Title: **Teaching absolute value inequalities to mature students.**

Authors:

Anna Sierpinska (Concordia University)
Georgeana Bobos (Concordia University)
Andreea Pruncut (Université de Montréal)

Address for correspondence:

Anna Sierpinska
Concordia University
Department of Mathematics and Statistics
1400 De Maisonneuve blvd. West
Montreal, QC, H3G 1M8
Canada
e-mail: anna.sierpinska@concordia.ca

NOTE : This file contains the last version of the paper under the above title before its publication in 2011, in Volume 78, issue 3 of *Educational Studies in Mathematics*. The paper should be cited as :

Sierpinska, A., Bobos, G. & Pruncut, A. (2011). Teaching absolute value inequalities to mature students. *Educational Studies in Mathematics* 78(3), 275-305. DOI: 10.1007/s10649-011-9325-2.

The published version of the paper can be dowloaded from the webpage of the journal at : http://www.springerlink.com/content/0013-1954/78/3/

ANNA SIERPINSKA, GEORGEANA BOBOS AND ANDREEA PRUNCUT

TEACHING ABSOLUTE VALUE INEQUALITIES TO MATURE STUDENTS

Abstract:

This paper gives an account of a teaching experiment on absolute value inequalities, whose aim was to identify characteristics of an approach that would realize the potential of the topic to develop theoretical thinking in students enrolled in prerequisite mathematics courses at a large, urban North American university. The potential is demonstrated in an epistemological analysis of the topic. It is also shown that this potential is not realized in the way the topic is presently taught in prerequisite mathematics courses. Three groups of students enrolled in such courses were each exposed to one of three approaches we conceived for teaching the topic, labeled the Procedural (PA), the Theoretical (TA) and the Visual (VA) approaches. The design of the three lectures was constrained by institutional characteristics of college level courses, and informed by epistemological and didactical analyses of the topic. It was found that following the VA lecture, which proposed two equally valid mathematical techniques (graphical and analytic), one of which could be used to test the validity of results obtained by the other, students were more likely to engage in some aspects of theoretical thinking. They displayed more reflective and systemic thinking than other groups, and were better equipped to deal with the logical intricacies of absolute value inequalities. VA afforded students a synthetic grasp of the inequalities, and a flexibility of thought not easily available to PA and TA students. However, without sufficient attention to tasks not easily solved by graphical means, VA approach provided students with a way to avoid the challenges of systemic and analytic thinking, some of which were more apparent in TA students. PA students expectedly behaved more as procedural knowers, but we saw interesting examples of engagement with theoretical thinking while dealing with the procedures proposed in the PA lecture.

INTRODUCTION

Students of prerequisite mathematics courses[i] can be frustrated with their fast pace and overloaded curricula (Sierpinska, Bobos, & Knipping, 2008), but if we look at the mathematical content of the prerequisite courses we find that it is not so much the content that is overwhelming as the number of conceptually disconnected types of tasks into which it has been divided (Sierpinska & Hardy, 2010), and absence of a theoretical organization (Barbé, Bosch, Espinoza, & Gascón, 2005). Consequently, students have no control over the validity of their solutions and therefore “need the teacher to tell them if they are right or wrong” (Sierpinska, 2007). They are not given a chance to develop *theoretical thinking* in mathematics, which is essential for gaining autonomy as learners.

To gain an idea of how teaching mathematics in prerequisite courses could be improved, we conducted a teaching experiment on absolute value inequalities – a topic typically taught in prerequisite College Algebra courses. This topic has been chosen not only because of the intrinsic epistemological significance of the notion of absolute value, especially in the theoretical foundation of number, and its role in applications but also because, as suggested in previous research [(Brumfiel, 1980); (Chiarugi, Fracassina, & Furinghetti, 1990); (Denton, 1975); (Duroux, 1983); (Gagatsis & Thomaidis, 1994); (Monaghan & Ozmantar, 2006); (Perrin-Glorian, 1995); (Perrin-Glorian, 1997); (Wilhelmi, Godino, & Lacasta, 2007)], what can be learned from it exceeds the purpose of teaching any particular mathematical concept.

Students in prerequisite courses rarely aim at specializing in mathematics. They are therefore more likely to encounter absolute value inequalities in the context of Statistics (e.g. when the median is presented as the minimum deviation location (Weisberg, 1992)) or physics (e.g. in error tolerance estimation) than in the context of epsilon-delta argumentation in Real Analysis. In their future studies, these students are usually not expected to process absolute value inequalities algebraically themselves; they are only expected to use the inequalities that are given to them in lectures in choosing the appropriate statistic in their processing of concrete numerical data and using the formulas by substituting numerical values for variables. Analytic proofs of the inequalities are rarely if ever given in statistics lectures for non-mathematical students. At most, an informal explanation is given (Hanley, Platt, Chung, & Bélisle, 2001). This deprives students of any theoretical control over the mathematics they are applying. They may remain unaware of the hypothetical character of theorems that include the formulas they are using and therefore of the limitations of their applicability. Developing a more theoretical grasp of processing at least certain simple cases of inequalities with absolute value would give them a chance to apply mathematics in their future studies and professions in a more critical way.

In our research, therefore, we were *looking for characteristics of an approach to teaching absolute value inequalities* in *College Algebra* courses for students who take them as prerequisite for programs other than specialization in mathematics *that would promote students' theoretical thinking about the topic*. The challenge, for us, was to do this while complying with the regular format of prerequisite courses, characterized by particularly rigid institutional constraints: the courses are short and intensive, classes are large, and student-teacher interaction is often limited to the lecture-and-assess style of communication. Our research consisted in a teaching experiment, where three approaches to teaching absolute value inequalities were tried.

We start by explaining in what sense we consider our study to be a "teaching experiment". We then present the theoretical framework that disciplined the design of the experiment and analysis of the results. This is followed by epistemological and didactic analyses of the notion of absolute value underlying our design of the experiment and analysis of its results. Remaining sections contain an outline of the three lectures we experimented with, research procedures, results and conclusions.

The paper is accompanied by "Supporting documentation" files[ii] containing slides of the three lectures we experimented with, students' solutions to exercises, and transcripts of interviews with them.

METHODOLOGY: TEACHING EXPERIMENT

As mentioned, our research started from the intention to teach absolute value inequalities in prerequisite mathematics courses in a way that would promote theoretical thinking. Three approaches to teaching the topic – "Procedural" (PA), "Theoretical" (TA) and "Visual" (VA) – were tried: they were communicated to students as three lectures followed by a set of exercises to be solved individually. Thus, we "experimented with" each approach, looking for the good and not so good aspects of each, from the point of view of our teaching goals. Such "experimenting" is very common in the practice of teaching. To some extent, it could be argued that, when researchers engage in teaching experiments, they transcend the teacher-researcher divide: there is always an expectation that students go through a learning cycle. While teachers rely mostly on their craft knowledge to encourage such developments in students, however, when researchers use teaching experiment methodologies, they circumscribe both the design and the evaluation of the teaching intervention within explicit conceptual boundaries. The design of our lectures was thus informed by theoretical conclusions derived from epistemological and didactical analyses of the notion of absolute value. It resembled classical experimental design only in the following sense:

> [T]he researcher selects one or more samples from a target population and subjects it or them to various treatments. The effect of one treatment is compared to the effects of others, with the intention of specifying differences between or among them. [(Steffe & Thompson, 2000), p. 270, in reference to (Campbell & Stanley, 1966)]

Our analysis of students' problem solving behaviors and of their responses in the interviews following the intervention departed from traditional notion of experiment in social sciences. We did not "suppress[…] conceptual analysis in the conduct of research" and we did not assume that "an experimental manipulation would *causally* affect other variables – such as measures of students' mathematical achievement – quite *apart from the individuals* involved in the treatment" (Steffe & Thompson, ibid., p. 270-271). In analyzing data, exploring students' mathematical activity was of primary interest for us. Furthermore, we withheld from formulating a hypothesis to be proved or disproved, or from considering students treated with any one of the approaches as a "control group". This exploratory nature brought our research closer to the way teaching experiment methodology is used in mathematics education research, as described in (Steffe & Thompson, 2000). We did not contrast any of the approaches with some prefabricated teaching/learning ideal: we did not, for instance, consider the group treated with the "Procedural Approach" (PA) as our control group, although this approach was the closest to the teaching of absolute value common in college algebra courses, which we criticize. In fact, we do not *a priori* reject procedural knowledge as not worth having or developing in students. For us, procedural understanding is still an *understanding*[iii], and not "memorizing" and "performing pointless operations on meaningless symbols" [(Porter & Masingila, 2000), p. 165].

THEORETICAL FRAMEWORK

Our theoretical framework is grounded in Anthropological Theory of the Didactic (ATD) [(Chevallard, 1999); (Chevallard, 2002); (Hardy, 2009), (Sierpinska, Bobos, & Knipping, 2008)], and a model of theoretical thinking in mathematics [(Sierpinska, Nnadozie, & Oktaç, 2002); (Sierpinska, 2005)].

ATD was helpful in taking account of institutional aspects of teaching prerequisite mathematics courses in North America, and in systematizing our epistemological and didactic analyses of the mathematical topic of our experiment. We framed these analyses in terms of mathematical and didactic praxeologies, that is, different tasks, techniques, technologies and theories constituting the mathematical and didactic organizations surrounding the notion of absolute value and absolute value inequalities.

The theoretical thinking (TT) model was behind both the conception of our research and the interpretation of the results. Three main features of TT are postulated: TT is "reflective", "systemic" and "analytic". Reflective thinking is expressed by an investigative attitude towards mathematical problems: reflecting back on one's solution; seeking a different, e.g. more economical approach; noticing relations with previously solved problems. It is the opposite of just applying a learned procedure and forgetting about the problem when solved. Reflective thinkers are more likely to hold the epistemological position of "constructed knowers" than "procedural knowers" (ibid., p. 85), a distinction borrowed from (Belenky, Clinchy, Goldberger, & Tarule, 1997).

Theoretical thinking is "systemic" in the sense of thinking about systems of concepts. It is definitional, based on proofs, and hypothetical. *Definitional* means that concepts are defined by reference to other concepts within the system. Decisions about the truth of a statement are made by means of *proofs* which rely on accepted definitions, conceptual and logical relations within a system and not on images evoked by terms, common beliefs or "gut feeling". *Hypothetical* refers to being aware of the conditional character of mathematical statements [(Sierpinska, Nnadozie, & Oktaç, 2002), p. 35- 37].

Analytic thinking refers to sensitivities to formal symbolic notations and specialized terminology ("linguistic sensitivity"), and to the structure and logic of mathematical language ("meta-linguistic sensitivity") (ibid.).

Engaging with theoretical thinking may help avoid both "conceptual" and "procedural" errors in the sense of Porter & Masingila (2000, p. 172):

> Procedural errors were comprised of syntax errors and errors in carrying out procedures, while conceptual errors included such things as the selection of inappropriate procedures, misinterpretation of mathematical terms, and errors in logic. [(Porter & Masingila, 2000), p. 172]

In particular, analytic sensitivity helps avoid syntax and logical errors, and sharpens attention to the technical meaning of mathematical terms. On the other hand, selecting appropriate procedures and carrying them out is certainly enhanced by systemic thinking. This is why our model of theoretical thinking is not based on a distinction between "procedural" and "conceptual" knowing in the sense of the above-mentioned article.

EPISTEMOLOGICAL ANALYSIS

In mathematical praxeology, absolute value is engaged in at least two types of tasks. One is the "epistemic" task of re-conceptualizing the notion of number as absolute measure within the realm of the notion of number as representing directed change. The other is the "pragmatic" task of processing analytical expressions of mathematical relationships.[iv] We start with a discussion of the pragmatic task.

The pragmatic task

Absolute value function serves to process analytic expressions of relationships between the magnitudes of certain variable quantities. The task includes transformations both ways: longer expressions are "compactified", and – as in solving absolute value inequalities – absolute value notation is undone. Techniques of this processing of expressions use one of the several equivalent characterizations of absolute value, their logical consequences such as, $|X| < a \Leftrightarrow -a < X < a$, for $a > 0$, properties such as the triangle inequality, and rules of algebra and logic underlying, in particular, the technique of reasoning by cases. These characterizations, properties and techniques of reasoning play the role of "technology" in the mathematical praxeology associated with processing absolute value expressions, whereas the logical and algebraic laws underlying the reasoning and proofs belong to the "theory" part.

There are several characterizations of absolute value, each with its merits as a definition in some context (Wilhelmi, Godino, & Lacasta, 2007). It is this idea that, over thirty years ago, led Brumfiel (1980) to highlighting the benefits of discussing several definitions of absolute value (he listed five[v]) with students. Time constraints prevented us from following this advice in our lectures. Brumfiel's discussion of the value of the definitions for solving particular kinds of problems included considerations of what Wilhelmi et al. much later (2007) called *epistemic*, *cognitive* and *instructional* dimensions of the didactic effectiveness of a definition. One of the five definitions was the *piecewise-linear function definition*:

$$|x| = \begin{cases} x \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$$

The other four were the square root definition ($|x| = \sqrt{x^2}$ [vi]), the maximum function definition ($|x| = Max\{x, -x\}$) and two definitions based on the notion of distance. Definitions based on the notion of distance are important in applications and in mathematical theory, in particular in generalizations of absolute value to norms in higher dimensions and general vector spaces, and in generalizations of limits and continuity in topology. In our lectures, we used the distance intuition of absolute value in the introduction. We formalized the notion, however, in the form of the piecewise-linear function definition, because it was useful as a basis of the reasoning by cases technique we wanted to teach. Some authors criticized this definition as hard to understand for students who have trouble with piecewise defined functions in general (Gagatsis & Thomaidis, 1994). The square root function definition avoids these difficulties but requires teaching quadratic inequalities leading to higher instructional and cognitive costs.

The piecewise-linear function definition suggests a technique for processing absolute value expressions based on "reasoning by cases" (*RBC*). For example, reasoning directly from this definition, an inequality such as $|x-1| < |x+1|$ can be seen as a disjunction of four cases:

1. $x-1 \geq 0\ and\ x+1 \geq 0\ and\ x-1 < x+1$
2. $x-1 \geq 0\ and\ x+1 < 0\ and\ x-1 < -(x+1)$
3. $x-1 < 0\ and\ x+1 \geq 0\ and - (x-1) < x+1$
4. $x-1 < 0\ and\ x+1 < 0\ and - (x-1) < -(x+1)$

Each case is a conjunction of three conditions. Two of these are "interval conditions", resulting from the application of the definition to the absolute value expressions in the inequality. The third inequality is the form that the given inequality takes in the intervals defined in the interval conditions. We call this the *complete RBC* technique[vii].

This reasoning can be simplified to avoid considering the case where the interval conditions are contradictory (case 2 in the example above). If the inequality contains only two absolute value expressions, say, $|f(x)|$ and $|g(x)|$, both linear, and *a* and *b* (with $a < b$) are the zeros of *f* and *g* then it is enough to consider three cases, corresponding to the form of the inequality in the intervals $x < a$, $a \leq x < b$ and $x \geq b$. The first interval corresponds to the conjunction of interval conditions $x < a\ and\ x < b$; the second interval – to the conjunction of $x \geq a\ and\ x < b$, and the third – to $x \geq a\ and\ x \geq b$. We call this the *simplified RBC technique*.

Teaching students the complete *RBC* technique (with the simplified technique being taught or discovered by the students later, and justified using a proof) has the potential of engaging them in theoretical thinking. The technique is completely transparent; no element of the reasoning is hidden from the students. Hence, students have, in principle, all they need to maintain full control over the technique and thus study the topic from the position of constructed knowers. Since all meanings in the technique are derived from the definition, "definitional" thinking is highlighted. The definition is a disjunction of two conditional statements – an embodiment of hypothetical thinking. All steps of the technique are logically justified; they are deduced from the definition, and refer to known algebraic rules. Therefore, the technique is based on systemic thinking. Moreover, the subtle interplay of logical connectives – conjunction within the cases; disjunction of the cases – and the technical meanings of "and" and "or", call for linguistic and logical sensitivities – that is, analytical thinking.

The epistemic task: re-conceptualization of the notion of number within a larger domain

In individual cognitive development (as in history of mathematics), number is first understood as a measure of the magnitude of something, relative to a conventional unit. This "something" can represent a change of the magnitude, such as increase or decrease, without, however, taking into account the direction of change. It represents an "absolute" measure of the change, which is all we are interested in in Arithmetic, the mathematics of states. The development of Analysis – the mathematics of motion – required, however, that not only the magnitude but also the direction of change be taken into account.

Taking into account the direction of change resulted in a new concept of number, sometimes called "directed number". For the sake of economy of mathematical thought, mathematicians were interested in *embedding* the previous notion of number into the new one, so that directed numbers could contain an isomorphic image of the measuring number, with its properties intact. Absolute value is a notion that allowed mathematicians to construct the isomorphic image of the old notion of number within the new one.

Here is a description of the construction. Let "*AMN*" represent the absolute measure numbers, and "*DN*" – directed numbers. Thinking of *AMN* as isomorphic with a part of *DN* can be modeled by two functions, f and g (Figure 1). The function f maps absolute measure numbers into directed numbers: in *DN* an absolute measure is treated as a positive number. By way of the function g, the absolute measure number appears as a particular aspect of the directed number, namely – its absolute value. The two functions account for an isomorphism between *AMN* and a part of *DN*: $g\big(f(x)\big) = g(x) = x$ because, in *DN*, x is treated as a positive directed number; $f\big(g(x)\big) = f(|x|) = |x| = x$, because, in *AMN*, there is no distinction between a number and its absolute value, so $|x|$ is identical with x.

$$AMN \xrightarrow{f} DN$$
$$x \mapsto x$$
$$DN \xrightarrow{g} AMN$$
$$x \mapsto |x|$$

Figure 1. Construction of an isomorphism between AMN and a part of DN

This isomorphism – as any isomorphism in mathematics – points, simultaneously, to a *structural similarity* between *AMN* and part of *DN*, and a *difference in the nature* of these objects. From the perspective of *DN*, a symbol like, for example, "+3" represents one single whole, a number in itself. From the perspective of *AMN*, this symbol represents two objects: a number (3) and a sign (+).

Cauchy (1821/1968) understood well the cognitive challenge that this extension of the notion of number represents. His remarks about the difference between "nombres" and "quantités" in his *Cours d'Analyse* (extensively quoted by Duroux (1983)) are evidence for this understanding. In Cauchy's text, the word "nombres" is reserved for what we have denoted here by *AMN*, while the name for our *DN* is "quantités".

In this conceptualization of the notion of number, *absolute* value makes sense only when a number's *relative* value is simultaneously considered. That is, when the value of the number depends on certain conditions. Therefore, the concept of absolute value is meaningful in the context of conditional or hypothetical reasoning on expressions containing letters used as variables, and not only constants, or letters used as placeholders or unknowns. When, however, students are first introduced to directed numbers, tasks they are given allow them to continue thinking of number as absolute measure, and conceive of the new "directed number" not as an entity in itself, but as a *compound object*, made of number in the old sense and a sign ("Sign+*AMN*" conception of number). For example, students would be given a rule such as, "when adding two directed numbers with different signs, subtract the one with smaller absolute value from the one with larger absolute value and supply the result with the sign of the number with larger absolute value". This rule applies only to processing expressions with constants, and not variables whose value is relative. In algebra, students may continue thinking this way, because the presentation of techniques by means of "worked out examples" and step-by-step procedures allows them to avoid using letters as variables and engaging in conditional reasoning. Using letters as variables is known to be difficult (Küchemann, 1981), and teachers and textbook authors try to facilitate students' learning. This way, however, they induce obstacles. Thinking of number as a compound object underlies common students' mistakes such as interpreting a letter in an algebraic expression as representing a non-negative number [(Duroux, 1983); (Chiarugi, Fracassina, & Furinghetti, 1990); (Gagatsis & Thomaidis, 1994)]. Number as absolute measure has no sign since it ignores direction. Thus, the letter variable x, which appears to represent a single entity, must refer to absolute measure, a number without a sign. The symbol "$-x$" then necessarily refers to a "negative number", and the statement "$|x| = -x$, if $x < 0$" could be understood as allowing the absolute value to be negative sometimes.

DIDACTIC ANALYSIS

Prerequisite *College Algebra* courses focus almost entirely on the pragmatic tasks of processing algebraic expressions, whose types are distinguished only by the kind of algebraic expressions to be transformed (expanded, or simplified). Absolute value inequalities appear as just one type of expressions to be processed.

In textbooks used in *College Algebra* courses, the topic of absolute value inequalities is usually divided into two types of tasks: solving inequalities $|ax + b| < c$ and $|ax + b| > c$ [e.g., (Martin-Gay, 2005)]. The technique for solving these inequalities is usually one we call "*PROP*", because it is based on certain "properties" that play the role of a technology in this praxeology. Theory is absent, which is common in many if not all North American college level mathematics courses [(Hardy, 2009), (Sierpinska & Hardy, 2010)]. Another technique exists in the less official praxeology of teaching the subject in North America, and whose traces we have found in students' work; we call it "Systematic Numerical Testing" (*SNT*). In this praxeology, the "theoretical block" (technology and theory) is absent altogether. We describe both techniques below.

The PROP technique

In (Martin-Gay, 2005), the notion of absolute value is introduced informally using the metaphor of distance and supplemented with the "property", "If a is a positive number, then $|X| = a$ is equivalent to $X = a \text{ or } X = -a$" (Property 0). There is no formal definition. Only the above mentioned two types of absolute value inequalities are considered. The *PROP* technique is based on two "properties": $|X| < a \Leftrightarrow -a < X < a$ (where $a > 0$), (Property 1) and $|X| > a \Leftrightarrow$

$X < -a \; or \; X > a$ (Property 2), that are not proved: they cannot be, because there is no definition to found the proof on. The technique is demonstrated on "worked out examples" for each type of inequality.

One of the examples is titled "Solve $|m - 6| < 2$":

> Solve $|m - 6| < 2$
> Replace *X* with *m* – 6 and *a* with 2 in the preceding property, and we see that $|m - 6| < 2$ is equivalent to $-2 < m - 6 < 2$.
> Solve this compound inequality for *m* by adding 6 to all three parts.
>
> $-2 < m - 6 < 2$
>
> $-2 + 6 < m - 6 + 6 < 2 + 6$ — Add 6 to all three parts
>
> $4 < m < 8$ — Simplify
>
> The solution set is $(4, 8)$ and its graph is shown [a representation of the interval on a number line follows] (Martin-Gay, 2005: 536)

In this solution, the notion of number as a measure endowed with a sign is sufficient and conditional reasoning is evacuated. Solving inequalities of the form $|ax + b| > c$ is taught similarly.

The technique conceals the logical symmetry of the two types of inequalities: there is conjunction of conditions in one inequality and disjunction in the other. If the "properties" on which the techniques are based were proved, or if the inequalities were solved using a definition of absolute value and *Reasoning by Cases*, a common pattern of reasoning would be revealed, using disjunction in both inequalities[viii], and this would open the way to generalization.

The SNT technique

Property 0 suggests solving inequalities $|X| < a$ and $|X| > a$ by first solving the equations $X = a$ and $X = -a$, obtaining two numbers and then deciding whether the solution of the inequality lies between these numbers or on either side of them. Without a definition of absolute value spelling out the conditions under which each of the possibilities $X = a$ and $X = -a$ occur, the decision is usually based on numerical testing. We called this technique, the *Systematic Numerical Testing* technique.[ix]

The technique "works well" in the sense of producing correct answers to the kinds of inequalities students are usually confronted with but it does not require students to engage in theoretical thinking. Students do not learn why this technique works and what assumptions about the functions involved in the inequality make it work. The technique does not apply to inequalities $f(x) < g(x)$ where the functions f and g do not intersect, which is the case, for example, of the inequality $|x + 3| \leq -3|x - 1|$[x], or in situations where the functions are not continuous[xi] or not defined everywhere on the real numbers[xii]. The technique is also useless for inequalities involving parameters.

DESIGN OF THREE LECTURES ON INEQUALITIES WITH ABSOLUTE VALUE

In our design of a lesson on absolute value inequalities, we tried to satisfy institutional constraints such as: (a) short lecture time allotted to the topic in the course (40-50 minutes); (b) lecture-and-assess format of interaction, and (c) showing the solution technique on an example. Some of the choices underlying the mathematical organization of the lectures have already been presented and justified above. In this section, we describe some more details of the three lectures[xiii].

For reasons presented in our analysis of the technique based on a certain property of absolute value (*PROP*) and the systematic numerical testing (*SNT*), neither was used as a condition in our experiment. We were aware, however, that the participating students' previous exposure to these techniques could influence their perception of the lectures in our study and understanding of the reasoning by cases techniques presented in them.

Overview of the PA, TA and VA approaches

Originally, we planned to design only two lectures. In one ("Theoretical approach", TA), the solution technique would be logically derived from the definition of absolute value: the complete *RBC*. In the other ("Procedural approach", PA) – the formal definition would still be given, but the solution technique – simplified *RBC* – would be presented as a sequence of steps to follow, like in the usual approach in prerequisite courses. In the design process, however, it occurred to us that the TA lecture was very artificial. It did satisfy the conditions of theoretical thinking, but it did not reflect the actual mathematical behavior of experts. Theoretical thinking is not the only kind of thinking involved in ordinary mathematical activity (Sierpinska, 2005). Therefore, we designed a third lecture, where the analytical technique of the TA was supplemented with a graphical technique ("Visual approach", VA). The VA lecture was faithful to the spirit of "economy of mathematical thought" (Castela, 2004). A mathematician usually has more than one technique to solve a class of problems and uses the least laborious one in a given situation. Using a different technique to solve the same problem may also serve as means to verifying the solution, supporting reflecting and proving.

The main difference between PA and the other two lectures was that PA made no explicit logical link between the definition of absolute value and the technique of solution. The main difference between TA and VA was that, in TA, logical analysis was presented as a means to getting an answer in an exercise, whereas in VA, logical analysis served to validate a result obtained in a visual way.

Outline of the lectures

All lectures introduced the notion of absolute value through a situation of evaluating the error of a measurement, leading to the idea of "magnitude of the difference between two numbers", and its generalization to the idea of "magnitude of a number", called its "absolute value". Beyond elementary school, "number" in mathematics courses usually refers to directed numbers, but even at the university level not all students may be aware of the distinction between this meaning and the everyday use of number as absolute measure. The aim therefore of the first part of the lecture was to bring about this awareness, by presenting students with the absurdity of speaking of measurement error in terms of directed numbers. In the introductory situation, two people, Jane and Joe, measure the length of an object. Jane's result is 55 mm and Joe's result is 58 mm. Another character, Tom, knows the true length of the object: 56 mm. He calculates the difference between the true length and the results of the measurement. He gets 1 for Jane and -2 for Joe, and claims that, since $1 > -2$, Jane made a bigger mistake than Joe. The lecturer ends the story with the question, "Do you agree with Tom?" Thus Tom stretches the notion of directed number to absurdity by calculating the error of a measurement as a *directed difference* between the exact value and the result of the measurement, and not as the magnitude of this difference. This reasoning makes the notion of magnitude of the difference between quantities – its "absolute value" in technical terms – sound like common sense. This was the intuitive basis of the notion of absolute value in the lectures.

This was followed by the piecewise linear function definition of absolute value. To avoid association between "–*a*" and negative numbers, the expression "opposite of the number" in reference to pairs such as "–2" and "–(–2)", or "2" and "–2" was used, instead of "negative two", commonly used by teachers and students.

Next we presented an illustration of an application of the formal definition: “We apply this definition to calculate absolute values of concrete numbers and to simplify expressions with letters involving absolute values”. There were two “examples”:

Example 1. To calculate the value of the expression $||-7+5|-|12\cdot 2-4||$.

Example 2. To find all numbers x such that $|x-1|<|x+2|$.

The solution of Example 1 was the same in all three lectures. The discussion of Example 2 in all lectures started by numerical testing of the inequality for $x=5$ and $x=-3$. The motivation given for proceeding to another way of solving the inequality was the impossibility of numerically checking it for all real numbers.

Lectures differed in the proposed solutions of Example 2. In TA, complete *RBC* (reasoning by cases) was presented as a direct application of the definition of absolute value to the expressions in the inequality, resulting in logical analysis of four possible cases. PA lecture used the simplified *RBC* technique, and the shortcut was not theoretically justified; the technique was presented as a sequence of steps to follow. VA presented the graphical and the complete *RBC* techniques. All lectures ended with the same set of exercises (Figure 2).

1. Calculate: $||16-24|-|6-56||$

In exercises 2- 6, solve the given inequality

2. $|x-1|<|x+1|$
3. $|x+3|<-3|x-1|$
4. $|2x-1|<5$
5. $|2x-1|>5$
6. $|50x-1|<|x+100|$

Figure 2. List of exercises at the end of the lectures

Justification of the choice of exercises

Our general purpose in the exercises was to see if students engage in theoretical thinking when dealing with the pragmatic task of processing expressions with absolute value that would not be all of the same type as the worked out example in the lecture. We chose the worked out example in the lectures and the exercises so that *PROP* (technique based on a property of absolute value) or *SNT* (systematic numerical testing) would not be directly applicable and students who learned them before would not be able to ignore the lectures. *PROP* applies directly only to exercises 4 and 5. *SNT* could be successfully applied to solving exercises 2, 4, 5 and 6 in our experiment, but applying it mechanically to exercise 3 was risky. Applying *SNT* a student would solve the equation $|x+3|=-3|x-1|$. Using Property 0 but forgetting about the assumption $a>0$, the student would solve $x + 3 = -3(x - 1)$ and $x + 3 = 3(x - 1)$, obtaining two "critical numbers", namely 0 and $\frac{3}{2}$, although, in fact, the functions on the left and right-hand side of the inequality do not intersect.

Exercise 1 was easy and we believed all will be able to do it; it was meant to appease any "test anxiety" that a participant may feel. A student unable to deal with this exercise would not possess even the Sign+*AMN* conception of number (measuring number endowed with a sign) and would thus not be fit to study absolute value inequalities. The second exercise was like Example 2 in the lecture. It was meant to help students understand the lectured technique better and learn it. It was also a test of the students' interpretation of the lecture.

The inequality in Exercise 3 was an obvious contradiction for anybody with an understanding that absolute value is a non-negative number, and able of grasping the structure of the inequality. Solving this exercise by mechanical application of the steps of the general technique could be a symptom of unreflective thinking.

Exercises 4 and 5 also addressed reflective thinking. The structure of the inequalities in these exercises was different from the inequality in Example 2 in the lecture and the method could not be applied without some adaptation, just by "following the steps". Moreover, these exercises were similar to those that participants could have learned to solve in their previous studies; therefore, students would have to deal with possible interference of previous knowledge. Lastly, in inequalities in exercises 4 and 5 only the direction of the inequality was different. It was possible to use the results of exercise 4 to solve exercise 5, if only the solver reflected back on the structures of the two inequalities.

The structure of the inequality in exercise 6 was similar to Example 2, but it was numerically more complicated, and not easy to solve using the graphical method. If solving all the previous inequalities was a process of learning the lectured techniques, students could test their understanding on this more challenging version of Example 2. For us, this inequality was mainly putting to the test students' analytic thinking. In VA, especially, students could avoid using analytic thinking until exercise 6.

Justification of the choice of presenting the solution technique on an example

All lectures demonstrated a solution technique on an example, which is usual in prerequisite mathematics courses. However, our lectures did not contain examples of all types of exercises students were then asked to solve. Only one example of inequality was solved, and students were expected to abstract the essential, generalizable elements of the example and then figure out solutions to other types of inequalities, using not only the solved example but also the definition. We made an effort to choose an "inductive example" [(Warnick, 2008), p. 34]. The process of abstraction that we hoped students would engage in was to consist in going back and forth between the practical activity of solving the exercises and the more theoretical reflection on the technique used, somewhat in the manner described in [(Ozmantar & Monaghan, 2007) (p. 93); see also (Monaghan & Ozmantar, 2006)].

Another reason for choosing to present the technique on an example was this: had we presented the method for solving an inequality of the type $|x-a|<|x-b|$ in general terms rather than on an example, we would have derived a formula for the solution ($x<\frac{a+b}{2}$ in case $a<b$ and $x>\frac{a+b}{2}$ in case $a>b$; no solution in case $a=b$). This formula would be good only to solve exercise 2. Students could be frustrated when seeing that they had not been given formulas for solving other exercises. Derivation of a formula could obliterate the students' interest in the process by which it was derived. By attending to the process, we thought, the students were better equipped to deal with inequalities not being of exactly the same type as the example in the lecture. We were leaving the responsibility of generalizing or adapting it to other types of inequalities to the students. The problem of solution of inequalities was thus left open to further investigation, inviting students to engage in reflective thinking.

RESEARCH PROCEDURES

We distributed about 600 invitations to participate in our research in classes of the prerequisite college-level mathematics courses in one university. We obtained an opportunistic sample of this population, as eighteen volunteers responded to our call. The volunteers were assigned to the approaches in a cyclic fashion: the first volunteer who responded was assigned to PA, the second to TA, the third to VA, the fourth to PA again, etc. This way, we could ensure that we have the same number of participants in each approach.

We had no control over the representativeness of our sample in terms of aspects such as age and gender distribution or the range of the prerequisite mathematics courses they were taking at the time of the experiment. Half of the participants were 21 years old or younger and 28% (5)

were between 22 and 25. The proportion is normally reversed in the prerequisite courses. Only four of the participants were female. At the time of the interviews, students were enrolled in elementary linear algebra and single-variable calculus courses, and all except four were taking these courses for admission into Computer Science or Engineering. The remaining four were applying for admission into a Business School.

In view of these methodological limitations, the strength of our study must be sought mainly in the *qualitative* analyses of the experiment, and not in the quantitative results. The quantitative results may serve, however, as a basis for useful conjectures.

Students were interviewed individually. After signing a consent form and responding to a short personal questionnaire, the student was asked to listen to and watch a prerecorded lecture (20-30 minutes) and mark, on paper copies of the slides, spots that were unclear or frustrating. Participants were told they will be able to ask for clarifications or share their comments after the lecture. Students could not interrupt the lecture. After the lecture, the interviewers (one or two of the authors) would ask the student to explain the marks. The interviewer could not use explanations borrowed from a different approach than the one experimented. Next, the student was given about 40 minutes to solve the exercises. The last part of the session (about 20-30 minutes) was an interview with the student mainly about the solutions. All conversations between the interviewers and the students were audio-recorded and then transcribed[xiv].

Analysis of data in this research was based on interpretation of students' written work and interviews with them. The interpretation was disciplined by (a) the adopted model of theoretical thinking, and (b) a *triangulation* procedure. We first tried to understand individual students' solutions of the exercises, based on their written work and interview transcripts. We would work independently first, and then meet to compare our interpretations and conjectures about how a given student might have been thinking, and to decide which behavior could count as symptom of a given aspect of theoretical thinking.

RESULTS

For the purpose of the presentation of the results, we have coded the participants using the acronym of the lecture they were given to listen (PA, TA or VA) and numbers 1 to 6, since, for each lecture, there were six students: PA-1,…PA-6, TA-1,…,TA-6, and VA-1,…,VA-6.

Participants' background

Participants' prior education regarding absolute value inequalities (AVI) and reasoning by cases (*RBC*), and knowledge of number and algebra could play a role in their performance. We did not pre-test the participants, but were able to obtain some information based on interviews and written solutions. We took into account also participants' grades in the mathematics courses they were taking at the time of the experiment.

Most students behaved as if their notion of number was directed number (*DN*), but one student in each group (PA-1, TA-3 and VA-1) appeared to believe that "$-x$" represents a negative number, suggesting they held the "absolute measure number endowed with a sign" (Sign+*AMN*) conception. The background notion of number therefore does not give any of the groups any advantage over the others.

Regarding basic algebraic skills, if we give 3 points to a student who is confident in his or her algebraic skills and makes no systematic mistakes in processing inequalities such as not changing the direction of the inequality when multiplying by a negative number ("Good" algebraic skills), 2 points to a confident student who does make such mistakes ("Medium") and 1 point to a student who lacks confidence and avoids doing algebra as much as possible ("Low"), then the sums of points for PA, TA and VA students are 14, 11 and 13 respectively. This puts the TA group at a disadvantage. VA students had lower algebraic skills than PA, but they could compensate by graphing.

A group with many students with low level understanding of variable – e.g. as a placeholder only (Küchemann, 1981) – would put the group at a disadvantage. Most students, however, used letters as variables in the sense of arbitrary element of a set when thinking about the relative values of two algebraic expressions in an interval, and as elements of a formal language (with no reference to anything outside the formal system) when they were processing the inequalities. Only one student (TA-5) used letters exclusively as placeholders. TA-5's solutions consisted in substituting some numbers into the inequalities and stating if the statement is true or false for that particular number. Sometimes she generalized from there, saying, e.g. "true for 1, 2 and all positive numbers", but not always. One student (VA-1) used letters exclusively as elements of a formal language. His solutions consisted in re-writing the inequalities without the absolute value brackets, and processing them until he obtained an expression with x on one side and a number on the other. In exercise 2, he left his solution at $x - 1 < x + 1$, saying he did not know what to do with it. Further simplification would lead to an expression without the letter x and this would not look like a solution to him. The notion of variable, therefore, gives the PA group an advantage over the other two, but it does not discriminate between VA and TA.

Another aspect of algebraic skills is the notion of inequality as a propositional function, whose truth value is conditional upon the values of the variable(s). This understanding is necessary to conceive of the solution of an inequality as the set of all values of the variable(s) for which the expression becomes a true statement. This understanding was implicit in all students except for TA-1, who could not understand why the lecturer was testing an inequality for various values of the variable. He was saying, "[this inequality] is a mathematical fact… so it must be true". He seemed to understand inequalities as formulas such as, e.g. a formula for the area of a triangle, or laws in physics. Again, the fact that there was a student with this kind of conception in TA puts this group at a disadvantage.

Four PA students remembered having studied absolute value inequalities (AVI) before; one, PA-2, remembered it well, three, PA-3, 4, 6, only very vaguely. All recalled being taught "solving by cases". In TA, three students (TA-2, 4 and 6) recalled having been taught AVI, but not with the technique presented in the lecture. All VA students remembered being taught AVI, but only one remembered it well and remembered he was taught the *RBC* technique presented in the lecture. Having seen AVI before gives some advantage to VA over PA, but this advantage is moderated by the fact that five VA students have never seen *RBC* before, while three PA students did. Moreover, only one student in each group remembered this knowledge well enough to use it in solving the exercises. TA group appeared disadvantaged in this respect as well, relative to both PA and VA.

The averages of students' average grades in mathematics courses taken at the time of the experiments in groups PA, TA and VA were, respectively, about 80%, 68% and 72% (more details in the next section). In principle, therefore, the PA group was mathematically the "best" to start with, while VA was medium. TA, again, was disadvantaged. As we will see in the next section, average grades in mathematics courses were not a good predictor of the "winner in the competition" between PA and VA. TA's performance was the lowest, but this could be predicted not only based on the grades, but also on all the other factors mentioned above.

On the seven factors we took into account in students' backgrounds, the PA group had strict advantage over the other two on four factors, and was better or equal to both or VA in the other three cases. TA was disadvantaged relative to both PA and VA in 5 cases. In general, therefore, we could say that, relative to their background, PA had advantage over the other groups, TA was disadvantaged and VA was in the middle.

PARTICIPANTS' PERFORMANCE ON THE EXERCISES

Participants' overall performance on exercises is presented in Table 1[xv]. The VA group produced more correct solutions than the other groups, and PA did better than TA. Table 1 also contains information about students' performance in mathematics courses they were taking at the time of the interviews.

Only three out of the eighteen participants obtained correct answers in all exercises: PA-6, VA-4 and VA-5. All three used techniques presented in the lectures. PA-6 used the simplified *RBC* technique. VA-4 used the complete *RBC* technique in exercises 2, 3 and 6, and *PROP* in exercise 4 and 5. VA-5 used *RBC* in exercises 2 and 6, *PROP* in 4 and 5, started exercise 3 with *RBC* but interrupted the process and wrote a structural proof. He supported and controlled his thinking with rough graphical sketches.

The VA group did not perform better because they were "better students", based on their achievement in the mathematics courses they were taking at the time of the interviews (Table 1). VA also did not perform better because more VA than PA participants have already studied absolute value inequalities, or even *RBC*, and used it to solve at least exercises 2 and 6. In fact, more PA students than VA students have seen *RBC* before and tried to use it in their solutions. Only one TA student has already seen *RBC* before, but this did not help him to solve exercises 2 and 6 correctly.

Table 1. Participants' performance on the exercises and in mathematics courses

Student	Ex.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Sum	Average (in %) on exercises	Average grade (in %) in Math courses
PA-1	1 (1)	1	0 (2)	1	1	0	4	67	70
PA-2	1	0	1	0	1	1	4	67	95
PA-3	1	0	0	0	0	0	1	17	66
PA-4	1	0	1	0	0	0	2	33	73
PA-5	1	1	0	1	1	0	4	67	79
PA-6	1	1	1	1	1	1	6	100	95
PA group	**6**	**3**	**3**	**3**	**4**	**2**	**21**	**58.5**	**79.7**
TA-1	1	1	0	0	0	0	2	33	87
TA-2	1	0	1	1	1	0	4	67	48
TA-3	1	0	0	1	0	0	2	33	62
TA-4	1	1	1	1	1	0	5	83	85
TA-5	1	0	0	0	0	0	1	17	47
TA-6	1	0	1	1	0	0	3	50	80
TA group	**6**	**2**	**3**	**4**	**2**	**0**	**17**	**47.2**	**68.2**
VA-1	0	0	0	0	0	0	0	0	No rec.
VA-2	0	1	1	1	1	0	4	67	30
VA-3	1	1	1	1	1	0	5	83	82
VA-4	1	1	1	1	1	1	6	100	89
VA-5	1	1	1	1	1	1	6	100	No rec.
VA-6	1	1	0	1	1	1	5	83	95
VA group	**4**	**5**	**4**	**5**	**5**	**3**	**26**	**72.2**	See Note (3)

Notes: (1) "1" stands for correct answer. (2) "0" stands for incorrect answer. (3) We had no record for VA-1 and VA-5. But it was clear from the interview that VA-1 was a rather poor student, and VA-5 appeared to be good. Assuming a generous 60% average for VA-1 and a modest 75% for VA-5, the average grade for the VA group would be 71.8%.

Familiarity with reasoning by cases, as in general, having seen a technique before, does not necessarily make it easier to re-learn it, as is well known. PA-3 and 4, for example, had very vague memories of the *RBC* technique. Both must have studied the complete technique, since they distinguished four cases and not three as in the PA lecture. Their "cases" made little sense, however, from the point of the given definition of absolute value[xvi]. VA students who had studied *RBC* before appeared to make better use of their previous experience. We could conjecture that if students re-learn a technique within a praxeology with explicit theoretical block (as in VA) then the recall of that technique is more accurate, and leads to a consolidation and refinement of previous knowledge.

In analyzing students' solutions and interviews, we identified behaviors that could be interpreted as symptoms of theoretical thinking ("positive behaviors") or of deficiency in theoretical thinking ("negative behaviors"). We describe the behaviors in the following sections.

A rough measure of theoretical thinking performance was obtained by assigning numbers to the behaviors. If a positive behavior appeared in a participant, the participant was assigned 1; if a negative behavior appeared, the participant was assigned -1; if the behavior did not occur, the participant was assigned 0. The totals for each group represented a rough measure of the group's overall theoretical thinking performance. It was not necessary to divide the total by the number of participants in each group for comparison, because the numbers of participants in the groups were the same. The totals obtained were: –7 for the PA group, 7 for TA, and 15 for VA.

Reflective thinking

We evaluated students' "reflective thinking" by distinguishing between behaviors that we called, in reference to (Belenky, Clinchy, Goldberger, & Tarule, 1997), procedural (*PB*) and constructed (*CB*). We define the behaviors in Table 2.

Table 2. Procedural and constructed knowing behaviors

Procedural behaviors	Constructed behaviors
[***PB1***] Solving exercise 3 by following the same technique as in exercise 2, without noticing that it could be solved faster in a different way.	[***CB1***] Noticing that it was not necessary to solve exercise 3 using the same technique as in exercise 2, and finding a faster way.
[***PB2***] In solving exercise 5, repeating actions already performed in exercise 4, e.g. the solution of the equation $2x - 1 = 0$ or of the equation $2x - 1 = 5$, after having already done it in exercise 4.	[***CB2***] Noticing a relationship between ex. 4 and 5 and using elements of the solution of exercise 4 to solve exercise 5 in writing or mentioning the possibility orally in the interview.
[***PB3***] Including certain elements of the lecture in a solution (e.g. a definition of absolute value applied to the expressions in the given inequality; a number line diagram such as in PA; a graph of the absolute value functions as in VA; an interval condition) but then not using these elements in finding the answer; the elements appear to play a "ceremonial" or "ritualistic" role only.	[***CB3***] All elements included in the written solution are used in the solution.

Table 3 presents individual and group measures of students' reflective thinking based on *PB* and *CB*. Some students displayed both procedural and constructed behaviors in the same exercise: written solution was procedural, but an alternative structural solution was presented in the interview. For example, PA-2 used *RBC* to solve exercise 3 in writing, but when discussing exercise 3, he said, "while I checked it, I [thought], maybe to calculate it is a waste of time…. We can see this is a false condition; absolute value is always greater or equal zero but this number [–3] is always less than zero, so this is false".

In the interviews, students displayed additional positive or negative symptoms of reflective thinking, but this depended on individual characteristics of the interviewer-student interactions, and could not be used as part of a common measure. In most cases those additional symptoms only confirmed the profile already inferred from the *PB* and *CB* analysis of their solutions.

In total, PA students were more likely to display procedural than constructed behaviors. In the TA group, procedural and constructed behaviors almost balanced each other, and VA students were more than twice as inclined to engage in constructed than in procedural behaviors.

Table 3. Procedural and constructed behaviors in PA, TA and VA groups.

Student	***PK1***	***PK2***	***PK3***	**PK-total**	***CK1***	***CK2***	***CK3***	**CK-total**	***PK+CK***
PA-1	–1[1]	–1	–1	–3	0	0	0	0	–3
PA-2	–1	–1	0[2]	–2	1[3]	0	1	2	0
PA-3	–1	0	–1	–2	0	0	0	0	–2
PA-4	–1	–1	–1	–3	1	0	0	1	–2
PA-5	–1	–1	–1	–3	0	1	0	1	–2
PA-6	–1	–1	0	–2	1	1	1	3	1
Total PA	**–6**	**–5**	**–4**	**–15**	**3**	**2**	**2**	**7**	**–8**
TA-1	–1	–1	0	–2	0	0	1	1	–1
TA-2	0	0	0	0	1	1	1	3	3
TA-3	–1	–1	–1	–3	0	0	0	0	–3
TA-4	0	0	0	0	1	1	1	3	3
TA-5	–1	0	0	–1	0	0	1	1	0
TA-6	–1	–1	0	–2	0	0	1	1	–1
Total TA	**–4**	**–3**	**–1**	**–8**	**2**	**2**	**5**	**9**	**1**
VA-1	–1	–1	–1	–3	0	0	0	0	–3
VA-2	0	0	0	0	1	1	1	3	3
VA-3	0	0	0	0	1	1	1	3	3
VA-4	–1	0	0	–1	0	0	1	1	0
VA-5	0	0	0	0	1	0	1	2	2
VA-6	–1	0	0	–1	0	1	1	2	1
Total VA	**–3**	**–1**	**–1**	**–5**	**3**	**3**	**5**	**11**	**6**

Note 1: "–1" = student displayed a *PB*. Note 2: "0" = student did not display the behavior. Note 3: "1" = student displayed a *CB*.

These results led us to the following conjectures: no explicit logical links between definitions and techniques (as in PA) encourages *PB* rather than *CB*. Making the logical links carefully and precisely in a lecture (as in TA) may raise students' awareness of the logical structure of mathematics, but students may still feel restrained in their thinking: knowing there is a logic behind the techniques, but lacking the confidence to adapt it to one's needs. Using logical analysis as a tool in validating a result obtained by graphical means (as in VA) may encourage *CB* more effectively, but the development and consolidation of confidence in using analytic tools for validation purposes requires engaging students in more tasks where the graphical means alone are not sufficiently reliable (as was the case of exercise 6).

Systemic thinking

In this section, we report on students' defining, proving and hypothetical thinking behaviors.

We treated as positive symptoms of definitional thinking the following behaviors:

DT-w: Student's written solution of at least one inequality contains an explicit application of the formal definition to the absolute value expressions

DT-o: Student refers to formal definition in oral reasoning.

Our main concern about students in prerequisite mathematics courses was that many of them "need the teacher to tell them if they are right or wrong" (Sierpinska, Bobos, & Knipping, 2007). Therefore, we focused on behaviors suggesting students' autonomy with respect to the correctness of their solutions, and treated them as positive symptom of proving. We distinguished these behaviors by the mathematical means students used to reduce their uncertainty with respect to their solutions, obtained otherwise than by these means:

P-numerical testing: plugging numbers into the initial inequality;

P-graphing-physical: graphing on paper

P-graphing-mental: mentally visualizing a graph

P-structural: reasoning focused on structural properties of the inequality, and the functions involved in it

P-RBC: using *RBC*

We characterized hypothetical thinking by one negative and one positive behavior:

HT-interval-conditions-not-taken: Written solution shows intention to use *RBC* but does not take into account the interval conditions when analyzing cases

HT-conditional-statements: Uses conditional statements (some form of "if… then" statements) in discussing his/her solutions

Students' systemic behaviors are represented in Table 4.

Table 4. Systemic behaviors

Student	Definitional Thinking		Proving					HT		TOTAL
	DT-w	DT-o	P-num	P-gr-physical	P-gr-mental	P-struc	P-RBC	HT-ic-not	HT-condit.	
PA-1	0	0	0	0	0	0	0	–1	1	**0**
PA-2	1	0	0	1	0	1	0	0	0	**3**
PA-3	0	0	0	0	0	0	0	–1	0	**–1**
PA-4	0	0	0	0	0	1	0	–1	0	**0**
PA-5	0	0	0	0	0	0	0	–1	1	**0**
PA-6	0	1	1	0	0	1	0	0	0	**3**
PA-total	**1**	**1**	**1**	**1**	**0**	**3**	**0**	**–4**	**2**	**5**
TA-1	1	0	0	0	0	0	0	0	0	**1**
TA-2	1	0	1	0	0	0	0	0	1	**3**
TA-3	1	0	1	0	0	0	0	–1	1	**2**
TA-4	0	1	0	0	0	1	0	0	1	**3**
TA-5	0	0	0	0	0	0	0	0	0	**0**
TA-6	1	0	0	0	0	0	0	0	0	**1**
TA-total	**4**	**1**	**2**	**0**	**0**	**1**	**0**	**–1**	**3**	**10**
VA-1	0	0	0	0	0	0	0	0	0	**0**
VA-2	0	0	0	0	1	1	0	0	0	**2**
VA-3	1	1	1	1	0	1	0	0	0	**5**
VA-4	0	0	1	0	0	0	0	0	0	**1**
VA-5	0	0	1	1	0	0	0	0	0	**2**
VA-6	0	0	0	0	0	0	1	0	0	**1**
VA-total	**1**	**1**	**3**	**2**	**1**	**2**	**1**	**0**	**0**	**11**

We give details of students' definitional, proving and hypothetical thinking behaviors in the sections below.

Definitional thinking

In all but three cases of DT-w, applications of the definition had the standard form. The exceptions were TA-2, TA-3 and VA-3[xvii].

VA-3 said he did not understand the analytical technique in the lecture. He used graphing in exercises 2 and 3, and systematic numerical testing to solve 4 and 5. He tried *RBC* in exercise 6. In applying the definition, he wrote:

$$|50x - 1| = 50x - 1 \; for \; 50x - 1 > 0$$
$$|50x - 1| = -50x + 1 \; for \; 50x - 1 < 0$$

This shows he understood the definition correctly (although he missed the value for $x = \frac{1}{50}$).

TA-2 and TA-3 also struggled with *RBC* but ended up showing an understanding of the definition. For example, TA-3 was writing expressions such as

$$2x - 1 \geq 0 \implies 2x - 1, \qquad 2x - 1 < 0 \Rightarrow -(2x - 1)$$

in exercise 4. This is clearly intended as an application of the definition, but the syntax is inaccurate, and the statements would be incomprehensible if we did not know the context. This student achieved an understanding of the definition after an initial struggle with his association of the negative sign with negative numbers:

> … here it says 'in general the absolute value of a number *x* is equal to *x* for *x* greater than zero, and negative *x* for *x* less than zero'…. You see that's a little hard to understand, you know, like if it was negative *x.* [You'd think that $-x$] it's going to get a negative value. OK, now if you just take the place of that [position on the number line] it makes sense but just after [seeing it for] the first time… (TA-3, interview after viewing the lecture).

In TA and VA lectures, the model solution of Example 2 based on the complete *RBC* technique contained an explicit application of the formal definition to the absolute value expressions. A TA or VA student who tried to apply this technique in an exercise and included explicit application of the definition in the written solution might have been merely following the model solution rather than engaging in definitional thinking. However, TA-2's, TA-3', and VA-3's struggles with the definition (crowned with success), removed our doubts about their definitional thinking. We were less sure about TA-1 and TA-6, because both displayed two procedural and no constructed behaviors. We were especially doubtful about definitional thinking in TA-1, because he also seemed to treat *RBC* as an exercise in processing expressions of a formal language. He had trouble understanding expressions with variables, without plugging in some concrete numbers into them and seeing what they mean this way. He seemed to be saying that numbers can be positive or negative, but for letters it doesn't even make sense to talk about being positive or negative in conditional terms[xviii]. This could have made the formal definition meaningless for him. Numerical testing would be more suitable to his way of thinking than reasoning by definition on which *RBC* was based. Indeed, starting from exercise 3, he would begin with *RBC,* writing and simplifying the "cases" but falling short of later combining them by disjunction, and replacing this step by numerical testing of the results of simplification of cases. The last step of the *RBC* technique appears natural if the formal definition is understood as a disjunction of two possibilities. For TA-1, however, this step was meaningless and he reverted to what was natural for him, namely numerical testing. His technique was a combination of *SNT* and *RBC*, consisting in "testing" the cases somewhat like the intervals marked by the critical numbers in *SNT*. The cases were tested, first, for internal consistency (if there was an inconsistency, the case was rejected), and, if the case was internally consistent, it was tested numerically (a number satisfying the condition was plugged into the given inequality; if the result was true, the case was kept; if not – it was rejected). If only one case was left, its result was stated as the answer. Accidentally, TA-1's solutions always ended with only one consistent case that also satisfied the inequality (because of algebraic mistakes).

In spite of these doubts, we decided to keep TA-1 in our list of students who displayed the DT-w behavior (definitional thinking expressed in writing), because he did display this behavior. To remove just this one student from the list, we would have to make our operationalization of Definitional Thinking much more complicated.

The cases of DT-o (definitional thinking expressed orally) were less doubtful as symptoms of definitional thinking. References to the definition obviously occurred as parts of mathematical reasoning. The deepest reflection on the definition was found in TA-4. When listening to the lecture, the student "found [the definition] a bit annoying". He said he was thinking, "when I see absolute value, I say, oh yeah, sure, I know that! So, why all this! This looks complicated, why get so nasty on absolute value?" Later on, however, when solving exercise 4 (using systematic numerical testing), he said he recalled the definition, and solved not only the equation $2x - 1 = 5$ but also $-(2x - 1) = 5$:

> **Interviewer**: In Exercise 4, you not only solve the equation $2x - 1 = 5$, but also the equation $-(2x - 1) = 5$. Why?
> **TA-4**. Well, when I checked that, as I said, I found it [the definition] a bit annoying, but, I observed that, OK, here we negate, and quickly reminded, when studying the absolute value function and its notation, that (pause), because it's tricky, when we have the positive and the negative, because *the negative somehow becomes positive* and I wanted to check here my 3, if I put here -3, what's going on (pause)…. Because here, well, in fact, it's like an equivalent of the absolute value, because if I have -3, here it will give me -4, -5; -5 is not equal to 5, but with the addition of the negative sign, the result will become positive.

TA-4 shows here an understanding of the role that the minus sign plays in the second part of the definition: turning the negative into the positive.

Students who did not display DT-w or DT-o also sometimes mentioned the definition but not as part of a reasoning. Rather, they would speak about their difficulty understanding it, without efforts of overcoming the difficulty. For example, PA-5 complained about the lecture not giving enough explanation of the definition: “It says it’s negative x but there is no explanation for it”. She did not apply the definition in her solutions. Rather, she was processing the inequalities formally, “putting pluses and minuses in front” of the expressions within absolute value brackets.

Proving

Altogether three PA (PA-2, 4, 6), three TA (TA 2, 3, 4) and five VA (all except VA-1) students engaged in one or more of the “proving behaviors” used to characterize this aspect of their thinking. Numerical testing and structural reasoning were the most popular behaviors: six students used each. Graphing was next, with four students: one PA student (PA-2 who learned the graphing method in previous studies, not in the lecture) and three VA students (VA-2, VA-3, VA-5). Only one student used *RBC* to check his answer: VA-6 first solved exercise 2 by graphing and then checked his solution by *RBC*.

Of the three VA students who used graphing as means of checking their solutions, VA-2 only imagined the graph in exercise 4 to make sure his solution (obtained by numerical testing) was correct. He used actual graphing as solution technique in exercises 2 and 3. In exercise 4, he found, by numerical testing, that –2 and 3 “evaluate 5” (make the function $|2x - 1|$ equal to 5) and, by imagining the graph of the function and of the constant 5, he assured himself that the solution set will be between these two numbers.

Many students were using numerical testing in the experiment, but not always for the purpose of verifying their answers. It was used for obtaining a solution in all or some exercises by seven students (PA-3, PA-4, PA-5, TA-1, TA-4, TA-5, and VA-2). In PA-4, PA-5, and TA-1 numerical testing was part of a combination of *RBC* with *SNT*.

Structural reasoning as means of control was used mainly in exercise 3 (PA-2, 4, 6; VA-2, 3). Only one student – TA-4 – used this type of reasoning extensively in other exercises as well. This proving behavior pervaded all TA-4’s thinking as he explained his solutions to us in the interview, which surprised us because his written solutions, based on numerical testing, appeared unsophisticated. The interview revealed that he was very concerned about the validity of his solutions, and, since he used numerical testing as a solution method, he needed other means to reduce his uncertainty. He used reasoning that took into account the structural characteristics of the given inequality and the functions involved in it. For example, he said, about his solving of exercise 6:

> So here I tried a method similar to number 2, trial and error. So if it's 0, it doesn't tell us much because -1, yeah, -1 is smaller than 100, but here, if we look quickly, 100 here, so how do we get 100 there? So, it's using 2. OK, so we might look at 2, what, what's happening, so $100 - 1 = 99$; and here, $2 + 100 = 102$. OK. But if I use something bigger, because this here $[x + 100]$, if we write it as a function, *it's evolving slowly* because of 1, each time we have one to the x, but here $[50x - 1]$,… we have 50, so we must be careful, because it *gets quickly bigger*.(TA-4, interview)

Reasoning focused on the structure of the inequality as a means of control of his solutions led him also to find a structural proof of exercise 3:

> And here [in exercise 3] I was quite puzzled, because I tested. I tested for 0, for a negative and for a positive and I observed, well, this, because this will be positive because it is absolute value, multiplied by a negative, it gives only a negative, and here it gives only a positive. So it is impossible that this is smaller than… (TA-4, interview)

We decided that structural reasoning was used in the service of verification of solution and not as a solution method also for five other students (PA-2, 4 and 6; VA-2, 3), although it was not always obvious. For example, PA-2 and PA-6 presented the argument more as a *procedural shortcut* than a verification tool, although this argument certainly reassured them about the validity of the solution they obtained by *RBC* (which was correct in both cases).

Only two students used structural reasoning as a solution method in exercise 3, and not as a verification tool: TA-2 and VA-5. They both struggled with understanding *RBC*, confused with the logical connectives involved in combining the cases. The structural argument appeared to liberate them from the complications of the *RBC* procedure; mid-way through their *RBC* solutions, they noticed that there exists *a theoretical shortcut* and interrupted the tedious method.

We found one borderline behavior which we did not count as a symptom of proving but which was quite interesting. PA-5 solved the exercises by following the demonstrated procedure and she was not checking her solutions in any of the ways we took into account, but she had the habit of justifying her particular actions by reference to algebraic rules.

> I had to get x alone, divide by –2 and every time you divide by negative you switch the sign around. And because anything with numerator 0 ends up being 0, you get $x > 0$. (PA-5, interview)

This may not qualify as proving or theoretical thinking yet, but, her personal praxeology contains a partial theoretical block: there is a "technology" in her praxeology.

Hypothetical thinking

The piecewise linear function definition of absolute value implies two conditional statements: if $x \geq 0$ then $|x| = x$, and, if $x < 0$ then $|x| = -x$. Understanding this definition, as well as understanding the *RBC* technique which is based on it, involves therefore an awareness of the conditional character of mathematical statements, which is an essential characteristic of what we have called "hypothetical thinking". Ignoring the assumptions about the positive or negative values of expressions within the absolute value brackets (the "interval conditions") in processing the cases were, therefore, negative symptoms of hypothetical thinking. We found this behavior in four PA students and one TA student. VA students did not make this mistake, but they could avoid making the mistake by choosing the graphical method. Therefore the ratio of those who did not take into account the interval conditions to those who used *RBC* could be a better indicator of negative hypothetical thinking behavior in groups. The ratios were, 4 : 6 in PA, 1 : 4 in TA and 0 : 4 in VA. Even with this more careful indicator, the VA group did better than the other two, and TA did better than PA. The gap between TA and PA only appears smaller than with just the count of number of students who made the mistake in each group.

We looked for positive symptoms of hypothetical thinking in the interviews, and it appeared that using conditional statements in mathematical reasonings, or discussing the conditions for something to be true, not just once, but repeatedly, say, at least three times, could be a good indicator of hypothetical thinking. We took the threshold of three times, because this would account for making conditional statements systematically in solving at least one of the exercises 2 and 6, where using *RBC* was most likely.

We found traces of this positive behavior in the discourses of two PA students, three TA students and no VA students. The PA students and one of the TA students used such statements only a minimal number of times. In particular, PA-1 made conditional statements only in explaining her reasoning in exercise 2, and even here they were not exactly explicit and consciously used *qua* conditional statements. She may have been using the interval conditions as mere labels for the cases, formally, just as she was formally "putting negative in front" rather than taking the opposite of an expression in intervals where it had negative value:

> I put $x - 1 = 0$, $x + 1= 0$, so $x = 1$, $x = -1$, and then I put it in a chart and I solved if x is smaller than -1, if it's in between and [if it's] greater than 1. And I solved it in these three cases. In the first case, [then] both are negative, which means I put negative in front, before the bracket and I came out with that. (PA-interview about exercise 2)

The most frequent use of conditional statements was found in TA-4's discourse. Hypothetical thinking seemed to go hand in hand with his pervasive proving behavior in the interview.

> Here, I found it a bit easier, because you don't have, like, a variable with a constant term, and then a variable and another constant term, so here I solve it as an equation, *as if it were* [hypothetically], an equality and I observe, if we have three, so $6 - 1$, 5, so it's equal, so it's not what we are searching. Or, if this is set up as a negative, it's -2, so it's $-4 - 1, -5$, so 5 and it's also equal. And if we try something between these -2 and 3, but excluding the specific value of -2 and 3, like 0, 1, and 1 is smaller than 5. But anything below or above, it will make something that will be bigger than 5. (TA-4, explaining his thinking in exercise 4)

His hypothetical thinking went beyond the technical use of *if… then* clauses, as it could be in an application of the *RBC* technique; he was obviously thinking in terms of what happens if we assume this or that, or if we proceed as if the object we are dealing with had this or that characteristic.

To conclude, we can say that PA did not perform well on systemic thinking compared with the other two groups, but also notice that VA did not score much better than TA on this aspect of theoretical thinking.

Analytical thinking

The context of inequalities with absolute value certainly puts to the test students' linguistic and logical sensitivities. *RBC* is based on a disjunction of several sets of conjunctions. Combining results of analysis of cases by disjunction follows from the fact that the definition of absolute value is logically equivalent to $(x \geq 0 \; and \; |x| = x) \; or \; (x < 0 \; and \; |x| = -x)$. This can be proved formally, but we rarely do that in teaching, hoping that students' logical sensitivity will be enough for them to grasp this equivalence intuitively and use conjunctions and disjunctions correctly when applying *RBC*. This is not what happened in the experiment.

Students' weakness in analytical thinking related with using *RBC* was revealed in behaviors such as:

A-cases-listed: Not knowing how to combine the cases in at least one exercise and leaving the cases uncombined (PA-3, TA-2; 3, VA-5)

A-combining-replaced: Replacing the step of combining the cases by another operation, e.g. numerical testing of the cases, in some exercises (PA-4; 5, TA-1: the already mentioned students who appeared to confuse *RBC* with *SNT* and their techniques were a combination of the two)

A-combining-by-conjunction: Combining the cases by conjunction in some exercises (PA-1, TA-2; 3, TA-6)

A-combining-inconsistent: Inconsistent use of logical connectives within a single exercise (TA-2; 3, VA-6)

Thus four PA students, four TA students and two VA students displayed at least one of the above-listed behaviors. The ratio of students showing these weaknesses to those that used *RBC* in their solutions were, therefore, 4 : 6 in PA, 4 : 4 in TA and 2 : 4 in VA. The worst performance was thus in the TA group and the best in VA, although the difference between VA and PA is not big. In our global evaluation of students' theoretical thinking behaviors, we will not use the relative indicators, but the straightforward count of negative analytical thinking behaviors in each group, that is, –4 in PA, –4 in TA, and –2 in VA.

TA students were aware of the technical character of the words "and" and "or" but did not help them in understanding their correct use in solving the inequalities. They appeared to have lost faith in their intuitive thinking and became confused about the formal one. Unaware of the technical character of the two logical connectives, PA students used what they thought made sense in each case. VA students were aware of the special status of the words, and did experience confusion but dealt more successfully with it than TA students. When unsure which logical operation to use, they would use numerical testing, or visualizing the graphs (VA-5). VA-6, whose use of logical connectives was quite erratic in exercise 3, managed to figure out their proper use by exercise 6.

Summary table: Overall group theoretical thinking performance

By adding up the measures obtained in analyzing symptoms of students' reflective, systemic and analytic thinking, we obtained a rough measure of the "overall group theoretical thinking performance" in our experiment. (Table 5).

Table 5. Overall group theoretical thinking performance

Aspects of Theoretical Thinking	**PA**	**TA**	**VA**
REFLECTIVE	–8	1	6
SYSTEMIC	5	10	11
ANALYTIC	–4	–4	–2
Totals	**–7**	**7**	**15**

We note VA's higher theoretical thinking performance overall than the other groups, and TA's higher performance than PA, in spite of PA's background advantage, and TA's disadvantage at the start.

CONCLUSIONS

In our research we were looking for characteristics of teaching absolute value inequalities that would promote students' theoretical thinking about the topic. Keeping in mind the fact that the experiment was conducted on only 18 students, we can offer only a few cautious conjectures.

The VA approach, which contained not only a theoretical justification of the technique it presented (as TA), but also met "the current standards of mathematical practice" (Balacheff, 2010, pp. 129-130) by offering an economical alternative technique for solving simpler inequalities, emerged victorious among the three groups of six students. Compared with other groups, VA students averaged better in obtaining correct answers; were more likely to reflect on a problem, noticing relationships with other problems and possibilities of reasoning out the solution conceptually without applying a general procedural technique; less likely than PA to engage in "ritualistic" behaviors. They were more likely to verify their answers and did so in a larger variety of ways; only in VA did a student use an analytical technique (*RBC*) to check a result. They were less likely to forget or remain unaware of the conditional character of the absolute value definition and the implications it has for reasoning about absolute value inequalities. The support of graphical visualization, whether physical or mental, allowed some of them to grasp the relationship represented in an inequality more globally, without having to rely on accurate application of a formal processing of algebraic expressions, which appeared to be the sole support for some PA and TA students. TA students often controlled their formal processing by the logical links between the *RBC* technique and the definition. PA students appeared to rely on their memory or tried to follow the example in the lecture as best they could. While linguistic and logical sensitivity was raised in both TA and VA students, and made them confused about the correct use of "and" and "or" in reasoning, VA students were more successful in dealing with the uncertainty.

What relevance for a better performance of VA students could there be in the fact that one of the two techniques in the lecture was graphical? In a research on absolute value equations and inequalities reported in (Chiarugi, Fracassina, & Furinghetti, 1990), students performed best on tasks set in a geometric context. Educators seem to believe in the appeal of the geometric context even without any systematic research and this is perhaps what motivates the introduction of absolute value visually through the metaphor of distance. This approach was highly advocated in the 1989 reforms of mathematics teaching in France. Perrin-Glorian (1995) reported, however, that while students following the reformed program did very well (80% success rate) on inequalities of the type $|ax + b| < c$ and $|ax + b| > c$, they did not do so well on more complex tasks. Perrin-Glorian conjectured that the geometric introduction made students rely too much on visualization which is not operational when the problem is numerically or algebraically more complicated. The behavior of VA students in our research corroborates this conjecture; without exercise 6, several VA students would not have taken up the challenges of systemic and analytic thinking.

A priori, PA did not deprive students of a chance to engage with theoretical thinking. The lecture contained a definition of absolute value and the given technique was only a small shortcut away from direct application of the definition. No one in the PA group, however, asked why the technique required finding the zeros of the absolute value expressions, and analyzing the inequality within the intervals determined by these numbers. They were interested in knowing precisely *what* to do in each step, and not in knowing *why* the step was there. This attitude could not lead to systemic thinking.

Several students in the PA group have already seen absolute value inequalities before and some have even seen reasoning by cases techniques. Those, however, who displayed few aspects of theoretical thinking, were not successful on the exercises. They could not reconstruct their previous knowledge; they remembered only scraps of it and these scraps interfered with their understanding of the technique presented in the lecture. In general, students enrolled in prerequisite mathematics courses rarely see absolute value and absolute value inequalities for the first time. They may only not have been successful the first time around. They may have developed habits of thought and conceptions that were responsible for their failure. We conjecture that if the topic is taught the same way they learned it before, with the same type of tasks, same techniques and no theory, their mathematical behavior will fall into the same routine and they will repeat their mistakes. It seems that expanding both the range of inequalities and of the techniques to deal with them, and including a theoretical justification of these techniques gives these students a chance to become aware of the shortcomings of their previous ways of thinking and overcome some of the causes of their previous lack of success in mathematics.

The main difference between VA and the other two approaches was that VA students had been shown two techniques of solving absolute value inequalities. This might have encouraged them to reflect on the most appropriate technique to use when starting an exercise – an activity that already belongs to theoretical thinking. With one technique only shown in both PA and TA, this reflection moment was not suggested. Based on their experience in prerequisite mathematics courses, students could think that the exercises are meant for practicing just this one demonstrated technique. In TA, the technique was logically derived from the definition of absolute value. TA students who decided to use reasoning by cases appeared to mind this link the first time they tried to apply it, but we noticed that repetition of the technique in five exercises led to gradual detachment from it. Theoretical thinking can only occur as a result of choice among several possible ways of thinking; if there is no choice, thinking becomes procedural or unreflective. Thus, eventually, TA students could start behaving like PA students, omitting certain essential elements of the technique in their solutions (e.g. initial assumptions, such as the interval conditions), confusing reasoning by cases with other techniques, and, by not resolving their uncertainty about conjunctions and disjunctions, start using them erratically, and perhaps even stop perceiving their use as an issue. The VA student who decided to completely ignore the graphical method and used only the complete reasoning by cases technique in all inequalities (VA-4) also displayed signs of unreflective behavior. Although he was quite sure of his mastery of the technique, he did not stop to see if he could use a shortcut in a particular task, or if the technique could be simplified somehow. The case of student TA-4, on the other hand, shows the power of theoretical thinking; a deep reflection on the definition of absolute value given in the lecture, together with the proving and hypothesizing activity which pervaded all of his thinking, compensated for his lack of algebraic skills and allowed him to solve four inequalities accurately and find a good approximation in the fifth one. TA-4's theoretical thinking activity could perhaps be partly attributed to the theoretical flavor of the TA lecture, but probably in the sense that it triggered into action a habit developed in his Human Sciences program in college where, he recalled, the meaning of concepts was constantly debated and the validity of arguments had to be defended against opposite views.

Discussions over the meaning of a concept or the choice of a technique are rare in the prerequisite mathematics courses, where students are usually presented with a single technique to solve a type of exercises. TA-4 complained about it in the interview:

> **Interviewer**: About the courses you are taking, you said [your experience is] "mildly enjoyable". Why not "very"?
> **TA-4**: Well, I've never seen a course in math… where you would have passions and hot debates of issues and things. It's just, 'sit down, this is the way it works'. OK, perhaps in some six hundred level courses of math [master's level], in seminars, but no, not in the two hundred [prerequisite courses][1], no, no, you have the textbook, you have the teacher, there might be an error, but still, there is no point in debating the theory. Of course, you can debate with yourself, to understand, yeah, but…

This research might encourage instructors of prerequisite mathematics courses to adopt approaches where solution techniques are conceptually connected with their theoretical underpinnings and merits of alternative techniques are discussed. Just changing the mathematical organization of the lectures, however, might not be enough. Some theoretical thinking behaviors were revealed in our research because students were invited to explain their thinking in the interviews and this forced them to reflect on their solutions and rationalize their actions. It seems reasonable to assume that, to consolidate as a habit, a way of thinking must be exteriorized and made accessible to others for interpretation and constructive criticism. Therefore, an implementation of a VA approach may require changing also the exclusively lecture-and-assess format of prerequisite courses.

Instructors of prerequisite mathematics courses might also remain unmoved by this research, saying that they are not interested in developing theoretical mathematical thinking in the context of absolute value inequalities because – and we have heard this argument quite often – these students will generally not go on to study mathematics at any higher level and will be at most passive users of ready-made formulas, most of them already pre-programmed in computer systems. In particular, most will not engage in processing complex analytic expressions

[1] The prerequisite mathematics courses at the university were numbered 200-209.

involving the “pragmatic task” of compactifying and expanding absolute value expressions. They will not study Real, Complex, and Functional Analyses or Topology where absolute value is both a foundation of the notion of number, a technical tool, and a basis of generalization of the notions of length and distance. For us, however, this argument is all the more reason to teach students this topic (and any other) so that they have a chance to develop theoretical thinking habits in the process. Because this is what they will need in any profession, and not the non-transferable skill of solving two types of absolute value inequalities using certain very limited techniques. Theoretical thinking, as we have seen on the example of TA-4, may allow them to successfully deal with novel situations, even when lacking certain technical knowledge and skills. It may push them to seek knowledge and understanding of this technical knowledge and skills on their own, as it did for TA-4, who told us how he found an interest in mathematics after having been thoroughly disenchanted with it in high school:

> I had bad experience with mathematics [in high school] and so I said, oh no, math it’s out of my life. And then I observed in *Political Science Review* – because I am more interested in political science – that, yes, you can have very interesting mathematical applications and then if you completely avoid math then, well, you’ve got philosophy, with its pompous blah-blah, Plato revisited, so, OK, what kind of job you get after studying that? Teaching? I’m sorry, but the answer is, No. And if you make some policy analysis for efficiency, and efficiency is a more economics concept, so in economics you got more math. But the problem is that, in high school, it won’t really interest you, the math, because math teachers are more with the natural sciences teachers, and so the applications of math in high school are not in economics… so [that you think] for social science, math is absolutely irrelevant... It’s only later on that I observed that it’s a very important methodological thing. (TA-4, interview)

On this we end our paper. The last word in this research should belong to a student.

ACKNOWLEDGEMENTS

1. Research reported in this paper was funded by the Social Sciences and Humanities Research Council of Canada, grant number 410-2006-1911.
2. We wish to thank the eighteen students who accepted to participate in our research.
3. We are indebted to the anonymous reviewers for their insightful comments and helpful suggestions.

REFERENCES

Balacheff, N. (2010). Bridging knowing and proving in mathematics: A didactical perspective. In G. Hanna, H. N. Jahnke, & H. Pulte, *Explanation and proof in mathematics. Philosophical and educational perspectives* (pp. 115-135). New York: Springer.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: the case of limits of functions in Spanish high schools. *Educational Studies in Mathematics 59*, 235-268.

Belenky, M., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1997). *Women's ways of knowing. The Development of self, voice and mind.* New York: Basic Books, Inc. .

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education 10*, 301-309.

Brumfiel, C. (1980). Teaching the absolute value function. *Mathematics Teacher 73(1)*, 24-30.

Campbell, D., & Stanley, J. C. (1966). *Experimental and quasi-experimental designs for research.* Chicago: Rand McNally.

Castela, C. (2004). Institutions influencing mathematics students' private work: A factor of academic achievement. *Educational Studies in Mathematics 57(1)*, 33-63.

Cauchy, A.-L. (1821/1968). *Cours d'Analyse de l'Ecole Royale Polytechnique.* Darmstadt: Wissenschaftliche Buchgesellschaft.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques 19(2)*, 221-266.

Chevallard, Y. (2002). Organiser l'étude. 1. Structures et fonctions. *Actes de la 11e école d'été de didactique des matheématiques* (pp. 3-22). Grenoble: La Pensée Sauvage.

Chiarugi, I., Fracassina, G., & Furinghetti, F. (1990). Learning difficulties behind the notion of absolute value. *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 3* (pp. 231-238). Oaxtepec, Mexico: CINVESTAV.

Chiarugi, I., Fracassina, G., & Furinghetti, F. (1990). Learning difficulties behind the notion of absolute value. *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 3* (pp. 231-238). Oaxtepec, Mexico: CINVESTAV.

Denton, B. (1975). The modulus function. *Mathematics Teaching 73*, 48-51.

Duroux, A. (1983). La valeur absolue. Difficultés majeures pour une notion mineure. *Petit x 3*, 43-67.

Gagatsis, A., & Thomaidis, I. (1994). Une étude multidimensionnelle du concept de valeur absolue. In M. Artigue, R. Gras, C. Laborde, & P. Tavignot, *Vingt Ans de Didactique de Mathématiques en France* (pp. 343-348). Grenoble: La Pensée Sauvage.

Hanley, J., Joseph, L., Platt, R.W., Chung, M.K., & Bélisle, P. (2001). Visualizing the median as the minimum-deviation location. *The American Statistician, 55*(1), 1-3.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level Calculus courses. *Educational Studies in Mathematics 72(3)*, 341-358.

Inhelder, B., Cellerier, G., & Ackermann, E. (1992). *Le cheminement des découvertes de l'enfant.* Neuchâtel: Delachaux et Niestlé.

Küchemann, D. (1981). Algebra. In K. Hart, M. Brown, D. Küchemann, D. Kerslake, G. Ruddock, & M. McCartney, *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.

Major, J. (2008). Developing an understanding of school mathematical notions among math students on the example of the concept of absolute value of real number (fragment of research findings). *Acta Mathematica 11*, 141-146.

Major, J., & Powązka, Z. (2007). Some remarks on definition of the absolute value of a real number. *Scientific issues, Mathematics XII*, 283-288.

Martin-Gay, K. (2005). *Beginning and intermediate algebra.* Upper Saddle River, NJ: Pearson/Prentice Hall.

Monaghan, J., & Ozmantar, M. F. (2006). Abstraction and consolidation. *Educational Studies in Mathematics*, 233-258.

Ozmantar, M., & Monaghan, J. (2007). A dialectical approach to the formation of mathematical abstractions. *Mathematics Education Research Journal 19(2)*, 89-112.

Perrin-Glorian, M.-J. (1995). The absolute value in secondary school. A case study of 'institutionalisation process'. *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 78-81). Recife, Brazil: Federal University of Pernambuco.

Perrin-Glorian, M.-J. (1997). Pratiques d'institutionnalisation en classe de seconde. L'exemple de la valeur absolue. *Cahier de didactique des mathématiques de l'IREM Paris 7, 29*.

Porter, M., & Masingila, J. (2000). Examining the effect of writing on conceptual and procedural knowledge in Calculus. *Educational Studies in Mathematics 42*, 165-177.

Sierpinska, A. (2005). On practical and theoretical thinking and other false dichotomies in mathematical education. In M. Hoffmann, J. Lenhard, & F. Seeger, *Activity and sign - grounding mathematics education* (pp. 117-135). New York: Springer.

Sierpinska, A. (2005). On practical and theoretical thinking and other false dichotomies in mathematics education. In M. Hoffmann, J. Lenhard, & F. Seeger, *Activity and sign - Grounding mathematics education* (pp. 117-135). New York: Springer.

Sierpinska, A. (2007). 'I need the teacher to tell me if I am right or wrong'. *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 45-64). Seoul: PME.

Sierpinska, A., & Hardy, N. (2010). Unterrichten wir noch Mathematik? In M. Nuehrenboerger, Schwartzkopf, R, Boettinger, C., & Soebeke, E. , *Mathematik im Denken der Kinder* (pp. 94-100). Seelze: Friedrich-Verlag.

Sierpinska, A., Bobos, G., & Knipping, C. (2007). A study of university students' frustration in pre-university level, prerequisite mathematics courses: emotions, positions and perception of achievement. *Didactica Mathematicae 30*, 47-76.

Sierpinska, A., Bobos, G., & Knipping, C. (2008). Sources of students' frustration in pre-university level, prerequisite mathematics courses. *Instructional Science 36*, 289-320.

Sierpinska, A., Nnadozie, A., & Oktaç, A. (2002). *A study of relationships between theoretical thinking and high achievement in linear algebra.* Retrieved February 25, 2010, from Anna Sierpinska: http://www.asjdomain.ca/Sierpinska-TT-Report.pdf

Skemp, R. (1978). Relational vs instrumental understanding. *Arithmetic Teacher 26(3)*, 9-15.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh, & A. E. Kelly, *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Vérillon, P. (2000). Revisiting Piaget and Vigotsky: In search of a learning model for technology education. *Journal of Technology Studies, 26*(1).

Warnick, B. (2008). *Imitation and education. A philosophical inquiry into learning by example.* Albany, N.Y.: State University of New York.

Weisberg, H. (1992). *Central tendency and variability (Sage University Paper Series on Quantitative Applications in the Social Sciences, series no 07-083).* Newbury Park, CA: Sage.

Wilhelmi, M., Godino, J., & Lacasta, E. (2007). Didactic effectiveness of mathematical definitions. The case of absolute value. *International Electronic Journal of Mathematics Education (www.iejme.com) 2(2)*.

---

[i] Students applying for certain university programs (science, engineering, business school, psychology, nursing, and others) are required to take secondary school or college level mathematics courses such as College Algebra, Vectors and Matrices, Pre-calculus and Calculus of one variable if they had not taken them before or obtained low grades in them.

[ii] Supporting documentation for the research can be viewed using links on A. Sierpinska's web page at http://www.annasierpinska.wkrib.com, under the rubric "Research" (http://www.annasierpinska.wkrib.com/index.php?page=research), in the section "2006-10 Experimenting with approaches to teaching inequalities with absolute value":

Lecture slides: http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Lecture-Slides.pdf
Student' solutions (raw data):
Procedural Approach (PA) students' solutions:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Solutions-PA.pdf
Theoretical Approach (TA) students' solutions:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Solutions-TA.pdf
Visual Approach (VA) students' solutions:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Solutions-VA.pdf
Transcripts:
Transcripts of interviews with students in the PA group:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Transcripts-PA.pdf
Transcripts of interviews with students in the TA group:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Transcripts-TA.pdf
Transcripts of interviews with students in the VA group:
http://www.annasierpinska.wkrib.com/pdf/Sierpinska.et.al-Abs-Val-Transcripts-VA.pdf

[iii] It is the realization – thanks to Stieg Mellin-Olsen – that "instrumental knowing" is an *understanding* and not a "non-understanding" that led Skemp to his well-known distinction between "instrumental" and "relational" *understanding*: "Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have *in the past* described as 'rules without reasons', without realizing that for many pupils and their teachers the possession of such a rule, and the ability to use it, was what they meant by 'understanding'." (Skemp, 1978)

[iv] The distinction between "epistemic" and "pragmatic" tasks was inspired by a terminology introduced by Inhelder et al. (Inhelder, Cellerier, & Ackermann, 1992) and then used by Vérillon (Vérillon, 2000) in reference to two types of aims of task-situated instrumented actions: the pragmatic actions aim (mainly) at "transform[ing]… a part of the environment", while the epistemic actions aim at "affording knowledge".

[v] There are other characterizations of the absolute value function, for example, as solutions of certain functional equations, see (Major, 2008); (Major & Powązka, 2007). These characterizations are interesting in the context of proving their equivalence with other definitions of absolute value but are not useful for solving inequalities with absolute value. They also require an advanced understanding of functions which is not the case for most students in the prerequisite mathematics courses.

[vi] This view of absolute value seems, in fact, more susceptible than the first one to be used spontaneously in making a complex conditional statement more concise. For example, in a task such as, "In a right-angled triangle with one side equal to $2\sqrt{a}$ and hypotenuse equal to $a+1$, $a>0$, find the length of the other side" (Chiarugi, Fracassina, & Furinghetti, 1990), students are not told to use absolute value, yet if they make the common mistake of omitting absolute value and writing their answer as $a-1$ instead of $|a-1|$, the teacher has an opportunity to discuss the "epistemic value" of absolute value with them. The square root definition does not solve, however, the problem of mistakes such as deducing $x>3$ from $x^2>9$ [(Gagatsis & Thomaidis, 1994); (Biza, Nardi, & Zachariades, 2007)].

[vii] More generally, in *RBC*, an absolute value inequality involving *n* absolute value expressions $|f_i(x)|$, where the functions f_i are linear, is seen as equivalent to a disjunction of 2^n cases, each being a conjunction of interval conditions and the form the inequality takes under these interval conditions, as prescribed by the definition of absolute value applied to the expressions $|f_i(x)|$.

[viii] Using the mentioned definition, the proofs of Properties 1 and 2 would exhibit reasoning by cases, which always involves disjunction of the conditions describing the cases:

$$|x|<a \Leftrightarrow (x\geq 0 \; and \; x<a) \; \boldsymbol{or} \; (x<0 \; and -x<a) \Leftrightarrow x\in[0,a)\cup(-a,0) \Leftrightarrow x\in(-a,a)$$

$$|x|>a \Leftrightarrow (x\geq 0 \; and \; x>a) \; \boldsymbol{or} \; (x<0 \; and -x>a) \Leftrightarrow x\in(a,\infty)\cup(-\infty,-a)$$

[ix] From http://www.mathmotivation.com/lectures/Absolute-Value-Inequalities.pdf (downloaded December 9, 2009):

[Y]ou replace the inequality symbol with =, solve this equation to find the critical numbers, plot the critical numbers, and test the intervals. For example, the inequality |x – 2| < 3 may be solved by first solving |x – 2| = 3 to get x = 5 and x = –1 . Then plot the critical numbers x = 5 and x= –1 on the number line and check the intervals. [Here, a diagram is plotted, with interval (–infinity, –1) labeled "Interval One", interval (–1, 5) labeled "Interval Two", and interval (5, infinity) labeled "Interval Three"] The test value of Interval Two, x = 0 results in a true statement when substituted into |x – 2| < 3 whereas the test values of Interval One and Interval Three, x = –2 and x = –1 result in false statements. So Interval Two makes up the solution, i.e. –1 < x < 5.

[x] See also the inequality $\sqrt{x-4}<\sqrt{2x-5}$, where the two functions do not intersect and therefore "critical numbers" cannot be found, but the inequality has a not-empty solution set, namely $(4,+\infty)$.

[xi] Consider, for example, the inequality $-|x+1|+1<\frac{x^2-3x+2}{x-1}$.

[xii] Consider the inequality $f(x)>\frac{1}{4}$, where *f* is a function defined, on the domain $(-\infty,-1)\cup(1,+\infty)$, by the rule $f(x)=|x^2-\frac{3}{4}|$.

[xiii] Links to the slides of the lectures are given in Note (ii).

[xiv] Links to the transcripts of interviews with students are given in Note (ii).
[xv] Students' written solutions are available through links given in Note (ii)
[xvi] See the supporting documentation files with students' written solutions and subsequent interviews.
[xvii] These students have not seen *RBC* before.
[xviii] Commenting, after the lecture, about the slide with the analytic solution of Example 2, he said, after complaining that the lecture was going too fast for him: "According to me, I usually put values, like… 2-1…, to digest and then I come here, I say, less than 0, it will have to be something like –3, -4, but with x you don't see the point. It's just like you just have been given a bunch of mathematical things and you just, OK, whatever. (TA-1, commenting on slide 7 of the lecture)."