Convex Formulation of Controller Synthesis for Piecewise-Affine Systems

SINA KAYNAMA

A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Applied Science (Electrical and Computer Engineering) at

Concordia University

Montréal, Québec, Canada

September 2012

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis proposal prepared

By: **SINA KAYNAMA**

Entitled: **Convex Formulation of Controller Synthesis for Piecewise-Affine Systems**

and submitted in partial fulfilment of the requirements for the degree of

Master of Applied Science (Electrical and Computer Engineering)

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________ Dr. L. A. Lopes, Chair

______________________ Dr. Y. Zhang, External Examiner

______________________ Dr. A. Aghdam, Examiner

______________________ Dr. L. Rodrigues, Supervisor

Approved by ______________________

Dr. W. E. Lynch, Chair

Department of Electrical and Computer Engineering

____

Dr. Robin A. L. Drew

Dean, Faculty of Engineering and Computer Science

ABSTRACT

Convex Formulation of Controller Synthesis for Piecewise-Affine Systems

SINA KAYNAMA

This thesis is divided into three main parts. The contribution of the first part is to present a controller synthesis method to stabilize piecewise-affine (PWA) slab systems based on invariant sets. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast as a set of Linear Matrix Inequalities (LMIs) by proper choice of an invariant set which is a target sliding surface. The method has two steps: the design of the attractive sliding surface and the design of the controller parameters. While previous approaches to PWA controller synthesis are cast as Bilinear Matrix Inequalities (BMIs) that can, in some cases, be relaxed to LMIs at the cost of adding conservatism, the proposed method leads naturally to a convex formulation. Furthermore, the LMIs obtained in this work have lower dimension when compared to other methods because the dimension of the closed-loop state space is reduced.

In the second part of the thesis, it is further shown that the proposed approach is less conservative than other approaches. In other words, it will be shown that for every solution of the LMIs resulting from previous approaches, there exists a solution for the LMIs obtained from the proposed method. Furthermore, it will be shown that while previous convex controller synthesis methods have no solutions to their LMIs for some examples of PWA systems, the approach proposed in this thesis yields a solution for these examples.

The contribution of the last part of this thesis is to formulate the PWA time-delay synthesis problem as a set of LMIs. In order to do so, we first define a sliding surface,

then control laws are designed to approach the specified sliding surface and ensure that the trajectories will remain on that surface. Then, using Lyapunov-Krasovskii functionals, sufficient conditions for exponential stability of the resulting reduced order system will be obtained.

Several applications such as pitch damping of a helicopter (2nd order system), rover path following example (3rd order system) and active flutter suppression (4th order system) along with some other numerical examples are included to demonstrate the effectiveness of the approaches.

“Equations are more important to me,

because politics is for the present,

but an equation is something for eternity.”

— Albert Einstein

ACKNOWLEDGEMENTS

It would not have been possible to write this thesis without the help and support of a number of people. I consider myself extremely lucky to have the opportunity to work under the incredible supervision and mentorship of Professor Luis Rodrigues. His expertise in the field, his encouragement, guidance, patience and support from the initial to the final level of this work, have been instrumental in my development as a researcher.

I would like to express my gratitude to the rest of my examination committee Professors Amir G. Aghdam, Youmin Zhang, Luiz A. Lopes for the constructive feedback that they gave me about the thesis. I am also heartily thankful to Dr. Behzad Samadi for his guidance and support which have been a blessing through every step that I have taken from my Bachelor's to Masters's studies.

I had the honour to be part of a great research group at Hybrid Control Systems (HYCONS) Lab. Miad Moarref, Amin Zavieh, Camilo Ossa, Gavin Kenneally, Mehdi Abedinpour Fallah, Azita Malek and Hadi Karimi have all made my stay at Concordia University an extremely interesting experience. I would like to thank Miad Moarref in particular, for all our brainstorming conversations which have been inspiring and informative and I wish him the best of luck with his PhD studies. Also, I would like to thank the professors and the administrative and technical support staff in the department who have played a vital role in my success.

Last, but by no means least, I would like to thank my family–Mom,Dad, Kaveh (and Kimia), Shahab (and Blerina) for all their never-ending love and support which cannot be put into words.

To my parents.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Motivation

Many well-established analysis and design techniques exist for linear systems. Linear system theory has been used in industrial engineering applications for decades. However, sometimes either the controller or the system under control or both, may not be a linear system, and therefore linear system theory cannot necessarily be applied. Furthermore, increase in demands on closed-loop systems characteristics have led attention to *nonlinear control theory*. Nonlinear control theory allows one to study how to apply existing linear methods to more general control systems and more importantly it provides controller designers with novel nonlinear control methods that cannot be analyzed using linear system theory.

Many of the nonlinear systems encountered in practice involve a coupling between continuous dynamics and discrete events. Systems in which these two kinds of dynamics coexist and interact are usually called *switched or hybrid control systems*. One will find a good introduction to hybrid systems in [5], and [6] offers more details on switched systems. A great deal of attention and effort in hybrid systems have been focused on the modeling, stability and control design methods [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

One important subclass of hybrid systems are *piecewise-affine* (PWA) systems. A PWA system is a hybrid system with affine continuous dynamics within different discrete modes. PWA systems are a very important and powerful modeling class for practical applications involving nonlinear dynamics because a wide variety of nonlinearities are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as piecewise-affine functions [20, 21, 22, 23]. There is also an intimate relation between PWA systems and linear parameter-varying (LPV) systems, in which the focus is on stabilization with additional performance objectives such as in terms of L_2-gain properties. PWA control can be seen as related to LPV control with the important difference that the scheduling of the controllers does not depend continuously on the value of a varying parameter. Instead, the controller gains switch discontinuously among a finite number of possible values of that parameter.

PWA systems can be used to approximate a wide variety of nonlinear systems. Currently, PWA systems are receiving wide attention due to the fact that the PWA framework provides a way to describe dynamical systems exhibiting switching between a multitude of linear dynamic regimes [24, 25, 26, 27]. Several promising methods have emerged for analysis and synthesis of PWA control systems such as those proposed in [28, 23, 17, 22, 29, 30, 31, 32, 33] and references therein.

However, there are only a few controller synthesis methods for PWA systems that can be cast as a convex optimization program. Convex optimization is a special class of mathematical optimization that studies the problem of minimizing convex functions over convex sets. An important advantage of formulating a problem as a convex optimization program is that there exist several reliable solution methods that can be embedded in a computer-aided design or analysis tools. Efficiency and reliability of finding solutions to these problems has made them one of the most popular topics in many different areas including control systems. However, formulating a problem as a convex optimization might not always be trivial and sometimes it is almost impossible due to a non-convex nature of the problem. Unfortunately, synthesis of PWA controllers also naturally leads to non-convex problems. These problems are $\mathscr{N}\mathscr{P}$ hard and therefore solving them is a non-trivial task.

Based on the above motivation, the first goal of this thesis is to develop a new approach to obtain a convex formulation for PWA synthesis.

One of the challenges faced by controller designers is dealing with *time-delay* systems. Many practical systems are subject to state delay. Time-delay is commonly encountered in various engineering systems, such as chemical processes, hydraulic, pneumatic and economic systems. This usually results in unsatisfactory performance and is frequently a source of instability, so control of time-delay systems is practically important. Some other examples of time-delay systems include power systems [34] and communication networks [35]. Time-delays can cause poor performance or even instability if their effect is neglected in control design. On the other hand, as it was mentioned in previous paragraphs, there are many advantages to work with nonlinear systems, especially PWA systems. As it was already pointed out, PWA systems provide a powerful modeling class for practical applications involving nonlinear dynamics. Therefore, *piecewise-affine time-delay* systems can be considered as an important tool for modeling nonlinear time-delay systems.

Some of the existing results for stability of time-delay systems can be found in references [36, 37, 38, 39, 40]. There are also a few novel contributions on the analysis of PWA time-delay systems in the literature such as [41, 42]. Although some of these approaches lead to convex problems, to the best of the author's knowledge, none of them addresses the controller synthesis problem for PWA time-delay systems. Therefore, designing a PWA state feedback controller for a PWA time-delay system and formulating it as a convex feasibility and/or optimization problem is still an open problem.

Based on the above motivation, the last goal of this thesis is to propose a convex formulation of the PWA time-delay controller synthesis problem for the case of a known constant delay.

1.2 Literature Survey

1.2.1 Piecewise-Affine Systems

Previous work on PWA Systems

Contributions of the Russian physicist, Aleksandr Aleksandrovich Andronov (1901-1952), to control theory and nonlinear dynamics can be considered as the very first appearance of PWA systems in control engineering. A brief summary of his research can be found in [43]. Later on, the theory of PWA systems was also used in the analysis and synthesis of nonlinear electrical circuits with most work done up until the 1970's [44, 45].

In the early 1980's, control and observation of piecewise-linear (PWL) systems over finite time intervals based on piecewise-linear algebra was proposed by Sontag [46]. He then suggested there might be a possibility of developing a systematic approach to numerical nonlinear regulation, based on piecewise-linear (PWL) approximations. Pettit [47] combined ideas and known results from linear systems, convex set theory, and computational geometry to create a new analysis tool for studying PWL systems.

In the early 90's, investigation on Lyapunov asymptotic stability of switched systems was proposed by Peleties, et. al., [48]. In the late 90's Boyd and Ghaoui [49] proposed an approach in which one can synthesize a linear state feedback for Lyapunov stability of a linear differential inclusion (DI) by solving a set of Linear Matrix Inequalities (LMIs) which is a convex problem. The more recent work on the analysis of PWA systems based on Lyapunov functions and LMIs can be found in [50, 51, 52, 53, 54, 48, 17, 55].

Several promising methods have emerged for Lyapunov based analysis of PWA systems such as those proposed in [52, 53, 54, 17, 29]. One of the very first steps towards controller synthesis of PWL systems was taken by Rantzer and Johansson in [20]. They extended the use of piecewise-quadratic cost functions from stability analysis of PWL systems in [54] to performance analysis and optimal control. In that work, the lower bounds on the optimal control cost are obtained by semidefinite programming based on the Bellman inequality. An upper bound to the optimal cost is also obtained by another convex optimization problem using the given control law. However, the method does not guarantee that the control law is stabilizing. Furthermore, as it is mentioned in [20]

"*This control law is simple but may be discontinuous and give rise to* ***[divergent]*** *sliding modes*" [20].

However, it is suggested in [20] that one can avoid sliding motions, which may occur at the boundaries of the partitions, by linear interpolation between resulting vector inputs.

In [1, 55], a synthesis method based on Bilinear Matrix Inequalities (BMIs) has been proposed for state and output feedback stabilization of PWA systems. The method has the advantage of guaranteeing that sliding modes are not generated at the switching and the controllers are therefore provably stabilizing. Another important feature of this method for practical implementation of the controllers is that continuity of the control input can also be guaranteed at the switching. However, BMI problems are not convex problems and thus, are not easy to be solved efficiently.

The tracking problem for a class of PWA systems was addressed in [33], and also [56]. Pavlov and Van de Wouw [57] show that for certain classes of PWA systems (both continuous and discontinuous) the controller design is characterized in terms of LMIs only if linear feedback is used:

"*Clearly, for the case of linear feedback, LMI conditions are now available ...*" [57].

Another LMI-based state feedback controller is designed based on a based on a piecewise-quadratic Lyapunov function in [58]. However, the controllers should be linear and moreover this approach should be applied only to PWL systems:

"*Note that Proposition 1 does not apply to piecewise affine systems that have multiple equilibria and therefore, the method in Theorem 1 does not apply in this case. This is a limitation of the proposed method.*" [58].

Previous Work on PWA Slab systems

PWA slab systems [30] are a subclass of PWA systems where the regions partitioning the domain are slabs. Hassibi and Boyd in reference [23] show that sufficient conditions for quadratic stabilization using piecewise-linear state feedback for PWA slab systems can be cast as a convex optimization problem. Unfortunately, if affine terms are included in the controller, the convex structure is apparently destroyed, making it hard to solve the problem globally:

"*it does not seem that the condition for stabilizability can be cast as an LMI*" [23].

Rodrigues and Boyd [30] introduce sufficient conditions for asymptotic stability of closed-loop piecewise-affine slab systems using piecewise-affine state feedback control laws. The resulting conditions form a non-convex problem and it is mentioned there:

"this synthesis problem cannot be formulated as one convex program ... "

However, under certain additional assumptions and relaxing the problem, reference [30] shows that one can develop algorithms to approximately solve these resulting non-convex problems with optimality guarantees.

References [4, 59] present algorithms for state feedback design of PWA systems based on LMIs which can be efficiently solved using software packages such as SeDuMi [60] and YALMIP [61]. In fact, to the best of the author's knowledge, the methods proposed in [4, 59] are the only ones that can formulate PWA state feedback as a set of LMIs. The method in [4] shows that one can avoid solving the Bilinear Matrix Inequalities (BMIs) proposed in [30] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility. In [59] a backstepping approach is developed for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin.

1.2.2 Piecewise-Affine Time-Delay Systems

Although, PWA systems are recently receiving significant attention, there are only a few contributions toward PWA time-delay systems. On the other hand, stability analysis for switched systems with time-delay can be found in many references such as [62, 63, 64] (reference [62] also develops sufficient conditions for exponential stability of linear time-delay systems with a class of switching signals).

The stability problem for PWA time-delay systems was first addressed in Kulkarni's work, [65], where a piecewise-quadratic Lyapunov function was used to derive linear matrix inequalities (LMIs) for stability analysis following Johansson's approach in [66]. PWA uncertain systems with unknown time-delay were investigated in [41]. In reference [41] LMI-based conditions for asymptotic stability were derived following the approach of Rodrigues and How [55].

Resemblance of the sampled-data PWA systems and PWA time-delay systems have recently resulted in novel contributions in the field. Analysis of sampled-data PWA systems consist of a continuous-time plant in feedback connection with a discrete-time emulation of a continuous time state feedback controller. However, the discrete-time controller can also be modeled as a continuous-time controller with time varying delay. Reference [2] studies the stability of sampled-data PWA systems using Lyapunov-Krasovskii functionals. The paper provides a set of LMIs as sufficient conditions for exponential convergence of the sampled-data system to an invariant set containing the origin.

To the best of the author's knowledge, references [65, 41, 2, 67] are the only available conducted research on stability analysis of PWA time-delay systems. Furthermore, none of the above mentioned references address the controller synthesis problem for such systems. Consequently, there is no convex formulation for controller synthesis of PWA time-delay systems in the existing literature.

1.3 Contributions

1.3.1 Contributions on Piecewise-Affine Controller Synthesis

One objective of this thesis is to develop a new controller synthesis method for PWA systems based on convex optimization. Considering the lack of such a powerful synthesis tool for PWA systems in the literature, this thesis addresses the following research questions:

- How can one formulate the PWA synthesis problem as a set of LMIs?

- Is it possible to have the problem formulated in lower dimensions and reduce the complexity of the LMIs?

- How much less conservative is the proposed approach compared to the methods available in the literature?

One of the most important contributions of this thesis is to use invariant set ideas to formulate the PWA synthesis problem as a set of LMIs. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast directly as a set of LMIs by proper choice of an invariant set which is a target sliding surface. It is further shown that the dimension of the LMIs obtained in this thesis is lower than in the other convex methods in the literature because the dimension of the state space is reduced, which further simplifies the synthesis problem. Furthermore, it will be also shown that for every solution of the LMIs resulting from previous approaches there exists a solution for the LMIs obtained from the proposed method. Finally, it will be shown that while previous convex controller synthesis methods have no solutions to their LMIs for some examples of PWA systems, the approach proposed in this thesis yields a solution for these examples.

1.3.2 Contributions on Piecewise-Affine Time-Delay Controller Synthesis

Unfortunately, controller synthesis of PWA time-delay systems has not received many research contributions. Therefore, the second objective of this thesis is to develop a controller synthesis method for PWA time-delay systems based on convex optimization. This thesis addresses the following research questions:

- Can the problem of PWA time-delay controller synthesis be cast as a linear matrix inequality problem?

- Are the proposed control laws still in PWA state feedback form?

An important contribution of this thesis is to formulate the controller synthesis problem for PWA slab systems for the case of a known constant time delay as a set of LMIs. In order to do so, we first define a sliding surface and then control laws are designed to approach the specified sliding surface and ensure that the trajectories will remain on that surface. Using Lyapunov-Krasovskii functionals, sufficient conditions for exponential stability of the resulting reduced order system will be proposed. Moreover, the designed control laws are still PWA state feedback controllers.

1.4 Publications

- The proposed methods in this thesis are mainly based on the following paper

S. Kaynama, B. Samadi and L. Rodrigues, "A Convex Formulation of Controller Synthesis for Piecewise-Affine Slab Systems Based on Invariant Sets" accepted for publication *in Proceedings of the 51th IEEE Conference on Decision and Control,* Maui, Hawaii, December 10-13, 2012.

Chapter 2

Previous Work on Piecewise-Affine Systems

2.1 Introduction

As it was mentioned in Chapter 1, piecewise-affine (PWA) systems are an important subclass of hybrid systems. A PWA system is a hybrid system with affine or linear dynamics within different discrete modes. It was also mentioned that PWA systems are a very important and powerful modeling class for practical applications involving nonlinear dynamics because a wide variety of nonlinearities are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as piecewise-affine functions [20, 21, 22, 23]. PWA systems can also be used to approximate a wide variety of nonlinear systems. Currently, PWA systems are receiving wide attention due to the fact that the PWA framework provides a way to describe dynamic systems exhibiting switching between a multitude of linear dynamic regimes [24, 25, 26, 27].

Although several promising methods have emerged for analysis of PWA control systems (see [28, 23, 17, 22, 29, 30, 31, 32, 33] and references therein), there are only a few controller synthesis methods for PWA systems that can be cast as a convex optimization program.

This chapter provides a brief review of piecewise-affine (PWA) systems and the available convex approaches towards their controller synthesis.

2.2 Mathematical Preliminaries

2.2.1 Review of Piecewise-Affine Systems

Piecewise-affine systems inherently involve a partition of the state space into regions with different affine dynamics. Therefore, PWA systems will be characterized by a partition of a subset of the state space $\mathscr{X}$ into a set of regions $\mathscr{R}_i$ such that the dynamics within each region are affine and strictly proper of the form

$$\dot{x}(t) = A_i x(t) + a_i + B_i u(t), \quad x(t) \in \mathscr{R}_i \tag{2.1}$$

$$y(t) = C_i x(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ the control input and a forward invariant set $\mathscr{X} \subset \mathbb{R}^n$ is partitioned into M polytopic cells $\mathscr{R}_i$, $i \in \mathscr{I} = \{1, \ldots, M\}$ such that $\cup_{i=1}^{M} \overline{\mathscr{R}}_i = \mathscr{X}$, $\mathscr{R}_i \cap \mathscr{R}_j = \emptyset$ where $\overline{\mathscr{R}}_i$ denotes the closure of $\mathscr{R}_i$ (see [22] for generating such partition).

Following [22], each cell is constructed as the intersection of a finite number (p_i) of half spaces:

$$\mathscr{R}_i = \left\{ x \mid H_i^T x + g_i > 0 \right\} \tag{2.3}$$

where $H_i = \begin{bmatrix} h_{i1} & h_{i2} & \cdots & h_{ip_i} \end{bmatrix} \in \mathbb{R}^{n \times p_i}$, $g_i = \begin{bmatrix} g_{i1} & g_{i2} & \cdots & g_{ip_i} \end{bmatrix}^T \in \mathbb{R}^{p_i}$ and $>$ represent an elementwise inequality. Each polytopic cell has a finite number of facets and vertices. Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathscr{N}_i$=*level-1* neighboring cells of $\mathscr{R}_i$. It is assumed that vector c_{ij} and the scalars d_{ij} exist such that the facet boundary between cells $\mathscr{R}_i$ and $\mathscr{R}_j$ is contained in the hyperplane described by $\{x \in \mathbb{R}^n \mid c_{ij}^T x - d_{ij} = 0\}$, for $i = 1, \ldots, M$ and $j \in \mathscr{N}_i$. A parametric description of the boundaries can then be obtained as [23] (see Figure 2.1)

$$\overline{\mathscr{R}}_i \cap \overline{\mathscr{R}}_j \subseteq \{x \mid x = F_{ij} s + l_{ij}\}, \quad s \in \mathbb{R}^{n-1} \tag{2.4}$$

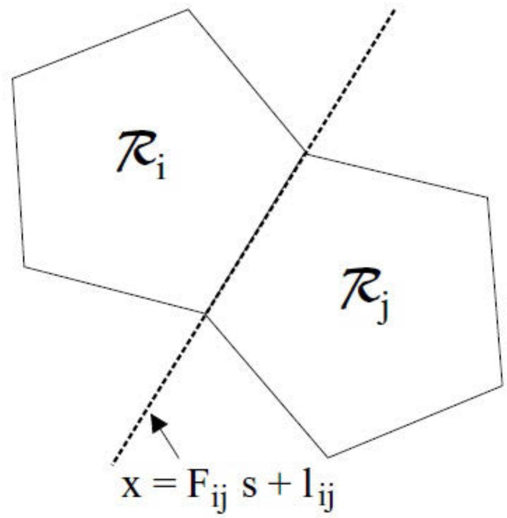

Figure 2.1: Two *level-1* neighboring cells and their boundary, [1]

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where F_{ij} is a full rank matrix whose columns span the null space of c_{ij}^T and $I_{ij} \in \mathbb{R}^n$ is a particular solution of $c_{ij}^T x = d_{ij}$ given by

$$I_{ij} = c_{ij}(c_{ij}^T c_{ij})^{-1} d_{ij}. \tag{2.5}$$

A slab region is defined as

$$\mathscr{R}_i = \{x \mid \beta_i < \lambda^T x < \beta_{i+1}\} \tag{2.6}$$

where $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$ and $\beta_i,\ \beta_{i+1} \in \mathbb{R},\ i = 1, \ldots, M$. The slab region $\mathscr{R}_i$ can also be cast as a degenerate ellipsoid

$$\mathscr{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \tag{2.7}$$

where

$$L_i = 2\lambda^T/(\beta_{i+1} - \beta_i), \tag{2.8}$$

$$l_i = -(\beta_{i+1} + \beta_i)/(\beta_{i+1} - \beta_i). \tag{2.9}$$

A PWA system whose regions are slabs is called a PWA slab system [30].

2.2.2 Schur Complement and Matrix Inversion Lemma

In this part, we will introduce some lemmas that will be frequently used in the rest of this thesis.

Lemma 2.2.1. *Schur Complement (negative semi-definite case): Consider a matrix* $X \in \mathbb{R}^n$ *partitioned as*

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \tag{2.10}$$

If

$$C \leq 0 \tag{2.11}$$

$$A - BC^{\dagger}B^T \leq 0 \tag{2.12}$$

$$B^T(I - CC^{\dagger}) = 0 \tag{2.13}$$

where $C^{\dagger}$ *is the pseudo inverse of matrix* C*, then, conditions (2.11), (2.12), and (2.13) are equivalent to*

$$X \leq 0. \tag{2.14}$$

Proof. See reference [68]. □

Remark 2.2.1. *Note that, if* C *in (2.11) is strictly less than zero, then* $C^{\dagger} = C^{-1}$ *and condition (2.13) is automatically verified.*

Lemma 2.2.2. *Schur Complement (negative definite case): Consider a matrix* $X \in \mathbb{R}^n$ *partitioned as*

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \tag{2.15}$$

If

$$C < 0 \tag{2.16}$$

$$A - BC^{-1}B^T < 0 \tag{2.17}$$

then, conditions (2.16) and (2.17) are equivalent to

$$X < 0. \tag{2.18}$$

Proof. See reference [68]. □

Lemma 2.2.3. *Matrix Inversion Lemma (Sherman-Woodbury-Morrison formula): For a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and matrices $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times n}$, the following equality is true,*

$$(A+BC)^{-1} = A^{-1} - A^{-1}B(I+CA^{-1}B)^{-1}CA^{-1} \tag{2.19}$$

where I is the identity matrix of appropriate dimension.

Proof. See references [68, 69]. □

2.3 Review of Piecewise-Affine Controller Synthesis

While the analysis of PWA control systems is a well-studied subject (see [28, 23, 17, 22, 29, 30, 31, 32, 33]), unfortunately, their controller synthesis has not received many research contributions due to the nonconvexity nature of the problem. In this section, based on references [20, 23, 57, 30, 59, 4], we will briefly review the available convex approaches towards PWA controller synthesis.

2.3.1 Approach From Rantzer and Johansson

Consider piecewise-affine systems of the form

$$\begin{aligned} \dot{x} &= A_i x + a_i + B_i u, \quad x(t) \in \mathscr{R}_i \\ y &= C_i x + c_i + D_i u \end{aligned} \tag{2.20}$$

where $\mathscr{R}_i$ was previously defined in Section 2.2. Rantzer and Johansson in [20] introduce the following notation

$$\begin{aligned}
\overline{\Lambda}_i &= \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \\
\overline{B}_i &= \begin{bmatrix} B_i \\ 0 \end{bmatrix} \\
\overline{C}_i &= \begin{bmatrix} C_i & c_i \end{bmatrix} \\
\overline{D}_i &= D_i \\
\overline{x} &= \begin{bmatrix} x \\ 1 \end{bmatrix}.
\end{aligned} \tag{2.21}$$

The cells are also assumed to be approximated by polyhedrons such that

$$\overline{E}_i\overline{x} \geq 0 \quad x \in \mathscr{R}_i \tag{2.22}$$

where

$$\overline{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix} \tag{2.23}$$

and $E_i \in \mathbb{R}^{n\times n}$ and $e_i \in \mathbb{R}^n$. The boundary of the cells then, will have the following form

$$\overline{F}_i\overline{x} = \overline{F}_j\overline{x} \quad x \in \mathscr{R}_i \cap \mathscr{R}_j \tag{2.24}$$

where

$$\overline{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix} \tag{2.25}$$

with $F_i \in \mathbb{R}^n$ and a scalar f_i.

Reference [20] considers the following general form of optimal control problem:

$$\begin{aligned}
min \qquad & L = \int_0^\infty l(x,u)dt \\
s.t. \qquad & \dot{x}(t) = f(x(t),u(t)) \\
& x(0) = x_0.
\end{aligned}$$

It is mentioned there, the optimal cost $V^*(x_0)$ for this problem can be characterized in terms of the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \inf_u \left(\frac{\partial V^*}{\partial x} f(x,u) + l(x,u) \right). \tag{2.26}$$

Reference [20] first, shows that every V satisfying the following inequality, is a lower bound on the optimal cost

$$0 \leq \frac{\partial V}{\partial x} f(x,u) + l(x,u), \quad \forall x, u. \tag{2.27}$$

Rantzer and Johansson then, show for the case L is piecewise-quadratic, maximization the lower bound in (2.27) implies a convex optimization problem in V with an infinite number of constraints parameterized by x and u. The following lemma from [20] shows how the maximization of the lower bound can be done numerically in terms of piecewise-quadratic cost function of the form

$$J(x_0,u) = \int_0^\infty (\bar{x}^T \overline{Q}_i \bar{x} + u^T R_i u) dt. \tag{2.28}$$

Lemma 2.3.1. [20] *(Lower Bound on Optimal Cost): Assume existence of symmetric matrices T and U_i, such that U_i have nonnegative entries, while $P_i = F_i^T T F_i$ and $\overline{P}_i = \overline{F}_i^T T \overline{F}_i$ satisfy*

$$\begin{bmatrix} P_i \Lambda_i + \Lambda_i^T P_i + Q_i - E_i^T U_i E_i & P_i B_i \\ B_i^T P_i & R_i \end{bmatrix} > 0 \qquad 0 \in \mathscr{R}_i \tag{2.29}$$

$$\begin{bmatrix} \overline{P}_i \overline{\Lambda}_i + \overline{\Lambda}_i^T \overline{P}_i + \overline{Q}_i - \overline{E}_i^T U_i \overline{E}_i & \overline{P}_i \overline{B}_i \\ \overline{B}_i^T \overline{P}_i & R_i \end{bmatrix} > 0 \qquad 0 \neq \mathscr{R}_i \tag{2.30}$$

Then, every continuous piecewise $\mathscr{C}^1$ trajectory $x(t)$ with $x(\infty) = 0$, $x(0) = x_0$ satisfies

$$J(x_0,u) \geq \sup_{T,U_i} \bar{x}_0^T \overline{P}_0 \bar{x}_0. \tag{2.31}$$

Proof. See reference [20]. □

Lemma 2.3.1 gives a lower bound on the minimal value of the cost function J. Upper bounds are obtained by studying specific control laws. Consider the control law obtained by the minimization

$$\min_u \left(\frac{\partial}{\partial x} f(x,u) + l(x,u) \right). \tag{2.32}$$

Reference [20] further shows that the exact minimization of the expression (2.32), can be done analytically in analogy with ordinary linear quadratic control, using the notation

$$L_i = -R_i^{-1} B_i^T P_i \tag{2.33}$$

$$\overline{L}_i = -R_i^{-1} \overline{B}_i^T \overline{P}_i \tag{2.34}$$

$$A_i = A_i + B_i L_i \tag{2.35}$$

$$\overline{A}_i = \overline{A}_i + \overline{B}_i \overline{L}_i \tag{2.36}$$

$$Q_i = Q_i + P_i B_i R_i^{-1} B_i^T P_i \tag{2.37}$$

$$\overline{Q}_i = \overline{Q}_i + \overline{P}_i \overline{B}_i R_i^{-1} \overline{B}_i^T \overline{P}_i. \tag{2.38}$$

The minimizing control law can then be written as

$$u(t) = \overline{L}_i \bar{x}, \quad x \in \mathscr{R}_i. \tag{2.39}$$

Remark 2.3.1. *As Ranzter and Johansson also point out in [20], solving the matrix inequalities in Lemma 2.3.1, does not guarantee that the control law minimizing (2.32) is even stabilizing. It is also mentioned that " This control law is simple but may be discontinuous and give rise to sliding modes", and by "sliding mode" they meant divergent sliding mode which makes the system unstable.*

2.3.2 Approach From Hassibi and Boyd

Another Convex approach towards controller synthesis of piecewise-affine systems, was introduced by Hassibi and Boyd [23]. The following PWA system was considered in that work

$$\dot{x} = A_i x + a_i + B_i^{(1)} w + B_i^{(2)} u \tag{2.40}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and i as before, implies $x(t) \in \mathscr{R}_i$ where, in this work, reference [23], assumes that the region $\mathscr{R}_i$ can be outer approximated by a union of (possibly degenerate) ellipsoids, ε_{ij}. In other words, matrices L_{ij} and l_{ij} exist such that

$$\mathscr{R}_i \subseteq \bigcup \varepsilon_{ij} \quad \text{where} \quad \varepsilon_{ij} = \{x \mid \left\|L_{ij}x + l_{ij}\right\| < 1\}. \tag{2.41}$$

Using control signal of the form $u = K_i x$, the closed-loop state equations (2.40) become

$$\dot{x} = (A_i + B_i^{(2)} K_i)x + a_i + B_i^{(1)} w. \tag{2.42}$$

Now considering a candidate quadratic Lyapunov function of the form $V = x^T P x$ and introducing the new variables $Y_i = K_i Q$ where $Q = P^{-1}$, reference [23] proposes the following lemma.

Lemma 2.3.2. [23] *If there exist variables Q, Y_i and μ_{ij} satisfying*

$$Q > 0 \tag{2.43}$$

$$\mu_{ij} < 0 \tag{2.44}$$

$$\begin{bmatrix} \begin{pmatrix} A_i Q + Q A_i^T + \mu_{ij} a_i a_i^T \\ + B_i^{(2)} Y_i + Y_i^T B_i^{(2)T} \end{pmatrix} & \mu_{ij} a_i l_{ij}^T + Q L_{ij}^T \\ (\mu_{ij} a_i l_{ij}^T + Q L_{ij}^T)^T & -\mu_{ij}(I - l_{ij} l_{ij}^T) \end{bmatrix} < 0 \tag{2.45}$$

then, the piecewise-linear state feedback control command $u = K_i x$ stabilizes (2.40) with $K_i = Y_i Q^{-1}$.

Proof. See reference [23]. □

The following remark also is taken directly from reference ([23]).

Remark 2.3.2. [23]*"Another natural choice of input command would be one that is affine in the state x, i.e., $u = K_i(x)x + k_i(x)$. However, it doesn't seem that the condition for stabilizability using this type of input command can be cast as an LMI."*

2.3.3 Approach From Pavlov and Van de Wouw

For a class of PWA control systems Pavlov and Van de Wouw in reference [57] design state feedback controllers that make the closed-loop system input-to-state convergent. The conditions for such controller design are formulated in terms of LMIs.

Consider the following class of PWA system

$$\begin{aligned} \dot{x} &= A_i x + b_i + Bu + Dw \quad \text{if} x \in \mathscr{R}_i \\ y &= Cx + Ew \end{aligned} \tag{2.46}$$

with $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^k$, external input $w \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$. Input u corresponds to the feedback part of the controller. The input w includes external time-dependent inputs such as, for example, disturbances and feedforward control signals.

The following lemma form [57] provides conditions under which there exists a state feedback rendering the corresponding closed-loop system input-to-state convergent (see reference [57] for the definition of the input-to-state convergence).

Lemma 2.3.3. [57] *Consider the system (2.46). Suppose the right-hand side of (2.46) is continuous and the LMI*

$$P = P^T > 0 \tag{2.47}$$

$$A_i P + P A_i^T + BY + Y^T B^T < 0 \tag{2.48}$$

is feasible. Then the system (2.46) in closed-loop with the controller $u = K(x+v)$ with $K := YP^{-1}$ and (v, w) as inputs is input-to-state convergent.

Proof. See reference [57]. □

Pavlov and Van de Wouw also proposed an approach to design an observer for system (2.46). Using Lemma 2.3.3 and the designed observer (which will be also obtained by LMIs), they introduce another set of convex inequalities which if they are feasible, an output feedback can also be obtained. This output feedback makes the system (2.46) input-to-state convergent.

Remark 2.3.3. *Note that, in the considered PWA class (2.46), matrix B must be constant for all regions. Moreover, the designed controller is in linear state feedback form.*

2.3.4 Approach From Rodrigues and Boyd

Contrary to [23], Rodrigues and Boyd in reference [30], consider piecewise-affine state feedback controllers rather than piecewise-linear ones. The PWA control input is of the form

$$u = K_i x(t) + k_i, \quad x(t) \in \mathscr{R}_i. \tag{2.49}$$

Using (2.49) and (2.1) the closed-loop dynamics of a PWA system will be

$$\dot{x}(t) = \overline{A}_i x(t) + \overline{a}_i, \quad x(t) \in \mathscr{R}_i, \tag{2.50}$$

where

$$\overline{A}_i = A_i + B_i K_i, \tag{2.51}$$

$$\overline{a}_i = a_i + B_i k_i. \tag{2.52}$$

The following lemma gives sufficient conditions for asymptotic stability of closed-loop PWA slab systems.

Lemma 2.3.4. [30] *Consider the PWA slab system (2.1). Given* $\alpha > 0$*, if there exist* $Q = Q^T > 0$ *and* $\mu_i > 0$ *satisfying*

$$\overline{A}_i Q + Q\overline{A}_i^T + \alpha Q < 0 \qquad \text{if } 0 \in \overline{\mathscr{R}}_i, \tag{2.53}$$

$$\begin{bmatrix} \overline{A}_i Q + Q\overline{A}_i^T + \alpha Q - \mu_i \overline{a}_i \overline{a}_i^T & -\mu_i \overline{a}_i l_i^T + Q L_i^T \\ -\mu_i l_i \overline{a}_i^T + L_i Q & \mu_i (1 - l_i^2) \end{bmatrix} < 0 \qquad \text{otherwise} \tag{2.54}$$

where L_i*,* l_i*,* $\overline{A}_i$ *and* $\overline{a}_i$ *were defined in (2.8), (2.9), (2.51) and (2.52), respectively, then the origin is an exponentially stable equilibrium point.*

Proof. See reference [30]. □

Introducing new variables $Y_i = K_i Q$ and substituting (2.51) in inequalities (2.53) and (2.54), Rodrigues and Boyd in [30] introduce the following problem.

Definition 2.3.1. [30] *The piecewise-affine state feedback synthesis problem is: for fixed* $\alpha > 0$

$$\begin{aligned}
&\textit{find} && Q, Y_i, k_i, \mu_i \\
&\textit{s.t.} && Q = Q^T > 0, \quad \mu_i > 0, \\
& && A_i Q + Q A_i^T + B_i Y_i + Y_i^T B_i^T + \alpha Q < 0 \qquad\qquad\qquad \textit{if } 0 \in \overline{\mathscr{R}}_i \\
& && \begin{bmatrix} A_i Q + Q A_i^T + B_i Y_i + Y_i^T B_i^T + \alpha Q - \mu_i \bar{a}_i \bar{a}_i^T & -\mu_i \bar{a}_i l_i^T + Q L_i^T \\ -\mu_i l_i \bar{a}_i^T + L_i Q & \mu_i (1 - l_i^2) \end{bmatrix} < 0 \quad \textit{otherwise}
\end{aligned}$$

It is then mentioned that: *"In fact, it is clear from (2.54) that this synthesis problem cannot be formulated as one convex program because (2.54) is not an LMI if the parameters $k_i, i = 1, \ldots, M$ are unknown."* However, it is shown there, how the piecewise-affine state feedback synthesis problem for piecewise-affine slab systems using a globally quadratic Lyapunov function can be relaxed and solved to a point near the global optimum by a finite set of LMIs. Reference [30] presents three algorithms to approximately solve this problem (See [30] for more details).

Remark 2.3.4. *Note that, the constraint (2.54) in Lemma 2.3.4 is nonconvex. The non-convexity of BMIs (2.54) is due to the existence of the term*

$$-\mu_i \bar{a}_i \bar{a}_i^T, \tag{2.55}$$

which includes a product of unknown gains k_i. Therefore, controller synthesis for PWA slab systems is a non-convex problem.

2.3.5 Relaxation Approach From Samadi and Rodrigues

Another important attempt towards convex formulation of PWA controller synthesis problem was the method proposed in [4]. Reference [4] shows that one can avoid solving the

BMIs (2.54) by ignoring the negative definite term (2.55), which is a convex relaxation. More precisely, the following lemma taken from [4] gives sufficient conditions for asymptotic stability of the closed-loop PWA slab system (2.1) using the relaxation method.

Lemma 2.3.5. [4] *Consider the PWA slab system (2.1) and the PWA state feedback (2.49). Given $\varepsilon > 0$, if there exist $P = P^T > 0$ and $\zeta_i > 0$ satisfying*

$$\overline{A}_i P + P\overline{A}_i^T + \varepsilon P < 0 \qquad \text{if } 0 \in \overline{\mathscr{R}}_i, \tag{2.56}$$

$$\Gamma_i = \begin{bmatrix} \overline{A}_i P + P\overline{A}_i^T + \varepsilon P & -\zeta_i \overline{a}_i l_i^T + P L_i^T \\ -\zeta_i l_i \overline{a}_i^T + L_i P & \zeta_i (1 - l_i^2) \end{bmatrix} < 0 \qquad \text{otherwise} \tag{2.57}$$

where L_i, l_i, $\overline{A}_i$ and $\overline{a}_i$ were defined in (2.8), (2.9), (2.51) and (2.52), respectively, then the origin is an exponentially stable equilibrium point.

Proof. See reference [4]. □

Remark 2.3.5. *Note that, the conditions of Lemma 2.3.4 are sufficient conditions and therefore, conservatism has been already introduced to the problem. Reference [4] adds more conservativeness by ignoring the negative definite term (2.55). Unfortunately, the resulting conditions may lead to infeasibility.*

2.3.6 Backstepping Approach From Samadi and Rodrigues

Another convex approach in the literature was the work done by Samadi and Rodrigues in reference [59]. They address backstepping controller synthesis for a class of piecewise-affine systems. Consider PWA systems in the following strict feedback form

$$\begin{aligned}
\dot{x}_1 &= A_{i_1}^{(1)} x_1 + a_{i_1}^{(1)} + B_{i_1}^{(1)} x_2, && \text{for} \quad E_{i_1}^{(1)} x_1 + e_{i_1}^{(1)} > 0 \\
\dot{x}_2 &= A_{i_2}^{(2)} X_2 + a_{i_2}^{(2)} + B_{i_2}^{(2)} x_3, && \text{for} \quad E_{i_2}^{(2)} X_2 + e_{i_2}^{(2)} > 0 \\
&\vdots \\
\dot{x}_n &= A_{i_n}^{(n)} X_n + a_{i_n}^{(n)} + B_{i_n}^{(n)} u, && \text{for} \quad E_{i_n}^{(n)} X_n + e_{i_n}^{(n)} > 0
\end{aligned} \tag{2.58}$$

where $x_j \in \mathbb{R}_j^n$, $i_j = 1, \ldots, M_j$ and $X_j = \begin{bmatrix} x_1 & \ldots & x_j \end{bmatrix}^T$ for $j = 1, \ldots, n$.

The piecewise-affine controllers design procedure for this class of PWA systems can be discussed for two cases. The first case consists of the construction of a sum of squares (SOS) Lyapunov function for PWA systems with discontinuous vector fields. The second case is the construction of a piecewise polynomial Lyapunov function for PWA systems with continuous vector fields. Both cases were addressed in reference [59] and due to the similarity of their controllers design process, we will review only the first case here (see reference [59] for the second case design procedure).

Samadi and Rodrigues [59], propose the following controller design procedure for PWA system (2.58):

To design a PWA controller for (2.58), we start from the following subsystem

$$\dot{x}_1 = A_{i_1}^{(1)} x_1 + a_{i_1}^{(1)} + B_{i_1}^{(1)} x_2, \qquad \text{for} \quad E_{i_1}^{(1)} x_1 + e_{i_1}^{(1)} > 0 \tag{2.59}$$

with $i_1 = 1, \ldots, M_1$. Then, it is assumed that there exist an SOS Lyapunov function $V^{(1)}(x_1)$ and an affine controller $x_2 = \gamma^{(1)}(x_1) = K^{(1)} x_1 + k^{(1)}$ such that

$$-\nabla V^{(1)}.(A_{i_1}^{(1)} x_1 + a_{i_1}^{(1)} + B_{i_1}^{(1)} \gamma^{(1)}(x_1)) - \Gamma_{i_1}^{(1)}(x_1).(E_{i_1}^{(1)} x_1 + e_{i_1}^{(1)}) - \alpha V^1 \quad \text{is SOS} \tag{2.60}$$

where $\alpha > 0$ and $\Gamma_{i_1}^{(1)}(x_1)$ is an SOS vector function.

The second step is to design an affine controller for the following subsystem

$$\begin{aligned} \dot{x}_1 &= A_{i_1}^{(1)} x_1 + a_{i_1}^{(1)} + B_{i_1}^{(1)} x_2, \qquad \text{for} \quad E_{i_1}^{(1)} x_1 + e_{i_1}^{(1)} > 0 \\ \dot{x}_2 &= A_{i_2}^{(2)} X_2 + a_{i_2}^{(2)} + B_{i_2}^{(2)} x_3, \qquad \text{for} \quad E_{i_2}^{(2)} X_2 + e_{i_2}^{(2)} > 0 \end{aligned} \tag{2.61}$$

Considering the following Lyapunov function

$$V^{(2)}(X_2) = V^{(1)}(x_1) + \frac{1}{2}(x_2 - \gamma^{(1)}(x_1)).(x_2 - \gamma^{(1)}(x_1)), \tag{2.62}$$

reference [59] shows that the synthesis problem can be formulated as the following SOS program.

$$
\begin{aligned}
\text{Find} \quad & x_3 = \gamma^{(2)}(X_2) \\
\text{s.t.} \quad & -\nabla_{x_1} V^{(2)}.(A_{i_1}^{(1)} x_1 + a_{i_1}^{(1)} + B_{i_1}^{(1)} x_2) \\
& -\nabla_{x_2} V^{(2)}.(A_{i_2}^{(2)} X_2 + a_{i_2}^{(2)} + B_{i_2}^{(2)} x_3) \\
& -\Gamma_{i_1}^{(1)}(x_1).(E_{i_1}^{(1)} x_1 + e_{i_1}^{(1)}) \\
& -\Gamma_{i_2}^{(2)}(X_2).(E_{i_2}^{(2)} X_2 + e_{i_2}^{(2)}) - \alpha V^{(2)} \\
& \text{is SOS,} \\
& -\Gamma_{i_1}^{(1)}(x_1) \quad \text{and} \quad -\Gamma_{i_2}^{(2)}(X_2) \quad \text{are SOS}
\end{aligned}
\tag{2.63}
$$

where $i_1 = 1, \ldots, M_1$, $i_2 = 1, \ldots, M_2$ and

$$\gamma^{(2)}(X_2) = K^{(2)} X_2 + k^{(2)}. \tag{2.64}$$

If this SOS program is feasible then the procedure can be repeated for the next step. Assume that all SOS programs in the backstepping procedure are feasible, the final controller $u = \gamma^{(n)}(X_n)$ will not be used to construct the SOS Lyapunov function and reference [59] shows one can setup an SOS program to find a PWA control $\gamma^{(n)}(X_n)$.

Remark 2.3.6. *Note that, reference [59] does not formulate the controller synthesis of PWA systems with PWA controllers as linear matrix inequalities (LMIs), however, the problem is cast as a sum of squares (SOS) program which still is a convex problem.*

Remark 2.3.7. *Although the backstepping method proposed by reference [59] leads to a convex problem, one cannot control the way in which the trajectories converge to the origin.*

Motivated by the drawbacks of existing methods, the next chapters present a convex formulation of the synthesis problem using an invariant set approach.

2.4 Summary

This chapter of the thesis, based on the previous work from references [17, 22, 29, 30], briefly reviews PWA systems. The available convex approaches towards their controller synthesis are also reviewed in this chapter.

Rantzer and Johansson in [20] propose a convex approach towards PWA controller synthesis. However, it is not guaranteed that the control law is stabilizing. Hassibi and Boyd [23] show that sufficient conditions for quadratic stabilization using PWL state feedback for PWA slab systems can be cast as a convex optimization problem. However, if affine terms are included in the controller, the convex structure is destroyed. Under certain additional assumptions, Rodrigues and Boyd [30] show that one can develop algorithms to approximately solve these non-convex problems with optimality guarantees. Pavlov and Van de Wouw in [57] also formulate the PWA controller synthesis as LMIs, however, a linear feedback control law must be used.

Among all the available convex approaches towards PWA controller synthesis (references [20, 23, 30, 33, 4, 59]), the methods proposed in references [4, 59] are the only ones that can formulate this problem as a convex optimization/feasibility program when the controllers are in the PWA state feedback form. Samadi and Rodrigues [59] developed a backstepping approach for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin. The method proposed in reference [4] shows that one can avoid solving the bilinear matrix inequalities (BMIs) proposed in reference [30] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility.

The limitations from the previous sections motivate the work that will be presented in the next chapters of this thesis.

Chapter 3

A Convex Formulation of Piecewise-Affine Controller Synthesis

3.1 Introduction

As it was already mentioned in the previous chapters, several promising methods have emerged for analysis and synthesis of PWA control systems such as those proposed in [28, 23, 17, 22, 29, 30, 31, 32, 33] and references therein. Unfortunately, synthesis of PWA controllers naturally leads to non-convex problems. Solving these problems is therefore a non-trivial task. To the best of our knowledge, the methods proposed in [4, 59] are the only ones that can formulate PWA state feedback as a set of LMIs. The method in [4] shows that one can avoid solving the Bilinear Matrix Inequalities (BMIs) proposed in [30] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility. In [59] a backstepping approach is developed for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin. This limitation motivates the work that will be presented here. It will be shown that the synthesis procedure proposed in this thesis leads to a convex problem in a reduced state space and the closed-loop trajectories converge to the origin along a desired direction. We have considered the pitch control of a helicopter model, as an application of our method. Simulation results will show how the method proposed in this chapter can efficiently damp the pitch motion. Finally, through a numerical example the backstepping method [59] will be compared to the proposed method. It will be shown that while using backstepping method it is not possible to control the way in which the trajectories converge to the origin, the proposed approach provides us with a surface which trajectories will slide to the origin along that surface.

3.2 Preliminaries

Recalling from Chapter 2, the dynamics of a PWA system can be written as

$$\dot{x}(t) = A_i x(t) + a_i + B_i u(t), \quad x(t) \in \mathscr{R}_i \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ the control input and a forward invariant set $\mathscr{X} \subset \mathbb{R}^n$ is partitioned into M polytopic cells $\mathscr{R}_i,\ i \in \mathscr{I} = \{1, \dots, M\}$ such that $\cup_{i=1}^{M} \overline{\mathscr{R}}_i = \mathscr{X}$, $\mathscr{R}_i \cap \mathscr{R}_j = \emptyset$ where $\overline{\mathscr{R}}_i$ denotes the closure of $\mathscr{R}_i$ (see [22] for generating such partition).

A slab region is defined as

$$\mathscr{R}_i = \{x \mid \beta_i < \lambda^T x < \beta_{i+1}\} \tag{3.2}$$

where $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$ and $\beta_i,\ \beta_{i+1} \in \mathbb{R},\ i = 1, \dots, M$. The slab region $\mathscr{R}_i$ can also be cast as a degenerate ellipsoid

$$\mathscr{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \tag{3.3}$$

where

$$L_i = 2\lambda^T / (\beta_{i+1} - \beta_i), \tag{3.4}$$

$$l_i = -(\beta_{i+1} + \beta_i)/(\beta_{i+1} - \beta_i). \tag{3.5}$$

A PWA system whose regions are slabs is called a PWA slab system [30]. Using a PWA control input of the form

$$u = K_i x(t) + k_i, \quad x(t) \in \mathscr{R}_i \tag{3.6}$$

into system (3.1) yields the closed-loop dynamics

$$\dot{x}(t) = \overline{A}_i x(t) + \overline{a}_i, \quad x(t) \in \mathscr{R}_i, \tag{3.7}$$

where

$$\overline{A}_i = A_i + B_i K_i, \quad \overline{a}_i = a_i + B_i k_i. \tag{3.8}$$

The following lemma from Chapter 2, gives sufficient conditions for asymptotic stability of closed-loop PWA slab systems.

Lemma 3.2.1. [30] *Consider the PWA slab system (3.1). Given* $\alpha > 0$*, if there exist* $Q = Q^T > 0$ *and* $\mu_i > 0$ *satisfying*

$$\Omega_{i0} + \alpha Q < 0 \qquad \text{if } 0 \in \overline{\mathscr{R}}_i, \tag{3.9}$$

$$\Omega_i = \begin{bmatrix} \Omega_{i1} & \Omega_{i2} \\ \Omega_{i2}^T & \Omega_{i4} \end{bmatrix} < 0 \qquad \text{otherwise} \tag{3.10}$$

where

$$\begin{aligned}
\Omega_{i0} &= \overline{A}_i Q + Q\overline{A}_i^T \\
\Omega_{i1} &= \overline{A}_i Q + Q\overline{A}_i^T + \alpha Q - \mu_i \overline{a}_i \overline{a}_i^T \\
\Omega_{i2} &= -\mu_i \overline{a}_i l_i^T + Q L_i^T \\
\Omega_{i4} &= \mu_i (1 - l_i^2)
\end{aligned}$$

with L_i and l_i defined in (3.4) and (3.5), respectively, then the origin is an exponentially stable equilibrium point.

Proof. See reference [30]. □

Note that the constraint (3.10) is nonconvex. The nonconvexity of BMIs (3.10) is due to the existence of the term

$$-\mu_i \overline{a}_i \overline{a}_i^T \tag{3.11}$$

which includes a product of unknown gains k_i. Therefore controller synthesis for PWA slab systems is a non-convex problem. The method proposed in [4], shows that one can avoid solving the BMIs (3.10) by ignoring the negative definite term (3.11), which is a convex relaxation. Note that the conditions of Lemma 3.2.1 are sufficient conditions and therefore, conservatism has been already introduced to the problem. Reference [4] adds more conservativeness by ignoring the negative definite term (3.11). Unfortunately, the resulting conditions may lead to infeasibility. Motivated by the drawbacks of existing methods, the next section presents a convex formulation of the synthesis problem using an invariant set approach.

3.3 Controller Synthesis

Consider the following class of PWA slab systems

$$\dot{x}(t) = A_i x(t) + a_i + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u(t), \quad x(t) \in \mathscr{R}_i \tag{3.12}$$

where $u \in \mathbb{R}^p$, $B_{2_i} \in \mathbb{R}^{m \times p}$ and $m \in \mathscr{M} = \{1, \cdots, n-1\}$, $m \geq p$.

Remark 3.3.1. *Note that, the equations of motion of several physical systems of interest come naturally in this form, in particular if one writes the equations of motion of mechanical systems divided into the kinematics (without input forcing terms) and the dynamics (with input forcing terms). Moreover, the introduced PWA class is not limited to single-input single-output (SISO) systems.*

We can rewrite equations (3.12) in the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11_i} & A_{12_i} \\ A_{21_i} & A_{22_i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{1_i} \\ a_{2_i} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u, \quad x(t) \in \mathscr{R}_i \tag{3.13}$$

where $x_1 \in \mathbb{R}^{n-m}$, $x_2 \in \mathbb{R}^m$. Assume further that in this class of PWA systems, the slab regions are only functions of x_1. Therefore, the definition of slab regions (3.3) can be rewritten as

$$\mathscr{R}_i = \left\{ x \mid \|L_i x + l_i\| = \left\| \begin{bmatrix} L_{1i} & 0 \end{bmatrix} x + l_i \right\| = \|L_{1i} x_1 + l_i\| < 1 \right\} \tag{3.14}$$

where $L_{1i}^T \in \mathbb{R}^{n-m}$. This chapter proposes a new method to formulate PWA controller synthesis for system (3.13) as a convex feasibility problem. The main result is presented in the next theorem.

Theorem 3.3.1. *Assuming that either B_{2_i} is invertible or $B_{2_i} = B_2$ is full rank, the PWA controller*

$$\begin{aligned} u = &-(S_2 B_{2_i})^{-1}[S_1(A_{11_i}x_1 + A_{12_i}x_2 + a_{1_i}) \\ &+ S_2(A_{21_i}x_1 + A_{22_i}x_2 + a_{2_i}) \\ &+ \gamma \frac{S_1x_1 + S_2x_2}{\|S_1x_1 + S_2x_2\|}], \end{aligned} \tag{3.15}$$

for $x \in \mathscr{R}_i,\ i = 1,\ldots,M$, exponentially stabilizes system (3.13) defined in a forward invariant set $\mathscr{X}$ if given $\gamma > 0$ and $\alpha > 0$, there exist $Q = Q^T > 0,\ \mu_i > 0$, and $Y = S_1 Q$, satisfying the following LMIs

$$\omega_{i0} + \alpha Q < 0 \qquad \text{if } 0 \in \overline{\mathscr{R}}_i, \tag{3.16}$$

$$\omega_i = \begin{bmatrix} \omega_{i1} & \omega_{i2} \\ \omega_{i2}^T & \omega_{i4} \end{bmatrix} < 0 \qquad \text{otherwise} \tag{3.17}$$

$$\begin{aligned} \omega_{i0} &= A_{11_i}Q + QA_{11_i}^T - A_{12_i}S_2^\dagger Y - Y^T(S_2^\dagger)^T A_{12_i}^T \\ \omega_{i1} &= A_{11_i}Q + QA_{11_i}^T - A_{12_i}S_2^\dagger Y - Y^T(S_2^\dagger)^T A_{12_i}^T \\ &\quad + \alpha Q - \mu_i a_{1_i} a_{1_i}^T \\ \omega_{i2} &= -\mu_i a_{1_i} l_i^T + QL_{1i}^T \\ \omega_{i4} &= \mu_i(1 - l_i^2) \end{aligned}$$

where

$$S_2^\dagger = S_2^T(S_2S_2^T)^{-1} \tag{3.18}$$

Proof. Consider a surface of the form

$$\sigma(x) = Sx = 0 \tag{3.19}$$

where

$$S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{3.20}$$

with $S_1 \in \mathbb{R}^{p\times(n-m)}$ and $S_2 \in \mathbb{R}^{p\times m}$, in which P is the number of the inputs to (3.13). In order to make $\sigma(x) = 0$ an attractive invariant set, we define a candidate Lyapunov function of the form

$$V(\sigma(x)) = \frac{1}{2}\sigma^T(x)\sigma(x). \tag{3.21}$$

Note that, although $V(\sigma(x))$ is implicitly based on $x(t)$, it is not a Lyapunov function for x, but it is rather a Lyapunov function for $\sigma(x)$. As a function of $\sigma(x)$, $V(\sigma(x))$ is obviously positive definite because it is a norm. In order to have finite-time convergence to $\sigma(x) = 0$, according to [70] and [71] one needs to ensure

$$\dot{V}(\sigma(x)) \leq -\mu\,\|\sigma(x)\| \tag{3.22}$$

where $\mu > 0$. Note that, the Lie derivative of the Lyapunov function in (3.21) is

$$\dot{V}(\sigma(x)) = \frac{\partial V(\sigma(x))}{\partial \sigma(x)}\dot{\sigma}(x) = \sigma^T(x)\dot{\sigma}(x). \tag{3.23}$$

We design $\sigma(x)$ such that

$$\dot{\sigma}(x) = -\gamma\left(\frac{\sigma(x)}{\|\sigma(x)\|}\right) \tag{3.24}$$

with $\gamma \geq \mu > 0$, the time rate of change of the Lyapunov function in (3.21) will be

$$\begin{aligned} \dot{V}(\sigma(x)) &= -\gamma\sigma^T(x)\left(\frac{\sigma(x)}{\|\sigma(x)\|}\right) \\ &= -\gamma\|\sigma(x)\| \leq -\mu\,\|\sigma(x)\|, \end{aligned} \tag{3.25}$$

which verifies (3.22). Using (3.13), (3.19) and (3.20) one can write

$$\begin{aligned} \dot{\sigma}(x) = S\dot{x} &= S_1(A_{11_i}x_1 + A_{12_i}x_2 + a_{1_i}) \\ &+ S_2(A_{21_i}x_1 + A_{22_i}x_2 + a_{2_i}) + (S_2B_{2_i})u. \end{aligned} \tag{3.26}$$

Since B_{2_i} is either invertible or constant for all $i \in \mathscr{I}$ and full rank, $S_2B_{2_i}$ is invertible (for example with the choice $S_2 = B_2^T$ when $B_{2_i} = B_2$), and replacing the PWA control law (3.15) into (3.26) ensures that (3.25) is verified. Therefore the target surface $\sigma(x) = 0$ is made an attractive invariant set. We now show that the trajectories converge to this target surface in finite time. Observe that (3.25) is equivalent to

$$\dot{V}(\sigma(x)) = -\gamma\sqrt{2}V^{\frac{1}{2}}(\sigma(x)) \tag{3.27}$$

for the Lyapunov function defined in (3.21). This is a differential equation. Assuming $V(\sigma(x(t_0)))$ as the initial condition, the solution to (3.27) can be found as

$$V^{\frac{1}{2}}(\sigma(x(t))) = V^{\frac{1}{2}}(\sigma(x(t_0))) - \frac{\sqrt{2}\gamma}{2}(t - t_0). \tag{3.28}$$

One now can see that

$$\exists t_c \in \mathbb{R}, \quad \text{such that} \quad V(\sigma(x(t_c))) = 0 \tag{3.29}$$

where $t_c \geq t_0$ is the finite time of convergence to the surface. In fact, replacing $V(\sigma(x(t_c))) = 0$ in (3.28) yields

$$t_c = \sqrt{2}\gamma^{-1}V^{\frac{1}{2}}(\sigma(x(t_0))) + t_0. \tag{3.30}$$

Furthermore (3.27) and (3.29) imply that

$$\dot{V}(\sigma(x(t_c))) = -\gamma\sqrt{2}V^{\frac{1}{2}}(\sigma(x(t_c))) = 0, \tag{3.31}$$

which yields

$$V^{\frac{1}{2}}(\sigma(x(t))) = 0, \qquad \forall t > tc \tag{3.32}$$

and therefore

$$V(\sigma(x(t))) = 0, \qquad \forall t \geq tc. \tag{3.33}$$

Since the trajectories converge in finite time to the surface $\sigma(x) = 0$ and remain on that surface for all future times, using (3.19) and (3.20), for $t \geq t_c$ we can write

$$S_1 x_1 + S_2 x_2 = 0. \tag{3.34}$$

Assuming

$$x_2 = S_2^T Z \tag{3.35}$$

where $Z \in \mathbb{R}^p$, we can rewrite (3.34) as

$$Z = -(S_2 S_2^T)^{-1} S_1 x_1 \tag{3.36}$$

Hence

$$x_2 = -S_2^{\dagger} S_1 x_1 \tag{3.37}$$

where

$$S_2^{\dagger} = S_2^T (S_2 S_2^T)^{-1} \tag{3.38}$$

is the pseudo-inverse of the matrix S_2. Therefore, using (3.13) and (3.37) we can rewrite the dynamics of the PWA system (3.13) for $t \geq t_c$ as

$$x_2 = -S_2^{\dagger} S_1 x_1 \tag{3.39}$$

$$\dot{x}_1 = (A_{11_i} - A_{12_i} S_2^{\dagger} S_1) x_1 + a_{1_i}, \quad x \in \mathscr{R}_i. \tag{3.40}$$

Due to (3.39), if $x_1(t)$ exponentially converges to the origin, then $x_2(t)$ will also exponentially converge to the origin. Therefore, exponential stability of the reduced order system (3.40) ensures that the PWA slab system (3.13) is exponentially stable under the control law (3.15). However, exponential stability of the reduced order system (3.40) is guaranteed if the LMIs (3.16)–(3.17) hold, based on Lemma 3.2.1 using

$$\overline{A}_i := (A_{11_i} - A_{12_i} S_2^{\dagger} S_1) \tag{3.41}$$

$$\overline{a}_i := a_{1_i} \tag{3.42}$$

This finishes the proof □

Remark 3.3.2. *As one can see, Theorem 3.3.1 results in a set of LMIs. Moreover no relaxation is used in the proof. In fact since (3.42) is always a constant vector (in each region), the term (3.11) is known, which makes the problem convex.*

Remark 3.3.3. *Theorem 3.3.1 reduces the complexity of the LMIs that must be solved because transforming the closed-loop stability problem for system (3.13) into a stability problem for system (3.40) makes the dimension of the closed-loop state space smaller than*

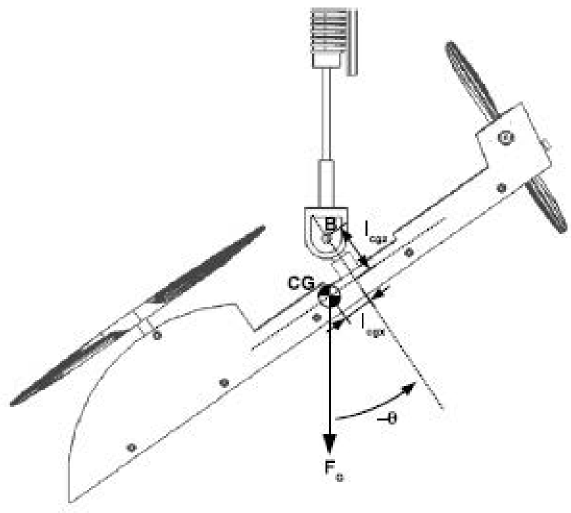

Figure 3.1: Pitch Model of The Helicopter From [2]

the dimension of the open-loop state-space. The control methods in [4, 59] do not perform this transformation and therefore are more complex because of two reasons: i) they lead to BMIs and ii) the dimension of the state space is larger.

3.4 Application and Numerical Example

3.4.1 Application to a Helicopter Pitch Model

A two degree of freedom model of a helicopter, taken from [2], will be considered as an application in this section. In this example, a simplified version of the pitch model of the helicopter (Figure 3.1) is considered. This model is described by the following equations:

$$\dot{x}_1 = x_2 \tag{3.43}$$

$$\dot{x}_2 = \frac{1}{I_{yy}}(-ml_{cgx}g\cos(x_1) - ml_{cgz}g\sin(x_1) - F_{vM}x_2 + u) \tag{3.44}$$

where x_1 and x_2 represent the pitch angle and pitch rate, respectively. The values of the parameters can be found in reference [2]. First, the PWA approximation $\hat{f}(x_1)$ of

$$f(x_1) = -ml_{cgx}g\cos(x_1) - ml_{cgz}g\sin(x_1) \tag{3.45}$$

is computed based on a uniform grid for x_1 (see reference [22]). A PWA model is then obtained by replacing $f(x_1)$ by $\hat{f}(x_1)$ in (3.44). The PWA model is described by the following equations:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5.3058 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 22.2968 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_1 \tag{3.46}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8.1786 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_2 \tag{3.47}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10.5751 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_3 \tag{3.48}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1.9210 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -12.4780 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_4 \tag{3.49}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 10.7980 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 29.2108 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_5 \tag{3.50}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and regions are defined as

$$\begin{cases} \mathscr{R}_1 & = \left\{x \in \mathbb{R}^2 \mid -\pi < x_1 < -\frac{3\pi}{5}\right\} \\ \mathscr{R}_2 & = \left\{x \in \mathbb{R}^2 \mid -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}\right\} \\ \mathscr{R}_3 & = \left\{x \in \mathbb{R}^2 \mid -\frac{\pi}{5} < x_1 < \frac{\pi}{5}\right\} \\ \mathscr{R}_4 & = \left\{x \in \mathbb{R}^2 \mid \frac{\pi}{5} < x_1 < \frac{3\pi}{5}\right\} \\ \mathscr{R}_5 & = \left\{x \in \mathbb{R}^2 \mid \frac{3\pi}{5} < x_1 < \pi\right\}. \end{cases}$$

Note that, this approximation belongs to the class of PWA systems defined in (3.12). To design the controllers, we first define

$$\gamma = 0.5 \qquad \alpha = 0.5, \tag{3.51}$$

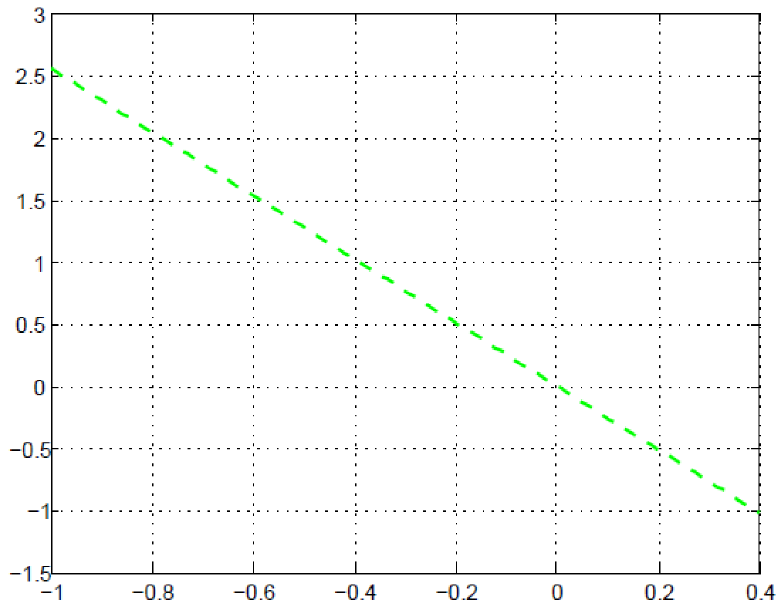

Figure 3.2: Designed sliding surface

and then we assign

$$S_2 = B_{2_i}^{-1} = 0.0283. \tag{3.52}$$

Using (3.51), (3.52) and solving LMIs (3.16) and (3.17), S_1 is obtained as

$$S_1 = 0.0724. \tag{3.53}$$

Therefore, the sliding surface defined in (3.19) for this problem is

$$\sigma(x) = \begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x. \tag{3.54}$$

Figure 3.2 shows this sliding surface. After computing $\sigma(x)$ and using (3.15), we are able

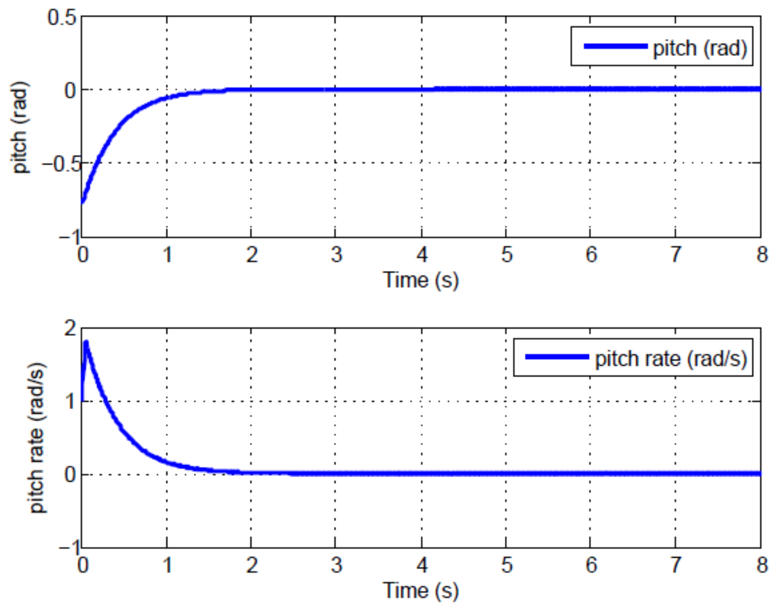

Figure 3.3: Simulation results for the closed-loop PWA system

to derive control laws for all five regions. These controllers are as in the following

$$u = -\begin{bmatrix} 0.1503 & 0.0683 \end{bmatrix} x - 0.5 \frac{\begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x \right\|} - 0.6316 \qquad \text{if } x \in \mathscr{R}_1 \tag{3.55}$$

$$u = -\begin{bmatrix} -0.2317 & 0.0683 \end{bmatrix} x - 0.5 \frac{\begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x \right\|} + 0.0884 \qquad \text{if } x \in \mathscr{R}_2 \tag{3.56}$$

$$u = -\begin{bmatrix} -0.2996 & 0.0683 \end{bmatrix} x - 0.5 \frac{\begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0930 & 0.0283 \end{bmatrix} x \right\|} + 0.1311 \qquad \text{if } x \in \mathscr{R}_3 \tag{3.57}$$

$$u = -\begin{bmatrix} 0.0544 & 0.0683 \end{bmatrix} x - 0.5 \frac{\begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x \right\|} + 0.3535 \qquad \text{if } x \in \mathscr{R}_4 \tag{3.58}$$

$$u = -\begin{bmatrix} 0.3059 & 0.0683 \end{bmatrix} x - 0.5 \frac{\begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0724 & 0.0283 \end{bmatrix} x \right\|} + 0.8275 \qquad \text{if } x \in \mathscr{R}_5 \tag{3.59}$$

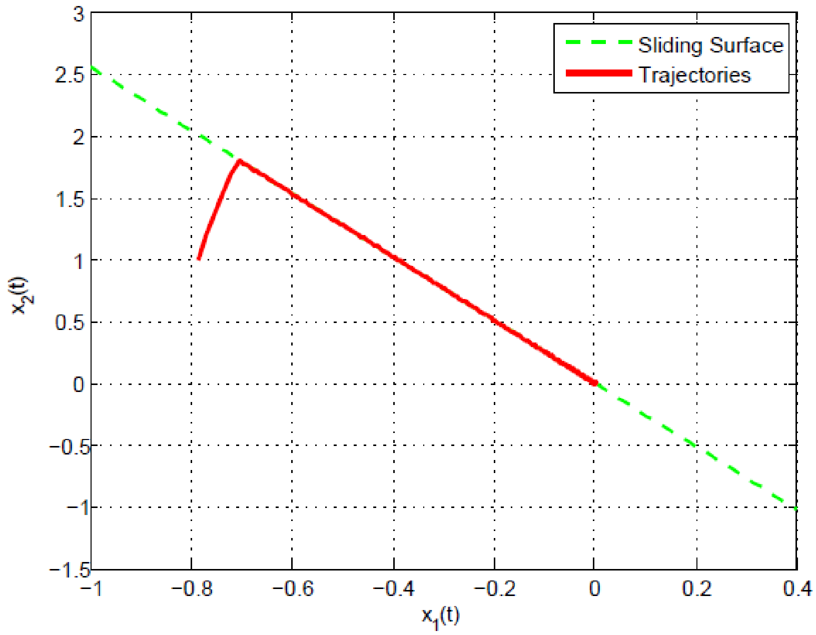

Figure 3.4: Trajectories of the PWA system and the designed sliding surface

Figure 3.3 shows the simulation results for this example with

$$x(0) = \begin{bmatrix} -\pi/4 & 1 \end{bmatrix}^T,$$

as the initial conditions. Figure 3.4 also shows the trajectories of the closed-loop PWA system. As one can see, the trajectories converge to the sliding surface and then slide to the origin.

3.4.2 Numerical Example

In order to make a comparison between the proposed method and the backstepping method (see Section 2.3.6), we consider the following PWA system from reference [59]

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -0.25 & 0.05 \\ -20 & -30 \end{bmatrix} x + \begin{bmatrix} 0 \\ 24 \end{bmatrix} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} u \quad \text{if} \quad x \in \mathscr{R}_1 \\
\dot{x} &= \begin{bmatrix} 0.1 & 0.05 \\ -20 & -30 \end{bmatrix} x + \begin{bmatrix} -0.07 \\ 24 \end{bmatrix} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} u \quad \text{if} \quad x \in \mathscr{R}_2 \\
\dot{x} &= \begin{bmatrix} -0.2 & 0.05 \\ -20 & -30 \end{bmatrix} x + \begin{bmatrix} 0.11 \\ 24 \end{bmatrix} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} u \quad \text{if} \quad x \in \mathscr{R}_3
\end{aligned} \tag{3.60}$$

where $L = 2000$ in this work and PWA regions are

$$\begin{cases}
\mathscr{R}_1 & = \left\{ x \in \mathbb{R}^2 \mid -L < x_1 < 0.2 \right\} \\
\mathscr{R}_2 & = \left\{ x \in \mathbb{R}^2 \mid 0.2 < x_1 < 0.6 \right\} \\
\mathscr{R}_3 & = \left\{ x \in \mathbb{R}^2 \mid 0.6 < x_1 < L \right\}.
\end{cases}$$

Using the backstepping method proposed in reference [59], the PWA controllers which stabilize the system to the origin are as follows

$$\begin{aligned}
u &= \begin{bmatrix} -0.1216 & 1.2572 \end{bmatrix} x + 0.03870 \quad \text{if} \quad x \in \mathscr{R}_1 \\
u &= \begin{bmatrix} -0.20165 & 1.2603 \end{bmatrix} x - 0.0033 \quad \text{if} \quad x \in \mathscr{R}_2 \\
u &= \begin{bmatrix} -0.13739 & 1.2567 \end{bmatrix} x + 10^{-5} \quad \text{if} \quad x \in \mathscr{R}_3.
\end{aligned} \tag{3.61}$$

Figure 3.5 shows the simulation results for the PWA closed-loop system (3.60) using PWA controllers (3.61) with $x(0) = \begin{bmatrix} 0.1 & -3 \end{bmatrix}^T$ as the initial conditions. The trajectories of the closed-loop PWA system also is shown in Figure 3.6.

As it was shown in Figure 3.5, one can stabilize the PWA system (3.60) to the origin, using the backstepping method [59]. However, Figure 3.6 shows one cannot control the way in which the trajectories converge to the origin. Therefore, in the rest of this section we design PWA controllers based on the proposed method in this thesis to make a comparison between both approaches.

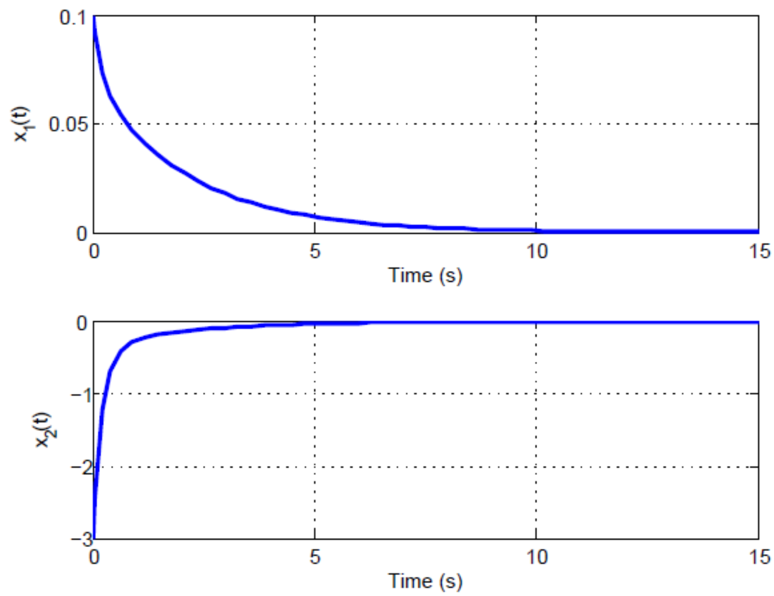

Figure 3.5: Simulation results, using the backstepping method

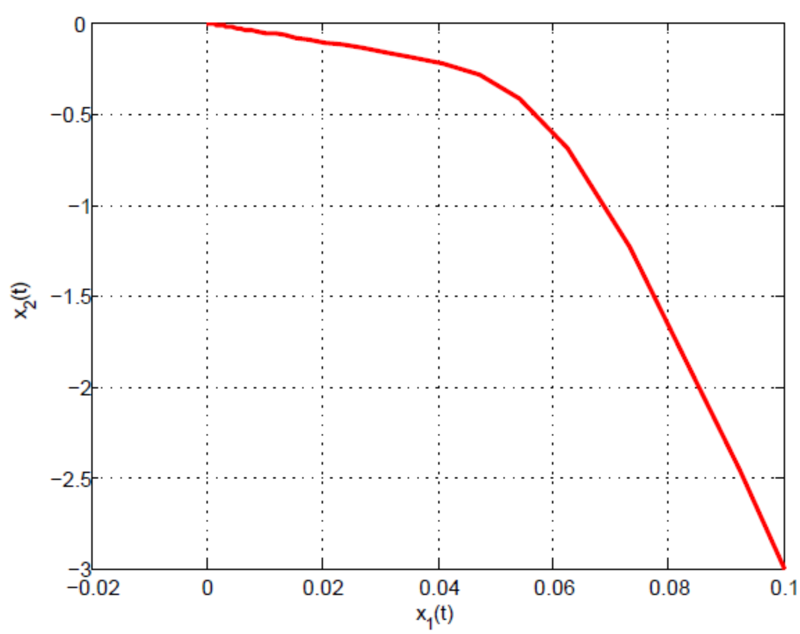

Figure 3.6: Trajectories, using the backstepping method

The procedure of the design is similar to the previous section. We first define

$$\gamma = 15 \qquad \alpha = 0.1, \tag{3.62}$$

we then assign

$$S_2 = B_{2_i}^{-1} = 0.05. \tag{3.63}$$

Using (3.62), (3.63) and solving LMIs (3.16) and (3.17), S_1 is obtained:

$$S_1 = 6.2625. \tag{3.64}$$

Therefore, the sliding surface defined in (3.19) for this problem will be

$$\sigma(x) = \begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x. \tag{3.65}$$

Now using (3.15), the PWA control laws for all three regions will be obtained as in the following

$$u = -\begin{bmatrix} -2.5656 & -1.1869 \end{bmatrix} x - 15 \frac{\begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x}{\left\| \begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x \right\|} - 1.2 \qquad \text{if} \quad x \in \mathscr{R}_1 \tag{3.66}$$

$$u = -\begin{bmatrix} -0.3738 & -1.1869 \end{bmatrix} x - 15 \frac{\begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x}{\left\| \begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x \right\|} - 0.7616 \quad \text{if} \quad x \in \mathscr{R}_2 \tag{3.67}$$

$$u = -\begin{bmatrix} -2.2525 & -1.1869 \end{bmatrix} x - 15 \frac{\begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x}{\left\| \begin{bmatrix} 6.2625 & 0.05 \end{bmatrix} x \right\|} - 1.8889 \quad \text{if} \quad x \in \mathscr{R}_3. \tag{3.68}$$

Figure 3.7 shows the simulation results for the closed-loop system with $x(0) = \begin{bmatrix} 0.1 & -3 \end{bmatrix}^T$. Figure 3.8 also shows that trajectories of the system first converge to the sliding surface and then slide to the origin along that surface.

3.5 Summary

The contribution of this chapter is to use invariant set ideas to formulate the PWA synthesis problem as a set of LMIs. Inspired by the theory of sliding modes, sufficient stabilization

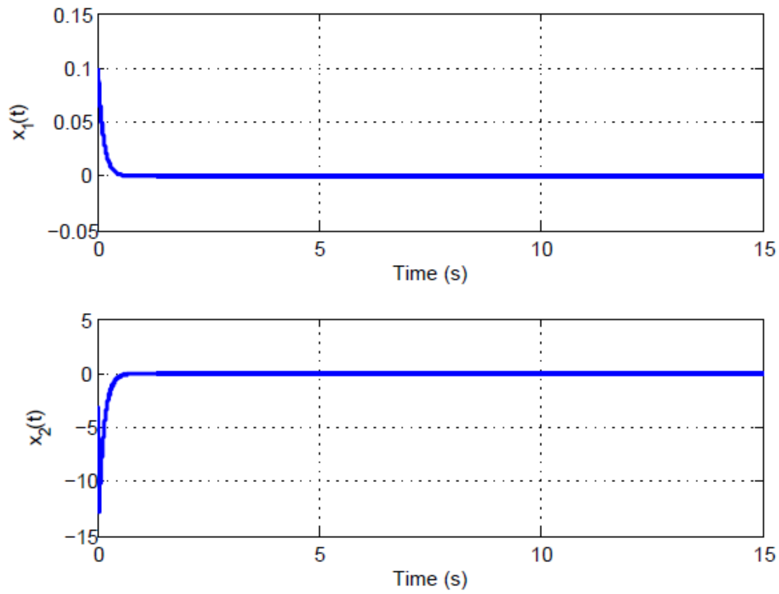

Figure 3.7: Simulation results, using the proposed method

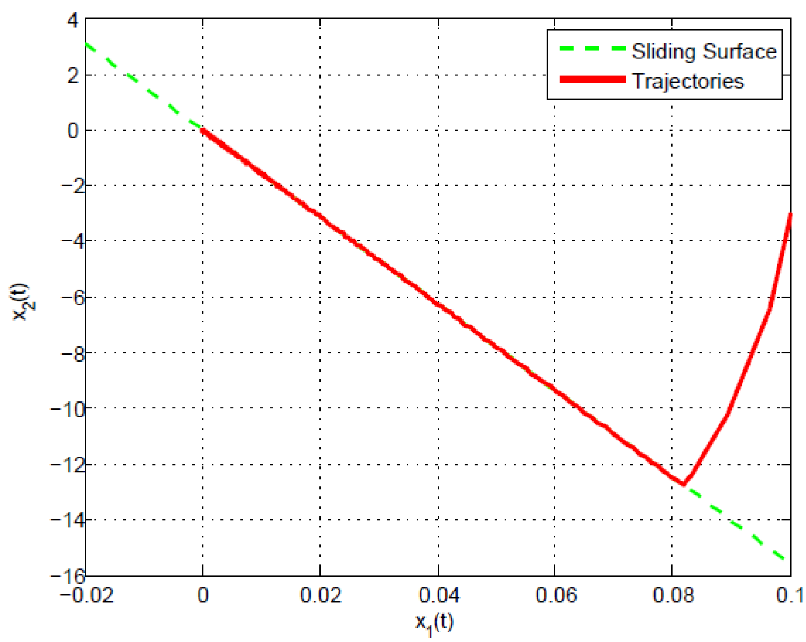

Figure 3.8: Trajectories, using the proposed method

conditions are cast directly as a set of LMIs by proper choice of an invariant set which is a target sliding surface. It is shown that the dimension of the LMIs obtained in this work is lower than in the other convex methods in the literature because the dimension of the state space is reduced, which further simplifies the synthesis problem. Application to pitch control of helicopter, showed the effectiveness of the approach and a numerical example showed while using backstepping method one cannot control the way in which the trajectories converge to the origin, the proposed approach provides us with a surface on which trajectories will slide to the origin. However, the drawback of the method can occur in the implementation phase because the actuators are not completely perfect and they may have delays and other imperfections. This, can lead to chattering which is a rapid motion of the control signal caused by the switching rule. In general, chattering must be eliminated from the controller and this can be achieved by smoothing out the control discontinuity in a thin boundary layer neighboring the sliding surface.

Chapter 4

Conservatism of the Piecewise-Affine Controller Synthesis

4.1 Introduction

As it was frequently mentioned, the methods proposed in [4, 59] are the only ones that can formulate piecewise-affine state feedback as a set of LMIs. In [59] a backstepping approach is developed for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin. The method in [4] shows that one can avoid solving the Bilinear Matrix Inequalities (BMIs) proposed in [30] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility. This limitation motivated the work presented in the previous chapter. It was shown that the synthesis procedure proposed there led to a convex problem in a reduced state space and the closed-loop trajectories converged to the origin along a desired direction. However, the contribution of this chapter is to make a comparison between the conservatism of the approach in [4] and the approach presented in the previous chapter.

It will be shown in this chapter that the proposed approach in Chapter 3 is less conservative than the proposed method in reference [4]. We will also consider unicycle path following problem and active flutter suppression (AFS), which is an interesting and hard control problem in aerospace systems, as applications of our method. Unicycle and flutter are inherently nonlinear phenomena. However, one can approximate the nonlinearities by PWA functions using for example the method detailed in [22]. Simulation results will demonstrate how the difference in the conservatism of the approaches will lead to different results.

4.2 Reduced Conservatism of the Proposed Approach

As it was mentioned in the previous sections, the relaxation approach in [4] shows that one can avoid solving BMIs (3.10) by ignoring the negative definite term (3.11). More precisely, the following lemma taken from [4] (see also Section 2.3.5) gives sufficient conditions for asymptotic stability of the closed-loop PWA slab system (3.1) using the relaxation method.

Lemma 4.2.1. [4] *Consider the PWA slab system (3.1) and the PWA state feedback (3.6). Given $\varepsilon > 0$, if there exist $P = P^T > 0$ and $\zeta_i > 0$ satisfying*

$$\Gamma_{i1} < 0 \qquad \textit{if } 0 \in \overline{\mathscr{R}}_i, \tag{4.1}$$

$$\Gamma_i = \begin{bmatrix} \Gamma_{i1} & \Gamma_{i2} \\ \Gamma_{i2}^T & \Gamma_{i4} \end{bmatrix} < 0 \qquad \textit{otherwise} \tag{4.2}$$

where

$$\Gamma_{i1} = \overline{A}_i P + P\overline{A}_i^T + \varepsilon P$$

$$\Gamma_{i2} = -\zeta_i \overline{a}_i l_i^T + PL_i^T$$

$$\Gamma_{i4} = \zeta_i (1 - l_i^2)$$

with

$$\overline{A}_i = A_i + B_i K_i,$$

$$\overline{a}_i = a_i + B_i k_i$$

L_i, l_i, $\overline{A}_i$ and $\overline{a}_i$ defined in (3.4), (3.5), and (3.8), respectively, then the origin is an exponentially stable equilibrium point.

Proof. See reference [4]. □

Therefore, one concludes that one might be able to synthesize a PWA state feedback controller (3.6) for (3.13) using the results of Lemma 4.2.1. Note that, Theorem 3.3.1 and Lemma 4.2.1 state sufficient conditions and, consequently, they both are conservative approaches. However, the following theorems show that the approach proposed in Theorem 3.3.1 is less conservative than the relaxation approach of Lemma 4.2.1 for PWA slab system (3.13).

Theorem 4.2.1. *For the class of systems (3.13) with full rank B_{2_i}, for every $P = P^T > 0$, $\varepsilon > 0$, $\zeta_i > 0$ satisfying (4.1), (4.2), there exist $Q = Q^T > 0$, $\alpha > 0$, $\mu_i > 0$ and $Y = S_1 Q$ satisfying (3.16), (3.17).*

Proof. Suppose (4.1) and (4.2) hold. Since Γ_i is negative definite and symmetric, using the Schur complement (see Lemma 2.2.2), the following must also hold

$$\Gamma_{i4} < 0 \tag{4.3}$$

$$\Lambda_i = \Gamma_{i1} - \Gamma_{i2}(\Gamma_{i4})^{-1}\Gamma_{i2}^T < 0. \tag{4.4}$$

Note that, since l_i and ζ_i are scalars, Γ_{i4} is also a scalar and (4.4) can be rewritten as

$$\Lambda_i = \Gamma_{i1} - (\Gamma_{i4})^{-1}\Gamma_i^* < 0 \tag{4.5}$$

where

$$\begin{aligned} \Gamma_i^* = \Gamma_{i2}\Gamma_{i2}^T = \zeta_i^2 l_i^2 \overline{a}_i \overline{a}_i^T - \zeta_i l_i \overline{a}_i L_i P \\ - \zeta_i l_i P L_i^T \overline{a}_i^T + P L_i^T L_i P, \end{aligned} \tag{4.6}$$

is a symmetric matrix.

For system (3.13) one can rewrite $\overline{A}_i$, $\overline{a}_i$, and P as

$$\begin{aligned}
\overline{A}_i &= A_i + B_i K_i \\
\overline{a}_i &= a_i + B_i k_i = \begin{bmatrix} a_{1_i} \\ a_{2_i} + B_{2_i} k_i \end{bmatrix} \\
P &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\end{aligned} \tag{4.7}$$

where $P_{11} = P_{11}^T \in \mathbb{R}^{(n-m)\times(n-m)} > 0$, $P_{12} = P_{21}^T \in \mathbb{R}^{(n-m)\times(m)}$, and $P_{22} = P_{22}^T \in \mathbb{R}^{m\times m}$. Now, using (4.7), (3.13) and, (3.14) one can write Γ_{i_1} and Γ_i^* as

$$\begin{aligned}
\Gamma_{i_1} = &\begin{bmatrix} A_{11_i}P_{11} + A_{12_i}P_{21} & A_{11_i}P_{12} + A_{12_i}P_{22} \\ A_{21_i}P_{11} + A_{22_i}P_{21} & A_{21_i}P_{12} + A_{22_i}P_{22} \end{bmatrix} \\
&+ \begin{bmatrix} A_{11_i}P_{11} + A_{12_i}P_{21} & A_{11_i}P_{12} + A_{12_i}P_{22} \\ A_{21_i}P_{11} + A_{22_i}P_{21} & A_{21_i}P_{12} + A_{22_i}P_{22} \end{bmatrix}^T \\
&+ \begin{bmatrix} 0 \\ B_{2_i}H_i \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2_i}H_i \end{bmatrix}^T + \varepsilon \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},
\end{aligned} \tag{4.8}$$

$$\Gamma_i^* = \begin{bmatrix} \Gamma_{i_1}^* & \Gamma_{i_2}^* \\ \Gamma_{i_2}^{*T} & \Gamma_{i_4}^* \end{bmatrix} \tag{4.9}$$

where

$$\begin{aligned}
H_i &= K_i P \\
\Gamma_{i_1}^* &= \zeta_i^2 l_i^2 (a_{1_i} a_{1_i}^T) - \zeta_i l_i (a_{1_i} L_{1i} P_{11}) \\
&\quad - \zeta_i l_i (P_{11} L_{1i}^T a_{1_i}^T) + P_{11} L_{1i}^T L_{1i} P_{11} \\
\Gamma_{i_2}^* &= \zeta_i^2 l_i^2 (a_{1_i} k_i^T B_{2_i}^T + a_{1_i} a_{2_i}^T) - \zeta_i l_i (a_{1_i} L_{1i} P_{12}) \\
&\quad - \zeta_i l_i (P_{11} L_{1i}^T a_{2_i}^T + P_{11} L_{1i}^T k_i^T B_{2_i}^T) \\
&\quad + P_{11} L_{1i}^T L_{1i} P_{12}
\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
\Gamma_{i_4}^* &= \zeta_i^2 l_i^2 (a_{2_i} k_i^T B_{2_i}^T + a_{2_i} a_{2_i}^T + B_{2_i} k_i k_i^T B_{2_i}^T + B_{2_i} k_i a_{2_i}^T) \\
&- \zeta_i l_i (a_{2_i} L_{1i} P_{12} + B_{2_i} k_i L_{1i} P_{12}) \\
&- \zeta_i l_i (a_{2_i} L_{1i} P_{12} + B_{2_i} k_i L_{1i} P_{12})^T \\
&+ P_{21} L_{1i}^T L_{1i} P_{12}.
\end{aligned} \tag{4.11}
$$

Note that, one can also rewrite the symmetric matrix Λ_i as

$$
\Lambda = \begin{bmatrix} \Lambda_{i1} & \Lambda_{i2} \\ \Lambda_{i2}^T & \Lambda_{i4} \end{bmatrix}. \tag{4.12}
$$

Now since (4.5) holds, the following inequality must also hold:

$$
\Lambda_{i1} < 0 \tag{4.13}
$$

or equivalently

$$
\begin{aligned}
\Lambda_{i1} &= A_{11_i} P_{11} + A_{12_i} P_{21} + P_{11} A_{11_i}^T + P_{21}^T A_{12_i}^T + \varepsilon P_{11} \\
&- \zeta_i^{-1} (1 - l_i^2)^{-1} (\zeta_i^2 l_i^2 (a_{1_i} a_{1_i}^T) - \zeta_i l_i (a_{1_i} L_{1i} P_{11}) \\
&- \zeta_i l_i (P_{11} L_{1i}^T a_{1_i}^T) + P_{11} L_{1i}^T L_{1i} P_{11}) < 0.
\end{aligned} \tag{4.14}
$$

Now we define

$$
P_{11} = Q \tag{4.15}
$$

$$
P_{21} = -S_2^\dagger Y \tag{4.16}
$$

$$
\zeta_i = \mu_i \tag{4.17}
$$

$$
\varepsilon = \alpha \tag{4.18}
$$

and replace them in (4.14) which yields

$$
\begin{aligned}
T_i &= A_{11_i} Q + Q A_{11_i}^T - A_{12_i} S_2^\dagger Y - Y^T (S_2^\dagger)^T A_{12_i}^T + \alpha Q \\
&- \mu_i^{-1} (1 - l_i^2)^{-1} (\mu_i^2 l_i^2 (a_{1_i} a_{1_i}^T) - \mu_i l_i (a_{1_i} L_{1i} Q) \\
&- \mu_i l_i (Q L_{1i}^T a_{1_i}^T) + Q L_{1i}^T L_{1i} Q) < 0.
\end{aligned} \tag{4.19}
$$

Therefore, taking into account that l_i is a scalar,

$$
T_i - \mu_i a_{1_i} a_{1_i}^T = \omega_{i1} - \omega_{i2} \omega_{i4}^{-1} \omega_{i2}^T < 0 \tag{4.20}
$$

because $T_i < 0$ and

$$-\mu_i a_{1_i} a_{1_i}^T < 0. \tag{4.21}$$

Moreover, using (4.3)

$$\mu_i(1-l_i^2) = \zeta_i(1-l_i^2) < 0. \tag{4.22}$$

Note that, (4.20) and (4.22) imply that LMIs (3.17) are verified using Schur complement. Moreover, for the case where $0 \in \overline{\mathscr{R}}_i$,

$$\Gamma_{i1} = \overline{A}_i P + P\overline{A}_i^T + \varepsilon P < 0, \tag{4.23}$$

implies that the following inequality must also hold

$$A_{11_i} P_{11} + A_{12_i} P_{21} + P_{11} A_{11_i}^T + P_{21}^T A_{12_i}^T + \varepsilon P_{11} < 0. \tag{4.24}$$

Again, using (4.15) to (4.18),

$$A_{11_i} Q + Q A_{11_i}^T - A_{12_i} S_2^{\dagger} Y - Y^T (S_2^{\dagger})^T A_{12_i}^T + \alpha Q < 0, \tag{4.25}$$

which implies that LMI (3.16) is verified. Therefore, for any solution to LMIs (4.1) and (4.2), there will also be a solution to LMIs (3.16) and (3.17). □

Theorem 4.2.2. *The converse of theorem 4.2.1 does not hold.*

Proof. Consider the following simple second order PWA slab system in class (3.13) defined in forward invariant set $\mathscr{X} = \{x \in \mathbb{R}^2 \mid -0.4 < x_1 < 1\}$:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_1, \tag{4.26}$$

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_2 \tag{4.27}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } x \in \mathscr{R}_3, \tag{4.28}$$

where

$$\mathscr{R}_1 = \left\{x \in \mathbb{R}^2 \mid -0.4 < x_1 < 0.4\right\}$$

$$\mathscr{R}_2 = \left\{x \in \mathbb{R}^2 \mid 0.4 < x_1 < 0.6\right\}$$

$$\mathscr{R}_2 = \left\{x \in \mathbb{R}^2 \mid 0.6 < x_1 < 1\right\}.$$

Stability of closed-loop PWA system (4.26), (4.27) and, (4.28) with PWA state feedback (3.6) can not be achieved by the relaxation method. In other words, the conditions of Lemma 4.2.1 for such a system are not satisfied for any $P = P^T > 0,\ \varepsilon > 0,\ \zeta_i > 0$. One can verify this fact by examining the Schur complement in (4.3) and (4.4). Note that, Γ_{i1} in (4.2) must be negative definite. Therefore, from (4.8)

$$A_{11_i}P_{11} + A_{12_i}P_{21} + P_{11}A_{11_i}^T + P_{21}^T A_{12_i}^T + \varepsilon P_{11} < 0 \tag{4.29}$$

Rewriting (4.29) for the closed-loop PWA system (4.27) with the controller defined in (3.6) yields

$$(2+\varepsilon)P_{11} < 0. \tag{4.30}$$

Since $\varepsilon > 0$, (4.30) implies

$$P_{11} < 0 \tag{4.31}$$

which contradicts the positive definiteness of P_{11}. Therefore (4.2) and subsequently conditions of Lemma 4.2.1 are not verified.

In the following we will show that the stability of the closed-loop PWA system (4.26), (4.27) and, (4.28) can be guaranteed by the proposed controller synthesis. In order to do so, we will show that LMIs (3.16) and (3.17) in Theorem 3.3.1 are verified for this system.

Assuming and replacing

$$\alpha = 1 \tag{4.32}$$

$$Q = 2, \tag{4.33}$$

in LMIs (3.16) and (3.17), we will have

$$\omega_{i0} + \alpha Q = -2 < 0 \qquad \text{if} \qquad x \in \mathscr{R}_1 \tag{4.34}$$

and

$$\begin{bmatrix} \omega_{i1} & \omega_{i2} \\ \omega_{i2}^T & \omega_{i4} \end{bmatrix} = \begin{bmatrix} -6 & -10 \\ -10 & -72 \end{bmatrix} < 0 \qquad \text{if} \quad x \in \mathscr{R}_2 \tag{4.35}$$

with

$$L_{1i} = L_{12} = 10, \quad l_i = l_2 = -5 \quad \text{and} \quad \mu_i = \mu_2 = 3 \tag{4.36}$$

and finally

$$\begin{bmatrix} \omega_{i1} & \omega_{i2} \\ \omega_{i2}^T & \omega_{i4} \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 10 & -60 \end{bmatrix} < 0 \qquad \text{if} \quad x \in \mathscr{R}_3 \tag{4.37}$$

with

$$L_{1i} = L_{13} = 5, \quad l_i = l_3 = -4 \quad \text{and} \quad \mu_i = \mu_3 = 4. \tag{4.38}$$

where L_{1i} and l_{1i} $(i = 1,\ 2,\ 3)$ are obtained from (3.4), (3.5) and (3.14). Therefore (3.16) and (3.17) are verified. □

4.3 Application

Based on the previous discussions, this sections illustrates how the proposed method in Chapter 3 is less conservative than the other available convex approaches through some popular applications. This section is divided into two parts. First we consider the problem of active flutter suppression (AFS), and then, we consider unicycle path following problem.

4.3.1 Active Flutter Suppression

In this section, we consider the problem of Active Flutter Suppression (AFS) to demonstrate how the proposed method works. The flutter model for a two fold airfoil (Figure 4.1) is taken from [3] and is given by

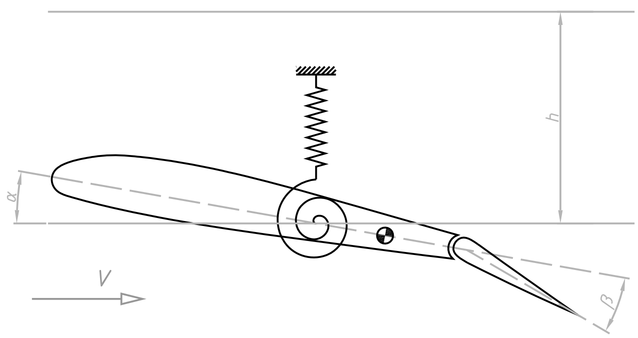

Figure 4.1: Airfoil model for AFS problem, taken from [3]

$$
\begin{aligned}
\dot{x} = &\begin{bmatrix} 0 & I \\ -M^{-1}(K_0+K_\mu) & -M^{-1}(C_0+C_\mu) \end{bmatrix} x \\
&+ \begin{bmatrix} 0 \\ -M^{-1}\begin{bmatrix} 0 \\ x_2 k_\alpha(\alpha) \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ \mu M^{-1} \end{bmatrix} Bu
\end{aligned}
\tag{4.39}
$$

where

$$
C_\mu = \begin{bmatrix} \rho U b C_{L\alpha} & \rho U b^2 C_{L\alpha}(\frac{1}{2}-a) \\ \rho U b^2 C_{m\alpha} & -\rho U b^3 C_{m\alpha}(\frac{1}{2}-a) \end{bmatrix}
$$

$$
C_0 = \begin{bmatrix} C_h & 0 \\ 0 & C_\alpha \end{bmatrix}
$$

$$
M = \begin{bmatrix} m & m x_\alpha b \\ m x_\alpha b & I_\alpha \end{bmatrix}
$$

$$
K_0 = \begin{bmatrix} k_h & 0 \\ 0 & 0 \end{bmatrix}
$$

$$K_{\mu} = \begin{bmatrix} 0 & \rho U^2 b C_{L\alpha} \\ 0 & -\rho U^2 b^2 C_{m\alpha} \end{bmatrix}$$

$$B = \begin{bmatrix} -\rho b C_{L\beta_1} & -\rho b C_{L\beta_2} \\ \rho b^2 C_{m\beta_1} & \rho b^2 C_{m\beta_2} \end{bmatrix} U^2$$

U is the airspeed, $C_{L\alpha}$ and $C_{m\alpha}$ are aerodynamic lift and moment coefficients, ρ is the air density, $C_{L\beta}$ and $C_{m\beta}$ are lift and moment coefficients per control surface deflection, respectively, m is the mass of the airfoil, I_{α} is the mass moment of inertia about the elastic axis, C_h and C_{α} are plunge and pitch structural damping coefficients, respectively, and L and M are the aerodynamic lift and moment about the elastic axis. Structural stiffness is represented by k_h and k_{α} for plunge and pitch motions, respectively. The source of nonlinearity is the torsional stiffness, which is

$$k_{\alpha}(\alpha) = k_{\alpha_0} + k_{\alpha_1}(\alpha) + k_{\alpha_2}(\alpha^2) + k_{\alpha_3}(\alpha^3) + k_{\alpha_4}(\alpha^4) \tag{4.40}$$

All the model parameters are taken from [3] and are available in the appendix. A controller is now designed for a PWA approximation of the system (4.39). Therefore, first we approximate the AFS nonlinear system by a PWA model using the method explained in [22]. The slab regions used for the approximation are defined by

$$\begin{cases} \mathscr{R}_1 & = \{x \in \mathbb{R}^4 \mid 0.6 < x_2 = \alpha < 1\} \\ \mathscr{R}_2 & = \{x \in \mathbb{R}^4 \mid -1 < x_2 = \alpha < -0.6\} \\ \mathscr{R}_3 & = \{x \in \mathbb{R}^4 \mid -0.6 < x_2 = \alpha < -0.2\} \\ \mathscr{R}_4 & = \{x \in \mathbb{R}^4 \mid -0.2 < x_2 = \alpha < 0.2\} \\ \mathscr{R}_5 & = \{x \in \mathbb{R}^4 \mid 0.2 < x_2 = \alpha < 0.6\}. \end{cases}$$

The dynamics equations in all regions will not be presented here for lack of space but, for example, the dynamics of the PWA model for AFS in $\mathscr{R}_5$ are

$$\begin{aligned}\dot{x} = &\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -293.27 & 2272.13 & -5.90 & -0.40 \\ 1885.94 & -69573.59 & 34.72 & 2.47 \end{bmatrix} x \\ &+ \begin{bmatrix} 0 \\ 0 \\ -471.47 \\ 13972.34 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -7606.78 & -7642.55 \\ 14250.44 & 9021.92 \end{bmatrix} u.\end{aligned} \tag{4.41}$$

This approximation belongs to the class of PWA systems defined in (3.12).

In order to make a comparison between the relaxation method [4] and the proposed method in Chapter 3, we design different controllers based on theses two different approaches. First, we try to solve the LMIs (4.1) and (4.2) from [4]. After trying different values for α, finally we found a solution to these LMIs. Figure 4.2 shows the simulation results for the same AFS model controlled by the method proposed in [4]. The simulation results in Figure 4.2 show that there is a high frequency oscillatory behavior of the state variables using the approach suggested in [4].

Finally, we followed the proposed method in Chapter 3 to illustrate how the proposed method works. The design process was as in the following:

To design the controllers, we first defined the parameters

$$\gamma = 2 \qquad \alpha = 0.01, \tag{4.42}$$

and then we assigned

$$S_2 = B_{2_5}^{-1} = \begin{bmatrix} 0.0002 & 0.0002 \\ -0.0004 & -0.0002 \end{bmatrix}. \tag{4.43}$$

Using (4.42), (4.43) and solving LMIs (3.16) and (3.17), the matrix S_1 is obtaineded as

$$S_1 = \begin{bmatrix} 0.00017 & -0.0198 \\ -0.0017 & 0.0320 \end{bmatrix}. \tag{4.44}$$

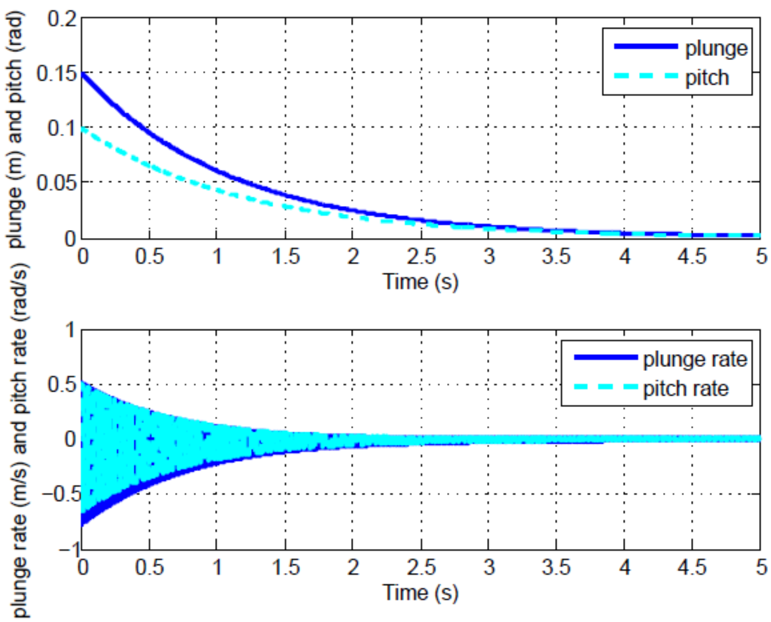

Figure 4.2: AFS state variables using controller in [4]

Therefore, the sliding surface defined in (3.19) for the AFS problem was

$$\sigma(x) = \begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x. \tag{4.45}$$

After computing $\sigma(x)$ and using (3.15), we were able to derive control laws for all five regions. For instance, the control input for the fifth region was

$$\begin{aligned} u = &\begin{bmatrix} -0.292 & 12.691 & -0.007 & 0.019 \\ 0.252 & -12.334 & 0.006 & -0.031 \end{bmatrix} x \\ &- \gamma \frac{\begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x \right\|} \\ &+ \begin{bmatrix} -2.545 \\ 2.471 \end{bmatrix} \end{aligned} \tag{4.46}$$

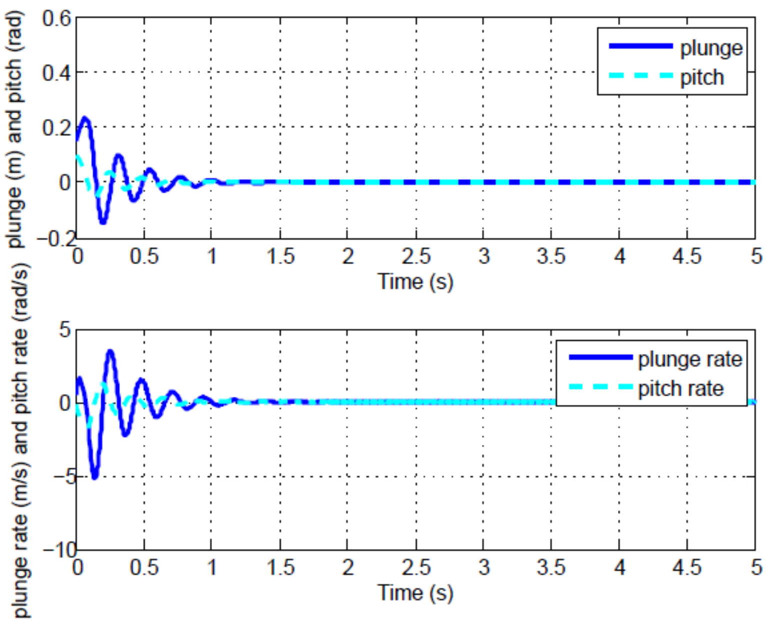

Figure 4.3: AFS state variables using proposed controller

Figure 4.3 shows simulation results for $x(0) = \begin{bmatrix} 0.15 & 0.1 & 0.5 & -0.2 \end{bmatrix}^T$ as the initial conditions. It can be clearly seen in the figure that flutter was effectively suppressed as desired.

Remark 4.3.1. *The active flutter suppression problem illustrates, while there is a solution to LMIs (4.1) and (4.2) -for this specific example- there is also a solution to LMIs (3.16) and (3.17). Moreover, the designed controllers based on this solution yield to even better simulation results.*

4.3.2 Unicycle Path Following

In this part, we consider the path following example from [30]. The objective of this example is to design a controller that makes a cart on the xy plane follow the straight line $y = 0$ with a constant velocity $u_0 = 1m/s$. It is assumed that a controller has already been designed to maintain a constant forward velocity. The carts path is then controlled by the torque T about the z-axis according to the following dynamics:

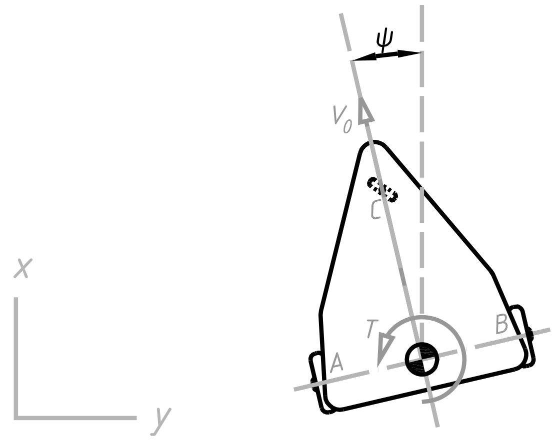

Figure 4.4: Unicycle Path Following Example

$$\begin{bmatrix} \dot{\psi} \\ \dot{y} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-k}{I} \end{bmatrix} \begin{bmatrix} \psi \\ y \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ u_0 \sin(\psi) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I} \end{bmatrix} T, \tag{4.47}$$

where ψ is the heading angle with time derivative r, $I = 1kgm^2$ is the moment of inertia of the cart with respect to the center of mass, $k = 0.01Nms$ is the damping coefficient, and T is the control torque. Assume the trajectories can start from any possible initial angle in the range $\psi_0 \in [-3\pi/5, 3\pi/5]$ and any initial distance from the line. The function $\sin(\psi)$ is approximated by a PWA function (see [22]) yielding a PWA slab system as follows

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 \\ 0.2891 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.4061 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} T \quad \text{if } x \in \mathscr{R}_1 \\ \dot{x} &= \begin{bmatrix} 0 & 0 & 1 \\ 0.9069 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.0180 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} T \quad \text{if } x \in \mathscr{R}_2 \end{aligned} \tag{4.48}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0.9927 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} T \qquad \text{if } x \in \mathscr{R}_3$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0.9069 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.0180 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} T \qquad \text{if } x \in \mathscr{R}_4 \tag{4.49}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0.2891 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.4061 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} T \qquad \text{if } x \in \mathscr{R}_5$$

with five regions defined in the following

$$\begin{cases} \mathscr{R}_1 & = \left\{ x \in \mathbb{R}^3 \mid -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5} \right\} \\ \mathscr{R}_2 & = \left\{ x \in \mathbb{R}^3 \mid -\frac{\pi}{5} < x_1 < -\frac{\pi}{15} \right\} \\ \mathscr{R}_3 & = \left\{ x \in \mathbb{R}^3 \mid -\frac{\pi}{15} < x_1 < \frac{\pi}{15} \right\} \\ \mathscr{R}_4 & = \left\{ x \in \mathbb{R}^3 \mid \frac{\pi}{15} < x_1 < \frac{\pi}{5} \right\} \\ \mathscr{R}_5 & = \left\{ x \in \mathbb{R}^3 \mid \frac{\pi}{5} < x_1 < \frac{3\pi}{5} \right\}. \end{cases}$$

First we attempted to derive control laws using the relaxation method [4]. Unfortunately, using different values for α in the range of

$$0.0001 < \alpha < 10,$$

we were not able to find any solution to the LMIs (4.1) and (4.2).

On the other hand, we can easily find a solution to LMIs (3.16) and (3.17). The controller design process, using the proposed method, for this example is as follows:

First we define

$$\gamma = 0.6 \qquad \alpha = 0.5, \tag{4.50}$$

and then we assign

$$S_2 = 0.1 B_{2_i}^{-1} = 0.1. \tag{4.51}$$

Using (4.50), (4.51) and solving LMIs (3.16) and (3.17), S_1 is obtained as

$$S_1 = \begin{bmatrix} 0.8655 & 0.8222. \end{bmatrix} \tag{4.52}$$

Therefore, the sliding surface defined in (3.19) for this problem is

$$\sigma(x) = \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x. \tag{4.53}$$

After computing $\sigma(x)$ and using (3.15), we are able to derive control laws for all five regions. These PWA controllers are as in the following

$$T_1 = -\begin{bmatrix} 2.377 & 0 & 8.645 \end{bmatrix} x - 6\frac{\begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x \right\|} + 3.339 \tag{4.54}$$

$$T_2 = -\begin{bmatrix} 7.456 & 0 & 8.645 \end{bmatrix} x - 6\frac{\begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x \right\|} + 0.148 \tag{4.55}$$

$$T_3 = -\begin{bmatrix} 8.162 & 0 & 8.645 \end{bmatrix} x - 6\frac{\begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x \right\|} \tag{4.56}$$

$$T_4 = -\begin{bmatrix} 7.456 & 0 & 8.645 \end{bmatrix} x - 6\frac{\begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x \right\|} - 0.148 \tag{4.57}$$

$$T_5 = -\begin{bmatrix} 2.377 & 0 & 8.645 \end{bmatrix} x - 6\frac{\begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.8655 & 0.8222 & 0.1 \end{bmatrix} x \right\|} - 3.339 \tag{4.58}$$

where T_1 for example, is the designed affine controller for region $\mathscr{R}_1$. Figure 4.5 shows the simulation results for this example with $x(0) = \begin{bmatrix} \pi/2 & 0.5 & 0 \end{bmatrix}^T$ as the initial conditions. Figure 4.6 also demonstrates how the unicycle converges to the line $y = 0$. The trajectories of the PWA closed-loop system are shown in Figure 4.7. As one can see, the trajectories of the system first converge to the sliding surface and then slide to the origin.

Remark 4.3.2. *The unicycle path following example, is in fact consistent with Theorem 4.2.2 for wide range of* $\alpha \in (0.001, 10)$. *In other words, while there was no solution to the LMIs (4.1) and (4.2), LMIs (3.16) and (3.17) yielded to a solution for arbitrary value of* α *within the same range.*

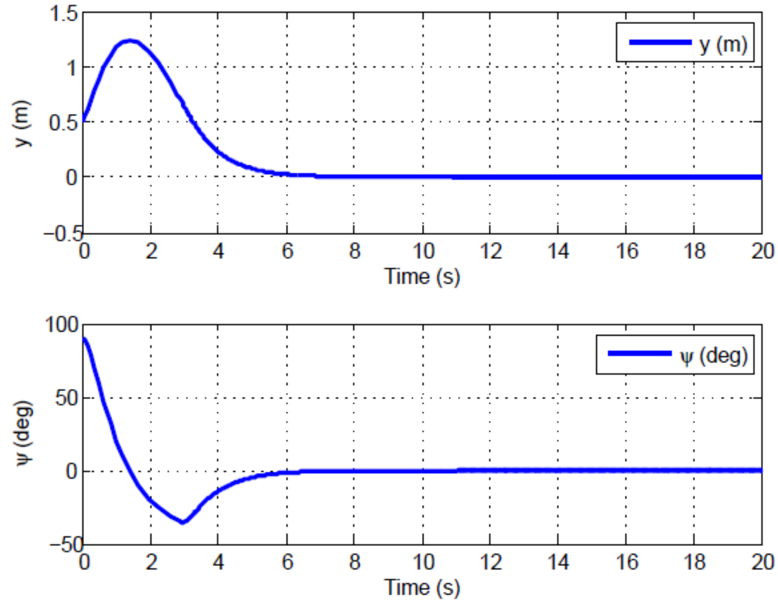

Figure 4.5: Time responses for unicycle path following problem

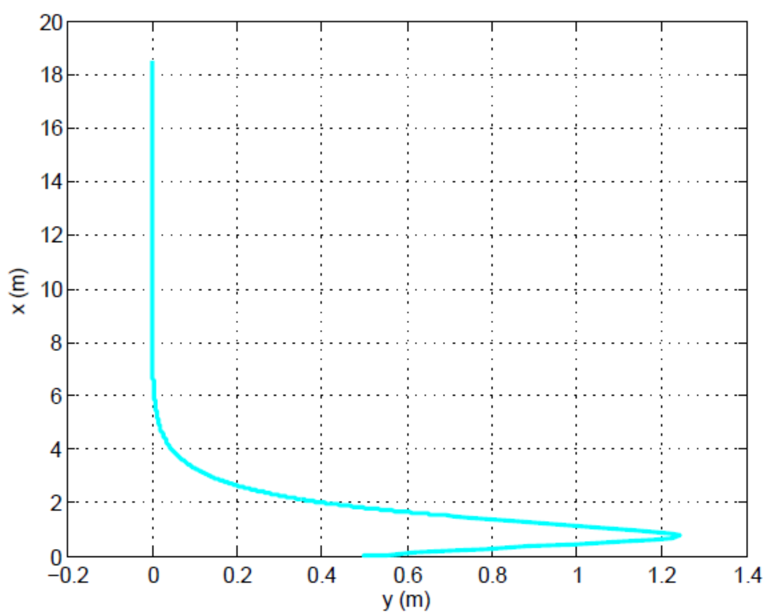

Figure 4.6: Distance of the unicycle from the y=0 line

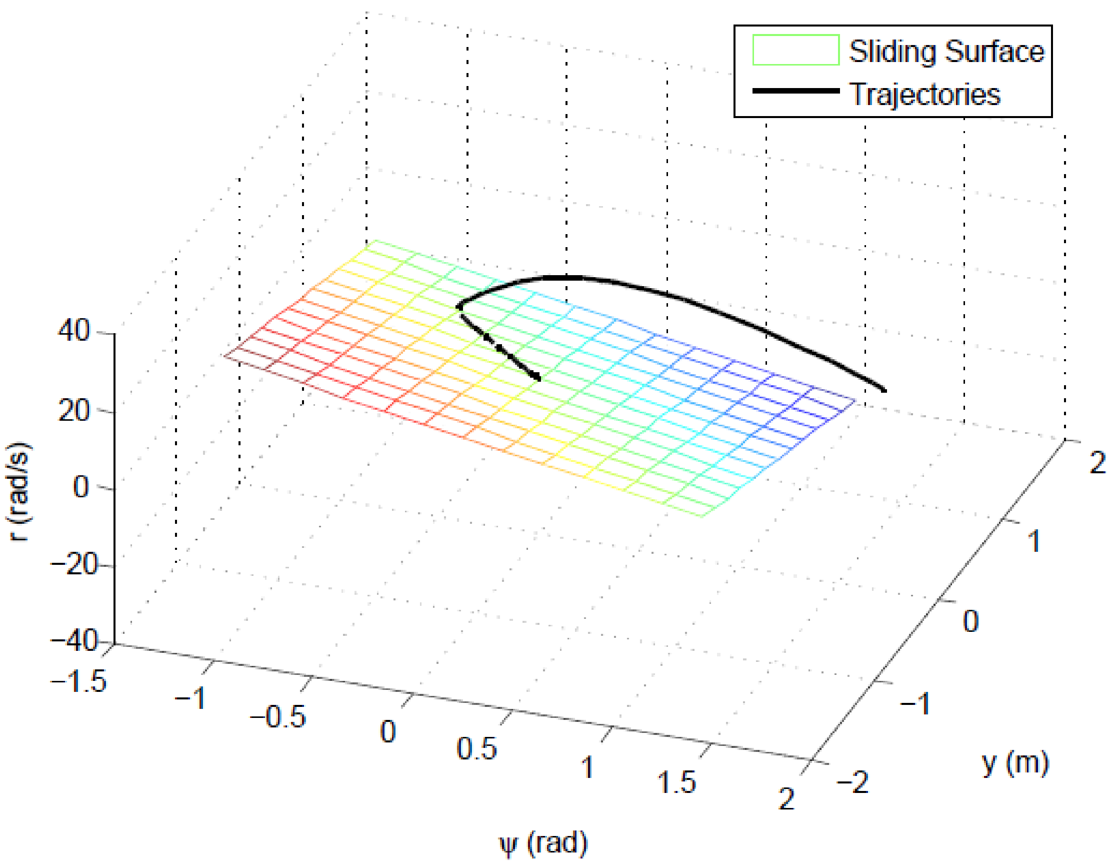

Figure 4.7: Trajectories of the unicycle and the designed sliding surface

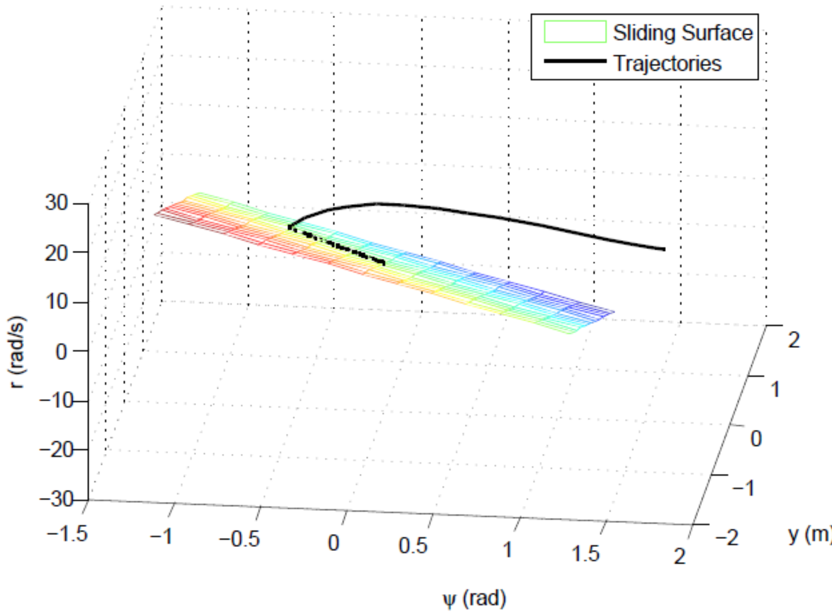

Figure 4.8: Trajectories of the unicycle and the designed sliding surface

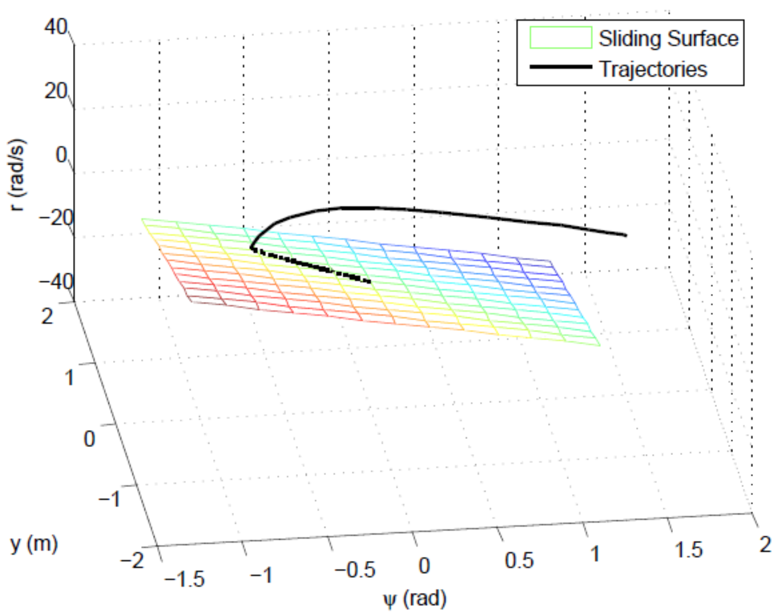

Figure 4.9: Trajectories of the unicycle and the designed sliding surface

4.4 Summary

This chapter shows that for every solution to the LMIs resulting from the previous LMI approaches, there exists a solution for the LMIs obtained from the proposed method. Furthermore, it is shown that while previous convex controller synthesis methods have no solutions to their LMIs for some examples of PWA systems, the approach proposed in this thesis yields a solution for these examples. Finally, the comparisons between the proposed method and the relaxation method is also demonstrated through some real-life applications. Application to active control of flutter suppression, which is considered a hard problem in aerospace control, showed while the relaxation approach led to a high frequency simulation results, the proposed approach was able to actively suppress flutter in a wing section. Finally, it was shown that the designed controllers using the proposed approach, made the cart trajectory converge to the desired straight line in the unicycle path following problem, whereas the relaxation approach led to no solutions to its LMIs. However, the PWA class that we are considering in this work is still conservative. The special structure of the matrix B_i, the invertibility of the matrix B_{2_i} and the partitioning of the slab regions based on only x_1 are some of the restrictions that we need to take into account for the defined class.

Chapter 5

Controller Synthesis of Piecewise-Affine Systems with Time-Delay

5.1 Introduction

While time-delay control of linear systems is a well-studied subject, unfortunately, its extension to piecewise-affine (PWA) systems has not received many research contributions. The only available conducted research in this area, investigate the analysis problem rather than the controller synthesis problem, see [65, 41, 2], and therefore, none of these mentioned references address the controller synthesis problem for such systems. Consequently, there is no convex formulation for controller synthesis of PWA time-delay systems in the existing literature. In this chapter of the thesis we will extend the proposed method in Chapter 3 to the case where a constant time-delay is involved in the dynamics of the PWA system and will formulate this problem as a convex program based on LMIs. The simulation results for a numerical example will also demonstrate the effectiveness of the approach.

5.2 Preliminaries

Consider a piecewise-affine system with time-delay described as

$$\dot{x}(t) = A_i x(t) + A_j^d x(t-\tau) + a_i + B_i u(t), \quad x(t) \in \mathscr{R}_i \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state at time t, $u(t) \in \mathbb{R}^p$ the control input and assume that a forward invariant set $\mathscr{X} \subset \mathbb{R}^n$ is partitioned into M polytopic cells $\mathscr{R}_i$, $i \in \mathscr{I} = \{1, \ldots, M\}$ such that $\cup_{i=1}^M \overline{\mathscr{R}}_i = \mathscr{X}$, $\mathscr{R}_i \cap \mathscr{R}_j = \emptyset$ where $\overline{\mathscr{R}}_i$ denotes the closure of $\mathscr{R}_i$ (see [22] for generating such partition). The constant τ is a positive known delay.

Following Chapter 3, a slab region is defined as

$$\mathscr{R}_i = \{x \mid \beta_i < \lambda^T x < \beta_{i+1}\} \tag{5.2}$$

where $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$ and $\beta_i,\ \beta_{i+1} \in \mathbb{R},\ i = 1, \ldots, M.$ The slab region $\mathscr{R}_i$ can also be cast as a degenerate ellipsoid

$$\mathscr{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \tag{5.3}$$

where

$$L_i = 2\lambda^T/(\beta_{i+1} - \beta_i), \tag{5.4}$$

$$l_i = -(\beta_{i+1} + \beta_i)/(\beta_{i+1} - \beta_i). \tag{5.5}$$

A PWA system whose regions are slabs is called a PWA slab system [30].

5.3 Controller Synthesis

Consider the following class of PWA slab systems with time-delay

$$\dot{x}(t) = A_i x(t) + A_j^d x(t-\tau) + a_i + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u(t), \quad x(t) \in \mathscr{R}_i \tag{5.6}$$

where $u \in \mathbb{R}^p$, $B_{2_i} \in \mathbb{R}^{m \times p}$ and $m \in \mathscr{M} = \{1, \cdots, n-1\},\ m \geq p.$

We can rewrite equations (5.6) for $x(t) \in \mathscr{R}_i$ in the following form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11_i} & A_{12_i} \\ A_{21_i} & A_{22_i} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} A^d_{11_j} & A^d_{12_j} \\ A^d_{21_j} & A^d_{22_j} \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} a_{1_i} \\ a_{2_i} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u(t), \tag{5.7}$$

where $x_1 \in \mathbb{R}^{n-m}$, $x_2 \in \mathbb{R}^m$. Assume further that in this class of PWA systems, the slab regions are only functions of x_1. Therefore, the definition of slab regions (3.3) can be rewritten as

$$\mathscr{R}_i = \left\{x \mid \|L_i x + l_i\| = \left\| \begin{bmatrix} L_{1i} & 0 \end{bmatrix} x + l_i \right\| = \|L_{1i} x_1 + l_i\| < 1 \right\} \tag{5.8}$$

where $L_{1i}^T \in \mathbb{R}^{n-m}$. This chapter proposes a new method to formulate PWA time-delay controller synthesis for system (5.7) as a convex feasibility problem.

Theorem 5.3.1. *Assuming that either B_{2_i} is invertible or $B_{2_i} = B_2$ is full rank, the PWA controller*

$$\begin{aligned} u = &-(S_2 B_{2_i})^{-1}[S_1(A_{11_i}x_1(t) + A_{12_i}x_2(t) + A^d_{11_j}x_1(t-\tau) + A^d_{12_j}x_2(t-\tau) + a_{1_i}) \\ &+ S_2(A_{21_i}x_1(t) + A_{22_i}x_2(t) + A^d_{21_j}x_1(t-\tau) + A^d_{22_j}x_2(t-\tau) + a_{2_i}) \\ &+ \gamma \frac{S_1 x_1(t) + S_2 x_2(t)}{\|S_1 x_1(t) + S_2 x_2(t)\|}], \end{aligned} \tag{5.9}$$

for $x(t) \in \mathscr{R}_i$, $i = 1, \ldots, M$, exponentially stabilizes system (5.7) defined in a forward invariant set $\mathscr{X}$ if given $\gamma > 0$, $\tau > 0$, $\varepsilon > 0$ and S_2, there exist $Q = Q^T > 0$, $\mu_i > 0$, and $Y = S_1 Q$, satisfying the following LMIs

- *If $0 \in \overline{\mathscr{R}}_i$*

$$\begin{bmatrix} \overline{\Omega}_{i_0} & \tau \begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} \\ \tau \begin{bmatrix} \overline{M}^T & 0 \end{bmatrix} & -\tau Q \end{bmatrix} < 0 \tag{5.10}$$

- *If $0 \notin \overline{\mathscr{R}}_i$*

$$\begin{bmatrix} \begin{pmatrix} A_{11_i}Q - A_{12_i}S_2^{\dagger}Y + QA_{11_i}^T \\ -Y^T S^{\dagger}{}_2^T A_{12_i}^T + \varepsilon Q - \mu_i a_{1i} a_{1i}^T \end{pmatrix} & (-\mu_i a_{1i} l_i^T + QL_{1i}^T) & A_{11_j}^d Q - A_{12_j}^d S_2^{\dagger} Y \\ (-\mu_i a_{1i} l_i^T + QL_{1i}^T)^T & \mu_i(1 - l_i^T l_i) & 0 \\ QA_{11_j}^d{}^T - Y^T S_2^{\dagger}{}^T A_{12_j}^d{}^T & 0 & -\varepsilon Q \end{bmatrix} < 0 \quad (5.11)$$

$$\begin{bmatrix} \left(-\begin{bmatrix} \overline{N} & -\overline{N} & 0 \end{bmatrix} - \begin{bmatrix} \overline{N}^T \\ -\overline{N}^T \\ 0 \end{bmatrix} \right) & \tau\omega_{2_i} & \tau\overline{N} \\ \tau\omega_{2_i}^T & -\tau Q & 0 \\ \tau\overline{N}^T & 0 & -\tau Q \end{bmatrix} \leq 0 \quad (5.12)$$

where

$$\overline{\Omega}_{i_0} = \begin{bmatrix} \begin{pmatrix} A_{11_i}Q - A_{12_i}S_2^{\dagger}Y + QA_{11_i}^T \\ -Y^T S^{\dagger}{}_2^T A_{12_i}^T + \varepsilon Q \end{pmatrix} & A_{11_j}^d Q - A_{12_j}^d S_2^{\dagger} Y & \tau QA_{11_i}^T - Y^T S^{\dagger}{}_2^T A_{12_i}^T \\ (A_{11_j}^d Q - A_{12_j}^d S_2^{\dagger} Y)^T & -\varepsilon Q & \tau (A_{11_j}^d Q - A_{12_j}^d S_2^{\dagger} Y)^T \\ \tau A_{11_i} Q - A_{12_i} S_2^{\dagger} Y & \tau A_{11_j}^d Q - A_{12_j}^d S_2^{\dagger} Y & -\tau Q \end{bmatrix}$$
$$+ \begin{bmatrix} -\begin{bmatrix} \overline{M} & -\overline{M} \end{bmatrix} - \begin{bmatrix} \overline{M}^T \\ -\overline{M}^T \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \quad (5.13)$$

$$\omega_{2_i} = \begin{bmatrix} QA_{11_i}^T - Y^T S_2^{\dagger}{}^T A_{12_i}^T \\ QA_{11_j}^d{}^T - Y^T S_2^{\dagger}{}^T A_{12_j}^d{}^T \\ a_{1_i}^T \end{bmatrix}$$

with

$$
\begin{aligned}
S_2^{\dagger} &= S_2^T (S_2 S_2^T)^{-1} \\
\overline{N} &= \overline{Q} N Q \\
\overline{M} &= \overline{Q}_0 M Q \\
\overline{Q} &= \begin{bmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\overline{Q}_0 &= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}
\end{aligned}
$$

and $N \in \mathbb{R}^{(2(n-m)+1)\times(n-m)}$ *and* $M \in \mathbb{R}^{2(n-m)\times(n-m)}$.

Proof. The initial procedure of the proof is almost similar to the proof of Theorem 3.3.1. Consider a surface of the form

$$\sigma(x(t)) = Sx(t) = 0 \tag{5.14}$$

where

$$S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{5.15}$$

with $S_1 \in \mathbb{R}^{p\times(n-m)}$ and $S_2 \in \mathbb{R}^{p\times m}$, where p is the number of the inputs to (5.7). In order to make $\sigma(x(t)) = 0$ an attractive invariant set, we define a candidate Lyapunov function of the form

$$V(\sigma(x(t))) = \frac{1}{2}\sigma^T(x(t))\sigma(x(t)). \tag{5.16}$$

Note that, although $V(\sigma(x(t)))$ is implicitly based on $x(t)$, it is not a Lyapunov function for $x(t)$, but it is rather a Lyapunov function for $\sigma(x(t))$. As a function of $\sigma(x(t))$, $V(\sigma(x(t)))$ is obviously positive definite because it is a norm. In order to have finite-time convergence to $\sigma(x(t)) = 0$, according to [70] and [71] one needs to ensure

$$\dot{V}(\sigma(x(t))) \leq -\mu \left\|\sigma(x(t))\right\| \tag{5.17}$$

where $\mu > 0$. Note that, the Lie derivative of the Lyapunov function in (5.16) is

$$\dot{V}(\sigma(x(t))) = \frac{\partial V(\sigma(x(t)))}{\partial \sigma(x(t))}\dot{\sigma}(x(t)) = \sigma^T(x(t))\dot{\sigma}(x(t)). \tag{5.18}$$

We design $\sigma(x(t))$ such that

$$\dot{\sigma}(x(t)) = -\gamma\left(\frac{\sigma(x(t))}{\|\sigma(x(t))\|}\right) \tag{5.19}$$

with $\gamma \geq \mu > 0$, and the time rate of change of the Lyapunov function in (5.16) will be

$$\begin{aligned}\dot{V}(\sigma(x(t))) &= -\gamma\sigma^T(x(t))\left(\frac{\sigma(x(t))}{\|\sigma(x(t))\|}\right)\\ &= -\gamma\|\sigma(x(t))\| \leq -\mu\,\|\sigma(x(t))\|,\end{aligned} \tag{5.20}$$

which verifies (5.17). Using (5.7), (5.14) and (5.15) one can write

$$\begin{aligned}\dot{\sigma}(x(t)) = S\dot{x}(t) = S_1(&A_{11_i}x_1(t) + A_{12_i}x_2(t) + A^d_{11_j}x_1(t-\tau) + A^d_{12_j}x_2(t-\tau) + a_{1_i})\\ &+ S_2(A_{21_i}x_1(t) + A_{22_i}x_2 + A^d_{21_j}x_1(t-\tau) + A^d_{22_j}x_2(t-\tau) + a_{2_i}) + (S_2B_{2_i})u(t)\end{aligned} \tag{5.21}$$

Since B_{2_i} is either invertible or constant for all $i \in \mathscr{I}$ and full rank, $S_2B_{2_i}$ is invertible (for example with the choice $S_2 = B_2^T$ when $B_{2_i} = B_2$), and replacing the control law (5.9) into (5.21) ensures that (5.20) is verified. Therefore, the target surface $\sigma(x(t)) = 0$ is made an attractive invariant set. We now show that the trajectories converge to this target surface in finite time. Observe that (5.20) is equivalent to

$$\dot{V}(\sigma(x(t))) = -\gamma\sqrt{2}V^{\frac{1}{2}}(\sigma(x(t))) \tag{5.22}$$

for the Lyapunov function defined in (5.16). This is a differential equation. Assuming $V(\sigma(x(t_0)))$ as the initial condition, the solution to (5.22) can be found as

$$V^{\frac{1}{2}}(\sigma(x(t))) = V^{\frac{1}{2}}(\sigma(x(t_0))) - \frac{\sqrt{2}\gamma}{2}(t-t_0). \tag{5.23}$$

One now can see that

$$\exists t_c \in \mathbb{R}, \quad \text{such that} \quad V(\sigma(x(t_c))) = 0 \tag{5.24}$$

where $t_c \geq t_0$ is the finite time of convergence to the surface. In fact, replacing $V(\sigma(x(t_c))) = 0$ in (5.23) yields

$$t_c = \sqrt{2}\gamma^{-1}V^{\frac{1}{2}}(\sigma(x(t_0))) + t_0. \tag{5.25}$$

Furthermore (5.22) and (5.24) imply that

$$\dot{V}(\sigma(x(t_c))) = -\gamma\sqrt{2}V^{\frac{1}{2}}(\sigma(x(t_c))) = 0, \tag{5.26}$$

which yields

$$V^{\frac{1}{2}}(\sigma(x(t))) = 0, \qquad \forall t > t_c \tag{5.27}$$

and therefore

$$V(\sigma(x(t))) = 0, \qquad \forall t \geq t_c. \tag{5.28}$$

Since the trajectories converge in finite time to the surface $\sigma(x(t)) = 0$ and remain on that surface for all future times, using (5.14) and (5.15), for $t \geq t_c$ we can write

$$S_1x_1(t) + S_2x_2(t) = 0 \qquad \forall t \geq t_c. \tag{5.29}$$

Moreover,

$$S_1x_1(t-\tau) + S_2x_2(t-\tau) = 0 \qquad \forall t \geq t_c + \tau. \tag{5.30}$$

Now assuming

$$x_2(t) = S_2^T Z(t) \tag{5.31}$$

$$x_2(t-\tau) = S_2^T Z(t-\tau) \tag{5.32}$$

where $Z(t) \in \mathbb{R}^p$, we can rewrite (5.29) and (5.30) as

$$Z(t) = -(S_2S_2^T)^{-1}S_1x_1(t) \tag{5.33}$$

$$Z(t-\tau) = -(S_2S_2^T)^{-1}S_1x_1(t-\tau) \tag{5.34}$$

for all $t \geq t_c + \tau$. Hence

$$x_2(t) = -S_2^{\dagger} S_1 x_1(t) \tag{5.35}$$

$$x_2(t-\tau) = -S_2^{\dagger} S_1 x_1(t-\tau) \tag{5.36}$$

where

$$S_2^{\dagger} = S_2^T (S_2 S_2^T)^{-1} \tag{5.37}$$

is the pseudo-inverse of the matrix S_2. Therefore, using (5.7) and (5.35) we can rewrite the dynamics of the PWA system (5.7) for $t \geq t_c + \tau$ as

$$x_2(t) = -S_2^{\dagger} S_1 x_1(t) \tag{5.38}$$

$$\dot{x}_1(t) = (A_{11_i} - A_{12_i} S_2^{\dagger} S_1) x_1(t) + (A^d_{11_j} - A^d_{12_j} S_2^{\dagger} S_1) x_1(t-\tau) + a_{1_i}, \quad x(t) \in \mathscr{R}_i. \tag{5.39}$$

Due to (5.38), if $x_1(t)$ exponentially converges to the origin, then $x_2(t)$ will also exponentially converge to the origin. Therefore, exponential stability of the reduced order system (5.39) ensures that the PWA slab system (5.7) is exponentially stable under the control law (5.9). Therefore, in the rest part of the proof, we show that one can ensure the exponential stability of $x_1(t)$ using a set of linear matrix inequalities.

Consider the following candidate Lyapunov-Krasovskii functional

$$V_T = V_1 + V_2 + V_3, \tag{5.40}$$

with

$$\begin{aligned} V_1 &= x_1^T(t) P x_1(t), \\ V_2 &= \int_{t-\tau}^{t} x_1^T(s) X x_1(s) ds \\ V_3 &= \int_{-\tau}^{0} \int_{t+s}^{t} \dot{x}_1^T(\theta) R \dot{x}_1(\theta) d\theta ds \end{aligned}$$

where P, X, and R are symmetric positive definite matrices in $\mathbb{R}^{n-m \times n-m}$.

Note that, V_1, V_2, and V_3 are all positive definite functions. Hence, V_T in (5.40) is also positive definite. To prove exponential stability of the trajectories of $x_1(t)$ to the origin, it is sufficient to show that the decreasing rate of the Lyapunov-Krasovskii functional (5.40) is negative in each region $\mathscr{R}_i$.

The time derivative of V_T is as follows

$$\dot{V}_T = \dot{V}_1 + \dot{V}_2 + \dot{V}_3. \tag{5.41}$$

Therefore, the decreasing rate of the Lyapunov-Krasovskii functional (5.40) consists of three different components.

The time derivative of V_1 is

$$\dot{V}_1 = \dot{x}_1^T P x_1 + x_1^T P \dot{x}_1. \tag{5.42}$$

Applying the Leibniz integral rule, the time derivative of V_2 will be

$$\dot{V}_2 = x_1^T X x_1 - x_1^T(t-\tau) X x_1(t-\tau) \tag{5.43}$$

In order to obtain the time derivative of V_3, we first apply the Leibniz integral rule

$$\dot{V}_3 = \int_{-\tau}^{0} \left(\dot{x}_1^T(t) R \dot{x}_1(t) - \dot{x}_1^T(t+s) R \dot{x}_1(t+s) \right) ds \tag{5.44}$$

Therefore,

$$\dot{V}_3 = \tau \dot{x}_1^T(t) R \dot{x}_1(t) - \int_{-\tau}^{0} \dot{x}_1^T(t+s) R \dot{x}_1(t+s) ds. \tag{5.45}$$

Now by a change of variable, equation (5.45) will have the following form

$$\dot{V}_3 = \tau \dot{x}_1^T(t) R \dot{x}_1(t) - \int_{t-\tau}^{t} \dot{x}_1^T(\theta) R \dot{x}_1(\theta) d\theta. \tag{5.46}$$

Note that, since $R > 0$, for any arbitrary time varying vector $h(t,\tau) \in \mathbb{R}^{n-m}$, we can write

$$\begin{bmatrix} \dot{x}_1^T(\theta) & h^T(t,\tau) \end{bmatrix} \begin{bmatrix} R & -I \\ -I & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_1(\theta) \\ h(t,\tau) \end{bmatrix} \geq 0 \tag{5.47}$$

where I is the identity matrix of order $(n-m)$. Inequality (5.47) yields

$$-\dot{x}_1^T(\theta) R \dot{x}_1(\theta) \leq -h^T(t,\tau)\dot{x}_1(\theta) - \dot{x}_1^T(\theta) h(t,\tau) + h^T(t,\tau) R^{-1} h(t,\tau). \tag{5.48}$$

Integrating both sides of (5.48) with respect to θ we will have

$$\begin{aligned}-\int_{t-\tau}^{t}\dot{x}_1^T(\theta)R\dot{x}_1(\theta)d\theta \leq &-\int_{t-\tau}^{t}h^T(t,\tau)\dot{x}_1(\theta)d\theta-\int_{t-\tau}^{t}\dot{x}_1^T(\theta)h(t,\tau)d\theta\\ &+\int_{t-\tau}^{t}h^T(t,\tau)R^{-1}h(t,\tau)d\theta.\end{aligned} \tag{5.49}$$

Therefore,

$$\begin{aligned}-\int_{t-\tau}^{t}\dot{x}_1^T(\theta)R\dot{x}_1(\theta)d\theta \leq &-h^T(t,\tau)(x_1(t)-x_1(t-\tau))-(x_1^T(t)-x_1^T(t-\tau))h(t,\tau)\\ &+\tau h^T(t,\tau)R^{-1}h(t,\tau).\end{aligned} \tag{5.50}$$

Finally by replacing $-\int_{t-\tau}^{t}\dot{x}_1^T(\theta)R\dot{x}_1(\theta)d\theta$ from (5.50) in equation (5.46), the time derivative of V_3 will satisfy the following inequality

$$\begin{aligned}\dot{V}_3 \leq\ &\tau\dot{x}_1^T(t)R\dot{x}_1(t)-h^T(t,\tau)(x_1(t)-x_1(t-\tau))-(x_1^T(t)-x_1^T(t-\tau))h(t,\tau)\\ &+\tau h^T(t,\tau)R^{-1}h(t,\tau).\end{aligned} \tag{5.51}$$

Substituting (5.42), (5.43) and (5.51) in equation (5.41), the decreasing rate of the candidate Lyapunov-Krasovskii functional will satisfy the following inequality

$$\begin{aligned}\dot{V}_T \leq\ &\dot{x}_1^TPx_1+x_1^TP\dot{x}_1+x_1^TXx_1-x_1^T(t-\tau)Xx_1(t-\tau)+\tau\dot{x}_1^T(t)R\dot{x}_1(t)\\ &-h^T(t,\tau)(x_1(t)-x_1(t-\tau))-(x_1^T(t)-x_1^T(t-\tau))h(t,\tau)\\ &+\tau h^T(t,\tau)R^{-1}h(t,\tau).\end{aligned} \tag{5.52}$$

For the case where $0 \notin \overline{\mathscr{R}}_i$, we define a new augmented vector ξ

$$\xi = \begin{bmatrix} x_1(t) \\ x_1(t-\tau) \\ 1 \end{bmatrix}. \tag{5.53}$$

Recalling the dynamics of $x_1(t)$ from (5.39), inequality (5.52) will change to

$$
\begin{aligned}
\dot{V}_T \leq & \xi^T \left(\begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \\ a_{1_i}^T \end{bmatrix} P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} P \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} & a_{1_i} \end{bmatrix} \right. \\
& + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} X \begin{bmatrix} 0 & I & 0 \end{bmatrix} \\
& + \tau \begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \\ a_{1_i}^T \end{bmatrix} R \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} & a_{1_i} \end{bmatrix} \\
& \left. - N \begin{bmatrix} I & -I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} N^T + \tau N R^{-1} N^T \right) \xi
\end{aligned}
\tag{5.54}
$$

for $x(t) \in \mathscr{R}_i$ and $x(t-\tau) \in \mathscr{R}_j$, where

$$\overline{A}_{1i} = (A_{11_i} - A_{12_i} S_2^{\dagger} S_1) \tag{5.55}$$

$$\overline{A^d}_{1j} = (A^d_{11_j} - A^d_{12_j} S_2^{\dagger} S_1) \tag{5.56}$$

and $h(t,\tau)$ was replaced by

$$h(t,\tau) = N^T \xi$$

with arbitrary matrix N of appropriate dimension. Therefore,

$$\dot{V}_T \leq \xi^T (\Psi_{1_i} + \tau \Psi_{2_i} + \Psi_{3_i} + \tau \Psi_{4_i}) \xi \tag{5.57}$$

for $x(t) \in \mathscr{R}_i$ and $x(t-\tau) \in \mathscr{R}_j$, where

$$\Psi_{1_i} = \begin{bmatrix} \overline{A}_{1i}^T P + P\overline{A}_{1i} + X & P\overline{A^d}_{1j} & Pa_{1_i} \\ \overline{A^d}_{1j}^T P & -X & 0 \\ a_{1_i}^T P & 0 & 0 \end{bmatrix}$$

$$\Psi_{2_i} = \begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \\ a_{1_i}^T \end{bmatrix} R \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} & a_{1_i} \end{bmatrix}$$

$$\Psi_{3_i} = -N \begin{bmatrix} I & -I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} N^T$$

$$\Psi_{4_i} = NR^{-1}N^T.$$

Note also that, from (5.8), slab regions are described as follows

$$\mathscr{R}_i = \{x \mid \|L_{1i}x + l_i\| < 1\}. \tag{5.58}$$

Therefore,

$$(L_{1i}x_1(t) + l_i)^T (L_{1i}x_1(t) + l_i) < 1$$

or equivalently

$$\xi^T \begin{bmatrix} L_{1i}^T L_{1i} & 0 & L_{1i}^T l_i \\ 0 & 0 & 0 \\ l_i^T L_{1i} & 0 & l_i^T l_i - 1 \end{bmatrix} \xi < 0 \tag{5.59}$$

with ξ defined in (5.53).

Therefore, using (5.57), (5.59) and $\mathbb{S}$-*procedure*, the sufficient conditions for exponential stability of the system (5.39) can be described as in the following matrix inequalities

$$P = P^T > 0 \tag{5.60}$$

$$\Psi_{1_i} + \tau\Psi_{2_i} + \Psi_{3_i} + \tau\Psi_{4_i} < \lambda_i \begin{bmatrix} L_{1i}^T L_{1i} & 0 & L_{1i}^T l_i \\ 0 & 0 & 0 \\ l_i^T L_{1i} & 0 & l_i^T l_i - 1 \end{bmatrix} \tag{5.61}$$

with previously defined Ψ_{1_i}, Ψ_{2_i}, Ψ_{3_i}, Ψ_{4_i} and with $\lambda_i > 0$. Rearranging inequality (5.61) yields

$$\overline{\Psi}_{1_i} + \tau\Psi_{2_i} + \Psi_{3_i} + \tau\Psi_{4_i} < 0 \tag{5.62}$$

where

$$\overline{\Psi}_{1_i} = \begin{bmatrix} \overline{A}_{1i}^T P + P\overline{A}_{1i} - \lambda_i L_{1i}^T L_{1i} + X & P\overline{A^d}_{1j} & Pa_{1_i} - \lambda_i L_{1i}^T l_i \\ \overline{A^d}_{1j}^T P & -X & 0 \\ a_{1_i}^T P - \lambda_i l_i^T L_{1i} & 0 & \lambda_i(1 - l_i^T l_i) \end{bmatrix}. \tag{5.63}$$

Using new variables $Q = P^{-1}$, $\mu_i = \lambda_i^{-1}$ and left multiplying inequality (5.62) by $\overline{Q}$ and right multiplying it by $\overline{Q}^T = \overline{Q}$ with

$$\overline{Q} = \begin{bmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.64}$$

and making $X = \varepsilon Q^{-1}$ and $R = Q^{-1}$ yields the equivalent conditions

$$Q = Q^T > 0 \tag{5.65}$$

$$\Xi_{1_i} + \Xi_{2_i} + \tau\Xi_{3_i} Q^{-1} \Xi_{3_i}^T + \tau\overline{N} Q^{-1} \overline{N}^T < 0 \tag{5.66}$$

where

$$\Xi_{1_i} = \begin{bmatrix} Q\overline{A}_{1i}^T + \overline{A}_{1i}Q + \varepsilon Q - \mu_i^{-1} Q L_{1i}^T L_{1i} Q^T & \overline{A^d}_{1j} Q & a_{1_i} - \mu_i^{-1} Q L_{1i}^T l_i \\ Q\overline{A^d}_{1j}^T & -\varepsilon Q & 0 \\ a_{1_i}^T - \mu_i^{-1} l_i^T L_{1i} Q^T & 0 & \mu_i^{-1}(1 - l_i^T l_i) \end{bmatrix} \tag{5.67}$$

$$\Xi_{2_i} = -\begin{bmatrix} \overline{N} & -\overline{N} & 0 \end{bmatrix} - \begin{bmatrix} \overline{N}^T \\ -\overline{N}^T \\ 0 \end{bmatrix} \tag{5.68}$$

$$\Xi_{3_i} = \begin{bmatrix} Q\overline{A}_{1i}^T \\ Q\overline{A^d}_{1j}^T \\ a_{1_i}^T \end{bmatrix} \tag{5.69}$$

$$\overline{N} = \overline{Q} N Q \tag{5.70}$$

and ε is a positive scalar.

Note that, the following matrix inequalities are sufficient conditions for (5.65) and (5.66):

$$Q = Q^T > 0 \tag{5.71}$$

$$\Xi_{1_i} < 0 \tag{5.72}$$

$$\Xi_{2_i} + \tau \Xi_{3_i} Q^{-1} \Xi_{3_i}^T + \tau \overline{N} Q^{-1} \overline{N}^T \leq 0 \tag{5.73}$$

In other words, (5.71), (5.72) and (5.73) imply (5.65) and (5.66). Using Schur complement (see Lemma 2.2.1 and Lemma 2.2.2), matrix inequalities (5.71), (5.72) and (5.73) can be recast as

$$\Xi_{1_i} < 0 \tag{5.74}$$

$$\begin{bmatrix} \begin{bmatrix} \Xi_{2_i} & \tau \Xi_{3_i} \\ \tau \Xi_{3_i}^T & -\tau Q \end{bmatrix} & \begin{bmatrix} \tau \overline{N} \\ 0 \end{bmatrix} \\ \begin{bmatrix} \tau \overline{N}^T & 0 \end{bmatrix} & -\tau Q \end{bmatrix} \leq 0 \tag{5.75}$$

with $\tau > 0$. Now what is left to do is to show that matrix inequalities (5.74) and (5.75) are equivalent to (5.11) and (5.12) and prove the inequalities for the case $0 \in \mathscr{R}_i$. Substituting (5.67) in (5.74) we will have

$$\begin{bmatrix} Q\overline{A}_{1i}^T + \overline{A}_{1i}Q + \varepsilon Q - \mu_i^{-1}QL_{1i}^T L_{1i}Q^T & \overline{A^d}_{1j}Q & a_{1_i} - \mu_i^{-1}QL_{1i}^T l_i \\ Q\overline{A^d}_{1j}^T & -\varepsilon Q & 0 \\ a_{1_i}^T - \mu_i^{-1} l_i^T L_{1i} Q^T & 0 & \mu_i^{-1}(1 - l_i^T l_i) \end{bmatrix} < 0 \tag{5.76}$$

Using the Schur complement, (5.76) is equivalent to

$$(1 - l_i^T l_i) < 0 \tag{5.77}$$

$$\begin{aligned} &\begin{bmatrix} Q\overline{A}_{1i}^T + \overline{A}_{1i}Q + \varepsilon Q - \mu_i^{-1}QL_{1i}^T L_{1i}Q^T & \overline{A^d}_{1j}Q \\ Q\overline{A^d}_{1j}^T & -\varepsilon Q \end{bmatrix} \\ &- \begin{bmatrix} a_{1_i} - \mu_i^{-1}QL_{1i}^T l_i \\ 0 \end{bmatrix} \mu_i (1 - l_i^T l_i)^{-1} \begin{bmatrix} a_{1_i}^T - \mu_i^{-1} l_i^T L_{1i} Q & 0 \end{bmatrix} < 0 \end{aligned} \tag{5.78}$$

Expressions (5.77) and (5.78) can be rearranged to the form

$$(1 - l_i^T l_i) < 0 \tag{5.79}$$

$$\begin{bmatrix} \begin{pmatrix} Q\overline{A}_{1i}^T + \overline{A}_{1i}Q + \varepsilon Q - \mu^{-1}QL_{1i}^T L_{1i}Q \\ -(a_{1i} - \mu_i^{-1}QL_{1i}^T l_i)\mu_i(1 - l_i^T l_i)^{-1}(a_{1i} - \mu_i^{-1}QL_{1i}^T l_i)^T \end{pmatrix} & \overline{A^d}_{1j}Q \\ Q\overline{A^d}_{1j}^T & -\varepsilon Q \end{bmatrix} < 0 \tag{5.80}$$

Again using Schur complement, conditions (5.79) and (5.80) are equivalent to

$$(1 - l_i^T l_i) < 0 \tag{5.81}$$

$$-\varepsilon Q < 0 \tag{5.82}$$

$$\begin{aligned} &Q\overline{A}_{1i}^T + \overline{A}_{1i}Q + \varepsilon Q - \mu_i^{-1}QL_{1i}^T L_{1i}Q \\ &- (a_{1i} - \mu_i^{-1}QL_{1i}^T l_i)\mu_i(1 - l_i^T l_i)^{-1}(a_{1i} - \mu_i^{-1}QL_{1i}^T l_i)^T \\ &+ \varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T < 0 \end{aligned} \tag{5.83}$$

Using Matrix Inversion Lemma (see Lemma 2.2.3), it was shown in reference [30] that

$$\begin{aligned}&Q\overline{A}_i^T+\overline{A}_iQ+\alpha Q-\mu_i^{-1}QL_i^TL_iQ\\&-(a_i-\mu_i^{-1}QL_i^Tl_i)\mu_i(1-l_i^Tl_i)^{-1}(a_i-\mu_i^{-1}QL_i^Tl_i)^T<0\end{aligned} \tag{5.84}$$

is equivalent to

$$\begin{aligned}&Q\overline{A}_i^T+\overline{A}_iQ+\alpha Q-\mu_ia_ia_i^T\\&-(-\mu_ia_il_i^T+QL_i^T)\mu_i^{-1}(I-l_il_i^T)^{-1}(-\mu_ia_il_i^T+QL_i^T)^T<0.\end{aligned} \tag{5.85}$$

The difference between conditions (5.83) and (5.84) is the fact that in (5.83) $\overline{A}_i=\overline{A}_{1i}$, $a_i=a_{1i}$, $L_i=L_{1i}$, $\alpha=\varepsilon$ and there is one extra term, namely, $\varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T$. However, following a similar procedure as the one used in reference [30] we can conclude that condition (5.83) is equivalent to

$$\begin{aligned}&Q\overline{A}_{1i}^T+\overline{A}_{1i}Q+\varepsilon Q+\varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T-\mu_ia_{1i}a_{1i}^T\\&-(-\mu_ia_{1i}l_i^T+QL_{1i}^T)\mu_i^{-1}(I-l_il_i^T)^{-1}(-\mu_ia_{1i}l_i^T+QL_{1i}^T)^T<0\end{aligned} \tag{5.86}$$

Using the fact that $1-l_i^Tl_i$ and $I-l_il_i^T$ are equivalent when l_i is a scalar, which is the case for piecewise-affine slab systems, inequality (5.86) can be further change to

$$\begin{aligned}&Q\overline{A}_{1i}^T+\overline{A}_{1i}Q+\varepsilon Q+\varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T-\mu_ia_{1i}a_{1i}^T\\&-(-\mu_ia_{1i}l_i^T+QL_{1i}^T)\mu_i^{-1}(1-l_i^Tl_i)^{-1}(-\mu_ia_{1i}l_i^T+QL_{1i}^T)^T<0.\end{aligned} \tag{5.87}$$

Therefore, conditions (5.81), (5.82) and (5.83) are equivalent to

$$(1-l_i^Tl_i)<0 \tag{5.88}$$

$$-\varepsilon Q<0 \tag{5.89}$$

$$\begin{aligned}&Q\overline{A}_{1i}^T+\overline{A}_{1i}Q+\varepsilon Q+\varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T-\mu_ia_{1i}a_{1i}^T\\&-(-\mu_ia_{1i}l_i^T+QL_{1i}^T)\mu_i^{-1}(1-l_i^Tl_i)^{-1}(-\mu_ia_{1i}l_i^T+QL_{1i}^T)^T<0\end{aligned} \tag{5.90}$$

Note that, conditions (5.88) and (5.90) are also equivalent to

$$\begin{bmatrix}\overline{A}_{1i}Q+Q\overline{A}_{1i}^T+\varepsilon Q-\mu_ia_{1i}a_{1i}^T+\varepsilon^{-1}\overline{A^d}_{1j}Q\overline{A^d}_{1j}^T & (-\mu_ia_{1i}l_i^T+QL_{1i}^T)\\(-\mu_ia_{1i}l_i^T+QL_{1i}^T)^T & \mu_i(1-l_i^Tl_i)\end{bmatrix}<0 \tag{5.91}$$

One can verify this by simply using the Schur complement. Inequality (5.91) can then be rearranged in the form

$$\begin{bmatrix} \overline{A}_{1i}Q + Q\overline{A}_{1i}^T + \varepsilon Q - \mu_i a_{1i} a_{1i}^T & (-\mu_i a_{1i} l_i^T + QL_{1i}^T) \\ (-\mu_i a_{1i} l_i^T + QL_{1i}^T)^T & \mu_i(1 - l_i^T l_i) \end{bmatrix} + \begin{bmatrix} \overline{A^d}_{1j} Q \\ 0 \end{bmatrix} \varepsilon^{-1} Q^{-1} \begin{bmatrix} Q\overline{A^d}_{1j}^T & 0 \end{bmatrix} < 0. \tag{5.92}$$

Finally, conditions (5.92) and (5.89) will be equivalent to the following matrix inequality

$$\Omega_{1_i} = \begin{bmatrix} \overline{A}_{1i}Q + Q\overline{A}_{1i}^T + \varepsilon Q - \mu_i a_{1i} a_{1i}^T & (-\mu_i a_{1i} l_i^T + QL_{1i}^T) & \overline{A^d}_{1j} Q \\ (-\mu_i a_{1i} l_i^T + QL_{1i}^T)^T & \mu_i(1 - l_i^T l_i) & 0 \\ Q\overline{A^d}_{1j}^T & 0 & -\varepsilon Q \end{bmatrix} < 0. \tag{5.93}$$

Hence (5.76) is equivalent to (5.93) and therefore exponential stability sufficient conditions (5.74) and (5.75) for system (5.39) will be equivalent to

$$\Omega_{1_i} < 0 \tag{5.94}$$

$$\begin{bmatrix} \begin{bmatrix} \Xi_{2_i} & \tau \Xi_{3_i} \\ \tau \Xi_{3_i}^T & -\tau Q \end{bmatrix} & \begin{bmatrix} \tau \overline{N} \\ 0 \end{bmatrix} \\ \begin{bmatrix} \tau \overline{N}^T & 0 \end{bmatrix} & -\tau Q \end{bmatrix} \leq 0. \tag{5.95}$$

Finally, we replace

$$\overline{A}_{1i} = (A_{11_i} - A_{12_i} S_2^{\dagger} S_1) \tag{5.96}$$

$$\overline{A^d}_{1j} = (A^d_{11_j} - A^d_{12_j} S_2^{\dagger} S_1) \tag{5.97}$$

$$S_1 Q = Y \tag{5.98}$$

in (5.94) and (5.95). Therefore, exponential stability of the reduced order system (5.39) is guaranteed if the LMIs (5.11) and (5.12) hold.

Note that, for the case $0 \in \overline{\mathscr{R}}_i$, affine term a_{1i} is zero. Therefore, using (5.52) and a new augmented vector ξ_0 as

$$\xi_0 = \begin{bmatrix} x_1(t) \\ x_1(t-\tau) \end{bmatrix} \tag{5.99}$$

we will have

$$
\begin{aligned}
\dot{V}_T \leq \xi_0^T \Bigg(& \begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \end{bmatrix} P \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} P \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} \end{bmatrix} \\
& + \begin{bmatrix} I \\ 0 \end{bmatrix} X \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} X \begin{bmatrix} 0 & I \end{bmatrix} \\
& + \tau \begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \end{bmatrix} R \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} \end{bmatrix} \\
& - M \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} M^T + \tau M R^{-1} M^T \Bigg) \xi_0
\end{aligned}
\tag{5.100}
$$

where

$$
\overline{A}_{1i} = (A_{11_i} - A_{12_i} S_2^{\dagger} S_1)
$$

$$
\overline{A^d}_{1j} = (A^d_{11_j} - A^d_{12_j} S_2^{\dagger} S_1)
$$

and $h(t,\tau)$ was replaced by

$$
h(t,\tau) = M^T \xi_0
$$

with arbitrary matrix M of appropriate dimension. Rearranging the above inequality, sufficient conditions for exponential stability of the reduced order system (5.39) will be

$$
\begin{aligned}
& \begin{bmatrix} \overline{A}_{1i}^T P + P\overline{A}_{1i} + X & P\overline{A^d}_{1j} \\ \overline{A^d}_{1j}^T P & -X \end{bmatrix} + \tau \begin{bmatrix} \overline{A}_{1i}^T \\ \overline{A^d}_{1j}^T \end{bmatrix} R \begin{bmatrix} \overline{A}_{1i} & \overline{A^d}_{1j} \end{bmatrix} \\
& - \begin{bmatrix} M & -M \end{bmatrix} - \begin{bmatrix} M^T \\ -M^T \end{bmatrix} + \tau M R^{-1} M^T < 0.
\end{aligned}
\tag{5.101}
$$

Using new variables $Q = P^{-1}$, $\mu_i = \lambda_i^{-1}$ and left multiplying inequality (5.62) by $\overline{Q}_0$ and right multiplying it by $\overline{Q}_0^T = \overline{Q}_0$ with

$$
\overline{Q}_0 = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}, \tag{5.102}
$$

matrix inequality (5.101) can be rewritten in the following form

$$\begin{aligned}&\begin{bmatrix}\overline{A}_{1i}^TQ+Q\overline{A}_{1i}+QXQ^T & \overline{A^d}_{1j}Q\\ Q\overline{A^d}_{1j}^T & -QXQ^T\end{bmatrix}+\tau\begin{bmatrix}Q\overline{A}_{1i}^T\\ Q\overline{A^d}_{1j}^T\end{bmatrix}R\begin{bmatrix}\overline{A}_{1i}Q & \overline{A^d}_{1j}Q\end{bmatrix}\\ &-\begin{bmatrix}\overline{M} & -\overline{M}\end{bmatrix}-\begin{bmatrix}\overline{M}^T\\ -\overline{M}^T\end{bmatrix}+\tau\overline{M}Q^{-1}R^{-1}Q^{-1}\overline{M}^T<0\end{aligned}\tag{5.103}$$

where

$$\overline{M}=\overline{Q}_0MQ.\tag{5.104}$$

Now using Schur complement, sufficient conditions from inequality (5.103) will be

$$\begin{bmatrix}\Omega_{i_0} & \tau\begin{bmatrix}\overline{M}Q^{-1}\\ 0\end{bmatrix}\\ \tau\begin{bmatrix}Q^{-1}\overline{M}^T & 0\end{bmatrix} & -\tau R\end{bmatrix}<0\tag{5.105}$$

where

$$\Omega_{i_0}=\begin{bmatrix}\left(\begin{matrix}\begin{bmatrix}\overline{A}_{1i}^TQ+Q\overline{A}_{1i}+QXQ^T & \overline{A^d}_{1j}Q\\ Q\overline{A^d}_{1j}^T & -QXQ^T\end{bmatrix}+\\ -\begin{bmatrix}\overline{M} & -\overline{M}\end{bmatrix}-\begin{bmatrix}\overline{M}^T\\ -\overline{M}^T\end{bmatrix}\end{matrix}\right) & \tau\begin{bmatrix}Q\overline{A}_{1i}^T\\ \\ Q\overline{A^d}_{1j}^T\end{bmatrix}\\ \tau\begin{bmatrix}\overline{A}_{1i}Q & \overline{A^d}_{1j}Q\end{bmatrix} & -\tau R^{-1}\end{bmatrix}.\tag{5.106}$$

Substituting (5.96), (5.97) and (5.98) in (5.105) , LMI condition (5.10) will be obtained with

$$X=\varepsilon Q^{-1}$$

$$R=Q^{-1}.$$

This finishes the proof. □

Remark 5.3.1. *Note that, since the structure of the controller (5.9) depends on $x_1(t-\tau)$ and $x_2(t-\tau)$, the delay considered in this work must be known. Moreover, the delay must be constant and must also be associated with the states of the system.*

Remark 5.3.2. *Note also that, although assuming an upper-bound on the delay will not affect the derivation of the LMIs, it will destroy the structure of the control signal (5.9) which depends on τ.*

Remark 5.3.3. *Although assuming unknown and/or time-varying delays would enlarge the class considered in this work, there are still some applications that the proposed method can be applied to, such as a water channel or liquid-level systems. In these applications the delays are caused by the connecting (long) pipes and therefore, are measurable and constant.*

5.4 Numerical Example

In order to illustrate how the proposed method work, a simple second order PWA time-delay system is considered in this section. Consider the following piecewise-affine time-delay system when

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } 0 \in \mathscr{R}_1 \\
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{if } 0 \in \mathscr{R}_2 \\
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } 0 \in \mathscr{R}_3
\end{aligned} \tag{5.107}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, τ is a constant known delay and

$$\begin{cases}
\mathscr{R}_1 &= \left\{ x \in \mathbb{R}^2 \mid -2 < x_1 < -1 \right\} \\
\mathscr{R}_2 &= \left\{ x \in \mathbb{R}^2 \mid -1 < x_1 < 1 \right\} \\
\mathscr{R}_3 &= \left\{ x \in \mathbb{R}^2 \mid 1 < x_1 < 2 \right\}.
\end{cases}$$

We first, consider the case when there is no time-delay involved in the dynamics of the PWA system. In other words we first study the case where $\tau = 0$ in state dynamics

(5.107). Assuming that the time-delay is zero, dynamics (5.107) will be equivalent to the following system

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0.1 & 1.1 \\ 1.1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } 0 \in \mathscr{R}_1 \\
\dot{x}(t) &= \begin{bmatrix} 0.1 & 1.1 \\ -0.9 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{if } 0 \in \mathscr{R}_2 \\
\dot{x}(t) &= \begin{bmatrix} 0.1 & 1.1 \\ 1.1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{if } 0 \in \mathscr{R}_3
\end{aligned} \tag{5.108}
$$

where regions were previously defined. Note that, in order to design control laws for system (5.108), one may consider two different approaches:

1. Applying the results of Theorem 3.3.1 to PWA system (5.108)

2. Applying the results of Theorem 5.3.1 to PWA time-delay system (5.107) with $\tau = 0$

Here, we consider both approaches. Applying the conditions of Theorem 3.3.1 to PWA system (5.108), the PWA controllers are designed. Figure 5.1 shows the simulation results for the closed-loop system with $x(0) = \begin{bmatrix} -1.5 & 0.2 \end{bmatrix}^T$ as the initial conditions.

Applying the results of Theorem 5.3.1 with $\tau = 0$ for PWA time-delay system (5.107) also yields to PWA controllers, which after being applied to the system the simulation results for the closed-loop system are obtained and shown in Figure 5.2 with $x(0) = \begin{bmatrix} 1.2 & -.5 \end{bmatrix}^T$ as the initial conditions.

In the next step, in order to show how the results of Theorem 5.3.1 work for the case where time-delay is involved, we consider PWA time-delay system (5.107) with $\tau = 5$ seconds. After applying the designed controllers to the system, the simulation results are obtained. Figure 5.3 and Figure 5.5 shows the simulation results with $x(0) = \begin{bmatrix} -1.5 & 0.2 \end{bmatrix}^T$ and $x(0) = \begin{bmatrix} 1.2 & -.5 \end{bmatrix}^T$, respectively. As you can see, these simulation results demonstrate that the trajectories of the system still converge to the origin in finite time in the presence of a constant time-delay.

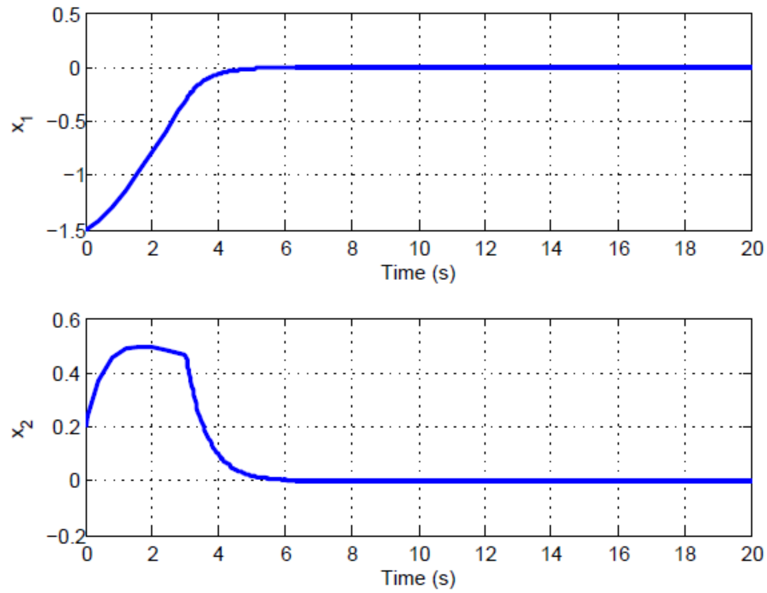

Figure 5.1: State variables, applying Theorem 3.3.1 to PWA system (5.108)

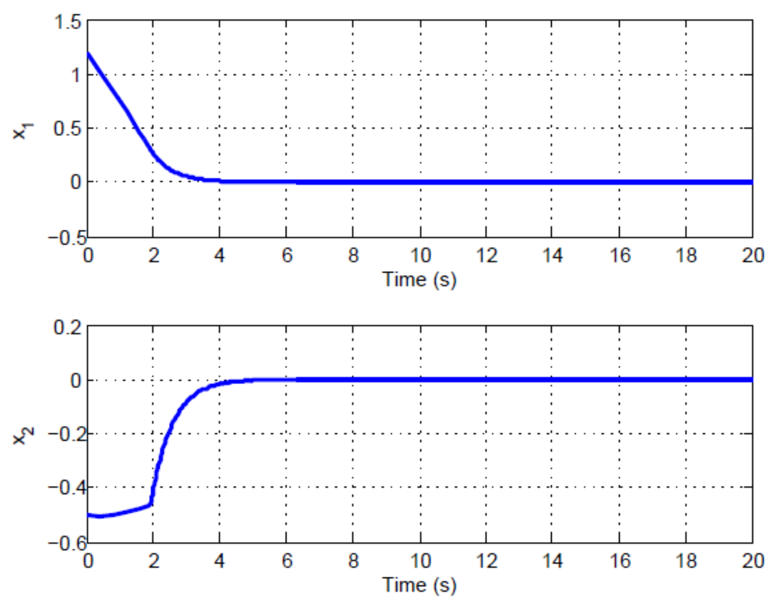

Figure 5.2: States variables, applying Theorem 5.3.1 to PWA system (5.107) with $\tau = 0$

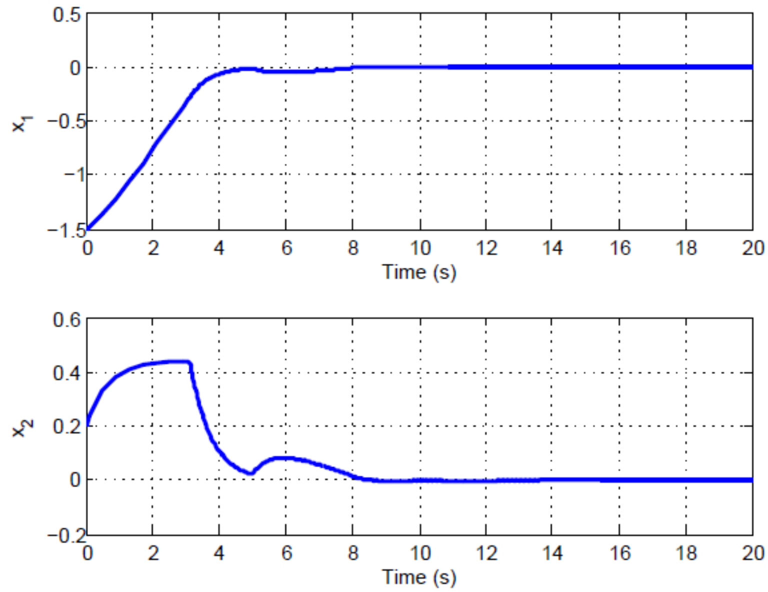

Figure 5.3: States for the case when time-delay is 5 seconds

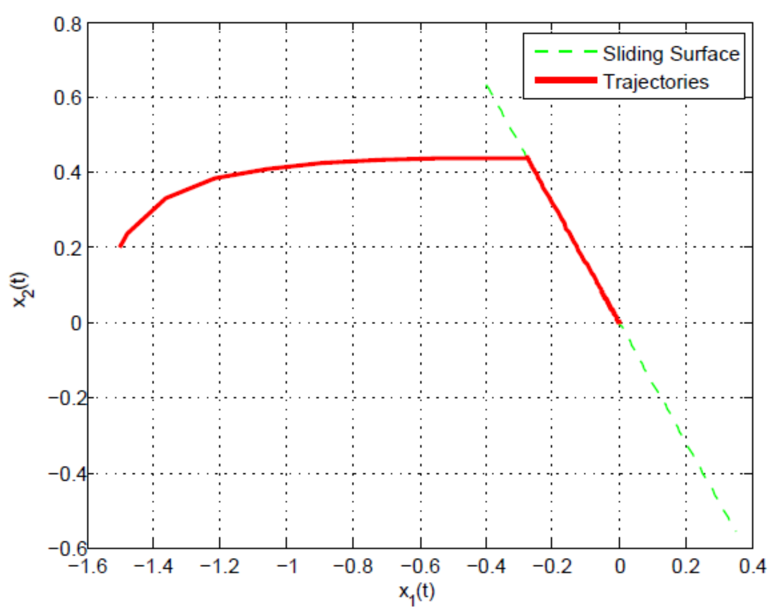

Figure 5.4: Trajectories for the case when time-delay is 5 seconds

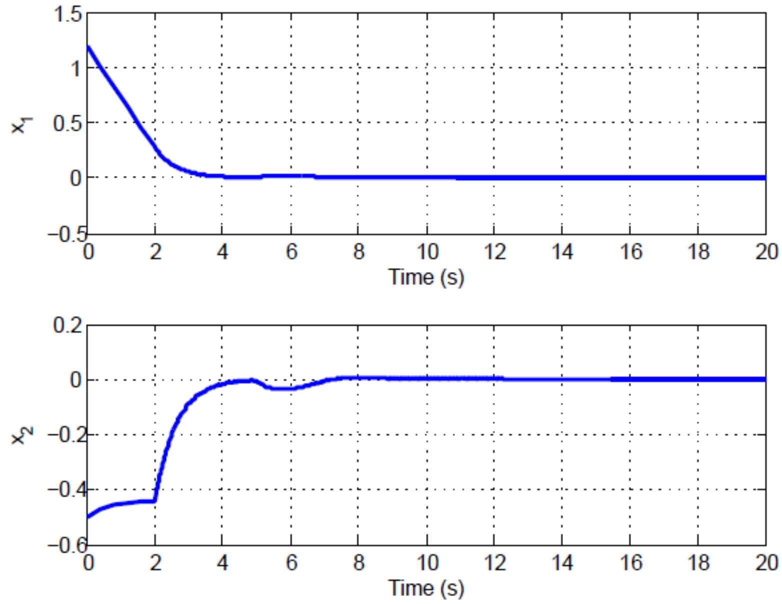

Figure 5.5: States for the case when time-delay is 5 seconds

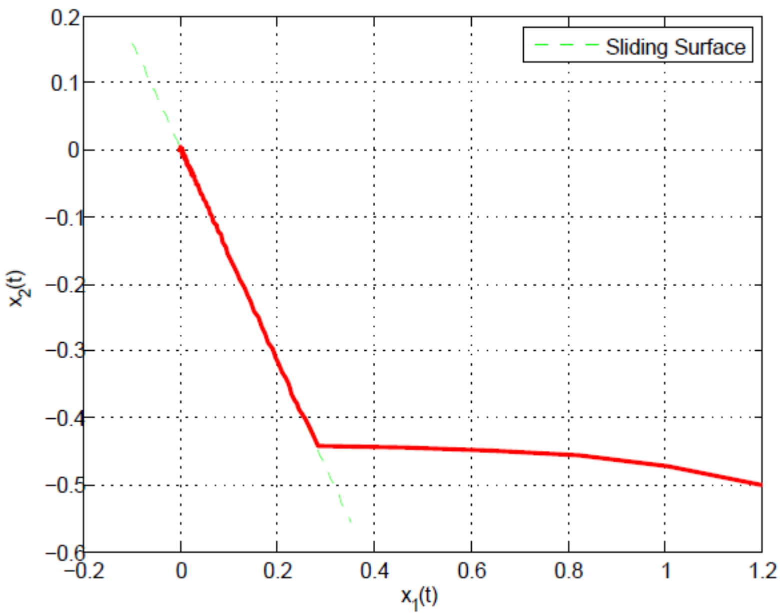

Figure 5.6: Trajectories for the case when time-delay is 5 seconds

5.5 Summary

The contribution of this chapter is to formulate the PWA time-delay synthesis problem as a set of LMIs. In order to do so, we first defined a sliding surface, then control laws were designed to make the trajectories approach the specified surface and ensure that the trajectories would remain on that surface. Then, using Lyapunov-Krasovskii functionals, sufficient conditions for exponential stability of the resulting reduced order system were proposed. Moreover, the designed control laws were still in PWA state feedback form. A numerical example demonstrated the effectiveness of the approach. However, considering the delay known and constant is one of the limitations of this approach. Moreover, the delay that we considered in this work is only due to the states of the system and if the delay appears in the input(s) and/or in the derivative of the states, the proposed method cannot be applied.

Chapter 6

Conclusion

The contributions of this thesis are summarized and potential extensions of the proposed methods are discussed in this chapter. The contributions of the first part of this thesis answered the following popular questions:

- Is it possible to directly formulate the piecewise-affine synthesis problem as a convex program?
- How much conservative is the proposed approach compared to the other methods?

The answer to the first question is "YES". Chapter 3 of this thesis for the first time proposed a novel approach that uses invariant set ideas to *directly* formulate the PWA synthesis problem as a set of Linear Matrix Inequalities (LMIs), which are convex problems. It was also shown that the dimension of the LMIs obtained in this work is lower than in the other convex methods in the literature.

Furthermore, in Chapter 4, it was shown that for every solution to the LMIs resulting from previous approaches, there exists a solution for the LMIs obtained from the proposed method. It was also shown that while previous convex controller synthesis methods have no solutions to their LMIs for some examples of PWA systems, the approach proposed in this thesis yields a solution for these examples. Therefore, the answer to the second question will be: "The proposed approach is less conservative than the other methods".

Although in Chapter 3 and Chapter 4 we addressed the first two questions, the following questions were remained:

- What will happen if the nonlinearities are associated with x_2 rather than x_1?
- Is it possible to come up with a larger class of PWA system that their controller syntheses can be similarly cast as a convex optimization problem?
- How can one extend the work to the tracking problem?

As it was shown in chapter 3, the proposed method only works for a special class of PWA system and furthermore one of the assumptions was the regions were partition based on x_1 (a subvector of the states) and therefore, no method proposed when regions partitioning was associated with x_2. Note also that, having information on trajectories of reference signals and defining a new sliding surface based on the error signals, it seems that the extension of the work to the tracking problem might also be possible.

The contributions of the last part of this thesis answered the following questions:

- Is it possible to directly formulate the PWA time-delay synthesis problem as a convex problem too?

The answer to this question is also “YES”. Chapter 5 of this thesis proposed an approach that used sliding mode control ideas to *directly* formulate the PWA synthesis problem as a set of LMIs. In order to do so, we first defined a sliding surface, then control laws were designed to make the trajectories approach the specified sliding surface and ensured that they would remain on that surface. Then, using Lyapunov-Krasovskii functionals, sufficient conditions for exponential stability of the resulting reduced order system were proposed. Moreover, the designed control laws were still in PWA state feedback form.

However, the following questions were remained:

- What will happen if the delay τ is unknown or time-varying?

- How can we come up with less conservative conditions?

- What will happen if the delay is associated with inputs or the derivative of the states?

As it was shown in chapter 5, the proposed time-delay method only works for the case of a known constant delay. In fact, since the designed control law included a term containing τ (the delay), having information about the value of the delay was crucial. Furthermore, since the derived conditions were sufficient conditions, conservatism was already introduced to the system and therefore, using more sufficient conditions during the proof, increased the conservatism of the proposed approach. Note also that, considering the case where the delay is associated with the derivative of the states and/or the inputs of the system, will further relax the conservativeness of the proposed approach.

Appendix

Parameters	Values
K_p	$\mathrm{diag}(0.5, 0.5)$
K_d	$\mathrm{diag}(0.05, 0.05)$
a	-0.45
U	$30\,\mathrm{m/s}$
M	$\begin{bmatrix} 12.387 & 0.418 \\ 0.418 & 0.065 \end{bmatrix}$
m	12.387
I_α	$0.065\,\mathrm{kgm}^2$
K_o	$\begin{bmatrix} 2844.4 & 0 \\ 0 & 0 \end{bmatrix}\,\mathrm{N/m}$
k_h	2844.4
K_μ	$\begin{bmatrix} 0 & 935.1 \\ 0 & -6.3 \end{bmatrix}\,\mathrm{kg/s}^2$
C_o	$\mathrm{diag}(27.43, 0.036)$

C_h	27.43
C_α	$0.036\,\mathrm{kgm^2/s}$
C_μ	$\begin{bmatrix} 31.17 & 3.99 \\ 0.21 & -0.027 \end{bmatrix}$
$K_\alpha(\alpha)$	$q\alpha$
q	$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T$
q_1	2.82
q_2	-62.322
q_3	3709.71
q_4	-24195.6
q_5	48756.954
μ	176.609
$C_{L\beta_1}$	3.358
$C_{L\beta_2}$	3.458
$C_{m\beta_1}$	-0.635
$C_{m\beta_2}$	-0.735
ρ	$1.225\,\mathrm{kg/m^3}$
$C_{m\alpha}$	$(0.5+a)C_{L\alpha}$
$C_{L\alpha}$	6.28

Bibliography

[1] L. Rodrigues, *Dynamic output feedback controller synthesis for piecewise-affine systems*. Stanford University, 2002.

[2] B. Samadi and L. Rodrigues, “Stability of sampled-data piecewise affine systems: A time-delay approach,” *Automatica*, vol. 45, no. 9, pp. 1995–2001, 2009.

[3] S. Afkhami and H. Alighanbari, “Nonlinear control design of an airfoil with active flutter suppression in the presence of disturbance,” *Control Theory & Applications, IET*, vol. 1, no. 6, pp. 1638–1649, 2007.

[4] B. Samadi and L. Rodrigues, “Controller synthesis for piecewise affine slab differential inclusions: A duality-based convex optimization approach,” in *in Proc. 46th IEEE Conference on Decision and Control*. IEEE, 2007, pp. 4999–5004.

[5] J. Lygeros, “Lecture notes on hybrid systems,” in *Notes for an ENSIETA workshop*, 2004.

[6] D. Liberzon, *Switching in systems and control*. Springer, 2003.

[7] H. Witsenhausen, “A class of hybrid continuous-time dynamic system,[j],” *IEEE Transaction on Control*, vol. 11, no. 6, pp. 665–683, 1966.

[8] F. Tossisi and A. Bemporad, “Hysdel-a tool for generating computational hybrid models for analysis and design problems,” *IEEE Trans. Control Syst. Technol*, vol. 12, pp. 235–249, 2004.

[9] T. Schlegl, M. Buss, and G. Schmidt, "A hybrid systems approach toward modeling and dynamical simulation of dextrous manipulation," *Mechatronics, IEEE/ASME Transactions on*, vol. 8, no. 3, pp. 352–361, 2003.

[10] G. Fourlas, K. Kyriakopoulos, and C. Vournas, "Hybrid systems modeling for power systems," *Circuits and Systems Magazine, IEEE*, vol. 4, no. 3, pp. 16–23, 2004.

[11] A. Bemporad, "Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form," *Automatic Control, IEEE Transactions on*, vol. 49, no. 5, pp. 832–838, 2004.

[12] V. Blondel and J. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems," *AUTOMATICA-OXFORD-*, vol. 35, pp. 479–490, 1999.

[13] M. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 475–482, 1998.

[14] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[15] J. Hespanha, "Uniform stability of switched linear systems: extensions of lasalle's invariance principle," *Automatic Control, IEEE Transactions on*, vol. 49, no. 4, pp. 470–482, 2004.

[16] A. Michel and B. Hu, "Towards a stability theory of general hybrid dynamical systems," *AUTOMATICA-OXFORD-*, vol. 35, pp. 371–384, 1999.

[17] S. Pettersson, "Analysis and design of hybrid systems," Ph.D. dissertation, Chalmers University of Technology, 1999.

[18] S. Prajna and A. Papachristodoulou, "Analysis of switched and hybrid systems-beyond piecewise quadratic methods," in *American Control Conference, 2003. Proceedings of the 2003*, vol. 4. IEEE, 2003, pp. 2779–2784.

[19] H. Ye, A. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 461–474, 1998.

[20] A. Rantzer and M. Johansson, "Piecewise linear quadratic optimal control," *Automatic Control, IEEE Transactions on*, vol. 45, no. 4, pp. 629–637, 2000.

[21] P. Julian, A. Desages, and O. Agamennoni, "High-level canonical piecewise linear representation using a simplicial partition," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 46, no. 4, pp. 463–480, 1999.

[22] L. Rodrigues and J. How, "Automated control design for a piecewise-affine approximation of a class of nonlinear systems," in *in Proc. American Control Conference.*, vol. 4. IEEE, 2001, pp. 3189–3194.

[23] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *in Proc. American Control Conference*, vol. 6. IEEE, 1998, pp. 3659–3664.

[24] V. Carmona, E. Freire, E. Ponce, and F. Torres, "On simplifying and classifying piecewise-linear systems," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 5, pp. 609–620, 2002.

[25] W. Heemels, M. Camlibel, and J. Schumacher, "On the dynamic analysis of piecewise-linear networks," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 3, pp. 315–327, 2002.

[26] P. Mosterman and G. Biswas, "Towards procedures for systematically deriving hybrid models of complex systems," *Hybrid Systems: Computation and Control*, pp. 324–337, 2000.

[27] J. Voros, “Modeling and parameter identification of systems with multisegment piecewise-linear characteristics,” *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pp. 184–188, 2002.

[28] M. Johansson, *Piecewise linear control systems: a computational approach*. Springer Verlag, 2003, vol. 284.

[29] L. Rodrigues, “Stability analysis of piecewise-affine systems using controlled invariant sets,” *Systems & control letters*, vol. 53, no. 2, pp. 157–169, 2004.

[30] L. Rodrigues and S. Boyd, “Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization,” *Systems & Control Letters*, vol. 54, no. 9, pp. 835–853, 2005.

[31] G. F. Jianbin Qiu, Tiejun Zhang and H. Liu, “Piecewise affine model based h-infinity static output feedback control of constrained nonlinear processes,” *IET Control Theory and Applications*, vol. 4, no. 11, pp. 2315–2330, Nov. 2010.

[32] A. Bemporad, G. Ferrari-Trecate, and M. Morari, “Observability and controllability of piecewise affine and hybrid systems,” *IEEE Transactions on Automatic Control*, vol. 45, no. 10, pp. 1864–1876, 2000.

[33] N. Van de Wouw and A. Pavlov, “Tracking and synchronisation for a class of pwa systems,” *Automatica*, vol. 44, no. 11, pp. 2909–2915, 2008.

[34] S. Bibian and H. Jin, “Time delay compensation of digital control for dc switchmode power supplies using prediction techniques,” *Power Electronics, IEEE Transactions on*, vol. 15, no. 5, pp. 835–842, 2000.

[35] W. Zhang, M. Branicky, and S. Phillips, “Stability of networked control systems,” *Control Systems Magazine, IEEE*, vol. 21, no. 1, pp. 84–99, 2001.

[36] E. Verriest, M. Fan, and J. Kullstam, “Frequency domain robust stability criteria for linear delay systems,” in *in Proceedings of the 32nd IEEE Conference on Decision and Control*. IEEE, 1993, pp. 3473–3478.

[37] S. WANG, B. CHEN, and T. LIN, “Robust stability of uncertain time-delay systems,” 1987.

[38] J. Su, “Further results on the robust stability of linear systems with a single time delay,” *Systems & Control Letters*, vol. 23, no. 5, pp. 375–379, 1994.

[39] Q. Han and K. Gu, “On robust stability of time-delay systems with norm-bounded uncertainty,” *Automatic Control, IEEE Transactions on*, vol. 46, no. 9, pp. 1426–1431, 2001.

[40] E. Fridman and U. Shaked, “Parameter dependent stability and stabilization of uncertain time-delay systems,” *Automatic Control, IEEE Transactions on*, vol. 48, no. 5, pp. 861–866, 2003.

[41] K. Moezzi, L. Rodrigues, and A. Aghdam, “Stability of uncertain piecewise affine systems with time delay: delay-dependent lyapunov approach,” *International Journal of Control*, vol. 82, no. 8, pp. 1423–1434, 2009.

[42] ——, “Stability of uncertain piecewise affine systems with time-delay,” in *American Control Conference, 2009. ACC’09.* IEEE, 2009, pp. 2373–2378.

[43] A. A. Andronov and S. E. Chaikin, *Theory of Oscillations*. Princeton University Press: Princeton, New Jersey, 1949.

[44] L. Chua, “Analysis and synthesis of multivalued memoryless nonlinear networks,” *IEEE Transactions on Circuit Theory*, vol. 14, no. 2, pp. 192–209, 1967.

[45] ——, “Efficient computer algorithms for piecewise-linear analysis of resistive nonlinear networks,” *IEEE Transactions on Circuit Theory*, vol. 18, no. 1, pp. 73–85, 1971.

[46] E. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346–358, 1981.

[47] N. Pettit and P. Wellstead, "Analyzing piecewise linear dynamical systems," *IEEE Control Systems Magazine*, vol. 15, no. 5, pp. 43–50, 1995.

[48] P. Peleties and R. DeCarlo, "Asymptotic stability of m-switched systems using lyapunov functions," in *Proceedings of the 31st IEEE Conference on Decision and Control*. IEEE, 1992, pp. 3438–3439.

[49] S. Boyd, L. El-Ghaoui, E. Feron, V. Balakrishnan, and E. Yaz, "Linear matrix inequalities in system and control theory," *Proceedings of the IEEE*, vol. 85, no. 4, pp. 698–699, 1997.

[50] M. Branicky, "General hybrid dynamical systems: Modeling, analysis, and control," *Hybrid Systems III*, pp. 186–200, 1996.

[51] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[52] J. Gonçalves, A. Megretski, and M. Dahleh, "Global analysis of piecewise linear systems using impact maps and surface lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2089–2106, 2003.

[53] S. Hedlund and M. Johansson, "A matlab toolbox for analysis of piecewise linear systems," 1999.

[54] M. Johansson and A. Rantzer, "Computation of piecewise quadratic lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.

[55] L. Rodrigues and J. How, "Observer-based control of piecewise-affine systems," *International Journal of Control*, vol. 76, no. 5, pp. 459–477, 2003.

[56] A. Pavlov, A. Pogromsky, N. Van De Wouw, and H. Nijmeijer, “On convergence properties of piecewise affine systems,” *International Journal of Control*, vol. 80, no. 8, pp. 1233–1247, 2007.

[57] A. Pavlov, N. van de Wouw, and H. Nijmeijer, “Convergent piecewise affine systems: analysis and design part i: continuous case,” in *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC’05.* IEEE, 2005, pp. 5391–5396.

[58] D. Ding and G. Yang, “State-feedback control design for continuous-time piecewise linear systems: an lmi approach,” in *American Control Conference, 2008.* IEEE, 2008, pp. 1104–1108.

[59] B. Samadi and L. Rodrigues, “Backstepping controller synthesis for piecewise affine systems: A sum of squares approach,” in *IEEE International Conference on Systems, Man and Cybernetics.* IEEE, 2007, pp. 58–63.

[60] J. F. Strum, “Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones,” *Optimization Methods and Software*, vol. 11-12, pp. 625–653, 1999. [Online]. Available: http://sedumi.ie.lehigh.edu/

[61] J. Löfberg, “YALMIP: A Toolbox for Modeling and Optimization in MATLAB,” in *Proc. CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://users.isy.liu.se/johanl/yalmip/

[62] Y. Sun, L. Wang, and G. Xie, “Stability of switched systems with time-varying delays: delay-dependent common lyapunov functional approach,” in *American Control Conference, 2006*. IEEE, 2006, pp. 6–pp.

[63] G. Zhai, Y. Sun, X. Chen, and A. Michel, “Stability and 𝓁 2 gain analysis for switched symmetric systems with time delay,” in *American Control Conference, 2003. Proceedings of the 2003*, vol. 3. IEEE, 2003, pp. 2682–2687.

[64] X. Sun, J. Zhao, and D. Hill, “Stability and l2-gain analysis for switched delay systems: A delay-dependent method,” *Automatica*, vol. 42, no. 10, pp. 1769–1774, 2006.

[65] V. Kulkarni, M. Jun, and J. Hespanha, “Piecewise quadratic lyapunov functions for piecewise affine time-delay systems,” in *American Control Conference, 2004. Proceedings of the 2004*, vol. 5. IEEE, 2004, pp. 3885–3889.

[66] M. Johansson, *Piecewise linear control systems*. Springer Verlag, 2003, vol. 284.

[67] M. Moarref and L. Rodrigues, “Asymptotic stability of sampled-data piecewise affine slab systems,” *Automatica*, 2012.

[68] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge Univ Pr, 2004.

[69] T. Kailath, *Linear systems*. Prentice-Hall Englewood Cliffs, NJ, 1980, vol. 1.

[70] H. Khalil and J. Grizzle, *Nonlinear systems*. Prentice hall, 1992, vol. 3.

[71] V. Utkin, “Sliding mode control: Mathematical tools, design and applications,” *Nonlinear and Optimal Control Theory*, pp. 289–347, 2008.