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Asymmetric Kernel Density Estimator for Length Biased Data

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Abstract

This article considers smooth density estimation based on length biased data that involves a random sample $X_1, \ldots, X_n$ based on a nonnegative random variable (r.v.) having a continuous probability density function (pdf) $g(x), x \in \mathbb{R}^+ = [0, \infty)$, where $g(x)$ is generated by another pdf $f(x), x \in \mathbb{R}^+$ given by $g(x) = \mu^{-1} x f(x), \quad x \in \mathbb{R}^+$, where $\mu = (E_g(X^{-1}))^{-1}$ is the harmonic mean of X with the pdf $g(.)$. Length-biased distributions (and in general weighted distributions) naturally occur in many statistical applications (see Patil and Rao[11]) where estimation of $f(x), x \in \mathbb{R}^+$ itself is of central importance.

The most commonly used estimator of the density function is the kernel estimator for the case of *i.i.d.* data that has been adapted to the length-biased setup, however, this approach may not be satisfactory as it may assign some probability to the negative region (see Chaubey, Sen and Li[5]). In order to avoid these problems, it would be better to adapt smooth density estimation technique for non-negative random variables. In the present article we investigate the adaptation of asymmetric kernel estimator proposed and studied in Chaubey, Sen and Sen[4] through smoothing of the usual empirical distribution function and the Cox's estimator. Our simulation study demonstrates that the asymmetric kernel estimators proposed here are good competitors to other estimators.

1 Introduction

In many applications the recorded observation may be assumed to have the probability density function $g(x)$, that is of the form

$$g(x) = \mu_w^{-1} w(x) f(x), \quad x \in \mathbb{R}^+, \tag{1.1}$$

where $f(x)$ is the original density, $w(x)$ is a non-negative known function called the weighting function,

$$\mu_w = 1/E_g(1/w(X)) \tag{1.2}$$

with $X \sim g(.)$. Patil and Rao (1978) cite several examples including those generated by PPS (probability proportional to size) sampling scheme (that is common in sample surveys), damage models and sub-sampling [see also Rao (1965), Patil and Ord (1975), Patil and Rao (1977) and Rao (1977)]. Here, we concentrate on the case when $w(x) = x$, a situation known as giving length-biased data and where typically the observations are non-negative. Thus we consider a random sample $\{X_1, \ldots, X_n\}$ be n nonnegative independent and identically distributed (i.i.d.) random variables (r.v.) having a continuous probability density function (pdf) $g(x), x \in \mathbb{R}^+ = [0, \infty)$ given by

$$g(x) = \mu^{-1} x f(x), \quad x \in \mathbb{R}^+ \tag{1.3}$$

where $\mu = (E_g(X^{-1}))^{-1}$ is the harmonic mean of X with the pdf $g(.)$, and estimation of $f(x), x \in \mathbb{R}^+$ itself is of central importance. Here it is tacitly assumed that $(0 <) < \mu < \infty$. Note that $\mu \leq E_g(X) = \mu^{-1} \int_0^\infty x^2 f(x)dx$, or equivalently, $\mu^2 \leq E_f(Y^2)$ where Y has the pdf f. Thus, assuming that Y has a finite 2$^{\text{nd}}$ moment insures that $\mu < \infty$ (even finite first moment of Y does so).

Bhattacharyya *et al.*(1988) studied the kernel density estimator for $f(x)$ obtained by using the corresponding estimator for $g(x)$ and the relation (1.3), replacing the unknown value μ by its harmonic mean estimator as proposed by Cox[7]. Cox[7] also gave a direct estimator of $F(x)$ that has been used in proposing an alternative density estimator by Jones (1991). In some aspects, Jones estimator performs much better than Bhattacharyya *et al.* estimator [see also Wu and Mao (1996)]. However, in general the kernels used are symmetric around zero the resulting density estimators may put a positive mass out side the support of $f(.)$ *i.e.* $\mathbb{R}^+ = [0, \infty)$.. This may also produce a large bias in the estimators for x near zero. This problem has long been recognized in density estimation in the context of *i.i.d.* data [see Silverman (1986)], however, it becomes more pertinent for the length-biased data, as the observations are necessarily non-negative.

In order to overcome this defect, many new methods estimating underlying density for non-negative random variables have been proposed particularly in recent years. Bagai and Prakasa Rao (1995) proposed replacing the symmetric kernel k by a pdf k^* with non-negative support. This certainly avoids the problem of positive mass in the negative region; however, only the first r order statistics are used for estimating $f(x)$, where $X_{(r)} < x \leq X_{(r+1)}$, $X_{(i)}$ denoting the i^{th} order statistic that is an undesirable feature. Chaubey and Sen (1996) proposed a density estimator as the derivative of a smooth version of the edf by adapting the so called Hille's (1948) smoothing lemma, which, in contrast to the proposal of Bagai and Prakasa Rao (1996), uses the whole data. This has been adapted to the length-biased set-up by Chaubey, Sen and Li[5].

An interesting class of estimators, based on *asymmetric kernels*, was proposed by

Chen [6] in the *i.i.d.* seup, using Gamma kernels. These are of the form

$$\hat{g}(x) = n^{-1} \sum_{i=1}^{n} K_{\rho_b(x),b}(X_i) \tag{1.4}$$

where

$$K_{\rho_b(x),b}(t) = \frac{t^{\rho_b(x)-1} e^{-t/b}}{b^{\rho_b(x)} \Gamma(\rho_b(x))}. \tag{1.5}$$

In his paper, Chen[6] gave two choices for $\rho_b(x)$. One is

$$\rho_b(x) = x/b + 1 \tag{1.6}$$

which leads to density estimator $\hat{f}_1(x)$; the other is

$$\rho_b(x) = \begin{cases} x/b & \text{if } x \geq 2b; \\ \frac{1}{4}(x/b)^2 + 1 & \text{if } x \in [0, 2b). \end{cases} \tag{1.7}$$

which leads to density estimator $\hat{f}_2(x)$. This paper also showed demonstrated through simulation that the $MISE$ of $\hat{f}_2$ is lower than that of $\hat{f}_1$.

Inspired by Chen's idea and using inverse Gaussian density

$$K_{IG(m,\lambda)}(y) = \frac{\sqrt{\lambda}}{\sqrt{2\pi y^3}} \exp\left(-\frac{\lambda}{2m}\left(\frac{y}{m} - 2 + \frac{m}{y}\right)\right), y > 0 \tag{1.8}$$

and reciprocal inverse Gaussian density

$$K_{RIG(m,\lambda)}(z) = \frac{\sqrt{\lambda}}{\sqrt{2\pi z}} \exp\left(-\frac{\lambda}{2m}\left(mz - 2 + \frac{1}{mz}\right)\right), z > 0 \tag{1.9}$$

as kernels, Scaillet [15] proposed the following two density estimators

$$\hat{g}_{IG}(x) = n^{-1} \sum_{i=1}^{n} K_{IG(x,1/b)}(X_i) \tag{1.10}$$

and

$$\hat{g}_{RIG}(x) = n^{-1} \sum_{i=1}^{n} K_{RIG(1/(x-b),1/b)}(X_i). \tag{1.11}$$

The Chen estimators do not suffer from boundary bias, but the Scaillet ones are inconsistent at $x = 0$. Moreover, there seems to be a problem with computing *asymptotic mean squared-error* (AMSE), and both the authors give two different AMSE formulae, one for $x/b \to \infty$ and the other for $x/b \to \kappa > 0$, where b is the bandwidth. One does not get a clear picture of what happens at a fixed $x \geq 0$.

Another class of asymmetric kernel estimators in the *i.i.d.* context has been proposed recently by Chaubey, Sen and Sen[4] that is motivated by generalization of the estimator in Chaubey and Sen[3]. This paper adapts the above smoothing estimator to the length-biased density estimation and proposes two new asymmetric kernel estimators one based on the usual empirical distribution function and the other one based on the Cox's estimator. For a comprehensive comparison, we also apply the ideas in Chen[6] and Scaillet[15] to obtain some other density estimators for length-biased data.

Along with some preliminary notions, the two proposed smooth estimators of $f(.)$ are given in Section 2 where their asymptotic properties are studies as well. The formulation of the density estimators motivated by Chen[6] and Scaillet[15] are also given in this section. The next section, Section 3 is devoted to numerical simulation studies and some discussion and the proofs are relegated to an appendix.

2 Preliminary Notions and New Smooth Density Estimators

There are basically two strategies to be used to obtain the smooth estimator of $f(x)$. One is based on a smooth version Cox's estimator $F_n(x)$ that is given below:

$$F_n(x) = \frac{\sum_{i=1}^n \frac{1}{X_i} I\{X_i \leq x\}}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.12}$$

The other one is to first estimate $g(x)$, that is obtained as the derivative of the smooth version of the empirical distribution function

$$G_n(x) = \frac{1}{n}\sum_{i=1}^n I\{X_i \leq x\}, \tag{2.13}$$

and then making some adjustments to obtain the estimator of underlying density $f(x)$. The key ingredient for the proposal is the following generalization of the Hille's lemma, in a slight different form of Lemma 1 given in Feller (1965, §VII.1).

Lemma 2.1 *Let u be any bounded and continuous function and $\Psi_{x,n}, n = 1, 2, ...$ be a family of distributions with mean $\mu_n(x)$ and variance $v_n(x)$ such that $\mu_n(x) \to x$ and $h_n^2(x) \to 0$. Then*

$$\tilde{u}(x) = \int_{-\infty}^{\infty} u(t) d\Psi_{x,n}(t) \to u(x). \tag{2.14}$$

The convergence is uniform in every subinterval in which $h_n(x) \to 0$ uniformly and u is uniformly continuous.

The smooth estimators of $F(x)$ and $G(x)$ are motivated by substituting $F_n(.)$ and $G_n(.)$, respectively for $u(.)$ in the above lemma. Here, we will also find alternative estimators using the ideas of Chen[6] and Scaillet[15].

2.1 Density Estimator Using Asymmetric Kernels Based on Cox's Estimator

Chaubey, Sen and Sen[4] suggested the choice of $\Psi_{x,n}$ through a family $Q_{(}.)$ of distributions with mean 1 and variance v_n. Thus, substituting $Q_{v_n}(t/x)$ for $\Psi_{x,n}(t)$ and $F_n(t)$ for $u(t)$ and in Lemma 2.1, we have the following smooth estimator of $F(x)$:

$$\widetilde{F}_n^+(x) = \int_0^\infty F_n(t) dQ_{v_n}(t/x). \tag{2.15}$$

This can be simplified (as in [4] to be,

$$\widetilde{F}_n^+(x) = 1 - \frac{\sum_{i=1}^n \frac{1}{X_i} Q_{v_n}(\frac{X_i}{x})}{\sum_{i=1}^n \frac{1}{X_i}}, \tag{2.16}$$

Thus a smooth estimator of f obtained by differentiating the above function is given by

$$\tilde{f}_n(x) = \frac{\frac{1}{x^2}\sum_{i=1}^n q_{v_n}(\frac{X_i}{x})}{\sum_{i=1}^n \frac{1}{X_i}} \tag{2.17}$$

where $q_{v_n}(t) = \frac{d}{dt} Q_{v_n}(t)$.

However, (2.17) may not be defined at $x = 0$, except in cases where $\lim_{x\to 0} \tilde{f}_n(x)$ exists. For example taking the $Q_v(.)$ to be the distribution function a Gamma distribution with appropriate parameters, this limit is zero, that is acceptable only if we are estimating $f(x)$ with $f(0) = 0$. This situation also occurs in estimating density with direct data [see Chaubey, Sen and Sen (2007)]. In their paper, they considered a perturbed version of the density estimator, replacing $Q_{v_n}(./x)$ by $Q_{v_n}(./(x+\varepsilon))$, $\varepsilon_n \downarrow 0$ as $n \to \infty$. This is equivalent to choosing $\Psi_{x,n}$ such that the corresponding mean is $x + \varepsilon_n \to x$ and the variance is $(x+\varepsilon_n)^2 v_n^2 \to 0$. Motivated by this idea, the perturbed version of (2.17) is given by

$$\tilde{f}_n^+(x) = \frac{\frac{1}{(x+\varepsilon_n)^2}\sum_{i=1}^n q_{v_n}(\frac{X_i}{x+\varepsilon_n})}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.18}$$

The following theorems provide the asymptotic properties of $\tilde{f}_n^+(x)$ parallel to the *i.i.d.* case studied in [4]. The strong consistency of $\tilde{f}_n^+(x)$ is given by the following theorem.

Thoerem 2.1 *If*

A. $f(\cdot)$ *is Lipschitz continuous on* $[0,\infty)$ *and* $E(X_1^{-1}) < \infty$;

B. $\sup_{x\geq 0}\int_0^\infty |\frac{d}{dx}[\frac{1}{x+\varepsilon_n} q_{v_n}(\frac{t}{x+\varepsilon_n})]| dt = o((\frac{\log\log n}{n^{1/2}})^{-1})$;

C. $\sup_{u>0, v>0} u q_v(u) < \infty$;

D. $v_n \to 0$, $\varepsilon_n \to 0$ *as* $n \to \infty$;

then we have

$$\sup_{x\geq 0} |\tilde{f}_n^+(x) - f(x)| \xrightarrow{a.s.} 0$$

as $n \to \infty$.

Regarding the asymptotic normality of $\tilde{f}_n^+(x)$, we have the following theorem.

Thoerem 2.2 *If*

E. $f(x)$ *is Lipschtiz continuous on* $[0,\infty)$ *and* $E(X_1^{-2})<\infty$*;*

F. $I_2(q) \triangleq \lim\limits_{v_n\to 0} v_n \int_0^\infty (q_{v_n}(t))^2 dt$ *exists;*

G1. for $1\le m\le 3$, $\int_0^\infty (q_{v_n}(t))^m dt = O(v^{1-m})$ *as* $v\to 0$*;*

G2. with $q^*_{m,v_n}(t) = \frac{(q_{v_n}(t))^m}{\int_0^\infty (q_{v_n}(w))^m dw}$, $1\le m\le 3$, *and as* $v_n\to 0$,

$$
\begin{aligned}
&(i) \qquad \mu_{m,v_n} = \int_0^\infty t q^*_{m,v_n}(t)dt = 1+O(v_n),\\
&(ii) \qquad \sigma^2_{m,v_n} = \int_0^\infty (t-\mu_{m,v_n})^2 q^*_{m,v_n}(t)dt = O(v_n^2)\\
&(iii) \qquad \sup_{0<v_n<\varepsilon} \int_0^\infty t^{4+\delta} q^*_{m,v_n}(t)dt < \infty, \textit{for some } \delta>0,\ \varepsilon>0;
\end{aligned}
$$

Then

(a) If $nv_n\to\infty$, $nv_n^3\to 0$, $nv_n\varepsilon_n^2\to 0$ *as* $n\to\infty$*, we have*

$$\sqrt{nv_n}(f_n^+(x)-f(x)) \to N\big(0, I_2(q)\frac{\mu f(x)}{x^2}\big), \textit{for } x>0.$$

(b) If $nv_n\varepsilon_n^2\to\infty$ *and* $nv_n\varepsilon_n^4\to 0$ *as* $n\to\infty$*, we have*

$$\sqrt{nv_n\varepsilon_n^2}(f_n^+(0)-f(0)) \to N\big(0, I_2(q)f(0)\big).$$

The proof of the two theorems are deferred to appendix.

Remark 1: It has been found appropriate in [4] to consider $q_{v_n}(x)$ being generated from a Gamma distribution, thus

$$q_{v_n}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

where $\alpha = \frac{1}{v_n^2}$ and $\beta = 1/v_n^2$. This choice will be used in the simulation studies.

Remark 2: We can show that

$$Bias[\tilde{f}_n^+(x)] = (xv_n^2+\varepsilon_n)f'(x) + \frac{x^2}{2}f''(x)v_n^2 + o(v_n^2+\varepsilon_n). \tag{2.19}$$

By the proof of Theorem 2.2, it is easy to show that

$$Var[\tilde{f}_n^+(x)] = \frac{I_2(q)\mu f(x)}{nv_n(x+\varepsilon_n)^2} + o((nv_n)^{-1}). \tag{2.20}$$

By (2.19) and (2.20), we have

$$MSE[\tilde{f}_n^+(x)] = [(xv_n^2+\varepsilon_n)f'(x)+\frac{x^2}{2}f''(x)v_n^2]^2+\frac{I_2(q)\mu}{nv_n}\frac{f(x)}{(x+\varepsilon_n)^2}.$$

So

$$AMISE[\tilde{f}_n^+] = \int_0^\infty [(xv_n^2+\varepsilon_n)f'(x)+\frac{x^2}{2}f''(x)v_n^2]^2dx+\frac{I_2(q)\mu}{nv_n}\int_0^\infty \frac{f(x)}{(x+\varepsilon_n)^2}dx. \tag{2.21}$$

Remark 3: Note that if we integrate (2.18) from 0 to ∞, we will obtain

$$\int_0^\infty \tilde{f}_n^+(x)dx = \frac{\sum_{i=1}^n \frac{Q_{v_n}(X_i/\varepsilon_n)}{X_i}}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.22}$$

If $\varepsilon_n \neq 0$, (2.22) is not equal to 1. In this case, f_n^+ is not a real density estimator which is integrated to unity. In order to overcome this defect, we divide $f_n^+(x)$ by $\int_0^\infty f_n^+(x)dx$, which leads to a corrected estimator

$$\tilde{f}_n^*(x) = \frac{\frac{1}{(x+\varepsilon_n)^2}\sum_{i=1}^n q_{v_n}(\frac{X_i}{x+\varepsilon_n})}{\sum_{i=1}^n \frac{Q_{v_n}(X_i/\varepsilon_n)}{X_i}}. \tag{2.23}$$

Since $\sum_{i=1}^n \frac{Q_{v_n}(X_i/\varepsilon)}{X_i} \to \sum_{i=1}^n \frac{1}{X_i}$ for a given sample, as $\varepsilon_n \to 0$, most of the asymptotic properties of $\tilde{f}_n^+$ still hold for $\tilde{f}_n^*$. We can establish the same theorems as Theorem 2.1 and 2.2 for $\tilde{f}_n^*$. But the biases of the two estimator are slightly different. Note that$\tilde{f}_n^*(x) = \frac{\tilde{f}_n^+}{1-\tilde{F}_n^+(\varepsilon_n)} \approx \frac{\tilde{f}_n^+}{1-F(\varepsilon_n)}$, then it is easy to show that

$$Bias(\tilde{f}_n^*(x)) = Bias(\tilde{f}_n^+(x))+\varepsilon_n f(0)f(x)+o(\varepsilon), \tag{2.24}$$

which will be used to find out the $AMISE$ of $\tilde{f}_n^*(x)$.

2.2 Smooth Density Estimators Based on Empirical Distribution Function

An alternative estimator of $f(x)$ is obtained by using Eq. (1.3) using a smooth estimator
$hatg(x)$ of $g(x)$ given by

$$\hat{f}(x) = \frac{\hat{g}(x)/x}{\int_{\mathbb{R}^+}(\hat{g}(x)/x)dx}. \tag{2.25}$$

Note that for the above estimator to be meaningful, $\hat{g}(x)$ must satisfy the following conditions:

(i) $\hat{g}(x) = 0$ for $x \leq 0$;
(ii) $\hat{g}(x)/x$ is integrable on $[0, \infty)$.

Using the smooth density estimator given in [4] with $\varepsilon_n = 0$, the above conditions may be satisfied. Hence we consider in place of $\hat{g}(x)$, the following estimator,

$$g_n(x) = \frac{1}{nx^2} \sum_{i=1}^{n} X_i q_{v_n}\left(\frac{X_i}{x}\right). \tag{2.26}$$

Note that $\int_0^\infty \frac{g_n(x)}{x} dx = \frac{1}{n} \sum_{i=1}^n \frac{1}{X_i}$, which is the Cox estimator of $1/\mu$ and using Eq. (2.25), a smooth estimator of $f(x)$ can be considered as

$$f_n(x) = \frac{\frac{1}{x^3} \sum_{i=1}^n X_i q_{v_n}\left(\frac{X_i}{x}\right)}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.27}$$

Note that the above estimator may be reasonable for a density $f(x)$ with $f(0) = 0$, but may need a modification general density functions. We use the idea perturbation as before to arrive at an acceptable estimator through this approach as given by

$$\hat{f}_n^+(x) = \frac{\frac{1}{(x+\varepsilon_n)^3} \sum_{i=1}^n X_i q_{v_n}\left(\frac{X_i}{x+\varepsilon_n}\right)}{\sum_{i=1}^n \frac{1}{X_i}} \tag{2.28}$$

The asymptotic properties of $\hat{f}_n^+(x)$ are presented in the following two theorems.

Thoerem 2.3 *If*
A. $v_n \to 0$, $\varepsilon_n \to 0$ *as* $n \to \infty$ *and* $E(X_1^{-1}) < \infty$;
B. $\sup_{x \geq 0} \int_0^\infty |\frac{d}{dx}[\frac{1}{x+\varepsilon_n} q_{v_n}(\frac{t}{x+\varepsilon_n})]| dt = o((\frac{\log\log n}{n^{1/2}})^{-1})$;
C. $\sup_{u>0, v>0} u q_v(u) < \infty$;
D. $g(\cdot)$ *is Lipschitz continuous on* $[0, \infty)$;
then we have, as $n \to \infty$,

$$\sup_{x>0} |\hat{f}_n^+(x) - f(x)| \xrightarrow{a.s.} 0 \tag{2.29}$$

Thoerem 2.4 *Assume the following regularity conditions as used in [4]:*
E. $g(\cdot)$ *is Lipschitz continuous on* $[0, \infty)$ *and* $E(X_1^{-2}) < \infty$;
F. $I_2(q) \triangleq \lim_{v \to 0} v \int_0^\infty (q_{v_n}(t))^2 dt$ *exists;*
G1. for $1 \leq m \leq 3$, $\int_0^\infty (q_{v_n}(t))^m dt = O(v^{1-m})$ *as* $v \to 0$;
G2. with $q^*_{m,v_n}(t) = \frac{(q_{v_n}(t))^m}{\int_0^\infty (q_{v_n}(w))^m dw}$, $1 \leq m \leq 3$, *and as* $v_n \to 0$,

$$(i) \quad \mu_{m,v_n} = \int_0^\infty t q^*_{m,v_n}(t) dt = 1 + O(v_n),$$

$$(ii) \quad \sigma^2_{m,v_n} = \int_0^\infty (t - \mu_{m,v_n})^2 q^*_{m,v_n}(t) dt = O(v_n^2)$$

$$(iii) \quad \sup_{0 < v_n < \varepsilon} \int_0^\infty t^{4+\delta} q^*_{m,v_n}(t) dt < \infty, \text{for some } \delta > 0,\ \varepsilon > 0.$$

Then

(a) If $nv_n \to \infty$*,* $nv_n\varepsilon_n \to \infty$*,* $nv_n^3 \to 0$*,* $nv_n\varepsilon_n^2 \to 0$ *as* $n \to \infty$*, we have*

$$\sqrt{nv_n}(\hat{f}_n^+(x) - f(x)) \to N\big(0, I_2(q)\frac{\mu f(x)}{x^2}\big), \text{for } x > 0.$$

(b) If $nv_n\varepsilon_n^2 \to \infty$ *and* $nv_n\varepsilon_n^4 \to 0$ *as* $n \to \infty$*, we have*

$$\sqrt{nv_n\varepsilon_n^2}(f_n^+(0) - f(0)) \to N\big(0, I_2(q)f(0)\big).$$

Remark 4: Note that we have

$$\begin{aligned} Bias[\hat{f}_n^+(x)] &= v_n^2 f(x) + (2v_n^2 x + \varepsilon_n)f'(x) \\ &\quad + v_n^2 \frac{x^2}{2} f''(x) + o(v_n^2 + \varepsilon_n) \end{aligned} \tag{2.30}$$

and

$$Var[\hat{f}_n^+(x)] = \frac{I_2(q)\mu f(x)}{nv_n(x+\varepsilon_n)^2} + o\big((nv_n)^{-1}\big). \tag{2.31}$$

So we have

$$MSE[\hat{f}_n^+(x)] \approx \frac{I_2(q)\mu f(x)}{nv_n x^2} + [v_n^2 f(x) + (2v_n^2 x + \varepsilon_n)f'(x) + v_n^2\frac{x^2}{2}f''(x)]^2. \tag{2.32}$$

Furthermore, we have

$$\begin{aligned} AMISE[\hat{f}_n^+(x)] &= \frac{I_2(q)\mu}{nv_n}\int_0^\infty \frac{f(x)}{(x+\varepsilon_n)^2}dx \\ &\quad + \int_0^\infty [v_n^2 f(x) + (2v_n^2 x + \varepsilon_n)f'(x) + v_n^2\frac{x^2}{2}f''(x)]^2 dx \end{aligned} \tag{2.33}$$

Note the fact that if $\varepsilon_n > 0$ the integral of (2.28) is less than 1. In this case, the density estimator seems a little left-shifted and slightly " lose " some weights. In order to get the " lost " weights back, we divide (2.28) by its integral $\int_0^\infty \hat{f}_n^+(x)dx$ and obtain the following corrected density estimator

$$\hat{f}_n^*(x) = \frac{\frac{1}{(x+\varepsilon_n)^3}\sum_{i=1}^n X_i q_{v_n}\big(\frac{X_i}{x+\varepsilon_n}\big)}{\int_0^\infty \frac{1}{(x+\varepsilon_n)^3}\sum_{i=1}^n X_i q_{v_n}\big(\frac{X_i}{x+\varepsilon_n}\big)dx}. \tag{2.34}$$

Since, as $\varepsilon_n \to 0$, $\int_0^\infty \frac{1}{(x+\varepsilon_n)^3}\sum_{i=1}^n X_i q_{v_n}\big(\frac{X_i}{x+\varepsilon_n}\big)dx \to \sum_{i=1}^n \frac{1}{X_i}$, we can have the same theorem as Theorem 2.3 and 2.4 for $\hat{f}_n^*(x)$. Furthermore, it is easy to show that

$$Bias(\hat{f}_n^*) = Bias(\hat{f}_n^+) + \varepsilon_n f(0)f(x) + o(\varepsilon_n). \tag{2.35}$$

2.3 Chen Density Estimators for Length Biased Data

An alternative form of (1.4) is

$$\hat{f}(x) = \int_0^\infty K_{\rho_b(x),b}(t)dG_n(t). \tag{2.36}$$

Recall that the empirical distribution for LB data is $F_n(x) = \frac{\sum_{i=1}^n \frac{1}{X_i} I\{X_i \le x\}}{\sum_{i=1}^n \frac{1}{X_i}}$. Substituting $F_n(x)$ in place of $G_n(x)$ in (2.36) will give us Chen density estimators for LB data as follows.

$$\hat{f}_C(x) = \int_0^\infty K_{\rho_b(x),b}(t)dF_n(t), \tag{2.37}$$

which can also be written as

$$\hat{f}_C(x) = \frac{\sum_{i=1}^n \frac{1}{X_i} K_{\rho_b(x),b}(X_i)}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.38}$$

Furthermore, since $\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} \xrightarrow{a.s.} \frac{1}{\mu}$, we can have

$$\begin{aligned} E\left(\hat{f}_C(x)\right) &\approx \int_0^\infty \frac{\mu}{y} K_{\rho_b(x),b}(y)g(y)dy \\ &= \int_0^\infty K_{\rho_b(x),b}(y)f(y)dy \\ &= E\left(f(\xi_x)\right) \end{aligned} \tag{2.39}$$

where ξ_x is a $\Gamma(\rho_b(x), b)$ random variable. Similar to Chen estimator for direct data, it is easy to show that $E\left(\hat{f}_C(x)\right) \to f(x)$ as $b \to 0$.

We use $\hat{f}_{C1}(x)$ and $\hat{f}_{C2}(x)$ to denote the density estimator under the $\rho_b(x)$'s choices (1.6) and (1.7) respectively.

2.4 Scaillet Density Estimators for Length Biased Data

Replacing Gamma kernels $K_{\rho(x),b}$ by inverse or reciprocal inverse Gaussian kernels as in [15] we can derive Scaillet density estimators for length-biased data as

$$\tilde{f}_{IG}(x) = \frac{\sum_{i=1}^n \frac{1}{X_i} K_{IG(x,1/b)}(X_i)}{\sum_{i=1}^n \frac{1}{X_i}}, \tag{2.40}$$

and

$$\tilde{f}_{RIG}(x) = \frac{\sum_{i=1}^n \frac{1}{X_i} K_{RIG(1/(x-b),1/b)}(X_i)}{\sum_{i=1}^n \frac{1}{X_i}}. \tag{2.41}$$

Remark 5: The other strategy based on regular empirical function might not be applicable to their method because they might cause the same problem as in Bhattacharyya *et al.*[2] estimator which has a large bias at the lower boundary.

3 Simulation Studies

In this section, we propose to compare various density estimators described in the previous sections through extensive simulation. First, we discuss the methods behind data-driven selection of smoothing parameters. Based on the selected parameters, a numerical comparison of estimators' local and global performances are carried out. For comparing different density estimators, first we generate the length biased sample. Based on the generated data, we obtain the optimum data dependent values of the smoothing parameters as described later. For our proposed estimators, we use Biased Cross Validation method and minimize the required BCV functions by `optimise` or `optim` in `R` to obtain the optimal solutions. For density estimators motivated by Chen and Scaillet's idea, we use UCV criterion to select parameters. With these chosen parameters, we compute $ISE(f_n, f) = \int_0^\infty [f_n(x) - f(x)]^2 dx$ and $SE\left(f_n(x), f(x)\right) = [f_n(x) - f(x)]^2$ at some chosen points. We obtain 1000 samples of ISE and SE and use the averages of them as approximations of $MISE$ and MSE, that are displayed in various tables.

3.1 Smoothing Parameter Selection

We basically use the Biased Cross Validation (BCV) criterion to select parameters ϵ_n and v_n involved for the new estimators as recommended in [4]. However, for the Chen and Scaillet estimators, this criterion is not feasible as it requires complicated derivatives the density estimators, that are not easily available. Chen[6] suggested using the Unbiased Cross Validation (UCV) function for an estimator $f_n(x; b, \mathcal{D})$ of $f(x)$ that involves data $\mathcal{D}$ and smoothing parameter b, that is given by

$$UCV(b) = \int_0^\infty f_n^2(x; b, \mathcal{D}) \lceil \S - \in \sum_{\rangle=\infty}^{\backslash} \{_{\backslash - \infty}(\mathcal{X}_\rangle; \lfloor, \mathcal{D}_\rangle)/\mathcal{Z}_\rangle,$$

where $Z_i = \sum_{j \neq i} \frac{X_i}{X_j}$.

Plugging in the corresponding Chen or Scaillet estimators for $f_n(x)$ in the above and minimizing with respect to b will give us the optimal solution for the parameter b in Chen and Scaillet estimators.

The Biased Cross-Validation function for $\tilde{f}_n^+(x)$ is given by

$$BCV(v_n, \varepsilon_n) = \int_0^\infty [(xv_n^2 + \varepsilon_n)\tilde{f}_n^{+'}(x) + \frac{x^2 v_n^2}{2} \tilde{f}_n^{+''}(x)]^2 dx + \frac{I_2(q)\hat{\mu}}{n v_n} \int_0^\infty \frac{\tilde{f}_n^+(x)}{(x + \varepsilon_n)^2} dx. \tag{3.42}$$

where $\hat{\mu} = \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i}\right)^{-1}$, being an estimator of μ. We can minimize (3.42) with respect to (v_n, ε_n) to find a choice of (v_n, ε_n).

Note that $\tilde{f}_n^*$ have the same asymptotic normality as $\tilde{f}_n^+(x)$ and slightly different bias. Therefore, according to (2.24), the BCV function for $\tilde{f}_n^*$ seems to be

$$\begin{aligned} BCV^*(v_n, \varepsilon_n) &= \frac{I_2(q)\hat{\mu}}{nv_n} \int_0^\infty \frac{\tilde{f}_n^+(x)}{(x+\varepsilon_n)^2} dx \\ &+ \int_0^\infty [(xv_n^2 + \varepsilon_n)\tilde{f}_n^{+'}(x) + \frac{x^2 v_n^2}{2}\tilde{f}_n^{+''}(x) + \varepsilon_n \tilde{f}_n^+(0)\tilde{f}_n^+(x)]^2 dx. \end{aligned} \tag{3.43}$$

Using the expression for $AMISE$ of $\hat{f}_n^+(x)$, we can obtain the following BCV for $\hat{f}_n^+(x)$

$$\begin{aligned} BCV(v_n, \varepsilon_n) &= \frac{I_2(q)\hat{\mu}}{nv_n} \int_0^\infty \frac{\hat{f}_n^+(x)}{(x+\varepsilon_n)^2} dx \\ &+ \int_0^\infty [v_n^2 \hat{f}_n^+(x) + (2v_n^2 x + \varepsilon_n)\hat{f}_n^{+'}(x) + v_n^2 \frac{x^2}{2}\hat{f}_n^{+''}(x)]^2 dx. \end{aligned} \tag{3.44}$$

Furthermore, using the relation of bias between $\hat{f}_n^+(x)$ and $\hat{f}_n^*(x)$, we can establish the following BCV function for $\hat{f}_n^*(x)$

$$\begin{aligned} BCV^*(v_n, \varepsilon_n) &= \frac{I_2(q)\hat{\mu}}{nv_n} \int_0^\infty \frac{\hat{f}_n^+(x)}{(x+\varepsilon_n)^2} dx + \int_0^\infty \Big[v_n^2 \hat{f}_n^+(x) \\ &+ (2v_n^2 x + \varepsilon_n)\hat{f}_n^{+'}(x) + v_n^2 \frac{x^2}{2}\hat{f}_n^{+''}(x) + \varepsilon_n \hat{f}_n^+(0)\hat{f}_n^+(x) \Big]^2 dx. \end{aligned} \tag{3.45}$$

3.1.1 Simulation for χ_2^2 and χ_6^2

These distributions have been studied elsewhere and would work as benchmark for our studies. The resulting length biased densities are those for χ_4^2 and χ_8^2 random variables, respectively. Simulation of these densities is done by simulating a Gamma density

$$f(x) = \frac{1}{2^{\alpha/2}\Gamma(2)} x^{\alpha/2-1} \exp\{-x/2\} I\{x > 0\}$$

with parameters $\alpha = 2$ and $\alpha = 6$. respectively. Note that estimator with inverse Gaussian kernel does not perform very well for direct data [see Kulasekera and Padgett (2006)]. Our computations showed similar findings for the length biased case and as a result the simulation based on the IG kernel is omitted.

Table 1: Simulated MISE for χ_2^2

Distribution	Estimator	Sample Size					
		30	50	100	200	300	500
	Chen-1	0.13358	0.08336	0.07671	0.03900	0.03056	0.02554
	Chen-2	0.11195	0.08592	0.05642	0.03990	0.03301	0.02298
	RIG	0.14392	0.11268	0.07762	0.06588	0.05466	0.04734
χ_2^2	Gamma(F)	0.06791	0.05863	0.03989	0.03135	0.02323	0.01589
	Gamma*(F)	0.02821	0.01964	0.01224	0.00796	0.00609	0.00440
	Gamma(G)	0.09861	0.07663	0.05168	0.03000	0.02007	0.01317
	Gamma*(G)	0.02370	0.01244	0.00782	0.00537	0.00465	0.00356

Table 2: Simulated MISE for χ_6^2

Distribution	Estimator	Sample Size					
		30	50	100	200	300	500
	Chen-1	0.01592	0.01038	0.00578	0.00338	0.00246	0.00165
	Chen-2	0.01419	0.00973	0.00528	0.00303	0.00224	0.00153
	RIG	0.01438	0.00871	0.00482	0.00281	0.00208	0.00148
χ_6^2	Gamma(F)	0.01109	0.00805	0.00542	0.00327	0.00249	0.00181
	Gamma*(F)	0.01141	0.00844	0.00578	0.00345	0.00264	0.00193
	Gamma(G)	0.01536	0.01063	0.00688	0.00398	0.00303	0.00213
	Gamma*(G)	0.01536	0.01063	0.00688	0.00398	0.00303	0.00213

Table 3: Simulated MSE for χ_2^2

Size	Estimator	x					
		0	0.1	1	2	5	10
n=30	I	0.1307	0.2040	0.0181	0.0044	0.0003	1.6×10^{-5}
	II	0.1187	0.2499	0.0173	0.0045	0.0012	8.7×10^{-5}
	III	0.2222	0.1823	0.0250	0.0074	0.0022	3.2×10^{-5}
	IV	0.1936	0.1447	0.0117	0.0042	0.0002	7.2×10^{-5}
	IV*	0.0329	0.0286	0.0090	0.0030	9.8×10^{-5}	1.5×10^{-4}
	V	0.1893	0.1720	0.0209	0.0066	0.0003	6.1×10^{-6}
	V*	0.0528	0.0410	0.0032	0.0020	8.4×10^{-5}	1.9×10^{-5}
n=50	I	0.1370	0.1493	0.0121	0.0030	0.0002	9.0×10^{-6}
	II	0.1279	0.1894	0.0112	0.0032	0.0008	4.6×10^{-5}
	III	0.2193	0.1774	0.0161	0.0046	0.0046	1.6×10^{-5}
	IV	0.1808	0.1365	0.0101	0.0036	0.0001	5.7×10^{-5}
	IV*	0.0196	0.0172	0.0070	0.0024	6.8×10^{-5}	1.4×10^{-4}
	V	0.1584	0.1440	0.0168	0.0060	0.0002	3.6×10^{-6}
	V*	0.0322	0.0236	0.0014	0.0012	4.8×10^{-5}	1.5×10^{-5}
n=100	I	0.1442	0.8201	0.0070	0.0017	0.0001	4.5×10^{-6}
	II	0.1142	0.1391	0.0054	0.0019	0.0005	2.3×10^{-5}
	III	0.2151	0.1631	0.0091	0.0030	0.0020	7.8×10^{-6}
	IV	0.1332	0.0823	0.0090	0.0032	0.0001	3.6×10^{-5}
	IV*	0.0105	0.0094	0.0047	0.0015	5.9×10^{-5}	1.0×10^{-4}
	V	0.1078	0.0980	0.0121	0.0051	0.0002	1.9×10^{-6}
	V*	0.0280	0.0184	0.0006	0.0007	3.8×10^{-5}	9.4×10^{-6}
n=200	I	0.3327	0.0901	0.0046	0.0012	6.6×10^{-5}	2.5×10^{-6}
	II	0.2111	0.0943	0.0027	0.0012	0.0003	1.3×10^{-5}
	III	0.2127	0.1896	0.0067	0.0019	0.0080	5.9×10^{-6}
	IV	0.0995	0.0782	0.0065	0.0024	7.4×10^{-5}	2.9×10^{-5}
	IV*	0.0137	0.0125	0.0031	0.0010	5.8×10^{-5}	8.2×10^{-5}
	V	0.0636	0.0560	0.0072	0.0038	0.0002	1.3×10^{-6}
	V*	0.0217	0.0134	0.0002	0.0005	2.9×10^{-5}	6.7×10^{-6}
n=300	I	0.2607	0.0690	0.0033	0.0009	4.9×10^{-5}	1.9×10^{-6}
	II	0.1565	0.0897	0.0018	0.0010	0.0002	1.0×10^{-5}
	III	0.2101	0.1643	0.0051	0.0026	0.0020	5.1×10^{-6}
	IV	0.0737	0.0577	0.0050	0.0019	4.8×10^{-5}	2.6×10^{-5}
	IV*	0.0194	0.0118	0.0001	0.0004	2.5×10^{-5}	5.5×10^{-6}
	V	0.0449	0.0363	0.0049	0.0030	0.0001	1.2×10^{-6}
	V*	0.0420	0.0167	0.0006	0.0003	1.5×10^{-5}	6.1×10^{-6}
n=500	I	0.2320	0.0732	0.0024	0.0006	3.1×10^{-5}	1.2×10^{-6}
	II	0.1124	0.0723	0.0012	0.0007	0.0001	6.7×10^{-6}
	III	0.2087	0.1681	0.0039	0.0040	6.4×10^{-5}	1.8×10^{-6}
	IV	0.0507	0.0365	0.0038	0.0015	2.9×10^{-5}	2.4×10^{-5}
	IV*	0.0038	0.0034	0.0016	0.0005	4.7×10^{-5}	5.1×10^{-5}
	V	0.0310	0.0205	0.0033	0.0023	0.0001	1.1×10^{-6}
	V*	0.0146	0.0088	0.0001	0.0003	2.0×10^{-5}	4.4×10^{-6}

I-Chen-1, II-Chen-2, III-RIG, IV-Gamma(F),
IV*-Corrected Gamma(F), V-Gamma(G), V*-Corrected Gamma(G)

Table 4: Simulated MSE for χ_6^2

Size	Estimator	x					
		0	0.1	1	4	10	20
n=30	I	0.0017	0.0018	0.0018	0.0019	0.0001	5.6×10^{-6}
	II	0.0018	0.0017	0.0011	0.0017	0.0002	2.6×10^{-6}
	III	5.6×10^{-5}	6.7×10^{-5}	0.0006	0.0017	0.0002	1.1×10^{-5}
	IV	0.0011	0.0010	0.0019	0.0012	8.5×10^{-5}	4.9×10^{-5}
	IV*	0.0015	0.0021	0.0020	0.0012	7.9×10^{-5}	4.2×10^{-5}
	V	0.0000	3.6×10^{-7}	0.0058	0.0008	0.0001	4.1×10^{-6}
	V*	0.0000	3.6×10^{-7}	0.0058	0.0008	0.0001	4.1×10^{-6}
n=50	I	0.0012	0.0013	0.0015	0.0012	0.0001	3.2×10^{-6}
	II	0.0013	0.0012	0.0008	0.0011	0.0001	1.3×10^{-5}
	III	4.6×10^{-5}	5.7×10^{-5}	0.0006	0.0011	0.0001	6.2×10^{-6}
	IV	0.0005	0.0005	0.0016	0.0008	5.5×10^{-5}	3.1×10^{-5}
	IV*	0.0006	0.0015	0.0016	0.0008	5.3×10^{-5}	3.2×10^{-5}
	V	0.0000	4.3×10^{-6}	0.0037	0.0004	7.9×10^{-5}	3.0×10^{-5}
	V*	0.0000	4.3×10^{-6}	0.0037	0.0004	7.9×10^{-5}	3.0×10^{-5}
n=100	I	0.0008	0.0009	0.0012	0.0006	4.8×10^{-5}	1.4×10^{-6}
	II	0.0008	0.0008	0.0007	0.0006	9.3×10^{-5}	5.5×10^{-6}
	III	3.4×10^{-5}	4.5×10^{-5}	0.0006	0.0006	9.9×10^{-5}	2.8×10^{-6}
	IV	0.0001	0.0001	0.0012	0.0006	2.8×10^{-5}	2.2×10^{-5}
	IV*	0.0001	0.0014	0.0012	0.0006	2.8×10^{-5}	2.2×10^{-5}
	V	0.0000	0.0022	0.0023	0.0002	4.9×10^{-5}	2.0×10^{-6}
	V*	0.0000	0.0022	0.0023	0.0002	4.9×10^{-5}	2.0×10^{-6}
n=200	I	0.0003	0.0004	0.0007	0.0003	2.7×10^{-5}	6.5×10^{-7}
	II	0.0003	0.0003	0.0004	0.0003	5.3×10^{-5}	1.9×10^{-6}
	III	2.0×10^{-5}	2.9×10^{-5}	0.0003	0.0003	5.6×10^{-5}	1.2×10^{-6}
	IV	3.0×10^{-5}	7.0×10^{-5}	0.0007	0.0003	1.4×10^{-5}	1.2×10^{-5}
	IV*	2.0×10^{-5}	0.0005	0.0007	0.0003	1.5×10^{-5}	1.2×10^{-5}
	V	0.0000	0.0007	0.0012	0.0001	3.0×10^{-5}	1.4×10^{-6}
	V*	0.0000	0.0007	0.0012	0.0001	3.0×10^{-5}	1.4×10^{-6}
n=300	I	0.0002	0.0003	0.0006	0.0002	1.9×10^{-5}	4.0×10^{-7}
	II	0.0002	0.0002	0.0004	0.0002	3.8×10^{-5}	1.0×10^{-6}
	III	1.4×10^{-5}	2.2×10^{-5}	0.0003	0.0003	4.0×10^{-5}	8.1×10^{-7}
	IV	1.4×10^{-5}	0.0001	0.0005	0.0002	1.0×10^{-5}	8.4×10^{-6}
	IV*	1.3×10^{-5}	0.0004	0.0005	0.0002	1.1×10^{-5}	8.8×10^{-6}
	V	0.0000	0.0011	0.0008	0.0001	2.3×10^{-5}	1.0×10^{-6}
	V*	0.0000	0.0011	0.0008	0.0001	2.3×10^{-5}	1.0×10^{-6}
n=500	I	0.0001	0.0002	0.0004	0.0001	1.3×10^{-5}	2.3×10^{-6}
	II	0.0001	0.0001	0.0003	0.0001	2.5×10^{-5}	4.7×10^{-7}
	III	9.4×10^{-6}	1.8×10^{-5}	0.0003	0.0001	2.6×10^{-5}	4.5×10^{-7}
	IV	9.2×10^{-6}	0.0001	0.0004	0.0001	6.8×10^{-6}	5.2×10^{-6}
	IV*	3.9×10^{-5}	0.0002	0.0004	0.0001	7.5×10^{-6}	5.9×10^{-6}
	V	0.0000	0.0006	0.0006	8.1×10^{-5}	1.6×10^{-5}	7.7×10^{-7}
	V*	0.0000	0.0006	0.0006	8.1×10^{-5}	1.6×10^{-5}	7.7×10^{-7}

I-Chen-1, II-Chen-2, III-RIG, IV-Gamma(F),
IV*-Corrected Gamma(F), V-Gamma(G), V*-Corrected Gamma(G)

Through the simulation results, we can see that, for χ^2_2 density, $\hat{f}_{C2}$ has smaller MSEs at the boundary and $MISE$s than $\hat{f}_{C1}$. This means $\hat{f}_{C2}$ performs better locally and globally than $\hat{f}_{C1}$, which adapts to the results of Chen (2000) in direct data case shows that $\hat{f}_2$ should have smaller $MISE$ than $\hat{f}_1$. Overall, the density estimators motivated by Chen or Scaillet's idea do not perform very well either globally or locally near lower boundary. Scaillet estimator has huge MSEs at the boundary and the largest $MISE$s. Therefore, they might not be suitable for estimating the density whose value does not equal to zero at the boundary. Two original gamma estimator even though they have two parameters, their behaviors quite differ from what are expected. This is due to the fact that, in this case the parameter ε_ns are usually not zero, which causes the estimators to " lose " some weights and not to be a valid density estimators [Their integrals from 0 to ∞ is less than 1]. In this example, two "stars " are two corrected gamma estimators, which perform best locally and globally. The boundary corrections are very necessary and effective. They reduce dramatically estimators' MSEs near the boundary and relatively slightly at the rest points. Therefore, two estimators have the smallest $MISE$s. The corrected gamma estimator based on F_n behave better near the boundary than the corrected gamma estimator based on G_n. However, at the points being far away from the origin, it is just the opposite. The estimator based G_n is better than the estimator based on F_n. Overall, estimator based on G_n has slightly smaller $MISE$s.

For χ^2_6, all estimators have comparable global results. At the lower boundary, RIG estimator, gamma estimators have similar results. In this case, original gamma estimators are almost as same as the corrected gamma estimators.

In the first example with density such that $f(0) > 0$, the corrected gamma estimators perform much better than the original estimators. In the second example with density such that $f(0) = 0$, the corrected estimators have similar local and global behaviors to the original ones. So we can use the corrected estimators to replace the original estimators without hesitation.

3.2 Simulation for Some Other Standard Distributions

We have simulated for the following standard distribution as well.

(i). Lognormal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi x}} \exp\{-(\log x - \mu)^2/2\} I\{x > 0\};$$

(ii). Weibull Distribution

$$f(x) = \alpha x^{\alpha-1} \exp(-x^\alpha) I\{x > 0\};$$

(iii). Mixtures of Two Exponential Distribution

$$f(x) = [\pi \frac{1}{\theta_1} \exp(-x/\theta_1) + (1-\pi)\frac{1}{\theta_2} \exp(-x/\theta_2] I\{x > 0\}.$$

Table 5: Simulated MISE for Lognormal with $\mu = 0$

Distribution	Estimator	Sample Size					
		30	50	100	200	300	500
Lognormal	Chen-1	0.12513	0.08416	0.05109	0.03450	0.02514	0.01727
	Chen-2	0.12327	0.08886	0.05200	0.03545	0.02488	0.01717
	RIG	0.14371	0.09733	0.05551	0.03308	0.02330	0.01497
	Gamma*(F)	0.06846	0.05614	0.03963	0.02640	0.01998	0.01470
	Gamma*(G)	0.16365	0.12277	0.07568	0.04083	0.029913	0.02035

Table 6: Simulated MISE for Weibull with $\alpha = 2$

Distribution	Estimator	Sample Size					
		30	50	100	200	300	500
Weibull	Chen-1	0.10495	0.06636	0.03884	0.02312	0.01700	0.01167
	Chen-2	0.08651	0.05719	0.03595	0.02225	0.01611	0.01111
	RIG	0.08530	0.05532	0.03227	0.01984	0.01470	0.01045
	Gamma*(F)	0.08358	0.06671	0.04935	0.03169	0.02652	0.01694
	Gamma*(G)	0.12482	0.08526	0.05545	0.03402	0.02731	0.02188

Table 7: Simulated MSE for Mixture of Two Exponential Distributions with $\pi = 0.4$, $\theta_1 = 2$ and $\theta_2 = 1$

Distribution	Estimator	Sample Size					
		30	50	100	200	300	500
Mixture	Chen-1	0.22876	0.17045	0.08578	0.06718	0.05523	0.03811
	Chen-2	0.17564	0.15083	0.07331	0.08029	0.04931	0.03808
	RIG	0.25284	0.20900	0.13843	0.10879	0.09344	0.07776
	Gamma*(F)	0.04147	0.02645	0.01375	0.00758	0.00532	0.00361
	Gamma*(G)	0.02534	0.01437	0.01091	0.01223	0.01132	0.00994

Table 8: Simulated MSE for Lognormal with $\mu = 0$

Size	Estimator	x					
		0	0.1	e^{-1}	1	5	8
n=30	I	0.1108	0.0618	0.1682	0.0211	2.0×10^{-4}	2.8×10^{-5}
	II	0.1045	0.0441	0.1616	0.0196	6.0×10^{-4}	6.4×10^{-5}
	III	0.0026	0.0494	0.2555	0.0207	3.0×10^{-4}	3.2×10^{-5}
	IV*	0.0090	0.1546	0.0607	0.0133	2.0×10^{-4}	9.1×10^{-5}
	V*	0.0007	0.7321	0.1230	0.0121	8.2×10^{-5}	9.4×10^{-6}
n=50	I	0.1123	0.0535	0.1199	0.0158	1.4×10^{-4}	1.1×10^{-5}
	II	0.1056	0.0442	0.1391	0.0110	3.7×10^{-4}	2.7×10^{-5}
	III	0.0027	0.0436	0.1719	0.0133	2.0×10^{-4}	1.5×10^{-5}
	IV*	0.0035	0.1349	0.0446	0.0111	1.7×10^{-4}	6.6×10^{-5}
	V*	0.0000	0.5482	0.0810	0.0080	4.7×10^{-5}	4.9×10^{-6}
n=100	I	0.1044	0.0486	0.1122	0.0086	5.1×10^{-5}	8.2×10^{-6}
	II	0.1038	0.0412	0.1248	0.0059	1.5×10^{-4}	1.3×10^{-5}
	III	0.0028	0.0383	0.0917	0.0064	7.2×10^{-5}	7.2×10^{-6}
	IV*	0.0033	0.1011	0.0261	0.0086	1.1×10^{-4}	4.7×10^{-5}
	V*	0.0000	0.3237	0.0417	0.0044	1.9×10^{-5}	2.1×10^{-6}
n=200	I	0.0854	0.0422	0.0310	0.0054	2.7×10^{-5}	2.9×10^{-6}
	II	0.0871	0.0387	0.0367	0.0036	6.5×10^{-5}	4.6×10^{-6}
	III	0.0024	0.0318	0.0385	0.0042	3.5×10^{-5}	3.3×10^{-6}
	IV*	0.0058	0.0717	0.0167	0.0058	9.2×10^{-5}	3.3×10^{-5}
	V*	0.0015	0.1780	0.0210	0.0026	1.0×10^{-5}	1.1×10^{-6}
n=300	I	0.0670	0.0394	0.0230	0.0034	1.7×10^{-5}	2.0×10^{-6}
	II	0.0739	0.0328	0.0256	0.0023	4.0×10^{-5}	3.0×10^{-6}
	III	0.0021	0.0295	0.0243	0.0027	2.2×10^{-5}	2.2×10^{-6}
	IV*	0.0033	0.1011	0.0261	0.0086	1.1×10^{-4}	4.7×10^{-5}
	V*	0.0013	0.1330	0.0152	0.0019	6.4×10^{-5}	8.2×10^{-6}
n=500	I	0.0519	0.0359	0.0158	0.0021	1.1×10^{-5}	1.3×10^{-6}
	II	0.0561	0.0277	0.0169	0.0017	2.1×10^{-5}	1.6×10^{-6}
	III	0.0018	0.0262	0.0167	0.0019	1.4×10^{-5}	1.3×10^{-6}
	IV*	0.0054	0.0577	0.0125	0.0042	7.5×10^{-5}	2.4×10^{-5}
	V*	0.0018	0.0902	0.0098	0.0012	4.5×10^{-6}	6.2×10^{-7}

I-Chen-1, II-Chen-2, III-RIG, IV*-Corrected Gamma(F), V*-Corrected Gamma(G)

Table 9: Simulated MSE for Weibull with $\alpha = 2$

Size	Estimator	x					
		0	0.1	$1/\sqrt{2}$	1	2	3
n=30	I	0.0856	0.1343	0.0720	0.0588	0.0030	1.9×10^{-4}
	II	0.0949	0.0555	0.0571	0.0398	0.0116	1.4×10^{-4}
	III	0.0025	0.0802	0.0549	0.0394	0.0095	4.6×10^{-4}
	IV*	0.0019	0.1049	0.0600	0.0682	0.0053	0.0022
	V*	0.0000	0.2852	0.0293	0.0336	0.0011	1.8×10^{-4}
n=50	I	0.0644	0.0576	0.0514	0.0349	0.0020	1.0×10^{-4}
	II	0.0679	0.0431	0.0382	0.0223	0.0077	7.1×10^{-4}
	III	0.0021	0.0208	0.0391	0.0218	0.0063	2.2×10^{-4}
	IV*	1.1×10^{-6}	0.0763	0.0475	0.0560	0.0048	0.0018
	V*	0.0000	0.1865	0.0194	0.0251	0.0008	1.4×10^{-4}
n=100	I	0.0468	0.0500	0.0261	0.0204	0.0012	4.3×10^{-5}
	II	0.0477	0.0332	0.0233	0.0129	0.0045	3.2×10^{-4}
	III	0.0017	0.0154	0.0238	0.0121	0.0039	1.0×10^{-4}
	IV*	0.0003	0.0577	0.0338	0.0425	0.0041	0.0013
	V*	0.0033	0.1282	0.0107	0.0180	0.0005	1.0×10^{-4}
n=200	I	0.0281	0.0349	0.0144	0.0109	0.0007	1.6×10^{-5}
	II	0.0287	0.0273	0.0140	0.0072	0.0024	1.1×10^{-4}
	III	0.0011	0.0161	0.0142	0.0067	0.0022	3.8×10^{-5}
	IV*	0.0004	0.0460	0.0181	0.0246	0.0032	7.5×10^{-4}
	V*	0.0106	0.0845	0.0070	0.0121	0.0004	7.5×10^{-5}
n=300	I	0.0228	0.0303	0.0099	0.0077	0.0004	8.8×10^{-6}
	II	0.0233	0.0237	0.0098	0.0050	0.0017	5.6×10^{-5}
	III	0.0009	0.0164	0.0101	0.0048	0.0016	2.6×10^{-5}
	IV*	3.3×10^{-7}	0.0400	0.0144	0.0197	0.0026	5.7×10^{-4}
	V*	0.0158	0.0668	0.0061	0.0102	0.0003	6.6×10^{-5}
n=500	I	0.0160	0.0218	0.0065	0.0050	0.0003	4.5×10^{-6}
	II	0.0156	0.0189	0.0067	0.0033	0.0010	2.5×10^{-5}
	III	0.0006	0.0148	0.0071	0.0033	0.0011	1.6×10^{-5}
	IV*	1.6×10^{-9}	0.0311	0.0068	0.0101	0.0019	2.3×10^{-4}
	V*	0.0283	0.0552	0.0059	0.0089	0.0002	6.5×10^{-5}

I-Chen-1, II-Chen-2, III-RIG, IV*-Corrected Gamma(F), V*-Corrected Gamma(G)

Table 10: Simulated MSE for Mixtures of Two Exponential Distributions with $\pi = 0.4$, $\theta_1 = 2$ and $\theta_2 = 1$

Size	Estimator	x					
		0	0.1	0.5	1	5	10
n=30	I	0.3499	0.3075	0.1033	0.0249	1.1×10^{-4}	2.6×10^{-6}
	II	0.3190	0.3181	0.0825	0.0245	4.5×10^{-4}	1.3×10^{-5}
	III	0.5610	0.4423	0.1245	0.0564	1.9×10^{-4}	2.9×10^{-6}
	IV*	0.0652	0.0549	0.0271	0.0098	3.4×10^{-4}	1.1×10^{-4}
	V*	0.0696	0.0539	0.0070	0.0065	5.4×10^{-5}	1.4×10^{-5}
n=50	I	0.3158	0.7921	0.0511	0.0128	6.4×10^{-5}	1.1×10^{-6}
	II	0.2848	0.7600	0.0551	0.0143	2.3×10^{-4}	2.3×10^{-6}
	III	0.5582	0.8473	0.0797	0.0364	9.1×10^{-5}	1.3×10^{-6}
	IV*	0.0489	0.0414	0.0187	0.0066	2.9×10^{-4}	7.7×10^{-5}
	V*	0.0500	0.0336	0.0032	0.0030	3.4×10^{-5}	9.2×10^{-6}
n=100	I	0.3253	0.2465	0.0257	0.0072	4.0×10^{-5}	5.0×10^{-7}
	II	0.2581	0.1926	0.0235	0.0082	1.1×10^{-4}	8.3×10^{-7}
	III	0.5433	0.4810	0.0505	0.0633	4.8×10^{-5}	5.8×10^{-7}
	IV*	0.0238	0.0200	0.0101	0.0036	2.0×10^{-4}	4.4×10^{-5}
	V*	0.0647	0.0317	0.0016	0.0014	1.4×10^{-5}	4.4×10^{-6}
n=200	I	0.8107	0.1730	0.0149	0.0040	2.6×10^{-5}	3.2×10^{-7}
	II	0.6067	0.1642	0.0158	0.0052	4.8×10^{-5}	3.5×10^{-7}
	III	0.5376	0.3734	0.0269	0.0589	2.7×10^{-5}	3.4×10^{-7}
	IV*	0.0125	0.0105	0.0055	0.0020	1.3×10^{-4}	2.4×10^{-5}
	V*	0.0870	0.0380	0.0011	0.0010	7.1×10^{-6}	1.7×10^{-6}
n=300	I	0.5566	0.0958	0.0110	0.0033	2.1×10^{-5}	2.8×10^{-7}
	II	0.3110	0.1213	0.0107	0.0044	3.3×10^{-5}	2.6×10^{-7}
	III	0.3110	0.1213	0.0107	0.0044	3.3×10^{-5}	2.6×10^{-7}
	VI*	0.0092	0.0075	0.0038	7.6×10^{-5}	0.0001	1.5×10^{-5}
	VII*	0.0806	0.0353	0.0011	0.0007	6.2×10^{-6}	1.0×10^{-6}
n=500	I	0.4798	0.0723	0.0075	0.0024	2.0×10^{-5}	2.3×10^{-7}
	II	0.2553	0.0950	0.0077	0.0034	1.8×10^{-5}	1.7×10^{-7}
	III	0.5273	0.2300	0.0176	0.0045	3.0×10^{-5}	3.9×10^{-7}
	IV*	0.0074	0.0047	0.0025	0.0010	7.1×10^{-5}	1.0×10^{-5}
	V*	0.0698	0.0311	0.0010	0.0006	6.1×10^{-6}	7.0×10^{-7}

I-Chen-1, II-Chen-2, III-RIG, IV-Gamma(F),
IV*-Corrected Gamma(F), V-Gamma(G), V*-Corrected Gamma(G)

Through the simulation results given in tables, we may draw the following conclusions. Two Chen estimators have similar performances at the edge. Usually $\hat{f}_{C2}$ have smaller $MISE$ than $\hat{f}_{C1}$. For direct data, Chen (2000) show that density estimator under parameter choice (1.7) has a better global performance than that under choice (1.6). This property might be adapted to LB data. The simulated MSEs show that the $\hat{f}_{C2}$ preform better than $\hat{f}_{C1}$ in the neighborhood of origin, so $MISE$ of $\hat{f}_{C2}$ is lower. However, the two Chen density estimators do not perform very well globally and locally near the origin comparing with other density estimators. They have relatively great $MISE$s and MSEs near the lower edge. Judged by the simulated MSEs at the boundary, the two estimators can not completely remove the bias at the edge, even in the case that underlying density such that $f(0) = 0$ [see simulation for χ_6^2, Lognormal].

Replacing gamma kernels with RIG kernels, Scaillet estimator have a great advantange in reducing MSEs at the origin for underlying density such that $f(0) = 0$ [see simulation for χ_6^2, Lognormal and Weibull distributions]. In some cases, the estimator even has zero error at the origin. So, under this circumstance, Scaillet estimator gives smaller $MISE$s than Chen estimators. It seems that RIG density is more suitable as kernels than gamma density in these cases. However, this advantage lost when estimating an underlying density where $f(0) \neq 0$. In such cases, Scaillet estimator has huge MSEs at the origin [see simulation for χ_2^2 and mixture of two exponential distributions]. According to the cases considered here, it seems reasonable to conclude that Scaillet estimator might be just suitable to estimate density with zero value at the lower boundary.

The corrected gamma estimator based on F_n performs very well locally and globally in the simulation for χ_2^2 and mixtures distributions. The parameter ε_n and boundary correction effectively reduce the bias at the boundary and result in the dramatic decrease of $MISE$s. For the rest distributions, this estimator has comparable $MISE$s to other estimators and satisfactory MSEs at the boundary. The BCV method is valid to decide whether the optimal solution of ε_n is zero or not. So that this estimator can accurately explore the characters of underly density at the boundary behind the data. Inspired by Scaillet estimator, in order to further reduce $MISE$, we can substitute gamma kernels with RIG or IG kernels. Actually, RIG or IG kernels are completely adapted to our estimator.

The corrected gamma estimator based on G_n has the smallest $MISE$s in the simulation for χ_2^2. However, the MSEs at the boundary are little worse than those for corrected gamma estimator based on F_n. Further simulation shows that, although this estimator has zero error at the boundary in some cases, it has relatively great MSEs in the neighborhood of origin. Thus this estimator may not be very stable in the neighborhood closer to the lower boundary. We may conjecture that, using the same smooth technique, the density estimator obtained by smoothing F_n performs better than that obtained by smoothing G_n. This seems to be true for kernel method as well, as concluded by Jones[9].

4 Appendix

4.1 Proof of Theorem 2.1

Proof: The proof involves the following representation of $\tilde{f}_n^+(x)$:

$$\begin{aligned}\tilde{f}_n^+(x) &= \frac{d}{dx}\int F_n(t)[\frac{1}{x+\varepsilon_n}q_{v_n}(\frac{t}{x+\varepsilon_n})]dt \\ &= \int F_n(t)\frac{d}{dx}[\frac{1}{x+\varepsilon_n}q_{v_n}(\frac{t}{x+\varepsilon_n})]dt. \end{aligned} \quad (4.46)$$

Next we use the uniform strong convergence of $F_n(x)$, namely,

$$\sup_{x\geq 0}|F_n(x)-F(x)| \xrightarrow{a.s.} 0, \quad (4.47)$$

and rest of the proof follows along the lines of the proof of Theorem 3 of Chaubey, Sen and Sen (2007).

4.2 Proof of Theorem 2.2

Proof: (a) Using Taylor's expansion, we can write

$$\tilde{f}_n^+(x) = \frac{1}{n}\sum_{i=1}^n Y_{in} - f(x)(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}) + o(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}) \quad (4.48)$$

where

$$Y_{in} = \frac{\mu}{(x+\varepsilon_n)^2}q_{v_n}(\frac{X_i}{x+\varepsilon_n}). \quad (4.49)$$

For $x > 0$, by the Law of the Iterated Logarithm,

$$\limsup_{n\to\infty}[\sqrt{nv_n}(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu})] \overset{a.s.}{=} O(\sqrt{v_n}\log\log n) \overset{a.s.}{\to} 0.$$

Hence, it is sufficient to consider the first term in (4.48). Under the conditions of the theorem, we can show that, as $v_n \to 0$ and $\varepsilon_n \to 0$,

$$E(Y_{1n}^3) = O(v_n^{-2}) \quad (4.50)$$

and

$$v_n E(Y_{1n}^2) \to I_2(q)\frac{\mu f(x)}{x^2}. \quad (4.51)$$

Since Y_{in} is nonnegative, we have

$$\begin{aligned}\frac{\sum_{i=1}^n E|Z_{in}|^3}{[\sum_{i=1}^n E(Z_{in}^2)]^{3/2}} &\leq \frac{E[Y_{1n}+E(Y_{1n})]^3}{\sqrt{n}[var(Y_{1n})]^{3/2}} \\ &= \frac{E(Y_{1n}^3)+3E(Y_{1n}^2)E(Y_{1n})+4(E(Y_{1n}))^3}{\sqrt{n}[E(Y_{1n}^2)-(E(Y_{1n}))^2]^{3/2}}. \end{aligned} \quad (4.52)$$

By (4.50), (4.51) and (4.52), we have

$$\frac{\sum_{i=1}^n E|Z_{in}|^3}{[\sum_{i=1}^n E(Z_{in}^2)]^{3/2}} = O(\frac{1}{\sqrt{nv_n}}) \to 0. \tag{4.53}$$

Then by the Theorem 7.1.2 of Chung (1974), we have

$$\frac{\sum_{i=1}^n Z_{in}}{(\sum_{i=1}^n Z_{in}^2)} \to N(0,1).$$

This implies

$$\sqrt{nv_n}\big[\frac{1}{n}\sum_{i=1}^n Y_{in} - E(Y_{1n})\big] \to N\big(0, I_2(q)\frac{\mu f(x)}{x^2}\big). \tag{4.54}$$

Further,

$$\begin{aligned}
\sqrt{nv_n}|E(Y_{1n}) - f(x)| &= \sqrt{nv_n}\Big|\int_0^\infty [f(t(x+\varepsilon_n)) - f(x)]tq_{v_n}(t)dt\Big| \\
&\leq \sqrt{nv_n}L\int_0^\infty |(t-1)x + t\varepsilon_n|tq_{v_n}(t)dt \\
&\leq \sqrt{nv_n}xL\int_0^\infty |t-1|tq_{v_n}(t)dt + \sqrt{nv_n\varepsilon_n^2}L\int_0^\infty t^2q_{v_n}(t)dt \\
&\leq \sqrt{nv_n}xL[\int_0^\infty (t-1)^2q_{v_n}(t)dt]^{1/2}[\int_0^\infty t^2q_{v_n}(t)dt]^{1/2} \\
&\quad + O(\sqrt{nv_n\varepsilon_n^2}) \\
&= O(\sqrt{nv_n^3}) + O(\sqrt{nv_n\varepsilon_n^2}). \qquad (4.55)
\end{aligned}$$

Using (4.48), (4.54) and (4.55), we can establish the part (a) of the theorem.

(b) For this part of the proof, we decompose $\tilde{f}_n^+(0)$ similar to (4.48) as,

$$\tilde{f}_n^+(0) = \frac{1}{n}\sum_{i=1}^n Y'_{in} - f(0)\big(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}\big) + o(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}) \tag{4.56}$$

where

$$Y'_{in} = \frac{\mu}{(\varepsilon_n)^2}q_{v_n}(\frac{X_i}{\varepsilon_n}). \tag{4.57}$$

We can show that

$$E(Y_{1n}^{'3}) = O(\varepsilon_n^4 v_n^{-2}) \tag{4.58}$$

and

$$\varepsilon_n^2 v_n E(Y_{1n}^{'2}) \to I_2(q)f(0). \tag{4.59}$$

Letting

$$Z'_{in} = \sqrt{\frac{v_n \varepsilon_n^2}{n}}\,[Y'_{in} - E(Y'_{in})] \tag{4.60}$$

and using (4.58), (4.59) and a similar inequality as in (4.52), we find

$$\frac{\sum_{i=1}^n E|Z'_{in}|^3}{[\sum_{i=1}^n E(Z'^2_{in})]^{3/2}} \le O(\frac{1}{\sqrt{n v_n \varepsilon_n^2}}) \to 0. \tag{4.61}$$

Then by Theorem 7.1.2 of Chung (1974), we have

$$\frac{\sum_{i=1}^n Z'_{in}}{(\sum_{i=1}^n Z'^2_{in})} \to N(0,1),$$

that implies

$$\sqrt{n v_n \varepsilon_n^2}\Big[\frac{1}{n}\sum_{i=1}^n Y'_{in} - E(Y'_{1n})\Big] \to N\big(0, I_2(q) f(0)\big). \tag{4.62}$$

Furthermore,

$$\begin{aligned}
\sqrt{n v_n \varepsilon_n^2}|E(Y'_{1n}) - f(0)| &= \sqrt{n v_n \varepsilon_n^2}\Big|\int_0^\infty [f(t\varepsilon_n) - f(0)] t q_{v_n}(t) dt\Big| \\
&\le \sqrt{n v_n \varepsilon_n^2} L \int_0^\infty |\varepsilon_n| t^2 q_{v_n}(t) dt \\
&= \sqrt{n v_n \varepsilon_n^2} L[\varepsilon_n(v_n^2 + 1)] \\
&= O(\sqrt{n v_n \varepsilon_n^4}). \qquad (4.63)
\end{aligned}$$

Next using (4.56), (4.62) and (4.63)establishes part (b) of the theorem.

4.3 Proof of Theorem 2.3

Proof: By Theorem 3 of Chaubey, Sen and Sen (2007), under the conditions of Theorem 2.3, we have

$$\sup_{x>0}|g_n^+(x) - g(x)| \xrightarrow{a.s.} 0 \tag{4.64}$$

where

$$g_n^+(x) = \frac{1}{n(x+\varepsilon_n)^2}\sum_{i=1}^n X_i q_{v_n}\Big(\frac{X_i}{x+\varepsilon_n}\Big).$$

On the other hand, by the strong law of large numbers, we have

$$\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} \xrightarrow{a.s.} \frac{1}{\mu}., \tag{4.65}$$

Hence, by (2.28), (4.64) and (4.65), we complete the proof.

4.4 Proof of Theorem 2.4

Proof: (a) Using Theorem in Chaubey, Sen and Sen (2007 we have

$$\sqrt{nv_n}(g_n^+(x) - g(x)) \to N\left(0, I_2(q)\frac{g(x)}{x}\right), \text{for } x > 0. \tag{4.66}$$

Further since $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}\right)$ converges to a normal distribution, the asymptotic normality of

$$\hat{f}_n^+(x) = \frac{g_n^+(x)}{(x+\varepsilon_n)\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i}} \tag{4.67}$$

is equivalent to that of

$$\frac{\mu g_n^+(x)}{(x+\varepsilon_n)}. \tag{4.68}$$

Thus using (4.66), it is easy to show that

$$\sqrt{nv_n}\left(\frac{\mu g_n^+(x)}{(x+\varepsilon_n)} - \frac{\mu g(x)}{x}\right) \to N\left(0, I_2(q)\frac{\mu f(x)}{x^2}\right), \text{for } x > 0, \tag{4.69}$$

that completes the proof of part (a).

(b) To prove part (b) of the theorem, we write

$$\hat{f}_n^+(0) = \frac{1}{n}\sum_{i=1}^n Y_{in}'' - f(0)\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}\right) + o\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\mu}\right) \tag{4.70}$$

where

$$Y_{in}'' = \frac{\mu X_i}{(\varepsilon_n)^3} q_{v_n}\left(\frac{X_i}{\varepsilon_n}\right). \tag{4.71}$$

We can show that

$$E(Y_{1n}''^3) = O(\varepsilon_n^4 v_n^{-2}) \tag{4.72}$$

and

$$\varepsilon_n^2 v_n E(Y_{1n}''^2) \to I_2(q) f(0). \tag{4.73}$$

Then the proof of part (b) follows from (4.72) and (4.73) along the lines of proof of part (b) of Theorem 2.1.

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