Examining the Impact of Instruction on First-Graders' Representation of Manipulatives and Knowledge of Symbols

Katarzyna Przednowek

A Thesis

in

The Department

of

Education

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Arts (Child Study) at

Concordia University

Montreal, Quebec, Canada

September 2012

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Katarzyna Przednowek

Entitled: Examining the Impact of Instruction on First-Graders' Representation of Manipulatives and Knowledge of Symbols.

and submitted in partial fulfillment of the requirements for the degree of

Master of Arts (Child Study)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee:

______________________________ Chair

Dr. Vivek Venkatesh

______________________________ Examiner

Dr. Hariclia Petrakos

______________________________ Examiner

Dr. Sandra Martin-Chang

______________________________ Supervisor

Dr. Helena Osana

Approved by ____

Richard Schmid
Chair of Department

______ 2012 ____

Brian Lewis
Dean of Faculty

ABSTRACT

Examining the Impact of Instruction on First-Graders' Representation
of Manipulatives and Knowledge of Symbols

Katarzyna Przednowek

Although "manipulatives" continue to be viewed as a promising solution to bridging the gap between abstract quantitative concepts and written symbols in mathematics instruction, literature on symbol use highlights that children have a difficult time grasping the relationship between the manipulatives and the concepts they stand for. Hence, manipulatives often fail to play a *supportive* role in understanding written numbers. This study investigated the effect of a two-component school-based instruction on first and second-graders' (M_{age}: 6.10 years, age range 6.4—7.7 years) understanding of the concepts represented by a set of manipulatives, as well as their use and understanding of written numbers. First, children either took part in explicit instruction outlining either a quantitative or a non-quantitative representation of the manipulatives. Second, all children took part in the same Addition Instruction, which connected the manipulatives to written numbers. The results indicated that explicitly introducing the manipulatives as representations of quantities resulted in a more comprehensive understanding of the concepts the manipulatives represent. A lack of significant results in children's use and understanding of addition procedures with written numbers emphasizes the need to implement manipulatives as tools that support children's *emerging* understanding of concepts within a mathematical domain. Limitations, future research, as well as scholarly and practical implications are discussed in light of the results.

Acknowledgments

I am deeply indebted to my supervisor, Dr. Helena Osana, as her continuous guidance, support, and warmhearted nature during these important years of my academic career has made the thesis writing process quite a rewarding endeavor. Your ability to delineate challenging statistical concepts is commendable.

I would like to extend my gratitude to Dr. Hariclia Petrakos for fulfilling numerous roles in both my academic and vocational undertakings in such a supportive manner. My sincere thanks are also due to Dr. Sandra Martin-Chang whose meticulous feedback challenged me to fully explore the potential of my thesis.

I would also like to thank my lab mates: Emmanuelle Adrien, Allyson Cooperman, Kim Desmarais, and Diana Royea, for their constant support, availability, and numerous brainstorming sessions. Furthermore, I would like to express thanks to all of the research assistants involved in the project. Your flexibility and genuine commitment greatly eased the taxing task of data collection.

Most importantly, I would like to thank my family and friends whose ongoing support helped me get through graduate school. Your limitless and wholehearted encouragement makes my pursuits and accomplishments all the more possible and meaningful.

Table of Contents

List of Figures viii

List of Tables ix

List of Appendices x

Statement of the Problem 1

Chapter 1 Literature Review 5

Definitions 5

Current State of Mathematics Education 8

Conceptual and Procedural Knowledge: Meaningful Learning 10

Developing Conceptual Understanding 18

Manipulative Use. 18

Difficulties with Manipulative Use. 22

Symbolic Development and Dual Representation 27

Representational Insight 27

Dual Representation: Manipulatives in Mathematics 30

Manipulatives: Characteristics 31

Manipulatives: Encoding and Use 32

Present Study 33

Chapter 2 Method 39

Participants 39

Design 39

Phase I: Procedural Addition Task. 42

Phase II: Encoding. 42

Phase III: Quantitative Representational Insight and Conceptual Addition Tasks 42

Phase IV: Addition Instruction. 43

Phase V: Dual representation and Addition Tasks 43

Description of the Interventions 43

Encoding Intervention. 43

Math Encoding. 45
Game Piece Encoding 49
Control. 51
Addition Instruction. 52
Instruments and Measures....57
Procedural Addition Task 57
Quantitative Representational Insight Task 58
Conceptual Representation Task....61
Procedure 62
Procedural Addition Task 63
Encoding Phase. 63
Quantitative Representational Insight and Conceptual Addition Tasks. ...64
Addition Instruction Phase....64
Data Analysis 65
Coding and Scoring....65
Procedural Addition Task. 66
Quantitative Representational Insight Task. 66
Conceptual Addition Task. 71
Reliability 73
Analysis....74
Chapter 3 Results 75
Participant Flow 75
Missing Data 75
Descriptive Statistics....76
The Effect of Encoding Condition on QRIT Scores after Addition Instruction 79
Read a Display 79
Construct a Display....80
Name Value of Individual Circle 82
The Effect of Encoding Condition on PAT and CAT Scores after Addition Instruction 84

	Procedural Addition Task	85
	Conceptual Addition Task	86
	Encoding Subtask	87
Chapter 4	Discussion	90
	Goals and Summary	90
	Limitations and Future Research	99
	Contributions to Literature	101
	Implications for Practice	102

References 103

Appendices 108

List of Figures

Figure

1 Study Design 41

2 "Jumpers" Game Board Set-up 50

3 "Jumpers" Game Moves 51

4 Set-up Manipulatives Board and Workbook 54

5 Procedures Addition Task (PAT) Interview Protocol 58

6 Quantitative Representational Insight Task (QRIT) Interview Protocol 59

7 Conceptual Addition Tasks (CAT) Interview Protocol 61

8 Scoring Rubric for Representation of Quantities under 10. 68

9 Scoring Rubric for Recomposition of Quantities 10 and over 70

10 Mean Scores for the Read a Display Subtask of the QRIT 80

11 Mean Scores for the Construct a Display Subtask of the QRIT 82

12 Mean Scores for the Name Value of Individual Circle Subtask of the QRIT 84

13 Mean Scores for the Correct Procedure Use Component of the PAT 86

14 Mean Scores for the Conceptual Justification of Addition Procedures Component of the CAT 87

15 Proportions of Perceived Quantitative Use Assigned to Manipulatives by Condition before and after Addition Instruction during the Encoding Task of the QRIT 89

List of Tables

Table

1 Number of Participants and Small Group Sizes by Condition during Encoding .45

2 Number of Participants and Small Group Sizes by Condition during Addition Instruction 53

3 Number of Equations Completed by Day of Addition Instruction 57

4 Means and (Standard Deviations) of pre-instruction and Posttest Scores across PAT, CAT, and QRIT components, as a function of Condition 77

5 Correlations Between Tasks of the QRIT, CAT, and PAT 78

6 Proportions of Perceived Use Assigned to Manipulatives by Condition before and after Addition Instruction during the Encoding Task of the QRIT 88

List of Appendices

Appendix

A Encoding Intervention: Math Encoding 108
B Encoding Intervention: Game Piece Encoding 117
C Encoding Intervention: Control. 122
D Addition Instruction 126
E Worksheet: Procedural Addition Task 130
F Scoring Sheet: Quantitative Representational Insight Task 132
G Scoring Sheet: Conceptual Addition Task 134

Statement of the Problem

A chief concern in mathematics education today is that children are learning mathematics with little conceptual understanding of the skills they are employing in the classroom (Kilpatrick, Swafford & Findell, 2001). Furthermore, instruction delivered in the average classroom continues to be mainly procedural, emphasizing students' use of sequential and routine actions on the formal notation system with little or no conceptual rationale (Stigler, Gonzales, Kawanaka, Knoll & Serrano, 1999).

Concrete mathematics tools, otherwise known as "manipulatives," have been repeatedly discussed in the literature and used in classrooms as a promising solution to bridging the gap between abstract mathematical concepts and formal notation systems (written numbers and signs), (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Osana & Pitsolantis, 2011; Wearne & Hiebert, 1988). Connections between the two are not always automatic, however, and even manipulatives often end up being used in a rote, procedural manner (see Hughes, 1986 in Uttal, Scudder, DeLoache, 1997; Resnick & Omanson, 1987; Uttal, 2003).

Developmental literature suggests that the difficulty with using manipulatives to enhance conceptual understanding of written symbols comes from the assumption most researchers and practitioners make: that manipulatives inherently carry with them the concepts they are supposed to represent (Ball, 1992; Clements, 1999). One of the derivations of this assumption perhaps comes from a "common teaching dilemma" described by Puchner et al. (2008): "often the teacher sees so clearly how the external representation depicts the idea they are trying to teach, they cannot imagine how the student would not easily form an accurate internal representation from the manipulative"

(p. 314). Essentially, teachers assume that the relationship between manipulatives and the concepts they are supposed to represent should be as easily apparent to their students as it is to them. Research shows that numerous factors impede children's grasp of representational relationships—that is, that one thing (symbol) stands for another (referent) (Uttal et al., 1997). These findings have been suggested as possible explanations for children's difficulty with representations in the world of mathematics. More specifically, the difficulty children show in understanding that a manipulative stands for a concept, and that both the concept and manipulative also stand for written symbols. This creates a difficulty at the practical level: children operate within two separate representational systems. Children may perform and conceptualize within the concrete system or within the written symbols system, without grasping that both represent the same ideas. Ideally, manipulatives should be used to *support* conceptual understanding of written numbers and the procedures carried out with them. Thus it is imperative that children grasp the symbolic relationship between the manipulatives and the concepts they stand for.

Unfortunately, research has yet to establish the best way to help children overcome these difficulties. Some scholars (e.g., McNeil & Uttal, 2009; Uttal et al., 1997) argue that direct instruction explicitly linking the referent to the representation (i.e., between the concept and the manipulative) will allow children to grasp the symbolic relationship between the two with more ease. In other words, children will understand the symbolic relationship with less difficulty if it is explicitly pointed out by the teacher, rather than having them attempt to discover it on their own. This line of reasoning, however, has not been tested empirically, and is by no means well understood.

The objective of this study was to examine whether explicitly telling students in the early elementary years how manipulatives are connected to their intended meaning would aid children in viewing the manipulatives as representations of mathematical quantities, which can then be used to support procedural skills. For this reason, during the first part of the study, one group of children took part in activities that developed a quantitative meaning of the manipulatives. The second group of children took part in activities that developed the meaning of the manipulatives as play objects so that they form a non-quantitative meaning for the manipulatives. All participants then took part in instruction that connected manipulatives to written symbols in the context of addition. Given that teachers continue to provide primarily procedural instruction within the classroom (see Stigler, et al., 1999 in Kilpatrick et al., 2001), the addition procedure was also procedural in nature.

I aimed to examine if the representational relationship between the manipulatives and the concepts they represent need to be explicitly taught prior to instruction, or whether children could grasp this relationship through the procedural instruction connecting manipulatives to written symbols. Furthermore, I also examined whether developing a conceptual foundation by explicitly linking concepts of quantity with manipulatives was sufficient in overcoming teachers' propensity for teaching within the formal mathematical system in a highly procedural manner. In other words, whether conceptual understanding developed prior to instruction would assist children in making sense of the procedures taught during instruction.

This study adds to the current literature by providing empirical evidence on the development of representational relationships within mathematics, through the use of manipulatives. From a practical perspective, the study supports teachers who use manipulatives in the classroom; the results speak to whether teachers need to introduce manipulatives to children in a specific way so that they understand the concepts that the manipulatives are assumed to represent.

Chapter 1: Literature Review

Definitions

Concepts. Hiebert and Lefevre (1986) defined conceptual knowledge as knowledge that is rich in relationships between units of knowledge. Both the individual pieces of information, as well as the links tying them together, are seen as equally important. In fact, these units of knowledge do not exist in isolation because the very nature of being identified as a conceptual unit of knowledge automatically denotes a relationship to other pieces of information. The concept of place value, for example, entails understanding that the position of a number within a multidigit number determines its worth. For example, the number 7 is worth the same as the 7 in 237 as they both represent 7 “ones.” They are both worth less than the 7 in 732 as it represents 7 “hundreds.” If a child has grasped this concept, then he has connected the idea that each digit has a specific quantity when used in isolation, with the idea that once a digit is used within a multidigit number, the quantity of each digit changes depending on its place.

Procedures. Hiebert and Lefevre (1986) identified two components of mathematical procedural knowledge, the first being the formal language, or the symbol representation system. This includes knowing what mathematical symbols look like and how they are configured, but does not necessarily involve understanding what the symbol means. Being able to write the shape of a “3” when asked to write the number three, for example, as opposed to a different marking, indicates having knowledge of the symbolic representational system, but does not, by itself, demonstrate an understanding of the quantity that the “3” represents.

The second component of procedural knowledge is the step-by-step use of algorithms or rules for solving mathematical tasks. When solving a multidigit addition problem such as $\begin{array}{r}23\\ \underline{+48}\end{array}$, for example, the first step involves adding the rightmost column (3+8), writing 1 underneath the line in the same column, regrouping a one to the top of the left hand column, and finally adding all three digits together (1+2+4) before writing down 7 under the line in the same column. Knowledge of the sequence of execution also does not automatically denote conceptual understanding, such as that the 3 and the 8 hold the same place value, that the regrouped "1" represents one ten, or that this is an addition problem and therefore the sum of the quantities will be greater than each of the quantities being added.

Representations. Goldin (2003) defined representations as "configurations of signs, characters, icons, or objects that somehow stand for, or 'represent' something else" (p. 276). In mathematics, representations are used to represent or "stand for" more abstract concepts and ideas. In order to learn from representations, the user must understand not only "that" the representation stands for something else, but also "how" it does so (Uttal & O'Doherty, in press).

***Concrete representations*.** Concrete, or physical, representations include two-dimensional objects, or visualizations (e.g., graphs, pictures, charts), as well as three-dimensional (3-D) objects (Uttal & O'Doherty, in press). Manipulatives are an example of a 3-D system of concrete representations that are designed to facilitate children's conceptual mathematical development. Children do not need to comprehend written representations of the same concepts in order to use manipulatives (Uttal, 2003). For example, the Dienes Blocks are a system of physical objects representing the concepts of quantities whereby a “single” block represents the quantity of “one,”[1] and a block the length of 10 “single” blocks represents the quantity of “ten,” and so on. Children do not need a conceptual understanding of the written number “10” in order to grasp the concept that the long block represents the quantity of “ten-ness”.

Symbolic representations. Referring back to Goldin’s (2003) definition of representations (2003), symbolic representations refer to the idea that a symbol “stands for” a referent. In the field of mathematics, symbols include the formal notation system composed of written numbers (e.g., 0 through 9), as well as symbols (=, -, +, etc.), which all denote mathematical concepts. Manipulatives have also been referred to as symbols as they too “stand for” concepts (Uttal et al., 1997).

Representational Insight. Representational insight refers to the “realization of the existence of a symbol-referent relation” (DeLoache, 1995, p. 110). The level of awareness of this relation varies, and has been documented as being difficult for young children to grasp and express (DeLoache, 1995).

Dual Representation Hypothesis. Any concrete symbol can be thought of or viewed in multiple ways, hence the term “dual representation” (DeLoache, 1987; 1991; 1995) For example, consider that a red plastic chip is being used as a concrete representation. By focusing on its physical characteristics, a child can view the chip as an object in and of itself: a red, round, plastic, chip. The chip can also be viewed as a representation of something else-- in this case a representation of the quantity of “ten.” The dual nature of the object can be viewed as sitting on a scale. Factors tipping the scale

[1] Quantities written out and placed in quotation marks (e.g., “ten”) refer to the concept of that quantity. Quantities written out with numbers (e.g., 10) refer to the symbolic representation of that quantity.

in favor of a child attending to the physical characteristic of the chip will take away from viewing the chip's representation of the quantity of "ten." Conversely, factors tipping the scale in favor of a child viewing the chip as a representation of a quantity will distance the child from solely focusing on the chip as a red, plastic object (Uttal et al., 1997).

Current State of Mathematics Education

State, national, and international assessments of mathematical knowledge conducted over the past 30 years, more recently including the 2007 Trends in International Mathematics and Science Study (TIMSS), indicate that although slightly improving, U.S. students continue to fall behind countries such as Hong Kong, Singapore, and Japan (Gonzales et al., 2008). More specifically, students' understanding of mathematical concepts and their performance in procedural computations is average at best (Gonzales et al., 2008), as students continue to have a limited understanding of *basic* mathematical concepts and show difficulty in their aptitude to apply mathematical skills to solve even the most basic of problems (Kilpatrick et al., 2001). This problem in the development of children's mathematical knowledge is fueled primarily by procedural classroom instruction in pre-kindergarten to eight-grade, which emphasizes the development of paper-and-pencil skills taught through demonstration of procedures and instilled by way of repeated practice (see Stigler, Gonzales, Kawanaka, Knoll & Serrano, 1999 in Kilpatrick et al., 2001).

Standards documents, such as the one published by the National Council of Teachers of Mathematics (NCTM) in 2000, have emphasized the need for children to learn with understanding or meaning. "Research has solidly established the important role of conceptual understanding in the learning of mathematics. By aligning factual knowledge and procedural proficiency with conceptual knowledge, students can become effective learners" (NCTM, 2000, p. 2).

The National Research Council (NRC) devised a model of key components in achieving mathematical proficiency (Kilpatrick, et al., 2001). This model is composed of five interwoven and interdependent strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The present study focused on the development of the first two strands. Conceptual understanding involves comprehension of mathematical concepts, operations, and relations, whereas procedural fluency includes the necessary skills to carry out procedures flexibly, accurately, efficiently, and appropriately.

The NRC emphasizes that "at the heart of mathematics in the prekindergarten, elementary, and middle school years are the concepts of *number* and *operations with numbers*" (Kilpatrick et al., 2001, p. 2), as proficiency with these concepts serves as a foundation to further education in mathematics and in related fields. The NRC stipulates that the concepts of *number* and *operations* are abstract, and therefore external representations, such as the formal notation system, can aid in their communication. Furthermore, the NCR highlights that "the usefulness of numerical ideas is enhanced when students encounter and use multiple representations for the same concept," (Kilpatrick et al., 2001, p. 2), hence understanding in mathematics involves the ability to represent concepts in multiple ways, such as with concrete representations, written symbols, and verbally. Additionally, the regularities of a number system (e.g., base-10 system for whole numbers) and the algorithms used to carry out numerical computations (procedures) can aid in the development of conceptual knowledge of *number* and *operation*, depending on how well they are understood by the student. The educational goals during early school years are for children to become proficient with number and operations on a conceptual and procedural level.

Conceptual and Procedural Knowledge: Meaningful Learning

Based on the current state of mathematics education, it is imperative that children's learning of mathematics be meaningful, which requires mathematics instruction that fosters both conceptual and procedural knowledge (Kilpatrick, et al., 2001; NCTM, 2000). Linking procedures with their underlying concepts not only enables children to assign meaning to symbols and algorithms, thereby enhancing procedure use, but also allows procedures to be retrieved with less effort (Hiebert & Lefevre, 1986; Kilpatrick et al., 2001).

Over the last few decades, in an effort to foster meaningful learning within North American classrooms, researchers have studied the effects of instruction that incorporates both conceptual and procedural understanding. Fuson and Briars (1990) examined the effects of what they called the Learning /Teaching approach on first- and second-graders' concepts of place value and multi-digit addition and subtraction procedures. The Learning/Teaching instruction that the researchers implemented first instilled conceptual understanding of place-value and then mapped these concepts to addition and subtraction procedures with written numbers. The study yielded significant positive results, as majority of the children improved their addition and subtraction procedures, their understanding of those procedures, as well as their understanding of place value concepts. Almost all of the children were able to explain tens for ones trading in the context of procedural operations indicating a good conceptual understanding of the addition and subtraction procedures.

Developing conceptual understanding was also fundamental in Wearne and Hiebert's (1988) application of the *cognitive processing theory* whereby fourth to sixth-grade students' conceptual understanding of written fractions was developed through a sequence of four processes: *connecting*, *developing*, *elaborating/routinizing*, and *abstracting.* The first two processes of the instruction developed conceptual understanding of written numbers, whereas the latter two focused on fostering procedural competence with written numbers. The results indicated that the majority of the children had assigned meaning to written symbols; over half of the children were able to use values of digits when explaining their addition procedures; and almost half showed flexibility in their understanding of the concepts, as they were able to apply their acquired knowledge to novel tasks. Children who applied semantic (conceptual) processes when completing transfer tasks were more likely to get correct results than children who applied non-semantic (procedural) rules. These findings point out that fostering conceptual understanding with fraction symbols is important to meaningful use of procedures.

Perry (1991) examined the effects of procedure (how to) versus principle-based (why) instruction on learning and transfer abilities of fourth and fifth-grade students on mathematical equivalence. During the first of two studies, children were randomly assigned to procedure alone or principle alone instruction conditions. Children in the procedure condition received instruction on a step-by-step procedure for solving the type of equivalence problem used in the pre-test ("a + b + c = a +__" or "a + b + c = __ + c").

Children in the principle instruction received an explanation of the addition principle governing equivalence equations (i.e., "the goal is to find a number in the blank so that both sides of the equation are equal"). Children solved the same type of equivalence problems after the instruction, assessing their procedure use. Transfer tasks examined conceptual knowledge in two ways: by assessing children's reasoning when justifying the correctness of problems solved by fictitious children and through their solutions of novel equivalence problems (e.g., "a x b x c = a x _"). The results of the first experiment showed no difference between the two groups in terms of learning, as both conditions showed success on procedural tasks. Children in the principle-based instruction, however, outperformed the procedure-based condition on the transfer tasks, showing a true understanding of the concept of equivalence.

The second study examined whether combining procedure instruction with principle instruction could foster both procedural and conceptual knowledge. More specifically, experiment two examined whether children would take advantage of the instruction that suited them more if given instruction containing both types of knowledge. The order of instruction for the two conditions differed. Principle plus procedure received the principle first, followed by the procedure to solve the problem. The procedure plus principle condition received the procedure first and then the principle.

The two groups were assessed using the same measures as in the first study. All four conditions across the two studies (study one: procedure alone and principle alone; study two: procedure then principle and principle then procedure) were compared. The results indicated that, once again, learning did not differ across all of the groups. A difference did exist in the transfer tasks, however. More specifically, children in the principle only condition outperformed the remaining groups on the transfer tasks. These results show that by providing children with a procedure, or an easily accessible approach, children may not consider why they are solving a problem in a given way, thereby making their newly acquired procedural knowledge context dependent (i.e., only applicable to problems that are isomorphically similar to ones where the procedure was initially learned). Furthermore, it seems that explaining the underlying concepts within a mathematical domain allows children to truly understand them, which is apparent when children not only correctly apply the acquired principle to novel tasks, but are also able to justify the use of a sequence of actions (procedure) to solve the tasks.

Additional evidence of the importance of conceptual knowledge for procedural performance comes from Rittle-Johnson and Alibali (1999), who examined the relationship between fourth and fifth-graders' conceptual and procedural understanding of mathematical equivalence (i.e., "a + b + c = a + _"). A pre-screening task identified "non-equivalent" children, those who could not solve equivalence problems correctly. Non-equivalent children were randomly assigned to one of three conditions: conceptual instruction, procedural instruction, or no instruction (control). The conceptual condition received principle-based instruction, explaining the principle of equivalence (based on Perry, 1991). The procedural instruction taught children a regrouping strategy for solving the equivalence problem (if a + b + c = a + _, then add b + c to find the missing number). Children were evaluated across three tasks: question, evaluation, and transfer. The question task assessed children's conceptual understanding of equivalence, as participants were required to determine whether fictitious children had correctly solved an equivalence problem. The evaluation task assessed children's procedural skills in solving equivalence problems. Finally, the transfer task determined children's ability to adapt acquired knowledge to solve novel tasks.

The results revealed that type of instruction did not yield significant differences in conceptual understanding or procedure use as determined by the question and evaluation tasks respectively. In other words, children across conceptual and procedural conditions improved across both types of knowledge. The difference between the two conditions was evident during the transfer task, however. The children in the concept-based instruction outperformed those in the procedure-based instruction on four out of five transfer tasks. Both groups were successful on the fifth task.

It was evident that both types of instruction were successful in allowing children to learn procedures for solving equivalence problems, as both groups attempted to apply them during the transfer task. Children in the conceptual group, however, were more likely to adapt procedures to solve novel problems, indicating greater conceptual understanding of equivalence problems. These results suggest that a bi-directional relationship exists between the acquisition of the two types of knowledge, as conceptual instruction generated procedure use and procedural instruction fostered conceptual understanding. The success of the conceptual instruction groups on the transfer task, however, seems to propose that a conceptual foundation has a greater impact on procedural knowledge than the reverse.

The relationship between conceptual and procedural knowledge was also examined in a longitudinal study conducted by Hecht and Vagi (2010) who studied the factors that play a role in the emergence of basic fraction skills in fourth grade students. Children's fraction skills were assessed across three types of problems prominent in the elementary school curriculum: computation of fraction algorithms (e.g., "½ + ¼ = _" Answer: ¾); estimation ("99/100 + 99/100 is close to: 1, 10, or 100?" Answer: 100); and word problems with fraction quantities. Based on performance on these fraction skills, participants were identified as either typically achieving or as having mathematics difficulties. A comparison of the two groups revealed that aside from differences in attentive classroom behavior, the groups differed in emerging fraction skills, and these differences were mediated by conceptual knowledge about fractions. Furthermore, the results indicate that the development of basic fraction skills and conceptual knowledge is bidirectional because conceptual knowledge exerted strong influences on all three fractions skills, and in turn, competence with basic fraction skills influenced subsequent conceptual knowledge. These findings not only emphasize the impact of conceptual understanding on procedure use, but also indicate that increased procedural skill may lead to greater conceptual knowledge.

Hiebert and Wearne (1996) conducted a longitudinal study examining the development of place value understanding as well as multidigit addition and subtraction in children from first to fourth grade. Prior to any instruction, children were categorized as either "understanders" or "non-understanders" based on their understanding of place value concepts. Once children received instruction on procedure use, understanders were more likely to use conventional procedures with meaning and were less likely to forget the learned procedures. Understanders were also more likely to create their own procedures or modify conventional procedures when solving tasks for which they had not received instruction. This evidence indicates that conceptual understanding is a fundamental goal for mathematics instruction, especially early on, as it not only predicts both future and concurrent procedural skills, but also allows children to participate more fully in learning mathematics.

Rittle-Johnson, Siegler, and Alibali (2001) proposed that conceptual and procedural knowledge develop and influence one another in an iterative fashion. Mainly, increase in one type of knowledge will lead to an increase in the second type of knowledge, which in turn will lead to an increase in the first. The authors conducted two experiments surrounding decimal fractions with fifth and sixth-grade students. Both types of knowledge were tested before and after the delivery of a computerized intervention for solving problems on a number line. The procedural test assessed a child’s ability to locate written decimals (e.g., 0.09) on a number line between 0 and 1. The conceptual assessment measured children’s knowledge of decimal related concepts including: (a) relative magnitude, (b) relations to fixed values, (c) continuous quantities, (d) equivalent values, and (d) plausible addition solutions. During the intervention, children played a computerized game called “Catch the Monster.” On the first two tasks, children were told that a monster was hiding behind a written decimal number (e.g., 0.509) and had to indicate the monster’s location by choosing from four marks on a number line. One third of the questions had scribe lines dividing the number line into 10 equal parts, whereas the second third did not. The final third of the questions asked children to choose the written decimal that corresponded with the monster’s location on the number line out of four possible options.

The first experiment provided correlational evidence for the process underlying the iterative model. The authors found that children’s initial conceptual knowledge predicted improvements in their procedural knowledge, which in turn predicted gains in their conceptual knowledge.

During the second experiment, fifth-grade and sixth-grade students' conceptual and procedural knowledge were similarly assessed before and after the "Catch the Monster" intervention. The level of support during the intervention varied according to condition. Children in the "prompted" condition received a visual prompt pointing out the tenths digit within the given decimal fraction. In the second condition, the "tenths marked" condition, the number line for the children in was divided into ten equal parts using scribe lines. The third condition had both of the prompts, and the control condition had neither one. The findings again supported the iterative model of conceptual and procedural knowledge development. Acquisition of both types of knowledge occurred in a bidirectional, or hand-over-hand fashion. Conceptual knowledge at pretest predicted gains in procedure use, with the most influence on transfer to novel problems, which in turn predicted gains in conceptual knowledge. The authors underlined that conceptual knowledge prior to intervention was key in the gains children made across both types of knowledge. The authors also emphasized that whether concepts or procedures develop first depends largely on the domain within which knowledge is being fostered. In this case, children had learned some decimal fraction concepts within their classroom instruction, but few procedures, and thus the authors posit that the students' foundational conceptual knowledge influenced gains in their procedural knowledge, which in turn influenced overall gains in their conceptual understanding of decimal fractions.

To recapitulate, research provides ample evidence that conceptual and procedural knowledge are necessary for meaningful leaning. Furthermore, although some research

(Hecht & Vagi, 2010; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001) supports an iterative model of development — that is, conceptual knowledge influencing procedural gains, and procedural knowledge subsequently influencing gains in conceptual understanding within a domain — fundamental conceptual understanding seems to play a critical role in subsequent growth across the two types of knowledge.

Developing Conceptual Understanding

Manipulative use. Conceptual understanding provides a critical foundation to meaningful learning in mathematics. Manipulatives, or concrete representations, are often used to teach and assess the concepts underlying both the written symbols and the operations performed on them (Rittle-Johnson & Siegler, 1998). In the above reviewed literature on the relationship between conceptual and procedural understanding, the researchers used concrete objects to teach concepts and assess their impact on procedures (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Wearne & Hiebert, 1988).

Manipulatives can indeed be helpful in this regard. Uttal, Liu, and DeLoache (2006), amongst many others, argue that young children have a difficult time grasping abstract concepts or ideas and therefore benefit from the representation of information in a more concrete way. As Ball (1992) emphasized, in the world of mathematics, there is a prevailing assumption that "Concrete is inherently good; abstract is inherently not appropriate—at least at the beginning, at least for young learners" (p. 16). Therefore, manipulatives are often used as a stepping stone, allowing children to represent abstract ideas in a concrete way before representing them with more elaborate representation systems, such as the formal notation system of written symbols used in mathematics.

Goldin (2003) offers a theoretical explanation for the benefits of making abstract concepts concrete. He suggests that manipulatives, and other concrete representations, form *external* representational systems which can represent or “stand for” *internal* representation systems, or the ideas and concepts formed in the mind by way of language, visual imagery, and so on. This includes counting out five blocks or writing out “5” to represent the concept of “five-ness,” for example. Conversely, internal representations can “stand for” external representations. For example, when presented with the equation 25-7 = _ a student can form a mental image of seven blocks being separated from a larger group of 25 to find the difference. Manipulatives, therefore, aid in the development of internal representations of concepts such as quantity, and conversely, can provide a way of externally representing ideas that are already emerging in a student’s mind.

Research provides illustrations of how concrete objects have been used to attach meaning to written symbols and procedures. For instance, mapping instructions (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Wearne & Hiebert, 1988) aim to develop and assign meaning to procedures through the use of manipulatives. First, the instruction connects concrete representations to the corresponding written symbols. Next, operations are carried out using both systems of representation. Finally, manipulatives are “shed” and problems are solved using only the formal system of written symbols. For example, children in the concept-based instruction in Hiebert and Wearne’s (1996) longitudinal study were taught about place value concepts by engaging in various mapping actions: (a) quantifying sets of concrete objects by groups of tens and analyzing multiple forms of representations of quantities (concrete representation, written symbols, and verbal representation); and (b) carrying out addition and subtraction procedures with written numbers with the support of manipulatives. Eventually, manipulatives were removed and children performed procedures using only written symbols. The concept-based lessons enabled children to grasp a greater level of conceptual understanding of place value with written symbols over time, but it is unclear which specific mapping action contributed most to students' learning.

Wearne and Hiebert (1988) also fostered children's conceptual understanding of written symbols (fractions) through the use of manipulatives. Through the *connecting process,* written symbols were connected to the underlying concepts of quantity (i.e., written symbol of 0.3 was connected to the quantity of three-tenths) by constructing links between individual symbols and manipulatives (base 10- blocks). During the *developing process*, children used manipulatives to conceptualize addition and subtraction problems and then represented them using symbols (1 long and 2 little blocks + 2 longs and 3 little blocks 1.2+2.3). During the *elaborating/ routinizing* process, children applied the conceptual knowledge acquired through the two previous processes and applied them to novel contexts (e.g., multiplication of decimals), and practiced procedures until they were executed without much cognitive effort. Eventually, during the *abstracting process*, the instruction moved away from using manipulatives to relying solely on the symbols to solve decimal problems. Once again, although results indicated that increased conceptual understanding enhanced procedure use, it is unclear which aspect of the four linking processes promoted this conceptual knowledge.

The Learning/Teaching mapping instruction implemented by Fuson and Briars (1990) with first- and second-grade students aimed to develop the relationship between the concepts (quantities expressed via spoken word) and the procedures (written numbers) by way of “physical embodiments,” or manipulatives. Manipulatives served as a way of physically representing the spoken words, and were connected in a step-by-step fashion to the written numbers. The focus of the first phase of the study was to connect manipulatives to block words, place value words, and written numbers. For example, children were taught to represent the spoken quantity of “one hundred twenty five” with one flat (a block representing 100), two longs, and five little blocks, and with the digit cards “125.” Furthermore, children were taught to make trades between blocks (10 little blocks for one long or 10 longs for one flat), therefore learning how to represent a specific quantity in multiple ways. Once children made the connections between blocks, numbers, and words, they were taught to add and subtract multi-digit quantities by moving the manipulatives and recording their actions with written numbers. Children were allowed to stop working with manipulatives once they were comfortable enough to solve problems exclusively with written numbers. Although the results indicate that the instruction allowed children to assign conceptual meaning to written numbers, it is difficult to establish which component of the Learning/Teaching approach is critical to its success.

Osana and Pitsolantis (2011) offer another example of developing understanding for written symbols by way of concrete representation. The instruction they employed with fifth and sixth graders made links between concepts and procedures at key points during the intervention based on a theory proposed by Hiebert (1984). The instruction made explicit links between pictorial and concrete representations (e.g., one-fourth of a square shaded in) and written representations of fractions (e.g., ¼) before advancing to procedural manipulations. Students receiving linking instruction not only improved on their conceptual knowledge of written fractions, but also on their connections between concepts and symbols. Once again, it is unclear which aspect of the instruction promoted the students' learning.

Difficulties with manipulative use. Throughout the reviewed literature it is evident that conceptual understanding of written symbols is conducted through manipulative use. None of the instruction reviewed above, however, focused on explicitly developing the knowledge that the manipulatives were supposed to convey, nor did any study assess whether children understood what the manipulatives stood for before any instruction began. That is, manipulatives, which are being connected to written symbols, are assumed to already carry these concepts. Researchers such as Clements (1999) and (1992) warn against this type of assumption, and advise that manipulatives be implemented with caution. As Clements (1999) explains, "although manipulatives have an important place in learning, their physicality does not carry the meaning of the mathematical idea" (p. 46) and therefore it should not be assumed that children will grasp the concepts the manipulatives are supposed to represent nor that the concepts will automatically transfer to written symbols.

This disconnect between concepts, manipulatives, and written symbols is evident in the research of Resnick and Omanson (1987), who examined the differences between understanding of place value conventions, written arithmetic operations, and operations with manipulatives of second and third-grade children who had already developed "buggy" subtraction algorithms. During the pilot study, the authors found that second- and third-graders who were weakest with the base-10 system in concrete form (i.e., using Base-10 blocks) made no errors using the carry procedure with written symbols.

Conversely, children who showed the strongest conceptual understanding of place value avoided using procedures and instead solved problems using mental computations.

To further explore the different kinds of understanding and to replicate the findings of the pilot study, 10 third-grade children participated in a subsequent study. The study was conducted in the form of interviews during which children's developing arithmetic knowledge, based on their regular mathematics instruction, was tracked at three time points during the school year. The interviews concentrated on tasks that aimed to expand and assess three main components of arithmetic knowledge: (a) conventions of decimal coding, (b) the principle of recomposition, and (c) procedures of addition and subtraction using tasks with manipulatives and tasks with written symbols.

The *conventions of decimal coding using manipulatives* included the following tasks: (a) discussing the name value of individual blocks (i.e., units, tens, and hundreds), (b) reading a display of blocks, and (c) constructing a display of a verbal quantity (i.e., not displayed using written symbols). The principle of *recomposition* was examined by: (a) asking children to show a quantity in two ways using the manipulatives, (b) examining whether children spontaneously traded blocks in subtraction with regrouping (e.g., 1 ten for 10 units), and (c) whether children could rebuild a display involving recomposition with more units or more tens.

The *conventions of decimal coding using written numbers* included the following tasks: comparing values of a single digit within a multidigit number and representing a single digit using manipulatives. Additional tasks required children to state the value of the carry mark in addition and the value of the "borrow" mark in subtraction. These tasks assessed both the principle of recomposition and conventions of decimal coding using written symbols. Lastly, children were asked to carry out the following procedures using written symbols: addition with carrying and subtraction with borrowing from the adjacent columns and across a zero.

Overall, the researchers found that the children had a better grasp of the value conventions using manipulatives rather than written symbols. Children could identify the value of an individual block in isolation (i.e., cube, long, and a flat), but had a much more difficult time reading a display that consisted of multiple denominations (e.g., two flats, 5 longs, and 3 cubes). Children seemed to understand that although blocks may be recomposed, the total value does not change. Furthermore, children were able to represent a multi-digit quantity, but showed difficulty using manipulatives to represent a single digit within a two to three digit number.

Children also encountered difficulty with written numbers. Despite being able to carry out addition and subtraction procedures correctly, the students struggled with assigning the appropriate value to both the carry and borrow marks. Furthermore, children showed they could decompose written numbers (separate into hundreds, tens, and ones) and carry out arithmetic in a column-by-column fashion, but they showed limited understanding that they were also decomposing quantities. It is evident from this investigation that many disconnects exist between concrete representations, representations with written symbols, and underlying conventions of place value—that is, gaps between the quantities represented by both symbolic systems.

Resnick and Omanson conducted another study in this series whereby fourth, fifth, and sixth-graders were taught to connect concrete representations with written representations in a step-by-step fashion. These children were chosen to take part in this study because of the "buggy" subtraction algorithms they had developed and were using in multi-digit subtraction. Participating in the "mapping instruction" was an attempt to fix these errors.

During the intervention, children used the manipulatives to display a multi-digit subtraction problem, and were taught to record it with written numbers. This included representing the first quantity with the manipulatives and then with written numbers, followed by representing the second quantity with manipulatives and then with written numbers. Finally, children manipulated the blocks to indicate they were subtracting the second quantity from the first, and represented each manipulation with written numbers. The manipulatives were then gradually faded out. Children practiced the recording procedure with written numbers while the experimenter manipulated the blocks. The final step completely shed the manipulatives and children solved subtraction problems solely with written numbers.

Although receiving the instruction allowed children to discuss the concept of regrouping in the context of subtraction, they could not apply it procedurally with the written numbers. This indicates that the children saw the concrete representations and written numbers as two separate systems, and that they had difficulty grasping that the two, in fact, represent the same concepts.

A possible reason for this difficulty may be that children tend to "conceive a stimulus or concept in a single way, and that they do not spontaneously (and sometimes even with prompting) consider alternate construals of the same stimulus" (Uttal, 2003, p. 108). This explanation echoes *centration,* a characteristic of preoperational thought in Piaget's theory of cognitive development. According to this notion, children up to about the age of seven have a difficult time *de-centering* or exploring all aspect of a visual stimulus. Instead of evaluating "perceptual events in a coordinated way with cognitions" (Wadsworth, 1971, p. 84), children at this age tend to focus solely on the perceptual characteristics of the stimulus. Perhaps prior experience with either representational system leads to difficulty perceiving the two systems as representations of the same concepts. Nevertheless, it is evident that understanding does not directly transfer from concrete representation to written symbols. As the Resnick and Omanson (1987) studies suggest, attention ought to be placed on the quantities each representational system stands for, rather than on the manipulatives and symbols.

Similar difficulties could be seen in Hughes' (1986) study of the relation between children's understanding of manipulatives and written representations (see Uttal, 2003; Uttal et al., 1997). Children had great difficulty using manipulatives to solve addition problems presented symbolically (e.g., 1+7 =). In the most extreme cases, children physically recreated the addition equation with the manipulatives, by arranging them to look like the symbols. Echoing Resnick and Omanson's (1987) findings, children in Hughes' study either used the concrete or written representations to solve problems, but had much difficulty connecting the two.

These examples reveal that children do not automatically see the relationship between manipulatives and written numbers, and that children may have a difficult time understanding that a concept may be expressed in multiple ways and through multiple representations (Uttal, 2003). As the NRC underlines (Kilpatrick et al., 2001), a rich conceptual understanding is evident when concepts are represented through multiple representational systems. Hence, children should be able to use manipulatives as a stepping-stone to written numbers, and be proficient with both as both systems represent the same underlying concepts. Once again, manipulatives are supposed to facilitate the learning of mathematical concepts that are necessary to use the written system meaningfully. If children do not grasp the relation between the manipulatives and the mathematical concepts they represent, then problem solving with the written system, while supported by unconnected manipulatives, will not be meaningful.

Symbolic Development and Dual Representation

Representational Insight

Manipulatives are believed to assist in mathematical learning because they can make complex information more cognitively manageable, allowing children to think about relations among the represented items in concrete terms (Uttal & O'Doherty, in press). For manipulatives to be effective at supporting mathematical learning, children must comprehend "that" and "how" the manipulatives represent their referent: as a concept and, or symbol (e.g., three chips can be viewed as representing the concept of "three-ness" and the written number "3") (Uttal et al., 1997; Uttal & O'Doherty, in press). As Uttal (2003) underlines, the "meaning and importance of the symbol lie in its relation to the referent" (p. 157), therefore, a symbol is only a symbol when it represents something else.

The process of understanding a symbolic relationship is not as automatic for children as it may be for adults (Clements, 1999; DeLoache, 1995; Uttal, 2003). DeLoache (1995) presented a theoretical model of children's symbol use and understanding, which points to a number of factors and mediating variables that affect (or have been hypothesized to affect) children's ability to use a symbol as a representation of a referent. In DeLoache's model*, domain knowledge*, *iconicity*, *salience as an object*, *instruction*, *and symbolization experience* make up the factors, whereas *perception of similarity*, *symbolic sensitivity*, *dual representation, representational insight*, and *mapping* make up the mediating variables between the factors that affect intended symbol use. R*epresentational insight* is particularly germane here for it is a pivotal component in children's emerging ability to use a symbol in the intended way. Representational insight is the ability to understand the relationship between symbols (in this case, manipulatives) and the concepts they are meant to represent. In other words, it refers to being able to perceive and mentally represent the relationship between the symbol and its referent (DeLoache, 1995).

Development of representational insight was also examined in a series of studies investigating young children's understanding of the symbolic relation between a scale model of a room and the larger room itself (DeLoache, 1987; 1989; 2000; Uttal, Schreiber & DeLoache, 1995). Each of the studies assessed whether two to three-year olds grasped the correspondence between the scale model and the large room during a task that encompassed retrieving a toy from the large room after viewing a miniature toy being hidden in the scale model. The children were explicitly told that the large toy is "in the same place" in the large room. Based on the results of each study, a number of factors negatively influenced children's representational insight between the model room (symbol) and the larger room (referent). These factors include a decreased level of explicitness in the instructions outlining the model-room relation (DeLoache, 1989) as well as increasing the time delay between hiding the toy in one place and retrieving it in another (Uttal et al., 1995).

Further studies examined the effect of the salience level of the model on the symbolic relation between the model and room. DeLoache (2000) placed the model behind a glass pane, therefore disabling children from manipulating it. Children in the windowpane condition performed significantly better on the search task than children who were allowed to manipulate the model.

In a follow up study, the researchers experimented with increasing the salience of the scale model for older children (3 years of age) who had outperformed the younger children (2.5 years of age) on the original scale model task (DeLoache, 2000). Children who were allowed to play with the scale model for 5 to10 minutes before beginning the search task performed significantly worse than children who went straight to the search task. Engaging with the model in a non-symbolic way made it more difficult for 3 year old children to view it in terms of its symbolic function (DeLoache, 2000)

In sum, the authors use the dual representation theory to explain these findings (DeLoache, 2000; Uttal et al., 1997). DeLoache (2000) defined “the existence of multiple mental representations of a single symbolic entity” (p. 330) as a dual representation, thus implying that symbols have both a concrete and abstract nature. In the case of the scale model research, the concrete nature of the scale model is that it is a small room, equipped with furniture, whereas the abstract nature of the model is that it is a symbol representing the larger room. Thus, achieving dual representation means grasping that an object can be understood as a thing in and of itself and simultaneously as a symbol representing something else (DeLoache, 1987).

The dual representation hypothesis dictates that the dual representation of an object can be viewed using a balancing scale metaphor (Uttal et al., 1997). The object in its own right is on one end of the scale and the object as a symbol on the other. Factors that emphasize the physical properties of the object will tip the scale in favor of viewing it as an object in and of itself, whereas factors emphasizing its symbolic relation to a referent will tip the scale in favor of viewing it as a symbol. As Itterson (1996) underlines, in order to use an object as a symbol, children need to see the symbol itself and also *see through* the symbol to its intended meaning. As illustrated by the scale model studies, the scale model's salience (placing it behind a glass pane) allowed children to foster its use as a symbol, allowing them to grasp dual representation, and therefore the symbolic relation between the symbol and what it stood for with more ease (DeLoache, 2000).

Dual Representation: Manipulatives in Mathematics Education

As the scale model studies show, younger children had considerable difficulty comprehending the seemingly simple relation between the two rooms. Thus, as Uttal and colleagues (1997) suggest, it is quite possible that "older children may have difficulty comprehending less transparent relations, such as those between manipulatives and the concepts they are designed to represent" (p. 44) and therefore it is evident that the "stands for" relationship between mathematical concepts and manipulatives should not be automatically assumed.

When using manipulatives in mathematics learning, grasping dual representation involves viewing the manipulatives as entities on their own as well as representations of abstract mathematical concepts. The dual representation hypothesis has been used to explain children's difficulty in mathematics, but only from a theoretical perspective (Uttal et al., 1997). As a number of factors influenced whether children used the scale model as a representation of the larger room, a number of factors may also influence children's use of manipulatives as symbols.

Manipulatives: Characteristics. Good manipulatives are ones that aid students in "building, strengthening, and connecting various representations of mathematical ideas" (Sarama & Clements, 2009, p. 146). As Uttal, O'Doherty, Newland, Hand and DeLoache (2009) underline, "highly attractive manipulatives make it harder for children to make a link between the objects themselves and their mathematical referents or to link what they learned from the manipulatives to mathematical concepts or other forms of more abstract representations" (p. 157). This includes refraining from using objects that are attractive or interesting to the children (including common objects such as toys or food) as they may get in the way of children's perception of the symbolic nature of the manipulatives (Uttal et al., 1997). Brown, McNeil, and Glenberg (2009) suggested using simple, bland concrete objects within the mathematics classroom because more realistic objects (e.g., animal or food shaped counters) may draw children's attention toward "superficial characteristics or irrelevant associations" (p. 161) and away from the concepts they are supposed to represent (e.g., quantities of whole numbers or fractions).

Although some research advocates for plain manipulatives, other indicates there may be some benefit to more attractive ones. McNeil, Uttal, Jarvin, and Sternberg (2009) examined the effects of perceptual richness of manipulatives on students' performance on word problems. Fifth-grade students were divided into three conditions by manipulative type. The perceptually rich condition received play-type bills and coins; the bland condition received bills and coins demarked with only the quantity represented by each (i.e., $1, 10¢, 1 ¢); and the third condition received no bills or coins.

The results indicated the students in the perceptually rich condition made the most number of arithmetic errors when solving the word problems. The perceptually rich condition made the fewest conceptual errors, however, suggesting that rich manipulatives may allow children to better conceptualize the problem they are trying to solve. It seems in the case of word problems, which attempted to draw on students' real-world knowledge, realistic manipulatives supported conceptual understanding, but at the same time they may be too distracting for children and result in an overabundance of total errors. These findings indicate that there are both benefits and drawbacks to using perceptually rich manipulatives, and thus the richness level of the manipulatives must be considered based on the goals of the instruction.

Manipulatives: Encoding and use. While there is a gap in the empirical literature of dual representation in mathematics, researchers nevertheless apply the dual representation theory to underscore the importance of encoding and using manipulatives appropriately (Uttal et al., 1997: Uttal et al., 2009). Uttal et al. (2009) emphasize, "manipulatives work best when used solely as tools for learning rather than focusing on the properties of the objects themselves" (p. 158). The chosen objects ought to be "privileged" because they should only be used as manipulatives (Uttal et al., 1997). Furthermore, allowing children to play with manipulatives may initially increase children's interest and attention to the objects, but at the same time may focus too much attention to the physical aspects of the objects and away from the intended symbolic representation (Uttal, 2003). Therefore, it seems likely that introducing manipulatives solely based on their symbolic function will allow children to tip the scale of dual representation towards seeing the manipulatives as symbols representing mathematical concepts rather than focusing on the objects themselves.

Furthermore, as Clements (1999) emphasized, children can manipulate concrete objects skillfully, and therefore in a procedural manner, without truly grasping the concepts they represent. Again, the physicality of the manipulatives does not carry the meaning of a mathematical idea and therefore concrete objects and their intended representation ought to be explicitly connected (Clements, 1999; Uttal, 2003). Children have trouble seeing past the concrete aspects of the materials that are being used to represent more abstract mathematical concepts, and therefore to overcome this "stumbling block," teachers may have to be "outright methodical in pointing out the connections between the representations that students construct and the more abstract concepts" (McNeil & Uttal, 2009, p. 139). Thus, it seems prudent that teachers explicitly assign meaning to the manipulatives themselves before further instruction takes place (Uttal et al., 1997). Explicitly linking manipulatives with their referents will allow children to develop those abstract mathematical concepts, which can then be used to assign meaning to written symbols.

Present Study

Meaningful learning occurs when conceptual knowledge is connected to procedures, thereby assigning meaning to both the written symbols and operations performed with them (Hiebert & Lefevre, 1986; Kilpatrick et al., 2001). Much research indicates that instruction fostering conceptual knowledge positively impacts procedural knowledge (Perry, 1991; Hiebert & Wearne, 1996; Wearne & Hiebert, 1988). Other research (Hecht & Vagi, 2010; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al,

2001) indicates that increasing procedural knowledge leads to greater conceptual understanding, but only by way of foundational conceptual understanding. Therefore, it seems that conceptual mathematical knowledge is fundamental to the development of both procedural and further conceptual knowledge, and is thus key to meaningful learning.

Manipulatives are seen as tools that support the understanding of abstract mathematical concepts because they allow children to represent mental concepts in more concrete terms. Research on symbolic development (Uttal, 2003; Uttal, et al., 1997), however, suggests that the relationship between manipulatives and their referents, the concepts they represent, may not be clear to children. Moreover, even though some children may be able to use manipulatives effectively to solve problems, this understanding does not automatically transfer to written numbers. Research shows that children view and use manipulatives and written numbers as separate systems, rather than seeing them as multiple representations of the same concepts (Resnick & Omanson, 1987; Uttal, 2003). As one of the goals in mathematical instruction is for children to represent abstract mathematical concepts with written numbers (Uttal, 2003; Kilpatrick et al., 2001), it is imperative that manipulatives play a *supportive* role in assigning conceptual meaning to the formal notation system.

Although much research makes parallel theoretical links between representational relationships using scale models and relationships using manipulatives in mathematics (Uttal, Liu, & DeLoache, 2006; Uttal et al., 1997), there is no direct evidence indicating that dual representation must be in place before connecting manipulatives with written symbols. Also, there is no empirical evidence demonstrating the best way to introduce manipulatives so that children encode them as symbols representing abstract mathematical concepts rather than as entities in their own right or non-mathematical representations (e.g., as a toy).

Furthermore, although research that links manipulatives with written numbers through instruction has been successful in developing conceptual understanding (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Osana & Pitsolantis, 2011; Wearne & Hiebert, 1988), the reality is that most mathematics classroom instruction is still quite procedural (Stigler et al., 1999). Therefore, it is necessary to distinguish which component of the instruction will foster conceptual understanding and therefore enhance procedure use: assigning meaning to manipulatives or the instruction connecting manipulatives to written numbers.

The objective of this study was therefore twofold. The first aim was to assess whether (a) explicit instruction outlining the quantitative meaning of the manipulatives prior to addition instruction mapping manipulatives to written numbers, would allow children to grasp the dual nature of the concrete symbols and therefore foster their representational insight between the manipulatives and quantitative concepts, or (b) whether children would be able to grasp this representational insight without having a quantitative dual representation of the manipulatives prior to the addition instruction. The second aim was to assess whether explicit instruction outlining the quantitative meaning of the manipulatives prior to the subsequent instruction on addition, would enhance children's use and understanding of written addition procedures.

First- and second-grade students from eight classrooms in the Montreal area were randomly assigned to one of three conditions: Math Encoding, Game Piece Encoding, and Control. The study consisted of an Encoding Phase and an Instruction Phase. During Encoding, the initial phase, manipulatives were introduced to small groups of children as either mathematics tools or play objects. In the Math Encoding condition, children were introduced to blue and red plastic chips, manipulatives that were unfamiliar to them, and explicit links were made between a chip and the quantity it represents (blue chips represent “one” and red chips represent “ten”). In the Game Piece Encoding condition, blue and red chips were introduced as pieces to a checkers-type board game in order to give the participants a non-quantitative representation of the manipulatives. Lastly, the Control group took part in storybook activities in order to control for time spent engaging with an instructor.

During the Instruction Phase, children across all of the conditions received small group instruction based on Fuson and Briar’s (1990) mapping instruction that connects manipulatives with written numbers in the context of multi-digit addition. Unlike Fuson and Briar’s Learning/Teaching instruction (1990), the addition instruction in this study was strictly procedural, focusing on the sequential steps of adding numbers vertically, reflecting the current condition of most mathematics instruction (Stigler et al., 1999). If children attached meaning to the manipulatives through concept-based instruction (Math Encoding condition of the Encoding Phase), then mapping the manipulatives to written symbols in the context of procedure-based addition instruction should be enough to develop children’s conceptual justification for using the procedures (both written symbols and operations performed on them).

The Quantitative Representational Insight Task (QRIT), a task I developed based on the work of Resnick and Omanson (1987), assessed whether children grasped the quantitative representation of the chips, and the extent to which they understood the mathematical concepts the chips represent. The QRIT was administered before and after Addition Instruction. All of the measures were administered during one-on-one interviews. Procedural and Conceptual Addition Tasks (PAT and CAT) were used to assess the effect of the Instruction Phase and were also administered before and after instruction.

The following research questions guided this study:

a. Will the students in the Math Encoding condition outperform both the Game Piece Encoding and Control conditions on the QRIT after instruction?

b. Will the students in the Math Encoding condition show significantly greater increases on the PAT and the CAT after instruction compared to both the Game Piece Encoding and Control conditions?

I hypothesized that explicitly connecting the concept of mathematical quantities to manipulatives before instruction would allow children to grasp the dual representation and therefore quantitative representational insight. Thus, I predicted that the Math Encoding condition would outperform Game Piece Encoding and Control conditions on the Quantitative Representational Insight Task, even after all groups received the instruction connecting manipulatives to written symbols.

Furthermore, I hypothesize that if mathematical quantities and manipulatives are explicitly connected during the Encoding Phase, and links between manipulatives and written representations were made through procedural instruction, children's understanding of mathematical quantities would allow them to assign meaning to written numbers and would therefore enable them to conceptually justify their rationale for using procedures with written addition. Thus, I predicted that Math Encoding would outperform Game Piece Encoding and Control conditions on both the Procedural and Conceptual Addition Tasks after instruction.

Chapter 2: Method

Participants

This study is a part of a larger study that involved 93 first- and second-grade students, of which 68 students (66 first-graders and two second-graders) formed the original sample of the current study. These students were from three different elementary schools in the Montreal area. School 1 consisted of four French-speaking first-grade classrooms, School 2 of two English-speaking first-grade classrooms, and School 3 consisted of a first-grade and a 1/2 split English-speaking classrooms.

The teachers from the respective classrooms were recruited in collaboration with the principals and mathematics consultants from two local school boards. All participating students received written consent from their parents to participate in the study, and were also asked to personally give consent to participate in project activities.

Design

The present study had an experimental design. It was carried out over 16 weeks and consisted of five phases as outlined in Figure 1. Students within each of the classrooms were randomly assigned to one of four conditions: Math Encoding, Game Piece Encoding, Control, and a fourth group, which will not be described any further as it was not a part of the current study. The participating students formed three small groups, one per condition, for a total of 24 small groups across the eight classrooms. The students stayed in their respective small groups for all experimental manipulations (Phases II and IV) and were individually interviewed for all other components of the study (Phases I, III, and V).

Trained research assistants delivered the experimental manipulations to each group in the language of their regular Mathematics instruction. There were a total of 4 small groups within each of the three conditions in which the language of delivered instruction was English and 4 small groups within each of the three conditions in which the language of delivered instruction was French. During the individual interview component, some children from the French-speaking classrooms chose to complete the entire interview in either French or English, or chose to switch languages part way. None of the participants from the English speaking classrooms chose to carry out any of the interviews in French.

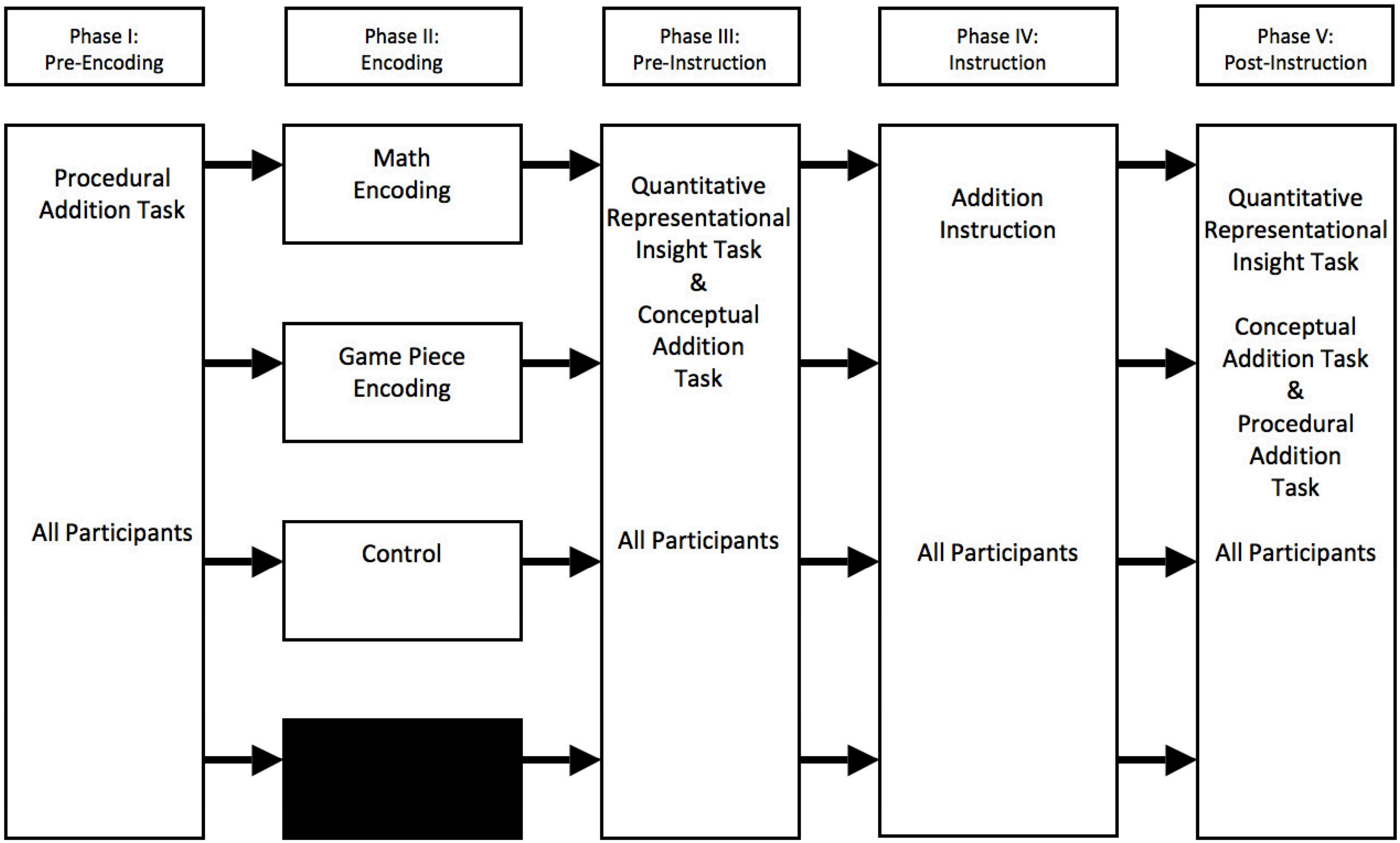

Figure 1. Design of the present study by phase. Black rectangle represents a fourth condition that is not included in the present study.

Phase 1: Procedural Addition Task. All participants were given a Procedural Addition Task (PAT) as a pretest to any instruction. I developed this task to assess students' ability to execute the standard addition procedure with regrouping. This task took approximately five minutes to administer.

Phase 2: Encoding. The Encoding phase consisted of lessons whereby children in the Math Encoding and Game Piece Encoding conditions were introduced to a set of manipulatives (blue and red plastic chips) in two distinct ways. In each case, the aim was to build a second representation of the chips: in one group as a math tool, and in the other group as a play object so that a non-quantitative representation was encoded. The participants had not been introduced to these manipulatives prior to the study. The Math Encoding group received direct instruction explicitly linking the chips to their respective quantities (blue as "one" and red as "ten"), whereas the Game Piece Encoding condition received instruction on explicitly encoding the chips as pieces in a game called *Jumpers*. The Control group took part in storybook activities. All conditions received the same amount of time with the manipulatives, or in the case of the Control group, with the storybook activities.

Phase 3: Quantitative Representational Insight and Conceptual Addition Tasks. During the third phase, all participants took part in individual interviews during which two measures were administered: the Quantitative Representational Insight Task (QRIT) and the Conceptual Addition Task (CAT). The QRIT assesses whether the participants have assigned the correct quantity to each colored chip after the Encoding Phase and served as a pretest to the Addition Instruction Phase. The CAT is a measure that assesses the participants' current level of conceptual understanding of addition procedures with written numbers.

Phase 4: Addition Instruction. Phase four of the study included instruction on addition, whereby connections were made between manipulatives and their written representations (number symbols). To avoid any direct encoding during instruction, which would eliminate the effects of the Encoding Phase, and to mirror the mainly procedural approach to teaching mathematics in North American classrooms, the instruction was designed to be strictly procedural. Thus, the instruction did not include any explanations of the quantities represented by the chips nor by the written numbers.

Phase 5: Quantitative Representational Insight and Addition Tasks. During the final phase of this study, the QRIT was administered to each student once more to assess whether children made gains in their quantitative representational insight after having received Addition Instruction. At this time, both the PAT and CAT were also administered to assess any changes in children's procedural and conceptual understanding of addition with written numbers respectively.

Description of Interventions

Encoding. During this phase of the study, children in the Math Encoding and Game Piece Encoding conditions took part in lessons that allowed each of the two groups to encode the manipulatives as a specific representation in addition to the physical objects that they are. Trained research assistants delivered the Encoding Phase to the small groups within each of the eight classrooms in two sessions. Both sessions ranged between 25 and 30 minutes and were delivered within 10 days of one another. Only children who fully took part in all phases of the study were included in the current sample and therefore data analysis. However, in order to maintain group size and to avoid single-participant groups as much as possible, Encoding was delivered to all the children within the original sample, even if he missed a session. That is to say, if a child missed Day 1 of Encoding, he still took part in Day 2. For this reason, group sizes varied across the two days of intervention.

Table 1 presents the small group sizes by condition for each of the two Encoding sessions. The current sample of each of the small groups (denoted with an "*n*" in the leftmost column of each of the conditions), ranged between 1 and 4 participants. The total small group (TSG) size for Day 1 and Day 2 of Encoding ranged between 1 and 4 children, depending on the number of original participants present in addition to the current sample.[2]

[2] There were a total of five single-participant groups on one or both days of Encoding.

Table 1

Number of Participants and Total Small Group Sizes by Condition during Encoding

	Math Encoding			Game Piece Encoding			Control		
Class	*n*	TSG Day 1	TSG Day 2	*n*	TSG Day 1	TSG Day 2	*n*	TSG Day 1	TSG Day 2
C1	3	4	4	4	4	4	3	3	4
C2	1	3	1	2	3	2	3	3	3
C3	2	4	2	4	4	4	3	3	3
C4	1	1	2	2	2	2	2	2	2
C5	1	1	2	1	2	1	3	3	3
C6	2	2	2	1	1	1	1	2	2
C7	3	4	4	1	2	2	2	3	3
C8	3	3	3	2	2	2	3	3	3
Total n	16			17			20		

Note. n = participants from current sample; TSG = Total Small Group size; C = Class.

Math Encoding. Instructors delivered Math Encoding to the small groups of children randomly assigned to this condition. Two plastic containers, one housing the red chips and the other the blue chips, were placed in front of the child. Math Encoding consisted of lessons making explicit links between the blue and red chips and concepts of quantity via spoken word (see Appendix A for Encoding Instructions). In order to establish representational insight between manipulatives and the concepts they represent, use or display of written numbers was not a part of this instruction. The lessons were spread across two days. The chips were referred to as "circles" in order to avoid any previous meaning children may have formed with objects that are referred to as "chips."

At the beginning of Day 1, the instructor introduced the blue circle as being worth the same as “one.” Children were then taught to represent quantities under “ten” with blue circles. Once the blue circle and quantities under “ten” were encoded, the instructor asked the children to represent the quantity of “ten” and then made the connection between 10 blue circles being worth the same as one red circle. The instructor explicitly told the children that one red circle is the same as the quantity of “ten,” and children physically traded one red circle for 10 blue circles. Next, children were taught to represent quantities “twenty” through “ninety” using only the red circles. The instructor led the small group in a final review of the value of each circle and reinforced the concepts of quantity with additional examples using all blue or all red circles.

At the start of Day 2 of Math Encoding, the instructor reviewed the following: (a) value of each colored circle, (b) representing quantities under “ten” using only blue circles, (c) trading a group of ten blue circles for a red circle, (d) representing quantities using only red circles, and (e) trading one red circle for 10 blue circles.

Next, the instructor taught the children bidirectional recomposition from blue circles to both red and blue circles (e.g., 39 blue circles is the same as 3 red and 9 blue, or 2 red and 19 blue, or 1 red and 29 blue circles) and red and blue circles to all blue circles (e.g., 2 red and 5 blue circles is the same as 25 blue circles). Representing quantities in numerous ways reinforces flexibility in the conceptual understanding of the quantities that the manipulatives represent. Furthermore, it provides a conceptual foundation to why numbers are regrouped in addition (Resnick & Omanson, 1987).

When teaching how to represent quantities, the instructor modeled to the children how to represent a specific quantity expressed verbally (e.g., “seven”), by placing the required number of circles on a counting mat (a legal sized, laminated, yellow paper). Once on the mat, the instructor counted out the circles by touching each one and repeating the total quantity once more to reinforce the concept of cardinality. When working with blue circles, each of which represent the quantity of "one," the instructor counted by ones. For example when representing the quantity of "seven," the instructor took out seven blue circles, placed them on the mat in a line, and touching the circles from left to right, counted: "one, two, three, four, five six, seven." The instructor then repeated the last number, "seven." The objective for counting each set out loud was to reinforce the concept of quantity with each example. Finally, the instructor asked the children "How much is this worth?" while gesturing to all of the displayed circles, in order to reinforce the total quantity represented.

When teaching children to represent quantities involving red circles, the instructor followed the same procedure as outlined above, but counted by tens when touching each of the red circles. For example, in order to represent the quantity "fifty," the instructor placed five red circles on the mat, and counted them by saying out loud: "ten, twenty, thirty, forty, fifty. Fifty." Once again, the instructor asked the children about the value of the total quantity.

When teaching children to represent quantities over "ten" using both colors (i.e., with multiple denominations), the instructor taught the children to count the red circles first, followed by the blue circles. For example, when representing the quantity of "twenty-three," the instructor counted out loud: "ten, twenty, twenty-one, twenty-two, twenty-three. Twenty-three."

During the recomposition examples, the instructor explicitly pointed out that one can trade 10 blue circles for a red circle because they are worth the same. The same connection was made between trading a red circle for 10 blue circles. Children were taught to maximize the recomposition of the quantity. For example, when representing “thirty-seven,” children initially represented the quantity with three red circles and seven blue circles. They then traded each of the red circles for 10 blue circles. The instructor also briefly guided the children through a discussion as to whether or not they could make any more trades based on what manipulatives were left on their mats, thereby underlining that a trade can only take place if red circles remained. The same applied for recomposition from all blue circles to red and blue circles, underlining the fact that a trade can only take place if groups of 10 blue circles remained.

Overall, the instruction followed a similar pattern across each concept taught. The instructor would first model how to count a quantity, by placing a specific number of circles on the mat and counting them appropriately. She would then ask the children to take out the same amount of circles and place them on the mat. Aloud, together with the instructor, each child would count his own circles. This was repeated for an additional three to four examples. During further examples, the instructor would ask the children to place a specific quantity on their mats and ask them to count out loud on their own. Children-led counting was repeated for an additional two to three examples. Once again, the instructor would always ask about the value of the total quantity whether the children had counted on their own, or along with the instructor.

Since the groups consisted of a maximum of four participants, the instructor could easily monitor their progress. If at any point a child made a mistake, the instructor provided corrective feedback. For example, if a child incorrectly represented a specific quantity (e.g., “seven”), the instructor reminded the child of the worth of the circle in question by saying: “Remember, the blue circle is worth the same as one. Let’s count again. One, two, three, four, five, six, seven. (pause) Seven. So we only need seven.” Another possible error was counting incorrectly either by ones or tens. For example, if a child incorrectly counted a representation constructed with the red chips (e.g., “thirty”), the instructor would say: “Remember, the red circle is worth the same as ten. Let’s count again. Ten, twenty, thirty. How much is this worth? (Wait for child to respond) That’s right, it’s worth thirty!”

Game Piece Encoding. The Game Piece Encoding condition consisted of introducing the blue and red chips as play objects. More specifically, the chips were introduced as game pieces to a game that borrows rules from Chinese Checkers (see Appendix B for the script used by the instructors in the Encoding Phase). In order to encode the chips as play objects, the children were discouraged from performing any other tasks with them, aside from those outlined in the instruction. At no point did the instructor discuss quantities or the notion of “how many” during the Encoding instruction.

The red and blue chips were presented in two separate containers, just as in the Math Encoding condition. They were introduced as red and blue “jumpers,” with each color denoting membership to one of two teams. The aim of the game is for each player to move his or her jumpers to the opposite side of a game board first. The instructor taught the children to set up the board by lining up the jumpers on their respective sides. When discussing the placement of the jumpers on the board, the instructor used language that deflected from counting, such as “fill up these back rows with red jumpers, and these back rows with blue jumpers,” rather than how many pieces were required to play or how many rows needed to be filled up. The instructor then proceeded to teach the participants the specific moves the jumpers can make by modeling how to move the game pieces on the board.

The board set-up is presented in Figure 2, and game moves are presented in Figure 3. The jumpers can travel on the board in two different ways. A player can *slide* a jumper to an empty square adjacent to the one currently on (including up, to each side, and diagonally) (Move A in Figure 3) or a player can *jump* a jumper over another piece (own piece or opposing team member’s piece), but only if the square after the piece being jumped over is open (Move B in Figure 3).

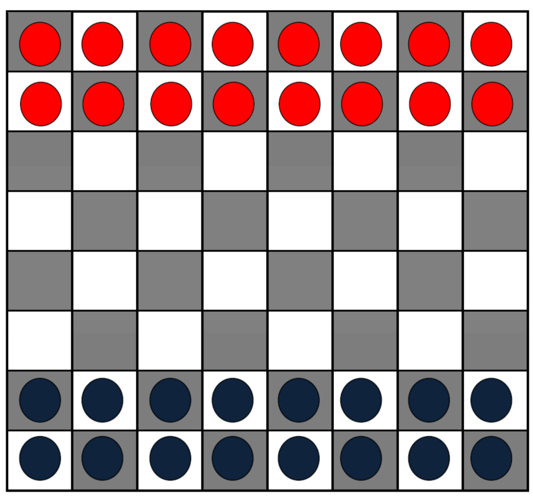

Figure 2. Set-up *“Jumpers”* game board.

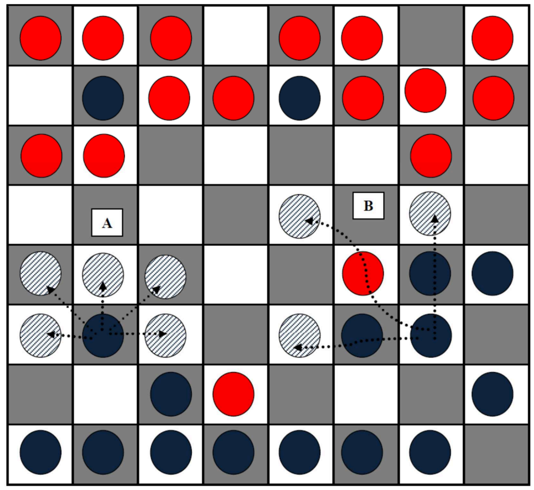

Figure 3. "Jumpers" game moves. A= *slide*; B = *jump.*

The instructor reviewed the rules with the children and then monitored the children while they played a game, reminding them of the rules if they made any errors in their moves. If children began mentioning numbers and quantities while playing, the instructor redirected their attention to the play aspect of the task by referring to the rules of the game, including the appropriate moves.

On Day 2 of the Encoding Phase, the instructor reviewed the parts of the game, including that the blue and red chips were jumpers, and the rules of the game. The children were then given an opportunity to play a full game. Once again, the instructor monitored the game and redirected the children when necessary.

Control. Students assigned to the Control condition took part in storybook activities (Appendix C for the instructions), which took place over two 50-minute periods. Day 1 included an introduction to a book called *Fancy Nancy and the Boy from Paris.* The book was chosen because it did not include any numbers or quantities, and it was available in both English and French. The instructor introduced the book by discussing the main character with the students, and then read the story aloud, asking engaging questions throughout. After the instructor finished reading the book, she led the children through a "Fancy Words" activity, discussing the meaning of some of the more complex words in the story. Time permitting, children were given blank paper and markers and were asked to draw their favorite part of the story.

At the start of Day 2, the instructor led the children in a recapitulation of the tale, and then read the story once more. Upon completion, the instructor led the children through another activity involving the "Fancy Words" in the story. This time they were given a worksheet that I designed and were asked to write the "Fancy Words" in fancy writing in one column and the regular words in regular writing in the other column (see Appendix C for worksheet). Time permitting, the instructor occupied the children with a coloring activity by giving them a coloring page of the Eiffel Tower and some markers.

If a child made references to any mathematical ideas throughout story reading or the associated activities, the instructor redirected their attention to any of the non-mathematical components of the story.

Addition Instruction. Participants across all three conditions received the same instruction on addition. It was delivered by trained research assistants to the same small groups of children as during the Encoding Phase and took place over two 40-to-50 minute periods. Similar to the Encoding Phase, Addition Instruction was delivered to all the children within the original sample in order to avoid single-participant groups, even despite a missed session. If a child missed Day 1 of Addition Instruction, he still took part in Day 2. For this reason, group sizes varied across the two days of intervention.

Table 2 presents the small group sizes by condition for each of the two Addition Instruction sessions. The current sample of each of the small groups (denoted with an "*n*" in the leftmost column of each of the conditions), ranged between 1 and 4 participants. The total small group (TSG) size for Day 1 and Day 2 of Addition Instruction ranged between 1 and 5 children, depending on the number of original participants present in addition to the current sample[3].

Table 2

Number of Participants and Small Group Sizes by Condition during Addition Instruction

	Math Encoding			Game Piece Encoding			Control		
Class	*n*	TSG Day 1	TSG Day 2	*n*	TSG Day 1	TSG Day 2	*n*	TSG Day 1	TSG Day 2
C1	3	5	5	4	5	5	3	5	5
C2	1	4	4	2	5	5	3	5	5
C3	2	4	5	4	5	5	3	5	5
C4	1	5	5	2	5	5	2	5	5
C5	1	2	2	1	2	2	3	3	3
C6	2	2	2	1	1	1	1	2	1
C7	3	3	4	1	1	1	2	3	2
C8	3	3	3	2	2	2	3	3	3
Total n	16			17			20		

Note. n = participants from current sample; TSG = Total Small Group size; C = Class.

The Addition Instruction aimed to connect concrete representations (chips) to numeric representations (written numbers) in the context of addition. This instruction was based on Fuson and Briars' (1990) Learning/Teaching Approach, which makes the connection between the addition procedure using manipulatives and the addition

[3] There were a total of three single-participant groups on one or both days of Addition Instruction.

procedure using written symbols. Although the Learning/Teaching Approach involves teaching children both procedures and concepts, the instruction in this study was strictly procedural and did not include explicit links between manipulatives and quantities (i.e., how much each concrete object represents), nor between numbers and quantities (i.e., how much each written number represents) (Appendix D for the script and numbers used by the instructor during the Addition Instruction).

The instructor introduced the participants to the materials used during the instruction, which included a small container of red chips, a small container of blue chips, a manipulatives board, and a written numbers workbook. The manipulatives board and workbook (Figure 4), on which children carried out addition using manipulatives and written numbers respectively, were based on those used in Fuson and Briars (1990).

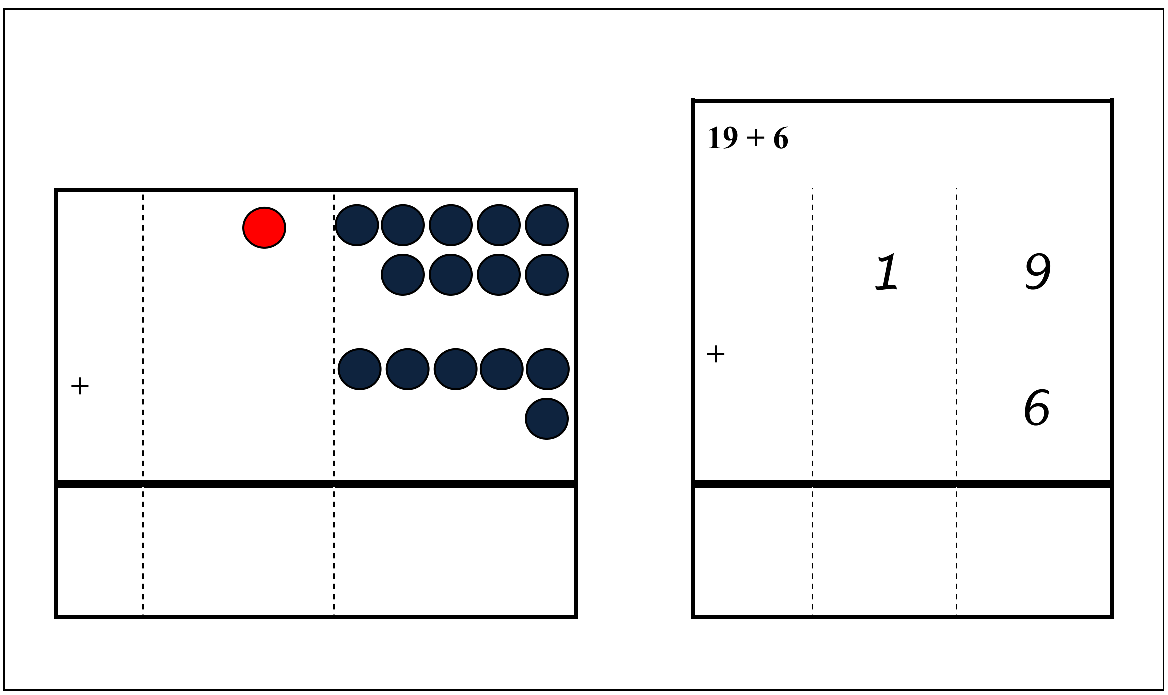

Figure 4. Set-up manipulatives board and workbook.

The manipulatives board held the manipulatives, whereas the corresponding numbers were written in the workbook positioned to the right of the manipulatives board.

Each child had his own board, workbook, and his own set of manipulatives. The assistant used her own board and manipulatives to model the addition procedures.

The instructor engaged in the following instructional sequence. First, she pointed to the written equation at the top of the worksheet in the workbook (e.g., 19+6) and read it aloud (e.g., “We are going to add 19 and 6”). Next, she represented each number concretely starting with the right column and moving to the left—that is, she placed nine blue chips in the right manipulatives column and one red chip in the left column. The instructor connected each step using the chips with written numbers in one of two ways: she pointed to the corresponding numbers already written out vertically on the workbook for the first few equations, and she wrote the corresponding numbers into the appropriate columns for the remainder. Instruction consisted of 10 equations for each of the two days of Instruction. Day 1 included four pre-written vertical equations and two pre-written equations on Day 2. The purpose of having numbers already pre-written for the first couple of equations was to gradually introduce the children to correct placement of numbers on the worksheet, working towards writing the vertical equations on their own.

If a student erred at any point during the instruction, the instructor simply said, “This is how you do it” and modeled the procedure to the child without any additional explanation. If a child questioned any of the procedures (e.g., “Why are we using this color?,” or “Why do I add these numbers?,”), the assistant said “Because this is the way I would like you to do it” or “Because this is the way we are going to learn it today.”

The instruction included addition of four types of problems; two types per each day of instruction. Day 1 consisted of single digits with a sum of less than 10 and single digits with a sum greater than or equal to nine. Day 2 consisted of double digits with the sum of the ones column being less than 10 and double digits with the sum of the ones column being greater than or equal to 10. Sums over 10 in the ones column involved trading 10 blue circles for one red one, which was also procedurally noted with a regrouped “1” placed above the tens column.

For each of the four types of problems, the instruction comprised two parts: (a) Imitation, during which students imitated the instructor’s instruction step by step; and (b) Structured Practice, during which students completed problems on their own while the instructor guided them through the sequence, ensuring no errors were being performed. Although I took many precautions to minimize variance within the delivery of the instruction (e.g., detailed script with specific error corrections), I anticipated factors such as group size, instructor-student dynamics, and learning speed would impact the amount of content the instructor would be able to get through in the allotted time. It was essential that children could practice each of the concepts taught, first through step-by-step imitation, and then on their own with the necessary amount of support. For this reason, the instructor delivered each concept by modeling two equations, and the children practiced an additional equation before the second type of equation was introduced. Once both types of equations were taught in this manner, children were able to practice up to four more equations; two per type of equation covered in the instruction period.

It was possible for the students to complete a maximum of 10 equations for the day. The number of equations completed during Addition Instruction by day, is presented in Table 3. During Day 1 of the instruction, 14 groups completed all 10 equations and 10 groups completed between 6 and 9 equations. During Day 2, 23 groups completed between 6 and 10 equations, and one group completed four equations. All students were exposed to all types of equations.

Table 3

Number of Total Equations Completed by Day of Addition Instruction

Total Completed Equations	Number of Groups	
	Day 1	Day 2
1	0	0
2	0	0
3	0	0
4	0	1
5	0	0
6	2	8
7	2	3
8	5	7
9	1	1
10	14	4
Total groups	24	24

Instruments and Measures

Procedures Addition Task. The Procedures Addition Task (PAT), a measure I designed for this study, aims to measure children's ability to perform the standard algorithm correctly. All items in the PAT were presented symbolically and children were required to complete each of the presented problems without needing to explain why or how they solved them. Figure 5 presents the interview protocol for the PAT. The interviewer presented each child with a total of three vertical addition problems on a single worksheet (see Appendix E for the worksheet used). The interviewer then asked each child to solve the problem by writing under the line.

Each question had a double-digit number on the top and a single digit number on the bottom. Two of the questions in the pre-instruction and all three questions in the post-instruction version of the measure required carrying a “1” to the tens column. This task assessed two things: (a) whether the child knew the procedure for carrying “1” ten to the tens column if the sum of the ones column was greater than nine, and (b) the child’s ability to line up the answer under the addition line in the appropriate columns.

Present each problem on a separate cue card.
I: I’m going to show you a worksheet with some problems on it. I’d like for you to solve them and write down your answer underneath the line *(point to the line of the first problem)*. Here’s a pencil and an eraser, just in case you need it. Remember, do the best that you can!

I: Ok, please solve these problems.

$$\begin{array}{r} 12 \\ \underline{+\ 6} \end{array}$$

$$\begin{array}{r} 17 \\ \underline{+\ 3} \end{array}$$

$$\begin{array}{r} 16 \\ \underline{+\ 9} \end{array}$$

Figure 5. Procedures Addition Task (PAT) interview protocol. I = Instructor.

Quantitative Representational Insight Task. The Quantitative Representational Insight Task (QRIT) is a measure I designed based on Resnick and Omanson’s (1987) block tasks and written tasks, and includes a series of questions assessing whether children attach quantitative meaning to the manipulatives. The QRIT was administered at two time points: before and after the Addition Instruction Phase. The QRIT was administered in an individual interview and consisted of four subtasks (Figure 6 for the interview protocol).

<table>
<tr><th>SUBTASKS</th><th>QRIT: Instructions</th></tr>
<tr><td>Encoding</td><td>Place the containers of red and blue circles on the mat.
I: Take a look at these (pointing / gesturing to the circles).
1. What are these?
2. What do you do with these?
3. Can you think of anything else to do with these?</td></tr>
<tr><td>Read a Display</td><td>I: For this activity, I’m going to give you a plastic bag with some circles in it. I’d like for you to put the circles on the mat.
(Hand child an open plastic bag with the predetermined amount for each question)
Ok, tell me how much this is worth.
1. 5 b (5) 3. 6 r (60)
2. 1 r 7 b (17) 4. 4 r 5 b (45)</td></tr>
<tr><td>Construct a Display</td><td>I: Now it’s your turn to use the circles. Place the containers in between the mat and the child.
Can you show me what __ would look like?
C: places circles on laminated mat
1. 8 3. 11*
2. 15 * 4. 26*

For questions marked *:
a) If the display is incorrect, ask: “Can you show me a different way?”
• If the second display is incorrect, move onto the next problem.
• If the second display is correct, follow the procedure for the correct displays (see b).

b) If the display is correct, and…
• …only blue circles or only red circles were used, ask: “Can you show me another way with both colors?”
• …both colors were used, ask: “Can you show me a different way with only one color?”</td></tr>
<tr><td>Name Value of Individual Circle</td><td>Move the containers beside you. Place one circle on the mat. Ask question.
Remove before placing second circle. Ask question again.
I: How much is this worth?
1. Red
2. Blue</td></tr>
</table>

Figure 6. Quantitative Representational Insight Task (QRIT) interview protocol.
I = Instructor; C=Child; r=red circle; b=blue circle.

During the first subtask, called *Encoding*, the interviewer presented the child with the now familiar containers of blue and red circles and, referring to the circles, asked “What are these?,” “What do you do with them?,” and “Can you think of anything else that can be done with them?,” to get a measure of how she perceived the manipulatives.

During the second subtask, called *Read a Display,* the interviewer asked the child about the worth of a predetermined amount of manipulatives in order to determine whether the child had grasped the quantity each color represents. The child was handed a plastic bag with a set number of circles and was be asked to take them out and place them on a laminated mat (which served as the work surface). The child took the circles out of the bag and placed them on the mat so that the layout of the circles was random. This reduced the possibility of the child perceiving the manipulatives as needing to be counted a certain way because of the way the instructor may have otherwise positioned them on the mat. The instructor presented four items, one including all blue circles (6 blue), two using red and blue circles (13: 1 red, 3 blue and 22: 2 red and 2 blue), and one using all red circles (8 red).

During *Construct a Display*, the third subtask, the instructor asked the child to show her what a certain quantity would look like (the pre-instruction items were: 5, 11, 26, 33, in that order). As it is possible that the child displays the correct number of circles without having fully grasped the different values of the red and blue circles (e.g., asked to show "twelve" and the child counts out 12 blue circles), the instructor asked the child to recompose the quantity—that is, represent the quantity in a different way. For example, if the child constructed a quantity with all blue circles (ones), the instructor prompted the child to show her that quantity once more, but using both red and blue circles, allowing to assess if the child had indeed assigned the appropriate quantity to each of the two colors (e.g., child should now display 1 red and 2 blue circles). If the child represented a quantity over 10 with both red and blue circles, the instructor would also ask the child to represent the display using all blue circles. If the child had grasped the relation between one red and 10 blue circles, then she should have been able to recompose the given quantity correctly.

The fourth and final subtask, referred to as *Name Value of Individual Circle,* directly asked the child to state the worth of the blue circle and then the red circle. This established (a) if the child had grasped the dual representation of the chips and therefore had moved past viewing them as objects in their own right, (b) if the child had grasped a quantitative representation of the chips, and (c) assessed specifically what quantity each color represents to the child.

Conceptual Addition Task. The Conceptual Addition Task (CAT) is a measure I designed based on Fuson & Kwon (1992) and assesses children's ability to understand what the written symbols represent within the context of the addition procedure.

Present each problem on a separate cue card.
I: I'm going to show you some problems solved by some kids that I know. I'm going to ask some questions about what they did. Ok?

This is what _____ did.
Did _____ get the right answer or the wrong answer? "Why is (insert child's response)?"

For questions with a carried 1:
If child does not discuss carried "1" independently:
a. Do you see this little "1" here? What does this mean?
b. Why do you think ___ put it there?
***Ask additional questions for clarification**
(i) Can you tell me a little bit more about that?
(ii) Can you explain that to me in another way?

Julie	Maggie	Carlos	Robby	Matthew
1		1		1
18	18	24	28	19
+ 9	+ 8	+ 8	+ 16	+ 2
27	16	32	34	111

Figure 7. Conceptual Addition Tasks (CAT) interview protocol. I = Instructor.

One at a time, the instructor showed the participant five addition problems solved by fictitious students and then asked about the correctness of the answer (see Figure 7 for Interview Protocol). Each problem was typed out on a separate cue card, and had a student's name and answer written out in child-like handwriting. All five equations required regrouping from the "ones" to the "tens" column. Two of the problems were solved correctly, and the remaining three incorrectly. The incorrect answers included common regrouping addition errors, including the "vanishing 10" (Fuson & Kwon, 1992) (see Maggie and Robby in Figure 7) whereby the regrouped "1" is missing, and writing down the regrouped "1" in the answer (see Matthew in Figure 7). Once the child stated whether the answer was correct or incorrect, the instructor asked, "Why did Matthew get the right/wrong answer?" in order to assess her conceptual understanding of the addition procedure. For questions with a regrouped "1," if a child did not spontaneously discuss this component in her justification of the correctness of the solution, the instructor would explicitly ask what the little "1" meant and also "why did Matthew put the little "1" here" while pointing to it on the card. A similar questioning strategy to assess children's conceptual understanding of addition and subtraction procedures was used by Fuson and Kwon (1992).

Procedure

Prior to any data collection, I went into each classroom and introduced myself and the other instructors so that the students become familiar with us. I explained that for the duration of the study, somebody from our research team would be meeting with each student individually and would also be doing some group activities in their classroom.

For each of the three individual interviews, students were taken out of their classroom and interviewed in a quiet place in the school. Before administering the first task, the research assistant received consent from the child. The child was only asked to sign the consent form once, but was reminded about the form and her choice to continue or stop at the start of every new meeting. Since the aim was to videotape how the children displayed and discussed their answers during individual interviews, the interviewer would ask the child's permission to videotape only their hands.

Procedural Addition Task. During the first phase, the PAT was administered to each student in combination with a few other measures that were not a part of the current study. Overall, the first interview took approximately 20 to 30 minutes to complete and was given to all the participants within approximately four weeks. During the fifth and final phase, the PAT was administered in the same individual format together with the QRIT and CAT post-instruction tests, taking place between 3 and 30 days after the final Addition Instruction session.

Encoding Phase. The Encoding Phase was carried out in a quiet area of the school in a small group format. Trained research assistants and I delivered the specific Encoding instructions to children in each of the three conditions to which they had been previously randomly assigned (Math Encoding, Game Piece Encoding, or Control). The Encoding phase spanned two 25 to 30 minute periods, and took place during the children's regularly scheduled mathematics class. The first session took place between 2 and 27 days after the pre-instruction PAT. The second session took place between 3 and 10 days after the first session.

Instructors were counterbalanced across the Encoding and Addition Phases for each condition within each of the eight classrooms. For example, let us consider the Math Encoding condition in one of the two English classrooms in School 2. Instructor 1 delivered Day 1, and instructor 2 delivered Day 2 of the Encoding Phase; instructor 3 delivered Day 1, and instructor 4 delivered Day 2 of Addition Instruction Phase. Thus, a different instructor delivered each of the four components of the intervention. Furthermore, as much as possible, the instructors were also rotated across conditions. For example, instructor 1 did not always deliver the Addition Instruction to only the small groups within the Game Piece Encoding condition, but would rather rotate (along with all the other instructors) across each of the conditions. In order to further reduce instructor effects, all of the instructors followed the exact same scripts for each of the three conditions.

Quantitative Representational Insight and Conceptual Addition Tasks. The timing of the delivery of pre-instruction and post-instruction depended on the availability of each of the classroom teachers. The QRIT and the CAT were administered in an individual interview format between 1 and 4 weeks of the final Encoding session. The post-instruction measures, along with the post-instruction PAT, were administered between 1 and 4 weeks of the final session of the Addition Instruction phase. The interview took place in a quiet location and took 30 to 50 minutes to complete. Once consent was reestablished, children's hands and verbal responses were once again videotaped during the interview.

Addition Instruction Phase. Along with the trained research assistants, I delivered the Addition Instruction (based on Fuson & Briars, 1990) to the participants in the same small groups to which they had been assigned at the start of the study. The Addition Instruction consisted of two full 40 to 50 minute mathematics periods, which were delivered in a quiet area of the classroom within the span of one week of one another. The first session of the Addition Instruction was delivered between 1 and 24 days of the pre-instruction test.

As previously described, instructors were counterbalanced across each of the days of Instruction in combination with the Encoding Phase, and also across each condition. Furthermore, as all three conditions received the same Addition Instruction, the instructors followed the exact same scripts for each of the small groups.

Data Analysis

Coding and Scoring. Coding and scoring of the measures were carried out in two stages: (a) during the individual interviews, and (b) while reviewing the videorecorded interviews after data collection. While the interviews took place, research assistants filled out paper coding sheets for tasks that did not require a justification component from the students. In contrast, components of measures requiring justification (part of the QRIT and majority of the CAT) were coded through a review of each interview video using a digital coding sheet. I assigned a numerical value to each of the codes, which I subsequently entered into SPSS.

If a child stated, “I don’t know” to any question presented by the instructor, she received a score of 0 for that component. If a child chose not to answer a question, she also received a score of 0, and the instructor inquired whether the child wanted to continue with the interview. At this point, if a child chose not to continue participating in the interview, no further scores were assigned and the maximum scores were calculated based solely on the questions that had been presented.

PAT. Each student completed the PAT worksheet during the individual interviews, which I later scored for correctness. Two components were scored for this measure (a) whether the child came up with the correct answer and (b) whether the child employed correct use of the standard addition procedure. Each child received a score of 1 if she solved the equation correctly and a score of 0 if she did not. To receive a score of 1 for the second component of the task, a child had to have solved each vertical equation by adding the addends in the ones and then the tens column, including regrouping one ten from the ones to the tens column and lining up the answer under the appropriate columns. Since the measure was scored solely based on written answers, if the child regrouped one ten within each equation by placing a regrouped “1” above the tens column, she was assigned a score of 1. If she did not place a regrouped “1” above the tens column, therefore not recording regrouping from one column to the next, she was assigned a score of 0. The PAT pre-instruction test included two questions that required regrouping, and one that did not. Since the question 12 + 6 in the pre-instruction test did not require regrouping, it was eliminated from the data set. The pre-instruction test had a maximum score of 4. All three questions in the post-instruction test required regrouping, and therefore a maximum score of 6 was possible.

QRIT. The QRIT was coded in two parts. The *Encoding Task* was coded from the videos, whereas *Read a Display, Construct a Display*, and *Name Value of Individual Circle* were coded during the interviews (see Appendix G for coding sheet). I assigned a numerical value to each of the codes.

During the *Encoding* subtask each child was presented with the manipulatives. She was first asked, “What are these?” and then what she could do with them. How the child had encoded the concrete objects and what she saw as their primary purpose were combined into one answer: use of manipulatives. The perceived uses of the circles resulted in three main categories: (a) quantitative use, (b) playing *Jumpers*, and (c) other uses. Quantitative uses included the following five subcategories: (a) to add and subtract, (b) to count, (c) to do Math, (d) to make numbers, and (e) to use as representations of specific quantities (for example, “ use as ones and tens”; “make groups of tens with them”). Only when children stated that the manipulatives were used for *Jumpers* were their responses categorized in the playing *Jumpers* use. Using the manipulatives to play other games (including *Bingo*, *Checkers*, *Backgammon*), to make pictures, to “cover things up,” to spin, or to use as money, were categorized as other uses. Subcategories were collapsed and only the three main categories are reported.

The *Read a Display* subtask was composed of four questions. If a child assigned the correct quantity to each circle when reading each display (e.g., the child stated that 2 red circles and 2 blue circles were worth “twenty-two”), she received a score of 1. If the child did not assign the correct quantity to each circle (e.g., the child stated that 2 red circles and 2 blue circles were worth “four”), she was given a score of 0. If a child assigned the right quantity to each circle, but miscounted the number of circles that were present in the individualized plastic bags, she was given a score of 1 and a counting error was noted. For example, one child took out eight red circles out of the baggie; he touched them one by one, but did not count out loud. He said the quantity was worth “seventy” instead of “eighty.” There were a total of two counting errors in the *Read a Display* subtask during the pre-instruction and two during the post-instruction test. The errors included miscounting the manipulatives by one blue circle or by one red circle. The maximum score for the task was 4.

The aim of the *Construct a Display* subtask was to assess whether, using the manipulatives, children were able to construct a representation of a quantity verbalized by the interviewer. There were two types of questions within this subtask: (a) representing quantities under 10 (one question) and (b) representing and recomposing quantities of 10 and over (three questions). A child received a score of 1 when she was able to construct a display of a quantity under 10 on the first trial (Scenario 1 in Figure 8). A child received a score of 1 when she was able to display, or recompose, a quantity of 10 and over in two ways within the first two trials (Scenario 1 in Figure 9).

Scenarios	Trial 1	Trial 2	Score
1	✓	-	1
2	✗	✗	0
3	✗	✓	.5

Figure 8. Scoring rubric for representation of quantities under 10. = correct representation; ß = incorrect representation.

Each child was allowed to make one error per question. Since quantities under 10 did not require recomposition, children were given a maximum of two trials to construct a correct display (Figure 8). Since quantities of 10 and over required a child to construct and then recompose each quantity, she was given three trials to complete the question (Figure 9). If at any point a child made two errors in a row, the interviewer moved onto the next question.

During the *Construct a Display* subtask, children made three types of errors: (a) counting, (b) representation, and (c) recomposition. Similar to the *Read a Display Task,* a counting error occurred when a child miscounted the number of circles she used while constructing a display of a quantity, while still assigning the correct quantity to each circle. For example, when asked to construct the quantity of “seventeen,” one child placed one red and 6 blue circles on the mat without counting out loud. When asked to show the quantity in a different way, she removed the red circle and traded it for 10 blue circles. She then placed them alongside the six circles she had left behind, thus constructing the final quantity with only 16 blue circles. Another child showed difficulty with correctly using the counting sequence, consistently skipping the number 22 when counting (e.g., “Nineteen, twenty, twenty-one, twenty-three.”), and therefore constructed the total quantity with one too few circles. When a counting error was identified in either the first construction or the recomposition of a given quantity, I scored the question as being correct, and disregarded the third construction. All counting errors were noted, marked as correct, and assigned a score of 1. There were a total of four counting errors in the pre-instruction test, and 11 in the post-instruction test. The errors consisted of misrepresenting a quantity by one or two red or blue circles. There was only one incident where a child was off by 10 blue circles when constructing the quantity of “thirty-three.” Her first representation of the quantity consisted of three red and three blue circles (thereby assigning the correct value to each of the circles). When asked to recompose the quantity using only blue circles, she placed 23 of them on the mat without counting aloud. Since she assigned the correct value to each of the circles, I noted her error as a counting error and allotted her a score of 1 for the item.

The second type of error, representation error, occurred when a child assigned an incorrect quantity to the circles when representing a quantity under 10 (e.g., a child displayed 5 red circles and 4 blue circles when representing the quantity of “9”). If a child erred on the first two trials, they were given a score of 0 and the interviewer moved onto the next question (Scenario 2 in Figure 8). If a child made an error on the first trial, but was successful on the second, she was assigned a score of .5, indicating that her representation was not consistent (Scenario 3 in Figure 8).

The last type of error, called a recomposition error, occurred when a child was unable to correctly recompose a quantity over nine. A score of 0 was given if the child was not able to construct two different representations within three trials (Scenarios 3, 4, and 5 in Figure 9). If the child was able to recompose a quantity within the three trials, but made an error, a score of .5 was given to indicate that the recomposition was not consistent (Scenarios 2 and 6 in Figure 9). The maximum score for the *Construct a Display* subtask was 4: 1 point for representing one quantity under 10, and 1 point for each of three correct recompositions of quantities over 10.

Scenarios	Trial 1	Trial 2	Trial 3	Score
1	✓	✓	-	1
2	✓	✗	✓	.5
3	✓	✗	✗	0
4	✗	✗	-	0
5	✗	✓	✗	0
6	✗	✓	✓	.5

Figure 9. Scoring rubric for recomposition of quantities 10 and over. = correct representation; ß = incorrect representation.

The final task of the QRIT, *Name Value of Individual Circle,* had a maximum possible score of 2. One point was given if a child answered that the blue circle was worth “one,” and the second if she answered that the red circle was worth “ten.”

CAT. One part of the CAT was scored during the interview, whereas the remainder was scored while reviewing the video recorded interviews. The correctness of the child’s response to the question “Did Matthew [the fictitious child] get the right answer or the wrong answer” was scored during the interview (Appendix F). A child was assigned a score of 1 if she responded correctly and a score of 0 if she responded incorrectly. The conceptual understanding of the addition procedure component was coded from the videos, and then assigned a numerical value.

The CAT was composed of a total of five questions: three with a regrouped “1,” and two with a “vanishing ten.” Questions with a regrouped “1” were coded across four components: (a) the child’s justification of why the fictitious student got the right or wrong answer, (b) whether the child spontaneously discussed the regrouped “1” in their justification, (c) the value the child assigned to the regrouped “1,” and (d) child’s justification for why the fictitious student placed the regrouped “1” above the tens column.

The justification of why the fictitious student got the right or wrong answer was scored across both types of questions, and therefore all codes reflecting children’s answers were pooled together. If the child discussed concepts of place value, including regrouping and adding numbers in the right column or the left column, to justify why he thought the fictitious student got the right or wrong answer, she received a score of 1. For example, when discussing why Julie got the right answer to the question 18+9=27, one student stated “You take away 1 from the 8 and you make it a group of 10 and then you have 7 (points to 8). Then you put the 1 here (points to regrouped 1) and it's a 2 here (points to the 2 in the answer).” If the child used the addition sequence, counted, or used mental math strategies to justify their answer (e.g., Julie go the right answer because 24 + 8 is 32), stated that the answer is smaller than an addend (e.g., 13+8=11 “Kelly got the wrong answer because 13 is higher than 11”), or indicated that the answer was too big through estimation, he was assigned a score of 0.

If a child spontaneously discussed the regrouped “1” in their answer (component b), she received a score of 1. If a child discussed the regrouped “1” incorrectly (e.g., “Tasha got the right answer because she put the little 1 here”) or did not spontaneously discuss it, she received a score of 0.

When discussing the worth of the regrouped “1” (component c), whether spontaneously or through a prompt from the interviewer, a worth of “ten” or “one group of ten” received a score of 1, whereas any other answer (e.g., “one”) received a score of 0.

If the child discussed regrouping ones to tens in their justification of why the students placed the regrouped “1” above the tens columns (component d), including answers referring to actions such as “carried over,” “moved from here to here,” “taking a group of ones and changing them for a ten,” and so on, essentially conveying that a quantity was regrouped from the ones column to the tens, she was assigned a score of 1. Answers that did not discuss regrouping a quantity from the ones to the tens column received a score of 0.

Questions with a “vanishing 10” were coded across two components: child’s justification of why the fictitious student got the right or wrong answer (as described above), and whether the child discussed the missing 10 in their justification. If the child spontaneously discussed a missing 10 in their justification of why the student got the wrong answer, she received a score of 1. For example, in the equation Kelly: 13+8=11, one child stated that Kelly got the wrong answer because “this one is 10 less than it's supposed to be; because 8+3 is 11 so you need a 1 here (points to the 1 in ones column in the answer) and a two here (points to the 1 in tens column in the answer), and there's already a 1 up here (points to the 1 in 13).” If the child did not spontaneously discuss the missing 10, she was assigned a score of 0.

The total maximum score for each of the questions with a regrouped “1” was 4, and the total maximum score for each of the questions with a “vanishing ten” was 2.

Reliability. Coding that took place during the interviews carried out by each of the interviewers. I scored this component on my own and, as it required no interpretation, no inter-rater reliability was established on these scores. One trained research assistant and I coded the video interview components. We established an inter-rater reliability of 94% on a random sample of 10 % of the 48 English video interviews.[4] Once inter-rater reliability was established, I coded all of remaining English interviews, whereas the research assistant coded all the interviews conducted in French. I assigned scores to the video data on my own.

[4] Inter-rater reliability for the video components was established only on English interviews because of my limited proficiency in the French language.

Analysis

Only certain components of each measure were analyzed.[5] For the PAT, both the *Correct Solution* and *Correct use of Standard Addition Procedure* components were analyzed separately. There was a maximum score of 2 for each component of the pre-instruction test and a maximum score of 3 for each component of the post-instruction test. As for the CAT, two components of the questions with a regrouped “1” were analyzed. I summed *Value Assigned to the Regrouped “1”* and *Justification for Placement of Regrouped “1”* in order to examine children’s *Conceptual Understanding of the Addition Procedure.* The combined score for each of the three examined questions was 2, for a total maximum score of 6. Lastly, the *Read a Display, Construct a Display*, and *Name Value of Individual Circle* subtasks of the QRIT were independently analyzed. The first two tasks had a maximum score of 4, and the last task had a maximum score of 2.

Overall, participants were assigned a score for each subtask by adding up the total number of points received and dividing by the total maximum possible scores to yield a percentage score for each of the subtasks within the dependent measures. All data were grouped by condition across all eight classrooms.

[5] The remaining components were analyzed as part of the larger study.

Chapter 3: Results

Participant Flow

The original sample of the current study was 68 (*N* = 68). Participants were excluded if they missed either day of Encoding, either day of Addition Instruction, or the post-instruction interview. Ten children missed one of the two days of Encoding: three missed Day 1, and seven missed Day 2. Four children missed one of the two days of Addition Instruction: two missed Day 1, and two missed Day 2. One child chose not to take part in the post-instruction interview. The final sample of participants thus included 53 students (M_{age}: 6.10 years, age range 6.4—7.7 years)[6]: 51 first-grade (24 female, 27 male) and two second-grade students (1 female, 1 male). Sixteen students, 9 male and 7 female, composed the Math Encoding group. Seventeen students, 8 male and 9 female, composed the Game Piece Encoding group, and 20 students, 11 male and 9 female, were in the Control group. The final sample consisted of 30 first-graders who received regular mathematics instruction in French, and 21 first- and two second-grader students who received regular mathematics instruction in English.

Missing Data

There was one exceptional case where, because of a recording error with the video camera during a pre-instruction interview, only data recorded on the coding sheets are available. I therefore excluded this participant from any analysis of data recorded on video.

Furthermore, although all of the instructors followed a script for each of the measures, there were still instances where a child was not asked a component of a task

[6] Because of administrative error, the mean age and the age range are based on a sample of 47 participants.

because of administrative error. Missing data were not assigned scores and were taken into account when calculating total percent scores. The only measure with missing data (other than the aforementioned recording error) was the CAT, with 12% missing in the pre-instruction and 5% missing in the post-instruction test.

Descriptive Statistics

The means and standard deviations of the key components of the PAT, CAT, and QRIT scores at pre- and post-instruction are presented as a function of condition in Table 4.

Table 4

Means and (Standard Deviations) of Pre-instruction and Post-instruction Scores across PAT, CAT, and QRIT components, as a function of Condition (N = 53)

	Math Encode (n = 16)		Game Piece Encode (n = 17)		Control (n= 20)	
	Pre	Post	Pre	Post	Pre	Post
Procedure Addition Task (PAT)						
Correct Solution	.69 (.44)	.65 (.37)	.65 (29)	.76 (.35)	.68 (.47)	.63 (.46)
Correct use of Standard Addition Procedure	.00 (0)	.17 (.37)	.00 (0)	.12 (.33)	.05 (.22)	.05 (.22)
Conceptual Addition Task (CAT)						
Conceptual Justification of Addition Procedure	.16 (.31)[a]	.27 (.34)[a]	.17 (.33)	.40 (.29)	.14 (.29)	.22 (.28)
Quantitative Representational Insight Task (QRIT)						
Read a Display	.80 (.29)	.91 (.22)	.25 (.00)	.59 (.37)	.25 (.00)	.40 (.31)
Construct a Display	.78 (.29)	.84 (.32)	.04 (.09)	.56 (.45)	.10 (.13)	.32 (.37)
Name Value of Individual Circles	.94 (.17)	.94 (.17)	.47 (.12)	.74 (.26)	.50 (.00)	.60 (.26)

Note. All scores are in percentages. Correct Solution maximum scores: pre-instruction= 2, post-instruction = 3; Correct use of Standard Addition Procedure maximum scores: pre-instruction = 4, post-instruction = 6; Conceptual Justification of Addition Procedure maximum pre-instruction and post-instruction scores = 6; Read a Display maximum pre-instruction and post-instruction scores = 4; Construct a Display maximum pre-instruction and post-instruction scores = 4; Name Value of Individual Circles maximum pre-instruction and post-instruction scores = 2.

[a] Because of a video recording error for one participant, tasks scored from videos have one fewer participant than tasks scored from scoring sheets. All PAT and QRIT: n = 16; CAT: n = 15.

As represented in Table 5, the *Correct Use of Standard Addition Procedure* of the PAT was correlated with the *Conceptual Justification of Addition Procedure* component of the CAT. All subtasks of the QRIT were correlated with one another. Neither component of the PAT, nor the sole component of the CAT, was found to correlate with any subtask of the QRIT.

Table 5

Correlations Between Components of the QRIT, CAT, and PAT

	1	2	3	4	5	6
Procedure Addition Task (PAT)						
1. Correct Solution	–					
2. Correct use of Standard Addition Procedure	.235	–				
Conceptual Addition Task (CAT)						
3. Conceptual Justification of Addition Procedure	.116	.360**	–			
Quantitative Representational Insight Task (QRIT)						
4. Read a Display	.132	.247	.099	–		
5. Construct a Display	.229	.224	.144	.869**	–	
6. Name Value of Individual Circle	.090	.103	.042	.848**	.883**	–

** $p < 0.01$ (2-tailed).

In order to address both of the primary hypotheses, the data were grouped by condition and analyzed in a 2 x 3 mixed ANOVA. Time (pre-instruction, post-instruction) was the within-group factor, and condition (Math Encoding, Game Piece Encoding, and Control) was the between-group factor. Separate analyses were conducted using each subtask of the QRIT, the CAT, and the PAT as dependent measures.

The Effect of Encoding Condition on QRIT Scores after Addition Instruction

To address the first research question regarding the effect of Encoding condition on the QRIT scores after the addition instruction, a 2 x 3 mixed ANOVA was conducted for each of the three components of the QRIT (*Read a Display, Construct a Display,* and *Name Value of Individual Circle*), with time (pre-instruction, post-instruction) as the within-group factor, and group (Math Encoding, Game Piece Encoding, and Control) as the between-group factor.

Read a Display. The comparison of the mean scores for the *Read a Display* subtask by group and time is presented in Figure 10. The results of a 2 x 3 mixed ANOVA revealed a main effect of time, $F(1, 50) = 21.765$, $p < .001$ for the *Read a Display* component of the QRIT. This indicates that regardless of condition, the students improved their performance from pre-instruction to post-instruction. The main effect of group was also significant, $F(2,50) = 36.757$, $p < .001$. Bonferroni post hoc comparisons indicated that the mean *Read a Display* score averaged across time was significantly higher for the Math Encoding condition ($M = .852$, $SE = .048$) than for both the Game Piece Encoding ($M = .419$, $SE = .046$, $p < .001$) and Control conditions ($M = .325$, $SE = .043$, $p < .001$). There was no significant difference between the Game Piece Encoding and Control conditions ($M = .094$, $SE = .063$, $p = .426$). No significant time x group interaction was found.

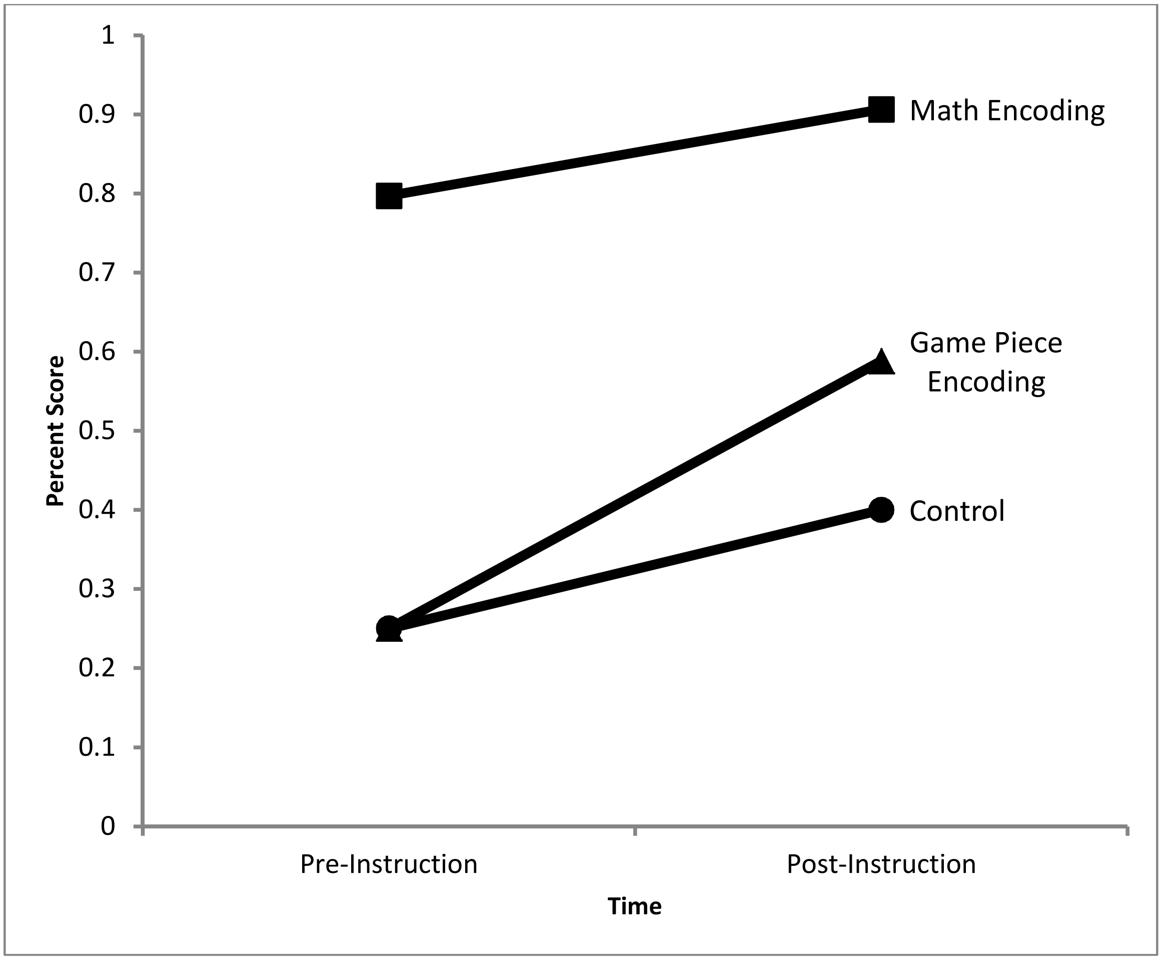

Figure 10. Mean scores for the *Read a Display* subtask of the QRIT.

Construct a Display. A comparison of the mean scores for the *Construct a Display* subtask by group and time is presented in Figure 11. The results of a 2 x 3 mixed ANOVA revealed a main effect of time, $F(1, 50) = 28.198$, $p = <.001$ for the *Construct a Display* subtask of the QRIT. This indicates that regardless of condition, the students improved their performance from pre-instruction to pro-instruction. There was also a significant main effect of group, $F(2,50) = 30.390$, $p < .001$. Bonferroni post hoc comparisons indicated that the mean *Construct a Display* score averaged across time was significantly higher for the Math Encoding condition ($M = .809$, $SE = .060$) than for both Game Piece Encoding ($M = .301$, $SE = .059$, $p < .001$) and Control condition ($M = .209$, *SE* = .054, $p < .001$). The Game Piece Encoding condition was not significantly different from the Control condition ($p = .759$).

The results revealed a time x group interaction, $F(2, 50) = 7.040$, $p = .002$, indicating that the effect of time was moderated by condition. Simple effects analyses demonstrated that the Math Encoding group ($M = 0.781$, $SE = .046$) outperformed both the Game Piece Encoding ($M = 0.044$, $SE = .044$, $p < .001$) and Control ($M = .100$, $SE = .041$, $p < .001$) groups at pre-instruction, with the latter two conditions not significantly different from each other ($p = .99$). At post-instruction, the Math Encoding group ($M = .836$, $SE = .096$) significantly outperformed the Control group ($M = .319$, $SE = .086$, $p < .01$) but not the Game Piece Encoding group (M = .559, SE = .093, $p = .131$). No significant difference was found between the Game Piece Encoding and Control groups at post-instruction ($p = .192$).

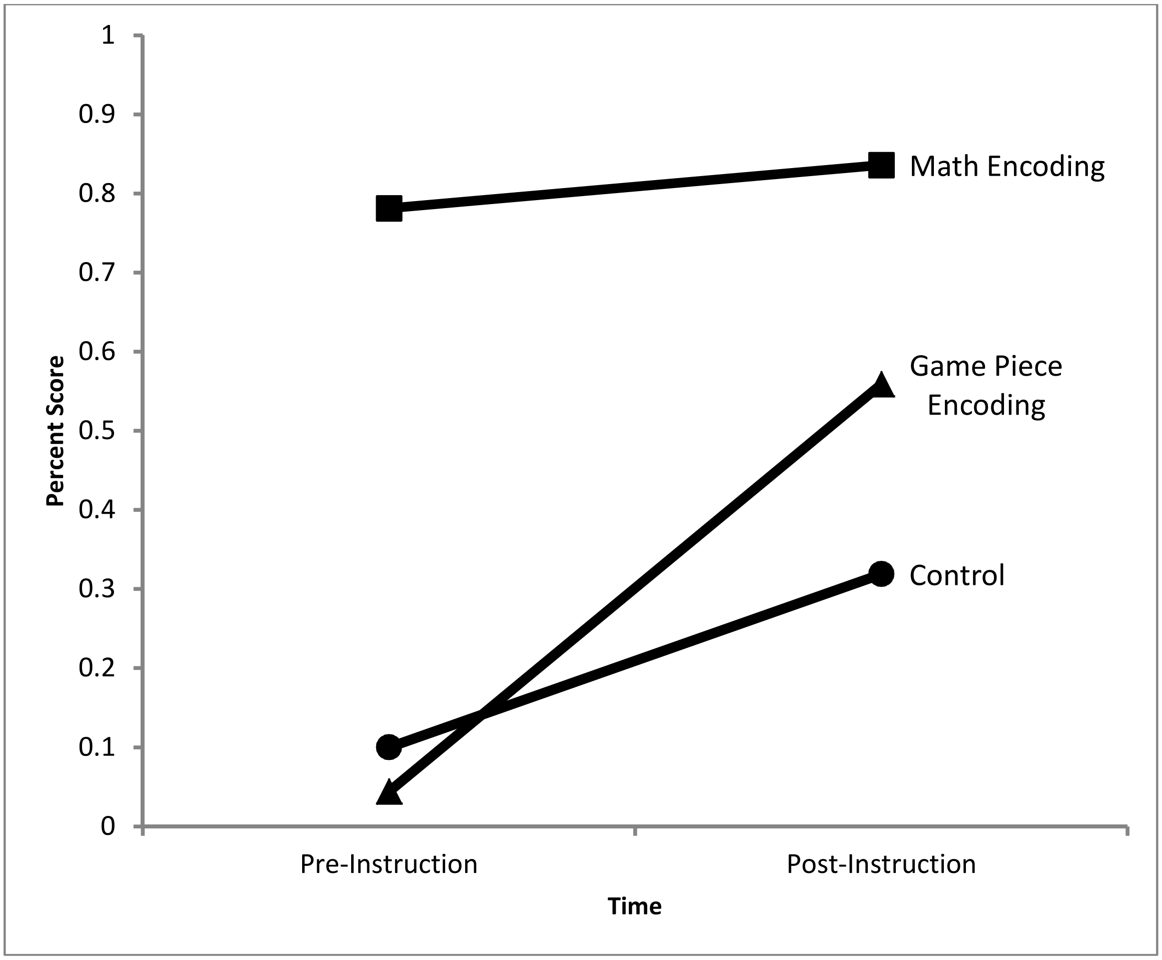

Figure 11. Mean scores for the *Construct a Display* subtask of the QRIT.

Name Value of Individual Circle. A comparison of the mean scores for the *Name Value of Individual Circle* subtask by condition and time is presented in Figure 12. The results of a 2 x 3 mixed ANOVA revealed a main effect of time, $F(1,50) = 13.578$, $p = .001$, which indicates that regardless of condition, the students improved their performance from pre-instruction to post-instruction. There was also a significant main effect of group, $F(2,50) = 36.746$, $p < .001$. Bonferroni post hoc comparisons indicated that the mean *Name Value of Individual Circle* score averaged across time was significantly higher for the Math Encoding condition ($M = .938$, $SE = .036$) than for both the Game Piece Encoding ($M = .603$, $SE = .035$, $p < .001$) and Control conditions ($M =$

.550, $SE = .032$, $p < .001$). There was no significant difference between the Game Piece Encoding and Control conditions ($p = .800$).

The results revealed a significant time x group interaction, $F(2, 50) = 5.195$, $p = .009$. Simple effects analyses demonstrated that at pre-instruction, the Math Encoding condition ($M = 0.938$, $SE = .029$) outperformed both the Game Piece Encoding ($M = 0.471$, $SE = .028$, $p < .001$) and Control ($M = .500$, $SE = .026$, $p < .001$) groups, with the latter two conditions not significantly different from each other ($p = .99$). At post-instruction, the Math Encoding group ($M = .938$, $SE = .059$) significantly outperformed the Control group ($M = .600$, $SE = .053$, $p < .01$) but not the Game Piece Encoding group ($M = .735$, $SE = .057$, $p = .053$). No significant difference was found between the Game Piece Encoding and Control groups at post-instruction ($p = .267$)

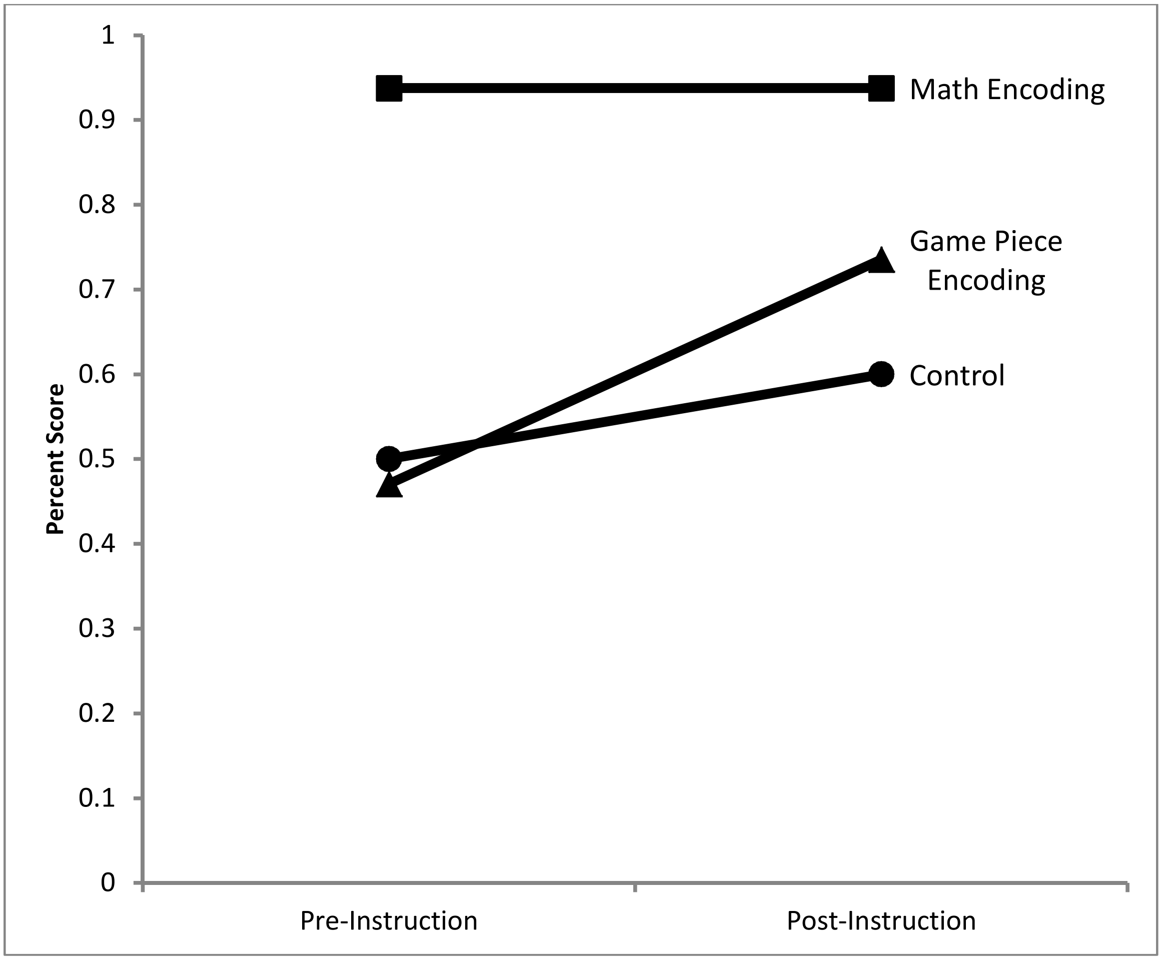

Figure 12. Mean scores for the *Name Value of Individual Circle* subtask of the QRIT.

The Effect of Encoding Condition on PAT and CAT Scores after Addition Instruction

To address the second research question regarding the effect of Encoding condition on PAT and CAT scores after the addition instruction, a 2 x 3 mixed ANOVA was conducted with time (pre-instruction, post-instruction) as the within-group factor, and group (Math Encoding, Game Piece Encoding, and Control) as the between-group factor. Separate analyses were conducted for the CAT and each subtask of the PAT.

PAT. The results indicate there was no main effect of time ($F(1, 50) = .043$, $p = .836$, nor group ($F(2,50) = .104$, $p = .902$) for the *Correct Solution* component of the PAT. Furthermore, no significant time x group interaction was found.

There was a significant main effect of time for the *Correct Use of Addition Procedure* for the PAT, ($F(1,50) = 6.266$, $p = .016$), indicating an improvement across time, regardless of condition. There were no differences between the groups averaged across time as was indicated by a lack of significant main effect of group ($F(2,50) = .136$, $p = .873$). Furthermore, no significant time x group interaction was found. A comparison of the mean scores for the *Correct Use of Addition Procedure* component by condition and time is presented in Figure 13.

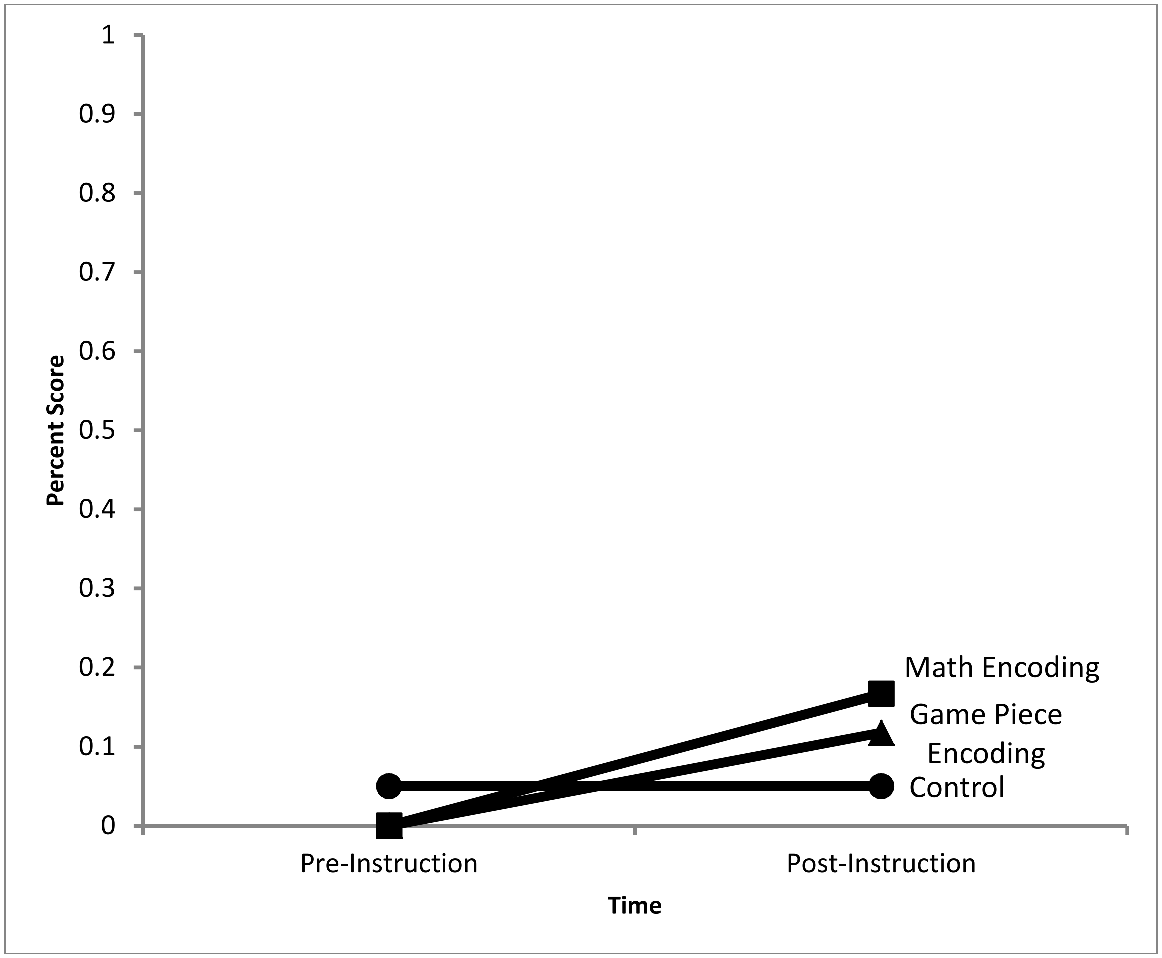

Figure 13. Mean scores for the *Correct Procedure Use* component of the PAT.

CAT. The results indicate there was a significant main effect of time for the *Conceptual Justification of Addition Procedure* component of the CAT ($F(1,49)$ = 10.407, p = .002). This indicates that the children improved their performance from pre-instruction to post-instruction, regardless of condition. There was no significant main effect of group ($F(2,49)$ = .763, p = .472). Furthermore, there was no significant time x group interaction. A comparison of the mean scores for the *Conceptual Justification of Addition Procedures* component by condition and time is presented in Figure 14.

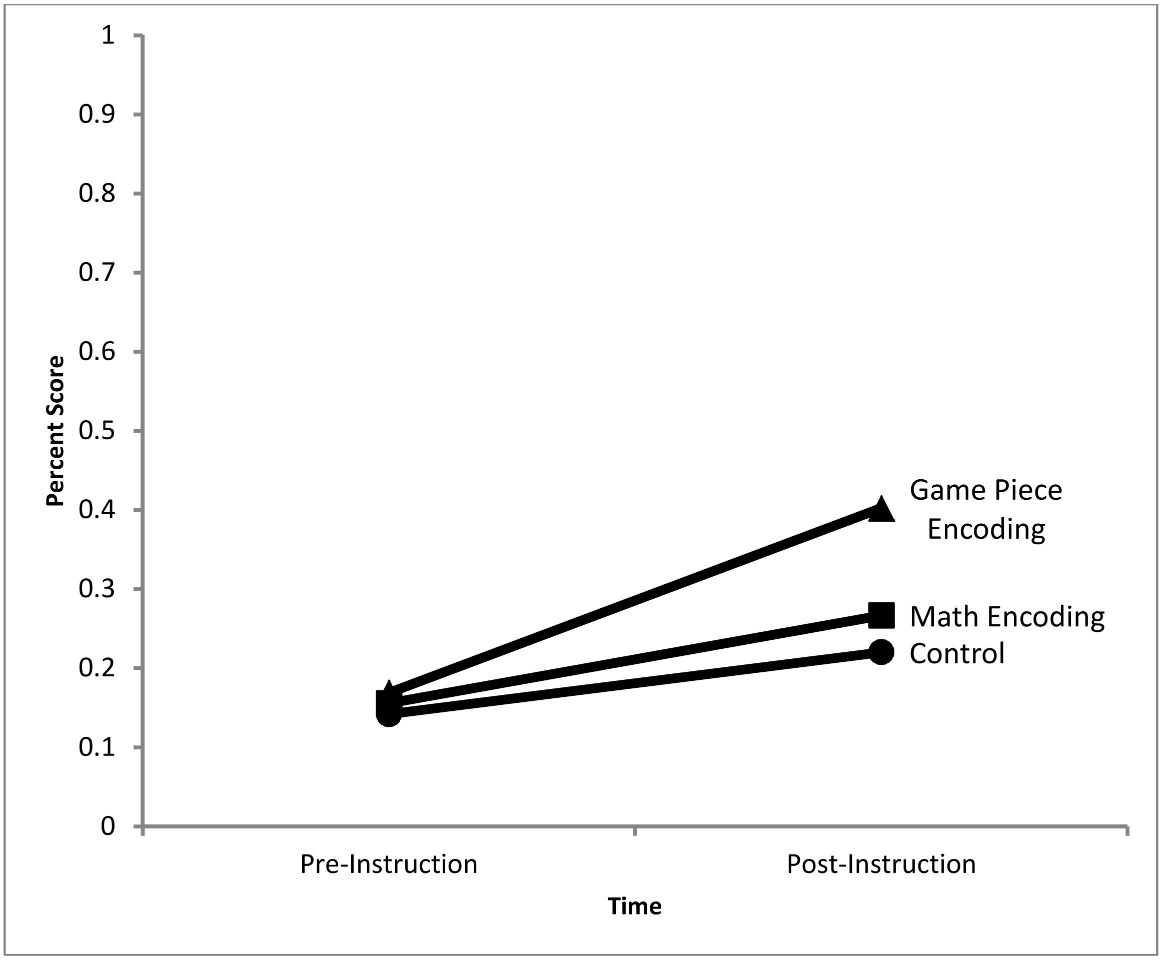

Figure 14. Mean scores for the *Conceptual Justification of Addition Procedures* component of the CAT.

Encoding Task

To examine how the children viewed the purpose of the manipulatives before and after the Addition Instruction, proportions of perceived uses were calculated for each condition and are reported in Table 6. The proportions of students viewing the manipulatives as having a quantitative use pre- and post-instruction are presented graphically in Figure 15. The proportion of students in the Game Piece Encoding condition and the Control condition who saw the manipulatives as having a quantitative use prior to the Addition Instruction (23% and 50%) increased after the instruction (65% and 75%), but not to the same extent as the Math Encoding condition. The Math

Encoding group had the largest proportion of children who saw the manipulatives as having a quantitative use at either time point (93% pre- and 87% post-instruction).

Not surprisingly, neither the Math Encoding nor Control condition perceived the manipulatives as “Jumpers” at either time point. Before using the manipulatives for addition, almost two-thirds of the Game Piece Encoding condition reported that manipulatives were to be used to play a game of “Jumpers,” whereas only one-third of the group still continued to perceive them in the same way after the instruction. The Math Encoding group had a slight increase (7% to 13%) and the remaining two conditions both decreased in their perception of the manipulatives as having a use other than for quantitative purposes or to play “Jumpers.”

Table 6

Proportions of Perceived Use Assigned to Manipulatives by Condition after Addition Instruction during the Encoding Subtask of the QRIT

Manipulative Use	Math Encoding (n =15)[a]		Game Piece Encoding (n = 17)		Control (n = 20)	
	Pre	Post	Pre	Post	Pre	Post
Quantitative use	.93	.87	.23	.65	.5	.75
Play “Jumpers”	0	0	.59	.29	0	0
Other uses	.07	.13	.18	.06	.5	.25

Note. Pre = Pre-instruction; Post = Post-instruction

[a] Because of a video recording error for one participant, the *Encoding* subtask of the QRIT has 15 participants (n = 15), one fewer than any non-video task (n = 16).

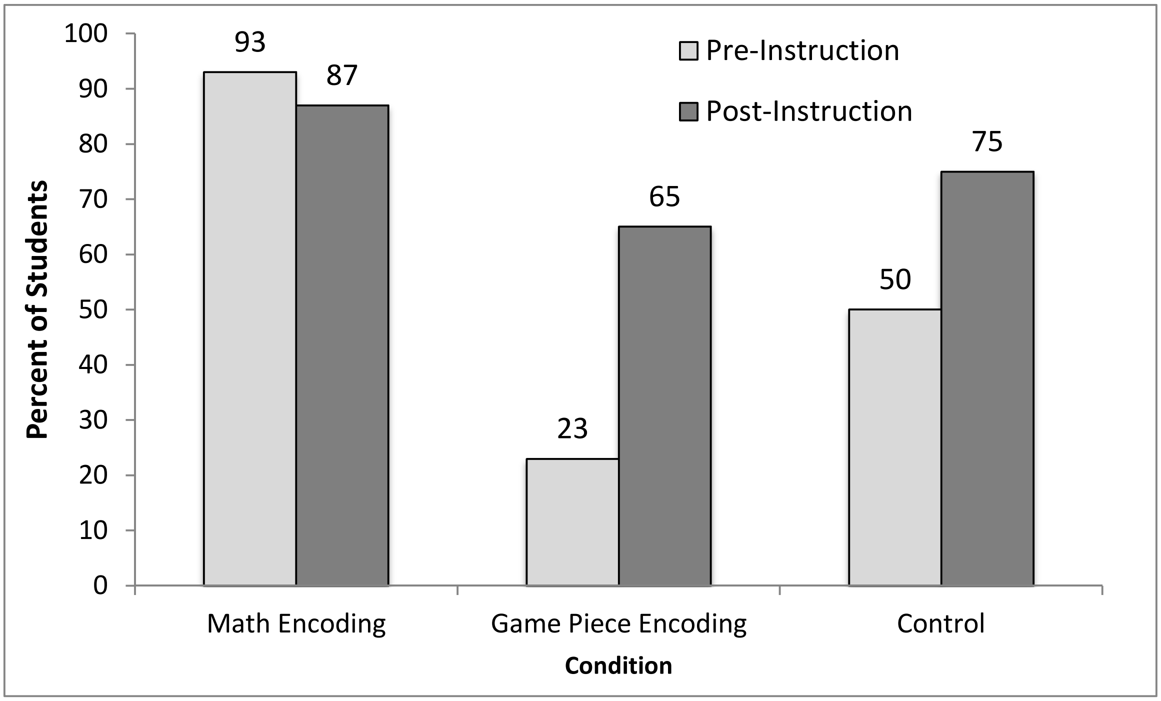

Figure 15. Proportions of students who assigned a quantitative use to the manipulatives by condition before and after Addition Instruction during the *Encoding* subtask of the QRIT.

Chapter 4: Discussion

Goals and Summary

The main objectives of this study were to assess whether explicitly outlining the relationship between manipulatives and the quantities they represent prior to linking the manipulatives to written symbols by way of instruction (a) allowed children to gain the intended representational insight, resulting in using the concrete objects in a symbolic way, and; (b) whether it allowed children to improve their use and understanding of addition procedures. An additional objective of this research was to provide empirical evidence to theoretical assertions surrounding children's development of dual representation and representational insight, based on research examining children's use of scale models (Uttal et al., 1997). Lastly, this research promised to shed some light for practitioners on the introduction and use of manipulatives to support students' meaningful learning in mathematics.

The findings of this study partially confirm the first hypothesis—that is, children who took part in the quantitative encoding instruction explicitly linking manipulatives and quantitative concepts seemed to grasp the quantitative duality of the manipulatives after the instruction. Specifically, they outperformed the Control group on all three subtasks of the QRIT, but outperformed the Game Piece Encoding group on only one of the QRIT subtasks (*Read a Display)*. Although the participants improved across all of the subtasks of the QRIT regardless of condition, only the Control condition was not able to acquire the quantitative representational insight of the manipulatives through the Addition Instruction to the same extent as the Math Encoding group. The Math Encoding group was more likely to read a display of quantities correctly than either of the groups.

Finally, children in the Math Encoding and Game Piece Encoding groups were just as likely to use the manipulatives as representations of “one” and “ten”—that is, they were just as likely assign the correct quantity to each type of manipulative, and as likely to correctly use the manipulatives to construct displays of quantities.

The results did not support my predictions of the relative benefits of the Math Encoding group on the Conceptual and Procedural Addition Tasks. Participating in instructions that fostered mathematical representational insight did not enable the Math Encoding group to better use or understand addition procedures with written numbers in comparison to the other two conditions. Although all of the conditions improved their use and conceptual justification of addition procedures, the results indicate that developing a conceptual understanding of the manipulatives did not give the Math Encoding Children an advantage either procedurally or in their conceptual understanding of the standard algorithm.

The results of the QRIT support the theoretical assumptions (Uttal et al., 1997) formulated on the scale model studies (DeLoache 1987; 1989; 2000; Uttal et al., 1995) that in order for children to use the manipulatives with meaning, teachers must explicitly outline the relationship between the manipulatives and what they stand for. Just as in the scale model studies, explicitly outlining the quantitative symbolic relationship during Encoding allowed children to develop a mathematical dual representation of the manipulatives—children viewed the manipulatives as objects, but also as representations of something else. The Math Encoding condition’s overall performance on the *Read, Construct,* and *Name Value* subtasks of the QRIT compared to the Control group indicated that they understood the symbolic relationship between the manipulatives and concepts they stood for. Furthermore, all three subtasks of the QRIT were correlated, suggesting were tapping into the same construct. Therefore, developing children's conceptual knowledge of mathematical quantities by explicitly telling them what the manipulatives represent, and then developing and instilling those quantitative concepts by way of representing and recomposing quantities in various ways, all lead to the correct quantitative use of the manipulatives. The Control group was not able to use the manipulatives correctly to the same extent as the Math Encoding group, therefore indicating that they had not grasped the relationship between the manipulatives and the quantities they represented solely on the basis of the Addition Instruction. This finding reaffirms Clements' (1999) and Ball's (1992) cautions against assuming that children will develop an understanding of the manipulatives simply by starting to work with them. It is evident that participating in instruction that connects manipulatives to written numbers does not overcome a lack of an introduction to the manipulatives prior to the instruction in developing a conceptual understanding of the quantities the manipulatives represent.

Although the Control condition was not able to "catch up" to the Math Encoding group in any of the QRIT subtasks, the Game Piece Encoding condition was able to do so for two out of the three subtasks (*Construct a Display* and *Name Value of Individual Circle)*. This finding was surprising as the children in the Game Piece Encoding group, who had initially developed a non-quantitative representation of the manipulatives, were able to grasp the quantitative representational insight to the same extent as the Math Encoding group through the combination of the Encoding and Addition Instruction.

A possible explanation of these results brings us back to DeLoache's (1995) theoretical model of symbol understanding, which stipulates that a number of factors affect and mediate children's symbol use. In the review of pertinent literature, I had discussed decreasing the symbol's salience as an object in order for children to grasp dual representation of that symbol. According to the model, decreasing salience improves children's ability to understand the representational insight between the symbol and its referent and further leads to using the symbol in the intended way. DeLoache also posits that *symbolization experience*, and therefore symbolic sensitivity, is another factor that affects children's use and understanding of symbols. Symbolization experience includes "both general experience with a variety of symbols and specific experience with any particular type of symbol" (DeLoache, 1995, p. 112) Empirical evidence supporting the impact of symbolic sensitivity on symbol use comes from the developmental literature. Two-and-a-half year old children who performed well on tasks that first required them to detect an easy symbol-referent relation also succeeded in detecting and using a more difficult symbol-referent relation (Marzolf & DeLoache, 1994). According to DeLoache (1995), symbolization experience leads to symbolic sensitivity, which is " a general expectation or readiness to look for and detect the presence of symbolic relations between entities" (DeLoache, 1995, p. 112). This developmental change allows children to grasp the dual representation with more ease, as children increase their focus on the abstract rather than concrete properties of the object.

In the current study, the Game Piece Encoding group received explicit instruction on the "Jumpers" game—that is, the children used the red and blue chips as pieces to a board game that belonged to two different teams and that could perform specific moves within the context of the game. The Game Piece Encoding assigned a game-piece meaning to the chips and, hence, was using the chips as symbols. According to

DeLoache's model (1995), it is possible that having symbolization experience with the chips prior to the Addition Instruction increased the children's symbolic sensitivity. In turn, the children were perhaps less focused on the concrete properties and more focused on the symbolic properties of the chips, thereby allowing them to acquire the quantitative dual representation of the chips with more ease.

It is important to underline that the Game Piece Encoding group was able to perform on the same level as the Math Encoding group on two out of three subtasks of the QRIT. The Math Encoding condition outperformed the Game Piece Encoding condition (along with the Control condition) on the *Read a Display* subtask. Perhaps there is something inherently different about this subtask in comparison to the *Construct a Display* and *Name Value* subtasks, despite their intercorrelations. It is also possible that the Game Piece Encoding group's performance across the subtasks was due to the order of the subtasks of the QRIT. Mainly, the subtasks were always administered in the same order, with the *Read a Display* preceding the remaining two subtasks. Perhaps the Game Piece Encoding group learned within the *Read a Display* subtask and then applied their knowledge to the remaining two subtasks. The Control group did not receive explicit instruction outlining neither the quantitative nor game piece symbol-referent relations. Hence, despite completing the subtasks in the same order as the remaining participants, it was not able to learn from the *Read a Display* subtask to the same extent as the Game Piece Encoding group perhaps did.

The purpose of the *Encoding* subtask of the QRIT was to establish how children viewed the manipulatives. I assumed that a quantitative perception of the manipulatives would allow the children to have an appropriate dual representation of the manipulatives (i.e., as objects and as representations of a quantities), and a non-quantitative perception would allow the children to have a non-quantitative dual representation. The proportions of the Game Piece Encoding and Control groups who saw the manipulatives as having a quantitative use greatly increased after the Addition Instruction. Even though the majority of each of the two groups viewed the manipulatives as having a quantitative purpose after the instruction, both proportions were still lower than those of the Math Encoding group at either time point.

When taking into account the significant results of the remaining subtasks of the QRIT, it is evident that although the majority of children in the Game Piece Encoding and Control groups changed their perception of the manipulatives, they still were not able to fully grasp the representational insight between the manipulative and the quantities they represented. This highlights a problem at a practical level: children may show they view the objects as a tool to use during mathematics lessons, but this does not automatically mean they will use them to their fullest capacity. This once again cautions against assumptions that children will automatically "pick up" the mathematical concepts once they just start manipulating concrete objects to do mathematics (Ball, 1992; Clements, 1999). As these results suggest, children might become aware that they use the objects in the context of mathematics class, but still not use them in the way they are intended—as a supportive tool in learning abstract concepts of mathematical quantities.

The non-significant results of the addition tasks (CAT and PAT) highlight some potential limitations of the study. Mainly, the Addition Instruction, which attempted to map and further develop conceptual understanding of quantity to written numbers, did not sufficiently necessitate the use of the manipulatives. The first day of the Addition

Instruction covered vertical addition of two types of problems: addition of two single digit numbers that did and did not require regrouping. Anecdotal evidence suggests that the children found these problems too easy. Although no formal data were collected on how children perceived the problems during the instruction, many children did not need to rely on the manipulatives to solve the written number problems as many of them stated the answers to the single-digit written addition problems before using the manipulatives. Therefore, they were able to solve the problems by relying on previously acquired skills and knowledge rather than on the manipulatives first.

Puchner and colleagues (2008) found similar results in their examination of teachers' use of manipulatives in elementary and middle school. In their classrooms, second- and third-graders were expected to use manipulatives to solve various mathematical problems. The researchers found that the manipulatives did not serve a supportive role in solving the problems, but rather became a completely disconnected part of the lesson. Puchner et al.'s explanation for this occurrence was that children did not rely on the manipulatives to solve the problems because they already knew how to apply the traditional algorithm. Any manipulative use became an end, rather than the means, to solving the problem. Children felt that they had to "do something" with the manipulatives and even went to the extent of attempting to replicate the results with the manipulatives based on the solution they obtained through the algorithm. Similarly, in the current study, it seemed that children had enough previous experience in solving the problems. As Puchner et al. (2008) pointed out, "when manipulatives are not utilized to foster emerging concepts, they become an end, rather than a possible means to an end" (p. 321). It seemed that in the present study, children did not rely on the manipulatives for assistance in solving the mathematical problems, but rather seemed to use them to display an answer they had achieved in another way (e.g., a previously learned procedure or invented strategy). Therefore, in order to make the manipulatives more useful in developing concepts of quantity represented by written symbols, solving the addition problems should not have been feasible by any other way except via the manipulatives.

Again, as Puchner and colleagues (2008) highlighted, manipulatives are useful when they assist in solving a challenging task, hence “manipulatives are a much more useful tool for testing out ideas that are slowly emerging within the student rather than understanding a concept after a procedure has been taught” (p. 321). Perhaps introducing problems with double-digit numbers in both the top and bottom of the addition algorithms on Day 1 may have been more conducive to thinking about how the manipulatives relate to the written symbols, rather than re-teaching the students a new way of adding smaller quantities for which they already had a strategy.

Day 2 of the Addition Instruction covered more challenging problems, which may have possibly forced the Math Encoding group to rely more heavily on their acquired representational insight with the manipulatives to solve the problems. Even if this were the case, one 50-minute period of instruction may have not been enough to see significant changes in their ability to use the concepts they learned during Encoding to increase their use and justification of addition procedures.

If the concept of quantity developed through manipulative use did not map onto written numbers, it is not surprising that children in the Math Encoding group did not outperform the others on the CAT and the PAT measures as predicted, nor that performance on the QRIT subtasks was not correlated with the CAT and the PAT. During the Addition Instruction, which intended to map the representation of quantities through manipulatives onto written symbols, the Math Encoding group did not need to rely on the representational insight they had developed between the manipulatives and quantities in order to solve problems with written numbers—they had perhaps reverted to using a procedure that was already familiar to them. The representational insight, therefore, did not give the Math Encoding group any advantage in justifying the incorrectly or correctly solved addition problems. Instead, the children seemed to revert to other, previously developed ideas and procedures to explain their answers. Essentially, I claim that manipulatives were not connected strongly enough to written numbers to use as a supportive tool in developing a conceptual understanding of addition procedures, which echoes the findings of other research where mapping instruction was not successful enough at connecting children's view of manipulatives and written numbers as multiple representations of the same concepts (Hughes, 1986; Resnick & Omanson, 1987).

As the literature on the development of conceptual and procedural knowledge shows, one type of knowledge fosters the development of the other (Hecht & Vagi, 2010; Hiebert & Wearne, 1996; Perry, 1991; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001; Wearne & Hiebert, 1988). Not surprisingly, therefore, conceptual understanding of addition procedures was correlated with procedure use in the present study. The PAT measured two components: whether children could obtain the correct answer when solving vertical addition problems, and whether they could employ the correct procedure in obtaining that answer. It was possible for children to end up with the correct answer without using the correct addition procedure, which involves lining up the numbers in the solution within the respective column and denoting the regrouping procedure with a little “1” above the tens column. As each group solved the majority of the questions correctly before and after the Addition Instruction, and there was no significant difference on the *Correct Solution* component within the groups across time, this suggests that the children referred to other, previously learned ways to solve the problems at both time points. Although data were not collected on how children solved the problems other than by their written solutions, based on the interviews I carried out and based on the reports from the research assistants, many children solved the problems by counting on their fingers. Although means for the *Procedure Use* component of the PAT improved over time, regardless of condition, the means were quite low in comparison to the means of the *Correct Solution* component. This suggests that children used strategies other than the addition procedure to solve the problems. Overall, the PAT did not provide a valid assessment of correct procedure use, as it did not force children to use the addition procedure to solve the problems.

Limitations and Future Research

This study raises several issues that need to be addressed in future research. The first, and most critical issue, is one raised by Puchner and colleagues (2008)—that is, manipulatives should serve to support children’s emerging understanding within a domain. Hence, the instruction connecting the manipulatives to written numbers should foster the development of emerging concepts by compelling children to rely on the conceptual knowledge developed with the manipulatives, rather than on previously acquired ideas and strategies. For this reason, future research should carry out a more thorough preliminary investigation of children’s current understanding and execution of addition concepts and procedures. This investigation will then guide the development of an intervention that is challenging enough for the children to foster those emerging concepts.

Second, future research should counterbalance the subtasks of the QRIT in order to control for order effects during pre- and post-instruction interviews. By doing so, any significant results will be more likely attributed to the Encoding difference, rather than a possible confounding factor such as the order in which the subtasks were presented.

Third, the measures assessing children's ability to use procedures and to justify their use must also be composed of tasks that will restrain their use of alternative ways of solving the problems. Similarly to the suggestions made for the Addition Instruction, the measures gauging use and understanding of addition with written symbols ought to necessitate children's use of the knowledge developed with the manipulatives.

Furthermore, a study that assesses children's knowledge of manipulatives prior to and after instruction makes it difficult to include manipulatives in any assessments. Introducing manipulatives prior to Addition Instruction increases the likelihood of children learning the meaning of the manipulatives prior to experimental manipulation. Conversely, including an assessment intended to measure children's understanding of the manipulatives without actually utilizing the manipulatives may not be sensitive enough to measure the intended constructs.

Additionally, as suggested by Fuson and Briars (1990), it may have been beneficial to implement a task that gauges children's understanding of the quantity represented by a written number by using the manipulatives. Such a task was designed and used by Kamii and Joseph (1988). A child was presented with a written number (e.g., 18). The interviewer first asked the child what the number was before proceeding with any subsequent questions. Once the child indicated she knew the name of the number ("eighteen"), the interviewer circled the 8 and asked the child to show with the manipulatives what "this part" meant. The same was done for the 1. This task taps into the child's understanding of the quantity represented by both the manipulative and written number and is more likely to require children to think of the written numbers in connection to the manipulatives, and therefore of the quantitative meaning assigned to them.

Lastly, taking into account the busy schedules of eight classrooms across four elementary schools resulted in some prolonged time spans between intervention and post-instruction data collection. In order to maximize the effect of the intervention, future research needs to ensure that students are tested with as few lapses as possible.

Contributions to the Literature

Uttal and colleagues (1997) proposed that because younger children had a difficult time perceiving the relationship between the scale model and the full-size room it stood for, older children would also have a difficult time perceiving the relationship between manipulatives and the mathematical concepts they stand for. Hence, the researchers proposed that children be told explicitly what the manipulatives stand for. The findings of the present study provide empirical support for this theoretical claim, mainly that the Math Encoding group grasped the quantitative representational insight across all of the subtasks of the QRIT. Furthermore, other mapping instructional interventions (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Wearne & Hiebert, 1988) fostered conceptual understanding of quantities by connecting manipulatives with spoken words and written symbols. The current study isolated the manipulative-concept of quantity relationship and the results indicate that explicitly teaching this component is key in gaining representational insight and therefore correct manipulative use.

Furthermore, partial results of this study may contribute to the literature on symbolic sensitivity. Mainly, exposing children to symbol-referent relationships may be beneficial to grasping any subsequent dual representations of an object, leading to the appropriate representational insight.

Implications for Practice

The current findings provide practitioners with guidance on how to introduce manipulatives in the classroom in a way that will promote correct quantitative use and reduce opportunities for children to develop other, non-quantitative representations. The results suggest that although introducing the objects as a symbol in a non-quantitative capacity may lead to an emerging acquisition of the quantitative representational insight of the manipulatives, explicitly outlining the quantitative relationship leads to a more comprehensive one.

On a final note, the lack of significant results for the CAT and PAT underlines the need to use the manipulatives to *support* emerging ideas, rather than employing them in unrelated, or disconnected ways. Therefore, it seems imperative that teachers gauge their students' current level of conceptual and procedural knowledge within a domain before employing manipulatives to further develop that knowledge.

References

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of mathematics education. *American Educator, 16*(2), 14-18, 46-47.

Brown, M., McNeil, N., & Glenberg, A. (2009). Using concreteness in education: Real Problems, potential solutions. *Child Development Perspectives, 3*, 160-164.

Clements, D. H. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood, 1*(1), 45-60

DeLoache, J. S. (1987). Rapid change in the symbolic functioning of very young children. *Science, 238*, 1556-1557.

DeLoache, J. S. (1989). Young children’s understanding of the correspondence between a scale model and a larger space. *Cognitive Development, 4,* 121-139.

DeLoache, J. S. (1995). Early understanding and use of symbols. *Current Directions in Psychological Science, 4*, 109–113.

DeLoache, J. S. (2000). Dual representation and young children's use of scale models. *Child Development, 71*(2), 329-338. doi:10.1111/1467-8624.00148

Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *Journal for Research in Mathematics Education, 21*(3), 180-206. doi:10.2307/749373

Fuson, K.C., & Kwon, Y. (1992). Korean children’s understanding of multidigit addition and subtraction. *Child Development, 63,*491-506.

Goldin, G. A. (2003). Representation in school mathematics: a unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. E. Schifter (Eds.), A research companion to principles and standards for school mathematics (pp. 275–285). Reston, VA: National Council of Teachers of Mathematics.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., and Brenwald, S. (2008). *Highlights from TIMSS 2007: Mathematics and Science Achievement of U.S. Fourth- and Eighth-Grade Students in an International Context* (NCES 2009–001 Revised). Washington, DC: National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.

Hecht, S.A., & Vagi, K.J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology, 102*(4), 843–859. doi: 10.1037/ a0019824

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 283-322). Hillsdale, NJ: Erlbaum.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert, J. Hiebert (Eds.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ England: Lawrence Erlbaum Associates, Inc.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*(3), 251-283.

Kamii, C., & Joseph, L. (1988). Teaching place value and double-column addition. *Arithmetic Teacher, 35*(8), 14-19.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Marzolf, D.P., & DeLoache, J.S. (1994). Transfer in young children's understanding of spatial representations. *Child Development, 65*, 1-15.

McNeil, N.M., & Uttal, D.H. (2009). Rethinking the use of concrete materials in learning: perspectives from development and education. *Child Development Perspectives, 3*(3), 137-139.

McNeil, N. M., Uttal, D. H., Jarvin, L., & Sternberg, R. J. (2009). Should you show me the money? Concrete objects both hurt and help performance on mathematics problems. *Learning and Instruction, 19*(2), 171-184. doi:10.1016/j.learninstruc. 2008.03.005

National Council of Teachers of Mathematics (2000). *Executive Summary: Principles and Standards for School Mathematics.* Retrieved from http://www.nctm.org/ uploadedFiles/Math_Standards/12752_exec_pssm.pdf

Osana, H. P., & Pitsolantis, N. (2011). Addressing the struggle to link form and understanding in fractions instruction. *British Journal of Educational Psychology*. Article published online: 7 November 2011. DOI: 10.1111/j.2044-8279.2011.02053.x

Perry, M. (1991). Learning and transfer: Instructional conditions and conceptual change. *Cognitive Development*, *6*, 449-468.

Puchner, L., Taylor, A., O'Donnell, B., & Fick, K. (2008). Teacher learning and mathematics manipulatives: a collective case study about teacher use of manipulatives in the elementary and middle school mathematics lessons. *School Science and Mathematics, 108*(7), 313-325. doi: 10.1111/j.1949-8594. 2008.tb17844.x

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser, R. Glaser (Eds.), *Advances in instructional psychology*, *Vol. 3* (pp. 41-95). Hillsdale, NJ England: Lawrence Erlbaum Associates

Rittle-Johnson, B.A., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91* (1), 175-189. doi: 10.1037/0022-0663.91.1.175

Rittle-Johnson, B.A., Siegler, R.S., & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An interative process. *Journal of Educational Psychology, 93*(2), 346-362. doi: 10.1037//0022-0663.96.2.346

Rittle-Johnson & Siegler (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skills; Studies in developmental psychology* (pp. 75-110). Hove, England: Psychology Press/Taylor & Francis

Sarama, J. & Clements, D.H. (2009). "Concrete: computer manipulatives in mathematics education. *Child Development Perspectives, 3*(3), 145-150.

Stigler, J.W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMMS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eight-Grade Mathematics Instruction in Germany, Japan, and United States.* Washington, DC: National Center for Education Statistics. Available: http://nces.ed.gov /timss.

Uttal, D. H. (2003). On the relation between play and symbolic thought: The case of mathematics manipulatives. In O. Saracho and B. Spodek (Eds), *Contemporary perspectives in early childhood* (pp. 97-113). Information Age Press.

Uttal, D.H., Liu, L.L., & DeLoache, J.S. (2006). Concreteness and symbolic development. In L. Balter and C.S. Tamis-LeMonda (Eds.), *Child psychology: a handbook of contemporary issues* (pp. 167-184). New York, NY: Psychology Press/Taylor & Francis.

Uttal, D. H., & O'Doherty, K. (in press). Comprehending and learning from visual representations: A developmental approach. In J. Gilbert, M. Reiner, & M. Nakhleh (Eds.), *Visualization: theory and practice in science education*. New York, NY: Springer.

Uttal, D. H., O'Doherty, K., Newland, R., Hand, L.L., & DeLoache, J.S. (2009). Dual representation and the linking of concrete and symbolic representations. *Child Development Perspectives, 3*(3), 156-159.

Uttal, D.H., Schreiber, J.C., & DeLoache, J.S. (1995). Waiting to use a symbol: the effects of delay on children's use of models. *Child Development, 66*(6), 1875-1889. doi: 10.1111/ j.1467-8624.1995.tb00971.x

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology, 18*(1), 37-54. doi:10.1016/S0193-3973(97)90013-7

Wadsworth, B.J. (1984). *Piaget's theory of cognitive and affective development* (Third ed.). New York, NY: Longman.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19*(5), 371-384. doi:10.2307/749172

Appendix A

Encoding Intervention: Math Encoding

<table>
<tr><th colspan="2">DAY 1</th></tr>
<tr><th colspan="2">1. Encoding: Blue circle</th></tr>
<tr><th>Instructions</th><th>Error Correction</th></tr>
<tr><td>Instructor (I): Take a look at all the things we have on the table.
We have a container of little plastic circles. There are some that are red (point to container) and there are some that are blue (point to container).
This is very important, please do not take any of the circles out of the box, until I tell you.
I’ll be showing you what to do with the circles, so I would like for you to look and listen to what I do and follow me when I say it’s your turn.</td><td></td></tr>
<tr><td>I: Pull out a blue circle and hold it up for everyone to see.
Take a look. I have a blue circle. The blue circle is worth the same as one (Pause)
I: How much is the blue circle (worth?)?
C: One</td><td></td></tr>
<tr><td>I: Please take out one blue circle from the container and hold it up.
I: How much is this worth?
C: One</td><td>I: No. The blue circle is worth the same as one. (Pause) How much is this worth?
C: One</td></tr>
<tr><th colspan="2">2. Represent 1-10: Blue circles</th></tr>
<tr><td>Pull out 2 blue circles and place them on the mat in front of you.
I: Let’s count the blue circles. Watch me first.
I models by touching left to right.
I: One-two. I have two.
Your turn, please take out two blue circles and place them on your mat.
C: take out circles.
I: Let’s count. (T counts with kids 1-2. Two)
I: How much is this worth?
C: Two</td><td>I: No. Remember, the blue circle is worth the same as one. (Pause) Let’s count again. One- two. (pause) Two.
How much is this worth?
C: Two</td></tr>
<tr><td colspan="2">Repeat for 5,8 and 3 blue circles</td></tr>
<tr><td>I: Now it’s your turn, please take out six blue circles and place them on your mat. Ok, now count them.
C: Count one-two-three-four-five-six
I: How much is this worth?
C: Six</td><td>I: No. Remember, the blue circle is worth the same as one. (Pause) Let’s count again.
I: point as child counts. One- two-three-four-five-six.
How much is this worth?
C: Three</td></tr>
<tr><td colspan="2">Repeat for 6,4 and 9 red circles</td></tr>
</table>

<table>
<tr><th colspan="2">3. Encoding: Red circle & Connect 10 Blue circles same as 1 Red</th></tr>
<tr><td>I: Place 10 blue circles on the mat in front of you.
I: Also place 10 blue circles on your mat
C: Take out 10 blue circles from the container and place them on their mats
I: Count the blue circles.
C: Count/touch the circles
I: How much is this worth?
C: 10</td><td>*Provide necessary feedback*</td></tr>
<tr><td>I: *Pull out a red circle and hold it up for everyone to see.*
Now, take a look. I have a red circle. The red circle is worth the same as 10. *(Pause)*
I: How much is the red circle worth?
C: 10</td><td>*I: The red circle is worth the same as ten. (Pause) How much is the red circle?*
C: Ten</td></tr>
<tr><td>I: Take a look at your mats. How much is that worth?
C: 10
I: You're right, it's 10.
One red circle *(hold up red circle)* is worth the same as 10 blue circles (*point to the blue circles on the mat*)
I: They are the same. *Sweep the blue circles to the side and replace them with a single red circle.*</td><td>*If children refer to the color & / circles "10 blue/ 10 blue circles.." etc.*
I: "The blue circle is worth the same as one. You have ten blue circles, so all together you have ten."</td></tr>
<tr><td>I: Now it's your turn. Take out a red circle...
C: take out a red circle
I: and trade with the 10 blue circles you have on your mat.
C: Sweep blue circles off the mat and replace with a single red circle.
I: Now, how much is this worth?
C: 10
I: You're right! It's still 10. You took 10 blue circles and you traded with 1 red circle. They're the same; they're both worth 10.</td><td>*If C say "One red/ red circle.." etc, I: "The red circle is worth the same as ten. So you have ten"*</td></tr>
<tr><th colspan="2">4. Representation 10-90: Red circles</th></tr>
<tr><td>*I: Pull out 2 red circles and place them on the mat in front of you.*
Let's count the red circles. Watch me first.
T models by touching left to right.
Ten-twenty. I have twenty.
Your turn, please take out two red circles and place them on your mat.
C: Place two red circles on their mats.
I: Let's count. *(T counts with kids; both T and C touch)* 10-20.
How much do you have in front of you?
C: Twenty</td><td>*I: Remember, the red circle is worth the same as ten. (Pause) Let's count again. Ten- Twenty. (pause) Twenty.*
How much is this worth?
C: Twenty</td></tr>
<tr><td>*Repeat for 4, 7, 5 and 9 red circles*</td><td></td></tr>
</table>

I: Now it's your turn, please take out three red circles and place them on your mat. C: Take out circles and place them on their mat. I: Can you count them? C: Count ten-twenty-thirty How much do you have in front of you? C: Thirty	*I: Remember, the red circle is worth the same as ten. (Pause) Let's count again. Ten-Twenty-Thirty. (pause) Thirty. How much is this worth?* *C: Thirty*
Repeat for 8 and 6 red circles	
Ensure when taking out larger quantities, T models lining them up in a way that will make counting easier (not necessarily a perfectly straight line, but close!)	
5. Additional Examples & Final Review	
I: *Places a blue circle on the mat and asks:* How much is the blue circle worth? C: one I: I'm going to place these on the mat. *Altogether, T places 8 blues on the mat.* I: How much is this worth? C: (count out independently or together) eight. I: *provide feedback*	*I: Remember, the blue circle is worth the same as one. (Pause) Let's count again. One-two-….8. (pause) Eight.* *How much is this worth?* *C: Eight.*
Repeat with 14 blue circles.	
I: *Places a red circle on the mat and asks:* How much is the red circle worth? C: Ten I: I'm going to place these on the mat. *Altogether, T places 6 blues on the mat.* I: How much is this worth? C: (count out independently or together). 10-20... 60. Sixty I: *provide feedback*	*I: Remember, the red circle is worth the same as ten. (Pause) Let's count again. Ten-Twenty-Thirty-Forty-Fifty-Sixty. (pause) Sixty.* *How much is this worth?* *C: Sixty*
Do an example with 9.	

Day 2	
1. Review	
Instructions	**Error Correction**
I: We had used these (*pointing to the circles)* last time we were together. This is very important. Just like last time, please do not take any of these out of the box, until I tell you. Watch me first and I will let you know when it is your turn. Let's go over what we did last week!	
I: *Pull out a blue circle and hold it up for everyone to see.* Take a look. I: How much is the blue circle worth? C: One	*I: No. The blue circle is worth the same as one. (Pause) How much is this worth? C: One*
I: *Place 6 blue on a mat.* I: Let's count together. How much is this worth? C: Count with T, while T points to each circle. One, two three...six. I: Yes, this is worth six.	
I: Now it's your turn, please take out 16 blue circles and place them on your mat. Can you count them? C: Count 1-2-3…16. I: How much is this worth? C: Sixteen.	
I: Now it's your turn, please take out 16 blue circles and place them on your mat. Can you count them? C: Count 1-2-3…16. I: How much is this worth? C: Sixteen.	
I: *Place 10 blue on a mat. Ask:* I: How much is this worth? C: Count with T, while T points to each circle. One, two three...ten. I: Yes, this is ten. How else can we show ten? C: With a red circle. I: You're right, 10 blues is worth the same as one red. *Sweep 10 blues off the mat and replace with one red.* T. How much is this worth? (*Pointing to the red)* *C:* Ten I: You're right! It's still 10. You took 10 blue circles and you traded with 1 red circle. They're the same; they're both worth 10.	*I: No. The red circle is worth the same as ten. (Pause) How much is this worth? C: Ten*
I: *Place 4 red on a mat. Ask:* I: Let's count together. How much is this worth? Remember, when we count the red circles, we count by 10s. C: count with T, while T points to each circle. Ten, twenty, thirty, forty. I: Yes, this is 40.	

I: Now it's your turn, please take out seven red circles and place them on your mat. Can you count them? C: Count 10-20-30…70. How much is this worth? C: Seventy I: *Place 1 red on a mat. Ask:* I: How much is this worth? C: Ten. I: Yes, this is ten. How else can we show ten? C: With blue circles circle. I: You're right, 1 red circle is worth the same as 10 blue circles. *Sweep 1 red off the mat and replace with 10 blues.* How much is this worth? (*Pointing to the blue)* *C:* Ten You're right! It's still 10. You took 1 red circle and you traded with 10 blue circles. They're the same; they're both worth 10.	*I: No. The blue circle is worth the same as one. (Pause) Let's count how much this is.(Count 1-10)* *C: Ten*

<table>
<tr><th colspan="2">2. Representation of quantities using blue circles and red circles</th></tr>
<tr><td>I: Pull out 2 red circles and 3 blue circles and place them on the mat in front of you.
Remember how we count the red circles? Watch me first.
T models by touching the red circle
Ten-twenty.
Now, remember how we count the blue circles? Watch me first.
T models by touching the blue circle
One-two-three.
Let's count the blue and red circles together. We always count the red circles first.
T models by touching the circles
Ten-twenty (pointing to the red circles) 21-22-23. (Pointing to the blue circles). Twenty-three.
I: Your turn. Please take out two red circles and three blue circles and place them on your mat.
I: Let's count. (T counts with children)
I: How much is this worth?
C: Twenty-Three.</td><td>I: Remember the blue circles are the same as 1 and the red circles are the same as 10.
(T counts the chips again).</td></tr>
<tr><td>I: Let's do a different example.
I: I want to show 11 with red and blue circles. What do I need?
C: 1 red and 1 blue circle.
I: Take out chips and place them on the mat. Count out the chips with the children.
Ten (pointing to the red circles) 11 (pointing to the blue circles).
Eleven.</td><td></td></tr>
<tr><td>I: Your turn. I want to show 45 with red and blue circles. What do I need?
C: 4 red and 5 blue circles.
I: Please take out four red circles and five blue circles and place them on your mat.
I: ** Individually ask children to count the circles (depending on time) Can you count them?
C: Ten-twenty-thirty-forty (pointing to the red circles) 41-42-43-44-45. Forty five (pointing to the blue circles).
I: How much is this worth?
C: Forty-Five.</td><td>I: Remember the blue circles are the same as 1 and the red circles are the same as 10.
(T counts the chips again).

I: We always count the red circles first.</td></tr>
<tr><td>I: Let's do another example. Please take out two red circles and seven blue circles and place them on your mat.
I: ** Individually ask children to count the circles (depending on time) Can you count them?
C: Ten-twenty (pointing to the red circles) 21-22-23-24...-27 (pointing to the blue circles).
I: How much is this worth?
C: Twenty-Seven.</td><td>We always count the red circles first.</td></tr>
</table>

3. Recomposition Over 10: All B to B & R	
I: Let's do a different example. I: I want to show 15 with blue circles. What do I need? C: 15 blue circles. I: *takes out 15 blue circles and places them on the mat.* I: How much is this worth? Let's count C: Count. 1, 2, 3, 4... 15. Fifteen. I: I'll show you another way you can show 15. Remember that a group of 10 blue circles are worth the same as 1 red circle. I trade a group of 10 blue circles (count out) for one red circle (sweep 10 blues, place them back in the container, and replace with one red) and I leave the rest on the mat. I: How much is this now? Let's count. C: T count with C: 10, 11, 12, 13, 14, 15. Fifteen. I: We still have fifteen. We took a group of 10 blue circles and traded it in for 1 red circle because they are worth the same.	*We always count the red circles first.*
I: Now it's your turn. Show me 12 with blue circles. *C:* Place 12 blue circles on the mats. I: How much is this worth? C: (Children count 1-2-3..12) 12 I: How can we show 12 in another way? C: trade a group of 10 blue circles for 1 red circle. I: ok, let's count (T with C) 1,2,3,4…10. And trade for one red circle. C: trade: place the group of 10 blue circles back to the container and take out one red circle and place it on their mats. I: Now how much do you have? ***** Individually ask children to count the circles (depending on time)*** C: Count 10, 11, 12. Twelve. I: Yes, we still have 12!	
I: Let's try another one. *Place 26 blue circles on mat* I: Let's count C: count 1-2-3…26. I: How much is this worth? C: 26 I: How can we show 26 in another way? C: trade a group of 10 blue circles for 1 red circle. I: ok, let's count. 1,2,3,4…10. And trade for one red circle. I: Now how much do we have? C: count 10, 11, 12…26. Twenty six I: Yes, we still have 26! I: How many blue circles do we have left? C: 1,2,3,4,5,6-16. I: Can we trade 10 blue circles for another red? C: yes I: ok, let's count. 1,2,3,4…10. And trade for one red circle. I: Now how much do we have? C: count 10, 20,21,22,23,24,25,26. Twenty six I: Yes, we still have 26!	

I: Now it's your turn. Show me 22 with blue circles. *C:* Place 22 blue circles on the mats. I: How much is this worth? *(point to 1-2 children's mats)* C: 22 I: How can we show 22 in another way? C: trade a group of 10 blue circles for 1 red. I: ok, let's count (T with C) 1,2,3,4…10. And trade for one red circle. C: *trade by placing the group of 10 blue circles to the side and take out one red circle and place it on their mats.* I: Now how much do we have? C: Count 10, 11, 12,13….22. Twenty-two.. I: Yes, we still have 22! I: How many blue circles do we have? C: 1,2,3,4,5,6-12. I: Can we trade a group of 10 blue circles for another red? C: yes I: ok, let's count. 1,2,3,4…10. And trade for one red circle. I: Now how much do we have? C: count 10, 20,21,22 Twenty-Two I: Yes, we still have 22!	
I: Now it's your turn again. Show me 14 with blue circles. What do you need? C: 14 blue circles. *C:* Place 14 blue circles on the mats. I: How much is this worth? C: Count out. 1-2-3-…14. Fourteen. I: How can we show 14 in another way? C: trade a group of 10 blue circles for 1 red. I: ok, let's count (T with C) 1,2,3,4…10. And trade for one red circle. C: *trade by placing the group of 10 blue circles to the side and take out one red circle and place it on their mats.* I: Now how much do we have? C: Count 10, 11, 12,…14. Fourteen. I: Yes, we still have 14! I: How many blue circles do we have? C: 1,2,3,4. I: Can we trade a group of 10 blue circles for another red? C: No I: Why can't we trade 10 blue circles for another red? C: Because we need a group of 10 blue circles to change it to 1 red circle. I: Good!	

Appendix B:

Encoding Instruction: Game Piece Encoding

DAY 1	
I: Take a look at all the things we have on the table. We have some plastic circles. There are some that are red (*point to container)* and there are some that are blue (*point to container)*. We also have a game board (*point to the board)* This is very important, please do not take any of the circles out of the box, until I tell you. I'll be showing you how to play the game, so I would like for you to look and listen to what I do and follow me when I say it's your turn. We're going to play a game called "Jumpers". There will be two teams: the red team and the blue team. (Point to each of the colors)	
1. Encoding blue and red circles as "Jumpers"	**Error Correction**
I: These *(hold up a blue circle)* blue circles are called blue *jumpers*. What are these? C: Blue *jumpers* (Place back in container). These *(hold up a red circle)* red circles are called red *jumpers*. What are these? C: red *jumpers* *(Place back in container).* The blue *jumpers* will be on a separate team and the red *jumpers* will be on a separate team.	**Representation:** "No, the blue circles are called blue *jumpers*". **Or** "No, the red circles are called red *jumpers*".
2. Setting up the Board	**Error Correction**
I: The first thing we will do is fill up these back rows (*point to the two back rows of one side of the board)* with the red *jumpers* and these back rows (*point to the two back rows of the opposite side of the board)* with the blue *jumpers*. (*Take out the red circles and line them up; take out the blue circles and line them up; DO NOT count the circles or the spots on the board)* I: Now we have two teams. A team of red *jumpers* (*point*) and a team of blue *jumpers* (*point)*.	**Begin discussion of quantity:** "we need to fill up the back rows of the board using all the red/ blue *jumpers*"
3. Rules of the Game	
I: Let me show you how we're going to play this game. The **point of the game** is to move all your *jumpers* to the other side of the board, before the other team moves their *jumpers* to your side. *(Point to the respective teams and sides)*. I: You can only move the *jumpers* forward. No moving the *jumpers* backward. *Jumpers* can only do two moves. You can either do a *slide* or a *jump*. -A *slide* is when you move a piece to an open square next to the jumper. Like this *(show with one jumper- moving one up) or like this (move one jumper to the side)* -A jump is when you jump over any color of a *jumper* as long as there's a space after the *jumper* you are jumping over.	

Let me show you. *(Move a couple of pieces on the board so that you can show different moves. Jump a jumper diagonally; jump a jumper to each side)* *(move all jumpers back)*	
4. Review of the Rules	**Error Correction**
I: Let's go over how we can move the *jumpers*. Can you move a *jumper* like this? *(slide a jumper backward)* C: No I: How about like this? *(jump a back jumper over the front row plus an extra space)* C: No I: So how do we move the *jumpers*? Who can show me a *slide*? *Pick a child to show/ explain a slide.* I: Who can show me a *jump*? *Pick a child to show/ explain a jump.* I: which way can we move the *jumpers*? C: Forward. I: So can I do this? (Lay out pieces so that you show a *slide* backward) C: No I: How come? C: Because you can only move the *jumpers forward*. I: Can I do this move? *(Lay out pieces so that you show a backward jump)* C: No I: You're right; you can only move the pieces forward. Like *this (show a correct slide)*, or like this *(show a correct jump).*	**Wrong move:** 'No, that's not the way we move the *jumpers*. Remember, a *slide* is… and a *jump* is…
5. Playing the Game	**Error Correction**
I: We're going to start playing a game. I'm going to divide you into two teams. You can help out the players on your team. So you can talk about what moves you can make next. *Divide the children into two teams. If odd number, monitor as a player in order to balance the teams. If even number, solely monitor.* I: Red always goes first. *Place one red chip in one closed fist, a blue in the other. Get a child from one team to choose a fist. If red, the team goes first.* *Children begin to play and teacher is there to monitor the game.*	**Wrong move:** See above. **Wrong direction:** "Remember, you always move your team's *jumpers* forward" and point out the direction. **Begin to discuss quantity:** See above.

DAY 2	
1. Review Game Components	**Error Correction**
I: last time you learned how to play the game called *Jumpers.* You'll get a chance to play the game today, but first I want to make sure you remember all the rules! Take a look at all the things we have on the table. *Point to the container of red circles.* What are these? C: Red *jumpers* I: and what are these? (*point to the blue circles)* C: Blue *jumpers* I: And what is this? (*point to the board)* C: A game board. Great!	**Representation:** "Remember, the blue circles are blue *jumpers.*" **Or** "Remember, the red circles are called red *jumpers*."
2. Review Rules	
I: Now let's talk about how to set up the board. Where do we line up each team of *jumpers*? C: At the back of the board. I: Great! Let's do that. C & I *line up the jumpers (DO NOT discuss how many pieces or how many squares. Instead, redirect to' filling out the back rows')* I: Let's go over how we can move the *jumpers*. Can you move a *jumper* like this? *(move a jumper onto a white square)* C: No I: How about like this? *(jump a back jumper over the front row plus an extra space)* C: No I: So how do we move the *jumpers*? Who can show me a *slide*? *Pick a child to show/ explain a slide.* I: Who can show me a *jump*? *Pick a child to show/ explain a jump.* I: which way can we move the *jumpers*? C: Forward. I: So can I do this? *(Lay out pieces so that you show a slide backward)* C: No I: Can I do this move? *(Lay out pieces so that you show a backward)* C: No I: You're right; you can only move the pieces forward. Like this *(show a correct slide),* or like this *(show a correct jump).*	

3. Playing the Game	**Error Correction**
I: We're going to start playing a game. Just like last time, I'm going to divide you into two teams: a red team and a blue team. You can help out the players on your team. So you can talk about what moves you can make next. *Divide the children into a blue team and a red team. If odd number, monitor as a player in order to balance the teams. If even number, solely monitor.* *Decide which side will go first. Children begin to play and teacher is there to monitor the game.*	**Wrong move:** 'No, that's not the way we move the *jumpers*. Remember, a *slide* is… and a *jump* is… **Wrong direction:** "Remember, you always move your team's *jumpers* forward" and point out the direction. **Begin to discuss quantity:** Bring children's attention to whichever is applicable in the context: the pieces, squares on board, actions etc.

Appendix C:

Encoding Intervention: Control Group

Day 1

Materials:

- Book**: Fancy Nancy and the Boy from Paris**
- *Fancy* words
- Sticky tack
- Bristol board
- Plain paper 1 x student
- Markers/crayons

<u>Throughout:</u> ***Monitor that children are <u>not making any references</u> to quantities, digits, shapes, etc. If any mathematical ideas arise, redirect to the task at hand (words, different parts of pictures, etc.).***

1. Pre-reading

Introduction:

I: Today we are going to read a story about Fancy Nancy.

- Do you know who Fancy Nancy is?
- Have you read any of her books? With who? (On own, with parents, grandparents, etc.)
- Fancy Nancy has many adventures. She is a girl who likes to dress up in fancy clothes. She also uses a lot of fancy words!

Do you know what the word *fancy* means?

I: We are going to look at the fancy words Fancy Nancy uses in this book, called Fancy Nancy and the Boy from Paris.

Using the printed word cards, read each "fancy" word that will be in the story and arrange it in on the Bristol board in front of the children (on table, etc).

Ask different children what they think each word means.

Tardy/Gorgeous/Terrified/Perplexed/Bonjour/Ami/Belle

1. Reading

- *Read the story* ***Fancy Nancy and the Boy from Paris*** *aloud.*
- *Throughout the story, ask questions:*

  How is Nancy feeling?

  What do you think might/will happen next?

  Questions regarding the environments presented within the story (classroom, playground, home, etc)

2. Post-reading

a) **Questions:** *Ask the children what they thought of the book*

   What was your favorite part of the story? Why?

   Discuss the difference between Paris, France and Paris, Texas; whether the children have ever been there, etc.

b) **Fancy words activity:**

   - *Ask the children what the words on the board mean (they should know more of them at this point). Tell them what the words mean – use the regular words from the last page of the book.*
   - *Ask the children if they can think of other fancy words and discuss their meaning.*

c) **<u>Drawing (extra activity):</u>**

   - *If there is extra time, provide the children with a piece of paper and crayons/markers. Ask them to draw a picture of their favorite part of the story. Collect their drawings afterwards (make sure to write their name on it).*

Day 2

Materials:
- Book**: Fancy Nancy and the Boy from Paris**
- *Fancy* words + regular words
- Sticky tack
- Bristol board
- Worksheets (1 x child and instructor)
- Markers/crayons

Throughout: ***Monitor that children are not making any references to quantities, digits, shapes, etc. If any mathematical ideas arise, redirect to the task at hand (words, different parts of pictures, etc.).***

1. Pre-reading

Review: Today we are going to read the story about Fancy Nancy again.
Do you remember who Fancy Nancy is?
- Last time I said that Fancy Nancy has many adventures. She is a girl who likes to dress up in fancy clothes, and she likes to use a lot of fancy words!

Who remembers what the word *fancy* means?
- Great! I want you to keep your ears open and listen for the fancy words that Nancy uses. We are going to do an activity after the story.

2. Reading

- *Read the story* ***Fancy Nancy and the Boy from Paris*** *aloud.*

Throughout the story, ask a few questions:
How is Nancy feeling?
What do you think might/will happen next?
Questions regarding the environments presented within the story (classroom, playground, home, etc.)

3. Post-reading

- Last time we had talked about the fancy words that Fancy Nancy uses in this book.
- Let's look at them now.
  - § *Using the printed word cards read a "fancy" word that was in the story and attach it to the Bristol board.*
  - § *Ask the children what the word means.*
  - § *After they answer, read off and attach the "regular word" next to the fancy word. (Talk about the fact that the last 3 words are in French)*

- * Tardy- late
- * Gorgeous- beautiful
- * Terrified- scared
- * Perplexed- mixed up
- * Bonjour- hello
- * Ami- friend
- * Belle- beautiful

- We are going to do an activity. We are going to write out the fancy words Nancy uses (*point to the fancy words on the board)* and also what they mean (*point to the regular words on the board)*. I have a worksheet here (*show children a worksheet)* and some markers.
- This is where you will write the fancy words (*point to the left column)* and over here is where you will write the regular words.

How can we write out the fancy words in a fancy way? (squiggly or bubble letters, underlines, etc.)
Great! Let me show you what we are going to do.
Write out first fancy word in first row of "fancy word" column. Make it fancy!
Write out regular word in regular letters in first row of the "regular word" column.
Take your time and write as many of the words as you can (*point to the board*).
Give each child a worksheet and some markers.

Coloring (Extra activity): If there's extra time, ask children to color a picture of the Eiffel Tower on the back of the worksheet.

Name: ________________
Grade: ________________

Fancy **NANCY**

and the
Boy from Paris

Fancy word	Regular word

Appendix D

Addition Instruction

• I= Instructor • Regular font: Spoken by Instructor	• *Italicized font:* Actions completed by Instructor • * =Instructor waits for students to complete the step
DAY 1	
1. Adding Single Digits (Imitation)	
I: Let's add three plus two (3+2) (*Points to the horizontal equation at the top of the worksheet)* I: Start with the 3 *(points to the written 3 in the horizontal equation)* I: Put 3 blue circles here (top right circle column) I: That (*Circle all the chips with your finger)* makes 3 *(trace the written 3 in the vertical equation with your finger). ** I: Add the 2 *(Point to the written 2 in the horizontal equation)* I: Put 2 blue circles here (middle right circle column) I: That (*Circle all the chips with your finger)* makes 2 *(trace the written 2 in the vertical equation with your finger).** I: Put all the blue circles together. *(Pull down all blue circles to the bottom)* I: Let's count them 1,2,3,4,5. There are five. I: Write 5 over here (bottom right number column of vertical equation)* I: *Points to the right number column* (3+2 is 5) I: *Points to the horizontal equation and say*, " 3+2 is 5" and *writes = 5 on the equation**	<u>Procedure</u> - Imitation: 2 equation - Structured Practice: 1 equations <u>Equations</u> 1) 3+2= (imitation) 2) 7+0= (imitation) 3) 5+4= (structured practice)
2. Adding single digits with sum greater than 9 (Imitation)	
I: Let's add four plus seven (4+7) (*Point the horizontal equation).* I: Start with the 4 (pointing to the written 4) I: Put 4 blue circles in top right circle column. I: That *(Circle all the chips with your finger)* makes 4. So I am going to write 4 here (in the right number column)* I: Add the 7 (pointing to the written 7) I: Put 7 blue circles in middle right circle column I: That *(Circle all the chips with your finger)* makes 7. So I am going to write 7 here (middle right number column)* I: Put all the blue circles together. *(Pulls down all blue circles to the bottom)* I: Let's count them 1,2,3,4,5,6,7,8,9,10,11. There are eleven.* I: Every time you have a group of 10 blue circles (*count out 10 blue circles)* you put the 10 blue circles back in the box (*place 10 blue circles in box)* and put a red circle here (*Put the red circle above the left circle column*) I: Write a little 1 above the top left number column* I: Count the blue circles, *Counts* "1. There is 1." I: Write 1 bottom right number column. * I: Put all the red circles together. (*Pull down all red circles to the bottom*) I: Count red circles. 1. There is 1. I: Writes1 bottom left number column. * I: *Points to the number column* and says, "4+7 is 11" I: *Points to the equation and says*, " 4+7 is 11" and *writes = 11 on*	<u>Procedure</u> - Imitation: 2 equations - Structured Practice: 1 equations <u>Equations</u> 1) 4+7= (imitation) 2) 5+5= (imitation) 3) 8+9= (structured practice)

*the equation**	
Additional structured practice examples: 4+4 6+8 0+3 7+5	
DAY 2	
1. Adding Double Digits (Imitation)	
I: Let's add twelve plus six *(point to the horizontal equation at the top of the worksheet)* I: Start with the 12 *(point to the written 12 in horizontal equation)* I: Put 1 red circle here (top left circle column) I: Put 2 blue circles here (top right circle column) I: That (*Circle all the chips with your finger)* makes 12 (P*oint to written 12 in vertical equation*)* I: Add the 6 *(point to the written 6 in equation)* I: Put 6 blue circles here (middle right circle column) I: That (*Circle all the chips with your finger)* makes 6 *(Point to written 6 in vertical equation)** I: Put all the blue circles together. *(Pull down all blue circles to the bottom)* * I: Let's count the blue circles 1,2,3,4,5,6,7, 8. There are eight (*Circle all the chips with your finger).* I: Write 8 here (bottom right column). * I: Put all the red circles together. *(Pull down all red circles to the bottom)* I: Let's count them 1. There is 1. I: Write 1 over here (bottom left number column)* I: *Point to the number column and say*, "*12 + 6* is 18" I: *Point to the equation and say*, "*12 + 6* is 18" and *write = 18 on the equation* *	<u>Procedure</u> - Imitation: 2 equations - Structured Practice: 1 equation <u>Equations</u> 1) 12+6= (imitation) 2) 44+13= (imitation) 3) 11+4= (structured practice)
2. Adding Double Digits; sum in singles is > 9 (Imitation)	
I: Let's add nineteen plus six *(point to the equation at the top of the worksheet)* I: Start with the 19 *(point to the written 19 in the horizontal equation)* I: Put 1 red circle here (top left circle column) I: Put 9 blue circles here (top right circle column) I: That *(Circle all the chips with your finger)* makes 19. Write 1 here (in the top left number column) and write 9 here (in the right number column)* I: Add the 6 *(point to the written 6 in the horizontal equation)* I: Put 6 blue circles here (middle right circle column) I: That *(Circle all the chips with your finger)* makes 6. I am going to write 6 here, under the 9 (middle right number column) * I: Put all the blue circles together. *(Pull down all blue circles to the bottom)* * I: Let's count them 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15. There are fifteen. I: Every time you have a group of 10 blue circles (*count out 10 blue circles)* you put the 10 blue circles back in the box (*place 10 blue circles in box)* and put a red circle here (*Put the red circle above the left circle column*) I: And now I am going to write a little 1 here (*Write 1 above the top left number column*) *	<u>Procedure</u> - Imitation: 1 equation - Structured Practice: 4 equations <u>Equations</u> 1) 19+6= (imitation) 2) 14+8= (Imitation) 3) 25+9= (structured practice)

I: Count the blue circles, "*Count:* 1,2,3,4,5. There are 5." I: Write 5 here (bottom right number column) * I: Put all the red circles together. (*Pull down all red circles to the bottom*)* I: Count the red circles, "Counts 1,2. There are 2." I: Write 2 over here (bottom left number column)* I: *Point to the number columns and say*, "(19+6 is 25)" I: *Point to the equation and say*, " 19+6 is 25" and *write = 25 on the equation**	
Additional structured practice examples: 34+24 16+25 46+1 13+7	

<u>Type of Example Instructions:</u>

- Demonstrate 2 equations while children imitate: Have the children sit around the instructor all facing their individual manipulatives board and workbook on the table. There will be one container of red circles and one of blue circles in front of each child. The instructor will go through the step by step instruction allowing children to imitate her actions.
- Structured practice examples: Instructor prompts for each step (i.e., "Now what do you do?"): The instructor will not demonstrate the steps with the students, unless children are having difficulty carrying out the procedure on their own.
- -If at any point the children ask why the instructor is carrying out an action, whether with the circles or written numbers, the instructor will simply say, "Because that is the way I want to show you how to do it."

Appendix E

Worksheet: Procedural Addition Task

Student Name:	School:	
Date:	Grade:	Teacher:

$$\begin{array}{r} 12 \\ +\quad 6 \\ \hline \end{array}$$

$$\begin{array}{r} 17 \\ +\quad 3 \\ \hline \end{array}$$

$$\begin{array}{r} 16 \\ +\quad 9 \\ \hline \end{array}$$

Appendix F

Coding Sheet: Quantitative Representational Insight Task

1) Quantitative Representational Insight Task (QRIT)

1. ENCODING

The child's answers will be recorded by the video camera.

2. READ A DISPLAY

Write down how much the child said each bag's content was worth.

a. 5 b _____

b. 1 r 7 b _____

c. 6 r _____

d. 4 r 5 b _____

3. CONSTRUCT A DISPLAY

Draw what the child constructed. Use "B" for blue chips and "R" for red chips. If there was no third construction, write an X in the box.

8	**15**	**11**	**26**
X			

Appendix G

Coding Sheet: Conceptual Addition Task

2) Addition Task – Conceptual Justification

QUESTION: Did ______ get the right answer or the wrong answer?

Circle what the child answered:

Julie	*YES*	*NO*
Maggie	*YES*	*NO*
Carlos	*YES*	*NO*
Robby	*YES*	*NO*
Matthew	*YES*	*NO*

The child's other answers will be recorded by the video camera.