Piecewise-Linear H_∞ Controller Synthesis with Applications to Inventory Control of Switched Production Systems ⋆

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Abstract

This paper focuses on the problem of inventory control of production systems. The main contribution of the paper is that production systems are modeled as constrained switched linear systems and the inventory control problem is formulated as a constrained switched H_∞ problem with a piecewise-affine (PWA) control law. The switching variable for the production systems modeled in this paper is the stock level. When the stock level is positive, some of the perishable produced parts are being stored and will deteriorate with time at a given rate. When the stock level is negative it leads to backorders, which means that orders for production of parts are coming in and there are no stocked parts to immediately meet the demand. A state feedback controller that forces the stock level to be kept close to zero (sometimes called a just-in-time policy), even when there are fluctuations in the demand, will be designed in this paper using H_∞ control theory. The synthesis of the state feedback controller that quadratically stabilizes the production dynamics and at the same time rejects the external demand fluctuation (treated as a disturbance) is cast as a set of linear matrix inequalities (LMIs). Two numerical examples are provided to show the effectiveness of the proposed method.

Key words: Switched production systems; Production planning; Inventory control; Piecewise-linear H_∞ control; Linear matrix inequalities.

1 Introduction

Nowadays, with the globalization of business, we are living in a world where the increasing competition between companies is dictating the business rules. Therefore, to survive, the companies are forced to focus seriously on how to produce high quality products at low cost and on how to respond quickly to rapid changes in the demand. The key competitive factors are the new technological advances and the ability to use them to quickly respond to rapid changes in the market. Production planning is one of the key ingredients that has a direct effect on the ability to quickly respond to rapid changes in the market. It is concerned with the optimal allocation of the production capacity of the system to meet the demand efficiently. In general this problem is not easy and requires significant attention.

Inventory control and production planning are classical, yet complex, subjects. Inventory, broadly defined as "quantity of commodity", serves basically as a buffer between two processes: supply and demand. It is necessary because there are obviously differences in rates and timing between supply and demand. Policies for inventory control and production planning involving forecasting are therefore extremely important in the management of companies. The problem of production planning has been tackled by many authors and many research results have been reported in the literature. Among them we

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quote the developments from [1,4,6–9,13,16–19] and the references therein. In these references, both stochastic and deterministic models have been proposed to handle the production planning and/or maintenance. Different approaches have been used to tackle production planning, such as, dynamic programming, linear programming, queuing theory, and Petri nets. Recently, a new concept of manufacturing and production planning control has emerged based on the availability of radio frequency identifiers (RFID) called auto-id based control [11].

This paper models deterministic production systems with switching. The switching corresponds to changes in the stock between two fundamentally different situations: having stocked products and running out of stocked products. In that sense, this work falls under the area of inventory control. It is proposed that the control policy (or decision making) should also include switching to cope with the switched nature of the system dynamics. Previous research on control theoretic methods applied to production systems has not addressed the case of switched production systems. Moreover, it has been mostly focused on the use of optimal control techniques leading to the solution of the Hamilton-Jacobi-Bellman equation. However, this equation is very hard to solve and numerical solutions can only be obtained for very simplified cases of very low state order. It is also very difficult to include state and control constraints. The solution method presented in this paper will depart considerably from previous research by showing how PWA control theory can be used to handle production planning of constrained switched production systems. The method developed in the paper can easily incorporate state and control constraints and is based on LMIs, which are convex constraints. Therefore they can be solved very efficiently, scaling very well with the size of the system. To the best of our knowledge the methodology we are using in this paper to formulate and solve the problem has never been used in production planning before. The main contribution of the paper is therefore that production systems are modeled as constrained switched systems and the inventory control problem is formulated as a switched H_∞ problem with a PWA control law.

The rest of the paper is organized as follows. In Section 2, the production planning problem is described for the case of production of a single part type and the problem assumptions are stated. The main contribution of the paper is presented in section 3. The inventory control problem is formulated as a switched H_∞ control problem posed as an optimization program subject to LMI constraints, which can be solved efficiently using currently available software packages. Section 4 presents two numerical examples to show the effectiveness of the proposed methodology to handle the inventory control problem. The paper then finishes by presenting possible extensions and the conclusions.

2 Problem Statement

As a preliminary step, let us consider the case of a production system producing one part type and formulate the inventory control problem. Let $x_1(t) \in I\!R, u(t) \in I\!R, d(t) \in I\!R$ denote the stock level, the production rate and the demand rate at time t, respectively. It is assumed that the demand rate $d(t)$ is composed of a known constant component $\hat{d}$ (maybe obtained by forecasting) plus an unknown time-varying (possibly fluctuating) component $w(t)$ with finite energy, i.e,

$$\int_0^\infty w^T(\tau)w(\tau)d\tau \leq w_{max} < \infty \tag{1}$$

so that a model for the demand rate is

$$d(t) = \hat{d} + w(t) \tag{2}$$

As mentioned in the introduction, the switching variable for the production systems modeled in this paper is the stock level $x_1(t)$. If $0 < x_1(t) < L$, where L is a warning limit value related to the maximum warehouse capacity c_{max} [1], some of the produced parts are being stored. It is assumed that the stocked parts may deteriorate with time at a rate factor ρ. The production model for $0 < x_1(t) < L$ is thus

$$\dot{x}_1(t) = -\rho x_1(t) + u(t) - d(t) \tag{3}$$

When $x_1(t) < 0$, it means that orders for production of parts are coming in and there are no stocked parts to immediately meet the demand, which leads to backorders. Therefore the production model for $x_1(t) < 0$ is simply

$$\dot{x}_1(t) = u(t) - d(t) \tag{4}$$

Finally, when $L < x_1(t) < c_{max}$ the production model is again

$$\dot{x}_1(t) = -\rho x_1(t) + u(t) - d(t) \tag{5}$$

with the additional constraint that the production rate $u(t)$ be such that it can guarantee that

$$y(t) \leq \bar{y}, \tag{6}$$

where in this case $y = x_1$ and $\bar{y}$ is a desired maximum value for the stock satisfying $\bar{y} \leq c_{max}$. These different dynamical systems can be described together as a switched linear system if the rejection rate is made model

[1] Typically L is a percentage of the maximum warehouse capacity decided by the management team as being a relevant warning value.

dependent as

$$\rho = \rho(x_1) = \begin{cases} \rho_1, \ x_1 \in \mathcal{R}_1 \\ \rho_2, \ x_1 \in \mathcal{R}_2 \\ \rho_3, \ x_1 \in \mathcal{R}_3 \end{cases}, \tag{7}$$

where, $\mathcal{R}_1 = \{x_1 \in I\!R \mid 0 < x_1 < L\}$, $\mathcal{R}_2 = \{x_1 \in I\!R \mid x_1 < 0\}$, $\mathcal{R}_3 = \{x_1 \in I\!R \mid L < x_1 < c_{max}\}$, $\rho_1 = \rho_3 = \rho, \ \rho_2 = 0$. Using (7), the switched system will have PWA dynamics described by

$$\dot{x}_1(t) = -\rho_i x_1(t) + u(t) - d(t), \ i = 1, 2, 3, \tag{8}$$

with $x_1(0) = x_1^0$. In order to capture the behavior of a production system, the production rate cannot obviously be negative and must also be bounded and limited so the following constraints will be introduced for all regions

$$0 \leq u(t) \leq \bar{u}, \tag{9}$$

where $\bar{u}$ is a known positive scalar corresponding to the maximum production rate and $\bar{u} \geq \max_t d(t)$ for feasibility.

Inventory control is a complex hierarchical control problem with several levels of decision-making and planning over several different time horizons. In this paper it will be assumed that the management top level long term planning has determined that the best policy to follow is one where the stock level is kept as close as possible to zero (this policy is sometimes called just-in-time - JIT). Therefore, defining the $\mathcal{L}_2$-norm of a $\mathcal{L}_2$ integrable signal z to be

$$\|z\|_2 \triangleq \left[\int_0^\infty z^T(t) z(t) dt\right]^{1/2} \tag{10}$$

the production control problem can now be stated.

Definition 1 *(*Switched Production Control Problem*): Given a production system with switched PWA dynamics (8), input constraints (9), output constraints (6) and demand rate given by (2) with a constant known factor $\hat{d}$ and a time-varying component $w(t)$ satisfying (1), design the production rate (control input) $u(t)$ such that the stock level $x_1(t)$ converges to zero when there is no time-varying demand component $w(t)$ and such that it is kept as close as possible to zero when $w(t) \neq 0$. As close as possible here means that the $\mathcal{L}_2$ gain of the closed-loop system, i.e, the $\mathcal{L}_2$-induced norm from $w(t)$ to $x_1(t)$, is minimized.*

The problem stated in definition 1 is a piecewise-linear H_∞ control problem that will be formulated as a set of Linear Matrix Inequalities (LMIs) in section 3. In what follows, we will state all problem assumptions and define a general model that for a particular realization agrees with (8). It is assumed that a PWA system is given with strictly proper dynamics in each region $\mathcal{R}_i$ of the form

$$\begin{cases} \dot{x}(t) = A_i x(t) + b_i + B_i u(t) + B_{w_i} w(t) \\ y(t) = C_i x(t) \end{cases} \tag{11}$$

where $x(t) \in I\!R^n, \ u(t) \in I\!R^m, \ w(t) \in I\!R^q$ and $y(t) \in I\!R^p$. Note that for system (8), assuming we have access to both the stock level $x_1(t)$ and the cumulative stock level $x_2(t) = \int_0^t x_1(s) ds$ and choosing the state vector $x(t) = [x_1(t) \ x_2(t)]^T$, the augmented dynamics fit the model (11) with

$$A_i = \begin{bmatrix} -\rho_i & 0 \\ 1 & 0 \end{bmatrix}, \ b_i = \begin{bmatrix} -\hat{d} \\ 0 \end{bmatrix} B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$B_{w_i} = -B_i, \ C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ n = 2, \ m = q = p = 1. \tag{12}$$

It is further assumed that each region $\mathcal{R}_i$ is polytopic. Therefore, following [10], each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{x \mid H_i^T x - g_i < 0\}, \tag{13}$$

where $H_i = [h_{i1} \ \ h_{i2} \dots h_{ip_i}]$, $g_i = [g_{i1} \ \ g_{i2} \dots g_{ip_i}]^T$. For example, for the PWA system (8),

$$H_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \ H_2 = [1 \ \ 0]^T, \ H_3 = H_1$$
$$g_1 = [0 \ \ L]^T, \ g_2 = 0, \ g_3 = [-L \ \ c_{max}]^T$$

Moreover the sets $\mathcal{R}_i$ $i \in \mathcal{I} = \{1, \dots, M\}$ partition a subset of the state space $\mathcal{X} \subset I\!R^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$ (see [15] for generating such a polytopic partition for a given nonlinear system). For system (11), we will use the following adapted definition of trajectories or solutions presented in [10].

Definition 2 [10] *Let $x(t) \in \mathcal{X}$ be an absolutely continuous function. Then $x(t)$ is a trajectory of the system (11) on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$ and Lebesgue measurable $u(t), \ w(t)$, the equation $\dot{x}(t) = A_i x(t) + b_i + B_i u(t) + B_{w_i} w(t)$ holds for all i for which $x(t) \in \overline{\mathcal{R}}_i$.* □

Remark 3 *Note that, as usual in PWA systems, the switching for system (11) is predefined to occur at the boundaries between regions, i.e, a switching occurs whenever the valid affine model of the dynamics changes. However, if a common Lyapunov function defined globally* *can be found for the system, which is the approach to be followed in section 3, stability is in fact proved for any switching (see [12]). In such case it is therefore irrelevant what happens to the trajectories at the boundary points.*

Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathcal{N}_i = \{$*level*-1 neighboring cells of $\mathcal{R}_i\}$. It can then be shown that vectors $c_{ij} \in I\!R^n$ and scalars d_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in I\!R^n \mid c_{ij}^T x - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{x = l_{ij} + F_{ij}s \mid s \in I\!R^{n-1}\} \tag{14}$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in I\!R^{n\times(n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}^T, and $l_{ij} \in I\!R^n$ is given by $l_{ij} = c_{ij}\left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. For example, for the PWA system (8) the boundary between regions $\mathcal{R}_1$ and $\mathcal{R}_2$ is described by $x_1 = 0$. For systems whose polytopic cells are slabs, called *piecewise-affine slab systems*, each $\mathcal{R}_i$ can be outer approximated by a degenerate ellipsoid ε_i, i.e, $\mathcal{R}_i \subseteq \varepsilon_i$ where

$$\varepsilon_i = \{x \in I\!R^n \mid \|E_i x + f_i\| \leq 1\}. \tag{15}$$

For *piecewise-affine slab systems* this covering is in fact exact, *i.e.,* $\varepsilon_i \subseteq R_i$ and $R_i \subseteq \varepsilon_i$. More precisely, for a slab region $\mathcal{R}_i = \{x \in I\!R^n \mid d_1 < c_i^T x < d_2\}$, the degenerate ellipsoid is described by $E_i = 2c_i^T/(d_2 - d_1)$ and $f_i = -(d_2+d_1)/(d_2-d_1)$. Finally, it is assumed that the control objective is to stabilize the origin, which is the case for system (11)–(12). To close this section we state a very important Lemma to be used in this paper.

Lemma 4 *(Schur complement) The LMI*

$$\begin{bmatrix} H & S^T \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0,\ H - S^T R^{-1} S > 0$$

where $H = H^T$, $R = R^T$ and S is a matrix with appropriate dimension.

3 Problem Formulation and Solution

This section formulates mathematically the piecewise-linear H_∞ control problem from definition 1 as an optimization program subject to LMI constraints. For system (11)–(12) we will use a PWA control law

$$u = K_i x + \hat{d},\ i = 1, \ldots, M \tag{16}$$

where, in this case, $M = 3$. Using (16) in (11)–(12) yields

$$b_i + B_i m_i + B_{w_i} w = B_{w_i} w. \tag{17}$$

Remark 5 *The method that will be developed in this section is not restricted to the system from definition 1. Rather, it is applicable to any system whose model is described by (11) and that verifies constraint (17). Note however that this relation might not be valid for a general PWA system, its validity depending obviously on the structure of b_i and B_i.*

The $\mathcal{L}_2$ gain of system (11) under the feedback law (16) is defined as

$$\sup_{\|w\|_2 \neq 0} \frac{\|y\|_2}{\|w\|_2} \tag{18}$$

where the $\mathcal{L}_2$ norm of $w(t)$, defined previously, is $\|w(t)\|_2^2 = \int_0^\infty w(t)^T w(t) dt$, and the supremum is taken over all nonzero trajectories of the system, starting from $x(0) = 0$. If the L_2 norm of a system is less than $\gamma > 0$ we say that the system has disturbance attenuation by a factor of at least γ. It can be shown (see [5,2] for details) that if there exists a quadratic function $V(x) = x^T P x$ with $P = P^T > 0$, and $\gamma > 0$ such that for all $t \geq 0$

$$\frac{d}{dt}V(x) + y^T y - \gamma^2 w^T w < 0 \tag{19}$$

then the $\mathcal{L}_2$ gain of the system (11) is less than γ. The function $V(x)$ is then called a Lyapunov function with supply rate $\gamma^2 w^T w - y^T y$ and we have by integration using (1)

$$\begin{aligned} V(x(t)) + \int_0^t y(\tau)^T y(\tau) d\tau &< V(0) + \gamma^2 \int_0^t w(\tau)^T w(\tau) d\tau \\ &\leq V(0) + \gamma^2 w_{max}. \end{aligned} \tag{20}$$

Therefore, assuming $x(0) = 0$, the trajectories of the closed-loop system are confined to the invariant ellipsoid

$$\mathcal{D} = \{x \in I\!R^n \mid x^T P x \leq \gamma^2 w_{max}\} \tag{21}$$

Using now (11) and (16) into (19), sufficient conditions for V being a Lyapunov function with supply rate $\gamma^2 w^T w - y^T y$ are thus $P = P^T > 0$ and

$$\begin{aligned} &[(A_i + B_i K_i)x + (b_i + B_i m_i + B_{w_i} w)]^T P x + \\ &+ x^T P [(A_i + B_i K_i)x + (b_i + B_i m_i + B_{w_i} w)] + \\ &\qquad x^T C_i^T C_i x - \gamma^2 w^T w < 0. \end{aligned} \tag{22}$$

Using (17) expression (22) can be recast as

$$\begin{aligned} &[(A_i + B_i K_i)x + B_{w_i} w]^T P x + \\ &+ x^T P [(A_i + B_i K_i)x + B_{w_i} w] + \\ &\qquad x^T C_i^T C_i x - \gamma^2 w^T w < 0. \end{aligned} \tag{23}$$

Expression (23) can be written in matrix form as

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \hat{S}_i + \hat{S}_i^T + C_i^T C_i & PB_{w_i} \\ (PB_{w_i})^T & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0, \tag{24}$$

where $\hat{S}_i = P(A_i + B_i K_i)$. Using Schur complement, inequality (24) is equivalent to

$$\hat{S}_i + \hat{S}_i^T + C_i^T C_i + \gamma^{-2} P B_{w_i} B_{w_i}^T P < 0 \tag{25}$$

Since $P > 0$, we can pre-multiply inequality (25) by Q^T and post-multiply by $Q = Q^T$ where $Q = P^{-1}$ to get

$$\bar{S}_i + \bar{S}_i^T + Q C_i^T C_i Q + \gamma^{-2} B_{w_i} B_{w_i}^T < 0 \tag{26}$$

where $\bar{S}_i = (A_i + B_i K_i) Q$. Finally, using the change of variables $Y_i = K_i Q$ and applying Schur complement again, inequality (26) is equivalent to

$$\begin{bmatrix} S_i + S_i^T + \eta B_{w_i} B_{w_i}^T & Q C_i^T \\ C_i Q & -I \end{bmatrix} < 0, \tag{27}$$

where $\eta = \gamma^{-2}$, $Y_i = K_i Q$, $S_i = A_i Q + B_i Y_i$, $i = 1 \ldots, M$. Inequalities (27) have the drawback of being conservative. In fact, the conditions for a given $i = 1, \ldots, M$ only have to be verified for $x \in \mathcal{R}_i$ rather than for all $x \in I\!\!R^n$. We will now derive less conservative conditions. Using (17) expression (22) can be recast for $x \in \mathcal{R}_i$ as

$$\begin{bmatrix} x \\ w \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i + C_i^T C_i & PB_{w_i} & 0 \\ (PB_{w_i})^T & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ 1 \end{bmatrix} < 0, \tag{28}$$

where $\bar{A}_i = A_i + B_i K_i$. If the condition $x \in \mathcal{R}_i$ in (28) is relaxed to $x \in \varepsilon_i$ and if expression (15) is used along with the $S - procedure$ [20,5] with multiplier $\lambda_i < 0$ yields the sufficient conditions $P = P^T > 0$ and

$$\begin{bmatrix} x \\ w \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P \bar{A}_i + C_i^T C_i & PB_{w_i} & 0 \\ (PB_{w_i})^T & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ 1 \end{bmatrix} < -\lambda_i \begin{bmatrix} x \\ w \\ 1 \end{bmatrix}^T \begin{bmatrix} E_i^T E_i & 0 & E_i^T f_i \\ 0 & 0 & 0 \\ f_i^T E_i & 0 & f_i^T f_i - 1 \end{bmatrix} \begin{bmatrix} x \\ w \\ 1 \end{bmatrix}. \tag{29}$$

Rearranging expression (29) the following sufficient conditions can be derived

$$P = P^T > 0, \lambda_i < 0, \; i = 1, \ldots, M, \tag{30}$$
$$\begin{bmatrix} \Omega_i & PB_{w_i} & \lambda_i E_i^T f_i \\ (PB_{w_i})^T & -\gamma^2 I & 0 \\ \lambda_i f_i^T E_i & 0 & -\lambda_i \left(1 - f_i^T f_i\right) \end{bmatrix} < 0$$

where $\Omega_i = \tilde{S}_i + \tilde{S}_i^T + C_i^T C_i + \lambda_i E_i^T E_i$, $\tilde{S}_i = \bar{A}_i^T P$. Inequalities (30) are nonlinear and hard to solve. The following theorem will present a set of LMIs equivalent to (30).

Theorem 6 *For piecewise-affine slab systems conditions (30) are equivalent to LMIs (31) with $\eta = \gamma^{-2}$, $Y_i = K_i Q$, $S_i = A_i Q + B_i Y_i$, $i = 1 \ldots, M$.*

$$Q = Q^T > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M,$$
$$\begin{bmatrix} S_i + S_i^T + \eta B_{w_i} B_{w_i}^T & Q C_i^T & Q E_i^T \\ C_i Q & -I & 0 \\ E_i Q & 0 & -\mu_i \left(1 - f_i f_i^T\right) \end{bmatrix} < 0 \tag{31}$$

Proof: Applying Schur complement two times, conditions (30) are equivalent to $P > 0$, $\lambda_i < 0$ and

$$1 - f_i^T f_i < 0$$
$$\bar{A}_i^T P + P \bar{A}_i + C_i^T C_i + \lambda_i E_i^T E_i + \gamma^{-2} P B_{w_i} B_{w_i}^T P + \lambda_i E_i^T f_i \left(1 - f_i^T f_i\right)^{-1} f_i^T E_i < 0$$

for $i = 1, \ldots, M$ where $\bar{A}_i = A_i + B_i K_i$. Setting $\mu_i = \lambda_i^{-1}$, $\eta = \gamma^{-2}$, pre-multiplying by Q^T and post-multiplying by $Q = Q^T$ where $Q = P^{-1}$ these conditions are equivalent to $Q > 0$, $\mu_i < 0$ and

$$1 - f_i^T f_i < 0$$
$$\tilde{\Omega}_i + \eta B_{w_i} B_{w_i}^T + Q C_i^T C_i Q + \mu_i^{-1} Q E_i^T E_i Q + \mu_i^{-1} Q E_i^T f_i \left(1 - f_i^T f_i\right)^{-1} f_i^T E_i Q < 0 \tag{32}$$

for $i = 1, \ldots, M$, where $\tilde{\Omega}_i = \bar{A}_i Q + Q \bar{A}_i^T$. Using the Matrix Inversion Lemma it was shown in [14] that

$$\tilde{\Omega}_i + \alpha Q + \mu_i^{-1} Q E_i^T E_i Q + \left(\bar{b}_i + \mu_i^{-1} Q E_i^T f_i\right) \mu_i \left(1 - f_i^T f_i\right)^{-1} \left(\bar{b}_i + \mu_i^{-1} Q E_i^T f_i\right)^T < 0 \tag{33}$$

is equivalent to

$$\tilde{\Omega}_i + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^T + \left(\mu_i \bar{b}_i f_i^T + Q E_i^T\right) \mu_i^{-1} \left(I - f_i f_i^T\right)^{-1} \left(\mu_i \bar{b}_i f_i^T + Q E_i^T\right)^T < 0 \tag{34}$$

The difference between conditions (32) and (33) is the fact that in (32) $\alpha = 0$, $\bar{b}_i = 0$ and there are two extra terms, namely, $QC_i^T C_i Q$ and $\eta B_{w_i} B_{w_i}^T$. However, following a similar procedure as the one used in [14] we can conclude that condition (32) is equivalent to

$$\bar{A}_i Q + Q\bar{A}_i^T + \eta B_{w_i} B_{w_i}^T + QC_i^T C_i Q + \\ QE_i^T \mu_i^{-1} \left(I - f_i f_i^T\right)^{-1} E_i Q < 0 \quad (35)$$

With the change of variables $Y_i = K_i Q,\ i = 1, \ldots, M$, and using the Schur complement and the fact that $1 - f_i^T f_i < 0$ and $I - f_i f_i^T < 0$ are equivalent when f_i is a scalar (which is indeed the case for piecewise-affine slab systems) yields the statement of the theorem, which finishes the proof. □

Remark 7 *Using the Schur complement we verify that the results from Theorem 6 can only be used when $|f_i| > 1, i = 1, \ldots, M$. If this is not the case, the LMIs (27) must be used instead of the LMIs (31).*

The results of this theorem will enable the determination of the controller gain matrices $K_i,\ i = 1, \ldots, M$, but there is no guarantee that the control input will meet the production rate bounds (9) that must always be satisfied. Extra constraints should then be added to the previous system of LMIs to force the control law to always satisfy the bounds. For this purpose, we recall first that the state $x(t) \in \mathcal{D},\ \forall t \geq 0$ where

$$\mathcal{D} = \{x \in I\!R^n \mid x^T Q^{-1} x \leq \gamma^2 w_{max}\} \quad (36)$$

In terms of the control input constraints, note that using (16), the constraints (9) are equivalent to

$$-\hat{d} \leq K_i x \leq \bar{u} - \hat{d} \quad (37)$$

Since it is known that $x(t) \in \mathcal{D},\ \forall t \geq 0$, an LMI will now be developed to guarantee that the constraints $M_i^j x \leq r^j,\ j = 1, \ldots, m$, are met for $x(t) \in \mathcal{D}$ and arbitrary $r = [r^1, \ldots, r^m]^T \in I\!R^m$ (an alternative similar reasoning can be found in [5]). For the right inequalities in (37), $M_i^j = K_i^j,\ r^j = \bar{u}^j - \hat{d}^j$, where $K_i^j,\ j = 1, \ldots, m$, are the rows of the feedback matrix K_i and $r^j,\ j = 1, \ldots, m$, are the components of the vector r. For the left inequalities in (37), $M_i^j = -K_i^j,\ r^j = \hat{d}^j$. We start by changing variables to $x = Qz$ so that in the new coordinates the constraints $Y_i^j z \leq r^j, j = 1, \ldots, m$ (where $Y_i = M_i Q$) must be met for

$$z \in \mathcal{D}_z = \{z \in I\!R^n \mid z^T Q z \leq \gamma^2 w_{max}\} \quad (38)$$

Notice now that for $Q > 0$, the set (38) is an ellipsoid (or possibly a ball) and the conditions $Y_i^j z \leq r^j, j = 1, \ldots, m$, represent each a half space. Therefore, $Y_i^j z \leq r^j$ will be met for each j and for all $z \in D_z$ if it is met for the same j and for $z^\star$, where $z^\star$ is the solution to the following optimization

$$\begin{aligned} &\max\ Y_i^j z \\ &s.t.\ z \in \mathcal{D}_z \end{aligned} \quad (39)$$

The solution to problem (39) is the point at which the hyperplane $Y_i^j z = r^j$ is a supporting hyperplane of the set $\mathcal{D}_z$. This solution can be obtained by changing variables to $\tilde{z} = Q^{\frac{1}{2}} z$, so that the ellipsoid becomes a ball. The optimality condition in this new space is that the normal vector to the hyperplane be parallel to the position vector of the point of tangency between the hyperplane and the ball. Changing variables again from $\tilde{z}$ to z yields

$$z^\star = \lambda Q^{-1} {Y_i^j}^T,\ \gamma^2 w_{max} = {z^\star}^T Q z^\star$$

for $\lambda > 0$. Combining these expressions yields

$$\lambda = \frac{\gamma\sqrt{w_{max}}}{\sqrt{Y_i^j Q^{-1} {Y_i^j}^T}},\ z^\star = \frac{\gamma\sqrt{w_{max}}}{\sqrt{Y_i^j Q^{-1} {Y_i^j}^T}} Q^{-1} {Y_i^j}^T$$

Therefore, $Y_i^j z \leq r^j$ is met for $z \in D_z$ and arbitrary $r^j \in I\!R$ if it is met for $z^\star$, i.e, if

$$\gamma\sqrt{w_{max}} \left(Y_i^j Q^{-1} {Y_i^j}^T\right) \leq r^j \sqrt{Y_i^j Q^{-1} {Y_i^j}^T} \quad (40)$$

Condition (40) will be verified if

$$\gamma^2 w_{max} Y_i^j Q^{-1} {Y_i^j}^T - {r^j}^2 \leq 0 \quad (41)$$

Using Schur complement, this sufficient condition is equivalent to the LMI

$$\begin{bmatrix} \frac{\eta (r^j)^2}{w_{max}} & Y_i^j \\ {Y_i^j}^T & Q \end{bmatrix} \geq 0, \quad (42)$$

where $\eta = \frac{1}{\gamma^2}$. Recall that $Y_i^j = M_i^j Q$ and either $M_i^j = K_i^j$ or $M_i^j = -K_i^j$. Note however that by using a Schur complement argument, Y_i^j can be replaced by $-Y_i^j$ in (42) without changing the inequality. Therefore, we can assume without loss of generality that $Y_i = M_i Q = K_i Q$. Finally, constraints (9) will be verified if the following LMIs are verified:

- If LMIs (42) are verified for $j = 1, \ldots, m$, with r^j replaced by $\bar{u}^j - \hat{d}^j$ and
- If LMIs (42) are verified for $j = 1, \ldots, m$, with r^j replaced by $\hat{d}^j$

Finally, the same reasoning can be applied to the output constraints (6) replacing K_i by C_i, Y_i by C_iQ and r by $\bar{y}$ yielding

$$\begin{bmatrix} \frac{\eta(\bar{y}^j)^2}{w_{max}} & (C_iQ)^j \\ (C_iQ)^T & Q \end{bmatrix} \geq 0 \tag{43}$$

Incorporating these constraints, a set of sufficient design conditions can now be stated in the following theorem. The result in the theorem will allow the design of an H_∞ piecewise-linear state feedback controller that stabilizes the system and guarantees the required disturbance rejection by a factor at least γ, guaranteeing also the verification of all input and output constraints.

Theorem 8 *Let γ be a positive constant. If there exist a symmetric matrix $Q > 0$ and matrices $Y_i,\ i = 1, \ldots, M$, such that the LMIs (31), (42) with $r = \bar{u} - \hat{d}$ and $r = \hat{d}$ and (43) hold, for $j = 1, \ldots, m$, then the system (11) verifying the constraint (17) under the controller (16) with $K_i = Y_iQ^{-1},\ i = 1, \ldots, M$ is exponentially stable, the closed-loop system satisfies a disturbance rejection level of at least $\gamma = 1/\sqrt{(\eta)} > 0$ and the control input and output satisfy the constraints (9), (6), respectively.*

Proof. Trivially follows from the above derivations and from the proof of Theorem 6. □

From a practical point of view, the controller that quadratically stabilizes the system and at the same time guarantees the maximum possible disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

Problem 1: (Piecewise-linear H_∞ control problem)

$$\begin{aligned} &\max\ \eta \\ &\quad s.t.\ Q = Q^T > 0,\ \eta > 0,\ \mu_i < 0, ,\ (31),\ (42),\ (43) \\ &\qquad -l_1 \prec Y_i^j \prec l_1,\ i = 1, \ldots, M,\ j = 1, \ldots, m, \end{aligned}$$

where $\succ$, $\prec$ mean component-wise inequalities and l_1 are given vector bounds.

Remark 9 *Note that there are $2m + p + 1$ LMIs to be solved for each region making a total of $M(2m + p + 1)$ LMIs for all M regions plus the extra LMI constraint $Q > 0$. Notice however that LMIs are convex constraints and therefore can be solved very efficiently using interior point methods. In fact, it has been shown that the complexity of solving multiple LMIs is not much more than solving one (see [5] and references therein for details). This is one of the strengths of the proposed method. However, the fact that a globally quadratic Lyapunov function is being used may potentially be conservative and alternative Lyapunov functions will be investigated in future work.*

Finally, the following corollary summarizes the results on the design of the controller that quadratically stabilizes any system in the class (11) verifying the constraint (17) and simultaneously guarantees the highest possible disturbance rejection level.

Corollary 10 *Let $\eta > 0,\ Q > 0,\ Y_i,\ i = 1, \ldots, M$, be the solution of problem 1. Then, the controller (16) with $K_i = Y_iQ^{-1},\ i = 1, \ldots, M$, quadratically stabilizes the class of production systems (11) with constraints (6), (9), (17) and, moreover, the closed-loop system satisfies a disturbance rejection level of at least $1/\sqrt{\eta}$.*

4 Numerical Examples

To illustrate the effectiveness of the developed results, we consider in this section two manufacturing production systems.

Example 1: Consider a manufacturing system producing one item. The problem data for this example can be found in table 1. For this problem, $x = [x_1(t)\ x_2(t)]^T \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}$. The corresponding matrices are

$$A_i = \begin{bmatrix} -\rho_i & 0 \\ 1 & 0 \end{bmatrix},\ b_i = \begin{bmatrix} -\hat{d} \\ 0 \end{bmatrix} B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ B_{w_i} = -B_i,\ C_i = [1\ \ 0],\ i = 1, 2, 3. \tag{44}$$

Notice that this system fits the general model (11) with constraint (17). Solving the optimization problem 1 using Yalmip [21] and SeDuMi [22] with LMIs (27) used for regions $\mathcal{R}_1$ and $\mathcal{R}_2$ and LMIs (31) used for region $\mathcal{R}_3$, yields $\eta = 0.0169$ and

$$K_1 = \begin{bmatrix} -1.51 \times 10^{-6} & -6.78 \times 10^{-7} \end{bmatrix}, \\ K_2 = \begin{bmatrix} -1.50 \times 10^{-6} & -6.75 \times 10^{-7} \end{bmatrix}, \\ K_3 = \begin{bmatrix} -1.51 \times 10^{-6} & -6.78 \times 10^{-7} \end{bmatrix}.$$

The disturbance rejection is guaranteed to be at least $\gamma = \eta^{-0.5}$.

$\rho_1 = \rho_3$	ρ_2	$\bar{u}$	$\hat{d}$	L	c_{max}	$\bar{y}$	w_{max}
0.01	0	2	1	90	100	100	10

Table 1
Production system data for example 1

To simulate the performance of this controller we will consider two cases:

(1) constant demand rate: in this case $d(t) = \hat{d} = 1$ and $w(t) = 0$. The simulation results are shown figure 1.

As expected, the production rate is maintained at a constant value $u(t) = \hat{d}$ and all the produced parts are supplied to meet the demand. The stock level therefore remains at zero as desired. Note that the control remains always between the given bounds.

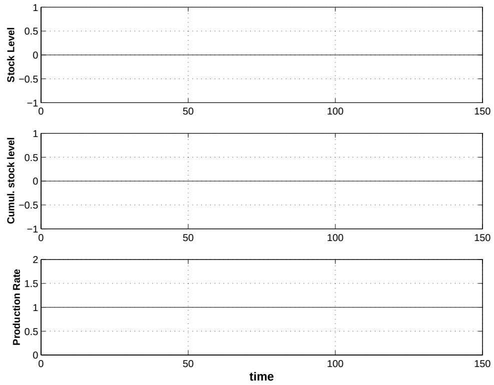

Fig. 1. Stock level, cumulative stock level and production rate for constant unitary demand rate.

(2) time varying demand rate: in this case the variable part of the demand rate is a rectangular pulse of the form $w(t) = step(t = 80) - step(t = 90)$. The simulation results are shown in figure 2. It can be seen that after the unexpected increase in the demand rate the stock decreases going into backorders and then the production rate compensates for the disturbance and the stock returns to zero. Note that again, in this case, the control input satisfies the imposed bounds. Although the controller was designed for $w_{max} = 10$, a simulation performed for the case $w(t) = 1.2\,(step(t = 80) - step(t = 90)) + \exp(-0.1t)\sin(0.5t)$ still shows convergence of the stock to zero and the controller is still within the bounds (see figure 3). This shows that the proposed design method provides some robustness to small increases in the energy of the time varying disturbance.

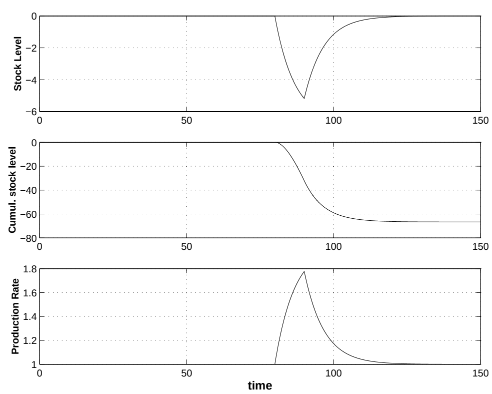

Fig. 2. Stock level, cumulative stock level and production rate for time varying demand rate.

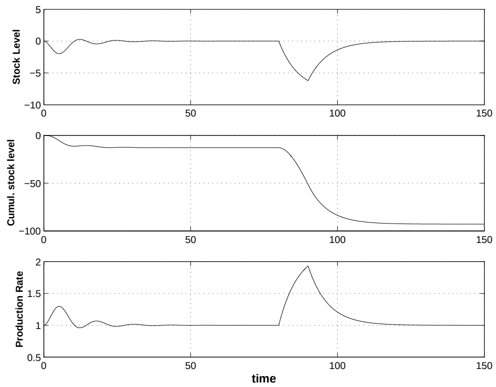

Fig. 3. Stock level, cumulative stock level and production rate for oscillating demand rate.

Example 2: Consider now a transfer production line consisting of two different facilities, each producing one product. The product manufactured at the first facility is supplied to the second facility to be incorporated in the second product that is then delivered to the costumer. Therefore, this example represents a cascaded transfer line. We assume that there is only deterioration of produced parts in the second stock. We also assume that the warehouse that stores the second product has limited capacity. However, for simplicity it is assumed that the warehouse that stores the first product has a very large capacity and that such capacity can be modeled as unlimited. The problem data for this example can be found in table 2. For this problem, $x(t) = [x_1(t)\;\; x_2(t)]^T \in I\!R^2$, $u(t) \in I\!R^2$ and $d(t) \in I\!R$. The corresponding matrices are given by

$$A_i = \begin{bmatrix} 0 & 0 \\ 0 & -\rho_i \end{bmatrix},\; b_i = \begin{bmatrix} 0 \\ -\hat{d} \end{bmatrix},\; B_i = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$B_{w_i} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\; C_i = [0\;\; 1]\,,\; i = 1, 2, 3. \qquad (45)$$

The regions are defined as in example 1 with the difference that now it is x_2 the state component responsible for the switching. Notice that this system also fits the general model (11) with constraint (17). Solving the optimization problem 1 using Yalmip [21] and SeDuMi [22] with LMIs (27) used for regions $\mathcal{R}_1$ and $\mathcal{R}_2$ and LMIs (31) used for region $\mathcal{R}_3$, yields $\eta = 0.0169$ and

$$K_1 = \begin{bmatrix} -1.56 \times 10^{-2} & -2.26 \times 10^{-2} \\ -8.87 \times 10^{-4} & -1.48 \times 10^{-1} \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -1.21 \times 10^{-2} & -7.20 \times 10^{-2} \\ -1.10 \times 10^{-8} & -1.50 \times 10^{-1} \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -1.56 \times 10^{-2} & -2.30 \times 10^{-2} \\ -9.13 \times 10^{-4} & -1.48 \times 10^{-1} \end{bmatrix}.$$

The disturbance rejection is guaranteed to be at least $\gamma = \eta^{-0.5}$.

$\rho_1 = \rho_3$	ρ_2	$\bar{u}$	$\hat{d}$	L	c_{max}	$\bar{y}$	w_{max}
0.01	0	[2, 2]	1	90	100	100	10

Table 2
Production system data for example 2

To simulate the performance of this controller we will consider again two cases:

(1) constant demand rate: in this case $d(t) = \hat{d} = 1$. The simulation results are shown figure 4. Again as expected, both production rates are maintained at a constant value $u(t) = \hat{d}$ and all the produced parts are supplied to meet the demand. The stock levels therefore remain at zero as desired. Note that both control inputs remain always between the given bounds.

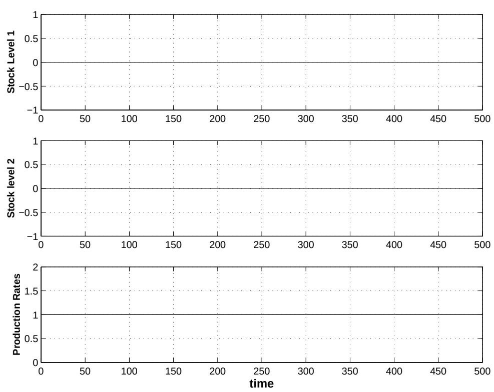

Fig. 4. Stock levels and production rates for constant unitary demand rate.

(2) time varying demand rate: in this case the variable part of the demand rate is again a rectangular pulse of the form $w(t) = step(t = 80) - step(t = 90)$. The simulation results are shown in figure 5. It can be seen that after the unexpected increase in the demand rate both stock levels decrease going into backorders and then the production rates compensate for the disturbance and both stock levels return to zero. Note that again, in this case, the control inputs satisfy the imposed bounds. A simulation performed for the case $w(t) = 1.2\,(step(t = 80) - step(t = 90)) + \exp(-0.1t)\sin(0.5t)$ still shows an adequate behavior of the controllers and convergence of the stocks to zero (see figure 6). Once again this shows that the proposed design method provides some robustness to small increases in the energy of the time varying disturbance.

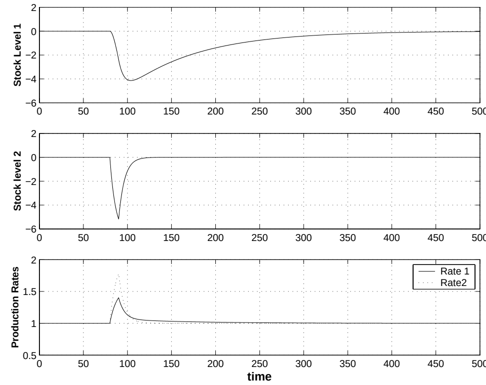

Fig. 5. Stock levels and production rates for varying demand rate.

5 Conclusions

This paper dealt with the inventory control problem for a deterministic production system with given deterministic demand rate plus an unknown fluctuating demand rate with finite energy whose bound is known. This problem has been modeled as a control problem of a switched (PWA) system and it has been solved using new results on piecewise-linear H_∞ control theory developed in the paper.

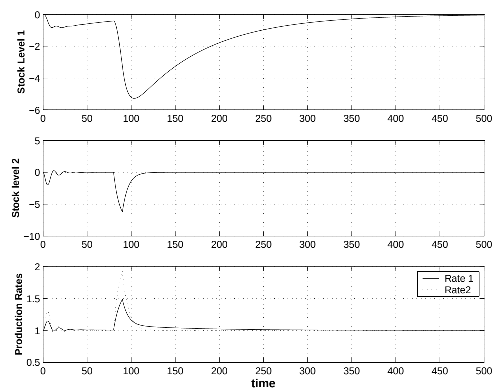

Fig. 6. Stock levels and production rates for oscillating demand rate.

The proposed approach has shown that switched control theory can be applied to inventory control problems. This model can be extended to include production of multiple parts using multiple machines, each machine producing one part. This can be easily dealt with because a system with multiple (say N) machines, each machine producing one part, is a decoupled system of N subsystems and each of the N subsystems can be solved independently, as presented in this paper.

The models presented in this paper deal with deterministic production systems. In practice, these models have some limitations since they do not include some important features such as, for example, breakdown of the machines. After breakdown, there is a no-activity time to get the machine repaired. To overcome this limitation in the model, future work can be aimed at extending it to switched stochastic production systems and to include breakdowns in the machines and even the preventive maintenance or corrective maintenance, as it was done in the literature for non-switched production models. For more details refer to [16], [4] and the recent book on stochastic manufacturing systems [17].

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