefine what it is to be a open and closed set in a metric space.

Definition 1.2.1 : *A **metric space** $\langle X,d\rangle$ is a space X together with a real-valued function $d : X \times X \to \mathbb{R}$, which measures the distance between pairs of points x and y in X. We require that d obeys the following axioms:*

i) $0 < d(x,y) \leq \infty \; \forall \, x,y \in \mathbf{X}, \; x \neq y$ and $d(x,x) = 0 \; \forall \, x \in \mathbf{X}$

ii) $d(x,y) = d(y,x) \; \forall \, x,y \in \mathbf{X}$

iii) $d(x,y) \leq d(x,z) + d(z,y) \; \forall \, x,y,z \in \mathbf{X}$

*Such a function d is called a **metric**.*

Examples of metrics on $\mathbb{R}$ are

1) $d(x,y) = |x - y|$ (Euclidean metric)

2) $d(x,y) = |x^3 - y^3|$

for $\mathbb{R}^2$

1) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ (Euclidean metric)

2) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ (Manhattan metric)

Notice that for both metrics in $\mathbb{R}^2$ as points move away from each other the distance always increases. (Note that we will be doing most of our work in the most familiar of the metric spaces, the Euclidean space.)

In topological spaces we had a notion of a open set and a closed set. In metric space we can also define open sets and closed sets under different rules of operation. The open set is defined as follows.

Definition 1.2.2 : *The Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. S is **open** if for each $x \in S$ there is an $\varepsilon > 0$ such that $B(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\} \subset S$.*

From the definition of a open set we have that in a metric space $\langle \mathbf{X}, d \rangle$, the set $\mathbf{X}$ will always be open. The proof is quite trivial and can be stated easily. Let us take a point $x \in \mathbf{X}$. Then $\forall\ \varepsilon > 0,\ B(x,\varepsilon) \subset \mathbf{X}$ since $x \in \mathbf{X}$ is nonempty and therefore the set $\mathbf{X}$ will be open. Before we can define what it is to be a closed set we will have to state the meaning of limits in a metric space.

Definition 1.2.3 : *A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric $\langle X, d \rangle$ is said to **converge** to a point $x \in X$ if, for any given number $\epsilon > 0$, there is an integer $N > 0$ so that*

$$d(x_n, x) < \epsilon \qquad \text{for all } n > N.$$

*In this case the point $x \in \mathbf{X}$, to which the sequence converges, is called the **limit** of the sequence, and we use the notation*

$$x = \lim_{n \to \infty} x_n.$$

Definition 1.2.4 : *Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. A point $x \in X$ is called a **limit point** of S if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points $x_n \in S \backslash \{x\}$ such that $\lim_{n \to \infty} x_n = x$.*

Now, that we have the definition of limit points consider a set S such that S contains all its limit points. Such a set S we would call closed.

Definition 1.2.5 : *Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. The **closure** of S, denoted $\bar{S}$, is defined to be $\bar{S} = S \cup \{\text{limits points of } S\}$. S is **closed** if it contains all of its* *limit points, that is,* $S = \bar{S}$ *. S is **perfect** if it is equal to the set of all its limit points.*

We know the complement of a open set is closed and the complement of a closed set is open in a general topological space. We may ask whether this holds in metric spaces. To answer this question consider a open set $S \subset \mathbf{X}$ in a metric space $\langle \mathbf{X}, d \rangle$ and suppose we have a sequence $\{x_n\} \in \mathbf{X} \backslash S$ with a limit $x \in \mathbf{X}$. If the limit x is in S then in any $B(x, \varepsilon)$ with $\varepsilon > 0$ we would have a $x_n \in \mathbf{X} \backslash S$, and S would not be open. This is a contradiction and therefore x must be contained in $\mathbf{X} \backslash S$ which proves that $\mathbf{X} \backslash S$ contains all its limit points and thus is a closed set.

To prove that the complement of $\mathbf{X} \backslash S$, meaning S, is open we must show that there exist $B(x, \varepsilon) \subset S$ for some $\varepsilon > 0$. If we assume that S is not open then for an $x \in S$ there is no $B(x, \varepsilon) \subset S$. Thus we can construct a sequence $x_n \in B(x, \frac{1}{n}) \cap (\mathbf{X} \backslash S)$ for every integer $n = 1, 2, 3, \ldots.$, which is a sequence in $\mathbf{X} \backslash S$ and has a limit in $\mathbf{X} \backslash S$. This contradicts $x \in S$ and therefore S is open.

Now let us further our definitions by considering the idea of subspaces. Take a subset S of a metric space $\langle \mathbf{X}, d \rangle$. If we restrict the metric d to subset S then we call S a metric space and a subspace of the metric space $\mathbf{X}$. This is to say that we take the distance between the points of S as we do for the space $\mathbf{X}$. An example of this is the space of $\mathbb{R}^2$ which is a Euclidean metric space, and the set $\langle x, 0 \rangle$ which is would be a subspace under the metric $\langle \mathbb{R}^2, d \rangle$. You may ask why concern ourselves with the subspaces? The reason is that the properties of the closure or whether a space is open or closed is all relative to what the space is contained in. Let us consider the metric space $\mathbb{R}$ and the open set $S = (0, 1)$. Then a subspace O which is the $(0, \frac{1}{2}]$ has a closure of $[0, \frac{1}{2}]$ in $\mathbb{R}$ and a closure of $(0, \frac{1}{2}]$ in S.

Proposition 1.2.1 : *Let* $\mathbf{X}$ *be a metric space and* $\mathbf{S}$ *a subspace of it. Then the closure of* $\mathbf{E}$ *relative to* $\mathbf{S}$ *is* $\bar{\mathbf{E}} \cap \mathbf{S}$*, where* $\bar{\mathbf{E}}$ *denotes the closure of* $\mathbf{E}$ *in* $\mathbf{X}$*. A set* $\mathbf{A} \subset \mathbf{S}$ *is closed relative to* $\mathbf{S}$ *if and only if* $\mathbf{A} = \mathbf{S} \cap \mathbf{F}$ *with* $\mathbf{F}$ *closed in* $\mathbf{X}$*. A set* $\mathbf{A} \subset \mathbf{S}$ *is open relative to* $\mathbf{S}$ *if and only of* $\mathbf{A} = \mathbf{S} \cap \mathbf{O}$ *with* $\mathbf{O}$ *open in* $\mathbf{X}$*.*

1.3 Equivalent and Homeomorphic Metric Spaces

In the embedding theorem we deal with functions that maps one metric space into another metric space. Therefore we will introduce some key concepts of mapping from one metric space to another. We start with the idea of what it means for one metric to be equivalent to another one. We define this as follows.

Definition 1.3.1 : *Two metrics d_1 and d_2 on a space X are **equivalent** if and only if there exist constants* $0 < c_1 < c_2 < \infty$ *such that*

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y), \qquad \forall (x,y) \in \mathbf{X} \times \mathbf{X}$$

An example of two equivalent metrics. Let $Y_1 = \{x \in R^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ and let d_1 = *Euclidean metric* and d_2 = *Manhattan metric*. These metrics d_1 and d_2 are equivalent in Y_1 but in the space $\mathbf{R}^2$ they are not (as then c_2 becomes infinite). We can ask when are two metric spaces equivalent. For example, let us take Y_1 and stretch one of corners to infinity and call this space Y_2. Are (Y_1, d_1) and (Y_2, d_2) equivalent metric spaces? We define equivalence between metric spaces as follows.

Definition 1.3.2 :*Two metrics spaces $\langle X_1, d_1 \rangle$ and $\langle X_2, d_2 \rangle$ are **equivalent** if there is a function $h : X_1 \to X_2$ that is one-to-one (i.e. h is invertible), such that the metric $\tilde{d}$ on X_1 defined by*

$$\tilde{d}(x,y) = d_2(h(x), h(y)), \qquad \forall (x,y) \in \mathbf{X}_1$$

is equivalent to d_1

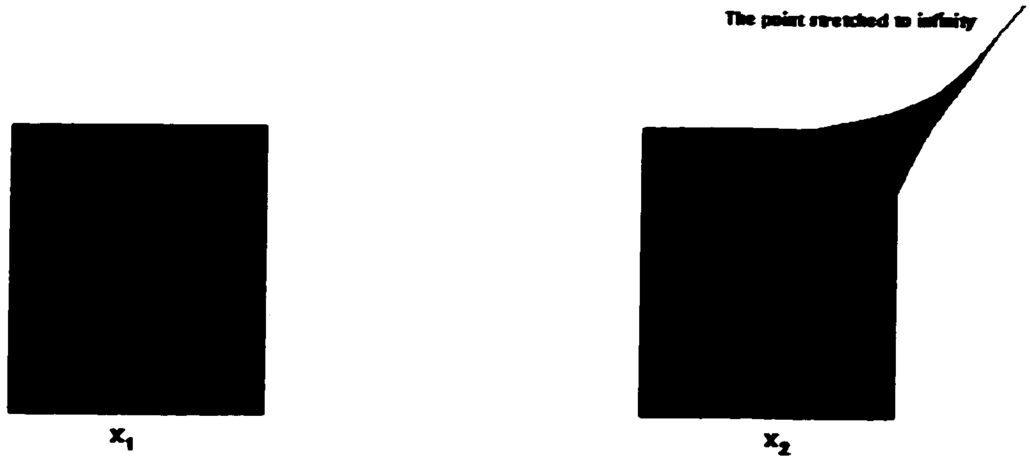
We can see the $\langle Y_1, d_1 \rangle$ and $\langle Y_2, d_2 \rangle$ are not equivalent. This follows from the fact that the definition of equivalence requires the deformation h to be bounded. Finally, we define a continuous function from one metric space to another.

Definition 1.3.3 :*A function $f : X_1 \rightarrow X_2$ from a metric space $\langle X_1, d_1 \rangle$ into a metric space $\langle X_2, d_2 \rangle$ is* ***continuous*** *if, for each $\epsilon > 0$ and $x \in X_1$, there is a $\delta > 0$ so that*

$$d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) < \epsilon.$$

If f is also one-to-one and onto, and thus invertible, and if also the inverse f^{-1} of f is continuous, then we say that f is a ***homeomorphism*** *between $\mathbf{X}_1$ and $\mathbf{X}_2$. In such a case we say that $\mathbf{X}_1$ and $\mathbf{X}_2$ are* ***homeomorphic.***

If we again reconsider $\langle Y_1, d_1 \rangle$ and $\langle Y_2, d_2 \rangle$, we know that they are not equivalent but we can see that they are homeomorphic.



The two spaces X_1 and X_2 are homeomorphic because they have the same topology.

Figure 1

1.4 Complete Metric Spaces

We are now at a point were we can separate and categorize metric spaces. One of these categories that will play a important role in what follows is the class of compact spaces. Before we can define compact spaces and their properties we will introduce notions of Cauchy sequences, convergence, and completeness.

Definition 1.4.1 : *A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric space $\langle X, d\rangle$ is called a* ***Cauchy sequence*** *if, for any given number $\epsilon > 0$, there is an integer $N > 0$ such that*

$$d(x_n, x_m) < \epsilon \qquad \forall\, n, m > N.$$

This can be pictured in ones mind as follows: as we move alone a sequence the points become closer and closer to each other. This does not necessarily mean that they are approaching a point. What can happen is that they are approaching a point that does not exist. Next we move to the idea of convergence.

From the definition of a convergent sequence it follows easily that any sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space $\langle \mathbf{X}, d\rangle$ that converges to a point $x \in \mathbf{X}$ is a Cauchy sequence. The inverse is not necessarily true. That is why we introduce a notion of a complete space.

Definition 1.4.2 : *A metric space $\langle X, d\rangle$ is* ***complete*** *if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit $x \in X$.*

Any Euclidean space $\mathbb{R}^n$ of any dimension $n \geq 1$ is a complete metric space.

1.5 Compactness

The idea of a compact space and of compact subspaces will play an important role in our definition of the embedding. Before we define compactness we must first consider a notion of subsequence. A subsequence of a sequence $\{x_n\}_{n=1}^{\infty}$ is a sequence of the form $x_{n_1}, x_{n_2} x_{n_3}, \ldots,$ where the n_j are natural numbers with $n_1 < n_2 < n_3 \ldots$. Now, we define the compact set.

Definition 1.5.1 : *Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. S is said to be* ***compact*** *if every infinite sequence $\{x_n\}_{n=1}^{\infty}$ in S contains a subsequence having a limit in S.*

Now that we have defined what is meant by a set being compact we will state the properties and proposition that come along with the sets being compact. One of these propositions is the relation between totally bounded sets and compact sets which states that any complete metric space which is closed and totally bounded is also compact and vice versa. To define the idea of totally bounded we must define what it is to be bounded.

Definition 1.5.2 : *Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. S is* ***bounded*** *if there is a point $a \in X$ and a number $\varepsilon > 0$ so that*

$$d(a,x) < \varepsilon \text{ for all } x \in X.$$

Definition 1.5.3 : *Let $S \subset X$ be a subset of a metric space $\langle X, d \rangle$. S is* ***totally bounded*** *if, for each $\varepsilon > 0$, there is a finite set of points $\{y_1, y_2, \ldots, y_n\} \subset S$ such that whenever $x \in X$, $d(x, y_i) < \varepsilon$ for some $y_i \in \{y_1, y_2, \ldots, y_n\} \subset S$. This set of points $\{y_1, y_2, \ldots, y_n\}$ is called an ε-net.*

We are now able to state and prove the main theorem of this section.

Theorem 1.5.1 : *Let $\langle X, d \rangle$ be a complete metric space. Let $S \subset X$. Then S is compact if* *and only if it closed and totally bounded.*

Proof Let S be closed and totally bounded and let $\{x_i \in S\}$ be an infinite sequence of points in the set S. Since S is totally bounded we can find a finite number of balls with radius 1 which cover S. Since we have a finite number of balls one of the balls must contain an infinite number of the points of the sequence $\{x_i \in S\}$. Let this ball be called B_1 and choose a point x_{N_1} that exist inside B_1. Since $S \cap B_1$ is again totally bounded we can continue in the same way to choose another point x_{N_2} in a ball B_2 such that $N_1 > N_2$ and B_2 contains an infinite number of points from $\{x_n\}$ and is of radius $\frac{1}{2}$. This can be continued and so as to create a nested sequence

$$B_1 \supset B_2 \supset B_3 \supset B_4 \supset B_5 \supset B_6 \supset B_7 \supset B_8 .... \supset B_n \supset ....$$

where B_n has the radius of $\frac{1}{2^{n-1}}$ and a we have a sequence of integers $\{N_n\}_{n=1}^{\infty}$ such that $x_{N_n} \in B_n$. It is easy to see that $\{x_{N_n}\}_{n=1}^{\infty}$ is a Cauchy sequence. Since S is closed and complete $\{x_n\}$ converges to a point x in S and therefore S is compact.

Now, let S be compact and suppose that there does not exist ε-net for S for an $\varepsilon > 0$. Then, there exist an infinite sequence of points in the sequence $\{x_n\} \in S$ with $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. We also know that the sequence must posses a convergent subsequence $\{x_{N_i}\}$ which is also a Cauchy sequence. Since $\{x_{N_i}\}$ is a Cauchy sequence we have a pair of integers N_i and N_j with $N_i \neq N_j$ so that $d(x_{N_i} x_{N_j}) < \varepsilon$. This is a contradiction, so S is closed and bounded. This completes the proof.

1.6 Connectedness

In this section we will discuss the concept of connectedness and what is meant by a space being connected. We start with the idea of a connected sets in a topological sense. To define connectedness we must first define disconnectedness.

Definition 1.6.1 : *If a set* $\mathbf{X}$ *is said to be* ***disconnected*** *if there exists a pair of open sets* O_1 *and* O_2 *such that*

1) $O_1 \cap O_2 = \emptyset$
2) $\mathbf{X} \subset O_1 \cap O_2$
3) $\mathbf{X} \cap O_1 \neq \emptyset$ *and* $\mathbf{X} \cap O_2 \neq \emptyset$

If a set $\mathbf{X}$ is not disconnected then it is called **connected**. We can also say that a space $\mathbf{X}$ is connected if and only if the only subsets of $\mathbf{X}$ that are both open and closed are the sets $\emptyset$ and $\mathbf{X}$. Connected can also be defined in a local sense with the definition of locally connected.

Definition 1.6.2 : *Let* $x \in X$. *Then,* X *is said to be* ***locally connected*** *at* x *if in every neighborhood* U *of* x *there exist a neighborhood* V *of* x *such that* $V \subset U$ *and* $U \cap V$ *is a connected set.* X *is locally connected if it is locally connected at each of its points.*

A metric space $\langle \mathbf{X}, d \rangle$ is connected if and only if the only two subsets of $\mathbf{X}$ that are simultaneously open and closed are $\mathbf{X}$ and $\emptyset$. A subset $S \subset \mathbf{X}$ is connected if the metric space $\langle S, d \rangle$ is connected. S is totally disconnected provided that the only nonempty connected subsets of S are subsets consisting of single points. We now proceed to the idea of pathwise-connected and disconnected subsets.

Definition 1.6.3 : *Let* $S \subset X$ *be a subset of a metric space* $\langle X, d \rangle$. *Then* S *is* ***pathwise-connected*** *if, for each pair of points* x *and* y *in* S, *there is a continuous function*

$f: [0,1] \to S$, from the metric space $([0,1], \text{Euclidean})$ into the metric space $\langle S, d\rangle$, such that $f(0) = x$ and $f(1) = y$. Such a function f is called a ***path*** *from x to y in S. S is* ***pathwise-disconnected*** *if it is not pathwise-connected.*

An example of a locally connected space is the set (Figure 2)

$$\left\{(x, \cos\tfrac{1}{x}) \mid x \neq 0\right\} \cap \{[-1,1] \times [0]\}.$$

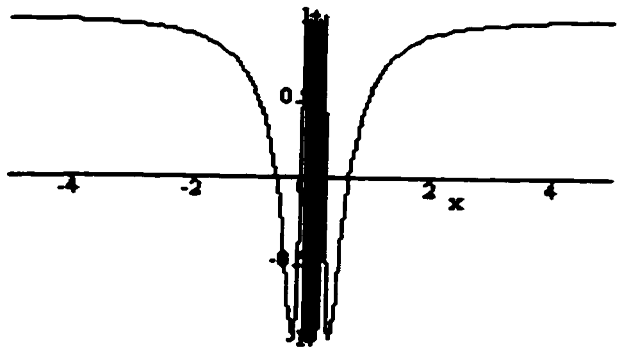


Figure 2

Let $S \subset \mathbf{X}$ be a subset of a metric space $\langle \mathbf{X}, d\rangle$. We can define two types of pathwise-connected sets, namely the simply connected and the multiply connected.

1) A pair of points $x, y \in S$ are simply connected if given any two paths f_0 and f_1 connecting x, y in S, we can continuously deform f_0 to f_1 without leaving the subset S.

2) A pair of points $x, y \in S$ are called multiply connected if they are not simply connected.

f_0 can be continuously deformed to f_1 if we can create a function g that continuously maps the Cartesian product $[0,1] \times [0,1]$ into S and such that function g can be expressed as $g(s,t)$ for $(0 \leq s, t \leq 1)$ where

1) $g(s,0) = f_0(s) (0 \leq s \leq 1)$

2) $g(s,1) = f_1(s)(0 \leq s \leq 1)$

3) $g(0,t) = x(0 \leq t \leq 1)$

4) $g(1,t) = y(0 \leq t \leq 1)$

2 Fractals

In the following chapter we will discuss the idea of a fractal space, the best space for us to work in. We will show most of our work in metric space $\mathbf{R}^2$ but a more general definition of compact subsets will be used in our fractal space.

2.1 Fractal Space

Definition 2.1.1 : *Let $\langle X, d\rangle$ be a complete metric space. Then $\mathcal{H}(X)$ denotes the space whose points are the compact subsets of $\mathbf{X}$, other than the empty set.*

We will now define a distance on the space $\mathcal{H}(\mathbf{X})$. For $x \in \mathbf{X}$ and $B \in \mathcal{H}(\mathbf{X})$ we define the "distance" $d(x,B) = \min\{d(x,y) : y \in B\}$. One may ask why should the minimum exist. This can be seen by looking at the function $f : B \to R$ defined by

$$f(y) = d(x,y) \text{ for all } y \in B.$$

f is continuous as a transformation from the metric space $\langle B,d\rangle$ to the metric space R. Let $P = \inf\{f(y) : y \in B\}$. We know that $f(y) \geq 0$ for all $y \in B$, and therefore P has a finite value. We now take a infinite sequence of points $\{y_n : n = 1,2,3...\} \subset B$ where $f(y_n) - P < \frac{1}{n}$ for each positive integer n. Since B is compact and y_n is a sequence in B, we have that $\{y_n : n = 1,2,3...\}$ would have a limit $\tilde{y} \in B$. Using the continuity of f we must have that $f(\tilde{y}) = P$ and therefore $\{d(x,y) : y \in B\}$ will have a minimum value.

Now we define the distance from a set $A \in \mathcal{H}(\mathbf{X})$ to a set $B \in \mathcal{H}(\mathbf{X})$ by $d(A,B) = \max\{d(x,B) : x \in A\}$. (Note: d is not a metric and is not symmetric).

Definition 2.1.2 : *Let $\langle X, d\rangle$ be a complete metric space. Then the Hausdorff distance between points A and B in $\mathcal{H}(\mathbf{X})$ is defined by*

$$h(A,B) = d(A,B) \vee d(B,A),$$

where $x \vee y$ denote the maximum of the two real numbers.

We will show that h is a metric on the space $\mathcal{H}(\mathbf{X})$. To prove h is a metric on the space $\mathcal{H}(\mathbf{X})$ we must show that h is a real-valued function and obeys the three axioms of a metric space. We know that h is a real-valued function by the fact $d(A,B) = \max\{d(x,B) : x \in A\}$ which is a real-valued function. Now we will prove the following three axioms of a metric space.

(1) Let $A,B \in \mathcal{H}(\mathbf{X})$. Then $h(A,B) = d(a,b)$ for some $a \in A, b \in B$, by the fact A and B are compact, so $0 \leq h(A,B) < \infty$.

Let $A \in \mathcal{H}(\mathbf{X})$. Then $h(A,A) = d(A,A) \vee d(A,A) = d(A,A) = \max\{d(x,A) : x \in A\} = 0$
Let $A,B \in \mathcal{H}(\mathbf{X})$. If $A \neq B$ then $h(A,B) = d(A,B) \vee d(B,A)$

$= \max\{d(x,B) : x \in A\} \vee \max\{d(x,A) : x \in B\} > 0.$

(2) Let $A,B \in \mathcal{H}(\mathbf{X})$. Then $h(A,B) = d(A,B) \vee d(B,A) = d(B,A) \vee d(A,B) = h(B,A)$.

(3) Let $A,B,C \in \mathcal{H}(\mathbf{X})$. Then for any $a \in A$

$d(a,B) = \min\{d(a,b) : b \in B\} \leq \min\{d(a,c) + d(c,b) : b \in B\} \forall c \in C$

$= d(a,c) + \min\{d(c,b) : b \in B\} \forall c \in C$, so we have that

$d(a,B) \leq \min\{d(a,c) : c \in C\} + \max\{\min\{d(c,b) : b \in B\} : c \in C\}$

$= d(a,C) + d(C,B)$, and then we know $d(A,B) \leq d(A,C) + d(C,B)$.

Similarly

$d(B,A) \leq d(B,C) + d(C,A)$, whence

$h(A,B) = d(A,B) \vee d(B,A) \leq d(B,C) \vee d(C,B) + d(A,C) \vee d(C,A)$

$= h(B,C) + h(A,C)$.

Therefore all three axioms are proved and h is a metric on the space $\mathcal{H}(\mathbf{X})$.

2.2 Contractions

In our discussions of fractals we have not really defined what it is to be a fractal or outline the structure of a fractal. This is due to the infancy of the idea of fractals and the complexity of the issues that surround the idea of fractals. One tool that can help us give some restrictions is the idea of contraction mappings.

Definition 2.2.1 : *A transformation $f : \mathbf{X} \to \mathbf{X}$ on a metric space $\langle \mathbf{X}, d\rangle$ is called* ***contractive*** *or a* ***contraction mapping*** *if there is a constant $0 \leq s < 1$ such that*

$$d(f(x),f(y)) \leq s \bullet d(x,y) \qquad \forall x,y \in \mathbf{X}.$$

Any such number s is called a contractivity factor for f.

For a more practical idea of contraction mappings we must discuss the idea of contraction mappings on complete metric space.

Definition 2.2.2 : *Let $f : \mathbf{X} \to \mathbf{X}$ be a transformation on a metric space. A point $x_f \in \mathbf{X}$ such that $f(x_f) = x_f$ is called a* ***fixed point*** *of the transformation f.*

As example, let us consider the mapping

$$w\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}$$

The point $\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$ is a fixed point of w. It is easy to check that

$$w\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}.$$

Theorem 2.2.1 : [(*The Contraction Mapping Theorem*).] *Let $f : \mathbf{X} \to \mathbf{X}$ be a contraction mapping on a complete metric $(\mathbf{X}, d)$. Then f possesses exactly one fixed point $x_f \in \mathbf{X}$ and, moreover, for any point $x \in \mathbf{X}$, the sequence $\{f^{\circ n}(x) : n = 0,1,2,\ldots\}$ converges to x_f. That is,*

$$\lim_{n\to\infty} f^{\circ n}(x) = x_f, \qquad \textit{for each } x \in \mathbf{X}.$$

To prove the contraction mapping theorem, we will first establish some inequalities. Let f be a contraction mapping and consider two iterations $f^3(x)$ and $f^2(x)$. Then $d(f^3(x), f^2(x)) \leq s \bullet d(f^2(x), f(x)) \leq s^2 \bullet d(f(x), x)$ for any $x \in X$. To generalize let

$$d\left(f^{\circ n}(x), f^{\circ n+c}(x)\right) \leq s^n \bullet d\left(f(x), f^c(x)\right)$$

be true for the iterations $f^{\circ n}(x)$ and $f^{\circ n+c}(x)$, where c is any positive number. Then

$$d\left(f^{\circ n+1}(x), f^{\circ n+1+c}(x)\right) \leq s \bullet d\left(f^{\circ n}(x), f^{\circ n+c}(x)\right) \leq s^{n+1} \bullet d\left(f(x), f^c(x)\right).$$

and we can state by induction that

$$d\left(f^{\circ n}(x), f^{\circ m}(x)\right) \leq s^{m \wedge n} d\left(x, f^{\circ |n-m|}\right)(x)$$

where the notation $m \wedge n$ stands for the smallest of the integers n and m.

We are now ready to prove the contraction mapping theorem

Proof To prove the contraction mapping theorem we will first prove that $f^{\circ n}(x)$ has a limit and that this limit is unique. Let us consider an $x \in \mathbf{X}$. Using the triangle inequality we have

$$d\left(x, f^{\circ k}(x)\right) \leq d(x, f(x)) + d\left(f(x), f^2(x)\right) + \cdots + d\left(f^{\circ(k-1)}(x), f^{\circ k}(x)\right)$$

$$\leq d(x,f(x)) + sd(x,f(x)) + s^2 d(x,f(x)) + \cdots + s^{k-1} d(x,f(x))$$
$$\leq (1 + s + s^2 + \cdots + s^{k-1}) d(x,f(x))$$
$$\leq (1 - s)^{-1} d(x,f(x)),$$

where $0 \leq s < 1$ is the contraction factor of f. Using the previously shown inequality we can rewrite the previous statement as

$$d\left(f^{\circ n}(x), f^{\circ m}(x)\right) \leq s^{m \wedge n} d\left(x, f^{\circ |n-m|}\right)(x) \leq s^{m \wedge n}(1 - s)^{-1} d(x,f(x)).$$

This shows that $\left\{f^{\circ n}(x)\right\}_{n=0}^{\infty}$ is a Cauchy sequence and thus, it converges to a point $x_f \in \mathbf{X}$, since $\mathbf{X}$ is complete.

$$\lim_{n \to \infty} f^{\circ n}(x) = x_f.$$

What is left to prove is that x_f is a fixed point of f and unique. It is quite easily seen that x_f is fixed by the fact that

$$f(x_f) = f(\lim_{n \to \infty} f^{\circ n}(x)) = \lim_{n \to \infty} f^{\circ (n+1)}(x) = x_f.$$

Continuity of f is proved below in Lemma 2.2.1.. To show that x_f is unique, we assume that there exists another fixed point y_f, $f(y_f) = y_f$. Given this, we have that

$$d(x_f, y_f) = d(f(x_f), d(y_f)) \leq sd(x_f, y_f)$$
$$\Rightarrow d(x_f, y_f) \leq sd(x_f, y_f)$$
$$\Rightarrow d(x_f, y_f) - sd(x_f, y_f) \leq 0$$
$$\Rightarrow (1 - s) d(x_f, y_f) \leq 0$$

by the fact that f is a contraction mapping. Therefore s is equal to zero and we have that $x_f = y_f$.

Some other information which can be drawn from the properties of contraction mapping are

Lemma 2.2.1 : *Let $w : X \to X$ be a contraction mapping on the metric space $\langle X, d\rangle$. Then w is continuous.*

Proof We know that if a function $f : \mathbf{X}_1 \to \mathbf{X}_2$ from a metric space $\langle \mathbf{X}_1, d\rangle$ into a metric space $\langle \mathbf{X}_2, d_2\rangle$ is continuous then, for each $\epsilon > 0$ and $x \in X_1$, there is a $\delta > 0$ so that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) \; < \epsilon.$$

Given that $d(f(x),f(y)) \; \leq sd(x,y)$ we can see that if $\delta = \frac{\epsilon}{s}$ we have $d(x,y) < \delta$ implies $d(f(x),f(y)) \; < \epsilon$ and therefore f is continuous.

Lemma 2.2.2 : *Let $w : X \to X$ be a contraction mapping on the metric space $\langle X, d\rangle$ Then w maps $\mathcal{H}(X)$ into itself.*

Proof Since that w is continuous for any nonempty subset S of a space $\mathcal{H}(\mathbf{X})$ we have that $w(S)$ is also nonempty. Let us take point $\{y_n\}$ such that $\{y_n = w(x_n)\}$, where $\{x_n\} \in \mathcal{H}(\mathbf{X})$ is a infinite sequence of points in S. We know that x_n has a convergent subsequence $\{x_{N_n}\}$since S is compact and therefore there exist an convergent subsequence $\{y_{N_n}\}$. Also since S is compact the subsequence converges to a point $\check{x} \in S$ and by the continuity of w the subsequence $\{y_{N_n}\}$ converges to $\check{y} = w(\check{x}) \in w(S)$. Therefore w maps $\mathcal{H}(\mathbf{X})$ into itself.

Lemma 2.2.3 : *Let $w : X \to X$ be a contraction mapping on the metric space $\langle X, d\rangle$ with a contractivity factor s. Then $w : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by*

$$w(B) = \{w(x) : x \in B\} \qquad \forall B \in \mathcal{H}(\mathbf{X})$$

is a contraction mapping on $(\mathcal{H}(\mathbf{X}), h(d))$ with contractivity factor s.

and finally

Lemma 2.2.4 : *Let $\langle X,d\rangle$ be a metric space. Let $\{w_n : n = 1,2,...,N\}$ be contraction mappings on $(\mathcal{H}(X), h(d))$. Let the contractivity factor for w_n be denoted by s_n for each n. Define $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by*

$$W(B) = w_1(B) \cup w_2(B) \cup ... \cup w_n(B) = \cup_{n=1}^{n} w_n(B) \qquad \text{for each } B \in \mathcal{H}(X).$$

Then W is a contraction mapping of $\mathcal{H}(\mathbf{X})$ with contractivity factor $s = \max\{s_n : n = 1,2,\ldots,N\}$.

Proof We demonstrate the claim for $N = 2$. An inductive argument then completes the proof. Let $B, C \in \mathcal{H}(\mathbf{X})$. We have

$$h(W(B), W(C)) = h(w_1(B) \cup w_2(B), w_1(C) \cup w_2(C))$$
$$\leq h(w_1(B) \cup w_1(C)) \vee h(w_2(B) \cup w_2(C)) \leq s_1 h(B,C) \vee s_2 h(B,C) \leq sh(B,C)$$

This completes the proof.

2.3 Iterated Function System (IFS)

We will now continue with the idea of contraction mappings and introduce Iterated Function Systems (IFS), a finite set of contraction mappings.

Definition 2.3.1 : *A (hyperbolic) iterated function system consists of a complete metric space $\langle X, d\rangle$ together with a finite set of contraction mapping $w_n : X \to X$, with respective contractivity factors s_n, for $n = 1,2,...,N$. The abbreviation **IFS** is used for "iterated function system." The notation for the IFS just announced is $\{X; w_n, n = 1,2,...,N\}$ and its contractivity factor is $s = max\{s_n : n = 1,2,...,N\}$*

What is meant by hyperbolic is that if linear map A on $\mathbb{R}^n$ has no eigenvalues of absolute value one it is hyperbolic. An example of an IFS is the mapping $W = \left\{\mathbb{R}; \frac{1}{4}x + \frac{3}{4}, \frac{1}{2}x, \frac{1}{4}x + \frac{1}{4}\right\}$. The action of this mapping can be seen in the following figure.

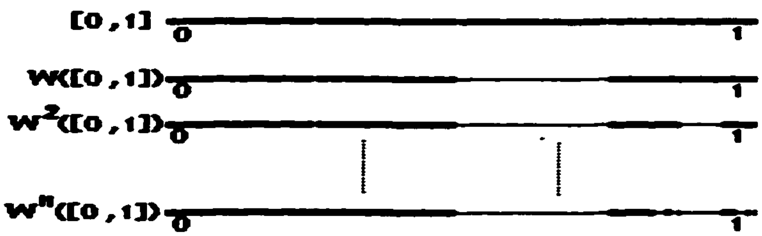


Figure 3

The contractivity factor is $\frac{1}{2}$ since this is the maximum of all contraction factors.

We can also define an IFS on a metric space in a more general way by adding a condensation transformation. Define a transformation $w_0 : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ on a metric space here $\langle \mathbf{X}, d\rangle$ by $w_0(B) = C$ for all $B \in \mathcal{H}(\mathbf{X})$ where $C \in \mathcal{H}(\mathbf{X})$. Then w_0 is called a condensation transformation. Now let us define a general IFS on a metric space.

Definition 2.3.2 : *Let $\{X; w_1, w_2, ..., w_n\}$ be a hyperbolic IFS with contractivity factor*

$0 \leq s < 1$. *Let* $w_o : \mathcal{H}(X) \to \mathcal{H}(X)$ *be a condensation transformation. Then* $\{X; w_0, w_1, w_2, ..., w_n\}$ *is called a* ***hyperbolic IFS with condensation****, with contractivity factor s.*

We summarize the facts about the IFS in the following theorem.

Theorem 2.3.1 : *Let* $\{X;\ w_n, n = 1,2,...,N\}$ *be a hyperbolic iterated function system with contractivity factor s. Then the transformation* $W : \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by*

$$W(B) = \cup_{n=1}^{n} w_n(B)$$

for all $B \in \mathcal{H}(\mathbf{X})$*, is a contraction mapping on the complete metric space* $(\mathcal{H}(\mathbf{X}), h(d))$ *with contractivity factor s. That is*

$$h(W(B), W(C)) \leq s \bullet h(B, C)$$

for all $B, C \in \mathcal{H}(\mathbf{X})$*. Its unique fixed point,* $A \in \mathcal{H}(\mathbf{X})$*, satisfies*

$$A = W(A) = \cup_{n=1}^{n} w_n(A)$$

and is given by $A = \lim_{n \to \infty} W^{\circ n}(B)$ *for any* $B \in \mathcal{H}(\mathbf{X})$*. A is called the attractor of the IFS W.*

2.4 Fractal Dimension

The idea of a fractal dimension comes from the need to distinguish between the fractal spaces. When we look at fractal spaces we see that upon magnification they do not loose complexity, they may even become more complex. So how are we to find the dimension of such a space? First, let us consider a line and separate it into $\frac{1}{n}$ pieces where n is a positive integer. When magnified n times we get an object that looks like the original object. The same goes for a square or cube but the square needs a magnification of n^2 with each piece having an area of $\frac{1}{n^2}$ and the cube needs a magnification of n^3 with each piece having an area of $\frac{1}{n^3}$. We now see that the exponents for the magnification of area are also the dimension of these objects. With this idea in mind we do the same to the fractal spaces in hope of finding the dimension.

Definition 2.4.1 : *Let $A \in \mathcal{H}(X)$, where (X,d) is a metric space. For each $\varepsilon > 0$ let $\mathcal{N}(A,\varepsilon)$ denote the smallest number of closed balls of radius $\varepsilon > 0$ needed to cover A. If*

$$D = \lim_{\varepsilon \to 0} \left\{ \frac{\ln \mathcal{N}(A,\varepsilon)}{\ln(1/\varepsilon)} \right\}$$

exists, then D is called the fractal dimension of A. We will also use the notation $D = D(A)$ and will say "A has fractal dimension D."

A more detailed explanation of what is written above follows. Let $B(x,\varepsilon)$ be a ball around x with a radius ε. For any $A \in \mathcal{H}(\mathbf{X})$ there exist a finite covering by such balls since $A \in \mathcal{H}(\mathbf{X})$ and is therefore compact. Let $\mathcal{N}(A,\varepsilon)$ denote the smallest positive integer V such that $A \subset \cup_{v=1}^{V} B(x,\varepsilon)$. Then

$$V\varepsilon^{-D} \approx \mathcal{N}(A,\varepsilon)$$
$$\ln(V\varepsilon^{-D}) \approx \ln(\mathcal{N}(A,\varepsilon))$$
$$\ln(V) + D\ln\left(\frac{1}{\varepsilon}\right) \approx \ln(\mathcal{N}(A,\varepsilon))$$

$$D \approx \frac{\ln(\mathcal{N}(A,\varepsilon)) - \ln(V)}{\ln\left(\frac{1}{\varepsilon}\right)}$$

since $\frac{\ln(V)}{\ln\left(\frac{1}{\varepsilon}\right)}$ *goes to zero as* $\varepsilon \to 0$ *we are left with* $D = \lim_{\varepsilon \to 0} \left\{ \frac{\ln \mathcal{N}(A,\varepsilon)}{\ln(1/\varepsilon)} \right\}$

We can also restate the above definition for a discrete variable ε_n.

Theorem 2.4.1 : *Let* $A \in \mathcal{H}(X)$, *where* $\langle X,d \rangle$ *is a metric space. Let* $\varepsilon_n = Cr^n$ *for real numbers* $0 < r < 1$ *and* $C > 0$, *and integers* $n = 1,2,3,\ldots$. *If*

$$D = \lim_{n \to \infty} \left\{ \frac{\ln \mathcal{N}(A,\varepsilon_n)}{\ln(1/\varepsilon_n)} \right\}.$$

then A has fractal dimension D.

Some other methods of finding the fractal dimension are as follows.

Theorem 2.4.2 : (*The Box Counting Theorem*) *Let* $A \in \mathcal{H}(\mathbf{R}^m)$, *where the Euclidean metric is used. Cover* $\mathbf{R}^m$ *by closed square boxes of side length* $(1/2^n)$, *as exemplified in figure I for* $n = 2$ *and* $m = 2$. *Let* $\mathcal{N}_n(A)$ *denote the number of boxes of side length* $(1/2^n)$ *which intersect the attractor. If*

$$D = \lim_{n \to \infty} \left\{ \frac{\ln(\mathcal{N}_n(A))}{\ln(2^n)} \right\},$$

exists, then A has fractal dimension D.

Theorem 2.4.3 : *Let* $\langle X,d \rangle$ *be a complete metric space. Let* $A \in \mathcal{H}(X)$. *Let* $\mathcal{N}(\varepsilon)$ *denote the minimum number of balls of radius* ε *needed to cover A. If*

$$D = \lim_{\varepsilon \to 0} \left\{ \sup \left\{ \frac{\ln \mathcal{N}(\tilde{\varepsilon})}{\ln(1/\tilde{\varepsilon})} : \tilde{\varepsilon} \in (0,\varepsilon) \right\} \right\}$$

exists, then D is called the fractal dimension of A.

2.5 Hausdorff-Besicovitch Dimension

The last method of finding a fractal dimension that we will discuss is called the Hausdorff-Besicovitch Dimension. This method is not readily used due to its difficulty in computation when working with experimental data.

When working with the Hausdorff-Besicovitch dimension we work in the Euclidean metric space $\langle \mathbb{R}^m, d \rangle$ where m is a positive number. Let $A \subset \mathbb{R}^m$ be bounded. Then we use the notation

$$diam(A) = \sup\{d(x,y) : x,y \in A\}.$$

Let $0 < \epsilon < \infty$, and $0 \leq p < \infty$. Let A denote the set of sequences of subsets $\{A_i \subset A\}$, such that $A = \cup_{i=1}^{\infty} A_i$. Then we define

$$\mathcal{M}(A,p,\varepsilon) = \inf\left\{\sum_{i=1}^{\infty} (diam(A_i))^p : \text{and } diam(A_i) < \varepsilon \quad \text{for } i = 1,2,3,\ldots\right\}$$

with

$$\mathcal{M}(A,p) = \sup\{\mathcal{M}(A,p,\varepsilon) : \epsilon > 0\}.$$

Theorem 2.5.1 : *Let m be a positive integer. Let A be a bounded subset of the metric space* $(R^m, Euclidean)$. *Let* $\mathcal{M}(A,p)$ *denote the function of* $p \in [0,\infty)$ *defined above. Then there is a unique real number* $D_H \in [0,m\}$ *such that*

$$\mathcal{M}(A,p) = \begin{cases} \infty & \text{if } p < D_H \text{ and } p \in [0,\infty) \\ 0 & \text{if } p > D_H \text{ and } p \in [0,\infty) \end{cases}$$

The corresponding real number D_H that occurs is called the Hausdorff-Besicovitch of the set A.

Theorem 2.5.2 : *Let m be a positive integer. Let $\{R^m; w_1, w_2, ... w_N\}$ be a hyperbolic IFS, and let A denote its attractor. Let w_n be a similitude of scaling factor s_n for each $n \in \{1,2,3,...,N\}$. If the IFS is totally disconnected or just-touching, then the Hausdorff-Besicovitch dimension $D_H(A)$ and the fractal dimension $D(A)$ are equal. In fact $D(A) = D_H(A) = D$, where D is unique solution of*

$$\sum_{n=1}^{N} |s_n|^D = 1, \ D \in [0,m].$$

If D is positive, then the Hausdorff D-dimensional measure $M(A, D_H(A))$ is a positive real number.

2.6 Lyapunov Exponents

Given a fixed point x_1 of a one-dimensional map f where $|f'(x)| = b > 1$, the orbits of any two points x,y near x_1 will separate at a rate of b for each iteration. Therefore if we look at a periodic point after k iteration we have to look at the derivative of the k^{th} iterations of the map which by the chain rule is the product the derivatives at the k points of the orbit. If this product of derivatives A is greater then 1 then the average rate of separation would be equal to $A^{\frac{1}{k}}$ per iterate. The Lyapunov number is introduced to quantify this average multiplicative rate of separation of the points very close to x_1. The Lyapunov exponents is just the natural logarithm of the Lyapunov number. Therefore a Lyapunov number of 2 (a Lyapunov exponent of $\ln 2$) for the orbit of x_1 would have all nearby points x double their distance with every iterate. If the Lyapunov number were $\frac{1}{2}$ the distance between the orbit x_1 and all nearby points would be halved.

Definition 2.6.1 : *Let f be a smooth map of the real line **R**. The **Lyapunov number** $L(x_1)$ of the orbit $\{x_1, x_2, x_3 ......\}$ is defined as*

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| ... |f'(x_k)|)^{\frac{1}{n}}$$

*if the limit exists. The **Lyapunov exponent** $h(x_1)$ is defined as*

$$h(x_1) = \lim_{n\to\infty} (1/n)[\ln|f'(x_1)| + ... + \ln|f'(x_k)|]$$

if this limit exists.

They are two forms of periodicity that we consider.

Definition 2.6.2 :*Let f be a smooth map. An orbit $\{x_1, x_2, ... x_n, ...\}$ is called **asymptotically periodic** if it converges to a period orbit as $n \to \infty$; this means that there exists a periodic orbit $\{y_1, y_2, ..., y_k, y_1, y_2, ...\}$ such that*

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

The orbit with the initial condition $x = \frac{1}{2}$ of $f(x) = 4x(1-x)$ is asymptotically periodic since after two orbits it goes to the fixed point 0. The term **eventually periodic** describes the case where the orbit lands precisely on the periodic orbit.

We now consider the Lyapunov exponent for the space of $\mathbf{R}^m$ where $m > 1$.

Definition 2.6.3 :*Let f be a smooth map on $\mathbf{R}^m$, let $J_n = Df^n(v_0)$, and for $k = 1, ..., m$ let r_k^n be the length of the k^{th} longest orthogonal axis of the ellipsoid $J_n U$ for an orbit with initial point v_0. Then r_k^n measures the contraction or expansion near the orbit of v_0 during the first n iterations. The kth **Lyapunov number** of v_0 is defined by*

$$L_k = \lim_{n \to \infty} (r_k^n)^{\frac{1}{n}}$$

if this limit exists. The ***kth Lyapunov exponents*** *of* v_0 *is* $h_k = \ln L_k$.

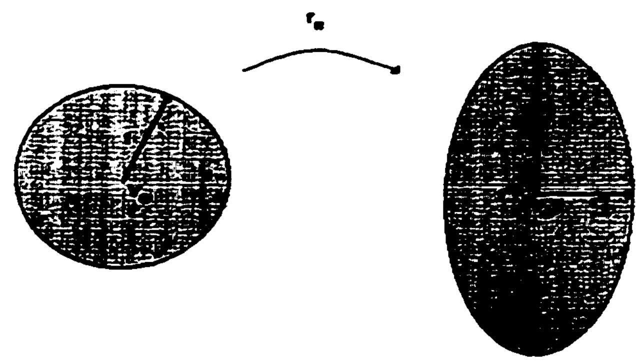


Figure 4

It is through the Lyapunov exponent that we are able to evaluate if a mapping is chaotic in its behavior.

Definition 2.6.3 *:Let f be a map of* $\mathbb{R}^m$, $m \geq 1$, *and let* $\{v_0, v_1, v_2, ...\}$ *be a bounded orbit of f. The orbit is* ***chaotic*** *if*

1. *it is not asymptotically periodic*
2. *no Lyapunov number is exactly one, and*
3. $L_1(v_0) > 1$.

3 Predicting the NYSE

In this are last and final chapter we will introduce and use the idea of embedding to create a space that embodies the characteristics of the New York Stock Exchange (NYSE). What we hope to show is that the NYSE does not just move in some random order but that it is chaotic and there is a method to its madness.

3.1 The Embedding Theorem

When discussing a system that is considered a dynamical systems we are talking of a set of rules from which the evolution of any point is dictated by a set of rules. An example of a dynamical system can be as simple as the equation expressed by the one-dimensional circle

$$f(x,y) = \sin^2(x) + \cos^2(y).$$

If the system is a set of equations where the rules are differential equations it is a **flow** and when the rules are discrete difference equations the system is referred to as a **map**. The evolution of a dynamical system is best described in its **phase space**, a coordinate system whose coordinates are all the variables that are entered into the mathematical formulation of the system. In short, the variables necessary to completely describe the state of the system at any given moment. If the system is a particle of mass m, then its state at any given moment is completely described by its speed v and the position r relative to some fixed point. Thus its phase space is two dimensional with coordinates v and r.

In studying the mathematical dynamical systems and of nonlinear deterministic systems we know that random looking behavior can arise from simple nonlinear systems. Such dynamics which we term chaotic exhibit complicated strange attractors that are fractal sets with positive Lyapunov exponents. If we are given a system recognizing the chaotic behavior is easy as producing its Fourier spectra of the evolution of one of the variables and as well it is fairly straightforward finding its Lyapunov exponents. However, when dealing with controlled experiments where we can not control all the variables or uncontrolled systems (like the weather) where the numbers of variables and mathematical formulation are not exactly known, the behavior of the system seems unformulated and the predictability of the system becomes extremely difficult. One of the most examined systems is that of a time series. For example Stock Market prices are expressed in a time series where at any given moment the stock has a designated value. But how do we unfold the prices to get back to the original system or at least reconstruct a system that simulates the original? And what of the problem of overlapping from the image projecting from a higher dimension to a lower dimension? These problems only increase the difficulty of prediction. For example, suppose that all trajectories in a phase space R^3 are attracted to a periodic cycle which we will call A and which is contained in R^3. Then there is a measurement map that measures the distance between any two points $x_1, x_2 \in R^3$ and expresses it in R^2.

$$M(x_1, x_2) = (y_1, y_2)$$

where y_1 and y_2 are expressed in R^2. This reconstructed space perceives the distance between points but how well does it preserve the properties of A? An example can be given quite easily by three diagrams. Let us take a mapping M that maps R^3 to R^2. In Figure 5 we can see the case where the information of R^3 is not lost to R^2 due to the one-to-one mapping and preserves differential information. In Figure 6 we see two points x_1 and x_2 in R^3 are projected to one point in R^2 and, therefore, we have a loss of some information since the it is not one-to-one. In Figure 7 we also see that we can have a one-one map yet still loss some information because of a loss of a differential point x in R^3 when project down into R^2, where it is not differential at x and therefore losses some ability to forecast information. The reason why M having a one-to-one property is so useful to us is that its future evolution which is completely specified by a point in phase space is not lost. What is meant by this is that if x is a point in phase space where an event occurs soon after this point, then the same event must occur soon after $M(x)$ if M is a one-to-one mapping. Although most of the interest lies in the case when A is a an attractor of a dynamical system, the main question can be posed more generally. Let A be a compact subset of Euclidean space R^k, and let M map R^k

to another Euclidean space R^n. Under what conditions can we be assured that A is "embedded" in R^n by a map M? This can be stated by the fact that A can be embedded by M if it is a smooth (Here, as in the remainder of the paper, the word smooth will be used to mean continuously differentiable, C^1) diffeomorphic from A onto its image $M(A)$.

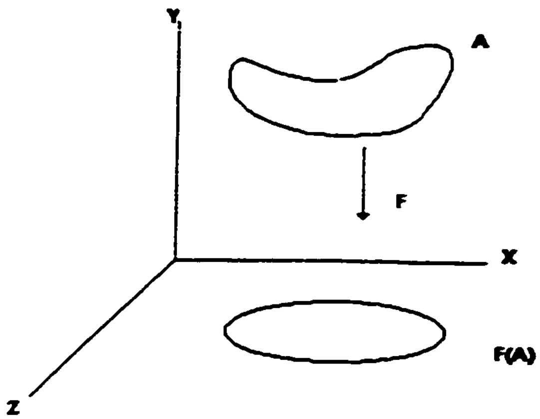


Figure 5

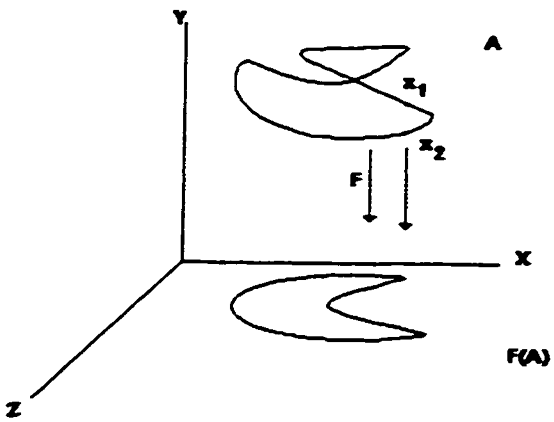


Figure 6

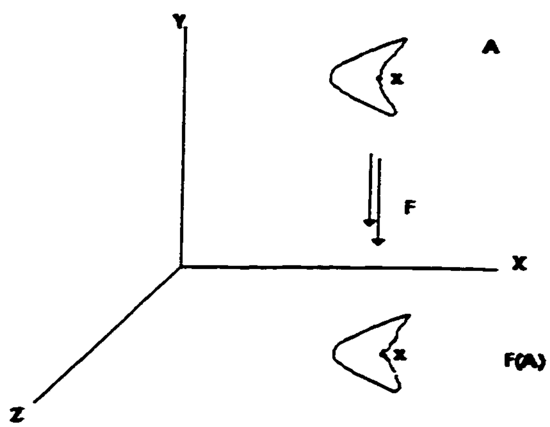


Figure 7

Definition 3.1.1 :*Let A be a compact subset of a euclidean space $\mathbb{R}^k$. Then a smooth map F on A is an **immersion** if the derivative map $DF(x)$ is one-to-one at every point $x \in A$.*

When working with the embedding theorem we have to assume two things. First that the mapping F is a flow in a Euclidean space $\mathbb{R}^k$ governed by an autonomous system of k

differential equations and secondly that all trajectories are asymptotic to an attractor A. By an autonomous system we mean a system of the form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n),\end{aligned}$$

i.e., there no time varying forces acting on the dynamical system from the outside and that the vector field f is stationary. This means that no two trajectories corresponding to two evolutions from two different initial conditions cross through the same point in phase space. Now we deal with the three theorems that make up the foundations of the embedding theorems we use.

Theorem 3.1.1 :(*Whitney Embedding Prevalence Theorem.*) *Let A be a compact smooth manifold of dimension d contained in $\mathbb{R}^k$. Almost every smooth map $\mathbb{R}^k \to \mathbb{R}^{2d+1}$ is an embedding of A.*

Theorem 3.1.2 :(*Fractal Whitney Embedding Prevalence Theorem.*) *Let A be a compact subset of $\mathbb{R}^k$ of box-dimension d, and let n be an integer greater than 2d. For almost every smooth map $F : \mathbb{R}^k \to \mathbb{R}^n$,*

1) *F is one-to one on A; and*
2) *F is an immersion on each compact subset C of a smooth manifold contained in A.*

Definition 3.1.2 :*If Φ is a flow on a manifold M, T is a positive number (called the delay), and $h : M \to \mathbb{R}$ is a smooth function, define the delay-coordinate map $F(h, \Phi, T) : M \to \mathbb{R}$ by*

$$F(h, \Phi, T)(x) = (h(x), h(\Phi_T(x)), h(x), h(\Phi_{2T}(x)), \ldots, h(x), h(\Phi_{(n-1)T}(x))).$$

Theorem 3.1.3 : *(Fractal Delay Embedding Prevalence Theorem.) Let Φ be a flow on an open subset U of R^k, and let A be a compact subset of U of box counting dimension d. Let $n > 2d$ be an integer, and let $T > 0$. Assume that A contains at most a finite number of equilibria. no periodic orbits of Φ of period T or 2T, at most finitely many periodic orbits of period $3T, 4T, ..., nT$, and that the linearization of each periodic orbit has distinct eigenvalues. Then for almost every smooth function h on U,*

1) *$F(h, \Phi, T)$ is one-to one on A; and*

2) *$F(h, \Phi, T)$ is an immersion on each compact subset C of a smooth manifold contained in A.*

The conclusion from the above theorem is that any smooth manifold of dimension m can be smoothly embedded in $n = 2m + 1$ dimensions. Also embedding the data in a dimension n where $n \geq 2m + 1$ preserves the topological properties of the attractor. More importantly the embedding is a diffeomorphism (a differentiable mapping with a differentiable inverse). Therefore we can recreate a space that simulates the original space and use our reconstructed space to predict future trends in the NYSE. This reconstructed space will be constructed using coordinates made from the observed variable (NYSE)

$$y(n) = [s(n), s(n + T), s(n + 2T), ...].$$

In this equation $s(n)$ will give value of the observed variable and T will be the time interval for a reasonably amount of time to pass to create independence between $s(n + kT)$ and $s(n + (k + 1)T), k \geq 0$.

3.2 Average Mutual Information

Let us consider data that is arranged in a time series. For example the temperature of the day or the value of a stock. These pieces of information are the product of many forces that can not be seen. Stock prices react to many outside variables such as the economic health of companies and countries. But obtainable forecasts do not begin to include all the other outside obstacles as volcanic eruptions and population. All we can say is that the price of a stock one day has a relation with the price of the next stock the next. When the price goes up people might buy or people might sell, no one is sure of the outcome but the relation between days, weeks or months does exist. So if we were to try to forecast any future trends how can this be done? One way to do this is to try to recreate the system that produces the prices of the stocks. As mentioned, given a time series we can try to create a phase space that simulates the output of our time series. The first step in constructing a phase space is to find the time interval that creates independence between values of observed variables. By the Fractal Delay Embedding Prevalence Theorem we must find the dimension of the data to create the phase space $y(n) = [s(n), s(n+T), s(n+2T), ...]$. The ideal would be that each of our coordinates would be independent of each other but since prices always affect the outcome of the next price we should be looking for a more lenient form of independence. This is where the idea of average mutual information comes in. We are looking for the time period in our interval when the prices come not totally independent but independent enough. This notion of information among measurements comes from Shanon's idea [see 6] of mutual information between two points a_i and b_i drawn form two sets A and B respectively. What is meant by mutual information is the amount that is learned by a_i about b_i. This is expressed as follows

$$\log_2\left[\frac{P_{AB}(a_i, b_j)}{P_A(a_i)P_B(b_j)}\right].$$

where $P_{AB}(a,b)$ is the joint probability density for measurements A and B resulting from the values of a_i and b_i. $P_A(a)$ and $P_B(b)$ are the individual probability densities for the measurements of A and B. Remembering the basic statistics, if we have A being completely independent of B, then $P_{AB}(a_i,b_j) = P_A(a_i)P_B(b_j)$ and the amount of mutual information is zero. To find the average of all the measurements between our sets we write

$$I_{AB} = \sum_{a_i,b_j} P_{AB}(a_i,b_j)\log_2\left[\frac{P_{AB}(a_i,b_j)}{P_A(a_i)P_B(b_j)}\right],$$

Given a time series $s(n)$ we can look for the associated information by letting A to be equal to $s(n)$ and B to $s(n+T)$, where T represents an amount of time.

$$I(T) = \sum_{s(n),s(n+T)} P(s(n),s(n+T))\log_2\left[\frac{P(s(n),s(n+T))}{P(s(n))P(s(n+T))}\right]$$

We will now use this idea on the index prices of the New York Stock Exchange (NYSE) to see if there is any form of Mutual Information shared between data that is expressed between the weekly data from the week of the 8th of January 1965 until the April 1st 1999. Before using the data we must first make sure that the data given is independent of any outside contamination. Looking at the NYSE we know that as the economy grows so will the stock prices. Therefore we have to detrend the index by filtering out the economic growth. The are many methods of achieving this goal, such as Ping Chen [see 7] method of filtering out the internal rate of growth over the period and Edgars Peters of filtering the Consumer Price Index. We have chosen the following method

$$s_i = \log(N_i) - (a\log(CPI_i) + c)$$

where

s_i = the detrended value of the NYSE
N_i = the NYSE price on week i
CPI_i = The CPI on week i.

The values of a and c are constant and were derived using the Least Squares Method on the NYSE against the CPI. The figure below shows the result of our detrending. Note the wave like cycles which appear to be non-periodic.

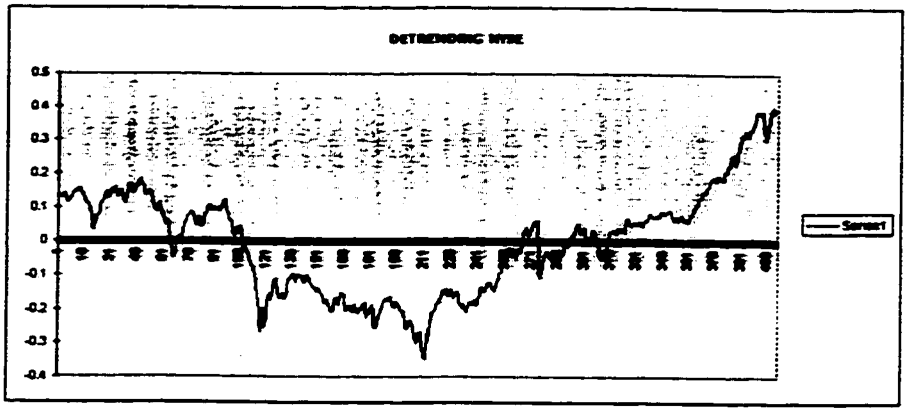


Figure 8

Now that we have our information we must find our time interval for independence. This is done by plugging in increasing values of T. We first test the mutual information of the sets $s(n)$ and $s(n+1)$ and then move on to $s(n)$ and $s(n+2)$ and so forth. As we let the value of T increase we look for the point where our information becomes independent enough. Below we map $s(n)$ versus $s(n+1)$. We can see that the line moves in a circular motion which shows us the first sign that there is a underlying symmetry to the information which at first seemed to have none.

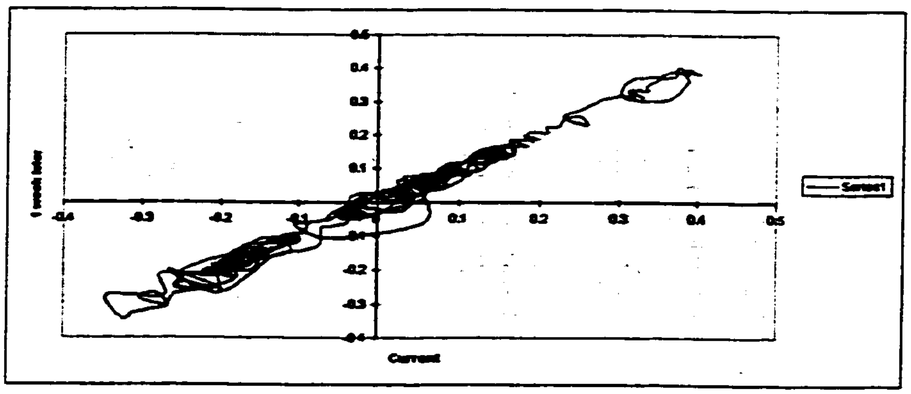


Figure 9: A 2 dimensional map of current detrended NYSE versus the one week later detrended value of the NYSE.

The question remains at which point will the value of T be enough to create independence if the function tends to zero. By the suggestion of Fraser [see 8] we choose the value of T to be the first minium of the average mutual information. A more detailed justification of this method is beyond the scope of this paper.

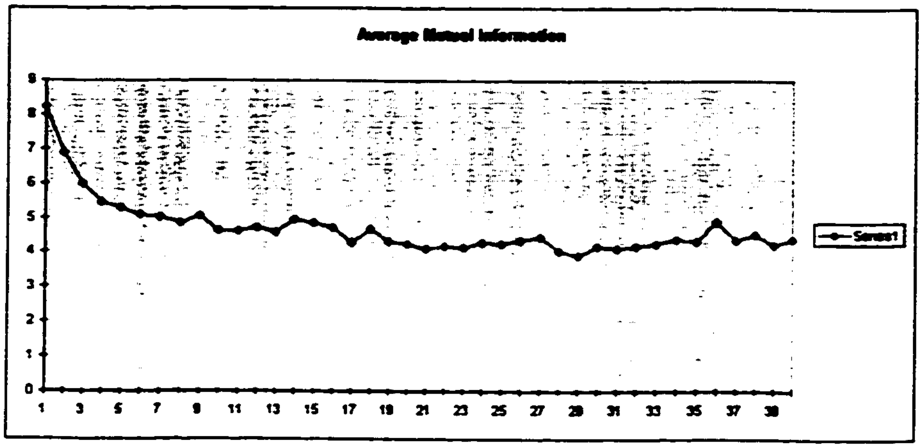


Figure 10: The first minimum occurs when $T = 8$. [Alog 1]

We are given that the time period for T should be 8. Below we map $s(n)$ versus $s(n+8)$

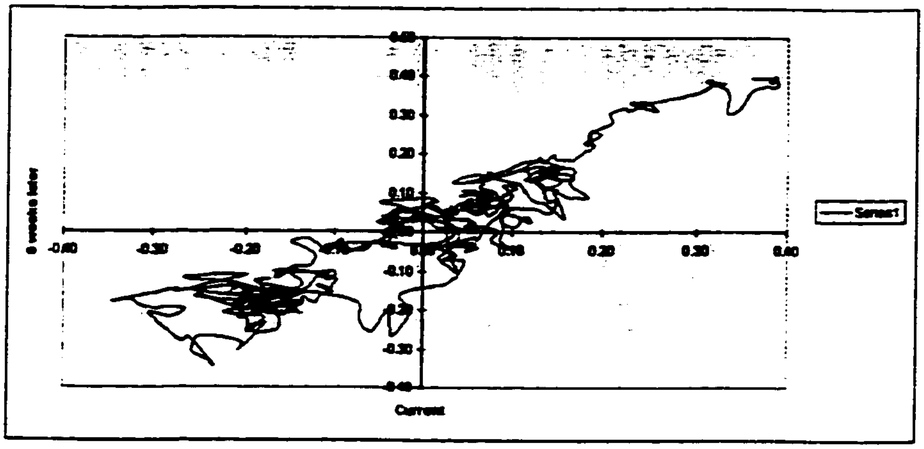


Figure 11

3.3 Dimension of the Phase Space

To be able to create a proper phase space we must have the dimension of our data. This is done using the most simple method to compute when considering immense amounts of data, the box counting method. We use this method by finding the natural density of the data. Taking into consideration that the volume occupied by a sphere of a certain radius r, in a dimension d, behaves as r^d we use the idea that we can get the dimension by looking at the density of points when we examine the distance between points in phase space. Image our data as a cloud of points. If we are to find the density of this cloud one would start with looking at each point in the data and then counting how many other points fall within a certain radius. This idea is expressed by the correlation function

$$C(q,r) = \frac{1}{M}\sum_{K=1}^{M}\left[\frac{1}{K}\sum_{K=1}^{K}\theta(r-|y(n)-y(k)|)\right]^{(q-1)}.$$

For simplicity sake we use the correlation function when $q = 2$ and $\theta(u)$ being the Heaviside function

$$\theta(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u < 0 \end{cases}.$$

Given that the radius r tends towards zero, we have

$$C(q,r) \approx r^{(q-1)D_q}$$

where D_q defines the fractal dimension when it exists. Therefore

$$D_q = \lim_{r\,small} \frac{\log[C(q,r)]}{(q-1)\log[r]}.$$

Since we are dealing with a finite number of points and we also can not literary take r towards zero. Therefore we concentrate on the slope of $\log[C(q,r)]$ versus $\log[r]$ to give us the limit of r.

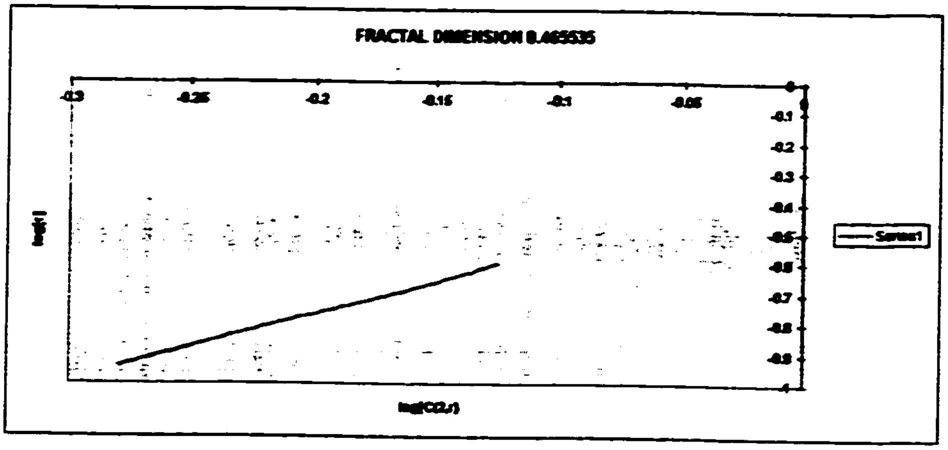


Figure 12: A mapping of $\log[C(q,r)]$ versus $\log[r]$ for the 65 values of r where $y(n) = [s(n), s(n+T), s(n+2T), s(n+3T)]$. Estimating the slope gives us that the fractal dimension is .466.

In the following figure we found that the fractal dimension linearizes around .95

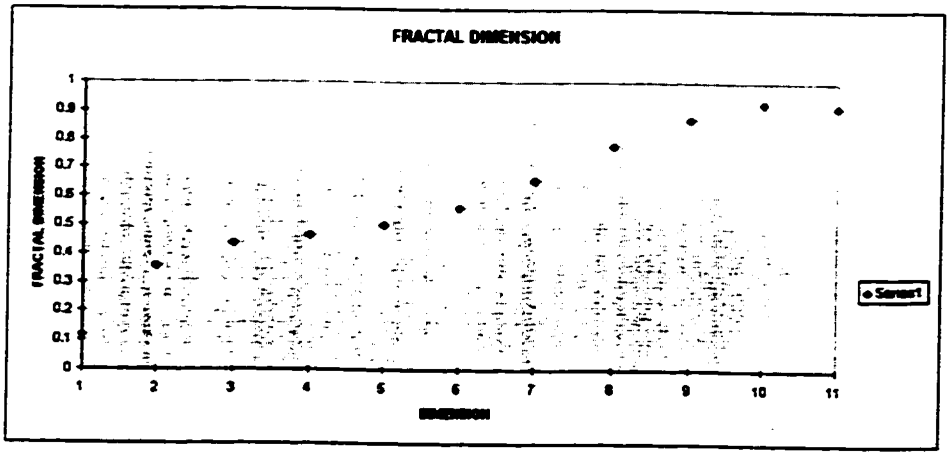


Figure 13: Fractal dimensions are found for each dimension of $y(n) = [s(n), s(n+T), s(n+2T), ...]$. We find that at dimension 10 the fractal dimension starts to linearize. Therefore we let are phase portrait be an array of dimension 10. [Alog 2]

3.4 Lyapunov Exponents of the NYSE

Given that we are dealing with a finite number of points we cannot take $n \to \infty$ as Definition 2.6.3 suggests. Therefore we must come up with a new method. If U is the unit sphere in $\mathbb{R}^m$ and A is an $m \times m$ matrix, then the orthogonal axes of the ellipsoid AU can be computed in a straightforward way.

Theorem 3.4.1 : *Let N be the unit disk in $\mathbb{R}^m$, and let A be an $m \times m$ matrix. Let $s_1^2, \ldots, s_m^2$ and $v_1, \ldots, v_m$ be the eigenvalues and unit eigenvectors, respectively, of the $m \times m$ matrix AA^T. Then*

1) *$v_1, \ldots, v_m$ are mutually unit vectors; and*
2) *the axes of the ellipse AN are $s_i v_i$ for $1 \leq i \leq m$.*

This allows us to take the square roots of the eigenvalues of the matrix AA^T to be the lengths of the axes. Combining this with Definition 2.6.3. we find the Lyapunov exponents of the transformation matrix A, in section 3.5, in 10 different places. An average of the Lyapunov exponents gives

$$\begin{aligned}
\lambda_1 &= 2.651 \\
\lambda_2 &= 2.2846 \\
\lambda_3 &= 1.649 \\
\lambda_4 &= 1.2835 \\
\lambda_5 &= 1.2053 \times 10^{-2} \\
\lambda_6 &= .25378 \\
\lambda_7 &= .76599 \\
\lambda_8 &= .94052 \\
\lambda_9 &= .91746 \\
\lambda_{10} &= .35882
\end{aligned}$$

Given that we have some $\lambda_i > 1$ we have that the NYSE is chaotic in its nature.

3.5 Estimating and Predicting the Time Series

Given that $T = 8$ and the dimension 10 we can create the following phase portrait

$$y[n] = [s(n), s(n+8), s(n+16), \ldots, s(n+72)]$$

Since our function is chaotic in nature we know that all orbits will follow some form of pattern as they move about the attractor. If we would like to know about a persons habits, we could look at his friends. Similarly, we will look at the closest "friends" of a point in the phase space to see where the point is going. What we mean by "friends" are the nearest neighbors. Take a point in the phase space and look at the k nearest points. Denote the k^{th} nearest point to $y(n)$ by $y^k(n)$ and the next point in the trajectory by $y(k, n+1)$. Note that $y(k, n+1)$ is not necessarily equal to $y^k(n+1)$. Let $z^k(n)$ be the distance between $y(n)$ and $y^k(n)$ and $z^k(k, n+1)$ the distance between $y(n+1)$ and $y(k, n+1)$. Then, the change from $z^k(n)$ to $z^k(k, n+1)$could be given by a matrix A.

$$z(k, n+1) = Jz^k(n)$$

For example if we have the dimension of $y(n)$ equal to 3 we then can write that

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial s(n)} & \frac{\partial F_1}{\partial s(n+T)} & \frac{\partial F_1}{\partial s(n+2T)} \\ \frac{\partial F_2}{\partial s(n)} & \frac{\partial F_2}{\partial s(n+T)} & \frac{\partial F_2}{\partial s(n+2T)} \\ \frac{\partial F_3}{\partial s(n)} & \frac{\partial F_3}{\partial s(n+T)} & \frac{\partial F_3}{\partial s(n+2T)} \end{bmatrix}.$$

Where F is the transformation such that $y(n+1) = F(y(n))$ for all n. Let $z_1(k, n+1)$ be the first coordinate of $z(k, n+1)$ then

$$z_1(k, n+1) = \frac{\partial F_1}{\partial s(n)} z_1^k(n) + \frac{\partial F_1}{\partial s(n)} z_2^k(n) + \frac{\partial F_1}{\partial s(n)} z_3^k(n)$$

and we can rewrite the equation as

$$\begin{bmatrix} z_1(1,n+1) \\ z_1(2,n+1) \\ z_1(3,n+1) \end{bmatrix} = \begin{bmatrix} z_1^1(n) & z_2^1(n) & z_3^1(n) \\ z_1^2(n) & z_2^2(n) & z_3^2(n) \\ z_1^3(n) & z_2^3(n) & z_3^3(n) \end{bmatrix} \begin{bmatrix} \frac{\partial F_1}{\partial s(n)} \\ \frac{\partial F_1}{\partial s(n+T)} \\ \frac{\partial F_1}{\partial s(n+2T)} \end{bmatrix}$$

where $z_r^k(n)$ is the k^{th} coordinate of the r^{th} closest neighbor. Similarly for the coordinates 2 and 3. We can write this system of equations as $A = BC$ and the entries of the matrix B are provided from the original time series

$$z_r^k = \{s(n_k + (r-1)T) - s(n + (r-1)T)\},$$

n_k is the n value associated with the kth neighbor to $y(n)$, and $a = 1, 2, ..., (d_E - 1)$ where d_E is the embedding dimension. Similarly, the entries of the matrix A are given by the equation

$$z_r(k, n+1) = \{s(n_k + 1 + (r-1)T) - s(n + 1 + (r-1)T)\}.$$

Therefore what is left is to invert B to obtain $\frac{\partial F_1}{\partial s(n)}, \frac{\partial F_1}{\partial s(n+T)}$, and $\frac{\partial F_1}{\partial s(n+2T)}$.

Using this idea we can try to predict the future outcomes of the NYSE by using A as a projection of future outcomes. Since B may not always be invertible we will use the Linear Least Square Solution by Householder Transformations.

3.6 Linear Least Square Solution

In Linear Least Square Solution by Householder Transformations we are given three matrices such as

$$A_{m\times 1} = B_{m\times n} C_{n\times 1}$$

and asked to find the matrix C when A and B are known. To do this we look to minimize C by using the norm $\|A - BC\|$. This is done by creating a Q such that

$$QB = R = \begin{bmatrix} \tilde{R} \\ \cdots \\ 0 \end{bmatrix}_{(m-n)\times n}$$

where we can write

$$\|A - BC\| = \|c - QBC\|$$

where $c = QA$, $Q^T Q = I$ and $\tilde{R}$ is an upper triangle matrix. In short the Householder transformation lets $B = B^{(1)}$ and $B^{(2)}, B^{(3)}, \ldots, B^{(n+1)}$ be defined as follows

$$B^{(k+1)} = P^{(k)} A^{(k)} \qquad \text{where } k = 1, 2, \ldots, n.$$

$P^{(k)}$ is a symmetric, orthogonal matrix of the form

$$P^{(k+1)} = I - \beta_k \mu^{(k)} \mu^{(k)^T}$$

where the elements of $P^{(k)}$ are derived so that $a_{i,k}^{(k+1)} = 0$ for $i = k + 1, \ldots, m$. Therefore, $P^{(k)}$ is generated as follows

$$\sigma_k = \left(\sum_{i=k}^{m} \left(a_{i,k}^{(k+1)} \right)^2 \right)^{\frac{1}{2}}$$

$$\beta_k = \left[\sigma_k\left(\sigma_k + \left|a_{i,k}^{(k+1)}\right|\right)\right]^{-1}$$

$$\mu_k^{(k)} = sgn\left(a_{i,k}^{(k+1)}\right)\left(\sigma_k + \left|a_{i,k}^{(k+1)}\right|\right)$$

$$\mu_k^{(k)} = a_{i,k}^{(k+1)} \text{ for } i > k.$$

A note worthy piece of information is that we must take enough nearest neighbors to make B a non-singular matrix.

3.7 The Prediction

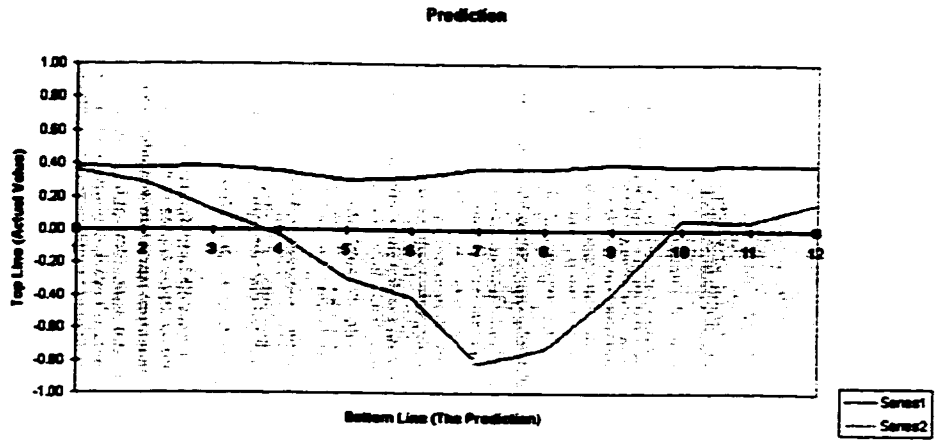


figure 14: Prediction over the a 12 week period
The top line represents the NYSE
The bottom line represents our prediction. [Alog 3]

From the above figure we can see that both lines move in convex manner. The bottom line moving in an emphasized manner of the actual value. The first and second week predictions are fairly accurate with the bottom line falling away for the weeks to follow. I believe that the prediction that is made using the Embedding Theorem and Householders method well give accurate answers for the first two weeks after this the prediction will starts to loss accuracy but will still hold the movements of the market to rise or fall. To prove this one must have one thing. Money to burn.

Algorithms in C++

Alog 1

Average mutual information with joint probability of s[n] and s[n+1]

```
#include <iostream.h>
#include <conio.h>
#include <math.h>

void main( ) {double answer, count = 0, turns = 0, P[3000], SUMIT[3000];
int TIME;

SUMIT[0]=1000;
// Probability for S[n] ALGOR 1//
for(int i = 1; i <= 412; i++)
{        P[i] = 0;
         for(int j = 1; j <= 412; j++)
         {        if(s[j] == s[i])
                           P[i] = P[i]+1;}
                  P[i]=P[i]/412;}

// Average mutual Information ALGOR 2//
for(int T = 1; T <= 40; T++)
{        SUMIT[T]=0;
         for(int n = 1; n <= 411; n++)
         {        if(n + T > 412) break;
                  for(int j = 1; j <= 412; j++)
                  {        if(j + T > 412) break;
                           if(s[n]==s[j] && s[n + T] == s[j + T])
                                    count=count+1;
                                    turns=turns+1;}
                           answer = count/turns;
```

```
                    count=0;turns=0;
                    double IT = answer * log10(answer/(P[n]*P[n+T]))/log10(2);
                SUMIT[T] = SUMIT[T] + IT;}
        cout<<"\n"<<SUMIT[T];
        if(SUMIT[T-1] < SUMIT[T]){ TIME = T-1;cout<<"\nTIME "<<TIME; getch();}}
return;}
```

Alog 2

Finding the Fractal Dimension

```
#include <iostream.h>
#include <conio.h>
#include <math.h>
void main( ){
int PHI;
double CQR = 0, CQRTOTAL = 0, CQTOTAL[71], R[71];
double DENUM;
double SUM1, SUM2 = 0;

for (int ADD = 0; ADD < 11; ADD++)
        {DENUM = 412 - ADD * 8;
        double r = -.60206, radius = .25, NUM1 = 0, NUM2 = 0, NUM3 = 0, NUM4 = 0;
        for( int COUNT = 1; COUNT <= 69; COUNT++)
        {       for( int COUNT1 = 1; COUNT1 <= DENUM; COUNT1++)
                {for( int COUNT2 = 1; COUNT2 <= DENUM; COUNT2++)
                {for(int COUNT3 = 0; COUNT3 <= ADD; COUNT3++)
                {SUM1 = s[COUNT2 + 8*COUNT3] - s[COUNT1 + 8*COUNT3];
                                SUM2 = SUM2 + SUM1*SUM1;}
                                if(radius - SUM2 >= 0 )
                                PHI = 1;
                        else
                                PHI = 0;
```

```
                    CQR = CQR + PHI;
                    SUM2 = 0;}
                CQRTOTAL = CQRTOTAL + CQR;
                CQR = 0;}

            CQRTOTAL = CQRTOTAL/(DENUM*DENUM);
            CQTOTAL[COUNT] = log10(CQRTOTAL); R[COUNT] = r;
            r = r - .005;
            radius = pow(10,r);
            CQRTOTAL = 0;}

        for(int COUNT = 1; COUNT <= 69; COUNT++)
        {       NUM1 = NUM1 + R[COUNT]*CQTOTAL[COUNT];
                NUM2 = NUM2 + R[COUNT];
                NUM3 = NUM3 + CQTOTAL[COUNT];
                NUM4 = NUM4 + R[COUNT]*R[COUNT];}

        double A = (69*NUM1 - NUM2*NUM3);
        double B = (69*NUM4 - NUM2*NUM2);
        cout<<"\n FRACTAL DIMENSION FOR "<< (ADD+1) <<" "<< A/B;}
        getch();
return;}
```

Alog 3

Linear Least Square Solution by Householder Transformations.

```
#include <iostream.h>
#include <conio.h>
#include <math.h>
```

```
double INNER_PRODUCT(int START, int ROW, double A[][16], double B[][16], int
COLUMN1, int COLUMN2)
{       double SUM= 0;

        for(int i = START; i <= ROW; i++)
                SUM = SUM + A[i][COLUMN1]*B[i][COLUMN2];

        return SUM; }

double INNER_PRODUCT1(int START, int ROW, double A[][16], double B[], int
COLUMN1)
{       double SUM= 0;

        for(int i = START; i <= ROW; i++)
                SUM = SUM + A[i][COLUMN1]*B[i];

        return SUM; }

double INNER_PRODUCT2(int START, int ROW, double A[][16], double B[], int
COLUMN1)
{       double SUM= 0;

        for(int i = START; i <= ROW; i++)
                SUM = SUM + A[COLUMN1][i]*B[i];

        return SUM; }

void main()
{//Nearest Neighbors
int VALUEN,NN[52], M = 20, DIM = 10, TIME = 8 ,POINT = 328, NUMBER = 400;
double SUM1, SUM2, TEMP, DISTANCE[52];

for(int z=1; z<=12; z++)
```

```
{       for(int COUNT = 1; COUNT <= M; COUNT++)
                DISTANCE[COUNT] = 10;

        for(int N = 1; N <= NUMBER; N++)
        {       if(N + (DIM-1)*TIME > NUMBER ) break;

        SUM2 = 0;
        {       for(int COUNT=0; COUNT < DIM; COUNT++)
                {       SUM1 = s[POINT + TIME*COUNT] - s[N + TIME*COUNT];
                        SUM2 = SUM2 + fabs(SUM1);}

                for(int COUNT = 1; COUNT <= M; COUNT++)
                {       if(SUM2 < DISTANCE[COUNT])
                        {       TEMP = DISTANCE[COUNT];
                                VALUEN = COUNT;
                for(int GREATEST = 1; GREATEST <= M; GREATEST++)
                {       if(TEMP < DISTANCE[GREATEST])
                        {       TEMP = DISTANCE[GREATEST];
                                VALUEN = GREATEST;}}
                                NN[VALUEN] = N;
                                DISTANCE[VALUEN] = SUM2;

                for(int i = 1; i <= M; i++)
                {       for(int j = 1; j < M; j++)
                        {       if(DISTANCE[j]>DISTANCE[i])
                                {       TEMP = DISTANCE[i];
                                        VALUEN = NN[i];
                                        DISTANCE[i]=DISTANCE[j];
                                        NN[i]=NN[j];
                                        DISTANCE[j]=TEMP;
                                        NN[j]=VALUEN;}}}break;}}}}

double QR[45][16];
```

```
double A[45][16];

for(int i = 1; i <= M-1; i++)
{        for(int COOR = 1; COOR <= DIM; COOR++)
                  A[i][COOR] = s[POINT + TIME*(COOR - 1)] - s[NN[i+1] + TIME*(COOR - 1)];}

         for(int i = 1; i <= M-1; i++)
         {        for(int COOR = 0; COOR < DIM; COOR++)
                  QR[i][COOR + 1] = A[i][COOR + 1];}

int JBAR,SUBI,PIVOT[20];
double
BETA,SIGMA,ALPHAK,ALPHA[20],QRKK,Y[20],Z[20],SUM[20],R[20],GAMMA;

//Finding Y
for(int j = 1; j <= DIM; j++)
{        SUM[j] = INNER_PRODUCT(1,M-1,QR,QR,j,j);
         PIVOT[j] = j;}

for(int k = 1; k <= DIM; k++)
{        SIGMA = SUM[k]; JBAR = k;
         for(int j = k + 1; j <= DIM; j++)
         {        if(SIGMA < SUM[j])
                  {        SIGMA = SUM[j];JBAR = j;}}

         if(JBAR != k)
         {        SUBI=PIVOT[k]; PIVOT[k]=PIVOT[JBAR]; PIVOT[JBAR]=SUBI;
                  SUM[JBAR]=SUM[k];SUM[k]=SIGMA;
                  for( int i = 1; i <= M-1; i++)
                  {        SIGMA=QR[i][k]; QR[i][k]=QR[i][JBAR];
                           QR[i][JBAR]=SIGMA;}}

         SIGMA = INNER_PRODUCT(k,M-1,QR,QR,k,k);
```

```
if(SIGMA < 1e-20 && SIGMA > -1e-20)
{cout<<"\n SINGULAR "<<k; break;}
QRKK = QR[k][k];

if(QRKK < 0 )
        ALPHA[k] = sqrt(SIGMA);
else
        ALPHA[k] = -sqrt(SIGMA);

ALPHAK =ALPHA[k];

BETA = 1/(SIGMA - QRKK*ALPHAK);

QR[k][k]=QRKK-ALPHAK;

for(int j = k + 1; j <= DIM; j++)
        Y[j]=BETA*INNER_PRODUCT(k,M-1,QR,QR,k,j);

for(int j = k + 1; j <= DIM; j++)
{       for(int i = k; i <= M-1; i++)
                QR[i][j] = QR[i][j] - QR[i][k]*Y[j];
                SUM[j] = SUM[j] - QR[k][j]*QR[k][j];}}

//Find R//
        for(int i = 1; i <= M-1; i++)
                R[i] = s[POINT + 72] - s[NN[i + 1] + 73];

        for(int j = 1; j <= DIM; j++)
        {       GAMMA = INNER_PRODUCT1(j, M-1, QR, R, j)/(ALPHA[j] * QR[j][j]);
        for(int i = j; i <= M-1; i++)
                R[i] = R[i] + GAMMA*QR[i][j];}
```

```
Z[DIM] = R[DIM]/ALPHA[DIM];

for(int i = DIM - 1; i >= 1; i--)
        Z[i] = -(INNER_PRODUCT2(i + 1, DIM, QR, Z, i) - R[i]) /
ALPHA[i];

for(int i = 1; i <= DIM; i++)
        Y[PIVOT[i]] = Z[i];

NUMBER = NUMBER + 1;
POINT = POINT + 1;

s[NUMBER] = (Y[1] * s[NN[2]] + Y[2] * s[NN[2] + 8] + Y[3] * s[NN[2] +
16]
            + Y[4] * s[NN[2] + 24] + Y[5] * s[NN[2] + 32]
            + Y[6] * s[NN[2] + 40]+ Y[7] * s[NN[2] + 48]
            + Y[8] * s[NN[2] + 56] + Y[9] * s[NN[2] + 64]
            + Y[10] * s[NN[2]+ 72] );

cout<<"\n "<< s[NUMBER];
getch();}
getch();
return:}
```

Data s[n]

DATE	NYSE	CPI		LOG(NYSE)	LOG(CPI)		s_i = log N_i-(a logCPI_i) +c)
650108	45.95	31.22		1.662285516	1.494432899		0.12
650205	46.98	31.25		1.671913012	1.494850022		0.13
650305	46.78	31.25		1.670060217	1.494850022		0.13
650402	46.66	31.28		1.668944734	1.495266744		0.13
650507	48.3	31.37		1.683947131	1.496514519		0.14
650604	46.74	31.45		1.669688708	1.49762065		0.13
650702	45.53	31.6		1.65829765	1.499687083		0.11
650806	46.13	31.63		1.663983455	1.500099192		0.12
650903	47.26	31.57		1.674493717	1.499274582		0.13
651001	48.21	31.63		1.683137131	1.500099192		0.14
651105	49.57	31.68		1.695218919	1.500785173		0.15
651203	49.32	31.74		1.693023068	1.501606922		0.15
660107	50.37	31.86		1.702171951	1.503245771		0.15
660204	50.64	31.86		1.704493697	1.503245771		0.16
660304	48.55	32.03		1.686189234	1.505556939		0.14
660401	48.76	32.14		1.688063697	1.507045872		0.14
660506	47.49	32.29		1.67660217	1.509068045		0.12
660603	46.54	32.32		1.667826379	1.509471352		0.11
660701	46.36	32.4		1.666143427	1.51054501		0.11
660805	45.5	32.52		1.658011397	1.512150537		0.10
660902	41.81	32.66		1.621280168	1.51401618		0.06
661007	39.37	32.75		1.595165415	1.515211304		0.03
661104	43.53	32.86		1.638788667	1.516667559		0.08
661202	43.44	32.89		1.637889817	1.517063873		0.07
670106	44.71	32.92		1.65040467	1.517459827		0.09
670203	47.73	32.92		1.678791434	1.517459827		0.11
670303	48.37	32.95		1.684576087	1.517855419		0.12
670407	49.1	33		1.691081492	1.51851394		0.13
670505	51.85	33.09		1.714748761	1.519696767		0.15
670602	49.56	33.18		1.695131298	1.520876382		0.13
670707	50.91	33.29		1.706803097	1.522313795		0.14
670804	53.22	33.43		1.72607487	1.524136377		0.15
670901	52.15	33.55		1.717254313	1.525692525		0.14
671006	54.05	33.61		1.732795698	1.526468512		0.16
671103	50.96	33.72		1.707229419	1.527887566		0.13
671201	52.52	33.81		1.720324717	1.529045171		0.14
680105	53.57	33.92		1.728921646	1.530455844		0.15
680202	51.67	34.04		1.713238462	1.531989551		0.13
680301	49.58	34.15		1.695306522	1.533390708		0.11
680405	51.79	34.3		1.714245911	1.53529412		0.13
680503	55.06	34.41		1.740836207	1.536684673		0.15
680607	57.11	34.53		1.75671216	1.538196578		0.17
680703	56.88	34.7		1.754959588	1.540329475		0.16
680802	54.18	34.87		1.733839001	1.542451947		0.14
680906	56.63	34.98		1.753046562	1.543819805		0.16

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