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Extension of local linear controllers to global piecewise affine controllers for uncertain nonlinear systems

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A two-step controller synthesis method is proposed in this paper for a class of uncertain nonlinear systems described by piecewise affine differential inclusions. In the first step, a robust linear controller is designed for the linear differential inclusion that describes the dynamics of the nonlinear system close to the equilibrium point. In the second step, a stabilizing piecewise affine controller is designed that coincides with the linear controller in a region around the equilibrium point. The proposed method has two objectives: global stability and local performance. It thus enables to use well known techniques in linear control design for local stability and performance while delivering a global piecewise affine controller that is guaranteed to stabilize the nonlinear system. To construct the required theoretical framework, a stability theorem for nonsmooth Lyapunov functions is presented and proved. The new method will be applied to two examples.

1 Introduction

Linear control theory provides a variety of well established tools to guarantee robust stability and performance (Doyle et al. 1990). This is, however, valid only locally if the controller is designed for the linearization of a nonlinear system. In fact, the linear controller may not even stabilize the nonlinear system if the initial condition is far from the linearization point. On the other hand, most of the methods in nonlinear control theory address global asymptotic stability but not necessarily performance. Designing a controller that has both a large region of attraction and a good local performance is therefore one of the most interesting research problems in nonlinear control theory (Murray 1996). Having this problem in mind, a two-step method is proposed in this paper to design a piecewise affine (PWA) controller for uncertain nonlinear systems described by piecewise affine differential inclusions (PWADI). The objective of the proposed method is to design a controller to satisfy a local performance requirement and to globally stabilize the nonlinear system. This is done by extending a linear controller designed for performance to a globally stabilizing PWA controller. One of the main advantages of this method is that it can be employed in many practical problems for which linear controllers currently exist without changing the local performance of the system.

The structure of the proposed method is shown in figure 1. In the first step, a robust linear controller is designed for the linear differential inclusion (LDI) that approximates the local behaviour of the nonlinear system in a neighbourhood of the desired operating point. Then, a PWA controller that coincides with the linear controller in a region around the equilibrium point and globally stabilizes the nonlinear system is designed in the second step. Since the design approach is based on finding a piecewise quadratic Lyapunov function, it is only approximate in the sense that there is no guarantee that a Lyapunov function can be found. If one is found, the global stability is achieved. Otherwise, the method is inconclusive. In spite of their approximate nature, Lyapunov-based methods for PWA controller design appear to work well in practice and are widely used in the literature (Hassibi & Boyd 1998, Rantzer & Johansson 2000, Feng 2002, Johansson 2003, Rodrigues & How 2003*a*,*b*).

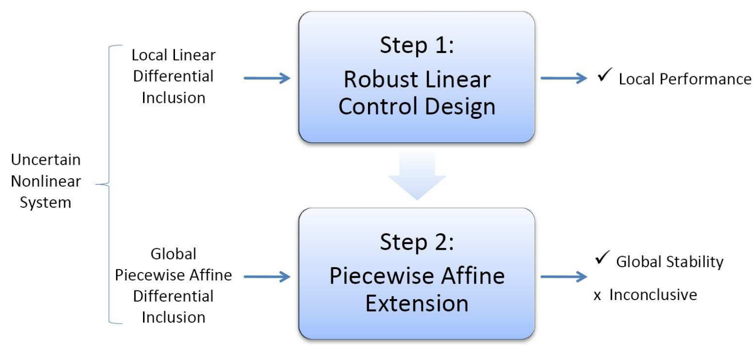

Figure 1. Structure of the proposed PWA controller design method

The literature on stability analysis and feedback control of continuous time PWA systems has concentrated on Lyapunov-based methods (Hassibi & Boyd 1998, Johansson 2003). Analysis of continuous time PWA systems by searching for a Lyapunov function to prove stability, are formulated in Hassibi & Boyd (1998) as convex programs involving Linear Matrix Inequalities (LMIs). These mathematical programs can then be solved efficiently using polynomial-time algorithms. It is shown in Rantzer & Johansson (2000) that given a PWA controller, an upper bound and a lower bound to a piecewise quadratic cost function can be obtained by semidefinite programming. Piecewise quadratic Lyapunov functions are demonstrated in the same reference to be a much richer class of Lyapunov function candidates than globally quadratic Lyapunov functions. Stability analysis of smooth nonlinear systems using PWADIs or PWA approximations is also presented in Johansson (2003). Based on the approach in Rantzer & Johansson (2000), it is shown in Feng (2002) that PWA controller design for an uncertain PWA system to establish global stability with H_∞ performance of the resulting closed-loop system can be formulated as a set of linear matrix inequalities (LMI). However, it is required that all local subsystems be stable which is a conservative assumption. Rodrigues & How (2003*b*) extend the stability analysis reported in Hassibi (2000) to obtain a synthesis method for PWA state and output feedback controllers. This reference also shows that other desired features can be included in the design, such as continuity of the control input, boundedness of the control gains and avoidance of attractive sliding modes. This method was later extended in Rodrigues & How (2003*a*) to stabilize nonlinear systems that can be approximated by PWADIs.

The main result of this paper is proved in Theorem 5.2. The contribution of this result is to provide the theoretical framework for extending a local linear controller to a global PWA controller based on piecewise quadratic Lyapunov functions. In previous research, Rodrigues & How (2003*b*) have also used piecewise quadratic Lyapunov functions to synthesize PWA controllers. However, the method of Rodrigues & How does not enable one to extend a local linear controller to a global PWA controller. Futhermore, it is assumed in Rodrigues & How (2003*b*) that there is one equilibrium point for the dynamic equations of each region. The equilibrium points of all regions are then selected *a priori* by solving an optimization problem. It is also required that each of the equilibrium points be the extrema of the corresponding sector of any candidate Lyapunov function. By contrast, Theorem 5.2 now shows that it is in fact not necessary to compute those equilibrium points. This has the important advantage of relieving the designer from this tedious and non-intuitive task.

Note that a piecewise quadratic function is not differentiable everywhere and therefore it is a nonsmooth function. Despite this fact, none of the previously existing approaches to PWA controller design have developed a nonsmooth theory nor have they considered using well-developed nonsmooth analysis theory in the literature e.g. Clarke et al. (1998). By contrast, in this paper, we depart from previous approaches to PWA controller design by providing a Lyapunov theorem for nonsmooth Lyapunov functions. The theorem has the advantage of including the standard Lyapunov stability theorem in Khalil (2002) as its special case for $\mathcal{C}^1$ Lyapunov functions. The proposed PWA controller in this paper has the additional advantage of coinciding locally with a robust linear controller designed using linear control methods. It combines local performance with global stability. One important application of the proposed method can thus be to extend the region of convergence of existing linear controllers for nonlinear systems.

The paper is organized as follows. An illustrative example is employed in section 2 to clarify the need for the proposed method. In section 3, a Lyapunov stability theorem for general nonsmooth Lyapunov functions is proved.

Continuous PWADIs are defined and sufficient conditions for monotonicity of piecewise smooth Lyapunov functions for continuous PWADIs are proved in section 4. Section 5 then explains the proposed method which consists of robust linear controller design and its PWA extension. Finally, PWA controllers are designed for two examples in section 6 and conclusions are drawn in section 7.

2 Illustrative example

In this section, the following nonlinear system is used to illustrate the design procedure:

$$\dot{x} = 0.5(1 - x^2) + u \tag{1}$$

The open loop system has two equilibrium points (figure 2), one at $x = -1$ (unstable) and the other one at $x = 1$ (stable). The goal is to design a controller so that for any $x(0) \in \mathcal{X} = \left[-4, 4\right]$, the trajectory of the system asymptotically converges to $x^\star = 1$. It is also required that for any $x(0) \in \left(0, 2\right)$, the following cost function

$$J = \int_0^\infty (Q(x-1)^2 + Ru^2)dt \tag{2}$$

be minimized where $Q = 2$ and $R = 1$.

To achieve this goal, continuous PWA functions $\sigma_1(x)$ and $\sigma_2(x)$ (figure 3) are first defined so that

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x) + u, \sigma_2(x) + u\} \tag{3}$$

where **conv** stands for the closed convex hull (Royden 1988) of a set and $\sigma_1(x)$ and $\sigma_2(x)$ are affine in x inside each of the following regions:

$$\mathcal{R}_1 = \left(-4, -2\right), \mathcal{R}_2 = \left(-2, 0\right), \mathcal{R}_3 = \left(0, 2\right), \mathcal{R}_4 = \left(2, 4\right) \tag{4}$$

In $\mathcal{R}_3$ (where $x^\star$ is located), the dynamics of the system are described by the following LDI,

$$\dot{x} \in \mathbf{conv}\{-1.6(x-1) + u, -0.4(x-1) + u\} \tag{5}$$

Defining $z = x - 1$, we have

$$\dot{z} \in \mathbf{conv}\{-1.6z + u, -0.4z + u\} \tag{6}$$

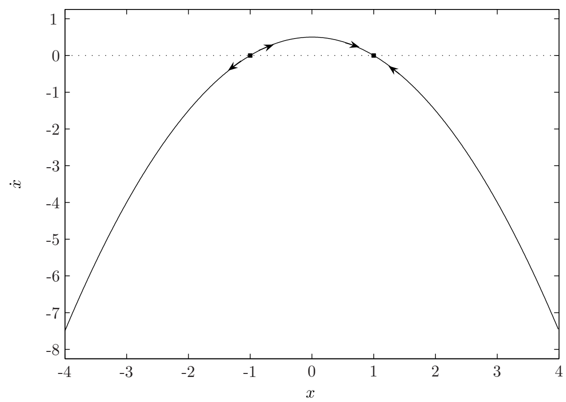

Figure 2. The trajectory of the open loop system.

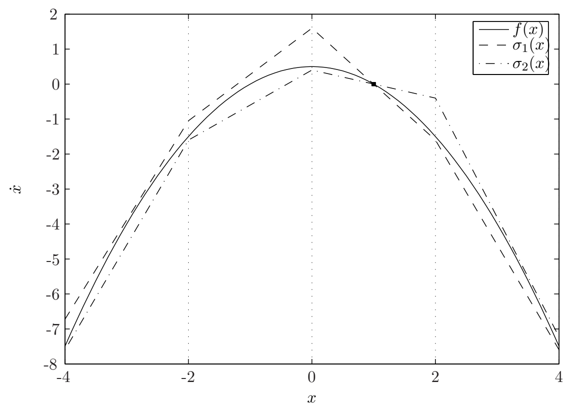

Figure 3. PWA differential inclusion

An LQR controller can be designed for (6) using the design method for robust linear controllers described in subsection 5.1. The resulting controller for $\mathcal{R}_3$ is then described by

$$u = -1.07x + 1.07 \tag{7}$$

Figure 4 shows the trajectory of the nonlinear system in feedback connection with the linear controller. It can be clearly seen that the system still has two equilibrium points. Therefore, although the closed-loop system locally satisfies the required performance measure, it is not globally stable. In the following sections, a method for extending the designed LQR controller to a PWA controller will be presented. It will be shown in section 6 that the resulting PWA controller has the same local performance and is globally stabling.

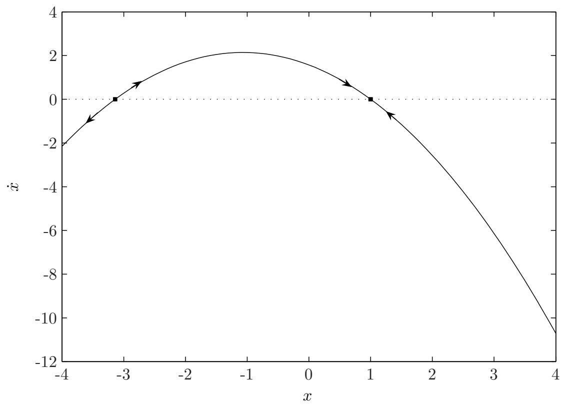

Figure 4. The trajectory of the system with the linear controller.

3 Stability analysis based on nonsmooth Lyapunov functions

In this section, a Lyapunov stability theorem is proved for nonsmooth Lyapunov functions. This theorem forms the theoretical framework for using piecewise smooth Lyapunov functions in stability analysis of nonlinear systems. There are other nonsmooth versions of Lyapunov theorems in the literature e.g. Rouche et al. (1977), Sontag (1983), Clarke et al. (1998), Ceragioli (1999). However, certain conditions in these theorems (such as, for example, the conditions on the Dini derivative or the proximal subdifferential of the Lyapunov function) are difficult to check in the case described in this paper. The objective of Theorem 3.1 will thus be to extend the standard Lyapunov stability theorem in Khalil (2002) to nonsmooth Lyapunov functions and to fit the framework needed in this paper. To the best of knowledge, this theorem in this exact form does not appear in the literature.

Consider the following autonomous nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, the initial state is $x(0) = x_0$ and $f : \mathcal{X} \to \mathbb{R}^n$ is bounded in $\mathcal{X} \subset \mathbb{R}^n$. The following theorem describes sufficient conditions for stability of system (8) in the sense of Lyapunov based on a continuous Lyapunov function that is not necessarily differentiable everywhere. Because of its importance, the theorem is proved here. The proof combines the proof of the standard Lyapunov theorem in Khalil (2002) for stability and the proof of the nonsmooth Lyapunov theorem in Clarke et al. (1998) for asymptotic stability.

Theorem 3.1 *For nonlinear system (8), if $f(x^\star) = 0$ and there exists a* *continuous function $V(x)$ such that*

$$V(x^\star) = 0 \tag{9}$$

$$V(x) > 0 \text{ for all } x \neq x^\star \text{ in } \mathcal{X} \tag{10}$$

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) \tag{11}$$

then $x = x^\star$ is a stable equilibrium point. Moreover if there exists a continuous function $W(x)$ such that

$$W(x^\star) = 0 \tag{12}$$

$$W(x) > 0 \text{ for all } x \neq x^\star \text{ in } \mathcal{X} \tag{13}$$

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) + \int_{t_1}^{t_2} W(x(\tau))d\tau \tag{14}$$

and

$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty \tag{15}$$

then all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$.

Proof Since $f(x^\star) = 0$ then $x = x^\star$ is an equilibrium point for system (8). For stability, we want to prove

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \;\text{ s.t. }\; \|x_0 - x^\star\| < \delta \Rightarrow \|x(t) - x^\star\| < \varepsilon, \forall t \geq 0 \tag{16}$$

Following Khalil (2002), we choose $r \in (0, \varepsilon]$ for a given $\varepsilon > 0$ such that

$$\mathcal{B}_r = \{x : \|x - x^\star\| \leq r\} \subset \mathcal{X} \tag{17}$$

Let $\alpha = \min_{\|x - x^\star\| = r} V(x)$. Then $\alpha > 0$ by (10). Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{x \in \mathcal{B}_r | V(x) \leq \beta\}$. Then Ω_β is in the interior of $\mathcal{B}_r$ (Figure 5). If $x_0 \in \Omega_\beta$ then (11) implies that $x(t) \in \Omega_\beta$ for all $t \geq 0$. As $V(x)$ is continuous and $V(x^\star) = 0$, there is a $\delta > 0$ such that $\|x - x^\star\| \leq \delta \Rightarrow V(x) < \beta$. Then

$$\mathcal{B}_\delta = \{x : \|x - x^\star\| \leq \delta\} \subset \Omega_\beta \subset \mathcal{B}_r \tag{18}$$

and $x_0 \in \mathcal{B}_\delta \Rightarrow x_0 \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in \mathcal{B}_r \Rightarrow x(t) \in \mathcal{B}_\varepsilon$. Therefore

$$\|x_0 - x^\star\| < \delta \Rightarrow \|x(t) - x^\star\| < r \leq \varepsilon, \forall t \geq 0 \tag{19}$$

This implies that $x = x^\star$ is a stable equilibrium point.

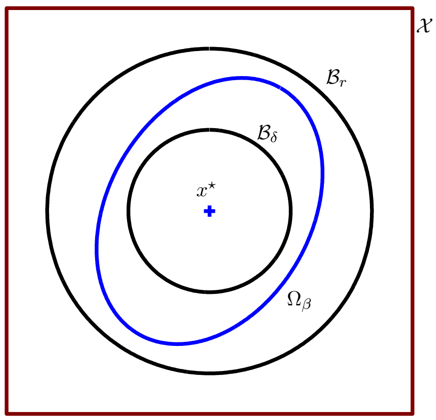

Figure 5. Geometric illustration of sets in the proof of Theorem 3.1.

To prove asymptotic stability, following Clarke et al. (1998), we show that $x(t)$ converges to $x^\star$ as $t \to \infty$. It follows from $x(0) = x_0$ and (14) that

$$V(x(t)) + \int_0^t W(x(t))d\tau \leq V(x_0) \tag{20}$$

Then (10), (13) and (20) imply that $V(x(t))$ and $\int_0^t W(x(t))d\tau$ are bounded. Because $V(x(t))$ is bounded, it follows from (15) that $\|x(t)\|$ is bounded. Since $f(x)$ is bounded in $\mathcal{X}$, $\dot{x}(t)$ is bounded and $x(t)$ satisfies a global Lipschitz condition on $t \in [0, +\infty)$ with constant L.

Assume that $x(t)$ fails to converge to $x^\star$. Then for some $\varepsilon > 0$ there exists a sequence of points t_i tending to infinity such that

$$\|x(t_i) - x^\star\| \geq \varepsilon, \ i = 1, 2, \ldots \tag{21}$$

Without loss of generality the sequence t_i can always be chosen such that

$$|t_{i+1} - t_i| > \frac{\varepsilon}{2L} \tag{22}$$

Since $\|x(t)\|$ is bounded, there exists a $\lambda > 0$ such that $\|x(t) - x^\star\| \leq \lambda$. Consider

$$\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]} = \{x : \ \frac{\varepsilon}{2} \leq \|x - x^\star\| \leq \lambda\} \tag{23}$$

$\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$ is not empty since $\lambda \geq \varepsilon > \frac{\varepsilon}{2}$. Let $\eta > 0$ be such that

$$x \in \mathcal{A}_{[\frac{\varepsilon}{2},\lambda]} \Rightarrow W(x) \geq \eta \tag{24}$$

Such η exists because of (13) and the fact that $\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$ is not empty. Consider t such that $|t-t_i| < \frac{\varepsilon}{2L}$. Since $x(t)$ is globally Lipschitz continuous with constant L, we have

$$\|x(t) - x(t_i)\| < \frac{\varepsilon}{2} \tag{25}$$

Inequalities (21), (25) and the following triangle inequality

$$\|x(t) - x^\star\| \geq \|x(t_i) - x^\star\| - \|x(t) - x(t_i)\| \tag{26}$$

imply $\|x(t) - x^\star\| > \frac{\varepsilon}{2}$ and consequently $x(t) \in \mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$. Therefore, from (24)

$$\int_{t_i-\frac{\varepsilon}{2L}}^{t_i+\frac{\varepsilon}{2L}} W(x(\tau))d\tau \geq \frac{\eta\varepsilon}{L} \tag{27}$$

and then using (22) and (13)

$$\int_{t_{i-1}}^{t_{i+1}} W(x(\tau))d\tau > \frac{\eta\varepsilon}{L} \tag{28}$$

This would imply that $\int_0^t W(x(t))d\tau$ diverges as $t \to \infty$, which is a contradiction with (20) and the conclusion that $\int_0^t W(x(t))d\tau$ is bounded. This proves that $x(t)$ converges to $x^\star$ as $t \to \infty$. □

Theorem 3.1, together with the monotonicity conditions in the following section, will be employed to construct a PWA controller synthesis method in section 5.

4 Piecewise affine differential inclusions and monotonicity conditions

In this section, sufficient conditions for monotonicity of nonsmooth functions $V(x)$ will be provided for the following differential inclusion

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x), \ldots, \sigma_{\mathcal{K}}(x)\} \tag{29}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $\sigma_\kappa(x)$ for $\kappa = 1, \ldots, \mathcal{K}$ are PWA functions defined in $\mathcal{X}$ as

$$\sigma_\kappa(x) = A_{i\kappa} x + a_{i\kappa}, x \in \overline{\mathcal{R}}_i \tag{30}$$

where $A_{i\kappa} \in \mathbb{R}^{n\times n}$, $a_{i\kappa} \in \mathbb{R}^n$ and $\mathcal{X}$ is partitioned into polytopic cells $\mathcal{R}_i, i = 1, \ldots, M$ such that

$$\begin{aligned} &\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}, \\ &\mathcal{R}_i \cap \mathcal{R}_j = \emptyset,\ i \neq j \end{aligned} \tag{31}$$

and $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. It is assumed that $\sigma_\kappa(x)$ for $\kappa = 1, \ldots, \mathcal{K}$ are continuous functions. Hence for neighbour regions $\mathcal{R}_i$ and $\mathcal{R}_j$

$$A_{i\kappa} x + a_{i\kappa} = A_{j\kappa} x + a_{j\kappa},\ \forall x \in \overline{\mathcal{R}}_i \cap \overline{\mathcal{R}}_j,\ \kappa = 1, \ldots, \mathcal{K} \tag{32}$$

In the following, the concept of generalized gradient is introduced. Note that the monotonicity condition on $V(x)$ can also be described by the Dini derivative (such as it is done in Rouche et al. (1977)). However, in this paper, the theorem of Rademacher (see page 93 in Clarke et al. 1998) is used to define the generalized gradient. This definition enables one to formulate the monotonicity condition for nonsmooth functions as Proposition 4.2. Then, the monotonicity condition for piecewise quadratic functions and PWA differential inclusions can be easily proved (Proposition 4.3).

Definition 4.1 (Clarke et al. 1998) For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the generalized gradient is defined as

$$\partial_C V(x) = \mathbf{conv}\{\lim_{i\to+\infty} \nabla V(x_i) : x_i \to x, x_i \notin N\} \tag{33}$$

where N is the set of measure zero where the gradient of V does not exist.

Proposition 4.2 *(Section 2.4 of Ceragioli 1999) Let $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \emptyset$ be continuous and let $V : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. V is nonincreasing along all solutions of*

$$\dot{x} \in \mathcal{F}(x) \tag{34}$$

if and only if

$$\forall x \in \mathbb{R}^n, \forall f \in \mathcal{F}(x), \max\{p.f : p \in \partial_C V(x)\} \leq 0 \tag{35}$$

where '.' stands for the inner product of two vectors.

The following proposition is a special realization of Proposition 4.2 for the case of PWA differential inclusions and piecewise differentiable functions.

PROPOSITION 4.3 *(Monotonicity of piecewise differentiable functions) Consider a Lipschitz continuous function $V : \mathcal{X} \rightarrow \mathbb{R}$ where*

$$V(x) = V_i(x), x \in \overline{\mathcal{R}}_i \tag{36}$$

and $V_i(x)$ is a $\mathcal{C}^1$ function. $V(x)$ is nonincreasing along all solutions of the differential inclusion (29) if for all $x \in \mathcal{R}_i$, $i = 1, \dots, M$ and $\kappa = 1, \dots, \mathcal{K}$

$$\nabla V_i(x).(A_{i\kappa}x + a_{i\kappa}) \leq 0 \tag{37}$$

Proof Let $x(t)$ be a solution of (29). The proof is divided into two parts. The first part considers the case when $x(t)$ is inside one of the regions and the second part addresses the case when $x(t)$ is at the boundary of two or more regions.

- If $x(t) \in \mathcal{R}_i$ for any $i \in \{1, \dots, M\}$,

  $$\partial_C V(x) = \{\nabla V_i(x)\} \tag{38}$$

  and $\dot{x} \in \mathcal{F}(x)$ where

  $$\mathcal{F}(x) = \mathbf{conv}\{A_{i1}x + a_{i1}, \dots, A_{i\mathcal{K}}x + a_{i\mathcal{K}}\} \tag{39}$$

  It follows from (37) that for any f in $\mathcal{F}(x)$,

  $$\nabla V_i(x).f \leq 0 \tag{40}$$

  This implies that (35) is satisfied in $\bigcup_{i=1}^{M} \mathcal{R}_i$.

- If $x(t) \in \bigcap_{i \in \mathcal{I}(x)} \overline{\mathcal{R}}_i$ where $\mathcal{I}(x) = \{i | x \in \overline{\mathcal{R}}_i\}$,

  $$\partial_C V(x) = \mathbf{conv}\{\nabla V_i(x) : i \in \mathcal{I}(x)\} \tag{41}$$

  From (37) and (32), it follows that for all l and j in $\mathcal{I}(x)$ and $\kappa = 1, \dots, \mathcal{K}$

  $$\nabla V_l(x).(A_{j\kappa}x + a_{j\kappa}) = \nabla V_l(x).(A_{l\kappa}x + a_{l\kappa}) \leq 0 \tag{42}$$

and therefore for any f in $\mathcal{F}(x) = \mathbf{conv}\{\sigma_1(x), \ldots, \sigma_{\mathcal{K}}(x)\}$

$$\nabla V_i(x).f \leq 0, \forall i \in \mathcal{I}(x) \tag{43}$$

Expressions (43) and (41) imply that

$$p.f \leq 0, \forall p \in \partial_C V(x), \forall f \in \mathcal{F}(x) \tag{44}$$

Hence, (35) is satisfied in $\bigcap_{i \in \mathcal{I}(x)} \overline{\mathcal{R}}_i$

In conclusion, (35) is satisfied in $\bigcup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}$ and by Proposition 4.2, $V(x)$ is nonincreasing along the solutions of (29) in $\mathcal{X}$. □

The importance of Proposition 4.3 lies in the fact that to check the monotonicity of $V(x)$, it is enough to check each quadratic piece $V_i(x)$ with the vector fields of all the subsystems in the same region. There is therefore no need to check it with the vector fields of the neighbour regions. Proposition 4.3 will be used in the proof of Theorem 5.2.

5 Extension of a Linear Controller to a PWA Controller

This section proposes a method to extend a local linear controller to a global PWA controller. The method consists of two steps. In the first step, a robust linear controller will be designed for the nonlinear system. In this step, the designer can benefit from well established methods for designing robust linear controllers to make the nonlinear system locally stable and to satisfy a performance requirement in a neighbourhood of the desired equilibrium point. In the second step, the objective is to design a PWA controller that coincides with the linear controller in the neighbourhood of the equilibrium point and guarantees global stability of the nonlinear closed-loop system.

Consider the following nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{45}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x,u), \ldots, \sigma_{\mathcal{K}}(x,u)\} \tag{46}$$

where $\sigma_\kappa(x,u)$ is defined as

$$\sigma_\kappa(x,u) = A_{i\kappa}x + a_{i\kappa} + B_{i\kappa}u, x \in \overline{\mathcal{R}}_i, \tag{47}$$

with $A_{i\kappa} \in \mathbb{R}^{n\times n}$, $a_{i\kappa} \in \mathbb{R}^n$, $B_{i\kappa} \in \mathbb{R}^{n\times m}$ for $i = 1,\ldots,M$. Polytopic regions $\mathcal{R}_i$ are constructed as the intersection of a finite number of half spaces

$$\mathcal{R}_i = \{x : E_i x + e_i \succ 0\}, \text{ for } i = 1,\ldots,M \tag{48}$$

where $E_i \in \mathbb{R}^{p_i\times n}$, $e_i \in \mathbb{R}^{p_i}$ and $\succ$ represents an elementwise inequality.

The objective is to stabilize system (45) to $x = x^\star$ while satisfying a performance requirement for x close to $x^\star$. The two steps of the proposed method will be presented in the following subsections.

5.1 *Step 1: Robust linear controller design*

The first step is to design a robust linear controller for the LDI describing local behaviour of the nonlinear system. Consider a region $\mathcal{R}_{i^\star}$ such that

$$x^\star \in \overline{\mathcal{R}}_{i^\star}$$

The dynamics of system (45) in this region can be described by the following LDI.

$$\dot{x} \in \mathbf{conv}\{A_{i^\star\kappa}x + a_{i^\star\kappa} + B_{i^\star\kappa}u : \ \kappa = 1,\ldots,\mathcal{K}\} \tag{49}$$

Changing variables to $z = x - x^\star$ and assuming $u = K_{i^\star}x + k_{i^\star}$ yields

$$\dot{z} \in \mathbf{conv}\{(A_{i^\star\kappa} + B_{i^\star\kappa}K_{i^\star})(z + x^\star) + a_{i^\star\kappa} + B_{i^\star\kappa}k_{i^\star} : \ \kappa = 1,\ldots,\mathcal{K}\} \tag{50}$$

To make $z = 0$ an equilibrium of the system, the following condition must be satisfied.

$$(A_{i^\star\kappa} + B_{i^\star\kappa}K_{i^\star\kappa})x^\star + a_{i^\star\kappa} + B_{i\kappa}k_{i^\star\kappa} = 0, \ \kappa = 1,\ldots,\mathcal{K} \tag{51}$$

The closed-loop dynamics of the system can then be written as

$$\dot{z} \in \mathbf{conv}\{(A_{i^\star\kappa} + B_{i^\star\kappa}K_{i^\star\kappa})z : \ \kappa = 1,\ldots,\mathcal{K}\} \tag{52}$$

The matrix gain $K_{i^\star}$ can be designed using robust linear control methodologies to satisfy desired design objectives. The affine term of the controller $k_{i^\star}$ is then computed as the solution of equation (51). The choice of the required performance measure depends on the application. In this work, a robust LQR is designed for the LDI (49). The following result is taken from Jadbabaie (1997).

Theorem 5.1 *(Jadbabaie 1997) Consider the cost function*

$$J = \int_0^\infty (z^T Q z + u^T R u) dt \tag{53}$$

where $Q \geq 0$ *and* $R > 0$ *for the following LDI*

$$\dot{z} \in \boldsymbol{conv}\{A_\kappa z + B_\kappa u : \kappa = 1, \ldots, \mathcal{K}\} \tag{54}$$

where $z \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$*. If there exist* S *and* Y *so that*

$$S > 0 \tag{55}$$

$$\begin{bmatrix} SA_\kappa^T + A_\kappa S + Y^T B_\kappa^T + B_\kappa Y & SQ^{1/2} & Y^T R^{1/2} \\ Q^{1/2} S & -I_n & 0 \\ R^{1/2} Y & 0 & -I_m \end{bmatrix} < 0 \tag{56}$$

for $\kappa = 1, \ldots, \mathcal{K}$ *, then for* $u = Kz$ *where* $K = YS^{-1}$*, we have*

$$J < z(0)^T S^{-1} z(0) \tag{57}$$

□

To avoid the dependency of the upper bound of the cost function on initial conditions of the system, it is proposed in Jadbabaie (1997) to assume that the initial condition is a random vector with zero mean and identity covariance, i.e.,

$$\begin{aligned} \mathbb{E}\{z(0)\} &= 0 \\ \mathbb{E}\{z(0)z(0)^T\} &= I \end{aligned} \tag{58}$$

It is shown in Jadbabaie (1997) that $\mathbf{tr}(S^{-1})$ ($\mathbf{tr}$ stands for trace of a matrix) is an upper bound on $\mathbb{E}\{J\}$. Therefore it is proposed in the same reference to solve the following optimization problem to minimize the upper bound on the cost function.

$$\begin{aligned} &\max \mathbf{tr}(S) \\ &\text{subject to } (55) \text{ and } (56) \end{aligned} \tag{59}$$

This optimization problem can be solved using SeDuMi Strum (2001) and Yalmip Löfberg (2004) to compute the controller gain $K_{i^\star}$. The affine term $k_{i^\star}$

can then be computed by solving (51). The controller in region $\mathcal{R}_{i^\star}$ can then be written as

$$u = \bar{K}_{i^\star}\bar{x}, \text{ where } \bar{K}_{i^\star} = \begin{bmatrix} K_{i^\star} & k_{i^\star} \end{bmatrix} \text{ and } \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{60}$$

The next step is to find a PWA controller that coincides with the linear controller (60) in $\mathcal{R}_{i^\star}$ and guarantees the stability of the closed-loop system in $\mathcal{X}$.

5.2 *Step 2: Piecewise-affine state feedback design*

The second step is to extend the robust linear controller to a PWA state feedback controller for the following differential inclusion that describes the global behaviour of the nonlinear system.

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x,u), \ldots, \sigma_{\mathcal{K}}(x,u)\} \tag{61}$$

where the $\sigma_\kappa(x,u)$ for $\kappa = 1, \ldots, \mathcal{K}$ are defined in (47). Considering the regions defined in (48), each region $\mathcal{R}_i$ can be outer approximated by a (possibly degenerate) quadratic curve ε_i

$$\mathcal{R}_i \subseteq \varepsilon_i = \{x : \bar{x}^T \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \bar{x} > 0\} \tag{62}$$

where $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ is a matrix with nonnegative entries and

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \\ 0 & 1 \end{bmatrix} \tag{63}$$

A parametric description of the boundaries between two regions $\mathcal{R}_i$ and $\mathcal{R}_j$ where $\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}}_j \neq \emptyset$ can also be obtained as (see Hassibi & Boyd (1998) and Rodrigues & How (2003*b*) for more details)

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}}_j \subseteq \{x : x = F_{ij}s + f_{ij},\ s \in \mathbb{R}^{n-1}\} \tag{64}$$

To stabilize the equilibrium point $x^\star$ of the nonlinear system (45) a PWA control input of the following form is considered

$$u = K_i x + k_i = \bar{K}_i \bar{x}, \text{ for } x \in \mathcal{R}_i \tag{65}$$

where

$$\bar{K}_i = \left[K_i \; k_i \right] \tag{66}$$

Consider the piecewise quadratic candidate Lyapunov function continuous at the boundaries and defined in $\mathcal{X}$ by the expression

$$V(x) = \bar{x}^T \bar{P}_i \bar{x}, \text{ for } x \in \overline{\mathcal{R}}_i \tag{67}$$

where $\bar{P}_i = \bar{P}_i^T \in \mathbb{R}^{(n+1)\times(n+1)}$ and

$$\bar{P}_i = \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \tag{68}$$

with $P_i \in \mathbb{R}^{n\times n}$, $q_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}$. To simplify the notation, define

$$\begin{aligned} &\bar{A}_{i\kappa} = \begin{bmatrix} A_{i\kappa} & a_{i\kappa} \\ 0 & 1 \end{bmatrix}, \bar{B}_{i\kappa} = \begin{bmatrix} B_{i\kappa} \\ 0 \end{bmatrix}, \bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix}, \\ &\bar{I} = \begin{bmatrix} I & -x^\star \\ -x^{\star\mathrm{T}} & x^{\star\mathrm{T}} x^\star \end{bmatrix}, \bar{x}^\star = \begin{bmatrix} x^\star \\ 1 \end{bmatrix} \end{aligned} \tag{69}$$

The following theorem describes sufficient conditions for the existence of a continuous piecewise quadratic Lyapunov function of the form (67) and a PWA controller of the form (65) that coincides with the robust linear controller in the region where $x^\star$ lies and guarantees global stability.

THEOREM 5.2 *Let there exist matrices $\bar{P}_i = \bar{P}_i^T$ defined in (68), $\bar{K}_i$ defined in (66), Z_i, $\bar{Z}_i$, $\Lambda_{i\kappa}$ and $\bar{\Lambda}_{i\kappa}$ that verify the following conditions for all $i = 1, \ldots, M$, $\kappa = 1, \ldots, \mathcal{K}$ and for a given decay rate $\alpha > 0$, desired equilibrium point $x^\star$, linear controller gain $\bar{K}_{i^\star}$ defined in (60) and $\epsilon > 0$*

- *Conditions on the PWA controller:*

$$\bar{K}_i = \bar{K}_{i^\star}, \text{ if } x^\star \in \overline{\mathcal{R}}_i \tag{70}$$

$$(\bar{A}_{i\kappa} + \bar{B}_\kappa \bar{K}_i)\bar{x}^\star = 0, \text{ if } x^\star \in \overline{\mathcal{R}}_i \tag{71}$$

$$(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{F}_{ij} = (\bar{A}_{j\kappa} + \bar{B}_{j\kappa}\bar{K}_j)\bar{F}_{ij}, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{72}$$

- *Continuity of the Lyapunov function:*

$$\bar{F}_{ij}^T(\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{73}$$

- *Positive definiteness of the Lyapunov function:*

$$\bar{P}_i\bar{x}^\star = 0, \text{ if } x^\star \in \overline{\mathcal{R}}_i \tag{74}$$

$$P_i > \epsilon I, \text{ if } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0 \tag{75}$$

$$\begin{cases} Z_i \in \mathbb{R}^{n\times n}, Z_i \succeq 0 \\ P_i - E_i^T Z_i E_i > \epsilon I \end{cases}, \text{ if } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0 \tag{76}$$

$$\begin{cases} \bar{Z}_i \in \mathbb{R}^{(n+1)\times(n+1)}, \bar{Z}_i \succeq 0 \\ \bar{P}_i - \bar{E}_i^T Z_i \bar{E}_i > \epsilon \bar{I} \end{cases}, \text{ if } x^\star \notin \overline{\mathcal{R}}_i \tag{77}$$

- *Monotonicity of the Lyapunov function:*

$$\begin{aligned} &\text{for } i \text{ such that } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0, \\ &P_i(A_{i\kappa} + B_{i\kappa}K_i) + (A_{i\kappa} + B_{i\kappa}K_i)^T P_i < -\alpha P_i \end{aligned} \tag{78}$$

$$\begin{aligned} &\text{for } i \text{ such that } x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0, \\ &\begin{cases} \Lambda_{i\kappa} \in \mathbb{R}^{n\times n}, \ \Lambda_{i\kappa} \succeq 0 \\ P_i(A_{i\kappa} + B_{i\kappa}K_i) + (A_{i\kappa} + B_{i\kappa}K_i)^T P_i + E_i^T \Lambda_{i\kappa} E_i < -\alpha P_i \end{cases} \end{aligned} \tag{79}$$

$$\begin{aligned} &\text{for } i \text{ such that } x^\star \notin \overline{\mathcal{R}}_i, \\ &\begin{cases} \bar{\Lambda}_{i\kappa} \in \mathbb{R}^{(n+1)\times(n+1)}, \ \bar{\Lambda}_{i\kappa} \succeq 0 \\ \bar{P}_i(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i) + (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)^T \bar{P}_i + \bar{E}_i^T \bar{\Lambda}_{i\kappa} \bar{E}_i < -\alpha \bar{P}_i \end{cases} \end{aligned} \tag{80}$$

Then for the following nonlinear system

$$\dot{x} = f(x) + g(x)u \text{ with } u = \bar{K}_i\bar{x} \text{ for } x \in \mathcal{R}_i, \tag{81}$$

all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$.

Proof The conditions on the PWA controller guarantee that it is an extension of the linear controller. The proof starts by analyzing each of these conditions in detail:

(i) Condition (70) makes the PWA controller coincide with the linear controller $u = \bar{K}_{i^\star}\bar{x}$ in the region(s) where $x^\star$ is located.

(ii) Condition (71) implies that $\sigma_\kappa(x^\star, \bar{K}_{i^\star}\bar{x}^\star) = 0$ for $\kappa = 1, \ldots, \mathcal{K}$. It follows from this and (46) that $\dot{x} = 0$ at $x = x^\star$ for the nonlinear system (81). Therefore $x^\star$ is an equilibrium point for (81) with $u = \bar{K}_i\bar{x}$ for $x \in \mathcal{R}_i$.

(iii) If $\overline{\mathcal{R}}_i \cap \overline{\mathcal{R}}_j \neq \emptyset$, using (64) for $x \in \overline{\mathcal{R}}_i \cap \overline{\mathcal{R}}_j$, we can write $\bar{x} = \bar{F}_{ij}\bar{s}$ where

$$\bar{s} = \begin{bmatrix} s \\ 1 \end{bmatrix} \tag{82}$$

Condition (72) then leads to $(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{F}_{ij}\bar{s} = (\bar{A}_{j\kappa} + \bar{B}_{j\kappa}\bar{K}_j)\bar{F}_{ij}\bar{s}$ for all $s \in \mathbb{R}^{n-1}$, which means that the PWA functions $\sigma_\kappa(x, \bar{K}_i\bar{x})$ for $\kappa = 1, \ldots, \mathcal{K}$ are continuous across the boundaries of the regions. Therefore, Proposition 4.3 can be used in the rest of the proof.

The main idea of the rest of the proof is to show that all conditions of Theorem 3.1 are satisfied for the nonlinear system (81) with $u = \bar{K}_i\bar{x}$ for $x \in \mathcal{R}_i$.

- *Conditions on the Lyapunov function:*
  (i) *Continuity:* Similarly to (72), continuity of $V(x)$ is guaranteed by (73).
  (ii) *Positive definiteness:* For $V(x)$ defined in (67) and i such that $x^\star \in \overline{\mathcal{R}}_i$, condition (74) implies that

$$V(x) = (x - x^\star)^{\mathrm{T}} P_i (x - x^\star) \text{ for } x \in \overline{\mathcal{R}}_i \text{ where } x^\star \in \overline{\mathcal{R}}_i \tag{83}$$

and therefore $V(x^\star) = 0$.

To prove that $V(x)$ is positive definite, we will show that

$$V(x) > \epsilon (x - x^\star)^{\mathrm{T}} (x - x^\star) \text{ for } x \neq x^\star \tag{84}$$

Consider $\mathcal{R}_i$ such that $x^\star \in \overline{\mathcal{R}}_i$ and $E_i x^\star + e_i \neq 0$. Inequality (84) is then implied by condition (75) and equation (83). Condition (75) can be relaxed for regions $\mathcal{R}_i$ where $x^\star \in \overline{\mathcal{R}}_i$ and $E_i x^\star + e_i = 0$. For these regions, it follows from condition (76) that

$$(x - x^\star)^{\mathrm{T}} P_i (x - x^\star) - (x - x^\star)^{\mathrm{T}} E_i^{\mathrm{T}} Z_i E_i (x - x^\star) > \epsilon (x - x^\star)^{\mathrm{T}} (x - x^\star) \tag{85}$$

Inequality (85) and $E_i x^\star + e_i = 0$ yields

$$(x - x^\star)^{\mathrm{T}} P_i (x - x^\star) - (E_i x + e_i)^{\mathrm{T}} Z_i (E_i x + e_i) > \epsilon (x - x^\star)^{\mathrm{T}} (x - x^\star) \tag{86}$$

However, we know that $E_i x + e_i > 0$ for $x \in \mathcal{R}_i$. Since all the entries of Z_i are nonnegative, we have $V(x) > \epsilon (x - x^\star)^{\mathrm{T}} (x - x^\star)$ for $x \neq x^\star$ in $\mathcal{R}_i$. Finally, for regions where $x^\star \notin \mathcal{R}_i$, it can be shown in a similar way that condition (77) implies that $V(x) > \epsilon (x - x^\star)^{\mathrm{T}} (x - x^\star)$ for $x \in \mathcal{R}_i$.

(iii) *Monotonicity:* Similarly, it follows from (78)-(80) that for $i = 1, \dots, M$ and $\kappa = 1, \dots, \mathcal{K}$

$$\nabla V_i(x).(A_{i\kappa}x + a_{i\kappa} + B_{i\kappa}(K_i x + k_i)) + \alpha V(x) \leq 0 \tag{87}$$

where $x \in \mathcal{R}_i$. This can be written as

$$\begin{bmatrix} \nabla V_i(x) \\ \alpha V(x) \end{bmatrix} . \begin{bmatrix} A_{i\kappa}x + a_{i\kappa} + B_{i\kappa}(K_i x + k_i) \\ 1 \end{bmatrix} \leq 0 \tag{88}$$

where, as before, the '.' stands for the inner product of two vectors. Therefore, invoking Proposition 4.3 for the following differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \begin{bmatrix} \mathbf{conv}\{\sigma_1(x,u), \dots, \sigma_{\mathcal{K}}(x,u)\} \\ 1 \end{bmatrix} \tag{89}$$

where t is the time variable, it follows that $V(x) + \int_0^t \alpha V(x(\tau))d\tau$ is nonincreasing along the trajectories of the differential inclusion (89). Therefore condition (14) in Theorem 3.1 with $W(x) = \alpha V(x)$ is satisfied for the nonlinear system (81).

The above results imply that all conditions of Theorem 3.1 are satisfied for the nonlinear system (81) and therefore all the trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$. □

Theorem 5.2 separates the conditions for positive-definiteness and monotonicty of the candidate Lyapunov function into three cases:

- The desired equilibrium point is inside the region or at some, but not all, of the boundaries.
- The desired equilibrium point is located at the intersection of all the boundaries.
- The desired equilibrium point is outside the region.

It also shows that there is no need to assume that there is one equilibrium point for the dynamic equations of each region and to select them *a priori* by solving an optimization problem (such as it was done in Rodrigues & How (2003*b*)).

Remark 1 One of the limitations of the method proposed in Theorem 5.2 is that it addresses the regulation problem for a fixed desired equilibrium point. It is however possible to apply the same method to a set of multiple desired equilibrium points at the cost of designing one controller for each equilibrium point.

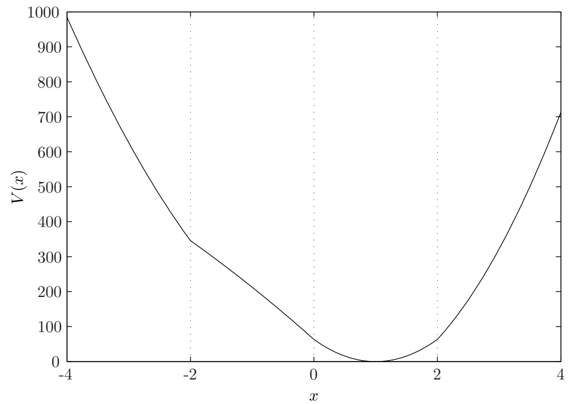

Figure 6. The computed Lyapunov function - Example 6.1

Remark 2 The conditions in Theorem 5.2 include bilinear matrix inequalities (BMI) which make the problem nonconvex. Toker & Özbay (1995) showed that the problem of checking the solvability of a BMI is $\mathcal{NP}$-hard. The complexity of the synthesis problem increases with the order of the system, the dimension of the partitioned space and the number of regions. However, PENBMI (Kocvara et al. 2004), a recent software package providing algorithms with local optimality guarantees, can be used in practice to search for a local solution to the problem.

6 Numerical Examples

Example 6.1 For the illustrative example in section 2, a PWA controller is designed to extend the region of convergence of the robust LQR controller. A feasible solution to the synthesis problem described in Theorem 5.2, was calculated using PENBMI (Kocvara et al. 2004) and Yalmip (Löfberg 2004). Figure 6 depicts the resulting piecewise quadratic Lyapunov function. The designed PWA controller (figure 7) is described by the following gains.

$$\begin{aligned} \bar{K}_1 &= \begin{bmatrix} -4.08 \; -0.437 \end{bmatrix}, \; \bar{K}_2 = \begin{bmatrix} -3.32 \; 1.07 \end{bmatrix} \\ \bar{K}_3 &= \begin{bmatrix} -1.07 \; 1.07 \end{bmatrix}, \; \bar{K}_4 = \begin{bmatrix} 2.45 \; -5.97 \end{bmatrix} \end{aligned} \tag{90}$$

Note that the PWA controller coincides with the linear LQR controller in $\big(0, 2\big)$. Figure 8 shows the trajectory of the closed-loop system consisting of the nonlinear system in feedback connection with the PWA controller. Notice that the closed-loop system has now only one equilibrium point in $\mathcal{X} = [-4, 4]$ and it is stable for all initial conditions in $\mathcal{X}$.

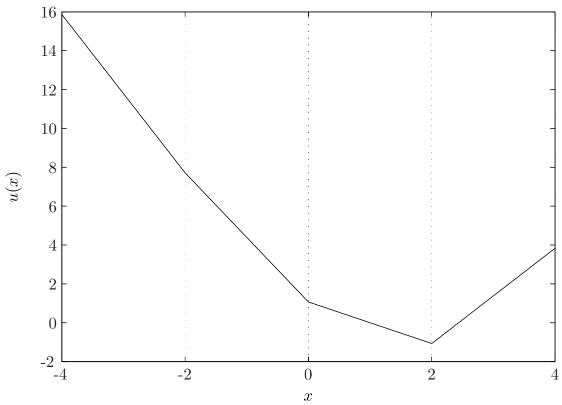

Figure 7. The designed PWA controller - Example 6.1

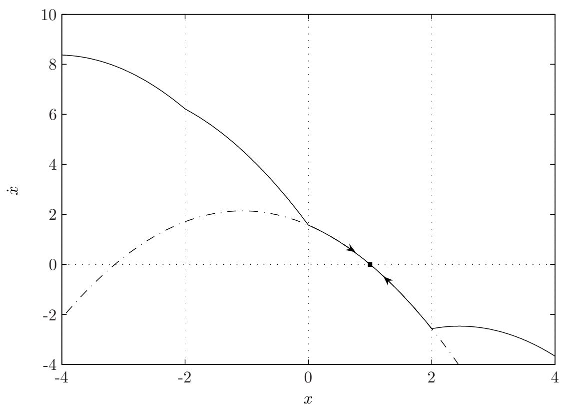

Figure 8. Trajectories of the closed-loop system with the PWA controller (solid) and the linear controller (dashed) - Example 6.1

Example 6.2 Consider the following simple PWA system (adopted from Rantzer & Johansson 2000, with slight modification)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.1x_2 + g(x_1) + u \end{aligned} \tag{91}$$

where $g(x_1)$ is the PWA function depicted in Figure 9. It is desired to stabilize the origin ($x_1 = x_2 = 0$) for this system. The local performance criterion is

$$J(x,u) = \int_0^\infty 4x_1^2(t) + 4x_2(t)^2 + u(t)^2 dt \tag{92}$$

At first, we designed a PWA controller by applying the synthesis method pro-

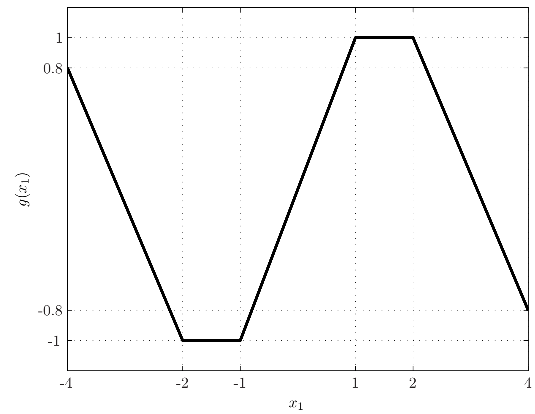

Figure 9. PWA function - Example 6.2

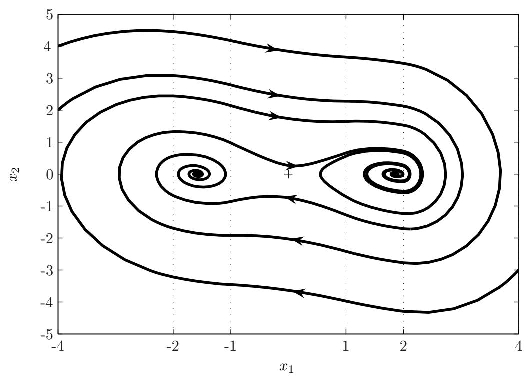

Figure 10. Trajectories of the closed-loop system for the PWA controller proposed in Rantzer & Johansson (2000) - Example 6.2

posed by Rantzer & Johansson (2000) using PWLTOOL (Hedlund & Johansson March 1999). Figure 10 shows the trajectories of the closed loop system. It can be seen that, in this case, the PWA controller designed by PWLTOOL does not stabilize the origin even locally.

We then employ Theorem 5.2 to stabilize the origin and to extend the following LQR controller with the cost function (92) to a PWA controller

$$u = -3.2361x_1 - 3.1376x_2 \tag{93}$$

Figure 11 depicts the trajectories of the closed loop system. The PWA controller stabilizes the origin while it coincides with the LQR controller (93) for the center region ($-1 < x_1 < 1$).

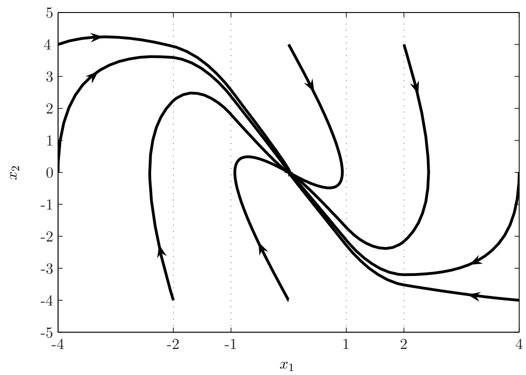

Figure 11. Trajectories of the closed-loop system for the proposed PWA controller - Example 6.2

Example 6.3 Consider the following second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 0.5x_2 - 0.5x_1^2 x_2 + u \end{aligned} \tag{94}$$

Figure 12 shows the trajectories of the open loop system. A linear controller $u = -198x_1 - 101x_2$ can extend the region of convergence to the origin as depicted in Figure 13. However, there still exist initial conditions in

$$\mathcal{X} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| -30 < x_1 < 30,\ -60 < x_2 < 60 \right\} \tag{95}$$

for which the trajectories of the system do not converge to the origin.

To design a PWA controller, the nonlinear system (94) should first be *included* by a PWADI. This is done by computing upper and lower PWA bounds on the nonlinear function $h(x) = 0.5x_1^2 x_2$ and then substituting the nonlinear function in (94) by its PWA bounds. Figure 14 shows the regions (triangles) for which the PWA bounds are computed.

A PWA controller was then designed that satisfies all the conditions of Theorem 5.2. The corresponding piecewise quadratic Lyapunov function is depicted in Figure 15. The trajectories in Figure 14 clearly show that the PWA controller enlarges the region of convergence.

7 Conclusions

This paper proposed a two-step synthesis method to achieve both local performance and global stability for a class of uncertain nonlinear systems. In this

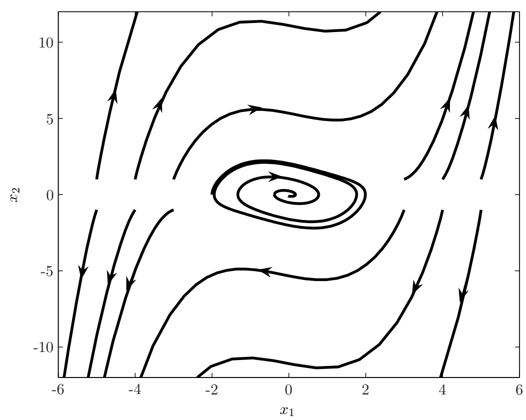

Figure 12. Trajectories of the open loop system - Example 6.3

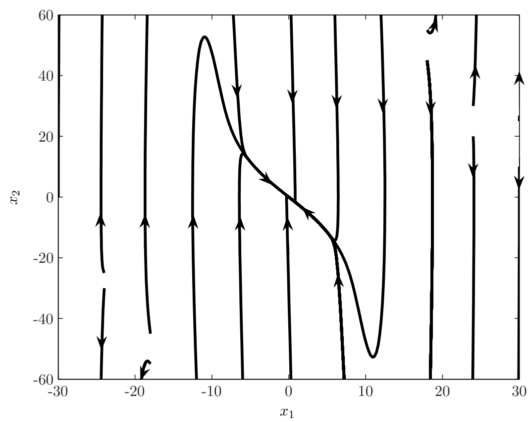

Figure 13. Trajectories of the closed-loop system for the linear controller - Example 6.3

method, a local robust linear controller is first designed for a neighbourhood of the desired equilibrium point to satisfy a local performance requirement. The local linear controller is then extended to a PWA controller to globally stabilize the nonlinear system. The PWA controller locally coincides with the linear controller. A stability theorem for nonsmooth piecewise quadratic Lyapunov functions was also presented that constructs the required theoretical framework.

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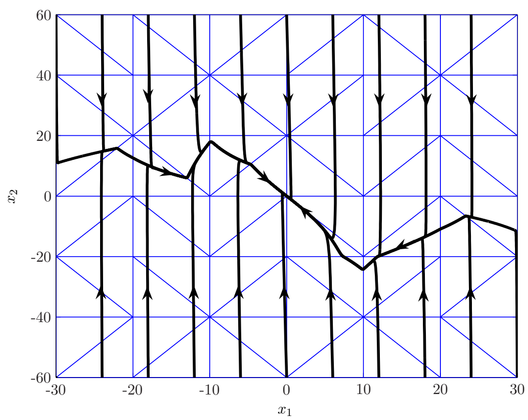

Figure 14. Trajectories of the closed-loop system for the PWA controller - Example 6.3

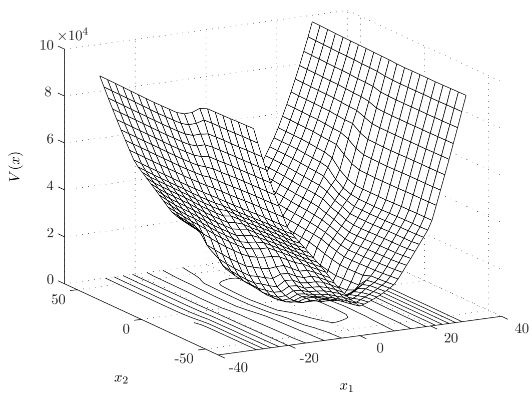

Figure 15. Piecewise quadratic Lyapunov function - Example 6.3

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