A Performance Analysis of Tandem Networks of Multiplexers with Binary Markovian Sources

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Abstract

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Currently, the modern networks including Internet and ATM are based on packet-switched technology. In this technology, the arriving packets are statistically multiplexed to achieve high bandwidth gain, also the output ports of routers may be modeled as multiplexers. As a result, in this type of networks, a packet goes through a number of multiplexers as it transits from source to destination. Because of this, the study of tandem networks of multiplexers is very important; as it will give us the information regarding the traffic shaping that occurs along the route. Due to the lack of exact analysis methods, most of the previous work in the analysis of tandem networks is based on either simulation or approximate models. This thesis presents an exact performance analysis of tandem networks with arbitrary number of multiplexers. Since, the traffic generated by multimedia sources in the real networks is correlated, the binary Markov *On/Off* source model is assumed for the input traffic to the network. The objective of the analysis is to determine the Probability Generating Function (PGF) of the queue length of each multiplexer in the tandem network as well as the corresponding performance measures. The complicated dependency among tandem multiplexers results in unknown boundary functions, determination of which is the main source of difficulty in the exact performance analysis. In this thesis, at first a straightforward solution technique is used to determine the PGF of the queue lengths and number of *On* sources for a tandem network with two multiplexers. Unfortunately, this solution does not extend to tandem networks with higher number of multiplexers. As a result, an alternative method has been developed, which determines the unknown boundary functions by using busy periods of multiplexers. The PGF of the queue length and number of *On* sources is obtained for each multiplexer in a tandem network with arbitrary number of multiplexers. Following that, the mean and variance of queue lengths as well as the packet delay at each multiplexer have been determined. A proof of this solution is given to show that the analysis is correct. Then, the solution has been extended to a more general tandem network, where each multiplexer is fed by multiple types of traffic. Finally, numerical results regarding the analysis are presented and compared with those of the simulation. The analysis shows network raffic gets smoother when it goes through higher number of multiplexers, this smoothing effect is more obvious in heterogeneous traffic case. It also shows that under constant traffic load, as the number of sources increases, the delay and queue length increase. The analysis results enable to explain the delicate interaction between traffic smoothing and source burstiness

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List of Symbols

$a_{i,j,k}$	Number of *On*-sources of type-j for multiplexer-i during slot k, multiple types of traffic case.
$a_{i,k}$	Number of *On*-sources of type-i during slot k, single type of traffic case.
a_j	A mathematical notation defined in (3.12) of Chapter 3.
$a(\omega)$	Transform of a_j, defined in (C.8) of Appendix C.
$b_{i,j,k}$	The total number of packets generated by type-j sources for multiplexer-i during slot k, multiple types of traffic case.
$b_{i,k}$	The total number of packets generated by type-i sources during slot k, single type of traffic case.
b_j	A mathematical notation defined in (3.13) of Chapter 3.
$b(\omega)$	Transform of b_j, defined in (C.8) of Appendix C.
$B_i(k)$	$B_i(k)=[X_i(k)]^{m_i}$
$\overline{B}_i^{(\ell)}(j)$	$\overline{B}_i^{(\ell)}(j)=\frac{1}{\ell!}\frac{d^\ell B_i(j)}{dz_i^\ell}\Big\vert_{z_i=y_i=0}$
$B_{n,j}(k)$	$B_{n,j}(k)=[X_{i,j}(k)]^{m_{i,j}}$
$C_{1i,2i}$	Mathematical notation, defined in (2.24) of Chapter 2.
$\widetilde{C}_{1i,2i}$	$\widetilde{C}_{1i,2i}=C_{1i,2i}\big\vert_{y_n=1}$.
$c_i(z_i)$	The PGF of $c_{j,i}$.

$c_{j,i}$	A variable that assumes the values of 1, 0 if j'th source from type-i is in *On* and *Off* states in the next slot respectively, given that this source is *On* in the present slot.
$c_{i,j}(z_i)$	The PGF of $c_{h,j,i}$.
$c_{h,i,j}$	A variable that assumes the values of 1, 0 if the h'th source from type-j that is feeding multiplexer-i in *On* and *Off* states in the next slot respectively, given that this source is *On* in the present slot.
$d_i(z_i)$	The PGF of $d_{j,i}$.
$d_{j,i}$	A variable that assumes the values of 1, 0 if j'th source from type-i is in *On* and *Off* states in the next slot respectively, given that this source is *Off* in the present slot.
$d_{i,j}(z_i)$	The PGF of $d_{h,j,i}$.
$d_{h,i,j}$	A variable that assumes the values of 1, 0 if h'th source from type-j that is feeding multiplexer-i is in *On* and *Off* states in the next slot respectively, given that this source is *Off* in the present slot.
$D_{1i,2i}$	Mathematical notation, defined in (2.25) of Chapter 2.
$\overline{D}_i$	Mean packet delay at i'th multiplexer.
$E_n(z_n)$	Mathematical notation, for single type traffic case, $E_n(z_n)$ is defined in (5.15). For multiple types traffic case, $E_n(z_n)$ is defined in (6.25).
$f_i(z_i)$	PGF of $f_{j,i,k}$.
f_j	Mathematical notation, defined in (3.7) of Chapter 3.
$f_{j,i,k}$	Number of packets generated by the j'th *On*-source of type-i during slot k.
$f_{i,j}(z_i)$	PGF of $f_{h,i,j,k}$.

$f_{h,i,j,k}$ Number of packets generated by the h'th *On*-source of type-j for multiplexer-i during slot k.

$F_n(z_n)$ Mathematical notation, for single type of traffic case, $F_n(z_n)$ is defined in (5.16). For multiple types of traffic case, $F_n(z_n)$ is defined in (6.26).

g_j, g'_j Mathematical notations, defined in (3.6) of Chapter 3.

$G_i(z_i)$ Mathematical notation, for single type of traffic, $G_i(z_i) = (\widetilde{C}_{2i}\lambda_{2i})^{m_i}$; for multiple types of traffic, $G_n(z_n) = \prod_{j=1}^{\tau_n}(\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}}$.

$h_{k+1-j,j}$ Mathematical notation, defined in (3.8) of Chapter 3.

$H_n(z_n)$ Mathematical notation, for single type of traffic, $H_i(z_i) = \lambda_{2i}^{m_i}$; for multiple types of traffic, $H_i(z_i) = \prod_{j=1}^{\tau_i}\lambda_{2i,j}^{m_{i,j}}$

$I_{k,j}$ Mathematical notation, defined in (3.10) of Chapter 3.

$\ell_{i,k}$ Queue length of multiplexer-i at the end of slot k.

m_i Number of type-i sources in the system, in single type of traffic case.

$m_{i,j}$ Number of type-j sources for multiplexer-i, in multiple types of traffic case.

$\overline{N}_i$ Mean queue length at i'th multiplexer.

$P_i(z_i)$ PGF of queue length at the i'th multiplexer.

$q_k(i_{n-1}, j_{n-1}, i_n, j_n)$ Joint probability of the number of packets in the queue and the number of *On* sources for multiplexers-(n-1) and n at the discrete time k.

$Q(z_{n-1}, y_{n-1}, z_n, y_n)$ The steady-state PGF of the queue lengths and the number of *On* sources for multiplexers-(n-1) and n.

$Q(1_{n-1}, 1_{n-1}, z_n, y_n, \omega)$ $Q(1_{n-1}, 1_{n-1}, z_n, y_n, \omega) = \sum_{k=0}^{\infty} Q_k(1_{n-1}, 1_{n-1}, z_n, y_n)\omega^k$.

$Q_k(z_{n-1}, y_{n-1}, z_n, y_n)$ The PGF of the queue lengths and the number of *On* sources for multiplexers-(n-1) and n at discrete time k.

$Q_k(z_{n-1}, Y_{n-1}, 0_n, 0_n)$ $Q_k(z_{n-1}, Y_{n-1}, 0_n, 0_n) = Q_k(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_n=0, y_n=0}$.

$Q_k(0_{n-1}, 0_{n-1}, z_n, Y_n)$ $Q_k(0_{n-1}, 0_{n-1}, z_n, Y_n) = Q_k(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=0, y_{n-1}=0}$.

$Q_k(1_{n-1}, 1_{n-1}, z_n, Y_n)$ $Q_k(1_{n-1}, 1_{n-1}, z_n, Y_n) = Q_k(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=1, y_{n-1}=1}$.

$Q_k(0_{n-1}, 0_{n-1}, 0_n, 0_n)$ $Q_k(0_{n-1}, 0_{n-1}, 0_n, 0_n) = Q_k(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=0, y_{n-1}=0, z_n=0, y_n=0}$.

$U_i(k)$ Mathematical notation, defined in (2.16) of Chapter 2.

$\overline{V_i}$ Variance of queue length at i'th multiplexer.

$X_i(k)$ Mathematical notation, defined in (2.15) of Chapter 2.

Y_i Mathematical notation, defined in (2.11) of Chapter 2.

$z_n^*(\omega)$ Unique root of the equation, $z_n - \lambda_{2n}^{m_n}\omega\Gamma_{n-1}(\lambda_{2n}^{m_n}\omega) = 0$.

$\Re_j$ Mathematical notation, defined in (3.9) of Chapter 3.

$1-\alpha_i$ The probability of a transition from active to idle state for type-i sources.

$1-\beta_i$ The probability of a transition from idle to active state for type-i sources.

$\phi_i(k)$ Mathematical notation, defined in (2.19) of Chapter 2.

$\lambda_{1i,2i}$ Solution of the equation $\lambda_i^2 - [\beta_i + \alpha_i f_i(z_i)]\lambda_i - (1-\alpha_i-\beta_i)f_i(z_i) = 0$.

ρ_i Traffic load generated by type-i sources, single type of traffic case.

$\rho_{i,j}$ Traffic load generated by type-j sources for multiplexer-i, multiple types of traffic case.

τ_i Number of the source types for multiplexer-i, in multiple types of traffic case.

$\xi_n(j)$	Prob(n'th multiplexer has a busy period $= j$ slots).
$\varphi^{(n)}(\ell)$	$\varphi^{(n)}(\ell)=\sum_{r=1}^{\ell}\frac{1}{z_n^r}\varphi_r^{(n)}(\ell) \quad , \ell \ge 1; \quad \varphi^{(n)}(0)=1$.
$\varphi_r^{(n)}(\ell)$	Prob(n'th multiplexer has r busy periods during an interval of ℓ slots).
$\Phi^{(n)}(\omega)$	Transform of $\varphi^{(n)}(\ell)$, $\Phi^{(n)}(\omega)=\sum_{\ell=1}^{\infty}\varphi^{(n)}(\ell)\omega^{\ell}$.
$\Gamma_i(\omega)$	PGF of the busy period of i'th multiplexer, $\Gamma_i(\omega)=\sum_{j=0}^{\infty}\xi_i(j)\omega^j$.
$\Theta_n(z_n)$	Mathematical notation, defined as $\Theta_n(z_n)=H_n(z_n)\Gamma_{n-1}(H_n(z_n))$.

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Chapter 1

Introduction

The field of communication networks has shown tremendous amount of growth in recent decades, which is making great changes in our lives. A communication network may be designed as a set of equipments and facilities that provide a service: the transfer of information between users located at various geographical points [1]. With the rapidly increasing volume of transferred information, the communication networks have become very complex. This has created many challenges in the network design and implementation, which led to a great research effort in the area of network modeling and performance analysis. The main objectives of the mathematical modeling of communication networks are the prediction of performance and resource requirements [2], such as the sizes of waiting buffers, the delay experienced by an arrival, and the availability of a service facility. In this chapter, first a brief introduction is given to the circuit and packet-switched networks. Then, the statistical multiplexer model of the packet-switched networks is introduced, and the tandem network model and the binary Markovian source model are presented. Afterwards, the previous research work on tandem networks in the literature is sumerized. Finally, the contributions and the outline of this thesis are presented.

1.1 Circuit and Packet-Switched Networks

Traditionally, there have been two types of networks: telephone and data networks. Telephone networks have been used to carry voice and the data networks to transport data. Voice and data are examples of real-time and non-real-time traffic respectively. Real-time traffic is delay sensitive but it is usually loss tolerant, while non-real-time traffic is loss sensitive but usually delay tolerant. The different characteristics of voice and data led to the development of two different types of networks.

Telephone networks use circuit-switching technology, which sets up a dedicated end-to-end path through allocation of bandwidth for each call. When the required resources are not available then new call requests are rejected, however accepted calls experience short deterministic delay, which is desired for real-time traffic. The circuit-switching technology is not appropriate for transmission of data since data sources are burstier, which results in the communication line being idle for most of the time. Because of this, data networks use packet-switched technology. In packet-switched networks, information is transmitted in short blocks, called packets. The packets travel from one node to the next one in the network until they reach from source to destination. Each intermediate node stores the incoming packets, makes routing decisions and then transmits them on the proper output links. In packet switching, there is no call blocking, but as packets travel from source to destination, they experience random delays [3]. In this type of networks, bandwidth efficiency is greater since calls are not assigned dedicated circuits but they are statistically multiplexed. However, statistical multiplexing and distributed routing decisions may cause congestion. Congestion increases delay, which is not appropriate for real-time traffic; also congestion may result in buffer overflows, and therefore causes loss of information.

Telephone and data networks have evolved independently of each other over the decades. The development of fiber optics as transmission medium has made enormous amounts of bandwidth available. This led to the expansion of both telephone and data networks as well as to the introduction of new services and transmission of multimedia traffic in the networks. Clearly, it is not economical to build and maintain a separate network for each type of service. It would be most beneficial to use a single infrastructure to serve all services. This led to the effort for development of the Broadband Integrated Services Digital Networks (B-ISDN) that would support transmission of the multimedia applications such as voice, video, data and other signals in a single fiber optics based network.

In telephony networks, Asynchronous Transfer Mode (ATM) has been designated as the target technology to meet the requirements of B-ISDN [3]. In ATM, the information is segmented into fixed-size packets, referred to as cells that are transmitted from source to destination in a virtual circuit. A pre-planned route is established before any cells are sent and all the packets between a pair of communicating parties follow the same route through the network, which is called virtual-circuit switching. ATM has been designed to inherit the best features of circuit and packet-switching.

On the other hand, data networks have shown tremendous growth with the introduction of the Internet. The Internet was designed to provide best-effort service for delivery of data traffic. The datagram approach is used in the Internet, where each packet is treated independently, so the packets with the same destination address do not necessarily all follow the same route and they may arrive out of sequence to their destination. As explained before, the data networks are not suitable for transmission of real-time traffic. However, the increasing popularity of the Internet has shifted the paradigm from "IP over everything" to "everything over IP". In order to manage the multitude of applications such as stream video, voice over IP, e-commerce and others, the Internet requires different QoS in addition to best-effort service, and new protocols are being proposed to provide requisite QoS to new applications on the Internet.

Nevertheless, ATM and the Internet are both based on packet-switched technology and their ultimate aim is to meet QoS requirements of different services. In packet-switched networks, statistical multiplexing is applied to obtain bandwidth efficiency, no matter whether they use virtual-circuit approach (connection oriented) or datagram approach (connectionless).

1.2 Statistical Multiplexing

Statistical multiplexing refers to sharing of expensive switching and transmission facilities among different packet streams in the network. As explained above, in packet-switched networks, packets are routed from source to destination, following the store and forward principle. When a packet reaches the nearby node, it is temporarily stored there until the transmission line to the next node becomes available. For this purpose, at each node, switching elements are installed to route the incoming packets to the appropriate output link. For those packets that cannot be transmitted immediately, buffer space has been provided at each switching port. Inside a switching element, packets from different input ports may go to the same output port. These packets will queue up in the output buffer and be transmitted according to some queueing disciplines. Thus, an output port of a switch may be modeled as a multiplexer (see Figure 1.1).

In a network, hundreds of sources may access a single link. Then statistical multiplexing is performed on the incoming packets to achieve high bandwidth gain, and buffering is required to absorb traffic fluctuations when the instantaneous rate of the aggregate incoming streams exceeds the capacity of the outgoing link. Again, this is multiplexing.

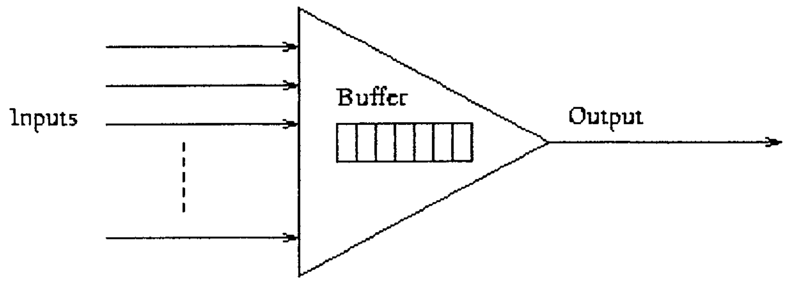


Figure 1.1 Statistical multiplexer model

Therefore, in order to implement efficient admission and flow control strategies to satisfy different QoS requirements, network designers need to acquire a very good understanding of the statistical multiplexing of the aggregate traffic generated by multimedia sources (with possibly different characteristics).

From a modeling point of view, a statistical multiplexer may be modeled as a deterministic server with a slotted time axis and correlated discrete-time arrival processes. Most often, the quantities of interest are the buffer occupancy (number of packets stored in the buffer, or equivalently, queue length) and the packet delay (or waiting time) experienced by the packets in the system.

1.3 Tandem Network Model

In packet-switched networks, packets are routed from source destination as shown in Figure 1.2. At each node, the packets are received, stored briefly and then passed on to the next node along the route when the transmission line becomes available. As the packets go through the network, the statistical properties of the traffic change. For example, the traffic becomes smoother and the long-range dependence of the traffic dissipate due to the statistical multiplexing gains [4]. Thus, the performance analysis at the network level is very important as this will enable more accurate determination of buffer requirements in the network and end-to-end delay.

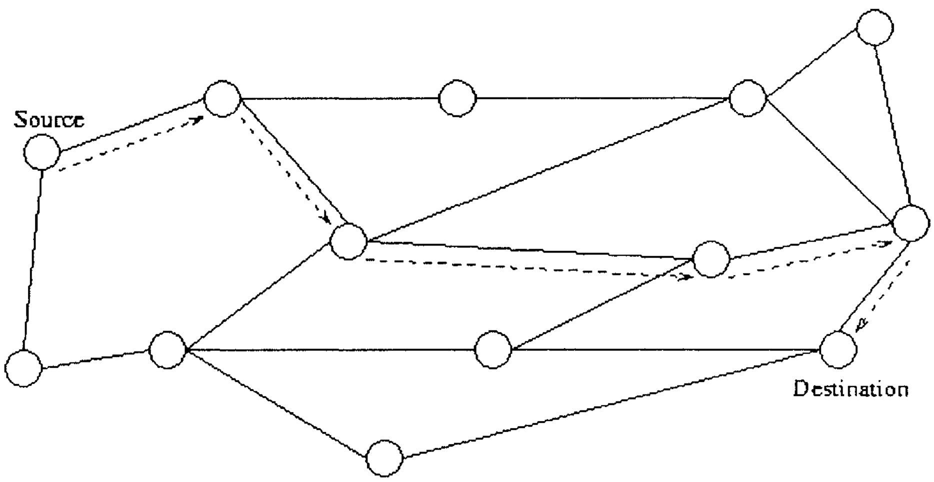


Figure 1.2 A diagram of a network

As packets go through the network from source to destination, the route that they follow is a series of switching nodes (which may be modeled as multiplexers) in tandem. The output of each multiplexer may leave the tandem network or enter the next multiplexer. Figure 1.3 is a general model of multiplexers in tandem. The input of each multiplexer in the tandem network consists of two parts: the a portion of the output from the previous multiplexer and external inputs from outside of the tandem network.

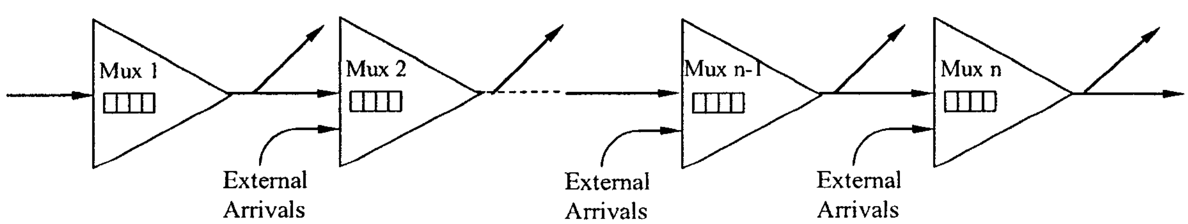


Figure 1.3 Tandem multiplexer model

1.4 Binary Markov *On/Off* Source Model

The performance evaluation of the networks requires accurate modeling of the network traffic. The current networks support various communication services, such as data, voice and video, each having different traffic characteristics. This has introduced significant changes in the way that uncorrelated traffic models (such as Poisson and Bernoulli) dominated the traditional performance analysis methods. In fact, when dealing with the traffic generated by multimedia sources, the uncorrelated random arrival process assumption becomes inadequate because of the dependencies in the stream of information. For these reasons, traffic characterization has been a major field of research during the past years due to its direct impact on the network performance analysis.

There have been many traffic models proposed in the literature for characterizing individual traffic sources or superposition of multiple sources. For instance, Poisson arrival process (continuous time case) and geometric inter-arrival process (discrete time case) are good models for data traffic; Interrupted Poisson Process (IPP) is a good model for voice traffic; and Markov Modulated Poisson Process (MMPP) can be sued to model data, voice and video traffic. A good review on traffic modeling can be found in [11].

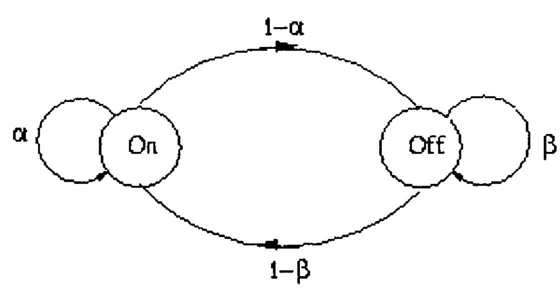


Figure 1.4 Binary Markov *On/Off* source model

Among these traffic models that have been used for different types of sources, one of the most versatile ones is the binary Markov *On/Off* model. This simple traffic model has been widely used for characterization of packet speech and data sources [12]. In this model (see Figure 1.4), active (*On*) periods alternate with idle (*Off*) periods according to a discrete-time Markov Chain. The transition from *On* to *Off* state occurs with probability $1-\alpha$, and from *Off* to *On* state occurs with probability $1-\beta$ from slot to slot. During an *On* period, the traffic source generates packets following some distribution; during *Off* periods, no packets are generated. As a result, the lengths of the *On* and *Off* periods are geometrically distributed.

The binary Markov *On/Off* model is good for capturing the correlation behavior of many traffic sources. Let us define the following:

p_{10}=Pr (a source is *On* | it was *Off* in the previous slot) $= 1 - \beta$,

p_{11}=Pr (a source is *On* | it was *On* in the previous slot) $= \alpha$,

The correlation index is usually defined as, $\Delta = p_{11} - p_{10} = \alpha + \beta - 1$ (see [10]). If $\alpha + \beta = 1$, then $\Delta = 0$, $p_{10} = p_{11}$; then the probability that a source is *On* does not depend on the status of the previous slot. So the source transitions between *On* or *Off* states follow a Bernoulli process, and hence the packets arrivals are independent from slot to slot. Thus choosing $\Delta \neq 0$ captures the correlation behavior of the traffic generated by a source. When $\alpha + \beta$ is high ($0 < \Delta < 1$), the packet arrivals have a positive correlation whereby packets have tendency to arrive in clusters. Alternatively, when $\alpha + \beta$ is low($-1 < \Delta < 0$), the packet arrivals have a negative correlation, where the packets are more dispersed in time scale .

More complicated models, such as the three-state Markov model, have been proposed for more accurate modeling. Queueing analysis with these types of models may be very complex and not mathematically tractable. Therefore, these models have rarely been applied in performance analysis; on the other hand, the binary Markov *On/Off* model has been frequently used for the modeling of voice, video and data traffic. In some cases, Gaussian distribution may be used as an approximation model for the traffic [15]. For example, [16] uses Gaussian Distributed and Autoregressive input as the approximation of superposition of *On-Off* voice sources.

Because of its versatility and flexibility, in this thesis work, the binary Markov *On/Off* model has been chosen as the basic model for the characterization of input traffic sources. Hence, the rest of the thesis will be mainly concerned with the analysis of tandem networks with correlated arrivals process, which consists of the superposition of many independent traffic streams generated by binary Markov sources.

1.5 Previous Work on Tandem Networks

Most analytical studies related to network performance focus on an isolated component in the network, such as a single multiplexer or a switching node. This is mainly because of the difficulties involving the performance analysis at the network level. The performance analysis of large-scale networks is still an intractable problem. However, the studies of a network connection, which typically consists of a number of queues in tandem, is of great importance since it might help us to understand the changes in the traffic characteristics at the interior of the network.

Due to lack of exact analysis methods, most of the previous work in the analysis of tandem networks has either focused on simulation experiments [5, 6, 17] or on some approximate models, whereby each node is analyzed in isolation, after fitting an approximate model to the departure process of each node [7 - 9, 15, 16]. The factors that complicate the exact analysis of networks may be listed as follows [10]:

- The arrival process of each node is often complicated and exhibits strong correlation. This correlation among arrivals makes the corresponding analysis far more complicated than that of an uncorrelated case.
- The interaction among the traffic streams in the tandem network gives rise to rather complicated, statistical dependence among the nodes. This dependence results in unknown boundary functions in the expression of the PGF for the system. The determination of the boundary function is generally very complex.
- As the independent Poisson model doean't hold for the network arrivals, the joint PGF of a tandem network does not posses a product-form solution, so direct application of the well-known combined iterative/decomposition methods becomes hard to justify.

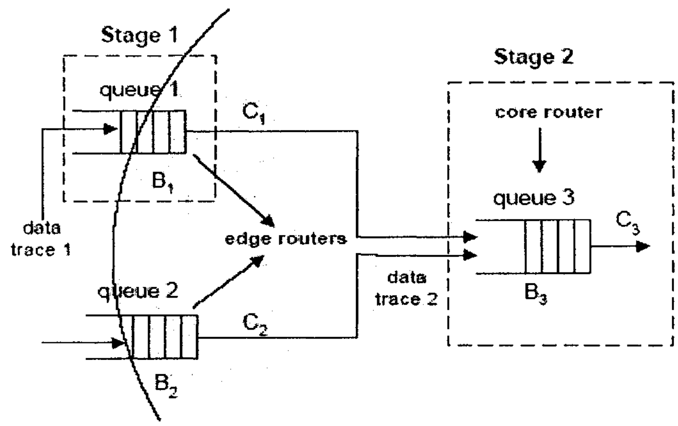


Figure 1.5 Network of two stage queues in the study of [17]

In [17], two types of queueing networks are considered: one with a single buffer and another with two buffers in tandem as shown in Figure 1.5. They present a discrete-event simulation methodology for the estimation of the correlation in the traffic.

A queueing network model of four-node ATM tandem is considered in [15]. The output process of each node is approximated by a renewal process, and the output of a node is fed to the next one. In addition, interfering traffic enters each node and leaves immediately. The interfering traffic in [15] consists of an M-stream, which is modeled by a Bernoulli process with batch arrivals, and a B-stream, which is modeled as a number of N discrete-time interrupted Poisson processes. By convolving the delay distribution at each switching node, an approximation for the end-to-end delay distribution is provided. The limitation of this approach lies in the characterization of the output process of each isolated switching node. In fact, the renewal approximation for the nodal departure process is hard to justify, as correlation is inherent in the output process of each node. This correlation has significant effect on the queueing behavior of the downstream nodes.

In the next, two works will be described, which use exact methods of analysis. In [18], a two-node tandem network model has been analyzed, where the number of external arrivals to the two queues is modulated by a single two-state Markov Chain. An expression for the PGF of the queue lengths distribution has been determined. However, the single two-state Markov Chain model for the external arrivals is too restrictive for modeling network traffic.

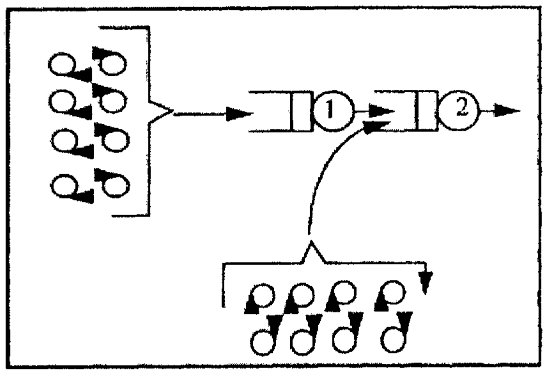


Figure 1.6 A two-node tandem network model in [19]

In [19], an exact analysis of a two-node tandem network with correlated arrivals has been presented. As shown in Figure 1.6, the node-1 is fed by the traffic generated by the superposition of m_1 independent and homogeneous Markov binary sources; the node-2 is fed with the output of node-1 as well as the traffic generated by m_2 independent and homogeneous Markov binary sources. A functional equation relating the joint PGF of the system between two consecutive slots was derived. From there the mean queue length for each node as well as the average packet delay in the network was determined. The limitation of [19] is that the boundary functions of the joint PGF are not determined, thus there is no explicit expression for the joint PGF although the performance measures have been determined.

1.6 Contributions of the Thesis

In this thesis, an exact performance analysis is presented for a tandem network with arbitrary number of multiplexers. A Markovian source model for the arrival process has been assumed that incorporate the correlation in the network traffic. The functional equation is derived, which describes the tandem network through the imbedded Markov Chain analysis. The unknown boundary functions have been determined, which leads to the solution of the functional equation: the joint PGF of the queue length and number of *On* sources for any multiplexer at the steady state. The correctness of the solution has been proven. From the joint PGF, it is feasible to derive the closed-form expressions for various performance measures of the tandem network, such as the mean and standard deviation of queue length, the mean packet delay for arbitrary number of sources and arbitrary number of multiplexers in the network. The solution has been extended to more general tandem networks with heterogeneous traffic feeding each of the multiplexers. In this multiple types of traffic case, the joint PGF of a multiplexer at the steady state, as well as the closed-form expressions of performance measures are also determined.

The numerical results regarding the analysis in the thesis are presented. The results show the smoothing effect of statistical multiplexing. As the traffic goes through higher number of multiplexers, it becomes smoother. As a result, both mean and variance of the delay and queue length are reduced. This reduction is more pronounced in the case of heterogeneous traffic feeding each multiplexer. The interaction between the smoothing effect and burstiness of traffic is explained. Finally, the simulation results are presented, which supports the analytical results of this thesis.

1.7 Outline of the Thesis

The outline of the rest of the thesis is as follows. In Chapter 2, the analytical model for the tandem network of multiplexers is presented. An embedded Markov chain analysis is used to derive the functional equation that relates the PGFs of the system between two consecutive slots, and then the functional equation is transformed into a new form, which is mathematically more tractable.

In Chapter 3, a straightforward method is used to determine the unknown boundary functions in the functional equation, and the solution of a tandem network with two multiplexers is given. The simulation and numerical results regarding the performance measures are also presented.

In Chapter 4, an alternative technique is used to determine the unknown boundary function, which involves the application of multiplexer busy periods. From there, the solutions for tandem networks with two and three multiplexers are obtained. Again, the simulation and numerical results regarding the three-multiplexer tandem network are presented.

In Chapter 5, the solution is extended to a tandem network with arbitrary number of multiplexers. And a proof of the solution is presented.

In Chapter 6, a general tandem network with heterogeneous traffic feeding each of the multiplexers is studied. The analytical model is presented and the solution is obtained. Then, the performance measures are determined, and the corresponding numerical results are presented.

In Chapter 7, the contributions and conclusions of this thesis are summerized. Finally in the Appendix A and B, some results related to the multiplexer busy periods are given, which are needed in the thesis. In Appendixes C and D, the details of mathematical derivations are given, which are referred in Chapters 3, 4 and 5.

Chapter 2

Tandem Network Modeling

In this chapter, first the tandem network and the source model are presented. Afterwards, a model of two multiplexers in tandem is considered, and the functional equation relating the joint PGF of the two queue lengths and the number of *On* sources are derived by using embedded Markov Chain analysis. Then, the functional equation is transformed into a new form, which is mathematically more tractable. Finally, the derivations are extended to general tandem networks with arbitrary number of multiplexers.

2.1 Tandem Network and Source Model

This thesis considers the performance modeling of n (n>1) multiplexers in tandem at the entrance of the network. Since these multiplexers are aggregating the traffic, the entire output of each multiplexer is fed to the next one together with the new arrivals. The new arrivals are generated by a number of Markovian sources. The tandem network model under consideration is shown in Figure 2.1, which will be modeled as a discrete-time queueing system.

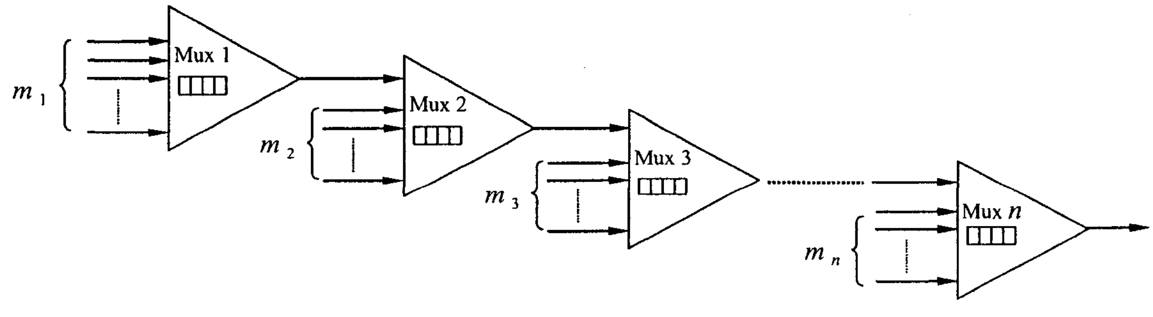


Figure 2.1 Tandem Network Model

It is assumed that each multiplexer has infinite buffer to store the arriving packets. The time axis is divided into intervals of equal lengths (slots) and a packet is transmitted at the slot boundaries. It is assumed that a packet cannot be transmitted during the slot that it arrives, and that a packet transmission time is equal to one slot.

The external packet arrivals to multiplexer-i ($1 \le i \le n$) is generated by m_i type-i sources as shown in Figure 2.2. Each type of sources are independent binary Markov sources alternating between *On* and *Off* states. For type-i sources, a transition from *On* to *Off* state occurs with probability ($1-\alpha_i$), and from *Off* to *On* state occurs with probability ($1-\beta_i$). As a result, the length of the *On* and *Off* periods are geometrically distributed. And the mean duration of *On* and *Off* periods are $\frac{1}{1-\alpha_i}$ and $\frac{1}{1-\beta_i}$ respectively. And the steady-state probabilities that a source is *On* and *Off* during a slot are $\frac{1-\beta_i}{2-\alpha_i-\beta_i}$ and $\frac{1-\alpha_i}{2-\alpha_i-\beta_i}$ respectively. It is assumed that an *On* source generates at least one packet during a slot, while an *Off* source generates no packets during a slot.

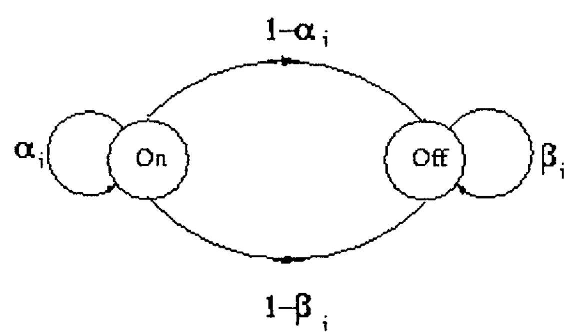


Figure 2.2 Source Model for Type-*i* Sources

Now let us introduce the following notations:

m_i = number of type-i sources feeding multiplexer-i.

$\ell_{i,k}$ = queue length of multiplexer-i at the end of slot k.

$a_{i,k}$ = number of *On*-sources of type-i during slot k.

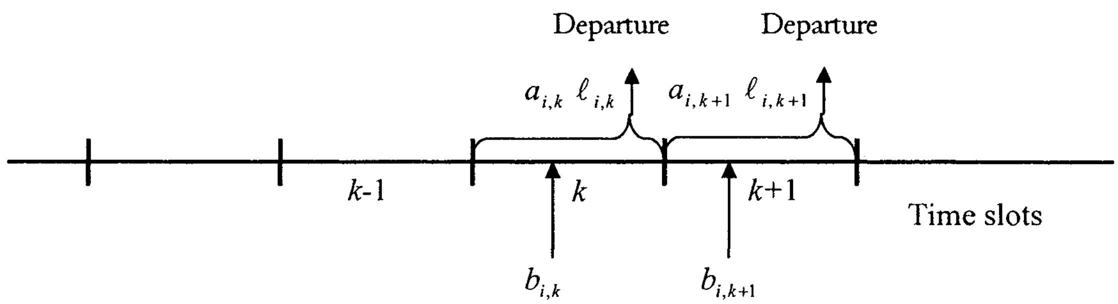
$f_{j,i,k}$ = number of packets generated by the j'th *On*-source of type-i during slot k. $f_{j,i,k}$ are independent identically distributed (i.i.d.) from slot to slot for type-i sources, with PGF $f_i(z_i)$.

$b_{i,k}$ = the total number of packets generated by type-i sources during slot k.

$c_{j,i}$ = a variable that assumes the values of 1, 0 if j'th source from type-i is in *On* and *Off* states in the next slot respectively, given that this source is *On* in the present slot.

$d_{j,i}$ = a variable that assumes the values of 1, 0 if j'th source from type-i is in *On* and *Off* states in the next slot respectively, given that this source is *Off* in the present slot.

The following figure shows the various random variables defined above,



The $c_{j,i}$, $d_{j,i}$ are i.i.d. Bernoulli random variables with the corresponding PGF given by:

$$c_i(z_i) = 1 - \alpha_i + \alpha_i z_i \ , \qquad d_i(z_i) = \beta_i + (1-\beta_i) z_i \tag{2.1}$$

From the above definitions, we have,

$$b_{i,k} = \sum_{j=1}^{a_{i,k}} f_{j,i,k} \ , \qquad a_{i,k+1} = \sum_{j=1}^{a_{i,k}} c_{j,i} + \sum_{j=1}^{m_i - a_{i,k}} d_{j,i} \tag{2.2}$$

The evolution of the first queue length is given by,

$$\ell_{1,k+1} = (\ell_{1,k} - 1)^+ + b_{1,k+1} \tag{2.3}$$

And the evolution of the i'th queue length (i >1) is given by,

$$\ell_{i,k+1} = (\ell_{i,k} - 1)^+ + b_{i,k+1} + u_{i,k} \quad , \quad 2 \le i \le n \tag{2.4}$$

where, $u_{i,k}$ is a random variable depending on whether the previous queue is empty or not,

$$u_{i,k} = \begin{cases} 1 & \text{if } \ell_{i-1,k} > 0 \\ 0 & \text{if } \ell_{i-1,k} = 0 \end{cases} \tag{2.5}$$

In the above equations the notation $(x)^+$ denotes $max(x, 0)$.

2.2 Tandem Network with Two Multiplexers

First, the simplest case of a tandem network with two multiplexers is considered. This system will be modeled as a discrete-time Markov Chain. Following that, the functional equation will be derived, which relates the PGF of the system between two consecutive slots. Afterwards, some preliminary results are given, so that the functional equation can be transformed into a new form that is mathematically more tractable.

2.2.1 Embedded Markov Chain Analysis

The state of the two multiplexer tandem network under consideration can be described by a four random variable set $(\ell_{1,k}, a_{1,k}, \ell_{2,k}, a_{2,k})$, where the notations $\ell_{1,k}, a_{1,k}, \ell_{2,k}$ and $a_{2,k}$ are defined in the previous section. Let us define $Q_k(z_1, y_1, z_2, y_2)$ as the joint PGF of $\ell_{1,k}, a_{1,k}, \ell_{2,k}$ and $a_{2,k}$, then,

$$Q_k(z_1, y_1, z_2, y_2) = E\left[z_1^{\ell_{1,k}} y_1^{a_{1,k}} z_2^{\ell_{2,k}} y_2^{a_{2,k}}\right] = \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{m_2} z_1^{i_1} y_1^{j_1} z_2^{i_2} y_2^{j_2} q_k(i_1, j_1, i_2, j_2) \tag{2.6}$$

$$\text{where, } q_k(i_1, j_1, i_2, j_2) = \Pr(\ell_{1,k} = i_1, a_{1,k} = j_1, \ell_{2,k} = i_2, a_{2,k} = j_2), \tag{2.7}$$

From (2.6), $Q_{k+1}(z_1, y_1, z_2, y_2)$ is given by,

$$Q_{k+1}(z_1, y_1, z_2, y_2) = E\left[z_1^{\ell_{1,k+1}} y_1^{a_{1,k+1}} z_2^{\ell_{2,k+1}} y_2^{a_{2,k+1}}\right] = \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{m_2} z_1^{i_1} y_1^{j_1} z_2^{i_2} y_2^{j_2} q_{k+1}(i_1, j_1, i_2, j_2) \tag{2.8}$$

Next, the relation between $Q_{k+1}(z_1, y_1, z_2, y_2)$ and $Q_k(z_1, y_1, z_2, y_2)$ will be derived. Let us substitute for $\ell_{1,k+1}, \ell_{2,k+1}$ in (2.8) from (2.3, 2.4), then we have,

$$Q_{k+1}(z_1, y_1, z_2, y_2) = E\left[z_1^{(\ell_{1,k}-1)^+ + b_{1,k+1}} y_1^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + b_{2,k+1} + u_{2,k}} y_2^{a_{2,k+1}}\right]$$

First, let us condition on $\ell_{1,k}, a_{1,k+1}, \ell_{2,k}, a_{2,k+1}$ and $u_{2,k}$; and substitute for $b_{1,k+1}, b_{2,k+1}$ from (2.2) in the above equation,

$$E\left[z_1^{(\ell_{1,k}-1)^+ + b_{1,k+1}} y_1^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + b_{2,k+1} + u_{2,k}} y_2^{a_{2,k+1}} \middle| \ell_{1,k}, a_{1,k+1}, \ell_{2,k}, a_{2,k+1}, u_{2,k} \right]$$

$$= z_1^{(\ell_{1,k}-1)^+} y_1^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} y_2^{a_{2,k+1}} E\left[z_1^{\sum_{j=1}^{a_{1,k+1}} f_{j1,k+1}} z_2^{\sum_{j=1}^{a_{2,k+1}} f_{j2,k+1}} \middle| \ell_{1,k}, a_{1,k+1}, \ell_{2,k}, a_{2,k+1}, u_{2,k} \right]$$

$$= z_1^{(\ell_{1,k}-1)^+} y_1^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} y_2^{a_{2,k+1}} [f_1(z_1)]^{a_{1,k+1}} [f_2(z_2)]^{a_{2,k+1}}$$

$$= z_1^{(\ell_{1,k}-1)^+} [y_1 f_1(z_1)]^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} [y_2 f_2(z_2)]^{a_{2,k+1}}$$

Then,

$$Q_{k+1}(z_1, y_1, z_2, y_2) = E\left[z_1^{(\ell_{1,k}-1)^+} [y_1 f_1(z_1)]^{a_{1,k+1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} [y_2 f_2(z_2)]^{a_{2,k+1}} \right]$$

Substituting for $a_{1,k+1}$ and $a_{2,k+1}$ from (2.2) in the above equation yields:

$$Q_{k+1}(z_1, y_1, z_2, y_2) = E\left[z_1^{(\ell_{1,k}-1)^+} [y_1 f_1(z_1)]^{\sum_{j=1}^{a_{1,k}} c_{j1} + \sum_{j=1}^{m_1 - a_{1,k}} d_{j1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} [y_2 f_2(z_2)]^{\sum_{j=1}^{a_{2,k}} c_{j2} + \sum_{j=1}^{m_2 - a_{2,k}} d_{j2}} \right] \tag{2.9}$$

Again, let us condition on $\ell_{1,k}, a_{1,k}, \ell_{2,k}, a_{2,k}$, $u_{2,k}$, and substitute from (2.1), we have,

$$E\left[z_1^{(\ell_{1,k}-1)^+} [y_1 f_1(z_1)]^{\sum_{j=1}^{a_{1,k}} c_{j1} + \sum_{j=1}^{m_1 - a_{1,k}} d_{j1}} z_2^{(\ell_{2,k}-1)^+ + u_k} [y_2 f_2(z_2)]^{\sum_{j=1}^{a_{2,k}} c_{j2} + \sum_{j=1}^{m_2 - a_{2,k}} d_{j2}} \middle| \ell_{1,k}, a_{1,k}, \ell_{2,k}, a_{2,k}, u_{2,k} \right]$$

$$= z_1^{(\ell_{1,k}-1)^+} [c_1(y_1 f_1(z_1))]^{a_{1,k}} [d_1(y_1 f_1(z_1))]^{m_1 - a_{1,k}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} [c_2(y_2 f_2(z_2))]^{a_{2,k}} [d_2(y_2 f_2(z_2))]^{m_2 - a_{2,k}}$$

$$= z_1^{(\ell_{1,k}-1)^+} \left(\frac{c_1(y_1 f_1(z_1))}{d_1(y_1 f_1(z_1))} \right)^{a_{1,k}} [d_1(y_1 f_1(z_1))]^{m_1} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} \left(\frac{c_2(y_2 f_2(z_2))}{d_2(y_2 f_2(z_2))} \right)^{a_{2,k}} [d_2(y_2 f_2(z_2))]^{m_2} \tag{2.10}$$

Next let us define:

$$Y_i = \frac{c_i(y_i f_i(z_i))}{d_i(y_i f_i(z_i))}, \quad B_i(1) = [d_i(y_i f_i(z_i))]^{m_i}, \quad B(1) = \prod_{i=1}^{2} B_i(1), \quad i = 1, 2, 3, \cdots n \tag{2.11}$$

Then (2.10) may be written as

$$E\left[z_1^{(\ell_{1,k}-1)^+} [y_1 f_1(z_1)]^{\sum_{j=1}^{a_{1,k}} c_{j1} + \sum_{j=1}^{m_1-a_{1,k}} d_{j1}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} [y_2 f_2(z_2)]^{\sum_{j=1}^{a_{2,k}} c_{j2} + \sum_{j=1}^{m_2-a_{2,k}} d_{j2}} \middle| \ell_{1,k}, a_{1,k}, \ell_{2,k}, a_{2,k}, u_{2,k} \right]$$
$$= B(1) z_1^{(\ell_{1,k}-1)^+} Y_1^{a_{1,k}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} Y_2^{a_{2,k}}$$

And (2.9) becomes

$$Q_{k+1}(z_1, y_1, z_2, y_2)$$
$$= E\left[B(1) z_1^{(\ell_{1,k}-1)^+} Y_1^{a_{1,k}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} Y_2^{a_{2,k}}\right]$$
$$= B(1) E\left[z_1^{(\ell_{1,k}-1)^+} Y_1^{a_{1,k}} z_2^{(\ell_{2,k}-1)^+ + u_{2,k}} Y_2^{a_{2,k}}\right]$$

In the next, let us remove the $()^+$ operation. From the definition in (2.6), the above equation may be expanded as,

$$Q_{k+1}(z_1, y_1, z_2, y_2)$$
$$= B(1) \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{m_2} z_1^{(i_1-1)^+} Y_1^{j_1} z_2^{(i_2-1)^+ + u_{2,k}} Y_2^{j_2} q_k(i_1, j_1, i_2, j_2)$$

$$Q_{k+1}(z_1, y_1, z_2, y_2)$$
$$= B(1) \sum_{i_1=1}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{m_2} z_1^{(i_1-1)} Y_1^{j_1} z_2^{(i_2-1)^+ + u_{2,k}} Y_2^{j_2} q_k(i_1, j_1, i_2, j_2)$$
$$+ B(1) \sum_{i_1=0}^{0} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{m_2} z_1^0 Y_1^{j_1} z_2^{(i_2-1)^+ + u_{2,k}} Y_2^{j_2} q_k(0, j_1, i_2, j_2)$$

$$Q_{k+1}(z_1, y_1, z_2, y_2)$$
$$= B(1) \sum_{i_1=1}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=1}^{\infty} \sum_{j_2=0}^{m_2} z_1^{(i_1-1)} Y_1^{j_1} z_2^{i_2-1+u_{2,k}} Y_2^{j_2} q_k(i_1, j_1, i_2, j_2)$$
$$+ B(1) \sum_{i_1=1}^{\infty} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{0} \sum_{j_2=0}^{m_2} z_1^{(i_1-1)} Y_1^{j_1} z_2^{u_{2,k}} Y_2^{j_2} q_k(i_1, j_1, 0, j_2)$$
$$+ B(1) \sum_{i_1=0}^{0} \sum_{j_1=0}^{m_1} \sum_{i_2=1}^{\infty} \sum_{j_2=0}^{m_2} z_1^0 Y_1^{j_1} z_2^{i_2-1} Y_2^{j_2} q_k(0, j_1, i_2, j_2)$$
$$+ B(1) \sum_{i_1=0}^{0} \sum_{j_1=0}^{m_1} \sum_{i_2=0}^{0} \sum_{j_2=0}^{m_2} z_1^0 Y_1^{j_1} z_2^0 Y_2^{j_2} q_k(0, j_1, 0, j_2)$$

Next, (2.5) may be used to remove the random variable $u_{2,k}$ from the above expression,

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$
$$= z_2 B(1)\sum_{i_1=1}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=1}^{\infty}\sum_{j_2=0}^{m_2} z_1^{(i_1-1)} Y_1^{j_1} z_2^{i_2-1} Y_2^{j_2} q_k(i_1,j_1,i_2,j_2)$$
$$+ z_2 B(1)\sum_{i_1=1}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{(i_1-1)} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(i_1,j_1,0,j_2)$$
$$+ B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=1}^{\infty}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{i_2-1} Y_2^{j_2} q_k(0,j_1,i_2,j_2)$$
$$+ B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(0,j_1,0,j_2)$$

The above equation may be written as,

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$
$$= \frac{1}{z_1} B(1)\sum_{i_1=1}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=1}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(i_1,j_1,i_2,j_2)$$
$$+ \frac{z_2}{z_1} B(1)\sum_{i_1=1}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(i_1,j_1,0,j_2)$$
$$+ \frac{1}{z_2} B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=1}^{\infty}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(0,j_1,i_2,j_2)$$
$$+ B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(0,j_1,0,j_2)$$

Next, letting the lower limits of i_1, i_2 in the summations start from 0 instead of 1 gives,

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$
$$= \frac{1}{z_1} B(1)\left\{\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(i_1,j_1,i_2,j_2) - \sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(0,j_1,i_2,j_2)\right.$$
$$\left. - \sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(i_1,j_1,0,j_2) + \sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(0,j_1,0,j_2)\right\}$$
$$+ \frac{z_2}{z_1} B(1)\left\{\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{i_1} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(i_1,j_1,0,j_2) - \sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(0,j_1,0,j_2)\right\}$$
$$+ \frac{1}{z_2} B(1)\left\{\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(0,j_1,i_2,j_2) - \sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{i_2} Y_2^{j_2} q_k(0,j_1,0,j_2)\right\}$$
$$+ B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{0} Y_1^{j_1} z_2^{0} Y_2^{j_2} q_k(0,j_1,0,j_2)$$

Combining the similar terms in the above gives,

$$
\begin{aligned}
&Q_{k+1}(z_1,y_1,z_2,y_2)\\
&=\frac{1}{z_1}B(1)\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(i_1,j_1,i_2,j_2)\\
&+\frac{z_2-1}{z_1}B(1)\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(i_1,j_1,0,j_2)\\
&+\frac{z_1-z_2}{z_1z_2}B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(0,j_1,i_2,j_2)\\
&+\frac{(z_2-1)(z_1-z_2)}{z_1z_2}B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(0,j_1,0,j_2)
\end{aligned}
\tag{2.12}
$$

As stated earlier in the assumptions, an *On* source generates at least one packet during a slot, which means that in (2.12) if $i_1=0$ then j_1 must be *zero*, if $i_2=0$ then j_2 must be *zero*. Therefore, we have $q_k(i_1,j_1,0,j_2)=0$ for all the cases of $j_2>0$; and $q_k(0,j_1,i_2,j_2)=0$ for all the cases of $j_1>0$; and $q_k(0,j_1,0,j_2)=0$ for all the cases of $j_1>0$ or $j_2>0$. So (2.12) becomes,

$$
\begin{aligned}
&Q_{k+1}(z_1,y_1,z_2,y_2)\\
&=\frac{1}{z_1}B(1)\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(i_1,j_1,i_2,j_2)\\
&+\frac{z_2-1}{z_1}B(1)\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{m_1}\sum_{i_2=0}^{0}\sum_{j_2=0}^{0} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(i_1,j_1,0,0)\\
&+\frac{z_1-z_2}{z_1z_2}B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{0}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{m_2} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(0,0,i_2,j_2)\\
&+\frac{(z_2-1)(z_1-z_2)}{z_1z_2}B(1)\sum_{i_1=0}^{0}\sum_{j_1=0}^{0}\sum_{i_2=0}^{0}\sum_{j_2=0}^{0} z_1^{i_1}Y_1^{j_1}z_2^{i_2}Y_2^{j_2}q_k(0,0,0,0)
\end{aligned}
$$

From the definition in (2.6), the above expression may be written as,

$$
\begin{aligned}
&Q_{k+1}(z_1,y_1,z_2,y_2)\\
&=B(1)\left\{\frac{1}{z_1}Q_k(z_1,Y_1,z_2,Y_2)+\frac{z_2-1}{z_1}Q_k(z_1,Y_1,0,0)\right.\\
&\qquad\left.+\frac{z_1-z_2}{z_1z_2}Q_k(0,0,z_2,Y_2)+\frac{(z_2-1)(z_1-z_2)}{z_1z_2}Q_k(0,0,0,0)\right\}\quad,\quad k\ge 0
\end{aligned}
\tag{2.13}
$$

This is the functional equation that relates the joint PGFs for two consecutive slots for a tandem network with two multiplexers. It is noted that $Q_k(z_1,Y_1,0,0)$ and $Q_k(0,0,z_2,Y_2)$ are referred as boundary functions and they are unknown.

Let $q(i_1, j_1, i_2, j_2)$ and $Q(z_1, y_1, z_2, y_2)$ denote the steady-state distribution and joint PGF of the system respectively, then we have $q(i_1, j_1, i_2, j_2) = \lim_{k\to\infty} q_k(i_1, j_1, i_2, j_2)$ and $Q(z_1, y_1, z_2, y_2) = \lim_{k\to\infty} Q_k(z_1, y_1, z_2, y_2)$. Therefore, the limiting form of the functional equation (2.13) at the steady-state is given by

$$
\begin{aligned}
&Q(z_1, y_1, z_2, y_2) \\
&= B(1)\left\{\frac{1}{z_1}Q(z_1, Y_1, z_2, Y_2) + \frac{z_2-1}{z_1}Q(z_1, Y_1, 0, 0)\right. \\
&\qquad \left. + \frac{z_1-z_2}{z_1 z_2}Q(0,0,z_2,Y_2) + \frac{(z_2-1)(z_1-z_2)}{z_1 z_2}Q(0,0,0,0)\right\}
\end{aligned}
\tag{2.14}
$$

It is noted that $Q(0,0,0,0)$ corresponds to the probability that the system is empty at the steady state.

2.2.2 Preliminary Results

In order to transform the functional equation in (2.13) into mathematically more tractable form, a number of preliminary results will be given here. More details regarding the preliminary results can be found in [10]. Let us define the following,

$$X_i(k+1) = X_i(1)[X_i(k)|_{y_i=Y_i}] \quad \text{with } X_i(0) = 1,\ X_i(1) = \beta_i + (1-\beta_i)y_i f_i(z_i) \tag{2.15}$$

$$U_i(k+1) = X_i(1)[U_i(k)|_{y_i=Y_i}] \quad \text{with } U_i(0) = y_i,\ U_i(1) = 1-\alpha_i + \alpha_i y_i f_i(z_i) \tag{2.16}$$

where $1 \le i \le n$.

$X_i(k)$ and $U_i(k)$ defined in the above have the recurrence relationships given below:

$$X_i(k) = [\beta_i + \alpha_i f_i(z_i)]X_i(k-1) + [1-\alpha_i-\beta_i]f_i(z_i)X_i(k-2)\ , \qquad k \ge 2 \tag{2.17}$$

$$U_i(k) = [\beta_i + \alpha_i f_i(z_i)]U_i(k-1) + [1-\alpha_i-\beta_i]f_i(z_i)U_i(k-2)\ , \qquad k \ge 2 \tag{2.18}$$

Let us define:

$$\phi_i(k) = \frac{U_i(k)}{X_i(k)} \quad \text{,then, } \phi_i(0) = y_i\ ,\quad \phi_i(1) = Y_i\ ,\quad \phi_i(k+1) = \phi_i(k)|_{y_i=Y_i} \tag{2.19}$$

$$B_i(k) = [X_i(k)]^{m_i},\quad B(k) = \prod_{i=1}^{2} B_i(k),\quad B_i^{\,j}(k) = B_i(k)|_{y_i=\phi_i(j)} \tag{2.20}$$

From (2.20) we have $B_i(1) = [X_i(1)]^{m_i}$, $B(1) = \prod_{i=1}^{2} B_i(1)$, which are the same as defined in (2.11). It is noted that $B_i^0(k) = B_i(k)$, and from (2.19, 2.20) it is easy to show that,

$$B_i(k+j) = B_i(j) B_i^{\,j}(k) \tag{2.21}$$

The homogeneous difference equations given in (2.17, 2.18) have the following characteristic equation,

$$\lambda_i^2 - [\beta_i + \alpha_i f_i(z_i)]\lambda_i - (1-\alpha_i-\beta_i) f_i(z_i) = 0$$

The roots of the above equation are given by,

$$\lambda_{1i,2i} = \frac{\beta_i + \alpha_i f_i(z_i) \mp \sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}}{2} \tag{2.22}$$

Then, the solutions of the difference equations are given by,

$$U_i(k) = D_{1i}\lambda_{1i}^k + D_{2i}\lambda_{2i}^k \quad , \quad X_i(k) = C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k \tag{2.23}$$

where,

$$C_{1i,2i} = \frac{1}{2} \mp \frac{2(y_i - y_i\beta_i - \alpha_i) f_i(z_i) + (\beta_i + \alpha_i f_i(z_i))}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}} \tag{2.24}$$

$$D_{1i,2i} = \frac{y_i}{2} \mp \frac{2(1-\alpha_i + \alpha_i y_i f_i(z_i)) - (\beta_i + \alpha_i f_i(z_i)) y_i}{2\sqrt{(\beta_i + \alpha_i f_i(z_i))^2 + 4(1-\alpha_i-\beta_i) f_i(z_i)}} \tag{2.25}$$

In the above expressions $\lambda_{1i}, C_{1i}, D_{1i}$ are taken with the negative sign and $\lambda_{2i}, C_{2i}, D_{2i}$ are taken with the positive sign.

From (2.20, 2.23), we have,

$$B_i(k) = (C_{1i}\lambda_{1i}^k + C_{2i}\lambda_{2i}^k)^{m_i} \tag{2.26}$$

2.2.3 New Form of the Functional Equation

With the above preliminary results, the original functional equation (2.13) can be transformed into a new form, which is mathematically more tractable. Because the steady-state distribution of a Markov Chain does not depend on its initial distribution, and this research work is only interested in the steady-state joint PGF, the initial state of the system does not matter. For simplicity, the following zero-initial conditions will be assumed,

$$Q_0(z_1,y_1,z_2,y_2)=1,\ Q_0(0,0,z_2,y_2)=1,\ Q_0(z_1,y_1,0,0)=1,\ Q_0(0,0,0,0)=1 \tag{2.27}$$

which means that the initial queue lengths are zero and all the sources are in *Off* states.

In the next, the functional equation (2.13) will be transformed into a new form. At first, it may be shown that the functional equation may be expressed as follows,

$$\begin{aligned}
&Q_{k+1}(z_1,y_1,z_2,y_2)\\
&=\frac{1}{(z_1z_2)^k}\Bigg\{z_2^{\,k}B(k+1)+(z_2-1)\sum_{j=1}^{k}z_2^{\,j}(z_1z_2)^{k-j}B(j)Q_{k+1-j}(z_1,\phi_1(j),0,0)\\
&\qquad+(z_1-z_2)\sum_{j=1}^{k}z_2^{\,j-1}(z_1z_2)^{k-j}B(j)Q_{k+1-j}(0,0,z_2,\phi_2(j))\\
&\qquad+(z_2-1)(z_1-z_2)\sum_{j=1}^{k}z_2^{\,j-1}(z_1z_2)^{k-j}B(j)Q_{k+1-j}(0,0,0,0)\Bigg\}
\end{aligned} \tag{2.28}$$

In the above, it is assumed that if the upper limit of a summation is less than its lower limit, then that summation is empty. The proof of the above result will be given in the following through induction. First, expanding $Q_{k+1}(z_1,y_1,z_2,y_2)$ in (2.13) for the first few values of k, we have,

I) For $k=0$

The functional equation (2.13) gives,

$$\begin{aligned}
&Q_1(z_1,y_1,z_2,y_2)\\
&=B(1)\Bigg\{\frac{1}{z_1}Q_0(z_1,Y_1,z_2,Y_2)+\frac{z_2-1}{z_1}Q_0(z_1,Y_1,0,0)\\
&\qquad+\frac{z_1-z_2}{z_1z_2}Q_0(0,0,z_2,Y_2)+\frac{(z_2-1)(z_1-z_2)}{z_1z_2}Q_0(0,0,0,0)\Bigg\}
\end{aligned}$$

From the zero-initial condition assumption in (2.27), we have

$$Q_1(z_1,y_1,z_2,y_2)=B(1)\left\{\frac{1}{z_1}+\frac{z_2-1}{z_1}+\frac{z_1-z_2}{z_1z_2}+\frac{(z_2-1)(z_1-z_2)}{z_1z_2}\right\}$$

or equivalently,

$$Q_1(z_1,y_1,z_2,y_2)=B(1) \tag{2.29}$$

Thus, it may be seen that (2.28) is true for $k=0$.

II) For $k = 1$

The functional equation (2.13) gives

$$Q_2(z_1,y_1,z_2,y_2)$$
$$= B(1)\left\{\frac{1}{z_1}Q_1(z_1,\phi_1(1),z_2,\phi_2(1)) + \frac{z_2-1}{z_1}Q_1(z_1,\phi_1(1),0,0)\right.$$
$$\left. + \frac{z_1-z_2}{z_1z_2}Q_1(0,0,z_2,\phi_2(1)) + \frac{(z_2-1)(z_1-z_2)}{z_1z_2}Q_1(0,0,0,0)\right\}$$

Substituting (2.20, 2.29) in the above, we have,

$$Q_2(z_1,y_1,z_2,y_2)$$
$$= B(1)\left\{\frac{1}{z_1}B^1(1) + \frac{z_2-1}{z_1}Q_1(z_1,\phi_1(1),0,0)\right.$$
$$\left. + \frac{z_1-z_2}{z_1z_2}Q_1(0,0,z_2,\phi_2(1)) + \frac{(z_2-1)(z_1-z_2)}{z_1z_2}Q_1(0,0,0,0)\right\}$$

Applying (2.21) in the above equation gives

$$Q_2(z_1,y_1,z_2,y_2)$$
$$= \frac{1}{z_1}B(2) + \frac{z_1-z_2}{z_1z_2}B(1)Q_1(0,0,z_2,\phi_2(1)) \tag{2.30}$$
$$+ \frac{z_2-1}{z_1z_2}B(1)Q_1(z_1,\phi_1(1),0,0) + \frac{(z_2-1)(z_1-z_2)}{z_1z_2}B(1)Q_1(0,0,0,0)$$

Thus, it may be seen that (2.28) is true for $k = 1$.

Now, let us assume that (2.28) is true for the order k, which is

$$Q_k(z_1,y_1,z_2,y_2)$$
$$= \frac{1}{(z_1z_2)^{k-1}}\left\{z_2^{\,k-1}B(k) + (z_2-1)\sum_{j=1}^{k-1} z_2^{\,j}(z_1z_2)^{k-1-j}B(j)Q_{k-j}(z_1,\phi_1(j),0,0)\right.$$
$$+ (z_1-z_2)\sum_{j=1}^{k-1} z_2^{\,j-1}(z_1z_2)^{k-1-j}B(j)Q_{k-j}(0,0,z_2,\phi_2(j)) \tag{2.31}$$
$$\left. + (z_2-1)(z_1-z_2)\sum_{j=1}^{k-1} z_2^{\,j-1}(z_1z_2)^{k-1-j}B(j)Q_{k-j}(0,0,0,0)\right\}$$

Then, it will be shown that (2.28) is also true for order $k+1$.

Substituting (2.31) in (2.13) gives

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$

$$=\frac{B(1)}{(z_1z_2)^k}\Bigg\{z_2{}^kB^1(k)++(z_2-1)\sum_{j=1}^{k-1}z_2{}^{j+1}(z_1z_2)^{k-1-j}B^1(j)Q_{k-j}(z_1,\phi_1(j+1),0,0)$$

$$(z_1-z_2)\sum_{j=1}^{k-1}z_2{}^j(z_1z_2)^{k-1-j}B^1(j)Q_{k-j}(0,0,z_2,\phi_2(j+1))$$

$$+(z_2-1)(z_1-z_2)\sum_{j=1}^{k-1}z_2{}^j(z_1z_2)^{k-1-j}B^1(j)Q_{k-j}(0,0,0,0)\Bigg\}$$

$$+\frac{B(1)}{z_1z_2}\Big\{(z_2-1)z_2Q_k(z_1,\phi_1(1),0,0)+(z_1-z_2)Q_k(0,0,z_2,\phi_2(1))+(z_2-1)(z_1-z_2)Q_k(0,0,0,0)\Big\}$$

From (2.21), after rearranging the terms, we have,

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$

$$=\frac{1}{z_1{}^k}B(k+1)+\frac{1}{(z_1z_2)^k}(z_2-1)\sum_{j=1}^{k-1}z_2{}^{j+1}(z_1z_2)^{k-1-j}B(j+1)Q_{k-j}(z_1,\phi_1(j+1),0,0)$$

$$+\frac{B(1)}{z_1}(z_2-1)Q_k(z_1,\phi_1(1),0,0)$$

$$+\frac{1}{(z_1z_2)^k}(z_1-z_2)\sum_{j=1}^{k-1}z_2{}^j(z_1z_2)^{k-1-j}B(j+1)Q_{k-j}(0,0,z_2,\phi_2(j+1))$$

$$+\frac{B(1)}{z_1z_2}(z_1-z_2)Q_k(0,0,z_2,\phi_2(1))$$

$$+\frac{1}{(z_1z_2)^k}(z_2-1)(z_1-z_2)\sum_{j=1}^{k-1}z_2{}^j(z_1z_2)^{k-1-j}B^1(j)Q_{k-j}(0,0,0,0)$$

$$+\frac{B(1)}{z_1z_2}(z_2-1)(z_1-z_2)Q_k(0,0,0,0)$$

Finally we obtain,

$$Q_{k+1}(z_1,y_1,z_2,y_2)$$

$$=\frac{1}{(z_1z_2)^k}\Bigg\{z_2{}^kB(k+1)+(z_2-1)\sum_{j=1}^{k}z_2{}^j(z_1z_2)^{k-j}B(j)Q_{k+1-j}(z_1,\phi_1(j),0,0)$$

$$+(z_1-z_2)\sum_{j=1}^{k}z_2{}^{j-1}(z_1z_2)^{k-j}B(j)Q_{k+1-j}(0,0,z_2,\phi_2(j))$$

$$+(z_2-1)(z_1-z_2)\sum_{j=1}^{k}z_2{}^{j-1}(z_1z_2)^{k-j}B(j)Q_{k+1-j}(0,0,0,0)\Bigg\}$$

The above is the same as (2.28). Thus, it has been shown that (2.28) is also true for order $k+1$. This completes the proof of (2.28). Actually, the solution of the functional equation in (2.13) is given by (2.28) except for the unknown boundary functions, $Q_k(0,0,z_2,y_2)$ and $Q_k(z_1,y_1,0,0)$.

2.3 Tandem Network with Arbitrary Number of Multiplexers

Next, a general tandem network with arbitrary number of multiplexers is considered. Assuming that the number of multiplexers in the network is greater than n ($n \geq 2$), then the behavior of the n'th multiplexer is not affected by the multiplexers on its downstream and the total effect of the multiplexers in its upstream has been summarized in the output of the (n-1)'st multiplexer. Thus, in order to determine the performance of the n'th multiplexer, one only needs to consider the joint performance of the (n-1)'st and n'th multiplexers.

The system that consists of (n-1)'st and n'th multiplexers may be modeled using a discrete-time four-dimensional Markov Chain as before. The state of the system under consideration can be defined by the set of four random variables, $(\ell_{n-1,k}, a_{n-1,k}, \ell_{n,k}, a_{n,k})$. Let $Q_k(z_{n-1}, y_{n-1}, z_n, y_n)$ denote the joint PGF of this system,

$$
\begin{aligned}
&Q_k(z_{n-1}, y_{n-1}, z_n, y_n) \\
&= E\left[z_{n-1}^{\ell_{n-1,k}} y_{n-1}^{a_{n-1,k}} z_n^{\ell_{n,k}} y_n^{a_{n,k}}\right] \\
&= \sum_{i_{n-1}=0}^{\infty} \sum_{j_{n-1}=0}^{m_{n-1}} \sum_{i_n=0}^{\infty} \sum_{j_n=0}^{m_n} z_{n-1}^{i_{n-1}} y_{n-1}^{j_{n-1}} z_n^{i_n} y_n^{j_n} q_k(i_{n-1}, j_{n-1}, i_n, j_n) \quad , \quad n \geq 2
\end{aligned}
\tag{2.32}
$$

where, $q_k(i_{n-1}, j_{n-1}, i_n, j_n) = \Pr(\ell_{n-1,k} = i_{n-1}, a_{n-1,k} = j_{n-1}, \ell_{n,k} = i_n, a_{n,k} = j_n)$,

Then $Q_{k+1}(z_{n-1}, y_{n-1}, z_n, y_n)$ is given by,

$$
\begin{aligned}
&Q_{k+1}(z_{n-1}, y_{n-1}, z_n, y_n) \\
&= E\left[z_{n-1}^{\ell_{n-1,k+1}} y_{n-1}^{a_{n-1,k+1}} z_n^{\ell_{n,k+1}} y_n^{a_{n,k+1}}\right] \\
&= \sum_{i_{n-1}=0}^{\infty} \sum_{j_{n-1}=0}^{m_{n-1}} \sum_{i_n=0}^{\infty} \sum_{j_n=0}^{m_n} z_{n-1}^{\ell_{n-1}} y_{n-1}^{j_{n-1}} z_n^{\ell_n} y_n^{j_n} q_{k+1}(i_{n-1}, j_{n-1}, i_n, j_n) \quad , \quad n \geq 2
\end{aligned}
\tag{2.33}
$$

Following the embedded Markov chain analysis of section 2.2.1, the relation between $Q_k(z_{n-1}, y_{n-1}, z_n, y_n)$ and $Q_{k+1}(z_{n-1}, y_{n-1}, z_n, y_n)$ may be determined,

$$
\begin{aligned}
&Q_{k+1}\left(z_{n-1}, y_{n-1}, z_n, y_n\right) \\
&= B_{n-1}(1)B_n(1)\left\{\frac{1}{z_{n-1}}Q_k\left(z_{n-1}, Y_{n-1}, z_n, Y_n\right) + \frac{z_n - 1}{z_{n-1}}Q_k\left(z_{n-1}, Y_{n-1}, 0_n, 0_n\right)\right. \\
&\quad \left. + \frac{z_{n-1} - z_n}{z_{n-1}z_n}Q_k\left(0_{n-1}, 0_{n-1}, z_n, Y_n\right) + \frac{(z_n - 1)(z_{n-1} - z_n)}{z_{n-1}z_n}Q_k\left(0_{n-1}, 0_{n-1}, 0_n, 0_n\right)\right\} \\
&\qquad , \quad k \geq 0, \; n \geq 2
\end{aligned} \tag{2.34}
$$

where, $Q_{k-j}(z_{n-1}, Y_{n-1}, 0_n, 0_n) = Q_{k-j}(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_n=0, y_n=0}$

$Q_{k-j}(0_{n-1}, 0_{n-1}, z_n, Y_n) = Q_{k-j}(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=0, y_{n-1}=0}$

$Q_{k-j}(0_{n-1}, 0_{n-1}, 0_n, 0_n) = Q_{k-j}(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=0, y_{n-1}=0, z_n=0, y_n=0}$

In the above equation, substituting 1 for z_{n-1} and y_{n-1} gives the joint PGF of the n'th multiplexer,

$$
\begin{aligned}
&Q_{k+1}\left(1_{n-1}, 1_{n-1}, z_n, y_n\right) \\
&= B_n(1)\left\{Q_k\left(1_{n-1}, 1_{n-1}, z_n, Y_n\right) + (z_n - 1)Q_k\left(1_{n-1}, 1_{n-1}, 0_n, 0_n\right)\right. \\
&\quad \left. + \frac{1 - z_n}{z_n}Q_k\left(0_{n-1}, 0_{n-1}, z_n, Y_n\right) - \frac{(z_n - 1)^2}{z_n}Q_k\left(0_{n-1}, 0_{n-1}, 0_n, 0_n\right)\right\} \quad , \quad k \geq 0
\end{aligned} \tag{2.35}
$$

where $Q_{k-j}(1_{n-1}, 1_{n-1}, z_n, Y_n) = Q_{k-j}(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=1,\, y_{n-1}=1}$,

$Q_{k-j}(1_{n-1}, 1_{n-1}, 0_n, 0_n) = Q_{k-j}(z_{n-1}, y_{n-1}, z_n, y_n)\big|_{z_{n-1}=1, y_{n-1}=1, z_n=0, y_n=0}$

This functional equation relates the joint PGFs of the n'th multiplexer in a tandem network for two consecutive slots. It is noted that $Q_k(0_{n-1}, 0_{n-1}, z_n, Y_n)$ is referred as a boundary function and that it is unknown.

Letting $q(i_n, j_n)$ and $Q(1_{n-1}, 1_{n-1}, z_n, y_n)$ denote the probability distribution and joint PGF of the of the n'th multiplexer at the steady state, we have $q(i_n, j_n) = \lim_{k\to\infty} q_k(i_n, j_n)$ and $Q(1_{n-1}, 1_{n-1}, z_n, y_n) = \lim_{k\to\infty} Q_k(1_{n-1}, 1_{n-1}, z_n, y_n)$.

Then the limiting form of the functional equation (2.35) at steady state is given by,

$$Q(1_{n-1},1_{n-1},z_n,y_n)$$

$$= B_n(1)\left\{Q(1_{n-1},1_{n-1},z_n,Y_n)+(z_n-1)Q(1_{n-1},1_{n-1},0_n,0_n)\right. \tag{2.36}$$

$$\left. +\frac{1-z_n}{z_n}Q(0_{n-1},0_{n-1},z_n,Y_n)-\frac{(z_n-1)^2}{z_n}Q(0_{n-1},0_{n-1},0_n,0_n)\right\} \quad , \quad k \geq 0$$

Next, as before the functional equation (2.35) will be transformed into a mathematically more tractable form. Again, zero-initial conditions are assumed,

$$Q_0(1_{n-1},1_{n-1},z_n,y_n)=1,\ Q_0(0_{n-1},0_{n-1},z_n,y_n)=1,$$

$$Q_0(1_{n-1},1_{n-1},0_n,0_n)=1,\ Q_0(0_{n-1},0_{n-1},0_n,0_n)=1 \tag{2.37}$$

Then, new form of (2.35) is given below,

$$Q_k(1_{n-1},1_{n-1},z_n,y_n)$$

$$= B_n(k)+(z_n-1)\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(1_{n-1},1_{n-1},0,0)$$

$$+\frac{1-z_n}{z_n}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},0_{n-1},z_n,\phi_n(j)) \tag{2.38}$$

$$-\frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},0_{n-1},0_n,0_n) \qquad k\geq 1, n\geq 2$$

The proof of the above equation is also through induction and the application of the preliminary results, which is similar to the proof of (2.28), therefore, the derivation will not be given here. Basically, the solution of the functional equation (2.35) is given by (2.38) except for the unknown boundary function, $Q_k(0,0,z_n,y_n)$.

Chapter 3

Performance Analysis of Two Multiplexers in Tandem

In this chapter, the performance of the simplest case of a tandem network with only two multiplexers is studied. The functional equation derived in the previous chapter is solved by using a straightforward approach. First, the two unknown boundary functions are determined, which lead to the determination of the joint PGF of the queue length and the number of *On* sources for the second multiplexer at the steady state. Then, from this joint PGF the marginal PGF of queue length as well as the corresponding performance measures are determined. Unfortunately, this solution technique does not extend to a tandem network with more than two multiplexers.

3.1 Determining the Boundary Functions

The functional equation describing two multiplexers in tandem is given in (2.28), which contains two unknown boundary functions, $Q_k(0,0,z_2,y_2)$ and $Q_k(z_1,y_1,0,0)$. In this section, these boundary functions are determined.

First, the boundary function $Q_k(z_1,y_1,0,0)$ is already available in the literature [10]. In the next, the derivation of this boundary function is explained. It is noted that $Q_k(1,1,0,0)$ is the probability that the second multiplexer is empty and all of its m_2 sources are in the *Off* state in slot k. Thus, given $Q_k(1,1,0,0)$, the first multiplexer must be empty in the previous slot and all of its m_1 sources must be in the *Off* state in slot k-1; otherwise the first multiplexer would output packets to the second one in slot k and then the second multiplexer cannot be empty in slot k, which conflicts with the given condition $Q_k(1,1,0,0)$. As a result, we have $Q_{k-1}(z_1=0, y_1=0 \mid \ell_{2,k}=0, a_{2,k}=0)$. From this zero initial condition at slot k-1, the first multiplexer reaches to slot k through a one-slot evolution, which is $[\beta_1+(1-\beta_1)y_1 f_1(z_1)]^{m_1}$. Therefore, $Q_k(z_1, y_1, 0,0)$ is given by

$$Q_k(z_1, y_1, 0,0) = Q_k(1,1,0,0)[\beta_1+(1-\beta_1)y_1 f_1(z_1)]^{m_1} \quad , \quad k \geq 1,$$

or equivalently

$$Q_k(z_1, y_1, 0,0) = B_1(1)Q_k(1,1,0,0) \quad , \quad k \geq 1 \tag{3.1}$$

Unfortunately, there is no such an easy way to derive the other unknown boundary function $Q_k(0,0,z_2,y_2)$. Next, this unknown boundary function will be expressed in terms of the known one in (3.1).

The functional equation in (2.28) may be written as,

$$\begin{aligned}
&Q_{k+1}(z_1, y_1, z_2, y_2) \\
&= \frac{1}{(z_1 z_2)^k}\Bigg\{ z_2^k B(k+1) + (z_2-1)\sum_{j=1}^{k} z_1^{k-j} z_2^k B(j) Q_{k+1-j}(z_1, \phi_1(j), 0,0) \\
&\qquad + \sum_{j=1}^{k} z_1^{k+1-j} z_2^{k-1} B(j) Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&\qquad - \sum_{j=1}^{k} z_1^{k-j} z_2^{k} B(j) Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&\qquad + (z_2-1)\sum_{j=1}^{k} z_1^{k+1-j} z_2^{k-1} B(j)B(j) Q_{k+1-j}(0,0,0,0) \\
&\qquad - (z_2-1)\sum_{j=1}^{k} z_1^{k-j} z_2^{k} B(j)B(j) Q_{k+1-j}(0,0,0,0)\Bigg\}
\end{aligned} \tag{3.2}$$

Substituting 0 for z_1, y_1 in the above equation results in the form of $\frac{0}{0}$ indeterminacy, therefore, L'Hopital's rule is applied, which gives,

$$
\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2) \\
&= \frac{1}{k!z_2^k}\Bigg\{ z_2^k \cdot k! \cdot \overline{B}_1^{(k)}(k+1)B_2(k+1) \\
&\quad + (z_2-1)\sum_{j=1}^{k} z_2^k B_2(j)\left[\frac{d^k}{dz_1^k}\left(z_1^{k-j}B_1(j)Q_{k+1-j}(z_1,\phi_1(j),0,0)\right)\Big|_{z_1=y_1=0} \right] \\
&\quad + \sum_{j=1}^{k} z_2^{k-1} \cdot k! \cdot \overline{B}_1^{(j-1)}(j)B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&\quad - \sum_{j=1}^{k} z_2^{k} \cdot k! \cdot \overline{B}_1^{(j)}(j)B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&\quad + (z_2-1)\sum_{j=1}^{k} z_2^{k-1} \cdot k! \cdot \overline{B}_1^{(j-1)}(j)B_2(j)Q_{k+1-j}(0,0,0,0) \\
&\quad - (z_2-1)\sum_{j=1}^{k} z_2^{k} \cdot k! \cdot \overline{B}_1^{(j)}(j)B_2(j)Q_{k+1-j}(0,0,0,0) \Bigg\}
\end{aligned}
\tag{3.3}
$$

where the notation $\overline{B}_i^{(\ell)}(j)$ denotes $\overline{B}_i^{(\ell)}(j) = \frac{1}{\ell!}\frac{d^\ell B_i(j)}{dz_i^\ell}\Big|_{z_i=y_i=0}$ (3.4)

Rearranging the terms in the above gives,

$$
\begin{aligned}
&Q_{k+1}\ (0,0,z_2,y_2) \\
&= \overline{B}_1^{(k)}(k+1)B_2(k+1) \\
&+ (z_2-1)\sum_{j=1}^{k} \frac{1}{k!} B_2(j)\left[\frac{d^k}{dz_1^k}\left(z_1^{k-j}B_1(j)Q_{k+1-j}(z_1,\phi_1(j),0,0)\right)\Big|_{z_1=y_1=0} \right] \\
&+ \sum_{j=1}^{k} \frac{1}{z_2} \overline{B}_1^{(j-1)}(j)B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&- \sum_{j=1}^{k} \overline{B}_1^{(j)}(j)B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j)) \\
&+ (z_2-1)\sum_{j=1}^{k} \frac{1}{z_2} \overline{B}_1^{(j-1)}(j)B_2(j)Q_{k+1-j}(0,0,0,0) \\
&- (z_2-1)\sum_{j=1}^{k} \overline{B}_1^{(j)}(j)B_2(j)Q_{k+1-j}(0,0,0,0)
\end{aligned}
\tag{3.5}
$$

Let us define the following additional notations:

$$
g_j = \overline{B}_1^{(j)}(j+1), \quad g_j' = \overline{B}_1^{(j)}(j) \tag{3.6}
$$

$$f_j = \frac{1}{z_2} g_{j-1} - g'_j \ , \tag{3.7}$$

$$h_{k+1-j,j} = (z_2 - 1)\frac{1}{k!}\left[\frac{d^k}{dz_1^k}\left(z_1^{k-j} B_1(j) Q_{k+1-j}(z_1, \phi_1(j), 0, 0)\right)\Big|_{z_1 = y_1 = 0} \right] \tag{3.8}$$

$$\Re_j = (z_2 - 1) f_j \tag{3.9}$$

$$I_{k,j} = h_{k+1-j,j} + \Re_j Q_{k+1-j}(0,0,0,0) \tag{3.10}$$

It is noted that the maximum power of z_2 in the denominators of f_j, g_j and $h_{k+1-j,j}$ are all j.

Then, equation (3.5) may be expressed as

$$Q_{k+1}(0,0,z_2,y_2) = g_k B_2(k+1) + \sum_{j=1}^{k} f_j B_2(j) Q_{k+1-j}(0,0,z_2,\phi_2(j)) + \sum_{j=1}^{k} I_{k,j} B_2(j) \tag{3.11}$$

Next, the expression $Q_{k+1-j}(0,0,z_2,\phi_2(j))$ will be eliminated in the above equation, and $Q_{k+1}(0,0,z_2,y_2)$ is expressed in terms of $Q_1(0,0,z_2,y_2)$ and the other boundary function. For this purpose, $Q_{k+1}(0,0,z_2,y_2)$ is determined for the first few values of k.

For k =1

$$Q_2(0,0,z_2,y_2) = g_1 B_2(2) + \sum_{j=1}^{1} f_j B_2(j) Q_{2-j}(0,0,z_2,\phi_2(j)) + \sum_{j=1}^{1} B_2(j) I_{1,j}$$
$$Q_2(0,0,z_2,y_2) = g_1 B_2(2) + f_1 B_2(1) Q_1(0,0,z_2,\phi_2(1)) + B_2(1) I_{1,1}$$

For k =2

$$Q_3(0,0,z_2,y_2) = g_2 B_2(3) + \sum_{j=1}^{2} f_j B_2(j) Q_{3-j}(0,0,z_2,\phi_2(j)) + \sum_{j=1}^{2} B_2(j) I_{2,j}$$

$$Q_3(0,0,z_2,y_2) = g_2 B_2(3) + f_1 B_2(1) Q_2(0,0,z_2,\phi_2(1)) + f_2 B_2(2) Q_1(0,0,z_2,\phi_2(2)) + \sum_{j=1}^{2} B_2(j) I_{2,j}$$

$$Q_3(0,0,z_2,y_2) = g_2 B_2(3) + f_1 B_2(1)\left[g_1 B_2^1(2) + f_1 B_2^1(1) Q_1(0,0,z_2,\phi_2(2)) + \sum_{j=1}^{1} B_2^1(j) I_{1,j} \right]$$
$$+ f_2 B_2(2) Q_1(0,0,z_2,\phi_2(2)) + \sum_{j=1}^{2} B_2(j) I_{2,j}$$

$$Q_3(0,0,z_2,y_2) = g_2 B_2(3) + f_1 g_1 B_2(3) + f_1^2 B_2(2) Q_1\big(0,0,z_2,\phi_2(2)\big) + f_1 \sum_{j=1}^{1} B_2(j+1) I_{1,j}$$

$$+ f_2 B_2(2) Q_1\big(0,0,z_2,\phi_2(2)\big) + \sum_{j=1}^{2} B_2(j) I_{2,j}$$

$$Q_3(0,0,z_2,y_2) = (g_2 + f_1 g_1) B_2(3) + (f_1^2 + f_2) B_2(2) Q_1(0,0,z_2,\phi_2(2))$$

$$+ f_1 \sum_{j=1}^{1} B_2(j+1) I_{1,j} + \sum_{j=1}^{2} B_2(j) I_{2,j}$$

For k =3

$$Q_4(0,0,z_2,y_2) = g_3 B_2(4) + \sum_{j=1}^{3} f_j B_2(j) Q_{4-j}\big(0,0,z_2,\phi_2(j)\big) + \sum_{j=1}^{3} B_2(j) I_{3,j}$$

$$Q_4(0,0,z_2,y_2) = g_3 B(4) + f_1 B_2(1) Q_3\big(0,0,z_2,\phi_2(1)\big) + f_2 B_2(2) Q_2\big(0,0,z_2,\phi_2(2)\big)$$

$$+ f_3 B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big) + \sum_{j=1}^{3} B_2(j) I_{3,j}$$

$$Q_4(0,0,z_2,y_2) = g_3 B(4) + f_1(g_2 + f_1 g_1) B_2(4) + f_1(f_1^2 + f_2) B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big)$$

$$+ f_1^2 \sum_{j=1}^{1} B_2(j+2) I_{1,j} + f_1 \sum_{j=1}^{2} B_2(j+1) I_{2,j} + f_2 g_1 B_2(4) + f_1 f_2 B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big)$$

$$+ f_2 \sum_{j=1}^{1} B_2(j+2) I_{1,j} + f_3 B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big) + \sum_{j=1}^{3} B_2(j) I_{3,j}$$

$$Q_4(0,0,z_2,y_2) = \big(g_3 + f_2 g_1 + f_1(g_2 + f_1 g_1)\big) B_2(4)$$

$$+ \big(f_1(f_1^2 + f_2) + f_1 f_2 + f_3\big) B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big)$$

$$+ f_1^2 \sum_{j=1}^{1} B_2(j+2) I_{1,j} + f_1 \sum_{j=1}^{2} B_2(j+1) I_{2,j} + f_2 \sum_{j=1}^{1} B_2(j+2) I_{1,j} + \sum_{j=1}^{3} B_2(j) I_{3,j}$$

$$Q_4(0,0,z_2,y_2) = \big(g_3 + f_2 g_1 + (f_1 g_1 + g_2) f_1\big) B_2(4)$$

$$+ \big(f_3 + f_2 f_1 + f_1(f_1^2 + f_2)\big) B_2(3) Q_1\big(0,0,z_2,\phi_2(3)\big)$$

$$+ (f_1^2 + f_2) \sum_{j=1}^{1} B_2(j+2) I_{1,j} + f_1 \sum_{j=1}^{2} B_2(j+1) I_{2,j} + \sum_{j=1}^{3} B_2(j) I_{3,j}$$

For k =4

$$Q_5(0,0,z_2,y_2) = g_4 B_2(5) + \sum_{j=1}^{4} f_j B_2(j) Q_{5-j}\big(0,0,z_2,\phi_2(j)\big) + \sum_{j=1}^{4} B_2(j) I_{4,j}$$

$$Q_5(0,0,z_2,y_2) = g_4 B_2(5) + f_1 B_2(1) Q_4(0,0,z_2,\phi_2(1)) + f_2 B_2(2) Q_3(0,0,z_2,\phi_2(2))$$
$$+ f_3 B_2(3) Q_2(0,0,z_2,\phi_2(3)) + f_4 B_2(4) Q_1(0,0,z_2,\phi_2(1)) + \sum_{j=1}^{4} B_2(j) I_{4,j}$$

$$\begin{aligned}
& Q_5(0,0,z_2,y_2) \\
&= g_4 B(5) + f_1\big(g_3 + g_2 f_1 + (f_1^2 + f_2) g_1\big) B_2(5) \\
&\quad + f_1\big(f_1(f_1^2 + f_2) + f_1 f_2 + f_3\big) B_2(4) Q_1(0,0,z_2,\phi_2(4)) \\
&\quad + f_1(f_1^2 + f_2)\sum_{j=1}^{1} B_2(j+3) I_{1,j} \\
&\quad + f_1^2 \sum_{j=1}^{2} B_2(j+2) I_{2,j} + f_1 \sum_{j=1}^{3} B_2(j+1) I_{3,j} \\
&\quad + f_2(g_2 + f_1 g_1) B_2(5) + f_2(f_1^2 + f_2) B_2(4) Q_1(0,0,z_2,\phi_2(4)) \\
&\quad + f_2 f_1 \sum_{j=1}^{1} B_2(j+3) I_{1,j} + f_2 \sum_{j=1}^{2} B_2(j+2) I_{2,j} + f_3 g_1 B_2(5) \\
&\quad + f_3 f_1 B_2(4) Q_1(0,0,z_2,\phi_2(4)) \\
&\quad + f_3 \sum_{j=1}^{1} B_2(j+3) I_{1,j} + f_4 B_2(4) Q_1(0,0,z_2,\phi_2(4)) + \sum_{j=1}^{4} B_2(j) I_{4,j}
\end{aligned}$$

$$\begin{aligned}
& Q_5(0,0,z_2,y_2) \\
&= \big[g_4 + f_1\big(g_3 + f_1(g_2 + f_1 g_1) + f_2 g_1\big)\big] B_2(5) \\
&\quad + f_2\big((g_2 + g_1 f_1) + g_1 f_3\big) B_2(5) \\
&\quad + \big[f_1\big(f_1(f_1^2 + f_2) + f_1 f_2 + f_3\big) + f_2(f_1^2 + f_2) + f_3 f_1 + f_4\big] B_2(4) Q_1(0,0,z_2,\phi_2(4)) \\
&\quad + \big(f_1(f_1^2 + f_2) + f_2 f_1 + f_3\big)\sum_{j=1}^{1} B_2(j+3) I_{1,j} \\
&\quad + (f_1^2 + f_2)\sum_{j=1}^{2} B_2(j+2) I_{2,j} + f_1 \sum_{j=1}^{3} B_2(j+1) I_{3,j} \\
&\quad + \sum_{j=1}^{4} B_2(j) I_{4,j}
\end{aligned}$$

Next, let us define a_j and b_j as follows,

$$a_1 = f_1,$$
$$a_2 = f_1 f_1 + f_2,$$
$$a_3 = f_1(f_1 f_1 + f_2) + f_2 f_1 + f_3,$$
$$a_4 = f_1\big[f_1(f_1 f_1 + f_2) + f_2 f_1 + f_3\big] + f_2(f_1 f_1 + f_2) + f_3 f_1 + f_4,$$
$$\vdots$$

$$b_1 = g_1,$$
$$b_2 = f_1 g_1 + g_2,$$
$$b_3 = f_1(f_1 g_1 + g_2) + f_2 g_1 + g_3,$$
$$b_4 = f_1[f_1(f_1 g_1 + g_2) + f_2 g_1 + g_3] + f_2(f_1 g_1 + g_2) + f_3 g_1 + g_4$$
$$\vdots$$

Let us define $a_0 = 1$, $b_0 = 1$, then a_j, b_j may be expressed as,

$$a_1 = f_1 a_0,$$
$$a_2 = f_1 a_1 + f_2 a_0,$$
$$a_3 = f_1 a_2 + f_2 a_1 + f_3 a_0,$$
$$a_4 = f_1 a_3 + f_2 a_2 + f_3 a_1 + f_4 a_0,$$
$$\vdots$$
$$b_1 = g_1 b_0,$$
$$b_2 = f_1 b_1 + g_2 b_0,$$
$$b_3 = f_1 b_2 + f_2 b_1 + g_3 b_0,$$
$$b_4 = f_1 b_3 + f_2 b_2 + f_3 b_1 + g_4 b_0,$$
$$\vdots$$

Therefore, it is concluded that

$$a_j = \sum_{i=0}^{j-1} f_{j-i} a_i \quad , j \geq 1, \quad \text{with } a_0 = 1 \tag{3.12}$$

$$b_j = g_j b_0 + \sum_{i=1}^{j-1} f_{j-i} b_i \quad , \quad j \geq 1, \text{ with } b_0 = 1 \tag{3.13}$$

One may note that the maximum powers of z_2 in the denominator of a_j and b_j are j. With the above definitions, equation (3.11) may be written as,

$$Q_{k+1}(0,0,z_2,y_2) = b_k B_2(k+1) + a_k B_2(k) Q_1(0,0,z_2,\phi_2(k)) + \sum_{i=0}^{k-1}\sum_{j=1}^{k-i} a_i B_2(j+i) I_{k-i,j} \quad , \quad k \geq 1 \tag{3.14}$$

Thus, $Q_{k+1}(0,0,z_2,y_2)$ has been expressed in terms of $Q_1(0,0,z_2,y_2)$ and the other boundary function.

From (2.29), we have,

$$Q_1(0,0,z_2,y_2) = B_2(1)\beta_1^{m_1} \tag{3.15}$$

Substituting this result in (3.14) gives,

$$Q_{k+1}(0,0,z_2,y_2) = b_k B_2(k+1) + a_k B_2(k+1)\beta_1^{m_1} + \sum_{i=0}^{k-1}\sum_{j=1}^{k-i} a_i B_2(j+i) I_{k-i,j} \quad , \quad k \ge 1 \tag{3.16}$$

From (3.10) we have,

$$I_{k-i,j} = h_{k+1-j-i,j} + \Re_j Q_{k+1-j-i}(0,0,0,0)$$

Substituting from (3.8), the above equation becomes,

$$I_{k-i,j} = \frac{z_2 - 1}{(k-i)!}\left[\frac{d^{k-i}}{dz_1^{k-i}}\left(z_1^{k-j-i} B_1(j) Q_{k+1-i-j}(z_1,\phi_1(j),0,0)\right)\Big|_{z_1=y_1=0}\right] + \Re_j Q_{k-1-i-j}(0,0,0,0)$$

Substituting the above result in (3.16) gives,

$$\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2) \\
&= b_k B_2(k+1) + a_k B_2(k+1)\beta_1^{m_1} \\
&\quad + (z_2 - 1)\sum_{i=0}^{k-1}\sum_{j=1}^{k-i}\frac{1}{(k-i)!} a_i B_2(j+i)\left[\frac{d^{k-i}}{dz_1^{k-i}}\left(z_1^{k-j-i} B_1(j) Q_{k+1-i-j}(z_1,\phi_1(j),0,0)\right)\Big|_{z_1=y_1=0}\right] \\
&\quad + \sum_{i=0}^{k-1}\sum_{j=1}^{k-i} a_i B_2(j+i)\Re_j Q_{k+1-i-j}(0,0,0,0) \qquad , \quad k \ge 1
\end{aligned}$$

Let us change the subscripts in the above equation as follows, $i \to i_2$, $j \to j_2$, and let $\ell_2 = k + 1 - i_2 - j_2$, then above equation becomes,

$$\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2) \\
&= b_k B_2(k+1) + a_k B_2(k+1)\beta_1^{m_1} \\
&\quad + (z_2 - 1)\sum_{i_2=0}^{k-1}\sum_{\ell_2=1}^{k-i_2}\frac{1}{(k-i_2)!} a_{i_2} B_2(k+1-\ell_2) \\
&\qquad\qquad * \left[\frac{d^{k-i_2}}{dz_1^{k-i_2}}\left(z_1^{\ell_2-1} B_1(k+1-i_2-\ell_2) Q_{\ell_2}(z_1,\phi_1(k+1-i_2-\ell_2),0,0)\right)\Big|_{z_1=y_1=0}\right] \\
&\quad + \sum_{i_2=0}^{k-1}\sum_{\ell_2=1}^{k-i_2} a_{i_2} B_2(k+1-\ell_2)\Re_{k+1-i_2-\ell_2} Q_{\ell_2}(0,0,0,0) \qquad , \quad k \ge 1
\end{aligned}$$

Next, exchanging the order of summations in the above gives,

$$
\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2)\\
&=b_k B_2(k+1)+a_k B_2(k+1)\beta_1^{m_1}\\
&\quad+(z_2-1)\sum_{\ell_2=1}^{k}\sum_{i_2=0}^{k-\ell_2}\frac{1}{(k-i_2)!}a_{i_2}B_2(k+1-\ell_2)\\
&\qquad\qquad *\left[\frac{d^{k-i_2}}{dz_1^{k-i_2}}\left(z_1^{\ell_2-1}B_1(k+1-i_2-\ell_2)Q_{\ell_2}(z_1,\phi_1(k+1-i_2-\ell_2),0,0)\right)\bigg|_{z_1=y_1=0}\right]\\
&\quad+\sum_{\ell_2=1}^{k}\sum_{i_2=0}^{k-\ell_2}a_{i_2}B_2(k+1-\ell_2)\Re_{k+1-i_2-\ell_2}Q_{\ell_2}(0,0,0,0)\qquad,\quad k\geq 1
\end{aligned}
$$

letting $r_2=k-i_2$ in the above equation, we have,

$$
\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2)\\
&=b_k B_2(k+1)+a_k B_2(k+1)\beta_1^{m_1}\\
&\quad+(z_2-1)\sum_{\ell_2=1}^{k}\sum_{r_2=\ell_2}^{k}\frac{1}{r_2!}a_{k-r_2}B_2(k+1-\ell_2)\\
&\qquad\qquad *\left[\frac{d^{r_2}}{dz_1^{r_2}}\left(z_1^{\ell_2-1}B_1(r_2+1-\ell_2)Q_{\ell_2}(z_1,\phi_1(r_2+1-\ell_2),0,0)\right)\bigg|_{z_1=y_1=0}\right]\\
&\quad+\sum_{\ell_2=1}^{k}\sum_{r_2=\ell_2}^{k}a_{k-r_2}B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\qquad,\quad k\geq 1
\end{aligned}
\tag{3.17}
$$

If $k=0$, from (2.29) we have $Q_1(0,0,z_2,y_2)=B_2(1)\beta_1^{m_1}$

This completes the derivation of expressing $Q_k(0,0,z_2,y_2)$ in terms of $Q_k(z_1,y_1,0,0)$. Substituting $Q_k(z_1,y_1,0,0)$ in (3.17) from (3.1), we have,

$$B_1(r_2+1-\ell_2)Q_{\ell_2}(z_1,\phi_1(r_2+1-\ell_2),0,0)=B_1(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)$$

Further,

$$
\begin{aligned}
&\frac{1}{r_2}\frac{d^{r_2}}{dz_1^{r_2}}\left(z_1^{\ell_2-1}B_1(r_2+1-\ell_2)Q_{\ell_2}(z_1,\phi_1(r_2+1-\ell_2),0,0)\right)\bigg|_{z_1=y_1=0}\\
&=\frac{1}{r_2}\frac{d^{r_2}}{dz_1^{r_2}}\left(z_1^{\ell_2-1}B_1(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\right)\bigg|_{z_1=y_1=0}\\
&=\frac{1}{(r_2-\ell_2+1)!}\frac{d^{r_2+1-\ell_2}}{dz_1^{r_2+1-\ell_2}}B_1(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\bigg|_{z_1=y_1=0}\\
&=\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)
\end{aligned}
$$

Substituting the above result in (3.17) gives

$$
\begin{aligned}
&Q_{k+1}(0,0,z_2,y_2)\\
&= b_k B_2(k+1) + a_k B_2(k+1)\beta_1^{m_1}\\
&+ (z_2-1)\sum_{\ell_2=1}^{k}\sum_{r_2=\ell_2}^{k} a_{k-r_2} B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0) \qquad (3.18)\\
&+ \sum_{\ell_2=1}^{k}\sum_{r_2=\ell_2}^{k} a_{k-r_2} B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0) \qquad , \quad k \ge 1
\end{aligned}
$$

Finally, the boundary function $Q_k(0,0,z_2,y_2)$ has been expressed only in terms of $Q_k(1,1,0,0)$ and $Q_k(0,0,0,0)$.

3.2 Joint Steady-State PGF of the Second Multiplexer

Since the first multiplexer is not affected by the second one, it behaves like a single multiplexer, which has already been studied in [22]. Therefore, only the performance of the second multiplexer needs to be studied. In this section, the joint steady-state PGF of the queue length and number of *On* sources for the second multiplexer will be determined.

After substituting 1 for z_1, y_1 in (3.2), the marginal PGF of the second multiplexer is obtained, which is given by

$$
\begin{aligned}
&Q_{k+1}(1,1,z_2,y_2)\\
&= \frac{1}{z_2^k}\Bigg\{ z_2{}^k B_2(k+1) + (1-z_2)\sum_{j=1}^{k} z_2^{k-1} B_2(j) Q_{k+1-j}(0,0,z_2,\phi_2(j))\\
&\quad + (z_2-1)\sum_{j=1}^{k} z_2^k B_2(j) Q_{k+1-j}(1,1,0,0)\\
&\quad + (z_2-1)(1-z_2)\sum_{j=1}^{k} z_2^{k-1} B_2(j) Q_{k+1-j}(0,0,0,0)\Bigg\}
\end{aligned}
$$

The above may be written as,

$$Q_{k+1}(1,1,z_2,y_2)$$
$$= B_2(k+1) + \frac{1-z_2}{z_2}\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j))$$
$$+(z_2-1)\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(1,1,0,0) \tag{3.19}$$
$$-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(0,0,0,0)$$

As may be seen, the above equation has only one boundary function, which has been determined in (3.18). In the following, this boundary function will be eliminated. First, $B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j))$ is formed from (3.18),

$$B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j))$$
$$= b_{k-j}B_2(k+1) + a_{k-j}B_2(k+1)\beta_1^{m_1}$$
$$+(z_2-1)\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2}B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)$$
$$+\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2}B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0) \quad , \quad k \ge j+1$$

From (3.17), if $k=j$, then $Q_{k+1-j}(0,0,z_2,y_2) = B_2(1)\beta_1^{m_1}$, thus

$$B_2(j)Q_{k+1-j}(0,0,z_2,\phi_2(j)) = B_2(k+1)\beta_1^{m_1} \quad , \quad \text{if } k=j$$

Substituting the above results in (3.19), we have,

$$Q_{k+1}(1,1,z_2,y_2)$$
$$= B_2(k+1) + \frac{1-z_2}{z_2}B_2(k+1)\beta_1^{m_1}$$
$$+\frac{1-z_2}{z_2}B_2(k+1)\sum_{j=1}^{k-1}(b_{k-j} + a_{k-j}\beta_1^{m_1})$$
$$-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2}B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0) \tag{3.20}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2}B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)$$
$$+(z_2-1)\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(1,1,0,0) - \frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(0,0,0,0) \quad , \quad k \ge 1$$

Let us define the transform of $Q_k(1,1,z_2,y_2)$ w. r. t. to discrete time k,

$$Q(1,1,z_2,y_2,\omega)=\sum_{k=0}^{\infty}Q_k(1,1,z_2,y_2)\omega^k \tag{3.21}$$

then,

$$Q(1,1,z_2,y_2,\omega)=Q_0(1,1,z_2,y_2)\omega^0+Q_1(1,1,z_2,y_2)\omega+\sum_{k=1}^{\infty}Q_{k+1}(1,1,z_2,y_2)\omega^{k+1},$$

Substituting (3.20) into the above equation, we have,

$$\begin{aligned}
&Q(1,1,z_2,y_2,\omega)\\
&=1+Q_1(1,1,z_2,y_2)\omega+\sum_{k=1}^{\infty}B_2(k+1)\omega^{k+1}+\frac{1-z_2}{z_2}\sum_{k=1}^{\infty}B_2(k+1)\beta_1^{m_1}\omega^{k+1}\\
&+\frac{1-z_2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_2(k+1)(b_{k-j}+a_{k-j}\beta_1^{m_1})\omega^{k+1}\\
&-\frac{(z_2-1)^2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j}a_{k-j-r_2}B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{k+1}\\
&+\frac{1-z_2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j}a_{k-j-r_2}B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\omega^{k+1}\\
&+(z_2-1)\sum_{k=1}^{\infty}\sum_{j=1}^{k}B_2(j)Q_{k+1-j}(1,1,0,0)\omega^{k+1}-\frac{(z_2-1)^2}{z_2}\sum_{k=1}^{\infty}\sum_{j=1}^{k}B_2(j)Q_{k+1-j}(0,0,0,0)\omega^{k+1}
\end{aligned} \tag{3.22}$$

Let us also define the following transforms:

$$Q(0,0,0,0,\omega)=\sum_{k=0}^{\infty}Q_k(0,0,0,0)\omega^k \tag{3.23}$$

$$Q(1,1,0,0,\omega)=\sum_{k=0}^{\infty}Q_k(1,1,0,0)\omega^k \tag{3.24}$$

Further, defining the probability distribution and PGF of the busy period for the first multiplexer as,

$$\xi_1(j)=\text{Prob(multiplexer-1 has a busy period of } j \text{ slots)},\ j=0,1,2,\ldots \tag{3.25}$$

$$\Gamma_1(\omega)=\sum_{j=0}^{\infty}\xi_1(j)\omega^j \tag{3.26}$$

Then from Appendix B, the derivatives of $B_1(k)$ has been expressed in terms of $\Gamma_1(\omega)$.

After some algebraic manipulation, details of which can be found in Appendix C, equation (3.22) may be written as,

$$
\begin{aligned}
&Q(1,1,z_2,y_2,\omega) \\
&= 1+\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\
&-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\
&-(z_2-1)^2[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\
&+(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}
\end{aligned}
\tag{3.27}
$$

In order to determine the steady-state PGF of the second multiplexer, the final-value theorem is applied,

$$Q(1,1,z_2,y_2)=Q_\infty(1,1,z_2,y_2)=\lim_{\omega\to 1}(1-\omega)Q(1,1,z_2,y_2,\omega)$$

Because,

$$\lim_{\omega\to 1}(1-\omega)Q(1,1,0,0,\omega)=Q(1,1,0,0)=1-(\rho_1+\rho_2)$$

where ρ_i is the traffic load generated by type-i sources, and $(\rho_1+\rho_2)$ is the total traffic load of the second multiplexer. From [22] we have,

$$\rho_i=m_i\sigma_i f_i'(1) \tag{3.28}$$

In the above, m_i is the number of type-i sources, σ_i is the probability that a type-i source is *On*, and $f_i'(1)$ is the mean number of packets that an *On* type-i source generates during a slot. σ_i may be expressed in terms of the source parameters,

$$\sigma_i=\frac{1-\beta_i}{2-\alpha_i-\beta_i} \tag{3.29}$$

After the application of final-value theorem to equation (3.27), we have,

$$Q(1,1,z_2,y_2)$$

$$= (z_2-1)(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}}$$

$$-(z_2-1)^2(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i})[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})]} \tag{3.30}$$

The above gives the joint PGF of the queue length and number of *On* sources for the second multiplexer at the steady state.

Substituting $y_2 = 1$ in (3.30) gives the marginal PGF of the queue length for the second multiplexer, which is given by

$$P_2(z_2) = (z_2-1)(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}}$$

$$-(z_2-1)^2(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i})[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})]} \tag{3.31}$$

where, $\widetilde{C}_{12} = C_{12}\big|_{y_2=1}$ and $\widetilde{C}_{22} = C_{22}\big|_{y_2=1}$ (3.32)

Next, the behavior of the second multiplexer is dicussed when the load of the first multiplexer approaches to zero. In this case, there are no arrivals from the first multiplexer to the second one, and therefore the second multiplexer should behave like a single multiplexer. In the following, it will be shown that the presented analysis supports this conclusion.

Since, given $\rho_1 = 0$, the first multiplexer is always idle, its busy period has zero duration with probability one. Thus from (3.26), we have $\Gamma_1(\omega) = 1$. Substituting this result in (3.27) and letting $y_2 = 1$ gives,

$$
\begin{aligned}
&Q(1,1,z_2,1,\omega)\\
&=1+\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&+(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega]}\\
&-(z_2-1)^2[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega]}
\end{aligned}
$$

From (3.32) and (2.24), we have,

$$\sum_{i=0}^{m_2}\binom{m_2}{i}\widetilde{C}_{12}^i\widetilde{C}_{22}^{m_2-i}=(\widetilde{C}_{12}+\widetilde{C}_{22})^{m_2}=1,$$

Therefore, $Q(1,1,z_2,1,\omega)$ may be written as,

$$
\begin{aligned}
&Q(1,1,z_2,1,\omega)\\
&=\sum_{i=0}^{m_2}\binom{m_2}{i}\widetilde{C}_{12}^i\widetilde{C}_{22}^{m_2-i}+\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&+(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega]}\\
&-(z_2-1)^2[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega]}
\end{aligned}
$$

Combining the first term with the second one, the third term with the fifth one in the above, we have,

$$
\begin{aligned}
&Q(1,1,z_2,1,\omega)\\
&=\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{\widetilde{C}_{12}^i\widetilde{C}_{22}^{m_2-i}}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&+(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\
&-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^i(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega]}
\end{aligned}
$$

The above may be written as,

$$Q(1,1,z_2,1,\omega)$$

$$=\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{\widetilde{C}_{12}^{i}\widetilde{C}_{22}^{m_2-i}}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^{i}(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}$$

$$+(z_2-1)Q(1,1,0,0,\omega)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^{i}(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}$$

$$-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^{i}(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}{(1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega]}$$

Combining the first, second and the forth terms in the above equation results in,

$$Q(1,1,z_2,1,\omega)$$

$$=\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{z_2(\widetilde{C}_{12})^{i}(\widetilde{C}_{22})^{m_2-i}\omega}{z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}$$

$$+(z_2-1)Q(1,1,0,0,\omega)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^{i}(\widetilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}$$

As expected the above expression corresponds to the single multiplexer result given in (23) of [22].

Next, let us determine the PGF of the queue length of the second multiplexer at $\rho_1=0$. Substituting $\rho_1=0$ and $\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i})=1$ in (3.31) gives,

$$P_2(z_2)=(z_2-1)(1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\widetilde{C}_{12}\lambda_{12})^{i}(\widetilde{C}_{22}\lambda_{22})^{m_2-i}}{z_2-\lambda_{12}^{i}\lambda_{22}^{m_2-i}}$$

Again, $P_2(z_2)$ corresponds to the single multiplexer result in (30) of [22].

In the above discussions, it has been shown that the second multiplexer behaves exactly like a single multiplexer if the load of the first multiplexer approaches to zero. This consistence gives further confidence that the previous analysis is correct.

3.3 Performance Measures of the Second Multiplexer

From the steady-state PGF of the queue length, it is easy to determine the corresponding performance measures, such as the mean and variance of queue length, mean packet delay, for the second multiplexer. First, the PGF of the queue length (3.31) is transferred into a more convenient form. Let us define the following,

$$E_2(z_2) = \sum_{i=1}^{m_2} \binom{m_2}{i} \frac{(\widetilde{C}_{12}\lambda_{12})^i (\widetilde{C}_{22}\lambda_{22})^{m_2-i}}{1-\lambda_{12}^i \lambda_{22}^{m_2-i}} \tag{3.33}$$

$$F_2(z_2) = \sum_{i=1}^{m_2} \binom{m_2}{i} \frac{(\widetilde{C}_{12}\lambda_{12})^i (\widetilde{C}_{22}\lambda_{22})^{m_2-i} \Gamma_1(\lambda_{12}^i \lambda_{22}^{m_2-i})}{(1-\lambda_{12}^i \lambda_{22}^{m_2-i})[z_2 - \lambda_{12}^i \lambda_{22}^{m_2-i} \Gamma_1(\lambda_{12}^i \lambda_{22}^{m_2-i})]} \tag{3.34}$$

Then, we have, $E_2(1) = 0, F_2(1) = 0$ (3.35)

and (3.31) may be expressed as,

$$\begin{aligned} & P_2(z_2) \\ & = (z_2 - 1)(1 - \rho_1 - \rho_2)\left[E_2(z_2) + \frac{(\widetilde{C}_{22}\lambda_{22})^{m_2}}{1-\lambda_{22}^{m_2}}\right] \\ & - (z_2 - 1)^2 (1 - \rho_1 - \rho_2)\left[F_2(z_2) + \frac{(\widetilde{C}_{22}\lambda_{22})^{m_2} \Gamma_1(\lambda_{22}^{m_2})}{(1-\lambda_{22}^{m_2})[z_2 - \lambda_{22}^{m_2}\Gamma_1(\lambda_{22}^{m_2})]}\right] \end{aligned} \tag{3.36}$$

Let us further define

$$H_2(z_2) = \lambda_{22}^{m_2}, \quad G_2(z_2) = (\widetilde{C}_{22}\lambda_{22})^{m_2} \tag{3.37}$$

$$\Theta_2(z_2) = H_2(z_2)\Gamma_1(H_2(z_2)) = \lambda_{22}^{m_2}\Gamma_1(\lambda_{22}^{m_2}) \tag{3.38}$$

Then, $H_2(1) = 1, G_2(1) = 1, \Theta_2(1) = 1$ (3.39)

As a result, (3.36) may be further expressed as,

$$\begin{aligned} P_2(z_2) = {} & (z_2 - 1)(1 - \rho_1 - \rho_2)\left[E_2(z_2) + \frac{G_2(z_2)}{1-H_2(z_2)}\right] \\ & - (z_2 - 1)^2 (1 - \rho_1 - \rho_2)\left[F_2(z_2) + \frac{G_2(z_2)\Gamma_1(H_2(z_2))}{[1-H_2(z_2)][z_2 - \Theta_2(z_2)]}\right] \end{aligned} \tag{3.40}$$

Multiplying both sides of the above equation with its denominator, we have,

$$\begin{aligned} & [1-H_2(z_2)][z_2 - \Theta_2(z_2)]P_2(z_2) \\ & = (z_2 - 1)(1 - \rho_1 - \rho_2)\big[[1-H_2(z_2)][z_2 - \Theta_2(z_2)]E_2(z_2) + [z_2 - \Theta_2(z_2)]G_2(z_2)\big] \\ & - (z_2 - 1)^2 (1 - \rho_1 - \rho_2)\big[[1-H_2(z_2)][z_2 - \Theta_2(z_2)]F_2(z_2) + G_2(z_2)\Gamma_1(H_2(z_2))\big] \end{aligned} \tag{3.41}$$

In the next, the performance measures will be expressed in terms of the derivatives of $H_2(z_2)$, $G_2(z_2)$, $E_2(z_2)$, $\Gamma_1(\omega)$ and $\Theta_2(z_2)$.

- Derivation of the mean queue length and packet delay

First, the mean queue length for the second multiplexer will be determined. Taking the third derivative of both sides of (3.41) with respect to z_2, and then substituting $z_2 = 1$; after noting (3.35, 3.39), an equation is obtained which contains $P_2'(1) = \left.\frac{dP_2(z_2)}{dz_2}\right|_{z_2=1}$,

$$-3H_2''(1)[1-\Theta_2'(1)] + 3H_2'(1)\Theta_2''(1) - 6H_2'(1)[1-\Theta_2'(1)]P_2'(1)$$
$$= -6(1-\rho_1-\rho_2)[\Gamma_1'(1)H_2'(1) + G_2'(1)] + 3(1-\rho_1-\rho_2)[-\Theta_2''(1) + 2(1-\Theta_2'(1))G_2'(1)]$$

where $G_2'(1) = \left.\frac{dG_2(z_2)}{dz_2}\right|_{z_2=1}$, $\Gamma_1'(1) = \left.\frac{d\Gamma_1(\omega)}{d\omega}\right|_{\omega=1}$,

$$\Theta_2'(1) = \left.\frac{d\Theta_2(z_2)}{dz_2}\right|_{z_2=1}, \quad \Theta_2''(1) = \left.\frac{d^2\Theta_2(z_2)}{dz_2^{\ 2}}\right|_{z_2=1},$$

$$H_2'(1) = \left.\frac{dH_2(z_2)}{dz_2}\right|_{z_2=1}, \quad H_2''(1) = \left.\frac{d^2H_2(z_2)}{dz_2^{\ 2}}\right|_{z_2=1}, \tag{3.42}$$

Solving the above equation for $P_2'(1)$, the mean queue length, $\overline{N}_2$, is determined,

$$\begin{aligned}\overline{N}_2 = P_2'(1) &= \frac{\Theta_2''(1)}{2[1-\Theta_2'(1)]} - \frac{1}{2H_2'(1)}\left[H_2''(1) + 2(1-\rho_1-\rho_2)G_2'(1)\right] \\ &+ \frac{1-\rho_1-\rho_2}{2H_2'(1)[1-\Theta_2'(1)]}\left[\Theta_2''(1) + 2\Gamma_1'(1)H_2'(1) + 2G_2'(1)\right]\end{aligned} \tag{3.43}$$

From the Little's Result, the mean packet delay, $\overline{D}_2$, that a packet experiences at the second multiplexer is given by,

$$\overline{D}_2 = \frac{\overline{N}_2}{\rho_1 + \rho_2} \tag{3.44}$$

- Derivation of the variance of queue length

The variance of the queue length of the second multiplexer requires the second order derivative of $P_2(z_2)$. After taking the forth order derivative of both sides of (3.41) with respect to z_2, and then substituting $z_2 = 1$ and noting (3.35, 3.39), the following equation containing $P_2''(1)$ is obtained:

$$-12H_2'(1)[1-\Theta_2'(1)]P_2'''(1)-12H_2''(1)[1-\Theta_2'(1)]P_2'(1)+12H_2'(1)\Theta_2''(1)P_2'(1)$$
$$-4H_2'''(1)[1-\Theta_2'(1)]+6H_2''(1)\Theta_2''(1)+4H_2'(1)\Theta_2'''(1)-24H_2'(1)[1-\Theta_2'(1)]P_2'(1)$$
$$-12H_2''(1)[1-\Theta_2'(1)]+12H_2'(1)\Theta_2''(1) \quad (3.45)$$
$$=-12(1-\rho_1-\rho_2)[\Gamma_1''(1)(H_2'(1))^2+\Gamma_1'(1)H_2''(1)+2\Gamma_1'(1)H_2'(1)G_2'(1)+G_2''(1)]$$
$$-4(1-\rho_1-\rho_2)\{6H_2'(1)[1-\Theta_2'(1)]E_2'(1)+\Theta_2'''(1)+3\Theta_2''(1)G_2'(1)-3[1-\Theta_2'(1)]G_2''(1)\}$$

where $$\Theta_2'''(1)=\left.\frac{d^3\Theta_2(z_2)}{dz_2^3}\right|_{z_2=1}, \quad G_2''(1)=\left.\frac{d^2G_2(z_2)}{dz_2^2}\right|_{z_2=1}, \quad (3.46)$$

$$E_2'(1)=\left.\frac{dE_2(z_2)}{dz_2}\right|_{z_2=1}, \quad H_2'''(1)=\left.\frac{d^3H_2(z_2)}{dz_2^3}\right|_{z_2=1}, \quad \Gamma_1''(1)=\left.\frac{d^2\Gamma_1(\omega)}{d\omega^2}\right|_{\omega=1}, \quad (3.47)$$

Solving (3.45) for $P_2''(1)$ gives,

$$P_2''(1)$$
$$=\frac{1}{6H_2'(1)[1-\Theta_2'(1)]}\Big\{2(1-\rho_1-\rho_2)\Theta_2'''(1)+3H_2''(1)\Theta_2''(1)$$
$$+6(1-\rho_1-\rho_2)[\Theta_2''(1)G_2'(1)+\Theta_2'(1)G_2''(1)+\Gamma_1'(1)H_2''(1)]\Big\} \quad (3.48)$$
$$+\frac{1}{3H_2'(1)}\Big\{-H_2'''(1)+6(1-\rho_1-\rho_2)H_2'(1)E_2'(1)-3H_2''(1)P_2'(1)\Big\}$$
$$+\frac{1}{3[1-\Theta_2'(1)]}\Big\{\Theta_2'''(1)+3\Theta_2''(1)P_2'(1)+3(1-\rho_1-\rho_2)[2\Gamma_1'(1)G_2'(1)+\Gamma_1''(1)H_2'(1)]\Big\}$$

The variance of the queue length for the second multiplexer, $\overline{V_2}$, may be expressed in terms of the derivatives of its PGF as,

$$\overline{V_2}=P_2''(1)+P_2'(1)-(P_2'(1))^2 \quad (3.49)$$

The expressions of performance measures require the derivatives of $H_2(z_2)$, $G_2(z_2)$, $E_2(z_2)$, $\Gamma_1(\omega)$ and $\Theta_2(z_2)$. Next, how to determine these derivatives is discussed. The derivatives of $H_2(z_2)$, $G_2(z_2)$ and $E_2(z_2)$ may be determined from their definitions in (3.33, 3.37) in a straightforward manner, therefore the details will not be given here. It is only noted that,

$$H_2'(1)=\frac{m_2(1-\beta_2)f_2'(1)}{2-\alpha_2-\beta_2}=\rho_2 \quad (3.50)$$

Next, the derivatives of $\Gamma_1(\omega)$ and $\Theta_2(z_2)$ will be expressed in terms of the derivatives of $z_1^*(\omega)$, which is the unique root of the equation $z_1 = \omega\lambda_{21}^{m_1}$. $\Gamma_1(\omega)$ is the PGF of the busy period of the first multiplexer, which has been determined in [23]. Let us repeat the result from [23],

$$\Gamma_1(\omega) = \frac{z_1^*(\omega)}{\omega}, \tag{3.51}$$

Equation (3.51) may be written as

$$\omega\Gamma_1(\omega) = z_1^*(\omega) \tag{3.52}$$

Taking the first three order derivatives of both sides the above equation with respect to ω, and substituting $\omega = 1$ gives,

$$\Gamma_1'(1) = \left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} - 1 \tag{3.53}$$

$$\Gamma_1''(1) = \left.\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1} - 2\Gamma_1'(1) \tag{3.54}$$

$$\Gamma_1'''(1) = \left.\frac{d^3 z_1^*(\omega)}{d\omega^3}\right|_{\omega=1} - 3\Gamma_1''(1) \tag{3.55}$$

Next, $\Theta_2'(1)$, $\Theta_2''(1)$, and $\Theta_2'''(1)$ will be determined. From (3.38, 3.52), we have,

$$\Theta_2(z_2) = z_1^*(H_2(z_2)) \tag{3.56}$$

Taking the first order derivative of $\Theta_2(z_2)$ in (3.56) with respect to z_2, we have,

$$\frac{d\Theta_2(z_2)}{dz_2} = \frac{dz_1^*(H_2(z_2))}{dH_2(z_2)}\frac{dH_2(z_2)}{dz_2}$$

Substituting $z_2 = 1$ in the above, and noting that $H_2(1) = 1$, we have,

$$\Theta_2'(1) = H_2'(1)\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} \tag{3.57}$$

Taking the second order derivative of $\Theta_2(z_2)$ in (3.56) with respect to z_2, we have,

$$\frac{d^2\Theta_2(z_2)}{dz_2^2} = \frac{d^2 z_1^*(H_2(z_2))}{d^2 H_2(z_2)}\left(\frac{dH_2(z_2)}{dz_2}\right)^2 + \frac{dz_1^*(H_2(z_2))}{dH_2(z_2)}\frac{d^2 H_2(z_2)}{dz_2^2}$$

Substituting $z_2 = 1$ in the above equation, we have,

$$\Theta_2''(1) = [H_2'(1)]^2 \left.\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1} + H_2''(1) \left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} \tag{3.58}$$

Taking the third order derivative of $\Theta_2(z_2)$ in (3.56) with respect to z_2, we have,

$$\frac{d^3\Theta_2(z_2)}{dz_2^3} = \frac{d^3 z_1^*(H_2(z_2))}{d^3 H_2(z_2)} \left(\frac{dH_2(z_2)}{dz_2}\right)^3 + 2\frac{d^2 z_1^*(H_2(z_2))}{d^2 H_2(z_2)} \frac{dH_2(z_2)}{dz_2} \frac{d^2 H_2(z_2)}{dz_2^2}$$
$$+ \frac{d^2 z_1^*(H_2(z_2))}{d^2 H_2(z_2)} \frac{dH_2(z_2)}{dz_2} \frac{d^2 H_2(z_2)}{dz_2^2} + \frac{dz_1^*(H_2(z_2))}{dH_2(z_2)} \frac{d^3 H_2(z_2)}{dz_2^3}$$

Substituting $z_2 = 1$ in the above equation, we have,

$$\Theta_2'''(1) = [H_2'(1)]^3 \left.\frac{d^3 z_1^*(\omega)}{d\omega^3}\right|_{\omega=1} + 3H_2'(1)H_2''(1) \left.\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1} + H_2'''(1) \left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} \tag{3.59}$$

Since the derivatives of $\Theta_2(z_2)$ has been expressed in terms of the derivatives of $z_1^*(\omega)$, then next the first three order derivatives of $z_1^*(\omega)$ at $\omega = 1$ will be determined. Following the notation of $H_2(z_2)$ defined in (3.37), $H_1(z_1)$ is defined as,

$$H_1(z_1) = \lambda_{21}^{m_1} \tag{3.60}$$

because $z_1^*(\omega)$ is the unique root of the equation $z_1 = \omega\lambda_{21}^{m_1}$, we have,

$$z_1^*(\omega) = \omega H_1(z_1)\big|_{z_1 = z_1^*(\omega)} = \omega H_1(z_1^*(\omega))\big| \tag{3.61}$$

Taking the first order derivative of both sides of (3.61) with respect to ω, we have,

$$\frac{dz_1^*(\omega)}{d\omega} = H_1(z_1^*(\omega)) + \omega \frac{dH_1(z_1^*(\omega))}{dz_1^*(\omega)} \frac{dz_1^*(\omega)}{d\omega} \tag{3.62}$$

Since the unique root of equation $z_1 = \omega\lambda_{21}^{m_1}$ at $\omega = 1$ is $z_1 = 1$, we have,

$$z_1^*(\omega)\big|_{\omega=1} = 1 \tag{3.63}$$

Substituting $\omega = 1$ in (3.62) and noting that $H_1(1) = 1$ gives,

$$\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} = \frac{1}{1 - H_1'(1)} \tag{3.64}$$

Taking the second order derivative of both sides of (3.61) with respect to ω, we have,

$$\frac{d^2 z_1^*(\omega)}{d\omega^2} = 2\frac{dH_1(z_1^*(\omega))}{dz_1^*(\omega)} \frac{dz_1^*(\omega)}{d\omega} + \omega \frac{d^2 H_1(z_1^*(\omega))}{d^2 z_1^*(\omega)} \left(\frac{dz_1^*(\omega)}{d\omega}\right)^2 + \omega \frac{dH_1(z_1^*(\omega))}{dz_1^*(\omega)} \frac{d^2 z_1^*(\omega)}{d\omega^2}$$

Substituting $\omega = 1$ in the above, and then solving the equation, we obtain,

$$\left.\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1} = \frac{1}{1-H_1'(1)}\left[2H_1'(1)\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1} + H_1''(1)\left(\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1}\right)^2\right] \tag{3.65}$$

Taking the third order derivative of both sides of (3.61) with respect to ω, we have,

$$\begin{aligned}\frac{d^3 z_1^*(\omega)}{d\omega^3} &= 3\frac{d^2 H_1(z_1^*(\omega))}{d^2 z_1^*(\omega)}\left(\frac{dz_1^*(\omega)}{d\omega}\right)^2 + 3\frac{dH_1(z_1^*(\omega))}{dz_1^*(\omega)}\frac{d^2 z_1^*(\omega)}{d\omega^2} \\ &+ \omega\frac{d^3 H_1(z_1^*(\omega))}{d^3 z_1^*(\omega)}\left(\frac{dz_1^*(\omega)}{d\omega}\right)^3 + 3\omega\frac{d^2 H_1(z_1^*(\omega))}{d^2 z_1^*(\omega)}\frac{dz_1^*(\omega)}{d\omega}\frac{d^2 z_1^*(\omega)}{d\omega^2} \\ &+ \omega\frac{dH_1(z_1^*(\omega))}{dz_1^*(\omega)}\frac{d^3 z_1^*(\omega)}{d\omega^3}\end{aligned}$$

Substituting $\omega = 1$ in the above, and then solving the equation, we obtain,

$$\begin{aligned}\left.\frac{d^3 z_1^*(\omega)}{d\omega^3}\right|_{\omega=1} = \frac{1}{1-H_1'(1)}\Bigg\{&3H_1''(1)\left(\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1}\right)^2 + 3H_1'(1)\left.\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1} \\ &+ H_1'''(1)\left(\left.\frac{dz_1^*(\omega)}{d\omega}\right|_{\omega=1}\right)^3 + 3H_1''(1)\left.\frac{dz_1^*(\omega)}{d\omega}\frac{d^2 z_1^*(\omega)}{d\omega^2}\right|_{\omega=1}\Bigg\}\end{aligned} \tag{3.66}$$

This completes the derivation of all the expressions needed for determining the mean and variance of queue length, as well as the mean packet delay.

3.4 Numerical Results

In this section, some numerical results regarding the analysis of this chapterare presented. The simulation results are also presented to show that the numerical results are correct. Because the behavior of the first multiplexer is not affected by the second one and it has been studied, the results are presented mainly for the second multiplexer. Unless otherwise stated, it will be assumed that an *On* source generates a single packet during a slot.

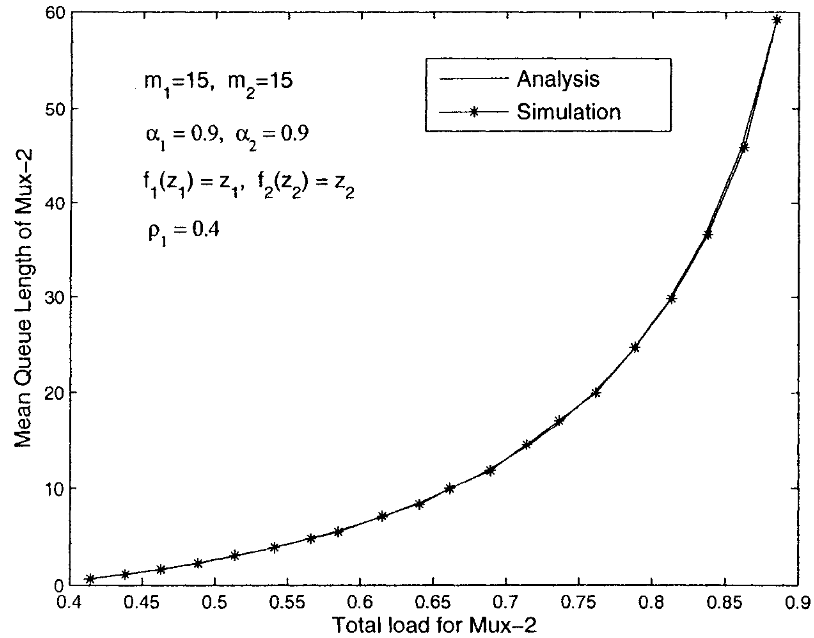


Figure 3.1 Analytical and simulation results: mean queue length of multiplexer-2 vs. its total load

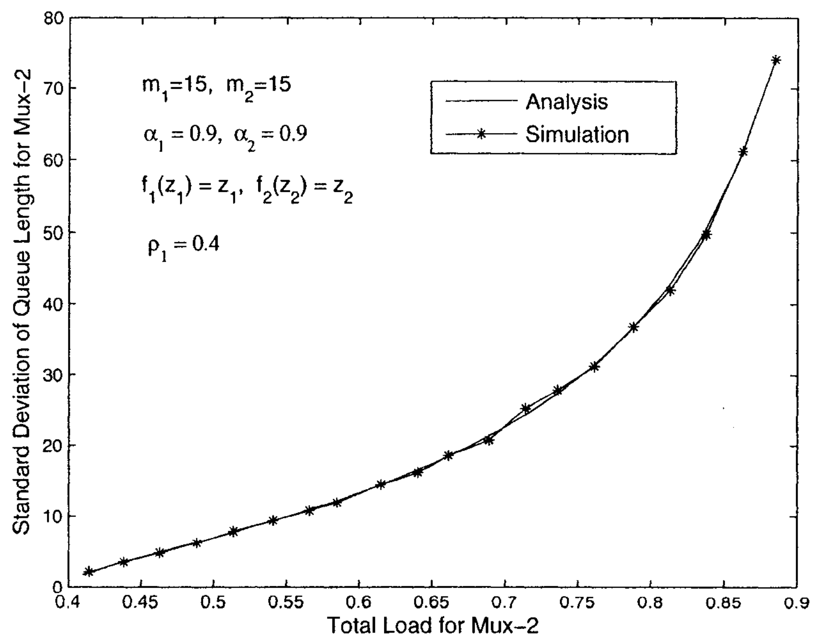


Figure 3.2 Analytical and simulation results: Standard deviation of queue length for multiplexer-2 vs. its total load

Figures 3.1 and 3.2 present both the analytical and simulation results for the mean and standard deviation of queue length against the total load of multiplexer-3. It may be seen that both mean and standard deviation increase with the load. Moreover, the analytical results match the simulation results exactly, which gives further support that presented analysis is correct.

In Figures 3.3 - 3.5, the mean queue length, mean packet delay and standard deviation of queue length are presented for multiplexer-2 respectively. The figures are plotted against the number of external sources feeding multiplexer-2, while its total load is kept constant. From (3.28, 3.29) it is concluded that the traffic load generated by one source is $\frac{(1-\beta_i)f_i'(1)}{2-\alpha_i-\beta_i}$. This expression may be written as $\frac{1/(1-\alpha_i)}{1/(1-\alpha_i)+1/(1-\beta_i)}f_i'(1)$, it may be seen that β_i increases as the single source traffic load decreases if α_i and $f_i'(1)$ are kept constant. From [25], the burstiness of a source is defined as the variance of the inter-arrival time of packets divided by the mean inter-arrival time squared, which is given by, $\frac{(1-\alpha_i)(\alpha_i+\beta_i)}{(2-\alpha_i-\beta_i)^2}$. In this expression, it may be seen that the burstiness increases as β_i increases while α_i is constant. The above discussion leads to the conclusion that as the traffic load of a source decreases, the burstiness of this source increases. Since the total load of multiplexer-2 is held constant, increasing the number of sources makes the traffic load generated by each source decrease; therefore, its burstiness increases. On the other hand, because of the statistical multiplexing, increasing the number of sources smoothes out the superposed traffic. From Figures 3.5 - 3.7, it may be seen that all curves rise with increasing number of sources, which means that burstiness overweighs traffic smoothing.

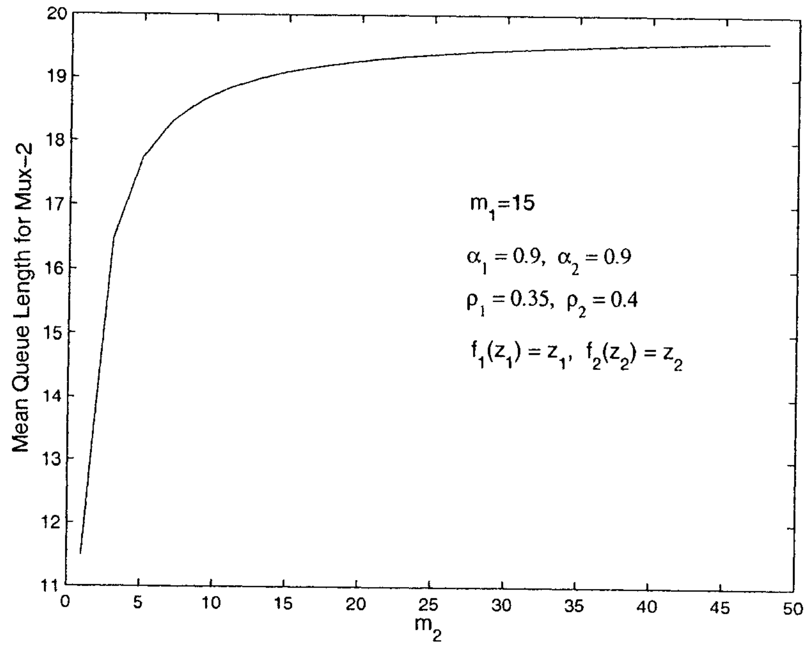


Figure 3.3. Mean queue length vs. the number of sources for multiplexer-2

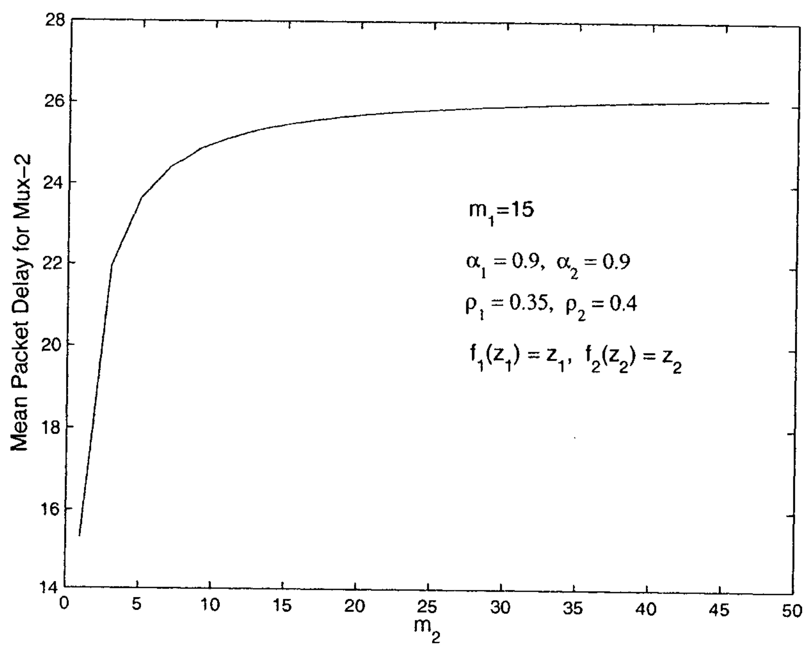


Figure 3.4 Mean packet delay vs. the number of sources for multiplexer-2

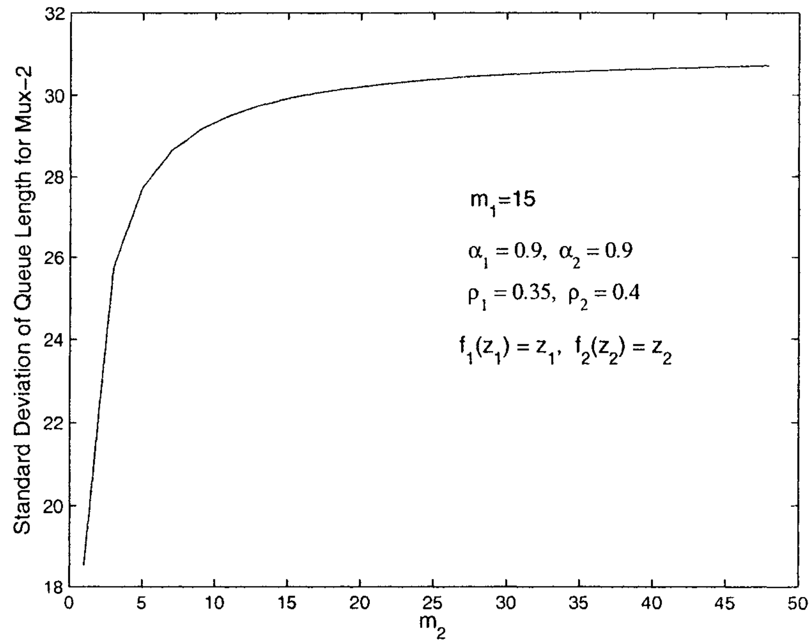


Figure 3.5 Standard deviation of queue length vs. the number of sources for multiplexer-2

Figures 3.6 and 3.7 present the mean and standard deviation of queue lengths versus its individual loads for multiplexer-i, $i = 1, 2$. For multiplexers 2, the input traffic from the preceding multiplexer is kept constant, thus increase in their traffic load is due to external traffic. The curves for multiplexers 2 is below the curve for multiplexer-1 except for heavy loading. This is due to the smoothing effect of statistical multiplexing; the traffic at the output of a multiplexer will be less bursty than its input traffic. Following the discussion in the previous paragraph, the sources feeding multiplexers-2 will be burstier than that of multiplexer-1 because each of them will generate less traffic. Further, under heavy traffic, the proportion of the input traffic of multiplexers 2, which have not already gone through smoothing, will increase, which explains the reversed positions of the curves.

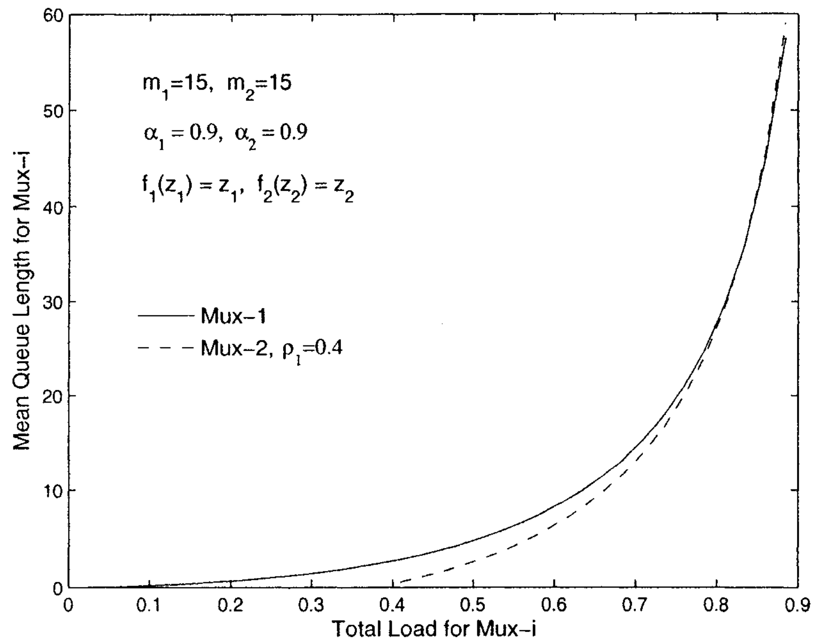


Figure 3.6 Mean queue length vs. its load for multiplexer-i

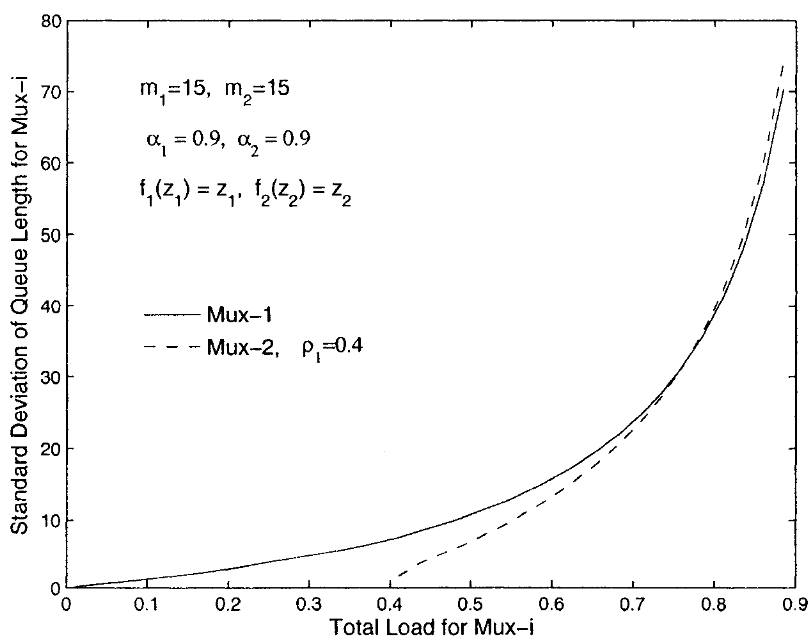


Figure 3.7 Standard deviation of queue length vs. its total load for multiplexer-i

Figure 3.8 presents the mean queue length of multiplexer-2 as m_1 or m_2 is changing while keeping the other constant at a value of 11. The traffic load of each multiplexer is held constant. It may be seen that two curves cross each other at $m_i = 11$. From the discussion of the previous paragraph, higher number of sources means that traffic is burstier when the load is kept constant. On the other hand, as the traffic goes through more multiplexers then it gets smoother. When $m_i < 11$, the solid line corresponds to $m_1 < m_2$ and the dashed line corresponds to $m_2 < m_1$. It may be seen that the solid line is higher than dashed line. This is because higher proportion of the traffic feeding multiplexer-2 has been smoothed out. The converse of this happens when $m_i > 11$.

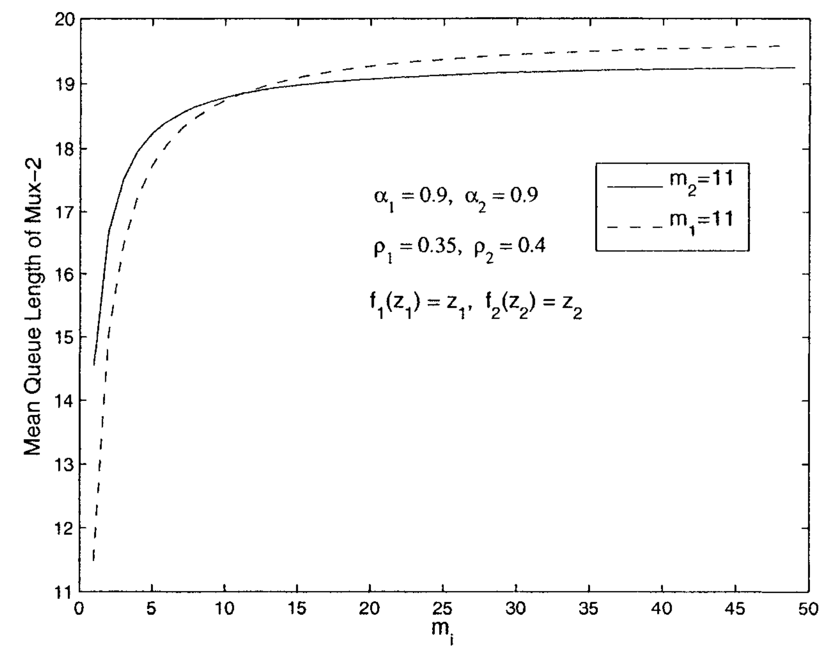


Figure 3.8 Mean queue length for multiplexer-2 vs. the number of sources for multiplexer-i

In Figures 3.9 - 3.11, mean queue length, mean packet delay, and standard deviation of the queue length are presented for multiplexer-2 versus its total load respectively for two different functions of $f_2(z_2)$. The results have been presented assuming that a type-2 *On* source generates two packets constantly during a slot, with $f_2(z_2) = z_2^2$, or generate geometrically distributed number of packets during a slot with mean equal to two, with $f_2(z_2) = z_2 /(2 - z_2)$. It may be seen that the results for deterministic packet generation are slightly lower than those for geometrical packet generation though both $f_2(z_2)$ have the same mean. This is due to that geometric sources are burstier than deterministic sources.

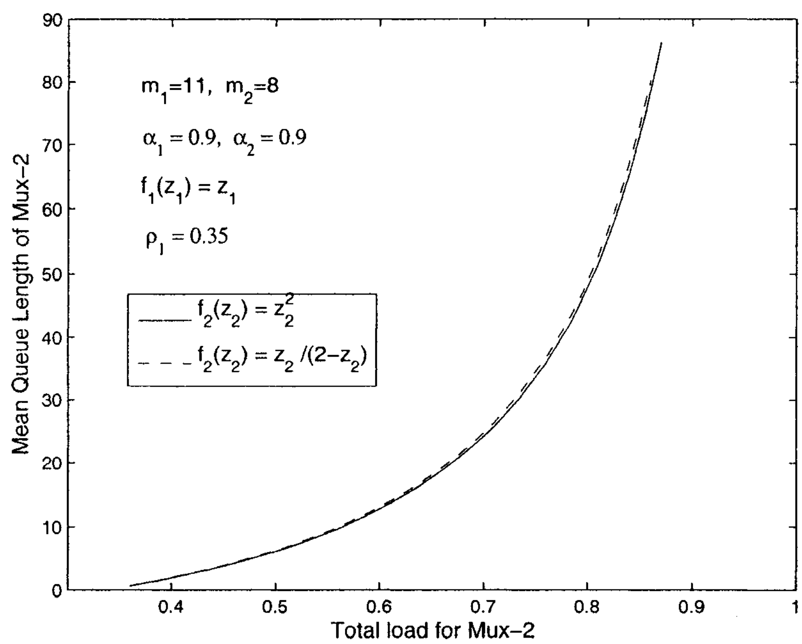


Figure 3.9 Mean queue length vs. its total load for multiplexer-2

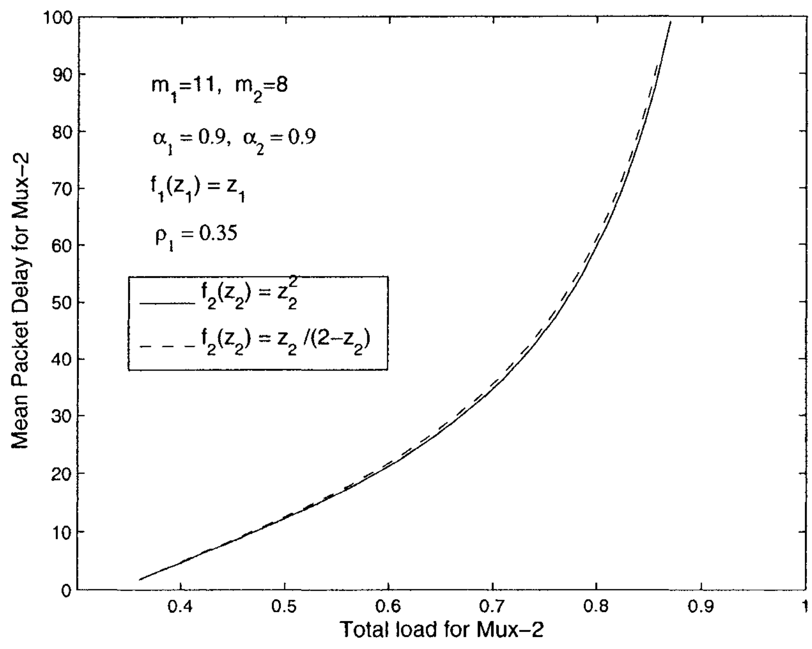


Figure 3.10 Mean packet delay vs. its total load for multiplexer-2

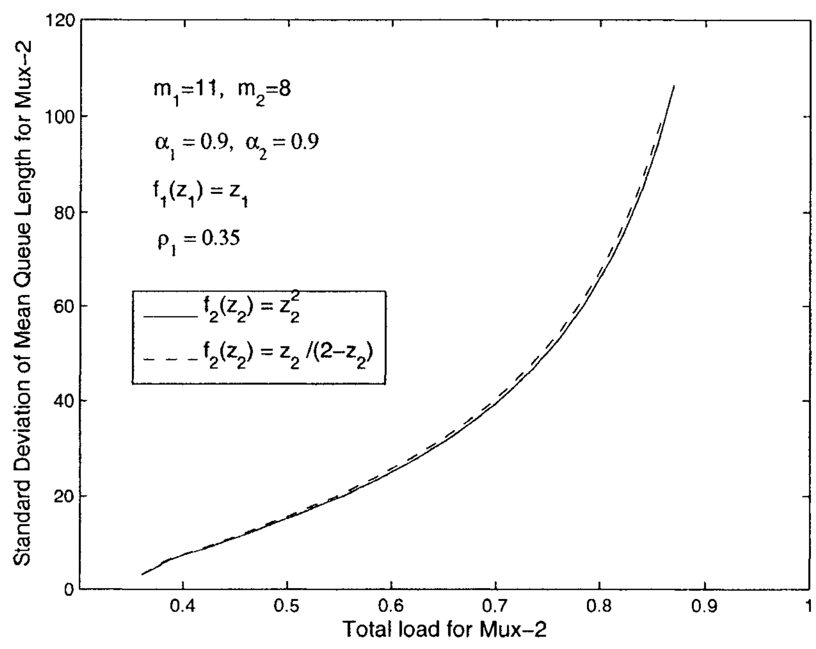


Figure 3.11 Standard deviation of queue length vs. its total load for multiplexer-2

Figures 3.12 and 3.13 present the mean and standard deviation of queue length respectively versus the total load for multiplexer-2 while ρ_1 or ρ_2 is varied and the other one is kept constant. It may be seen that both curves cross each other when the total load is 0.8. As stated above the traffic going through more multiplexers becomes smoother. When total load is less than 0.8, the curves corresponding $\rho_1 > \rho_2$ achieves lower values because higher proportion of the traffic feeding multiplexer-2 has been smoothed out. The converse of this happens when $\rho_1 < \rho_2$.

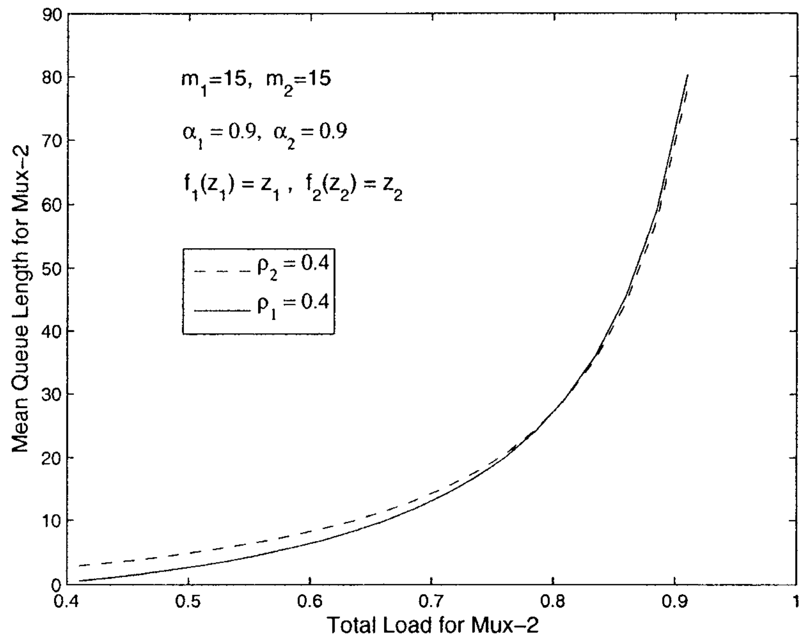


Figure 3.12 Mean queue length vs. its total load for multiplexer-2

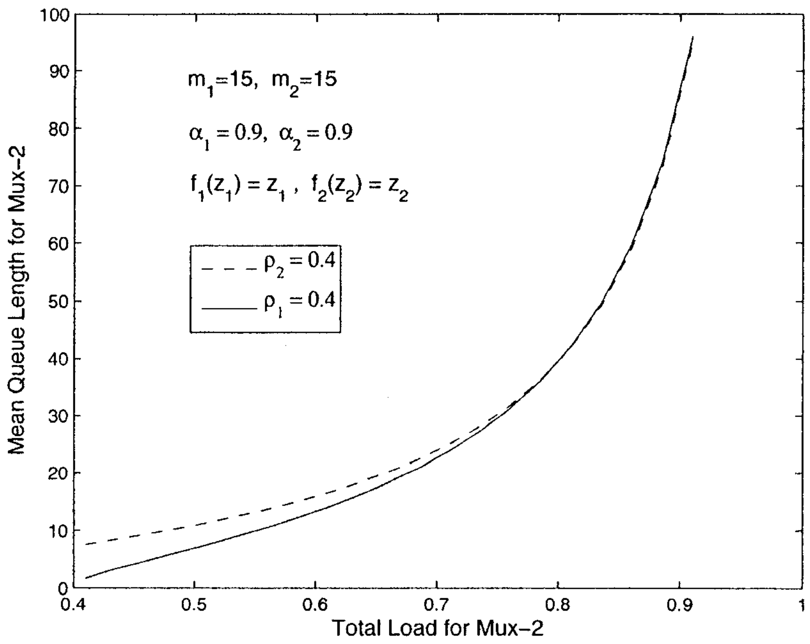


Figure 3.13 Standard deviation of queue length vs. its total load for multiplexer-2

Chapter 4

Alternative Performance Analysis of Tandem Networks

Unfortunately, it has not been possible to apply the solution technique of the previous chapter to tandem networks with more than two multiplexers. In this chapter, an alternative analysis is developed to study a tandem network with two and three multiplexers. The alternative analysis uses busy period of a multiplexer to determine the unknown boundary function. Fortunately, this analysis extends to a general tandem network with arbitrary number of multiplexers.

4.1 Two Multiplexers in Tandem

First, the alternative solution technique is applied to the two-multiplexer case. At first, how to determine the boundary function in the functional equation is explained, by using the alternative technique. Then, the steady-state PGF for the second multiplexer will be determined, which turns out to be the same as what has been obtained using the previous technique. After that, the PGF of the busy period of the second multiplexer will be determined, which is needed in studying a system with three multiplexers in tandem.

4.1.1 Determining the Boundary Function

As stated in section 3.2, because the first multiplexer is not affected by the second one, it behaves as a single multiplexer. So only the performance analysis of the second multiplexer is of interest. Next, the marginal PGF of the second multiplexer is repeated from (3.19),

$$
\begin{aligned}
&Q_k(1,1,z_2,y_2) \\
&= B_2(k) + (z_2 - 1)\sum_{j=1}^{k-1} B_2(j)Q_{k-j}(1,1,0,0) \\
&+ \frac{1-z_2}{z_2}\sum_{j=1}^{k-1} B_2(j)Q_{k-j}\left(0,0,z_2,\phi_2(j)\right) \\
&- \frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k-1} B_2(j)Q_{k-j}(0,0,0,0) \qquad k \geq 1
\end{aligned} \tag{4.1}
$$

It may be seen that in the above equation there is only one unknown boundary function. In the next, this boundary function is determined using the alternative method. The essence of this method is to write down $Q_k(0,0,z_2,y_2)$ through interpretation of the result for a single multiplexer.

First, let us consider the second multiplexer in isolation without the input from the first multiplexer. Then the evolution of this multiplexer would be same as a single multiplexer, which is given by, (see [22])

$$
\widetilde{Q}_k(0,0,z_2,y_2) = \frac{1}{z_2^{k-1}} B_2(k) + \frac{z_2 - 1}{z_2}\sum_{h=1}^{k-1} \frac{B_2(k-h)}{z_2^{k-h-1}} Q_h(1,1,0,0) \tag{4.2}
$$

where $\widetilde{Q}_k(0,0,z_2,y_2)$ denotes the PGF of the second multiplexer without the input from the first one. The terms of the summation on the RHS of (4.2) may be considered as PGFs conditioned on mutually exclusive events. These events correspond to the last time that the multiplexer queue was empty which may be at the end of any of the k slots. Assuming that the last time this event occurred at the end of h'th slot, then, its probability is given by $Q_h(1,1,0,0)$. $B_2(k-h)$ gives the PGF of the number of *On* sources and the number of packet arrivals from the last time the multiplexer queue was empty. z_2^{k-h-1} in the denominator corresponds to the PGF of the number of packet departures during the $(k-h)$ slots that the multiplexer was busy. One may note that z_2^{k-h-1} reveals that the multiplexer-2 has always been busy during this $(k-h)$ slots, which guarantees that the last time it was empty was at the end of the h'th slot. One may also note that the first term on the RHS corresponds to the event that the last time the queue length was zero was at the initial state.

Next, the arrivals from the first multiplexer will be considered. Equation (4.2) is modified using the above interpretation. Since the goal is to determine $Q_k(0,0,z_2,y_2)$, it is known that the first multiplexer is empty and all of the m_1 sources are in the *Off* state at slot *k*. First, the modification of the first term in (4.2) is considered. From the zero initial-condition assumption, it is known that the first multiplexer is also empty at slot 0. Let us assume that the first multiplexer has r busy periods in the interval from 0 to *k*'th slot, and the probability of this event is $\varphi_r^{(1)}(k)$ (see Appendix A); then, the first multiplexer will output *k* - *r* packets to the second multiplexer during these *k* slots, which has the PGF of z_2^{k-r}. Therefore, the first term on RHS of (4.2) will become,

$$\frac{1}{z_2^{k-1}}B_2(k)\varphi_r^{(1)}(k)z_2^{k-r} = \frac{1}{z_2^{r-1}}B_2(k)\varphi_r^{(1)}(k),$$

Now summing up over all possible number of busy periods that the first multiplexer may have during these *k* slots, we have,

$$\sum_{r=1}^{k}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k)B_2(k) \tag{4.3}$$

Next, the modification of the second term in (4.2) will be considered. The summation $\frac{z_2-1}{z_2}\sum_{h=1}^{k-1}\frac{B_2(k-h)}{z_2^{k-h-1}}Q_h(1,1,0,0)$ on the RHS of (4.2) corresponds to the evolution of multiplexer-2 from slot *h* to slot *k*. Given $Q_h(1,1,0,0)$, it is known that multiplexer-2 is empty at slot *h*, then multiplexer-1 must be empty at slot *h*-1. If the first multiplexer has *r* busy periods from slot *h*-1 to slot *k*, which has the probability of $\varphi_r^{(1)}(k-h+1)$; then multiplexer-1 will output *k*-*h*+1-*r* packets to multiplexer-2, which has the PGF of $z_2^{k-h+1-r}$. Thus, the summation becomes,

$$\frac{z_2-1}{z_2}\sum_{h=1}^{k-1}\frac{B_2(k-h)}{z_2^{k-h-1}}Q_h(1,1,0,0)\varphi_r^{(1)}(k-h+1)z_2^{k-h+1-r}$$
$$=(z_2-1)\sum_{h=1}^{k-1}\frac{B_2(k-h)}{z_2^{r-1}}Q_h(1,1,0,0)\varphi_r^{(1)}(k-h+1),$$

Again summing up over all possible number of busy periods, we have,

$$(z_2-1)\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{B_2(k-h)}{z_2^{r-1}}Q_h(1,1,0,0)\varphi_r^{(1)}(k-h+1)$$

Therefore, we finally have,

$$
\begin{aligned}
&Q_k(0,0,z_2,y_2) \\
&=\sum_{r=1}^{k}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k)B_2(k) \\
&+(z_2-1)\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k-h+1)B_2(k-h)Q_h(1,1,0,0) \quad , \qquad k\geq 1
\end{aligned}
\tag{4.4}
$$

From the above, the unknown boundary function $Q_{k-j}(0,0,z_2,\phi_2(j))$ in (4.1) may be expressed as,

$$
\begin{aligned}
&Q_{k-j}(0,0,z_2,\phi_2(j)) \\
&=\sum_{r=1}^{k-j}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k-j)B_1^{(j)}(k-j) \\
&+(z_2-1)\sum_{h=1}^{k-1-j}\sum_{r=1}^{k-h+1-j}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k-h+1-j)B_2^{(j)}(k-h-j)Q_h(1,1,0,0), \quad k\geq j+1
\end{aligned}
\tag{4.5}
$$

Thus, the unknown boundary function in (4.1) has been expressed in terms of $Q_h(1,1,0,0)$ and the probabilities of the busy periods for the first multiplexer.

4.1.2 Steady-State PGF of the Second Multiplexer

In this section, the steady-state PGF of the second multiplexer will be determined. Substituting (4.5) back into (4.1), we have,

$$
\begin{aligned}
&Q_k(1,1,z_2,y_2) \\
&=B_2(k)+(z_2-1)\sum_{j=1}^{k-1}B_2(j)Q_{k-j}(1,1,0,0) \\
&-\frac{z_2-1}{z_2}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k-j)B_2(k) \\
&-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_2^{r-1}}\varphi_r^{(1)}(k-j-h+1)B_2(k-h)Q_h(1,1,0,0) \\
&-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{k-1}B_2(j)Q_{k-j}(0,0,0,0) \quad , \qquad k\geq 1
\end{aligned}
\tag{4.6}
$$

Substituting the above into $Q(1,1,z_2,y_2,\omega)$ in (3.21) gives,

$$
\begin{aligned}
&Q(1,1,z_2,y_2,\omega) \\
&= 1+\sum_{k=1}^{\infty} B_2(k)\omega^k + (z_2-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_2(j)Q_{k-j}(1,1,0,0)\omega^k \\
&-(z_2-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_2^r}\varphi_r^{(1)}(k-j)B_2(k)\omega^k \\
&-(z_2-1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_2^r}\varphi_r^{(1)}(k-j-h+1)B_2(k-h)Q_h(1,1,0,0)\omega^k \\
&-\frac{(z_2-1)^2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_2(j)Q_{k-j}(0,0,0,0)\omega^k
\end{aligned}
\tag{4.7}
$$

A general form of the above PGF has been simplified in Appendix D. Applying the result in (D.13) with $n=2$, the above equation may be expressed as,

$$
\begin{aligned}
&Q(1,1,z_2,y_2,\omega) \\
&= 1+\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\
&+(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\
&-(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\
&-(z_2-1)^2[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]}
\end{aligned}
\tag{4.8}
$$

The steady-state PGF of the second multiplexer is determined through the application of the final-value theorem to (4.8),

$$Q(1,1,z_2,y_2)=Q_\infty(1,1,z_2,y_2)=\lim_{\omega\to 1}(1-\omega)Q(1,1,z_2,y_2,\omega)$$

Noting that

$$\lim_{\omega\to 1}(1-\omega)Q(1,1,0,0,\omega)=Q(1,1,0,0)=1-\rho_1-\rho_2$$

We have,

$$
\begin{aligned}
&Q(1,1,z_2,y_2) \\
&=(z_2-1)(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}} \\
&-(z_2-1)^2(1-\rho_1-\rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i})[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})]}
\end{aligned}
\tag{4.9}
$$

The above result gives the joint PGF of the second multiplexer. Substituting $y_2 = 1$ in (4.9) gives the PGF of the queue length of the second multiplexer, which is

$$P_2(z_2) = (z_2 - 1)(1 - \rho_1 - \rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}}{1 - \lambda_{12}^i\lambda_{22}^{m_2-i}} - (z_2 - 1)^2(1 - \rho_1 - \rho_2)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})}{(1 - \lambda_{12}^i\lambda_{22}^{m_2-i})[z_2 - \lambda_{12}^i\lambda_{22}^{m_2-i}\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i})]} \tag{4.10}$$

It may be found that the equations (4.8), (4.9) and (4.10) are the same as (3.27), (3.30) and (3.31) in the previous chapter respectively. This gives strong support that the alternative analysis presented above is correct.

4.1.3 PGF of the Busy Period of the Second Multiplexer

The performance analysis of a tandem network with three multiplexers will require the PGF of the busy period of the second multiplexer, $\Gamma_2(\omega)$. In this section, this PGF is derived. From (B.8) of Appendix B, we have,

$$\Gamma_2(\omega) = \frac{1}{\omega}\left(1 - \frac{1}{Q(1,1,0,0,\omega)}\right) \tag{4.11}$$

It may be seen that $\Gamma_2(\omega)$ is determined completely by $Q(1,1,0,0,\omega)$. Next, $Q(1,1,0,0,\omega)$ is determined by invoking the analytical property of the function $Q(1,1,z_2,1,\omega)$ within the poly disk $(|z_2| \le 1; |\omega| < 1)$. Substituting $y_2 = 1$ in (4.8), we have,

$$\begin{aligned} &Q(1,1,z_2,1,\omega) \\ &= 1 + \sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\ &+ (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\ &- (z_2 - 1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\ &- (z_2 - 1)^2[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(\tilde{C}_{12}\lambda_{12})^i(\tilde{C}_{22}\lambda_{22})^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2 - \lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \end{aligned} \tag{4.12}$$

Let $F_i(z_2)$ denote the following factor in the denominator of $Q(1,1,z_2,1,\omega)$,

$$F_i(z_2) = z_2 - \lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega) \quad ,0 \le i \le m_2 \tag{4.13}$$

It is noted that the number of distinct factors of $F_i(z_2)$ is equal to (m_2+1).

Next, it will be shown that each $F_i(z_2)$ has a single root within the open poly disk $(|z_2|<1; |\omega|<1)$ through the application of Rouche's theorem. Let us define,

$$h(z_2) = z_2, \quad g_i(z_2) = -\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega) \tag{4.14}$$

As shown in the Appendix A of [24], for the cases under consideration, λ_{12} and λ_{22} are analytic functions of z_2; and because $\Gamma_1(\omega)$ is a PGF of the busy period, $\Gamma_1(\omega)$ is also an analytic function of ω. As a result, $g_i(z_2)$ is analytic as required by the Rouche's theorem within the closed unit circle, $|z_n| \le 1$. It has also been shown in the Appendix A of [24] that $|\lambda_{12}|<1$ and $|\lambda_{22}| \le 1$ on $|z_2|=1$, then on the closed unit circle $(|z_2|=1; |\omega|<1)$, we have $|\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega|<1$. Because $\Gamma_1(\omega)$ is a PGF, we have $|\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)|<1$ on $|z_2|=1$. Therefore, on the closed unit circle $(|z_2|=1; |\omega|<1)$, we have,

$$|h(z_2)| = |z_2| = 1, \qquad |g_i(z_2)| = |\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)| < 1 \tag{4.15}$$

From the above we conclude,

$|h(z_2)| > |g_i(z_2)|$ on the closed unit circle $(|z_2|=1; |\omega|<1)$, and $h(z_2)$ and $g_i(z_2)$ are analytical functions within and on the unit circle. Therefore, Rouche's theorem applies, and as a result $h(z_2)$ and $F_i(z_2) = h(z_2) + g_i(z_2)$ have the same number of zeros within the open unit circle $(|z_2|<1; |\omega|<1)$. Since $h(z_2)$ has a single zero, $F_i(z_2)$ will also have a single zero within the open unit circle. Let z_i^* denote the zero of the i'th distinct denominator factor, $F_i(z_2)$.

Following the similar steps in the application of Rouche's theorem, it is easy to show that the denominator factor $(1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)$ has no zeros within the open unit circle $(|z_2|=1; |\omega|<1)$. So these z_i^* are the only roots of the common denominator of (4.12) within the open unit circle.

Because $Q(1,1,z_2,1,\omega)$ is an analytic function within the closed unit circle, it has to be bounded. Then, the roots of the denominator, z_i^*, must also be the roots of the numerator. Since we have $\lambda_{12}\big|_{z_2=0} = 0$; then for any i, $0 < i \le m_2$, the unique root of the denominator is $z_i^* = 0$, which also appears in the numerator as we also have $\tilde{C}_{12}\big|_{z_2=0} = 0$. Therefore these m_2 roots do not give us an equation to solve for $Q(1,1,0,0,\omega)$. Let us consider the term corresponding to $i = 0$ in (4.12), which is given below,

$$
\begin{aligned}
&Q_0(1,1,z_2,1,\omega)\\
&= 1 + \frac{(\tilde{C}_{22}\lambda_{22})^{m_2}\omega}{1-\lambda_{22}^{m_2}\omega} + (z_2-1)[Q(1,1,0,0,\omega)-1]\frac{(\tilde{C}_{22}\lambda_{22})^{m_2}\omega}{1-\lambda_{22}^{m_2}\omega}\\
&\quad -(z_2-1)\frac{(\tilde{C}_{22}\lambda_{22})^{m_2}\omega\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)}{(1-\lambda_{22}^{m_2}\omega)[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]}\\
&\quad -(z_2-1)^2[Q(1,1,0,0,\omega)-1]\frac{(\tilde{C}_{22}\lambda_{22})^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)}{(1-\lambda_{22}^{m_2}\omega)[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]}
\end{aligned}
$$

The above equation may be written as,

$$
\begin{aligned}
&Q_0(1,1,z_2,1,\omega)\\
&= \frac{1}{(1-\lambda_{22}^{m_2}\omega)[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]}\Big\{(1-\lambda_{22}^{m_2}\omega)[z_2-\lambda_{22}^{m_2}\Gamma_1(\lambda_{22}^{m_2}\omega)]\\
&\qquad + (\tilde{C}_{22}\lambda_{22})^{m_2}\omega[z_2-\lambda_{22}^{m_2}\Gamma_1(\lambda_{22}^{m_2}\omega)]\\
&\qquad + (z_2-1)[Q(1,1,0,0,\omega)-1](\tilde{C}_{22}\lambda_{22})^{m_2}\omega[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]\\
&\qquad - (z_2-1)(\tilde{C}_{22}\lambda_{22})^{m_2}\omega\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\\
&\qquad - (z_2-1)^2[Q(1,1,0,0,\omega)-1](\tilde{C}_{22}\lambda_{22})^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\Big\}
\end{aligned}
\tag{4.16}
$$

Letting $z_2^*(\omega)$ denote the unique root of the denominator of the above equation, that is $z_2^*(\omega) = z_0^*$. Then, $z_2^*(\omega)$ is the root of the following equation,

$$z_2 - \lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega) = 0,$$

Substituting $z_2^*(\omega)$ in the above equation gives,

$$z_2^*(\omega) - \lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\Big|_{z_2=z_2^*(\omega)} = 0 \tag{4.17}$$

As stated before, $z_2^*(\omega)$ must also be the root of the numerator of (4.16), therefore, we have,

$$\begin{aligned}&(1-\lambda_{22}^{m_2}\omega)[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]+(\widetilde{C}_{22}\lambda_{22})^{m_2}\omega[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]\\&+(z_2-1)[Q(1,1,0,0,\omega)-1](\widetilde{C}_{22}\lambda_{22})^{m_2}\omega[z_2-\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)]\\&-(z_2-1)(\widetilde{C}_{22}\lambda_{22})^{m_2}\omega\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\\&-(z_2-1)^2[Q(1,1,0,0,\omega)-1](\widetilde{C}_{22}\lambda_{22})^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\Big|_{z_2=z_2^*(\omega)}=0\end{aligned}$$

Substituting (4.16) in the above yields,

$$\begin{aligned}&-(z_2-1)(\widetilde{C}_{22}\lambda_{22})^{m_2}\omega\lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\Big|_{z_2=z_2^*(\omega)}\\&=(z_2-1)^2[Q(1,1,0,0,\omega)-1](\widetilde{C}_{22}\lambda_{22})^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega)\Big|_{z_2=z_2^*(\omega)}\end{aligned}$$

Solving the above equation for $Q(1,1,0,0,\omega)$ gives,

$$Q(1,1,0,0,\omega)=1+\frac{\lambda_{22}^{m_2}\omega\Big|_{z_2=z_2^*(\omega)}}{1-z_2^*(\omega)} \tag{4.18}$$

Finally, substituting (4.18) into (4.11) results in the PGF of the busy period of the second multiplexer,

$$\Gamma_2(\omega)=\frac{\lambda_{22}^{m_2}\Big|_{z_2=z_2^*(\omega)}}{1-z_2^*(\omega)+\lambda_{22}^{m_2}\omega\Big|_{z_2=z_2^*(\omega)}} \tag{4.19}$$

Next, let us determine $Q(1,1,0,0,\omega)$ and $\Gamma_2(\omega)$ at $\rho_1=0$. When $\rho_1=0$, according to the definition of a busy period, we have $\Gamma_1(\omega)=1$. So, $\Gamma_1(\lambda_{12}^{i}\lambda_{22}^{m_2-i})=1$. Substituting this result in (4.17) gives,

$$z_2^*(\omega)-\lambda_{22}^{m_2}\omega\Big|_{z_2=z_2^*(\omega)}=0$$

Thus, we have

$$z_2^*(\omega)=\lambda_{22}^{m_2}\omega\Big|_{z_2=z_2^*(\omega)}, \tag{4.20}$$

Substituting the above result in (4.18) gives us

$$Q(1,1,0,0,\omega)=\frac{1}{1-z_2^*(\omega)}$$

It may be seen that the above result is the same as the single multiplexer case in (26) of [22].

Substituting (4.20) in (4.19), the PGF of the busy period of the second multiplexer is obtained,

$$\Gamma_2(\omega) = \lambda_{22}^{m_2}\Big|_{z_2 = z_2^*(\omega)} = \frac{z_2^*(\omega)}{\omega}$$

Not surprisingly, it may be seen that for $\rho_1 = 0$, $\Gamma_2(\omega)$ is reduced to the single multiplexer case in (42) of [23].

4.2 Three Multiplexers in Tandem

Now, the alternative analysis is applied to a tandem network with three multiplexers. As the second multiplexer is not affected by the third one, it behaves exactly like the second multiplexer in a tandem system with only two multiplexers, which has just been studied in the preceding section. Therefore, one needs to study only the third multiplexer in this tandem network.

4.2.1 Steady-State PGF of the Third Multiplexer

In this section, the joint steady-state PGF of the queue length and the number of *On* sources of the third multiplexer is determined. This will be done following the same steps as in the previous section. First, the new technique is used to determine the unknown boundary function, and then the final value theorem is applied to determine the steady-state PGF of the third multiplexer.

Substituting n =3 in (2.38) gives the following transient PGF of the third multiplexer,

$$\begin{aligned}
&Q_k(1_2, 1_2, z_3, y_3) \\
&= B_3(k) + (z_3 - 1)\sum_{j=1}^{k-1} B_3(j) Q_{k-j}(1_2, 1_2, 0_3, 0_3) \\
&\quad + \frac{1 - z_3}{z_3}\sum_{j=1}^{k-1} B_3(j) Q_{k-j}\big(0_2, 0_2, z_3, \phi_3(j)\big) \\
&\quad - \frac{(z_3 - 1)^2}{z_3}\sum_{j=1}^{k-1} B_3(j) Q_{k-j}(0_2, 0_2, 0_3, 0_3) \qquad k \geq 1
\end{aligned} \tag{4.21}$$

The only unknown boundary function in the above is $Q_{k-j}(0_2,0_2,z_3,\phi_3(j))$. Analogous to (4.4), the boundary function $Q_k(0_2,0_2,z_3,y_3)$ may be written as,

$$
\begin{aligned}
&Q_k(0_2,0_2,z_3,y_3) \\
&=\sum_{r=1}^{k}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k)B_3(k) \\
&+(z_3-1)\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k-h+1)B_3(k-h)Q_h(1_2,1_2,0_3,0_3) \quad , \qquad k\geq 1
\end{aligned}
\tag{4.22}
$$

The above equation may be written as,

$$
\begin{aligned}
&Q_{k-j}(0_2,0_2,z_3,\phi_3(j)) \\
&=\sum_{r=1}^{k-j}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k-j)B_3^{(j)}(k-j) \\
&+(z_3-1)\sum_{h=1}^{k-1-j}\sum_{r=1}^{k-h+1-j}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k-h+1-j)B_3^{(j)}(k-h-j)Q_h(1_2,1_2,0_3,0_3) \\
&\qquad\qquad , \quad k\geq j+1
\end{aligned}
\tag{4.23}
$$

Substituting (4.23) back into (4.21), we have,

$$
\begin{aligned}
&Q_k(1_2,1_2,z_3,y_3) \\
&=B_3(k)+(z_3-1)\sum_{j=1}^{k-1}B_3(j)Q_{k-j}(1_2,1_2,0_3,0_3) \\
&-\frac{z_3-1}{z_3}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k-j)B_3(k) \\
&-\frac{(z_3-1)^2}{z_3}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_3^{r-1}}\varphi_r^{(2)}(k-j-h+1)B_3(k-h)Q_h(1_2,1_2,0_3,0_3) \\
&-\frac{(z_3-1)^2}{z_3}\sum_{j=1}^{k-1}B_3(j)Q_{k-j}(0_2,0_2,0_3,0_3) \quad , \qquad k\geq 1
\end{aligned}
\tag{4.24}
$$

Let us define the following transform,

$$
Q(1_2,1_2,z_3,y_3,\omega)=\sum_{k=0}^{\infty}Q_k(1_2,1_2,z_3,y_3)\omega^k \tag{4.25}
$$

Then from (4.24) we have,

$$Q(1_2,1_2,z_3,y_3,\omega)$$
$$=1+\sum_{k=1}^{\infty}B_3(k)\omega^k+(z_3-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_3(j)Q_{k-j}(1_2,1_2,0_3,0_3)\omega^k$$
$$-(z_3-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_3^r}\varphi_r^{(2)}(k-j)B_3(k)\omega^k \tag{4.26}$$
$$-(z_3-1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_3^r}\varphi_r^{(2)}(k-j-h+1)B_3(k-h)Q_h(1_2,1_2,0_3,0_3)\omega^k$$
$$-\frac{(z_3-1)^2}{z_3}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_3(j)Q_{k-j}(0_2,0_2,0_3,0_3)\omega^k$$

After substituting $n=3$ in (D.13), the above equation may be written as,

$$Q(1_2,1_2,z_3,y_3,\omega)$$
$$=1+\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}\omega}{1-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega}$$
$$+(z_3-1)[Q(1_2,1_2,0_3,0_3,\omega)-1]\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}\omega}{1-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega} \tag{4.27}$$
$$-(z_3-1)\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}\omega\lambda_{13}^i\lambda_{23}^{m_3-i}\omega\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)}{(1-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)[z_3-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)]}$$
$$-(z_3-1)^2[Q(1_2,1_2,0_3,0_3,\omega)-1]\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}\omega\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)}{(1-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)[z_3-\lambda_{13}^i\lambda_{23}^{m_3-i}\omega\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i}\omega)]}$$

In order to determine the steady-state PGF of the third multiplexer, the final-value theorem is applied to (4.27),

$$Q(1_2,1_2,z_3,y_3)=Q_\infty(1_2,1_2,z_3,y_3)=\lim_{\omega\to 1}(1-\omega)Q(1_2,1_2,z_3,y_3,\omega) \tag{4.28}$$

Noting,

$$\lim_{\omega\to 1}(1-\omega)Q(1_2,1_2,0_3,0_3,\omega)=Q(1_2,1_2,0_3,0_3)=1-\sum_{j=1}^{3}\rho_j \tag{4.29}$$

where ρ_j is the external arrival rates from type-j sources, which is given in (3.28).

Then, from (4.27, 4.28) we have,

$$Q(1_2,1_2,z_3,y_3)$$
$$=(z_3-1)(1-\sum_{j=1}^{3}\rho_j)\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}}{1-\lambda_{13}^i\lambda_{23}^{m_3-i}} \tag{4.30}$$
$$-(z_3-1)^2(1-\sum_{j=1}^{3}\rho_j)\sum_{i=0}^{m_3}\binom{m_3}{i}\frac{(C_{13}\lambda_{13})^i(C_{23}\lambda_{23})^{m_3-i}\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i})}{(1-\lambda_{13}^i\lambda_{23}^{m_3-i})[z_3-\lambda_{13}^i\lambda_{23}^{m_3-i}\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i})]}$$

The above is the joint steady-state PGF of the queue length and the number of *On* sources for the third multiplexer.

Substituting $y_3 = 1$ in (4.30), the PGF of the queue length for the third multiplexer is obtained,

$$P_3(z_3) = (z_3 - 1)(1 - \sum_{j=1}^{3} \rho_j) \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i}}{1 - \lambda_{13}^i \lambda_{23}^{m_3-i}}$$
$$- (z_3 - 1)^2 (1 - \sum_{j=1}^{3} \rho_j) \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i} \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i})}{(1 - \lambda_{13}^i \lambda_{23}^{m_3-i})[z_3 - \lambda_{13}^i \lambda_{23}^{m_3-i} \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i})]} \tag{4.31}$$

4.2.2 PGF of the Busy Period of the Third Multiplexer

In this section, the PGF of the busy period for the third multiplexer $\Gamma_3(\omega)$ will be determined. $\Gamma_3(\omega)$ is needed in the performance analysis of tandem networks with four multiplexers. From (B.8), we have,

$$\Gamma_3(\omega) = \frac{1}{\omega}\left(1 - \frac{1}{Q(1_2, 1_2, 0_3, 0_3, \omega)}\right) \tag{4.32}$$

In the above it may be seen that $\Gamma_3(\omega)$ is determined completely by $Q(1_2, 1_2, 0_3, 0_3, \omega)$. Next, $Q(1_2, 1_2, 0_3, 0_3, \omega)$ is determined through invoking the analytical property of the function $Q(1_2, 1_2, z_3, 1_3, \omega)$ within the closed poly disk $(|z_3| \le 1; |\omega| < 1)$.

Substituting $y_3 = 1$ in (4.27), we have,

$$Q(1_2, 1_2, z_3, 1_3, \omega)$$
$$= 1 + \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i} \omega}{1 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega}$$
$$+ (z_3 - 1)[Q(1_2, 1_2, 0_3, 0_3 \omega) - 1] \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i} \omega}{1 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega}$$
$$- (z_3 - 1) \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i} \omega \lambda_{13}^i \lambda_{23}^{m_3-i} \omega \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i} \omega)}{(1 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega)[z_3 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i} \omega)]}$$
$$- (z_3 - 1)^2 [Q(1_2, 1_2, 0_3, 0_3, \omega) - 1] \sum_{i=0}^{m_3} \binom{m_3}{i} \frac{(\widetilde{C}_{13}\lambda_{13})^i (\widetilde{C}_{23}\lambda_{23})^{m_3-i} \omega \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i} \omega)}{(1 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega)[z_3 - \lambda_{13}^i \lambda_{23}^{m_3-i} \omega \Gamma_2(\lambda_{13}^i \lambda_{23}^{m_3-i} \omega)]} \tag{4.33}$$

where, $\widetilde{C}_{13} = C_{13}\big|_{y_3=1}$ and $\widetilde{C}_{23} = C_{23}\big|_{y_3=1}$

The above equation has the same form as the one in (4.12). Therefore, following the derivation of the PGF of the busy period for the second multiplexer, we have,

$$Q(1_2,1_2,0_3,0_3,\omega) = 1 + \frac{\lambda_{23}^{m_3}\omega\big|_{z_3=z_3^*(\omega)}}{1-z_3^*(\omega)} \tag{4.34}$$

Substituting (4.34) into (4.32) gives the PGF of the busy period for the third multiplexer,

$$\Gamma_3(\omega) = \frac{\lambda_{23}^{m_3}\big|_{z_3=z_3^*(\omega)}}{1-z_3^*(\omega)+\lambda_{23}^{m_3}\omega\big|_{z_3=z_3^*(\omega)}} \tag{4.35}$$

4.2.3 Performance Measures of the Third Multiplexer

In the next, the mean and variance of queue length, and packet delay for the third multiplexer will be determined. Determination of these performance measures for the third multiplexer follows the same steps as those in section 3.3. First, $P_3(z_3)$ in (4.31) is transferred into a more convenient form. Let us define,

$$E_3(z_3) = \sum_{i=1}^{m_3}\binom{m_3}{i}\frac{(\widetilde{C}_{13}\lambda_{13})^i(\widetilde{C}_{23}\lambda_{23})^{m_3-i}}{1-\lambda_{13}^i\lambda_{23}^{m_3-i}} \tag{4.36}$$

$$F_3(z_3) = \sum_{i=1}^{m_3}\binom{m_3}{i}\frac{(\widetilde{C}_{13}\lambda_{13})^i(\widetilde{C}_{23}\lambda_{23})^{m_3-i}\Gamma_2(\lambda_{13}^i\lambda_{23}^{m_3-i})}{(1-\lambda_{13}^i\lambda_{23}^{m_3-i})[z_3-\lambda_{13}^i\lambda_{23}^{m_3-i}\Gamma_1(\lambda_{13}^i\lambda_{23}^{m_3-i})]} \tag{4.37}$$

Then we have, $E_3(1)=0, F_3(1)=0$ (4.38)

And $P_3(z_3)$ may be written as,

$$\begin{aligned} P_3(z_3) &= (z_3-1)(1-\sum_{j=1}^{3}\rho_j)\left[E_3(z_3)+\frac{(\widetilde{C}_{23}\lambda_{23})^{m_3}}{1-\lambda_{23}^{m_3}}\right] \\ &\quad -(z_3-1)^2(1-\sum_{j=1}^{3}\rho_j)\left[F_3(z_3)+\frac{(\widetilde{C}_{23}\lambda_{23})^{m_3}\Gamma_2(\lambda_{23}^{m_3})}{(1-\lambda_{23}^{m_3})[z_3-\lambda_{23}^{m_3}\Gamma_2(\lambda_{23}^{m_3})]}\right] \end{aligned}$$

Let us further define

$$H_3(z_3) = \lambda_{23}^{m_3}, \quad G_3(z_3) = (\widetilde{C}_{23}\lambda_{23})^{m_3} \tag{4.39}$$

$$\Theta_3(z_3) = H_3(z_3)\Gamma_2(H_3(z_3)) = \lambda_{23}^{m_3}\Gamma_2(\lambda_{23}^{m_3}) \tag{4.40}$$

Then we have $H_3(1)=1, G_3(1)=1, \Theta_3(1)=1$ (4.41)

As a result, $P_3(z_3)$ may be further written as,

$$\begin{aligned} P_3(z_3) &= (z_3-1)(1-\sum_{j=1}^{3}\rho_j)\left[E_3(z_3)+\frac{G_3(z_3)}{1-H_3(z_3)}\right] \\ &\quad -(z_3-1)^2(1-\sum_{j=1}^{3}\rho_j)\left[F_3(z_3)+\frac{G_3(z_3)\Gamma_2(H_3(z_3))}{[1-H_3(z_3)][z_3-\Theta_3(z_3)]}\right] \end{aligned} \tag{4.42}$$

In the next, the performance measures will be expressed in terms of the derivatives of $H_3(z_3)$, $G_3(z_3)$, $E_3(z_3)$, $\Gamma_2(\omega)$ and $\Theta_3(z_3)$.

- Derivation of the mean queue length and mean packet delay

First, the mean queue length of the third multiplexer will be determined. Both sides of (4.42) are multiplied with the common denominator, then the third derivative of both sides is taken with respect to z_3. After that, substituting $z_3=1$ and noting (4.38, 4.41) gives,

$$\begin{aligned} &-3H_3''(1)[1-\Theta_3'(1)]+3H_3'(1)\Theta_3''(1)-6H_3'(1)[1-\Theta_3'(1)]P_3'(1) \\ &=-6(1-\sum_{j=1}^{3}\rho_j)[\Gamma_2'(1)H_3'(1)+G_3'(1)]+3(1-\sum_{j=1}^{3}\rho_j)[-\Theta_3''(1)+2(1-\Theta_3'(1))G_3'(1)] \end{aligned}$$

Solving the above equation for $P_3'(1)$ gives the mean queue length of the third multiplexer,

$$\begin{aligned} \overline{N}_3 = P_3'(1) &= \frac{\Theta_3''(1)}{2[1-\Theta_3'(1)]}-\frac{H_3''(1)}{2H_3'(1)}+\frac{1-\sum_{j}^{3}\rho_j}{H_3'(1)[1-\Theta_3'(1)]}[\Gamma_2'(1)H_3'(1)+G_3'(1)] \\ &\quad +\frac{(1-\sum_{j}^{3}\rho_j)\Theta_3''(1)}{2H_3'(1)[1-\Theta_3'(1)]}-\frac{1-\sum_{j}^{3}\rho_j}{H_3'(1)}G_3'(1) \end{aligned} \tag{4.43}$$

$$\text{where } \Theta_3'(1)=\left.\frac{d\Theta_3(z_3)}{dz_3}\right|_{z_3=1}, \quad \Theta_3''(1)=\left.\frac{d^2\Theta_3(z_3)}{dz_3^{\,2}}\right|_{z_3=1} \tag{4.44}$$

$$H_3'(1)=\left.\frac{dH_3(z_3)}{dz_3}\right|_{z_3=1}, \quad H_3''(1)=\left.\frac{d^2H_3(z_3)}{dz_3^{\,2}}\right|_{z_3=1}, \tag{4.45}$$

$$G_3'(1)=\left.\frac{dG_3(z_3)}{dz_3}\right|_{z_3=1}, \quad \Gamma_2'(1)=\left.\frac{d\Gamma_2(\omega)}{d\omega}\right|_{\omega=1}, \tag{4.46}$$

From the Little's result, the mean packet delay at the third multiplexer is given by,

$$\overline{D}_3 = \overline{N}_3 \Big/ \sum_{j=1}^{3} \rho_j \tag{4.47}$$

- Derivation of the variance of queue length

The variance of the queue length requires the second order derivative of $P_3(z_3)$. After multiplying both sides of equation (4.42) with the common denominator, and then taking the forth derivative with respect to z_3, $z_3 = 1$ is substituted and then the equation is solved for $P_3''(1)$,

$$\begin{aligned}
&P_3''(1) \\
&= \frac{1}{6H_3'(1)[1-\Theta_3'(1)]}\left\{2(1-\sum_j^3 \rho_j)\Theta_3'''(1) + 3H_3''(1)\Theta_3''(1)\right. \\
&\qquad\qquad \left. + 6(1-\sum_j^3 \rho_j)[\Theta_3''(1)G_3'(1) + \Theta_3'(1)G_3''(1) + \Gamma_2'(1)H_3''(1)]\right\} \\
&+ \frac{1}{3H_3'(1)}\left\{-H_3'''(1) + 6(1-\sum_j^3 \rho_j)H_3'(1)E_3'(1) - 3H_3''(1)P_3'(1)\right\} \\
&+ \frac{1}{3[1-\Theta_3'(1)]}\left\{\Theta_3'''(1) + 3\Theta_3''(1)P_3'(1) + 3(1-\sum_j^3 \rho_j)[2\Gamma_2'(1)G_3'(1) + \Gamma_2''(1)H_3'(1)]\right\}
\end{aligned} \tag{4.48}$$

where $\Theta_3'''(1) = \left.\frac{d^3\Theta_3(z_3)}{dz_3^3}\right|_{z_3=1}$, $G_3''(1) = \left.\frac{d^2G_3(z_3)}{dz_3^2}\right|_{z_3=1}$,

$$E_3'(1) = \left.\frac{dE_3(z_3)}{dz_3}\right|_{z_3=1}, \quad H_3'''(1) = \left.\frac{d^3H_3(z_3)}{dz_3^3}\right|_{z_3=1}, \quad \Gamma_2''(1) = \left.\frac{d^2\Gamma_2(\omega)}{d\omega^2}\right|_{\omega=1}, \tag{4.49}$$

The variance of the queue length of the third multiplexer is given by,

$$\overline{V}_3 = P_3''(1) + P_3'(1) - (P_3'(1))^2 \tag{4.50}$$

Next, how to determine the derivatives of $H_3(z_3)$, $G_3(z_3)$, $E_3(z_3)$, $\Gamma_2(\omega)$ and $\Theta_3(z_3)$ is discussed. The derivatives of $H_3(z_3)$, $G_3(z_3)$ and $E_3(z_3)$ can be determined from their definitions in (4.36, 4.39) in a straightforward manner and the details will not be given here. It is only noted that,

$$H_3'(1) = \frac{m_3(1-\beta_3)f_3'(1)}{2-\alpha_3-\beta_3} = \rho_3 \tag{4.51}$$

Next, the first three derivatives of $\Gamma_2(\omega)$ and $\Theta_3(z_3)$ will be determined. These derivatives will be expressed in terms of the derivatives of $z_2^*(\omega)$, which is the unique root of the equation,

$$z_2 - \omega H_2(z_2)\Gamma_1(\omega H_2(z_2)) = 0, \tag{4.52}$$

From its definition, $\Gamma_2(\omega)$ is the PGF of the busy period of the second multiplexer, which has been determined in (4.19). Let us repeat this result,

$$\Gamma_2(\omega) = \frac{H_2(z_2^*(\omega))}{1 - z_2^*(\omega) + H_2(z_2^*(\omega))\omega}, \tag{4.53}$$

Taking the successive derivatives of the above equation and substituting $\omega = 1$, and then noting that $\Gamma_2(1) = 1$, $z_2^*(1) = 1$ gives,

$$\Gamma_2'(1) = \left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1} - 1 \tag{4.54}$$

$$\Gamma_2''(1) = \left.\frac{d^2 z_2^*(\omega)}{d\omega^2}\right|_{\omega=1} + 2(\Gamma_2'(1))^2 - 2\rho_2\left(\left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1}\right)^2 \tag{4.55}$$

$$\begin{aligned} \Gamma_2'''(1) = {} & \left.\frac{d^3 z_2^*(\omega)}{d\omega^3}\right|_{\omega=1} + 3\left(\Gamma_2''(1) + \frac{d^2 z_2^*(\omega)}{d\omega^2}\right)\left[\Gamma_2'(1) - \rho_2 \cdot \frac{dz_2^*(\omega)}{d\omega}\right]\Bigg|_{\omega=1} \\ & - 3H_2''(1)\cdot\left.\left(\frac{dz_2^*(\omega)}{d\omega}\right)^3\right|_{\omega=1} - 6\rho_2\Gamma_2'(1)\left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1} \end{aligned} \tag{4.56}$$

Now, the first three derivatives of $\Theta_3(z_3)$ will be determined. From (4.40), we have,

$$\Theta_3(z_3) = H_3(z_3)\Gamma_2(H_3(z_3)) \tag{4.57}$$

Taking the successive derivatives of $\Theta_3(z_3)$ with respect to z_3 and then substituting $z_3 = 1$ gives,

$$\Theta_3'(1) = \rho_3(1 + \Gamma_2'(1)) \tag{4.58}$$

$$\Theta_3''(1) = H_3''(1)[1 + \Gamma_2'(1)] + \rho_3^2[2\Gamma_2'(1) + \Gamma_2''(1)] \tag{4.59}$$

$$\Theta_3'''(1) = H_3'''(1)[1+\Gamma_2'(1)] + 3\rho_3 H_3''(1)[\Gamma_2''(1) + 2\Gamma_2'(1)] + \rho_3^3[\Gamma_3'''(1) + 3\Gamma_2''(1)] \tag{4.60}$$

Finally, the first three derivatives of $z_2^*(\omega)$ will be determined at $\omega = 1$. From equation (4.52), $z_2^*(\omega)$ is the unique root of the equation, $z_2 - \lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega) = 0$, then we have,

$$z_2^*(\omega) = \omega H_2(z_2)\Gamma_1(\omega H_2(z_2))\Big|_{z_2 = z_2^*(\omega)} \tag{4.61}$$

Taking the first order derivative of both sides of (4.65) with respect to ω, we have,

$$\begin{aligned}\frac{dz_2^*(\omega)}{d\omega} &= \left[H_2\left(z_2^*(\omega)\right) + \omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right]\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)\\ &+ \omega H_2\left(z_2^*(\omega)\right)\frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[H_2\left(z_2^*(\omega)\right) + \omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right]\end{aligned} \tag{4.62}$$

Since the unique root of the equation $z_2 - \lambda_{22}^{m_2}\omega\Gamma_1(\lambda_{22}^{m_2}\omega) = 0$ at $\omega = 1$ is $z_2 = 1$, we have,

$$z_2^*(\omega)\Big|_{\omega=1} = 1 \tag{4.63}$$

Substituting $\omega = 1$ in (4.62) and noting that $z_2^*(1) = 1$, $H_2(1) = 1$, $\Gamma_1(1) = 1$ gives,

$$\left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1} = [1+\Gamma_1'(1)]\left[1 + \rho_2\left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1}\right]$$

Solving the above equation, we obtain,

$$\left.\frac{dz_2^*(\omega)}{d\omega}\right|_{\omega=1} = \frac{1+\Gamma_1'(1)}{1-\rho_2[1+\Gamma_1'(1)]} \tag{4.64}$$

Taking the second order derivative of both sides of (4.65) with respect to ω, we have,

$$\frac{d^2 z_2^*(\omega)}{d\omega^2} = \left[2\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{dz_2^*(\omega)}{d\omega} + \omega \frac{d^2 H_2\left(z_2^*(\omega)\right)}{d^2 z_2^*(\omega)} \cdot \left(\frac{dz_2^*(\omega)}{d\omega}\right)^2 \right.$$

$$\left. + \omega \frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{d^2 z_2^*(\omega)}{d\omega^2} \right] \Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)$$

$$+ 2\left[H_2\left(z_2^*(\omega)\right) + \omega \frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{dz_2^*(\omega)}{d\omega} \right]^2 \frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)}$$

$$+ \omega H_2\left(z_2^*(\omega)\right) \frac{d^2\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d^2\left(\omega H_2\left(z_2^*(\omega)\right)\right)} \left[H_2\left(z_2^*(\omega)\right) + \omega \frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{dz_2^*(\omega)}{d\omega} \right]^2$$

$$+ \omega H_2\left(z_2^*(\omega)\right) \frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)} \left[2\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{dz_2^*(\omega)}{d\omega} + \omega \frac{d^2 H_2\left(z_2^*(\omega)\right)}{d^2 z_2^*(\omega)} \cdot \left(\frac{dz_2^*(\omega)}{d\omega}\right)^2 \right.$$

$$\left. + \omega \frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)} \cdot \frac{d^2 z_2^*(\omega)}{d\omega^2} \right]$$

Substituting $\omega = 1$ in the above equation, and then solve the equation for $\left.\frac{d^2 z_2^*(\omega)}{d\omega^2}\right|_{\omega=1}$ gives,

$$\left.\frac{d^2 z_2^*(\omega)}{d\omega^2}\right|_{\omega=1} = \frac{1}{1-\rho_2\left[1+\Gamma_1'(1)\right]} \left\{ \left[1+\Gamma_1'(1)\right]\left[2\rho_2 \cdot \frac{dz_2^*(\omega)}{d\omega} + H_2''(1) \cdot \left(\frac{dz_2^*(\omega)}{d\omega}\right)^2 \right] \right.$$

$$\left.\left. + [2\Gamma_1'(1) + \Gamma_1''(1)]\left[1 + \rho_2 \cdot \frac{dz_2^*(\omega)}{d\omega}\right]^2 \right\}\right|_{\omega=1} \tag{4.65}$$

Taking the third order derivative of both sides of (4.65) with respect to ω, we have,

$$
\begin{aligned}
&\frac{d^3 z_2^*(\omega)}{d\omega^3} \\
&=\left[3\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2+3\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}+\omega\frac{d^3H_2\left(z_2^*(\omega)\right)}{d^3z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^3\right. \\
&\quad\left.+3\omega\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^3z_2^*(\omega)}{d\omega^3}\right]\cdot\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right) \\
&+3\left[2\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}+\omega\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2\right. \\
&\quad\left.+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}\right]\cdot\left[H_2\left(z_2^*(\omega)\right)+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right]\frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)} \\
&+3\left[H_2\left(z_2^*(\omega)\right)+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right] \\
&\quad\cdot\left\{\frac{d^2\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d^2\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[H_2\left(z_2^*(\omega)\right)+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right]^2\right. \\
&\quad\quad+\frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[2\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}+\omega\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2\right. \\
&\quad\quad\quad\left.\left.+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}\right]\right\} \\
&+\omega H_2\left(z_2^*(\omega)\right)\left\{\frac{d^3\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d^3\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[H_2\left(z_2^*(\omega)\right)+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right]^3\right. \\
&\quad+3\frac{d^2\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d^2\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[H_2\left(z_2^*(\omega)\right)+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\right] \\
&\quad\cdot\left[2\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}+\omega\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}\right] \\
&\quad+\frac{d\Gamma_1\left(\omega H_2\left(z_2^*(\omega)\right)\right)}{d\left(\omega H_2\left(z_2^*(\omega)\right)\right)}\left[3\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2+3\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2}\right. \\
&\quad\quad+\omega\frac{d^3H_2\left(z_2^*(\omega)\right)}{d^3z_2^*(\omega)}\cdot\left(\frac{dz_2^*(\omega)}{d\omega}\right)^3+3\omega\frac{d^2H_2\left(z_2^*(\omega)\right)}{d^2z_2^*(\omega)}\cdot\frac{dz_2^*(\omega)}{d\omega}\cdot\frac{d^2z_2^*(\omega)}{d\omega^2} \\
&\quad\quad\left.\left.+\omega\frac{dH_2\left(z_2^*(\omega)\right)}{dz_2^*(\omega)}\cdot\frac{d^3z_2^*(\omega)}{d\omega^3}\right]\right\}
\end{aligned}
$$

Substituting $\omega = 1$ in the above equation, and then solving the equation for $\left.\frac{d^3 z_2^*(\omega)}{d\omega^3}\right|_{\omega=1}$, we obtain,

$$\left.\frac{d^3 z_2^*(\omega)}{d\omega^3}\right|_{\omega=1}$$

$$= \frac{1+\Gamma_1'(1)}{1-\rho_2[1+\Gamma_1'(1)]}\left[3H_2''(1)\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2 + 3\rho_2\frac{d^2 z_2^*(\omega)}{d\omega^2} + H_2'''(1)\left(\frac{dz_2^*(\omega)}{d\omega}\right)^3\right.$$

$$\left.\left. + 3H_2''(1)\frac{dz_2^*(\omega)}{d\omega}\cdot\frac{d^2 z_2^*(\omega)}{d\omega^2}\right]\right|_{\omega=1} \tag{4.66}$$

$$+\frac{3\Gamma_1''(1)+\Gamma_1'''(1)}{1-\rho_2[1+\Gamma_1'(1)]}\left.\left[1+\rho_2\frac{dz_2^*(\omega)}{d\omega}\right]^3\right|_{\omega=1}$$

$$+\frac{3[\Gamma_1''(1)+2\Gamma_1'(1)]}{1-\rho_2[1+\Gamma_1'(1)]}\left[1+\rho_2\frac{dz_2^*(\omega)}{d\omega}\right]\cdot\left.\left[2\rho_2\frac{dz_2^*(\omega)}{d\omega} + H_2''(1)\left(\frac{dz_2^*(\omega)}{d\omega}\right)^2 + \rho_2\frac{d^2 z_2^*(\omega)}{d\omega^2}\right]\right|_{\omega=1}$$

The expressions of (4.64, 4.65, 4.66) involve the derivatives of PGF of the busy period for the first multiplexer. This is because that the first multiplexer affects the third one through the second multiplexer.

Thus, all the unknowns have been determined in the expressions of the performance measures for the third multiplexer.

4.2.4 Numerical Results

In this section, some numerical results are presented regarding the performance of the third multiplexer. And simulation results are also presented to show the correctness of the analysis. Unless otherwise stated, it is assumed that each *On* source generates only one packet during a slot.

Figures 4.1 and 4.2 present both the analytical and simulation results for the mean and standard deviation of queue length against the total load of multiplexer-3. It may be seen that both mean and standard deviation increase with the load. Moreover, the analytical results match the simulation results perfectly, which gives further support that presented analysis is correct.

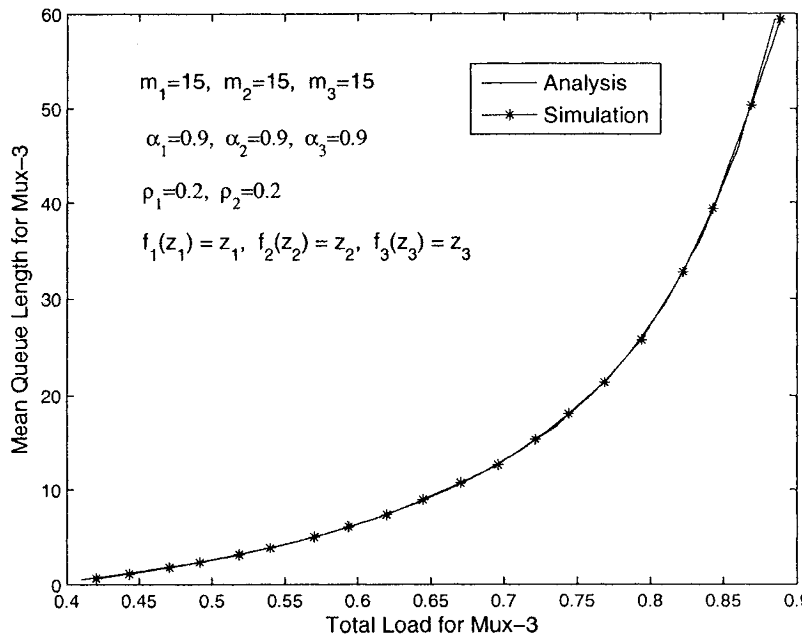


Figure 4.1 Mean queue length vs. its total load for multiplexer-3

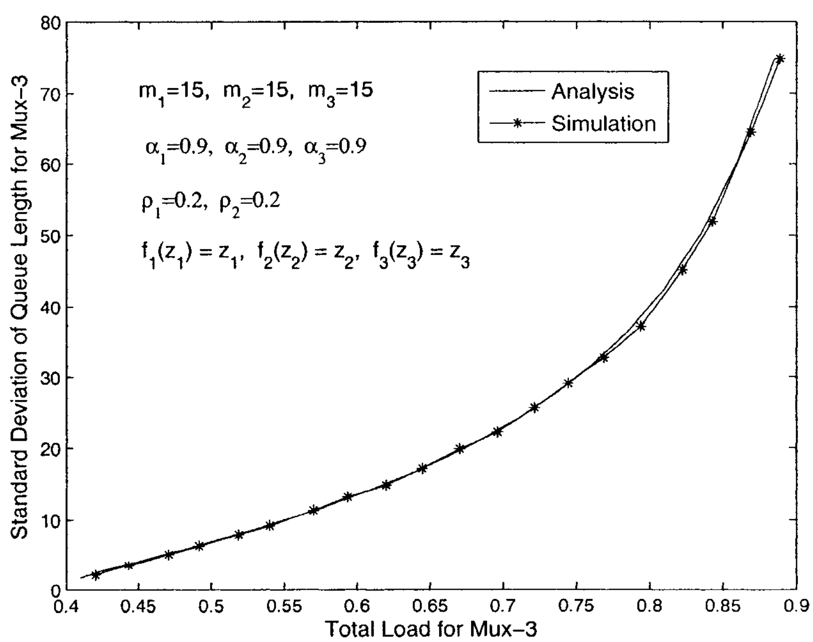


Figure 4.2 Standard deviation of queue length vs. the total load for multiplexer-3

In Figures 4.3 - 4.5, the mean queue length, mean packet delay and standard deviation of queue length are presented for multiplexer-3 respectively. The figures are plotted against the number of sources feeding multiplexer-3, while its total load is kept constant. As stated in section 3.4, when the traffic load generated by a source decreases, the burstiness of this source increases. Since the total load of multiplexer-3 is held constant, increasing the number of sources makes the traffic load generated by each source decrease; therefore, its burstiness increases. On the other hand, as a result of statistical multiplexing, increasing the number of sources smoothes out the superposed traffic. From Figures 4.5 - 4.7, it may be seen that all curves rise with the increase of number of sources, which means that burstiness overweighs traffic smoothing.

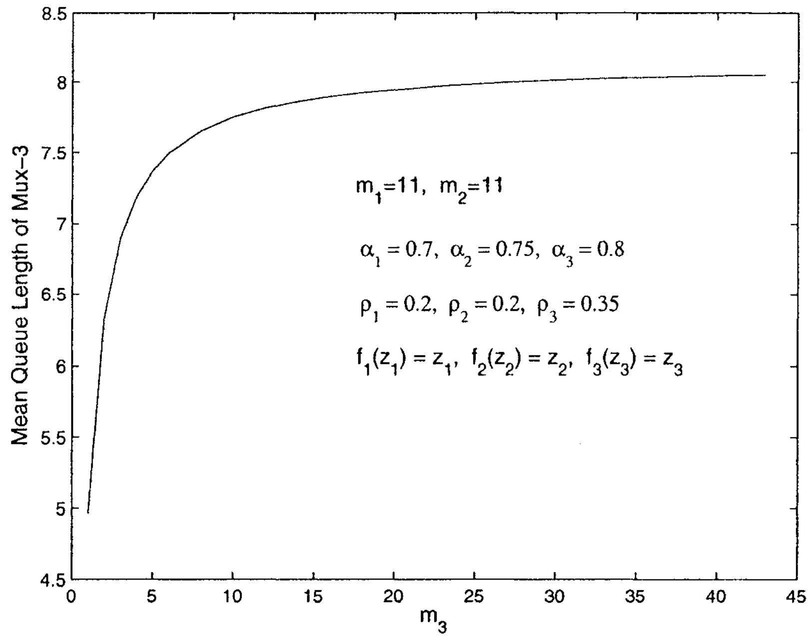


Figure 4.3 Mean queue length vs. the number of sources for multiplexer-3

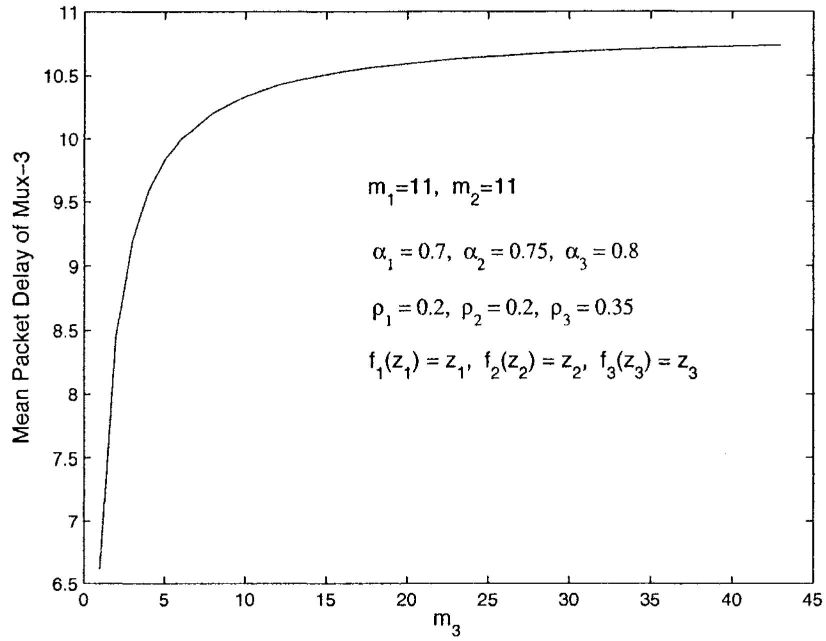


Figure 4.4 Mean packet delay vs. the number of sources for Multiplexer-3

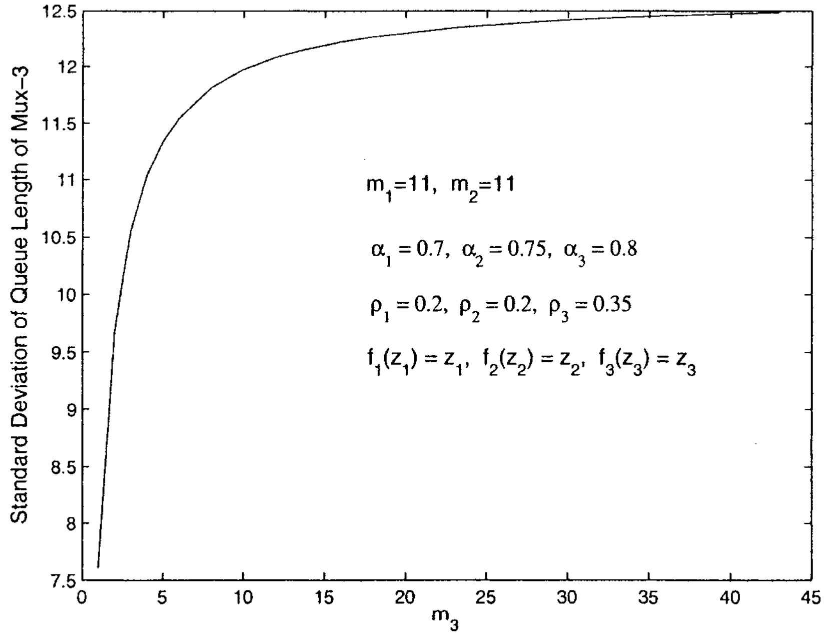


Figure 4.5 Standard deviation of queue length vs. the number of sources for multiplexer-3

Figures 4.6 and 4.7 present the mean and standard deviation of queue lengths versus their individual loads for multiplexer-*i*, $i = 1, \cdots 3$. For multiplexers 2 and 3, the input traffic from the preceding multiplexer is kept constant, thus increase in their traffic load is due to external traffic. The curves for multiplexers 2 and 3 are very close to each other and they are below the curve for multiplexer-1 except for heavy loading. This is due to the smoothing effect of statistical multiplexing; the traffic at the output of a multiplexer will be less bursty than its input traffic. Following the discussion in the previous paragraph, the sources feeding multiplexers 2 and 3 will be burstier than that of multiplexer 1 because each of them will generate less traffic. Further, under heavy traffic, the proportion of the input traffic of multiplexers 2, 3, which have not already gone through smoothing, will increase, which explains the reversed positions of the curves.

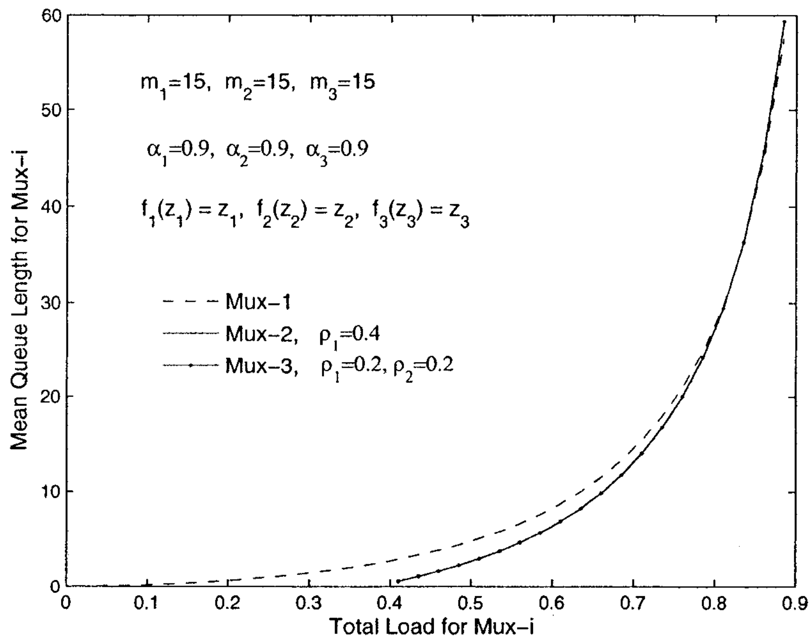


Figure 4.6 Mean queue length vs. its total load for multiplexer-*i*

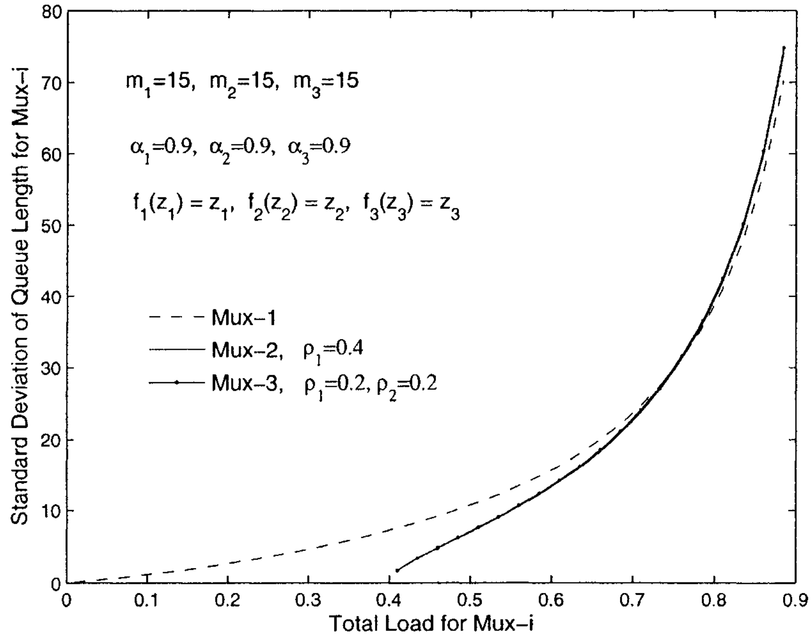


Figure 4.7 Standard deviation of queue length vs. its total load for multiplexer-*i*

Figure 4.8 presents the mean queue length of multiplexer-3 as the number of sources feeding one of the multiplexers is varied while keeping the other two constant at a value of 11. The traffic load of each multiplexer is also held constant. It may be seen that three curves cross each other at $m_i = 11$. From the discussion of the previous paragraph, higher number of sources mean that traffic is burstier when the load is kept constant. In addition, as the traffic goes through more multiplexers then it gets smoother. When $m_i < 11$, it may be seen that the solid line is higher than the line with asterisks, and which is higher than the dashed line. This is because higher proportion of the traffic feeding multiplexer-3 has been smoothed out than multiplexers-2 and 1; and higher proportion of the traffic feeding multiplexer-2 has been smoothed out than multiplexers-1. The converse of this happens when $m_i > 11$.

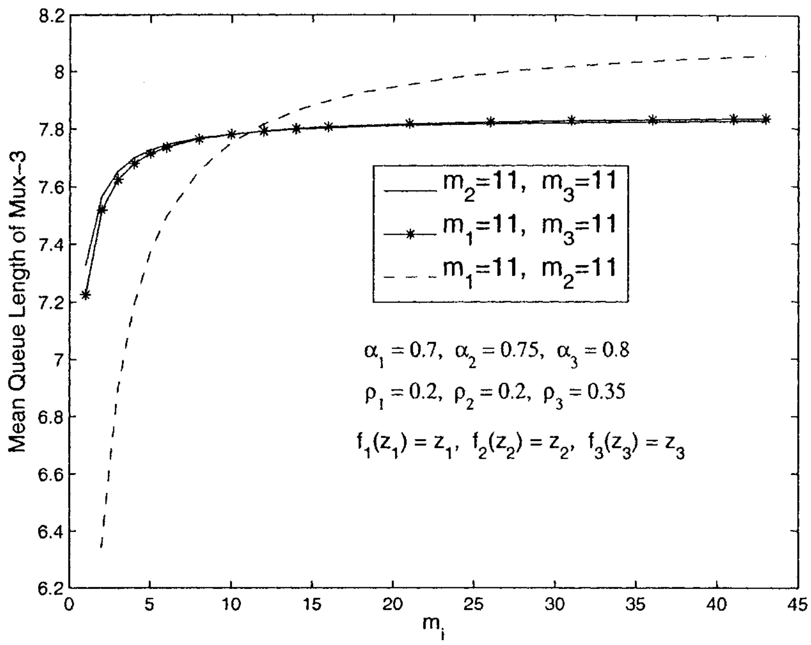


Figure 4.8 Mean queue length of multiplexer-3 vs. the number of sources of multiplexer-*i*

In Figures 4.9 - 4.11, the mean queue length, mean packet delay, and standard deviation of the queue length are presented for multiplexer-3 versus its total load respectively for two different functions of $f_3(z_3)$. The results have been presented assuming that a type-3 *On* source generates two packets constantly during a slot, with PGF $f_3(z_3) = z_3^2$, or generates geometrically distributed number of packets during a slot with mean equal to two, with PGF $f_3(z_3) = z_3/(2 - z_3)$. It may be seen that the results for deterministic packet generation are slightly lower than those for geometrical packet generation though both have the same mean. Again, this is due to that geometric sources are burstier than deterministic sources.

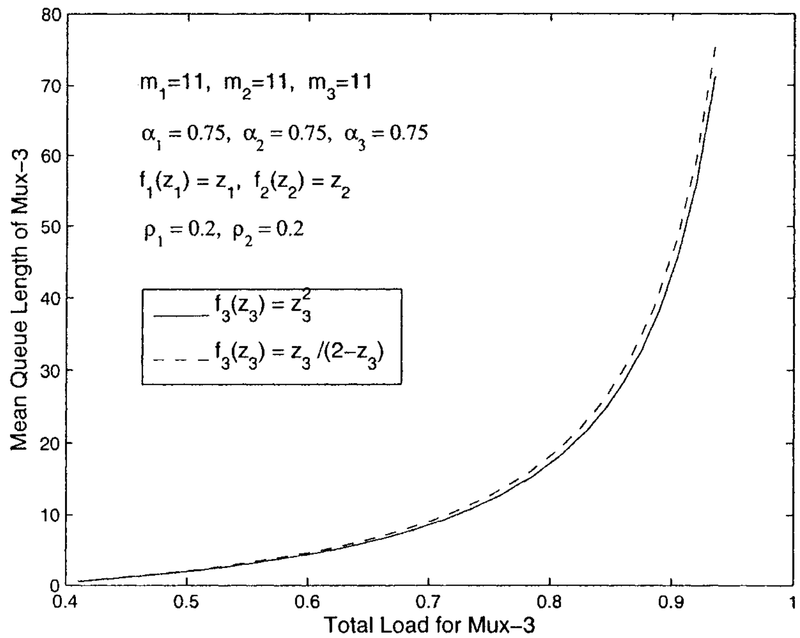


Figure 4.9 Mean queue length vs. its total load for multiplexer-3

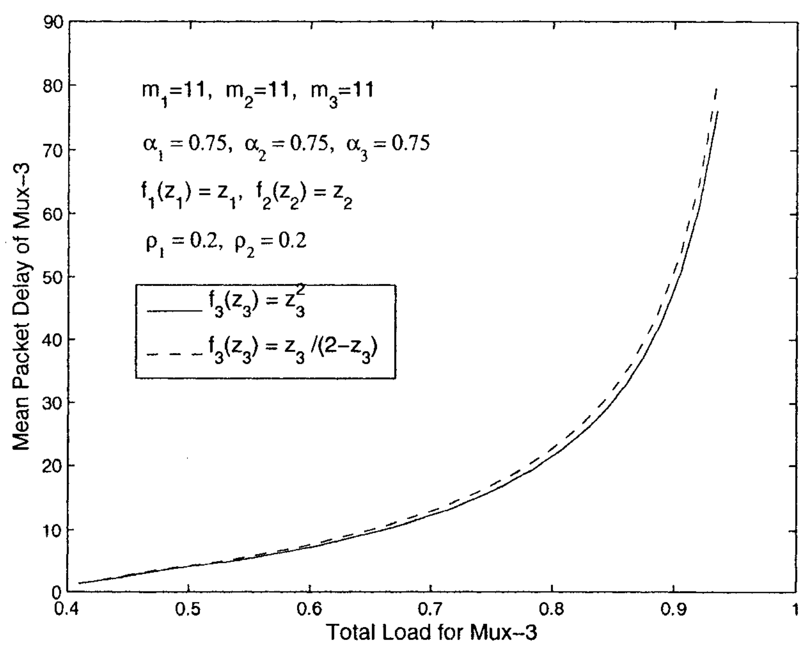


Figure 4.10 Mean packet delay vs. its total load for multiplexer-3

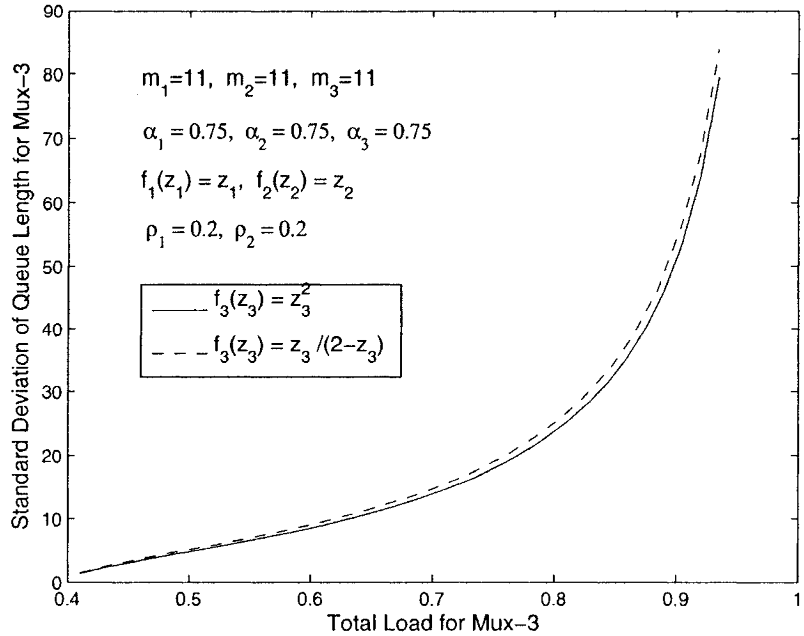


Figure 4.11 Standard deviation of queue length vs. its total load for multiplexer-3

Figures 4.12 and 4.13 present the mean and standard deviation of queue length respectively versus the total load for multiplexer-3 while ρ_1 or ρ_3 is varying and the other two type of traffic loads ρ_i are kept constant. It may be seen that both curves cross each other when the total load is 0.6. Again, this is explained through the smoothing effect of multiplexers. When total load is less than 0.6, the curves corresponding $\rho_1 > \rho_3$ achieves lower values because higher proportion of the traffic feeding multiplexer-3 is smoothed out. The converse of this happens when $\rho_1 < \rho_3$.

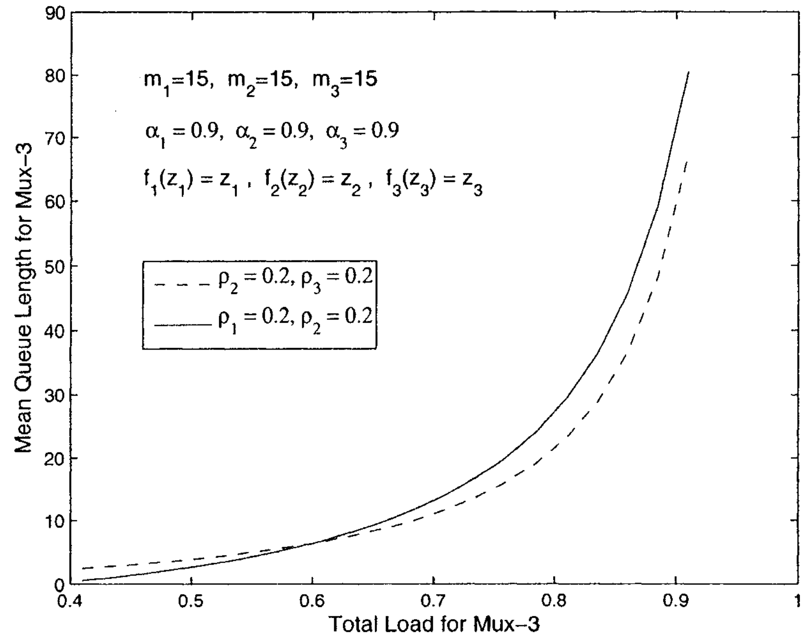


Figure 4.12 Mean queue length vs. its total load for multiplexer-3

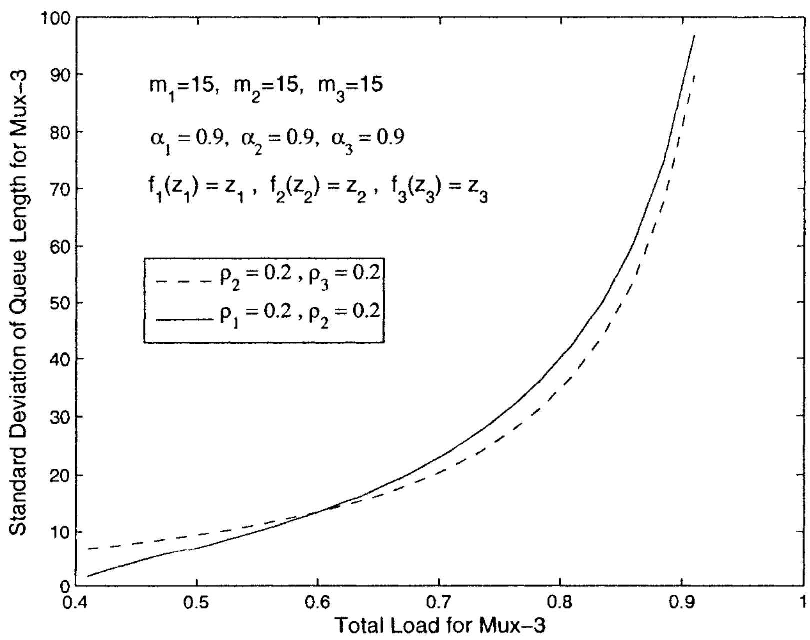


Figure 4.13 Standard deviation of queue length vs. its total load for multiplexer-3

Chapter 5

Performance Analysis of General Tandem Networks of Multiplexers

In this chapter, the performance analysis is presented for tandem networks with arbitrary number of multiplexers. The analysis will follow the same steps as in Chapter 4. The new technique will be used to determine the unknown boundary function and then the joint PGF, as well as the corresponding performance measures for each multiplexer in the tandem network. Afterwards, it will be shown that the solution satisfies the equilibrium form of the functional equation describing the system. This will give the proof that the new solution technique is correct.

5.1 Performance Analysis of General Tandem Networks

In Chapter 4, two special cases of tandem networks have be studied: two and three multiplexers in tandem. Now the solution will be extended to a general case: a tandem network with n (n>1) multiplexers. The unknown boundary function, the PGF of the busy period, and the steady-state PGF will be determined for the n'th multiplexer. The following analysis will follow the same steps as in Chapter 4.

First, let us repeat the joint PGF of the n'th multiplexer from (2.35),

$$\begin{aligned}
&Q_k(1_{n-1},1_{n-1},z_n,y_n) \\
&= B_n(k) + (z_n - 1)\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(1_{n-1},1_{n-1},0,0) \\
&+ \frac{1-z_n}{z_n}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}\left(0_{n-1},0_{n-1},z_n,\phi_n(j)\right) \\
&- \frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(0_{n-1},0_{n-1},0_n,0_n) \qquad k \geq 1, n \geq 2
\end{aligned} \tag{5.1}$$

Following the interpretation given in sections 4.1.1 and 4.2.1, the unknown boundary function in (5.1) may be written as follows,

$$Q_k(0_{n-1},0_{n-1},z_n,y_n)$$
$$=\sum_{r=1}^{k}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k)B_n(k) \tag{5.2}$$
$$+\frac{z_n-1}{z_n}\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_n^{r-2}}\varphi_r^{(n-1)}(k-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n),\quad k\geq 1$$

As before, it is noted that the first term on the RHS of (5.2) corresponds to the event that the last time the n'th multiplexer is empty is at the initial slot; and that the second term corresponds to the event that the last time the n'th multiplexer is empty is at the end of h'th slot. In the first term, $\varphi_r^{(n-1)}(k)$ corresponds to the probability that (n-1)'st multiplexer will have r complete busy periods during the k slots. And the PGFs of the packets received and transmitted by the n'th multiplexer during the k slots are given by $B_n(k)z_n^{k-r}$ and z_n^{k-1} respectively. For the second term, given that the n'th multiplexer is empty at the end of h'th slot, the (n-1)'st multiplexer must be empty in the previous slot. Thus in the above, $\varphi_r^{(n-1)}(k-h+1)$ corresponds to the probability that (n-1)'st multiplexer will have r complete busy periods during the (k-h+1) slots. Since, busy periods are separated by an idle slot, (n-1)'st multiplexer will not generate packets at its output during r of these slots. The PGFs of the packets received and transmitted by the n'th multiplexer during the (k-h) slots are given by $B_n(k-h)z_n^{k-h-r+1}$ and z_n^{k-h-1} respectively.

Substituting (5.2) in (5.1), we have,

$$Q_k(1_{n-1},1_{n-1},z_n,y_n)$$
$$=B_n(k)+(z_n-1)\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(1_{n-1},1_{n-1},0_n,0_n)$$
$$-\frac{z_n-1}{z_n}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-j)B_n(k) \tag{5.3}$$
$$-\frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n)$$
$$-\frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},0_{n-1},0_n,0_n)\quad,\qquad k\geq 1$$

Let us define the following transforms,

$$Q(1_{n-1},1_{n-1},z_n,y_n,\omega)=\sum_{k=0}^{\infty}Q_k(1_{n-1},1_{n-1},z_n,y_n)\omega^k$$

$$Q(1_{n-1},1_{n-1},0_n,0_n,\omega)=\sum_{k=0}^{\infty}Q_k(1_{n-1},1_{n-1},0_n,0_n)\omega^k \tag{5.4}$$

$$Q(0_{n-1},0_{n-1},0_n,0_n,\omega)=\sum_{k=0}^{\infty}Q_k(0_{n-1},0_{n-1},0_n,0_n)\omega^k$$

Then we have,

$$\begin{aligned}
&Q(1_{n-1},1_{n-1},z_n,y_n,\omega)\\
&=1+\sum_{k=1}^{\infty}B_n(k)\omega^k+(z_n-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(1_{n-1},1_{n-1},0_n,0_n)\omega^k\\
&-(z_n-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j)B_n(k)\omega^k\\
&-(z_n-1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^k\\
&-\frac{(z_n-1)^2}{z_n}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},0_{n-1},0_n,0_n)\omega^k
\end{aligned} \tag{5.5}$$

From Appendix D the above equation may be written as,

$$\begin{aligned}
&Q(1_{n-1},1_{n-1},z_n,y_n,\omega)\\
&=1+\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}\\
&+(z_n-1)[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}\\
&-(z_n-1)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]}\\
&-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]}
\end{aligned} \tag{5.6}$$

In order to determine the steady-state PGF of the n'th multiplexer, let us apply the final-value theorem to (5.6),

$$Q(1_{n-1},1_{n-1},z_n,y_n)=Q_\infty(1_{n-1},1_{n-1},z_n,y_n)=\lim_{\omega\to 1}(1-\omega)Q(1_{n-1},1_{n-1},z_n,y_n,\omega)$$

Noting that,

$$\lim_{\omega \to 1}(1-\omega)Q(1_{n-1},1_{n-1},0_n,0_n,\omega) = Q(1_{n-1},1_{n-1},0_n,0_n) = 1-\sum_{i=1}^{n}\rho_i$$

We have,

$$\begin{aligned}
&Q(1_{n-1},1_{n-1},z_n,y_n) \\
&= (z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}} \\
&\quad -(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i})[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})]}
\end{aligned} \tag{5.7}$$

The above is the joint PGF for the n'th multiplexer. Substituting $y_n = 1$ in the above, we obtain the PGF of the queue length of the n'th multiplexer, which is given by

$$\begin{aligned}
&P_n(z_n) = Q(1_{n-1},1_{n-1},z_n,1_{n-1}) \\
&= (z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}} \\
&\quad -(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i})[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})]}
\end{aligned} \tag{5.8}$$

In the above expressions, (5.7, 5.8), we may see that the PGF of the n'th multiplexer contains the PGF of the busy period of the (n-1)'th multiplexer, $\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})$. From (B.8) of Appendix B, we have,

$$\Gamma_{n-1}(\omega) = \frac{1}{\omega}\left(1-\frac{1}{Q(1_{n-2},1_{n-2},0_{n-1},0_{n-1},\omega)}\right) \tag{5.9}$$

It may be seen that $\Gamma_{n-1}(\omega)$ is totally determined by $Q(1_{n-2},1_{n-2},0_{n-1},0_{n-1},\omega)$. Since the determination of $Q(1_{n-2},1_{n-2},0_{n-1},0_{n-1},\omega)$ is equivalent to that of $Q(1_{n-1},1_{n-1},0_n,0_n,\omega)$, next we will determine $Q(1_{n-1},1_{n-1},0_n,0_n,\omega)$.

Substituting $y_n = 1$ in (5.6), we have,

$$
\begin{aligned}
&Q(1_{n-1},1_{n-1},z_n,1_n,\omega) \\
&=1+\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \\
&+(z_n-1)[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \\
&-(z_n-1)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}\omega\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]} \\
&-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(\widetilde{C}_{1n}\lambda_{1n})^i(\widetilde{C}_{2n}\lambda_{2n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]}
\end{aligned}
\tag{5.10}
$$

where, $\widetilde{C}_{1n}=C_{1n}\big|_{y_n=1}$ and $\widetilde{C}_{2n}=C_{2n}\big|_{y_n=1}$

Following the same process of the development in section 4.1.3 and 4.2.2, the expression $Q(1_{n-1},1_{n-1},0_n,0_n,\omega)$ may be determined by invoking the analytical property of the function $Q(1_{n-1},1_{n-1},z_n,1_n,\omega)$ inside the poly disk $(|z_n|<1;|\omega|<1)$, and through the application of Rouche's theorem. Therefore, the details will not be presented here and just the results are given,

$$
Q(1_{n-1},1_{n-1},0_n,0_n,\omega)=1+\frac{\lambda_{2n}^{m_n}\omega\big|_{z_n=z_n^*(\omega)}}{1-z_n^*(\omega)}
\tag{5.11}
$$

where $z_n^*(\omega)$ is the root of the equation

$$
z_n-\lambda_{2n}^{m_n}\omega\Gamma_{n-1}(\lambda_{2n}^{m_n}\omega)=0
\tag{5.12}
$$

From (5.11), we have,

$$
\Gamma_n(\omega)=\frac{1}{\omega}\left(1-\frac{1}{Q(1_{n-1},1_{n-1},0_n,0_n,\omega)}\right)=\frac{\lambda_{2n}^{m_n}\big|_{z_n=z_n^*(\omega)}}{1-z_n^*(\omega)+\lambda_{2n}^{m_n}\omega\big|_{z_n=z_n^*(\omega)}}
$$

Therefore,

$$
\Gamma_{n-1}(\omega)=\frac{\lambda_{2(n-1)}^{m_{n-1}}\big|_{z_{n-1}=z_{n-1}^*(\omega)}}{1-z_{n-1}^*(\omega)+\lambda_{2(n-1)}^{m_n}\omega\big|_{z_{n-1}=z_{n-1}^*(\omega)}}
\tag{5.13}
$$

5.2 Performance Measures of the *n*'th Multiplexer

In this section, the mean and variance of queue length will be determined, as well as mean packet for the *n*'th multiplexer. It may be seen that the PGF of the queue length of the *n*'th multiplexer in (5.8) has the same form as the PGF of the queue length of the second and third multiplexers in (3.31) and (4.31). Thus, the derivation of the performance measures for the *n*'th multiplexer will follow the same steps as in sections 3.3 and 4.2.3. First, let us write (5.8) in a more convenient form:

$$\begin{aligned} P_n(z_n) &= (z_n - 1)\left(1 - \sum_{i=1}^{n} \rho_i\right)\left[E_n(z_n) + \frac{G_n(z_n)}{1 - H_n(z_n)}\right] \\ &\quad - (z_3 - 1)^2\left(1 - \sum_{i=1}^{n} \rho_i\right)\left[F_n(z_n) + \frac{G_n(z_n)\Gamma_{n-1}(H_n(z_n))}{[1 - H_n(z_n)][z_n - \Theta_n(z_n)]}\right] \end{aligned} \tag{5.14}$$

where

$$E_n(z_n) = \sum_{i=1}^{m_n} \binom{m_n}{i} \frac{(\tilde{C}_{1n}\lambda_{1n})^i (\tilde{C}_{2n}\lambda_{2n})^{m_n - i}}{1 - \lambda_{1n}^i \lambda_{2n}^{m_n - i}} \tag{5.15}$$

$$F_n(z_n) = \sum_{i=1}^{m_n} \binom{m_n}{i} \frac{(\tilde{C}_{1n}\lambda_{1n})^i (\tilde{C}_{2n}\lambda_{2n})^{m_n - i} \Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n - i})}{(1 - \lambda_{1n}^i \lambda_{2n}^{m_n - i})[z_n - \lambda_{1n}^i \lambda_{2n}^{m_n - i} \Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n - i})]} \tag{5.16}$$

$$H_n(z_n) = \lambda_{2n}^{m_n}, \quad G_n(z_n) = (\tilde{C}_{2n}\lambda_{2n})^{m_n} \tag{5.17}$$

$$\Theta_n(z_n) = H_n(z_n)\Gamma_{n-1}(H_n(z_n)) = \lambda_{2n}^{m_n}\Gamma_{n-1}(\lambda_{2n}^{m_n}) \tag{5.18}$$

And we have, $E_n(1) = 0$, $F_n(1) = 0$, $H_n(1) = 1$, $G_n(1) = 1$, $\Theta_n(1) = 1$ (5.19)

- Derivation of the mean queue length and mean packet delay

Let us apply the same technique as in the previous chapter. First multiplying both sides of (5.14) with the common denominator, and then taking the third derivative of both sides with respect to z_n, afterwards substituting $z_n = 1$ gives,

$$\begin{aligned} & -3H_n''(1)[1 - \Theta_n'(1)] + 3H_n'(1)\Theta_n''(1) - 6H_n'(1)[1 - \Theta_n'(1)]P_n'(1) \\ &= -6(1 - \sum_{i=1}^{n} \rho_i)[\Gamma_{n-1}'(1)H_n'(1) + G_n'(1)] + 3(1 - \sum_{i=1}^{n} \rho_i)[-\Theta_n''(1) + 2(1 - \Theta_n'(1))G_n'(1)] \end{aligned}$$

Solving the above equation for $P_n'(1)$ gives the mean queue length of the *n*'th multiplexer, which is given by

$$\overline{N}_n = P_n'(1) = \frac{\Theta_n''(1)}{2[1-\Theta_n'(1)]} - \frac{H_n''(1)}{2H_n'(1)} + \frac{1-\sum_{i=1}^{n}\rho_i}{H_n'(1)[1-\Theta_n'(1)]}[\Gamma_{n-1}'(1)H_n'(1)+G_n'(1)]$$
$$+\frac{(1-\sum_{i=1}^{n}\rho_i)\Theta_n''(1)}{2H_n'(1)[1-\Theta_n'(1)]} - \frac{1-\sum_{i=1}^{n}\rho_i}{H_n'(1)}G_n'(1) \tag{5.20}$$

where $$\Theta_n'(1) = \left.\frac{d\Theta_n(z_n)}{dz_n}\right|_{z_n=1}, \quad \Theta_n''(1) = \left.\frac{d^2\Theta_n(z_n)}{dz_n{}^2}\right|_{z_n=1} \tag{5.21}$$

$$H_n'(1) = \left.\frac{dH_n(z_n)}{dz_n}\right|_{z_n=1}, \quad H_n''(1) = \left.\frac{d^2H_n(z_n)}{dz_n{}^2}\right|_{z_n=1}, \tag{5.22}$$

$$G_n'(1) = \left.\frac{dG_n(z_n)}{dz_n}\right|_{z_n=1}, \quad \Gamma_{n-1}'(1) = \left.\frac{d\Gamma_{n-1}(\omega)}{d\omega}\right|_{\omega=1}, \tag{5.23}$$

From the Little's result, the mean delay that a packet experiences at the n'th multiplexer is given by,

$$\overline{D}_n = \frac{\overline{N}_n}{\sum_{j=1}^{n}\rho_j} \tag{5.24}$$

- Derivation of the variance of the queue length

The variance of the queue length for the n'th multiplexer requires the second order derivative of $P_n(z_n)$. First, both sides of (5.14) are multiplied with the common denominator, and then the fourth derivative is taken with respect to z_n. After, $z_n = 1$ is substituted, and finally the resulting equation for $P_n''(1)$ is solved,

$$P_n''(1)$$
$$= \frac{1}{6H_n'(1)[1-\Theta_n'(1)]}\left\{2(1-\sum_j^n \rho_j)\Theta_n'''(1)+3H_n''(1)\Theta_n''(1)\right.$$
$$\left.+6(1-\sum_j^n \rho_j)[\Theta_n''(1)G_n'(1)+\Theta_n'(1)G_n''(1)+\Gamma_{n-1}'(1)H_n''(1)]\right\} \quad (5.25)$$
$$+\frac{1}{3H_n'(1)}\left\{-H_n'''(1)+6(1-\sum_j^n \rho_j)H_n'(1)E_n'(1)-3H_n''(1)P_n'(1)\right\}$$
$$+\frac{1}{3[1-\Theta_n'(1)]}\left\{\Theta_n'''(1)+3\Theta_n''(1)P_n'(1)+3(1-\sum_j^n \rho_j)[2\Gamma_{n-1}'(1)G_n'(1)+\Gamma_{n-1}''(1)H_n'(1)]\right\}$$

where $$\Theta_n'''(1)=\left.\frac{d^3\Theta_n(z_n)}{dz_n^3}\right|_{z_n=1}, \quad G_n''(1)=\left.\frac{d^2G_n(z_n)}{dz_n^2}\right|_{z_n=1}, \quad (5.26)$$

$$E_n'(1)=\left.\frac{dE_n(z_n)}{dz_n}\right|_{z_n=1}, \quad H_n'''(1)=\left.\frac{d^3H_n(z_n)}{dz_n^3}\right|_{z_n=1}, \quad \Gamma_{n-1}''(1)=\left.\frac{d^2\Gamma_{n-1}(\omega)}{d\omega^2}\right|_{\omega=1}, \quad (5.27)$$

The variance of the queue length for the n'th multiplexer, $\overline{V_n}$, may be expressed in terms of the derivatives of its PGF as

$$\overline{V_n} = P_n''(1)+P_n'(1)-(P_n'(1))^2 \quad (5.28)$$

In the next, the derivatives of $G_n(z_n)$, $E_n(z_n)$, $H_n(z_n)$, $\Gamma_{n-1}(\omega)$, and $\Theta_n(z_n)$ will be determined. Since the derivatives of $G_n(z_n)$, $E_n(z_n)$, $H_n(z_n)$ may be determined in an straightforward manner from their definitions in (5.15, 5.17), the details will not be given here. It only be noted that,

$$H_n'(1)=\frac{m_n(1-\beta_n)f_n'(1)}{2-\alpha_n-\beta_n}=\rho_n \quad (5.29)$$

Next, the derivatives of $\Gamma_{n-1}(\omega)$ will be expressed in terms of the derivatives of $z_{n-1}^*(\omega)$. $\Gamma_{n-1}(\omega)$ is the PGF of the busy period of the (n-1)'th multiplexer, which has been determined in (5.13). Taking the first three order derivatives of both sides of (5.13) with respect to ω and substituting $\omega=1$ gives,

$$\Gamma_{n-1}'(1)=\left.\frac{dz_{n-1}^*(\omega)}{d\omega}\right|_{\omega=1}-1 \quad (5.30)$$

$$\Gamma''_{n-1}(1)=\left.\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right|_{\omega=1}+2(\Gamma'_{n-1}(1))^2-2\rho_{n-1}\left(\left.\frac{dz^*_{n-1}(\omega)}{d\omega}\right|_{\omega=1}\right)^2 \tag{5.31}$$

$$\begin{aligned}\Gamma'''_{n-1}(1)=&\left.\frac{d^3 z^*_{n-1}(\omega)}{d\omega^3}\right|_{\omega=1}+3\left(\Gamma''_{n-1}(1)+\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right)\left[\Gamma'_{n-1}(1)-\rho_{n-1}\cdot\frac{dz^*_{n-1}(\omega)}{d\omega}\right]\Bigg|_{\omega=1}\\&-3H''_{n-1}(1)\cdot\left.\left(\frac{dz^*_{n-1}(\omega)}{d\omega}\right)^3\right|_{\omega=1}-6\rho_{n-1}\Gamma'_{n-1}(1)\left.\frac{dz^*_{n-1}(\omega)}{d\omega}\right|_{\omega=1}\end{aligned} \tag{5.32}$$

From (5.18), we have $\Theta_n(z_n)=H_n(z_n)\Gamma_{n-1}(H_n(z_n))$. Taking the first three order derivatives of $\Theta_n(z_n)$ with respect to z_n, and then substituting $z_n=1$, we have

$$\Theta'_n(1)=\rho_n(1+\Gamma'_{n-1}(1)) \tag{5.33}$$

$$\Theta''_n(1)=H''_n(1)[1+\Gamma'_{n-1}(1)]+\rho_n^2[2\Gamma'_{n-1}(1)+\Gamma''_{n-1}(1)] \tag{5.34}$$

$$\Theta'''_n(1)=H'''_n(1)[1+\Gamma'_{n-1}(1)]+3\rho_n H''_{n-1}(1)[\Gamma''_{n-1}(1)+2\Gamma'_{n-1}(1)]+\rho_n^3[\Gamma'''_n(1)+3\Gamma''_{n-1}(1)] \tag{5.35}$$

Finally, the first three order derivatives of $z^*_{n-1}(\omega)$ are determined. From (5.12) we have,

$$z^*_{n-1}(\omega)=\omega H_{n-1}(z_{n-1})\Gamma_{n-2}(\omega H_{n-1}(z_{n-1}))\big|_{z_{n-1}=z^*_{n-1}(\omega)} \tag{5.36}$$

Because the unique root of the equation $z_{n-1}-\omega H_{n-1}(z_{n-1})\Gamma_{n-2}(\omega H_{n-1}(z_{n-1}))=0$ at $\omega=1$ is $z_{n-1}=1$, we have,

$$z^*_{n-1}(\omega)\big|_{\omega=1}=1$$

Taking the first order derivative of both sides of (5.36) and then substituting $\omega=1$ and noting $z^*_{n-1}(\omega)\big|_{\omega=1}=1$, $H_{n-1}(1)=1$, $\Gamma_{n-2}(1)=1$, we have

$$\left.\frac{dz^*_{n-1}(\omega)}{d\omega}\right|_{\omega=1}=[1+\Gamma'_{n-2}(1)]\left[1+\rho_{n-1}\left.\frac{dz^*_{n-1}(\omega)}{d\omega}\right|_{\omega=1}\right]$$

Solving the above equation, we obtain,

$$\left.\frac{dz^*_{n-1}(\omega)}{d\omega}\right|_{\omega=1}=\frac{1+\Gamma'_{n-2}(1)}{1-\rho_{n-1}[1+\Gamma'_{n-2}(1)]} \tag{5.37}$$

Taking the second order derivative of both sides of (5.36) and then substituting $\omega = 1$ and noting $z^*_{n-1}(\omega)\big|_{\omega=1} = 1, H_{n-1}(1) = 1,\ \Gamma_{n-2}(1) = 1$, we obtain a equation. Solving this equation for $\left.\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right|_{\omega=1}$ gives,

$$\left.\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right|_{\omega=1} = \frac{1}{1-\rho_{n-1}\left[1+\Gamma'_{n-2}(1)\right]}\left\{\left[1+\Gamma'_{n-2}(1)\right]\left[2\rho_{n-1}\cdot\frac{dz^*_{n-1}(\omega)}{d\omega} + H''_{n-1}(1)\cdot\left(\frac{dz^*_{n-1}(\omega)}{d\omega}\right)^2\right] + \left[2\Gamma'_{n-2}(1)+\Gamma''_{n-2}(1)\right]\left[1+\rho_{n-1}\cdot\frac{dz^*_{n-1}(\omega)}{d\omega}\right]^2\right\}\Bigg|_{\omega=1} \tag{5.38}$$

Taking the third order derivative of both sides of (5.36) and then substituting $\omega = 1$ and noting $z^*_{n-1}(\omega)\big|_{\omega=1} = 1, H_{n-1}(1) = 1,\ \Gamma_{n-2}(1) = 1$, we obtain a equation. Solving the equation for $\left.\frac{d^3 z^*_{n-1}(\omega)}{d\omega^3}\right|_{\omega=1}$ gives,

$$\begin{aligned}
\left.\frac{d^3 z^*_{n-1}(\omega)}{d\omega^3}\right|_{\omega=1}
&= \frac{1+\Gamma'_{n-2}(1)}{1-\rho_{n-1}[1+\Gamma'_{n-2}(1)]}\left[3H''_{n-1}(1)\left(\frac{dz^*_{n-1}(\omega)}{d\omega}\right)^2 + 3\rho_{n-1}\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2} + H'''_{n-1}(1)\left(\frac{dz^*_{n-1}(\omega)}{d\omega}\right)^3 \right.\\
&\qquad \left. + 3H''_{n-1}(1)\frac{dz^*_{n-1}(\omega)}{d\omega}\cdot\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right]\Bigg|_{\omega=1} \\
&+ \frac{3\Gamma''_{n-2}(1)+\Gamma'''_{n-2}(1)}{1-\rho_{n-1}[1+\Gamma'_{n-2}(1)]}\left[1+\rho_{n-1}\frac{dz^*_{n-1}(\omega)}{d\omega}\right]^3\Bigg|_{\omega=1} \\
&+ \frac{3[\Gamma''_{n-2}(1)+2\Gamma'_{n-2}(1)]}{1-\rho_{n-1}[1+\Gamma'_{n-2}(1)]}\left[1+\rho_{n-1}\frac{dz^*_{n-1}(\omega)}{d\omega}\right] \\
&\quad \cdot\left[2\rho_{n-1}\frac{dz^*_{n-1}(\omega)}{d\omega} + H''_{n-1}(1)\left(\frac{dz^*_{n-1}(\omega)}{d\omega}\right)^2 + \rho_{n-1}\frac{d^2 z^*_{n-1}(\omega)}{d\omega^2}\right]\Bigg|_{\omega=1}
\end{aligned} \tag{5.39}$$

The expressions of (5.37, 5.38, 5.39) involve the derivatives of $\Gamma_{n-2}(\omega)$, which is the PGF of the busy period of the (n-2)'nd multiplexer. This is due to that the (n-2)'th multiplexer affects the n'th one through the (n-1)'th multiplexer. Since the performance of the n'th multiplexer is affected by all the preceding multiplexers, the PGF of the busy period of multiplexer-1, 2, ..., (n-1) is needed in order to determine the performance of the n'th multiplexer.

5.3 Proof of the Solution

In the performance analysis with the alternative solution technique, the unknown boundary function have been determined following an interpretation of the single multiplexer result. In this section, it will be proven that the solution obtained through this technique is correct. This is done by showing that the steady-state joint PGF given in (5.8) satisfies (2.36), the equilibrium form of the functional equation at the steady state, which is repeated below,

$$\begin{aligned} & Q(1_{n-1},1_{n-1},z_n,y_n) \\ & = B_n(1)\Big\{ Q(1_{n-1},1_{n-1},z_n,Y_n)+(z_n-1)Q(1_{n-1},1_{n-1},0_n,0_n) \\ & \quad +\frac{1-z_n}{z_n}Q(0_{n-1},0_{n-1},z_n,Y_n)-\frac{(z_n-1)^2}{z_n}Q(0_{n-1},0_{n-1},0_n,0_n)\Big\} \quad , \quad k \geq 0 \end{aligned} \tag{5.40}$$

First, the steady-state boundary function $Q(0_{n-1},0_{n-1},z_n,y_n)$ in the above is determined. Let us define the transform,

$$Q(0_{n-1},0_{n-1},z_n,y_n,\omega)=\sum_{k=0}^{\infty}Q_k(0_{n-1},0_{n-1},z_n,y_n)\omega^k$$

Then from (5.2) we have

$$\begin{aligned} & Q(0_{n-1},0_{n-1},z_n,y_n,\omega) \\ & = 1+\sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k)B_n(k)\omega^k \\ & \quad +(z_n-1)\sum_{k=2}^{\infty}\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^k \end{aligned}$$

Letting A_0, A_1 denote the separate parts of $Q(0_{n-1},0_{n-1},z_n,y_n,\omega)$:

$$A_0 = 1 + \sum_{k=1}^{\infty}\sum_{r=1}^{k}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k)B_n(k)\omega^k \tag{5.41}$$

$$A_1 = (z_n - 1)\sum_{k=2}^{\infty}\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^k \tag{5.42}$$

Then $Q(0_{n-1},0_{n-1},z_n,y_n,\omega)$ may be expressed as,

$Q(0_{n-1},0_{n-1},z_n,y_n,\omega) = A_0 + A_1$

In the next, we will determine both A_0, A_1.

- Determining A_0

Substituting from (A.1) in (5.41), we have

$$A_0 = 1 + z_n\sum_{k=1}^{\infty}\varphi^{(n-1)}(k)B_n(k)\omega^k$$

Substituting for $B_n(k)$ from (2.26), we have

$$A_0 = 1 + z_n\sum_{i=0}^{m_n}\sum_{k=1}^{\infty}\binom{m_n}{i}\varphi^{(n-1)}(k)(C_{1n}\lambda_{1n}^k)^i(C_{2n}\lambda_{2n}^k)^{m_n-i}\omega^k$$

From (A.3),

$$A_0 = 1 + z_n\sum_{i=0}^{m_n}\binom{m_n}{i}C_{1n}^iC_{2n}^{m_n-i}\Phi^{(n-1)}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)$$

From (A.5),

$$A_0 = 1 + z_n\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{z_n - \lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)} \tag{5.43}$$

- Determining A_1

Exchanging the order of summations in A_1 defined in (5.42), we have,

$$A_1 = (z_n - 1)z_n\sum_{h=1}^{\infty}\sum_{k=h+1}^{\infty}\sum_{r=1}^{k-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-h+1)B_n(k-h)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^k$$

Letting $\ell = k - h$, we have,

$$A_1 = (z_n - 1)z_n\sum_{h=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell+1)B_n(\ell)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{\ell+h}$$

Because $\left(\sum_{\ell=0}^{\infty}\sum_{r=1}^{\ell+1}\cdots\right) \Rightarrow \left(\sum_{\ell=0}^{\infty}\sum_{r=1}^{\ell+1}\cdots\right) - \left(\sum_{\ell=0}^{0}\sum_{r=1}^{1}\cdots\right)$,

$$A_1 = (z_n - 1)z_n \left\{ \sum_{h=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{r=1}^{\ell+1} \frac{1}{z_n^r} \varphi_r^{(n-1)}(\ell+1) B_n(\ell) Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^{\ell+h} \right.$$

$$\left. - \sum_{h=1}^{\infty} \frac{1}{z_n} \varphi_1^{(n-1)}(1) B_n(0) Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^h \right\}$$

From (5.6) we have,

$$A_1 = (z_n - 1)z_n \left\{ \sum_{h=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{r=1}^{\ell+1} \frac{1}{z_n^r} \varphi_r^{(n-1)}(\ell+1) B_n(\ell) Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^{\ell+h} \right.$$

$$\left. - \sum_{h=1}^{\infty} \frac{1}{z_n} B_n(0) Q_h(0_{n-1}, 0_{n-1}, 0_n, 0_n) \omega^h \right\}$$

From (A.1),

$$A_1 = (z_n - 1)z_n \left\{ \sum_{h=1}^{\infty} \sum_{\ell=0}^{\infty} \varphi^{(n-1)}(\ell+1) B_n(\ell) Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^{\ell+h} \right.$$

$$\left. - \sum_{h=1}^{\infty} \frac{1}{z_n} B_n(0) Q_h(0_{n-1}, 0_{n-1}, 0_n, 0_n) \omega^h \right\}$$

Substituting for $B_n(\ell)$ from (2.26) and noting that $B_n(0) = 1$, we have,

$$A_1 = -(z_n - 1) \sum_{h=1}^{\infty} Q_h(0_{n-1}, 0_{n-1}, 0_n, 0_n) \omega^h$$

$$+ (z_n - 1) z_n \sum_{i=0}^{m_n} \sum_{h=1}^{\infty} \sum_{\ell=0}^{\infty} \binom{m_n}{i} \varphi^{(n-1)}(\ell+1) (C_{1n} \lambda_{1n}^{\ell})^i (C_{2n} \lambda_{2n}^{\ell})^{m_n - i} Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^{\ell+h}$$

Substituting from (A.3),

$$A_1 = -(z_n - 1) \sum_{h=1}^{\infty} Q_h(0_{n-1}, 0_{n-1}, 0_n, 0_n) \omega^h$$

$$+ (z_n - 1) z_n \sum_{i=0}^{m_n} \sum_{h=1}^{\infty} \binom{m_n}{i} (C_{1n} \lambda_{1n}^{-1})^i (C_{2n} \lambda_{2n}^{-1})^{m_n - i} \Phi^{(n-1)}(\lambda_{1n}^i \lambda_{2n}^{m_n - i} \omega) Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n) \omega^{h-1}$$

From (5.4)

$$A_1 = -(z_n - 1)\left[Q(0_{n-1}, 0_{n-1}, 0_n, 0_n, \omega) - 1 \right]$$

$$+ (z_n - 1) z_n \left[Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1 \right] \sum_{i=0}^{m_n} \binom{m_n}{i} (C_{1n} \lambda_{1n}^{-1})^i (C_{2n} \lambda_{2n}^{-1})^{m_n - i} \Phi^{(n-1)}(\lambda_{1n}^i \lambda_{2n}^{m_n - i} \omega) \omega^{-1}$$

From (A.5)

$$A_1 = -(z_n - 1)[Q(0_{n-1}, 0_{n-1}, 0_n, 0_n, \omega) - 1]$$
$$+ (z_n - 1)z_n[Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^i C_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}{z_n - \lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)} \quad (5.44)$$

Now substituting (5.43, 5.44) back in $Q(0_{n-1}, 0_{n-1}, z_n, y_n, \omega)$ gives us

$$Q(0_{n-1}, 0_{n-1}, z_n, y_n, \omega)$$
$$= 1 + z_n \sum_{i=0}^{m_n}\sum_{r=1}^{\infty}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i (C_{2n}\lambda_{2n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}{z_n - \lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}$$
$$+ (z_n - 1)z_n[Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^i C_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}{z_n - \lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}$$
$$- (z_n - 1)[Q(0_{n-1}, 0_{n-1}, 0_n, 0_n, \omega) - 1]$$

Applying the final-value theorem to the above equation, we obtain the steady-state form of the boundary function, which is,

$$Q(0_{n-1}, 0_{n-1}, z_n, y_n) = \lim_{\omega \to 1}(1 - \omega)Q(0_{n-1}, 0_{n-1}, z_n, y_n, \omega)$$
$$= (z_n - 1)z_n Q(1_{n-1}, 1_{n-1}, 0_n, 0_n)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^i C_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)}{z_n - \lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i \lambda_{2n}^{m_n-i}\omega)} \quad (5.45)$$
$$- (z_n - 1)Q(0_{n-1}, 0_{n-1}, 0_n, 0_n)$$

Now it is ready to show that the steady-state joint PGF given in (5.8) satisfies the functional equation in (5.40), which describes the behavior of the n'th multiplexer in a general tandem network at the steady state. Substituting (5.8, 5.45) in the RHS of (5.40) gives us,

$$Q(1_{n-1},1_{n-1},z_n,y_n)$$

$$= B_n(1)\Bigg\{(z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}\Bigg|_{y_n=Y_n}$$

$$-(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i})[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})]}\Bigg|_{y_n=Y_n}$$

$$+(z_n-1)Q(1_{n-1},1_{n-1},0_n,0_n)$$

$$-(z_n-1)^2Q(1_{n-1},1_{n-1},0_n,0_n)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^iC_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}\Bigg|_{y_n=Y_n}$$

$$+\frac{(z_n-1)^2}{z_n}Q(0_{n-1},0_{n-1},0_n,0_n)$$

$$-\frac{(z_n-1)^2}{z_n}Q(0_{n-1},0_{n-1},0_n,0_n)\Bigg\}$$

Canceling out the two identical terms in the above equation and noting that,

$\sum_{i=0}^{m_n}\binom{m_n}{i}C_{1n}^iC_{2n}^{m_n-i}=(C_{1n}+C_{2n})^{m_2}=1$, and $Q(1_{n-1},1_{n-1},0_n,0_n)=1-\sum_{i=1}^{n}\rho_i$, where the first equation follows from (2.24),

we have,

$$Q(1_{n-1},1_{n-1},z_n,y_n)$$

$$= B_n(1)\Bigg\{(z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}\Bigg|_{y_n=Y_n}$$

$$-(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i})[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})]}\Bigg|_{y_n=Y_n}$$

$$+(z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}C_{1n}^iC_{2n}^{m_n-i}\Bigg|_{y_n=Y_n}$$

$$-(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^iC_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}\Bigg|_{y_n=Y_n}\Bigg\}$$

Combining the first term with the third one, and the second term with the forth one on the RHS of the above equation, we have,

$$Q(1_{n-1},1_{n-1},z_n,y_n)$$

$$= B_n(1)\left\{(z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{C_{1n}^i C_{2n}^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}\Bigg|_{y_n=Y_n}\right.$$

$$\left. -(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}\cdot\frac{C_{1n}^i C_{2n}^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}\Bigg|_{y_n=Y_n}\right\}$$

Since $B_n(1)=[X_n(1)]^{m_n}$, and only the factors $C_{1n}^i C_{2n}^{m_n-i}$ in the above contains y_n, the above equation may be written as,

$$Q(1_{n-1},1_{n-1},z_n,y_n)$$

$$= (z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{[(C_{1n}|_{y_n=Y_n})X_n(1)]^i[(C_{2n}|_{y_n=Y_n})X_n(1)]^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}$$

$$-(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})} \tag{5.46}$$

$$\cdot\frac{[(C_{1n}|_{y_n=Y_n})X_n(1)]^i[(C_{2n}|_{y_n=Y_n})X_n(1)]^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}$$

From (2.15, 2.22, 2.24), we have,

$$C_{1n}=\frac{X_n(1)-\lambda_{2n}}{\lambda_{1n}-\lambda_{2n}}\ ,\qquad C_{2n}=\frac{\lambda_{1n}-X_n(1)}{\lambda_{1n}-\lambda_{2n}} \tag{5.47}$$

Thus from (5.47) and (2.15), we have,

$$\left(C_{1n}|_{y_n=Y_n}\right)X_n(1)=\frac{\left(X_n(1)|_{y_n=Y_n}\right)X_n(1)-\lambda_{2n}X_n(1)}{\lambda_{1n}-\lambda_{2n}}=\frac{X_n(2)-\lambda_{2n}X_n(1)}{\lambda_{1n}-\lambda_{2n}}$$

Substituting for $X_2(2)$ from (2.17) in the above equation, we have,

$$\left(C_{1n}|_{y_n=Y_n}\right)X_n(1)=\frac{X_n(1)[\beta_n+\alpha_n f_n(z_n)]+(1-\alpha_n-\beta_n)f_n(z_n)-\lambda_{2n}X_n(1)}{\lambda_{1n}-\lambda_{2n}}$$

From (2.22) we have,

$$\lambda_{1n}+\lambda_{2n}=\beta_n+\alpha_n f_n(z_n)\ ;\quad \lambda_{1n}\lambda_{2n}=-(1-\alpha_n-\beta_n)f_n(z_n)$$

Thus

$$\left(C_{1n}|_{y_n=Y_n}\right)X_n(1)=\frac{X_n(1)[\lambda_{1n}+\lambda_{2n}]-\lambda_{1n}\lambda_{2n}-\lambda_{2n}X_n(1)}{\lambda_{1n}-\lambda_{2n}}=\lambda_{1n}\frac{X_n(1)-\lambda_{2n}}{\lambda_{1n}-\lambda_{2n}}$$

Substituting for $X_n(1)$, λ_{1n}, λ_{2n} from (2.15, 2.22) in the above, we have,

$$\left(C_{1n}\big|_{y_n=Y_n}\right)X_n(1)=\lambda_{1n}\left(\frac{1}{2}-\frac{2(y_n-y_n\beta_n-\alpha_n)f_n(z_n)+(\beta_n+\alpha_n f_n(z_n))}{2\sqrt{(\beta_n+\alpha_n f_n(z_n))^2+4(1-\alpha_n-\beta_n)f_n(z_n)}}\right)$$

Therefore, we have

$$\left(C_{1n}\big|_{y_n=Y_n}\right)X_n(1)=C_{1n}\lambda_{1n} \tag{5.48}$$

Following similar steps, it is easy to show that,

$$\left(C_{2n}\big|_{y_n=Y_n}\right)X_n(1)=C_{2n}\lambda_{2n} \tag{5.49}$$

Substituting (5.48, 5.49) into (5.46), we have,

$$\begin{aligned}&Q(1_{n-1},1_{n-1},z_n,y_n)\\&=(z_n-1)\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}}\\&\quad-(z_n-1)^2\left(1-\sum_{i=1}^{n}\rho_i\right)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i})[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i})]}\end{aligned} \tag{5.50}$$

We may see that (5.50) is same as (5.8), which shows that the steady-state joint PGF given in (5.8) satisfies the equilibrium form of the functional equation in (5.40). This proves that the solution is correct. Further, since the system is modeled as a Markov chain, it has a unique solution; therefore, there are no any other solutions.

Chapter 6

Tandem Networks with Multiple Types of Traffic

In the previous chapters, it has been assumed that the external arrivals to each multiplexer in the tandem network are generated by a single type of sources, although the external traffic for different multiplexers could be of different types. This assumption may not be suitable for real networks. Because in practice, the input traffic for each multiplexer may be generated by different types of sources, such as data, voice and video sources. In this chapter, the results are extended to a tandem network with multiple types of traffic sources feeding each of the multiplexers.

6.1 Network Modeling

It is assumed that the tandem network model consists of an arbitrary number of multiplexers, and each multiplexer is fed by a number of different types of sources. Each type of sources consist of a number of independent *On/Off* sources, and each *On* source generates packets according to arbitrary PGFs. The goal is to determine the steady-state PGF for the *n*'th ($n>1$) multiplexer. First, let us introduce the following new definitions regarding the more complicated system model:

τ_i = number of the source types for multiplexer-*i*, $i = 1, 2, \cdots, n$

$m_{i,j}$ = number of type-*j* sources for multiplexer-*i*, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, \tau_i$

$\rho_{i,j}$ = traffic load of type-*j* sources for multiplexer-*i*, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, \tau_i$

ρ_i = traffic load for multiplexer-*i*, $i = 1, 2, \cdots, n$; $\rho_i = \sum_{j=1}^{\tau_i} \rho_{i,j}$

$\alpha_{i,j}$, $\beta_{i,j}$ = parameters for type-*j* sources of multiplexer-*i*, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, \tau_i$.

$\ell_{i,k}$ = queue length for multiplexer-*i* at the end of slot *k*.

$a_{i,j,k}$ = number of *On*-sources of type-j for multiplexer-i during slot k.

$\overline{a}_{i,k} = (a_{i,1,k}, a_{i,2,k}, \cdots, a_{i,\tau_i,k})$

$f_{h,i,j,k}$ = number of packets generated by the h'th *On*-source of type-j for multiplexer-i during slot k. $f_{h,i,j,k}$ are independent identically distributed (i.i.d.) from slot to slot for type-j sources feeding multiplexer-i, with the PGF $f_{i,j}(z_i)$.

$b_{i,j,k}$ = the total number of packets generated by external sources of type-j that are feeding multiplexer-i during slot k.

$b_{i,k}$ = the total number of packets that are feeding multiplexer-i from external sources during slot k, $b_{i,k} = \sum_{j=1}^{\tau_i} b_{i,j,k}$

$c_{h,i,j}$ = a variable that assumes the values of 1, 0 if the h'th source from type-j that is feeding multiplexer-i in *On* and *Off* states in the next slot respectively, given that this source is *On* in the present slot.

$d_{h,i,j}$ = a variable that assumes the values of 1, 0 if h'th source from type-j that is feeding multiplexer-i is in *On* and *Off* states in the next slot respectively, given that this source is *Off* in the present slot.

The $c_{h,i,j}$, $d_{h,i,j}$ are i.i.d. Bernoulli random variables with the corresponding PGF given by:

$$c_{i,j}(z_i) = 1 - \alpha_{i,j} + \alpha_{i,j} z_i \,, \qquad d_{i,j}(z_i) = \beta_{i,j} + (1 - \beta_{i,j}) z_i \tag{6.1}$$

From the above definitions, we have

$$b_{i,j,k} = \sum_{j=1}^{a_{i,k}} f_{h,i,j,k} \,, \qquad a_{i,j,k+1} = \sum_{j=1}^{a_{i,k}} c_{h,i,j} + \sum_{j=1}^{m_{i,j} - a_{i,j,k}} d_{h,i,j} \tag{6.2}$$

The evolution of the first queue length is given by,

$$\ell_{1,k+1} = (\ell_{1,k} - 1)^+ + b_{1,k+1} = (\ell_{1,k} - 1)^+ + \sum_{j=1}^{\tau_1} b_{1,j,k} \tag{6.3}$$

And the evolution of the i'th queue length ($i > 1$) is given by,

$$\ell_{i,k+1} = (\ell_{i,k} - 1)^+ + b_{i,k+1} + u_{i,k} = (\ell_{i,k} - 1)^+ + \sum_{j=1}^{\tau_i} b_{i,j,k+1} + u_{i,k} \quad , \quad 2 \le i \le n \tag{6.4}$$

where $u_{i,k}$ is a random variable depending on whether the previous queue is empty or not,

$$u_{i,k} = \begin{cases} 1 & \text{if } \ell_{i-1,k} > 0 \\ 0 & \text{if } \ell_{i-1,k} = 0 \end{cases} \tag{6.5}$$

In the above equations the notation $(x)^+$ denotes *max*(*x*, 0).

As discussed in section 2.3, the total effect of the multiplexers preceding the n'th one has been summarized in the output of the (n-1)'st multiplexer. Thus, in order to determine the performance of the n'th multiplexer, only the joint system consisting of the (n-1)'st and the n'th multiplexers needs to be considered. The state of the two multiplexers can be defined by $(\ell_{n-1,k}, \bar{a}_{n-1,k}, \ell_{n,k}, \bar{a}_{n,k})$. Let us define $Q_k(z_{n-1}, \bar{y}_{n-1}, z_n, \bar{y}_n)$ as the joint PGF of the random variables $\ell_{n-1,k}$, $\bar{a}_{n-1,k}$, $\ell_{n,k}$ and $\bar{a}_{n,k}$, then we have

$$Q_k(z_{n-1}, \bar{y}_{n-1}, z_n, \bar{y}_n) = E\left[z_{n-1}^{\ell_{n-1,k}} \bar{y}_{n-1}^{\bar{a}_{n-1,k}} z_n^{\ell_{n,k}} \bar{y}_n^{\bar{a}_{n,k}}\right] \quad , \quad n \ge 2$$

where $\bar{y}_i = (y_{i,1}, y_{i,2}, \cdots, y_{i,\tau_i})$, and $\bar{y}_{n-1}^{\bar{a}_{n-1,k}} = (y_{n-1,1}^{a_{n-1,1,k}}, y_{n-1,2}^{a_{n-1,2,k}}, y_{n-1,3}^{a_{n-1,3,k}}, \cdots, y_{n-1,\tau_{n-1}}^{a_{n-1,\tau_{n-1},k}})$,

$$\bar{y}_n^{\bar{a}_{n,k}} = (y_{n,1}^{a_{n,1,k}}, y_{n,2}^{a_{n,2,k}}, y_{n,3}^{a_{n,3,k}}, \cdots, y_{n,\tau_n}^{a_{n,\tau_n,k}}),$$

Then, the joint PGF at slot (k +1) is given by,

$$Q_{k+1}(z_{n-1}, \bar{y}_{n-1}, z_n, \bar{y}_n) = E\left[z_{n-1}^{\ell_{n-1,k+1}} \bar{y}_{n-1}^{\bar{a}_{n-1,k+1}} z_n^{\ell_{n,k+1}} \bar{y}_n^{\bar{a}_{n,k+1}}\right]$$

Following the Markov chain analysis of section 2.2, the functional equation is obtained that relates the joint PGFs for two consecutive slots for the n'th multiplexer,

$$\begin{aligned} & Q_{k+1}\left(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n\right) \\ &= B_n(1)\Big\{Q_k\left(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{Y}_n\right) + (z_n - 1)Q_k\left(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n\right) \\ &\quad + \frac{1 - z_n}{z_n} Q_k\left(0_{n-1}, \bar{0}_{n-1}, z_n, \bar{Y}_n\right) - \frac{(z_n - 1)^2}{z_n} Q_k\left(0_{n-1}, \bar{0}_{n-1}, 0_n, \bar{0}_n\right)\Big\} \quad , \quad k \ge 0 \end{aligned} \tag{6.6}$$

where

$$B_n(1) = \prod_j^{\tau_n} B_{n,j}(1), \quad B_{n,j}(1) = [d_{n,j}(y_{n,j} f_{n,j}(z_n))]^{m_{n,j}}, \tag{6.7}$$

$$1_{n-1} = z_{n-1}\big|_{z_{n-1}=1} \quad , \quad \bar{1}_{n-1} = (y_{n-1,1} = 1, y_{n-1,2} = 1, \cdots, y_{n-1,\tau_n} = 1), \tag{6.8}$$

$$0_{n-1} = z_{n-1}\big|_{z_{n-1}=0} \quad , \quad \bar{0}_{n-1} = (y_{n-1,1} = 0, y_{n-1,2} = 0, \cdots, y_{n-1,\tau_{n-1}} = 0), \tag{6.9}$$

$$0_n = z_n\big|_{z_n=0} \quad , \quad \bar{0}_n = (y_{n,1} = 0, y_{n,2} = 0, \cdots, y_{n,\tau_n} = 0), \tag{6.10}$$

$$\bar{Y}_n = (y_{n,1} = Y_{n,1}, y_{n,2} = Y_{n,2}, \cdots, y_{n,\tau_i} = Y_{n,\tau_n}),\ Y_{n,j} = \frac{c_{n,j}\left(y_{n,j} f_{n,j}(z_n)\right)}{d_{n,j}\left(y_{n,j} f_{n,j}(z_n)\right)}, \tag{6.11}$$

It may be shown that the functional equation given in (6.6) can be expressed as follows,

$$\begin{aligned}
&Q_k(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n) \\
&= B_n(k) + (z_n - 1)\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n) \\
&+ \frac{1 - z_n}{z_n} \sum_{j=1}^{k-1} B_n(j) Q_{k-j}\left(0_{n-1}, \bar{0}_{n-1}, z_n, \bar{\phi}_n(j)\right) \\
&- \frac{(z_n - 1)^2}{z_n} \sum_{j=1}^{k-1} B_n(j) Q_{k-j}(0_{n-1}, \bar{0}_{n-1}, 0_n, \bar{0}_n) \qquad k \geq 1, n \geq 2
\end{aligned} \tag{6.12}$$

where,

$$\bar{\phi}_n(j) = (\phi_{n,1}(j), \phi_{n,2}(j), \cdots, \phi_{n,\tau_n}(j)),\ \ \phi_{i,j}(k) = \frac{U_{i,j}(k)}{X_{i,j}(k)} \quad , \quad B_{n,j}(k) = \left[X_{i,j}(k)\right]^{m_{i,j}} \tag{6.13}$$

And $U_{i,j}(k), X_{i,j}(k)$ are defined as,

$$X_{i,j}(k+1) = X_{i,j}(1)[X_{i,j}(k)\big|_{y_{i,j}=Y_{i,j}}] \ \text{ with } X_{i,j}(0) = 1,\ X_{i,j}(1) = \beta_{i,j} + (1 - \beta_{i,j}) y_{i,j} f_{i,j}(z_i)$$

$$U_{i,j}(k+1) = X_{i,j}(1)[U_{i,j}(k)\big|_{y_{i,j}=Y_{i,j}}] \ \text{ with } U_{i,j}(0) = y_{i,j},\ U_{i,j}(1) = 1 - \alpha_{i,j} + \alpha_{i,j} y_{i,j} f_{i,j}(z_i)$$

The proof of equation (6.12) can also be done through induction as in the case of (2.28), and the details will not be given here. In general, (6.12) gives the solution of the functional equation, except that the unknown boundary function, $Q_{k-j}\left(0_{n-1}, \bar{0}_{n-1}, z_n, \bar{\phi}_n(j)\right)$ need to be determined.

6.2 Performance Analysis

In this section, the unknown boundary function in (6.12) will be determined and then the steady-state PGF of the n'th multiplexer will be obtained. Afterwards, the performance measures will be determined.

Through the busy period of the (n-1)'st multiplexer, the unknown boundary function in (6.12) may be written as,

$$\begin{aligned} & Q_k(0_{n-1},\overline{0}_{n-1},z_n,\overline{y}_n) \\ & =\sum_{r=1}^{k}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k)B_n(k) \\ & +(z_n-1)\sum_{h=1}^{k-1}\sum_{r=1}^{k-h+1}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-h+1)B_n(k-h)Q_h(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n) \quad , \qquad k\geq 1 \end{aligned} \tag{6.14}$$

Substituting (6.14) in (6.12), we have

$$\begin{aligned} & Q_k(1_{n-1},\overline{1}_{n-1},z_n,\overline{y}_n) \\ & =B_n(k)+(z_n-1)\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n) \\ & -\frac{z_n-1}{z_n}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-j)B_n(k) \\ & -\frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^{r-1}}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n) \\ & -\frac{(z_n-1)^2}{z_n}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},\overline{0}_{n-1},0_n,\overline{0}_n) \quad , \qquad k\geq 1 \end{aligned} \tag{6.15}$$

The transform of the above equation is defined as ,

$$Q(1_{n-1},\overline{1}_{n-1},z_n,\overline{y}_n,\omega)=\sum_{k=0}^{\infty}Q_k(1_{n-1},\overline{1}_{n-1},z_n,\overline{y}_n)\omega^k \tag{6.16}$$

From (6.15) we have

$$
\begin{aligned}
&Q(1_{n-1},\overline{1}_{n-1},z_n,\overline{y}_n,\omega)\\
&=1+\sum_{k=1}^{\infty}B_n(k)\omega^k+(z_n-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n)\omega^k\\
&-(z_n-1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j)B_n(k)\omega^k\\
&-(z_n-1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n)\omega^k\\
&-\frac{(z_n-1)^2}{z_n}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}B_n(j)Q_{k-j}(0_{n-1},\overline{0}_{n-1},0_n,\overline{0}_n)\omega^k
\end{aligned}
\tag{6.17}
$$

Let us define

$$Q(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n,\omega)=\sum_{k=0}^{\infty}Q_k(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n)\omega^k$$

$$Q(0_{n-1},\overline{0}_{n-1},0_n,\overline{0}_n,\omega)=\sum_{k=0}^{\infty}Q_k(0_{n-1},\overline{0}_{n-1},0_n,\overline{0}_n)\omega^k$$

Following the algebraic manipulations similar to those in Appendix D, equation (6.17) may be written as,

$$
\begin{aligned}
&Q(1_{n-1},\overline{1}_{n-1},z_n,\overline{y}_n,\omega)\\
&=1+\sum_{\overline{i}=\overline{0}}^{\overline{m}_n}\frac{\prod_{j=1}^{\tau_n}\binom{m_{n,j}}{i_j}(C_{1n,j}\lambda_{1n,j})^{i_j}(C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\,\omega}{1-\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\,\omega}\\
&+(z_n-1)\left[Q(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n,\omega)-1\right]\sum_{\overline{i}=\overline{0}}^{\overline{m}_n}\frac{\omega\prod_{j=1}^{\tau_n}\binom{m_{n,j}}{i_j}(C_{1n,j}\lambda_{1n,j})^{i_j}(C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}}{1-\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}}\\
&-(z_n-1)\sum_{\overline{i}=\overline{0}}^{\overline{m}_n}\binom{m_{n,j}}{i_j}\frac{\left[\omega\prod_{j=1}^{\tau_n}\binom{m_{n,j}}{i_j}(C_{1n,j}\lambda_{1n,j})^{i_j}(C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\right]\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\cdot\Gamma_{n-1}\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)}{\left(1-\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\left[z_n-\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\cdot\Gamma_{n-1}\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\right]}\\
&-(z_n-1)^2\left[Q(1_{n-1},\overline{1}_{n-1},0_n,\overline{0}_n,\omega)-1\right]\\
&\quad\cdot\sum_{\overline{i}=\overline{0}}^{\overline{m}_n}\binom{m_{n,j}}{i_j}\frac{\left[\omega\prod_{j=1}^{\tau_n}\binom{m_{n,j}}{i_j}(C_{1n,j}\lambda_{1n,j})^{i_j}(C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\right]\cdot\Gamma_{n-1}\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)}{\left(1-\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\left[z_n-\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\cdot\Gamma_{n-1}\left(\omega\prod_{j=1}^{\tau_n}\lambda_{1n,j}^{i_j}\lambda_{2n,j}^{m_{n,j}-i_j}\right)\right]}
\end{aligned}
\tag{6.18}
$$

where $\sum_{\bar{i}=0}^{\bar{m}_n} = \sum_{i_1=0}^{m_{n,1}} \sum_{i_2=0}^{m_{n,2}} \cdots \sum_{i_{\tau_n}=0}^{m_{n,\tau_n}}$

$$\lambda_{1n,j} = \frac{\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n) - \sqrt{(\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))^2 + 4(1-\alpha_{n,j} - \beta_{n,j}) f_{n,j}(z_n)}}{2}$$

$$\lambda_{2n,j} = \frac{\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n) + \sqrt{(\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))^2 + 4(1-\alpha_{n,j} - \beta_{n,j}) f_{n,j}(z_n)}}{2}$$

$$C_{1n,j} = \frac{1}{2} - \frac{2(y_{n,j} - y_{n,j}\beta_{n,j} - \alpha_{n,j}) f_{n,j}(z_n) + (\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))}{2\sqrt{(\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))^2 + 4(1-\alpha_{n,j} - \beta_{n,j}) f_{n,j}(z_n)}}$$

$$C_{1n,j} = \frac{1}{2} + \frac{2(y_{n,j} - y_{n,j}\beta_{n,j} - \alpha_{n,j}) f_{n,j}(z_n) + (\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))}{2\sqrt{(\beta_{n,j} + \alpha_{n,j} f_{n,j}(z_n))^2 + 4(1-\alpha_{n,j} - \beta_{n,j}) f_{n,j}(z_n)}}$$

In order to determine the steady-state PGF of the n'th multiplexer, let us apply the final-value theorem to (6.18),

$$Q(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n) = Q_\infty(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n) = \lim_{\omega \to 1}(1-\omega) Q(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n, \omega)$$

We note that, $\lim_{\omega \to 1}(1-\omega) Q(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n, \omega) = Q(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n) = 1 - \sum_{i=1}^{n} \rho_i$

where

$$\rho_i = \sum_{j=1}^{\tau_i} \rho_{i,j} = \sum_{j=1}^{\tau_i} \frac{m_{i,j}(1-\beta_{i,j}) f'_{i,j}(1)}{2 - \alpha_{i,j} - \beta_{i,j}}, \quad i = 1, 2, \cdots n$$

then,

$$Q(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{y}_n)$$

$$= (z_n - 1)\left(1 - \sum_{i=1}^{n} \rho_i\right) \sum_{\bar{i}=0}^{\bar{m}_n} \frac{\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (C_{1n,j}\lambda_{1n,j})^{i_j} (C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}}{1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}}$$

$$- (z_n - 1)^2 \left(1 - \sum_{i=1}^{n} \rho_i\right) \sum_{\bar{i}=0}^{\bar{m}_n} \binom{m_{n,j}}{i_j} \frac{\left[\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (C_{1n,j}\lambda_{1n,j})^{i_j} (C_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\right] \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)}{\left(1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\left[z_n - \left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right) \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\right]}$$

(6.19)

The above is the joint PGF of the n'th multiplexer. Substituting $\bar{y}_n = \bar{1}_n$ in (6.19) gives the PGF of the queue length of the n'th multiplexer,

$$P_n(z_n) = Q(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{1}_n)$$

$$= (z_n - 1)\left(1 - \sum_{i=1}^{n} \rho_i\right) \sum_{\bar{i}=\bar{0}}^{\bar{m}_n} \frac{\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (\widetilde{C}_{1n,j}\lambda_{1n,j})^{i_j} (\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}}{1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}}$$

$$- (z_n - 1)^2 \left(1 - \sum_{i=1}^{n} \rho_i\right) \sum_{\bar{i}=\bar{0}}^{\bar{m}_n} \binom{m_{n,j}}{i_j} \frac{\left[\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (\widetilde{C}_{1n,j}\lambda_{1n,j})^{i_j} (\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\right] \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)}{\left(1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\left[z_n - \left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right) \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\right]} \tag{6.20}$$

where $\widetilde{C}_{1n,j} = C_{1n,j}\big|_{y_{n,j}=1}$ and $\widetilde{C}_{2n,j} = C_{2n,j}\big|_{y_{n,j}=1}$

Next, $\Gamma_n(\omega)$ will be determined. From Appendix B, $\Gamma_n(\omega)$ can be expressed in terms of $Q(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n, \omega)$. Following the derivation in section 4.2.2, the expression $Q(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n, \omega)$ may be determined by invoking the analytical property of function $Q(1_{n-1}, \bar{1}_{n-1}, z_n, \bar{1}_n, \omega)$ inside the poly disk $(|z_n| < 1; |\omega| < 1)$ and through the application of Rouche's theorem. Therefore, the details will not presented here and only the results are given,

$$Q(1_{n-1}, \bar{1}_{n-1}, 0_n, \bar{0}_n, \omega) = 1 + \frac{\prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}} \omega \Big|_{z_n = z_n^*(\omega)}}{1 - z_n^*(\omega)} \tag{6.21}$$

where $z_n^*(\omega)$ is the root of the equation,

$$z_n - \left(\omega \prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}}\right) \cdot \Gamma_{n-1}\left(\omega \prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}}\right) = 0 \tag{6.22}$$

From (B.8), we have,

$$\Gamma_n(\omega) = \frac{1}{\omega}\left(1 - \frac{1}{Q(1_{n-1},\bar{1}_{n-1},0_n,\bar{0}_n,\omega)}\right) = \frac{\left.\prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}}\right|_{z_n = z_n^*(\omega)}}{1 - z_n^*(\omega) + \omega \left.\prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}}\right|_{z_n = z_n^*(\omega)}} \tag{6.23}$$

6.3 Performance Measures for the *n*'th Multiplexer

In the next, the mean queue length, packet delay and the variance of the queue length will be determined for the n'th multiplexer. First, let us transfer (6.20) into a more convenient form:

$$\begin{aligned} P_n(z_n) &= (z_n - 1)\left(1 - \sum_{i=1}^{n} \rho_i\right)\left[E_n(z_n) + \frac{G_n(z_n)}{1 - H_n(z_n)}\right] \\ &- (z_3 - 1)^2\left(1 - \sum_{i=1}^{n} \rho_i\right)\left[F_n(z_n) + \frac{G_n(z_n)\Gamma_{n-1}(H_n(z_n))}{[1 - H_n(z_n)][z_n - \Theta_n(z_n)]}\right] \end{aligned} \tag{6.24}$$

where

$$E_n(z_n) = \sum_{i=1}^{\overline{m}_n} \frac{\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (\widetilde{C}_{1n,j}\lambda_{1n,j})^{i_j} (\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}}{1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}} \tag{6.25}$$

$$F_n(z_n) = \sum_{i=1}^{\overline{m}_n} \binom{m_{n,j}}{i_j} \frac{\left[\prod_{j=1}^{\tau_n} \binom{m_{n,j}}{i_j} (\widetilde{C}_{1n,j}\lambda_{1n,j})^{i_j} (\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}-i_j}\right] \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)}{\left(1 - \prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\left[z_n - \left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right) \cdot \Gamma_{n-1}\left(\prod_{j=1}^{\tau_n} \lambda_{1n,j}^{i_j} \lambda_{2n,j}^{m_{n,j}-i_j}\right)\right]} \tag{6.26}$$

$$H_n(z_n) = \prod_{j=1}^{\tau_n} \lambda_{2n,j}^{m_{n,j}}, \quad G_n(z_n) = \prod_{j=1}^{\tau_n} (\widetilde{C}_{2n,j}\lambda_{2n,j})^{m_{n,j}} \tag{6.27}$$

$$\Theta_n(z_n) = H_n(z_n)\Gamma_{n-1}(H_n(z_n)) \tag{6.28}$$

We note that

$$E_n(1) = 0,\ F_n(1) = 0,\ H_n(1) = 1,\ G_n(1) = 1,\ \Theta_n(1) = 1 \tag{6.29}$$

It may be seen that (6.24) has the same form as (5.14). So the performance measures of the n'th multiplexer may be determined in the same manner.

- Derivation the mean queue length and packet delay

First, both sides of (6.24) are multiplied with the common denominator, and then both sides are taken the third derivative with respect to z_n, afterwards substituting $z_n = 1$ gives,

$$-3H_n''(1)[1-\Theta_n'(1)] + 3H_n'(1)\Theta_n''(1) - 6H_n'(1)[1-\Theta_n'(1)]P_n'(1)$$
$$= -6(1-\sum_{i=1}^{n}\rho_i)[\Gamma_{n-1}'(1)H_n'(1) + G_n'(1)] + 3(1-\sum_{i=1}^{n}\rho_i)[-\Theta_n''(1) + 2(1-\Theta_n'(1))G_n'(1)]$$

Solving the above equation for $P_n'(1)$ gives the mean queue length of the n'th multiplexer,

$$\overline{N}_n = P_n'(1) = \frac{\Theta_n''(1)}{2[1-\Theta_n'(1)]} - \frac{H_n''(1)}{2H_n'(1)} + \frac{1-\sum_{i=1}^{n}\rho_i}{H_n'(1)[1-\Theta_n'(1)]}[\Gamma_{n-1}'(1)H_n'(1) + G_n'(1)]$$
$$+ \frac{(1-\sum_{i=1}^{n}\rho_i)\Theta_n''(1)}{2H_n'(1)[1-\Theta_n'(1)]} - \frac{1-\sum_{i=1}^{n}\rho_i}{H_n'(1)}G_n'(1) \qquad (6.30)$$

where $$\Theta_n'(1) = \left.\frac{d\Theta_n(z_n)}{dz_n}\right|_{z_n=1}, \quad \Theta_n''(1) = \left.\frac{d^2\Theta_n(z_n)}{dz_n^{\,2}}\right|_{z_n=1} \qquad (6.31)$$

$$H_n'(1) = \left.\frac{dH_n(z_n)}{dz_n}\right|_{z_n=1}, \quad H_n''(1) = \left.\frac{d^2H_n(z_n)}{dz_n^{\,2}}\right|_{z_n=1}, \qquad (6.32)$$

$$G_n'(1) = \left.\frac{dG_n(z_n)}{dz_n}\right|_{z_n=1}, \quad \Gamma_{n-1}'(1) = \left.\frac{d\Gamma_{n-1}(\omega)}{d\omega}\right|_{\omega=1}, \qquad (6.33)$$

From the Little's result, the mean delay that a packet experiences at the n'th multiplexer is given by,

$$\overline{D}_n = \frac{\overline{N}_n}{\sum_{j=1}^{n}\rho_j} \qquad (6.34)$$

- Derivation of the variance of the queue length

In order to determine the variance of the queue length for the n'th multiplexer, the second order derivative of $P_n(z_n)$ is required. Following the same steps as in Chapter 4, we have,

$$P_n''(1)$$
$$= \frac{1}{6H_n'(1)[1-\Theta_n'(1)]}\left\{2(1-\sum_j^n \rho_j)\Theta_n'''(1) + 3H_n''(1)\Theta_n''(1)\right.$$
$$\left. + 6(1-\sum_j^n \rho_j)[\Theta_n''(1)G_n'(1) + \Theta_n'(1)G_n''(1) + \Gamma_{n-1}'(1)H_n''(1)]\right\} \quad (6.35)$$
$$+ \frac{1}{3H_n'(1)}\left\{-H_n'''(1) + 6(1-\sum_j^n \rho_j)H_n'(1)E_n'(1) - 3H_n''(1)P_n'(1)\right\}$$
$$+ \frac{1}{3[1-\Theta_n'(1)]}\left\{\Theta_n'''(1) + 3\Theta_n''(1)P_n'(1) + 3(1-\sum_j^n \rho_j)[2\Gamma_{n-1}'(1)G_n'(1) + \Gamma_{n-1}''(1)H_n'(1)]\right\}$$

where $\Theta_n'''(1) = \left.\frac{d^3\Theta_n(z_n)}{dz_n^3}\right|_{z_n=1}$, $G_n''(1) = \left.\frac{d^2G_n(z_n)}{dz_n^2}\right|_{z_n=1}$,

$$E_n'(1) = \left.\frac{dE_n(z_n)}{dz_n}\right|_{z_n=1}, \quad H_n'''(1) = \left.\frac{d^3H_n(z_n)}{dz_n^3}\right|_{z_n=1}, \quad \Gamma_{n-1}''(1) = \left.\frac{d^2\Gamma_{n-1}(\omega)}{d\omega^2}\right|_{\omega=1}, \quad (6.36)$$

The variance of the queue length for the n'th multiplexer may be expressed in terms of the derivatives of its PGF as,

$$\overline{V}_n = P_n''(1) + P_n'(1) - (P_n'(1))^2 \quad (6.37)$$

The expressions of the mean and variance of queue length, as well as mean packet require the derivatives of $G_n(z_n)$, $E_n(z_n)$, $H_n(z_n)$, $\Gamma_{n-1}(\omega)$, and $\Theta_n(z_n)$. The derivation of these derivatives is the same as in the previous chapter. Here details are not given, but only it is noted that,

$$H_n'(1) = \sum_j^{\tau_n} \frac{m_{n,j}(1-\beta_{n,j})f_{n,j}'(1)}{2-\alpha_{n,j}-\beta_{n,j}} = \rho_n \quad (6.38)$$

6.4 Numerical Results

In this section, the numerical results are presented regarding the analysis of tandem networks with multiple types of traffic. The tandem network is assumed to consist of two multiplexers, and each multiplexer is fed by two types of Markovian sources. The mean queue length, mean packet delay and standard deviation of multiplexer-2 are presented versus its total load in Figures 6.1 - 6.3 respectively. In all of the three figures, two curves are presented. The solid-line curves correspond to varying the load of one type of sources feeding multiplexer-1, while keeping the load for other three types of sources constant. And the dashed-line curves correspond to varying the load of one type of sources feeding multiplexer-2, while keeping the load for other three types of sources constant. It may be seen that for both curves, the mean queue length, mean packet delay and standard deviation of queue length increase with the total load of multiplexer-2. It also may be seen that for the same total load of multiplexer-2, the solid-line curves are below than the dashed-line. As explained before, statistical multiplexing smoothes the traffic. The higher is the number of multiplexers that traffic goes through, the smoother the traffic becomes and results in smaller delay and queue length. In solid-line curves, higher proportion of traffic has gone through more multiplexers than in dashed-line, and this accounts for smoother traffic. In addition, it may be seen that the difference between solid and dashed lines is bigger than those figures in previous chapter. It shows that the smoothing effect is more significant

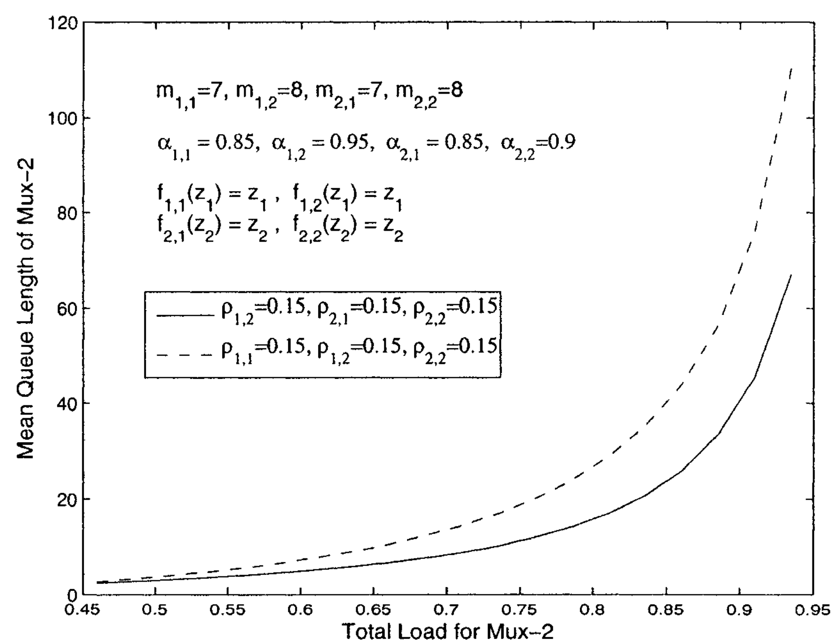


Figure 6.1 Mean queue length of multiplexer-2 vs. total load of multiplexer-2

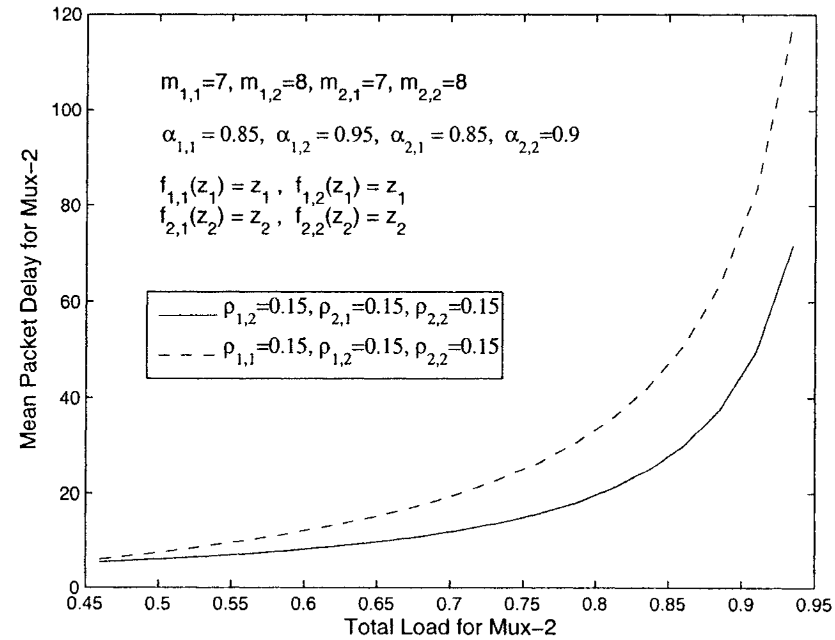


Figure 6.2 Mean packet delay of multiplexer-2 vs. total load of multiplexer-2

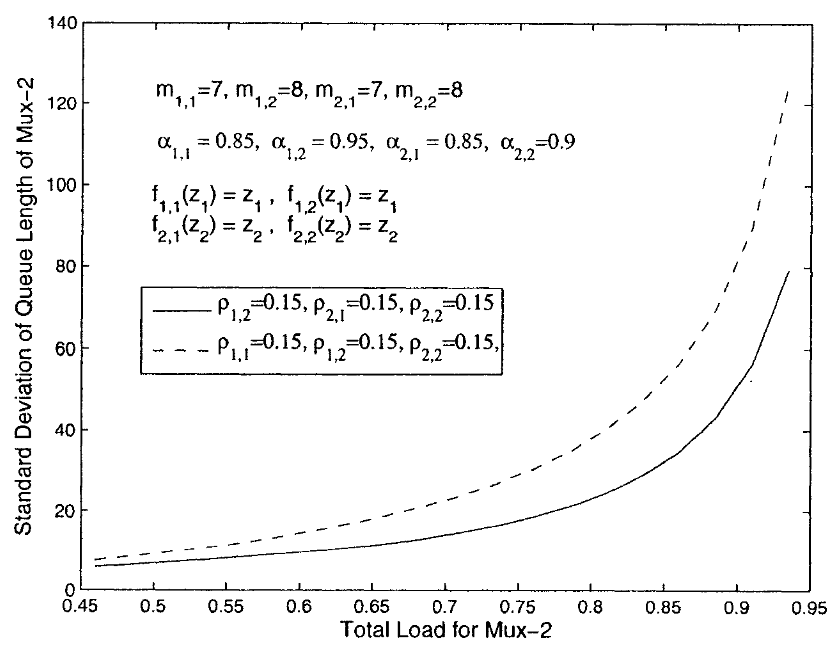


Figure 6.3 Standard deviation of queue length vs. total load for multiplexer-2

Chapter 7

Contributions and Conclusions

The modern networks, including the Internet and ATM are based on packet-switched technology. The long-term objective of modern networks is to provide satisfactory services for multimedia traffic such as voice, video and data with different QoS in a single network. In packet-switched networks, statistical multiplexing is performed to achieve bandwidth efficiency. The switch and router output ports may be modeled as multiplexers. As packets go through the network from source to destination, the route they pass through may be modeled as a number of multiplexers in tandem. The studies of a network connection, which typically consists of a number of multiplexers in tandem, is of great importance for network design since it helps to understand how the network traffic characteristics change in the interior of the network.

However, most analytical studies related to network performance focus on an isolated component in the network because of the difficulty of performance analysis at the network level. Due to lack of exact analysis methods, most previous studies on tandem networks have focused either on simulation or on approximate models.

This thesis studies an arbitrary number of multiplexers in tandem. Each multiplexer is fed by the output of the previous multiplexer as well as external traffic. External traffic is generated by a number of binary Markov *On/Off* sources. This Markovian source model has been known to be good at capturing the correlation in the multimedia traffic. In this thesis, an exact performance analysis of the studied tandem network model is presented. The functional equation describing the tandem network has been derived through the imbedded Markov Chain analysis. The unknown boundary functions is determined, as well as the solution of the functional equation: the joint PGF of queue length and number of *On* sources for any multiplexer at the steady state. The correctness of the solution has been proven by showing that it satisfies the functional equation in equilibrium. From the joint PGF, the closed-form expressions for the performance measures of network are derived, such as the mean and variance of queue length, as well as mean packet. Finally, the solution is extended to a more general tandem network with multiple types of traffic feeding each of the multiplexers. Again, the joint PGF of an arbitrary multiplexer at the steady state is determined, as well as present the closed-form expressions of performance measures. The analytical results are compared with those of the simulation, and show that they match each other very well.

The numerical results are presented regarding the analysis in the thesis. The analysis and numerical results show the smoothing effect of statistical multiplexing: as the traffic goes through higher number of multiplexers, it becomes smoother. From the numerical results, both mean and variance of the queue length and delay drop down. This drop is more obvious in the case of heterogeneous traffic feeding each multiplexer. The analysis enables to explain the delicate interaction between traffic smoothing and source burstiness. When traffic load is kept constant while the number of sources is increased, the analysis and results show that the burstiness may overweigh the smoothing effect, as a result, the mean queue length and packet delay may increase.

Appendix A

Some Results Related to Busy Periods of a Multiplexer

It is assumed that a busy period of a queue begins and ends with idle slots and two consecutive idle slots correspond to a busy period with zero duration (see Figure A.1). Further, two consecutive busy periods is separated by an idle slot and this slot is assumed to belong to the starting busy period. As a result of this definition, the busy periods are independent and identically distributed. More details of the definition of busy period can be found in [20, 23].

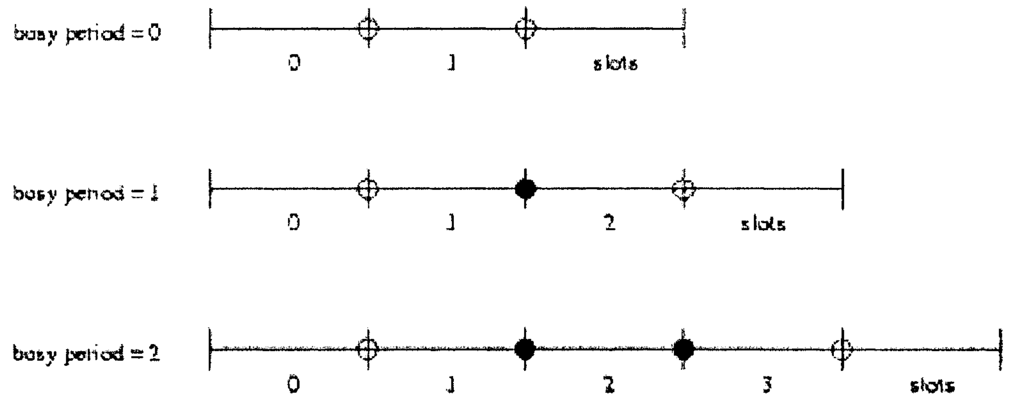


Figure A.1 Definition of a busy period

Following the notations in [23], let us define,

$\xi_n(j)$ = Prob(n'th multiplexer has a busy period = j slots), $j = 0, 1, 2, \ldots$

$\varphi_r^{(n)}(\ell)$ = Prob(n'th multiplexer has r busy periods during an interval of ℓ slots).

$$\varphi^{(n)}(\ell) = \sum_{r=1}^{\ell} \frac{1}{z_n^r} \varphi_r^{(n)}(\ell) \ , \ \ell \geq 1 \ , \ \text{and} \ \varphi^{(n)}(0) = 1 \quad \text{(A.1)}$$

We note that $\varphi^{(n)}(\ell)\big|_{z_n=1}$ corresponds to the probability that n'th multiplexer has integer number of busy periods during an interval of ℓ slots. Next, let us assume that the duration of the last busy period in the interval is $(\ell - j - 1)$ slots, then, $\varphi^{n}(\ell)$ may be expressed recursively as follows,

$$\varphi^{(n)}(\ell) = \frac{1}{z_n}\sum_{j=0}^{\ell-1} \xi_n(\ell - j - 1)\varphi^{(n)}(j) \tag{A.2}$$

Next let us define the following transform,

$$\Phi^{(n)}(\omega) = \sum_{\ell=1}^{\infty} \varphi^{(n)}(\ell)\omega^{\ell} \tag{A.3}$$

And define the PGF of the busy period of n'th multiplexer as

$$\Gamma_n(\omega) = \sum_{j=0}^{\infty} \xi_n(j)\omega^{j} \tag{A.4}$$

From (A.2) and (A.3), we have

$$\Phi^{(n)}(\omega) = \frac{1}{z_n}\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1} \xi_n(\ell - j - 1)\varphi^{(n)}(j)\omega^{\ell}$$

Exchanging the order of summations, we have

$$\Phi^{(n)}(\omega) = \frac{1}{z_2}\sum_{j=0}^{\infty}\sum_{\ell=j+1}^{\infty} \xi_n(\ell - j - 1)\varphi^{(n)}(j)\omega^{\ell}$$

Letting $i = \ell - j - 1$,

$$\Phi^{(n)}(\omega) = \frac{1}{z_n}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \xi_n(i)\varphi^{(n)}(j)\omega^{i+j+1}$$

From (A.3) and (A.4) we have

$$\Phi^{(n)}(\omega) = \frac{1}{z_n}\omega\Gamma_n(\omega)[1 + \Phi^{(n)}(\omega)]$$

Solving the above equation for $\Phi^{(n)}(\omega)$, we have

$$\Phi^{(n)}(\omega) = \frac{\omega\Gamma_n(\omega)}{z_n - \omega\Gamma_n(\omega)} \tag{A.5}$$

Appendix B

The Relation between $B_1(k)$ and $\Gamma_1(\omega)$

The PGF of the busy period of the first multiplexer, $\Gamma_1(\omega)$, may be expressed in terms of the derivatives of $B_1(k)$. In this appendix, the relation between $B_1(k)$ and $\Gamma_1(\omega)$ will be determined.

Because the first multiplexer is not affected by other multiplexers in the network, it behaves like a single multiplexer that has been studied in [22]. From equation (12) of [22], we have,

$$Q_k(z_1, y_1) = \frac{1}{z_1^k} B_1(k) + (z_1 - 1)\sum_{j=1}^{k} \frac{B_1(j)}{z_1^j} Q_{k-j}(0,0)$$

The above may be written as,

$$Q_k(z_1, y_1) = \frac{1}{z_1^k} B_1(k) + \sum_{j=1}^{k} \left(\frac{B_1(j)}{z_1^{j-1}} - \frac{B_1(j)}{z_1^j} \right) Q_{k-j}(0,0)$$

Substituting 0 for z_1 and y_1 results in $\frac{0}{0}$ indeterminacy, so L'Hopital's rule is applied,

$$Q_k(0,0) = \overline{B}_1^{(k)}(k) + \sum_{j=1}^{k} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j) \right] Q_{k-j}(0,0) \quad \text{(B.1)}$$

where the notation $\overline{B}_i^{(\ell)}(j)$ is defined as $\overline{B}_i^{(\ell)}(j) = \left. \frac{1}{\ell!} \frac{d^\ell B_i(j)}{dz_1^\ell} \right|_{z_i = y_i = 0}$

Define the transform of $Q_k(0,0)$ with respect to the discrete time k as,

$$Q(0,0,\omega) = \sum_{k=0}^{\infty} Q_k(0,0)\omega^k \quad \text{(B.2)}$$

Then from (B.1) we have,

$$Q(0,0,\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k + \sum_{k=0}^{\infty} \sum_{j=1}^{k} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j) \right] Q_{k-j}(0,0)\omega^k$$

Exchanging the order of summations,

$$Q(0,0,\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k + \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j)\right] Q_{k-j}(0,0)\omega^k$$

Letting $n = k - j$, the above becomes

$$Q(0,0,\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k + \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j)\right] Q_n(0,0)\omega^{n+j}$$

From (B.2),

$$Q(0,0,\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k + Q(0,0,\omega)\sum_{j=1}^{\infty} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j)\right]\omega^j$$

Therefore, we have

$$Q(0,0,\omega) = \frac{\sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k}{1 - \sum_{j=1}^{\infty} \left[\overline{B}_1^{(j-1)}(j) - \overline{B}_1^{(j)}(j)\right]\omega^j}$$

The above may be written as,

$$Q(0,0,\omega) = \frac{\sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k}{\sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k - \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k+1)\omega^{k+1}} \tag{B.3}$$

From equation (39) of [23] we have

$$\Gamma_1(\omega) = \frac{1}{\omega}\left(1 - \frac{1}{Q(0,0,\omega)}\right) \tag{B.4}$$

Substituting (B.3) in (B.4) gives

$$\Gamma_1(\omega) = \frac{\sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k+1)\omega^k}{\sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k} \tag{B.5}$$

Define

$$\Gamma_a(\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k+1)\omega^k \ , \quad \Gamma_b(\omega) = \sum_{k=0}^{\infty} \overline{B}_1^{(k)}(k)\omega^k \tag{B.6}$$

Then from (B.5) we have

$$\Gamma_1(\omega) = \frac{\Gamma_a(\omega)}{\Gamma_b(\omega)} \tag{B.7}$$

Because the derivation of (39) in [23] does not depend on the arrival process of the queue, the relation in (B.4) applies to any multiplexer in the tandem network, which is

$$\Gamma_n(\omega) = \frac{1}{\omega}\left(1 - \frac{1}{Q(0_n, 0_n, \omega)}\right) \tag{B.8}$$

Appendix C

Simplification of $Q(1,1,z_2,y_2,\omega)$ Given in (3.22)

In this appendix, $Q(1,1,z_2,y_2,\omega)$ given in (3.22) will be simplified. Let us write (3.22) as the following,

$$Q(1,1,z_2,y_2,\omega) = A_0 + A_1 + A_2 + A_3 \tag{C.1}$$

where

$$A_0 = 1 + Q_1(1,1,z_2,y_2)\omega + \sum_{k=1}^{\infty} B_2(k+1)\omega^{k+1} + \frac{1-z_2}{z_2}\sum_{k=1}^{\infty} B_2(k+1)\beta_1^{m_1}\omega^{k+1} \tag{C.2}$$

$$A_1 = \frac{1-z_2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_2(k+1)(b_{k-j} + a_{k-j}\beta_1^{m_1})\omega^{k+1} \tag{C.3}$$

$$\begin{aligned} A_2 = &-\frac{(z_2-1)^2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2} B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{k+1} \\ &+\frac{1-z_2}{z_2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j} a_{k-j-r_2} B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2} Q_{\ell_2}(0,0,0,0)\omega^{k+1} \end{aligned} \tag{C.4}$$

$$A_3 = (z_2-1)\sum_{k=1}^{\infty}\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(1,1,0,0)\omega^{k+1} - \frac{(z_2-1)^2}{z_2}\sum_{k=1}^{\infty}\sum_{j=1}^{k} B_2(j)Q_{k+1-j}(0,0,0,0)\omega^{k+1} \tag{C.5}$$

Next, the above terms will be determined one by one.

- Derivation of A_0

Substituting for $Q_1(1,1,z_2,y_2)$ in (C.1) from (2.29), we have,

$$A_0 = 1 + B_1(1)\omega + \sum_{k=1}^{\infty} B_2(k+1)\omega^{k+1} + \frac{1-z_2}{z_2}\sum_{k=1}^{\infty} B_2(k+1)\beta_1^{m_1}\omega^{k+1}$$

Combining the second and the third terms, we have,

$$A_0 = 1 + \sum_{k=1}^{\infty} B_2(k)\omega^k + \frac{1-z_2}{z_2}\beta_1^{m_1}\sum_{k=1}^{\infty} B_2(k+1)\omega^{k+1}$$

Expanding $B_2(k)$ and $B_2(k+1)$ using binomial theorem in (2.26), we get,

$$A_0 = 1 + \sum_{k=1}^{\infty}\sum_{i=0}^{m_2}\binom{m_2}{i}(C_{12}\lambda_{12}^{k})^{i}(C_{22}\lambda_{22}^{k})^{m_2-i}\omega^{k}$$
$$+\frac{1-z_2}{z_2}\beta_1^{m_1}\sum_{k=1}^{\infty}\sum_{i=0}^{m_2}\binom{m_2}{i}(C_{12}\lambda_{12}^{k+1})^{i}(C_{22}\lambda_{22}^{k+1})^{m_2-i}\omega^{k+1}$$

This results in the expression of A_0

$$A_0 = 1 + \sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^{i}(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}$$
$$+\frac{1-z_2}{z_2}\beta_1^{m_1}\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12}^{2})^{i}(C_{22}\lambda_{22}^{2})^{m_2-i}\omega^{2}}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega} \quad \text{(C.6)}$$

- Derivation of A_1

Exchanging the order of summations in A_1 gives,

$$A_1 = \frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}B_2(k+1)(b_{k-j}+a_{k-j}\beta_1^{m_1})\omega^{k+1}$$

Letting $n = k - j$, we have,

$$A_1 = \frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}B_2(n+j+1)(b_n+a_n\beta_1^{m_1})\omega^{n+j+1}$$

Expanding $B_2(n+j+1)$ by using (2.26), we have,

$$A_1 = \frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=0}^{m_2}\binom{m_2}{i}(C_{12}\lambda_{12}^{n+j+1})^{i}(C_{22}\lambda_{22}^{n+j+1})^{m_2-i}(b_n+a_n\beta_1^{m_1})\omega^{n+j+1}$$

The above may be written as,

$$A_1 = \frac{1-z_2}{z_2}\sum_{i=0}^{m_2}\sum_{n=1}^{\infty}\binom{m_2}{i}\frac{\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}(C_{12}\lambda_{12}^{n+1})^{i}(C_{22}\lambda_{22}^{n+1})^{m_2-i}(b_n+a_n\beta_1^{m_1})\omega^{n+1}$$

After some algebraic operation, we have,

$$A_1 = \frac{1-z_2}{z_2}\sum_{i=0}^{m_2}\sum_{n=1}^{\infty}\binom{m_2}{i}\frac{(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)^{2}}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}C_{12}^{i}C_{22}^{m_2-i}(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)^{n}(b_n+a_n\beta_1^{m_1})$$

The above may be expressed as,

$$A_1 = \frac{1-z_2}{z_2}\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{C_{12}^{i}C_{22}^{m_2-i}(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)^{2}}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}\left[b(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)-1+\beta_1^{m_1}a(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)-\beta_1^{m_1}\right] \quad \text{(C.7)}$$

where $a(\omega)=\sum_{n=0}^{\infty}a_n\omega^n$ and $b(\omega)=\sum_{n=0}^{\infty}b_n\omega^n$ (C.8)

Next, these above transforms will be determined. First, let us determine $a(\omega)$. Let us repeat (3.11) here,

$$a_j = \sum_{i=0}^{j-1} f_{j-i} a_i \text{ , } j \geq 1 \text{, and } a_0 = 1$$

then from (C.8) we have,

$$a(\omega) = 1 + \sum_{j=1}^{\infty} a_j \omega^j = 1 + \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} f_{j-i} a_i \omega^j$$

Exchanging the order of summations,

$$a(\omega) = 1 + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} f_{j-i} a_i \omega^j$$

Letting $n = j - i$, we have,

$$a(\omega) = 1 + \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} f_n a_i \omega^{n+i} = 1 + a(\omega) \sum_{n=1}^{\infty} f_n \omega^n$$

Therefore,

$$a(\omega) = \frac{1}{1 - \sum_{n=1}^{\infty} f_n \omega^n} \tag{C.9}$$

From (3.5, 3.6), we have

$$f_n = \frac{1}{z_2} \overline{B}_1^{(n-1)}(n) - \overline{B}_1^{(n)}(n) \tag{C.10}$$

Substituting the above in (C.9) gives us,

$$a(\omega) = \frac{1}{1 - \sum_{n=1}^{\infty} \frac{1}{z_2} \overline{B}_1^{(n-1)}(n) \omega^n + \sum_{n=1}^{\infty} \overline{B}_1^{(n)}(n) \omega^n} = \frac{1}{1 - \sum_{n=0}^{\infty} \frac{1}{z_2} \overline{B}_1^{(n)}(n+1) \omega^{n+1} + \sum_{n=1}^{\infty} \overline{B}_1^{(n)}(n) \omega^n}$$

From (B.6) in Appendix B, we have,

$$a(\omega) = \frac{1}{1 - \frac{1}{z_2} \Gamma_a(\omega) \omega + \Gamma_b(\omega) - 1} = \frac{z_2}{z_2 \Gamma_b(\omega) - \omega \Gamma_a(\omega)} = \frac{z_2 / \Gamma_b(\omega)}{z_2 - \omega \Gamma_a(\omega) / \Gamma_b(\omega)}$$

From (B.7) in Appendix B, we have,

$$a(\omega) = \frac{z_2 / \Gamma_b(\omega)}{z_2 - \omega \Gamma_1(\omega)} \tag{C.11}$$

Next, the expression $b(\omega)$ will be determined. Again, (3.12) is repeated here,

$$b_j = g_j b_0 + \sum_{i=1}^{j-1} f_{j-i} b_i \text{ , } j \geq 1 \text{, and } b_0 = 1$$

Then from (C.8) we have,

$$b(\omega) = 1 + \sum_{j=1}^{\infty} b_j \omega^j = 1 + \sum_{j=1}^{\infty} g_j \omega^j + \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} f_{j-i} b_i \omega^j$$

Exchanging the order of summations,

$$b(\omega) = 1 + \sum_{j=1}^{\infty} g_j \omega^j + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{j-i} b_i \omega^j$$

Letting $n = j - i$ and substituting for g_j from (3.5), we have,

$$\begin{aligned} b(\omega) &= 1 + \sum_{j=1}^{\infty} \overline{B}_1^{(j)}(j+1)\omega^j + \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} f_n b_i \omega^{n+i} \\ &= 1 + \sum_{j=1}^{\infty} \overline{B}_1^{(j)}(j+1)\omega^j + \left(b(\omega) - 1\right)\sum_{n=1}^{\infty} f_n \omega^n \end{aligned}$$

Therefore,

$$b(\omega) = \frac{1 + \sum_{j=1}^{\infty} \overline{B}_1^{(j)}(j+1)\omega^j - \sum_{n=1}^{\infty} f_n \omega^n}{1 - \sum_{n=1}^{\infty} f_n \omega^n}$$

Substituting (C.9) in the above equation gives,

$$b(\omega) = \frac{1 + \sum_{j=1}^{\infty} \overline{B}_1^{(j)}(j+1)\omega^j - \frac{1}{z_2}\sum_{n=0}^{\infty} \overline{B}_1^{(n)}(n+1)\omega^{n+1} + \sum_{n=1}^{\infty} \overline{B}_1^{(n)}(n)\omega^n}{1 - \frac{1}{z_2}\sum_{n=0}^{\infty} \overline{B}_1^{(n)}(n+1)\omega^{n+1} + \sum_{n=1}^{\infty} \overline{B}_1^{(n)}(n)\omega^n}$$

From (B.6) in Appendix B, we have,

$$b(\omega) = \frac{1 + \Gamma_a(\omega) - \beta_1^{m_1} - \frac{1}{z_2}\Gamma_a(\omega)\omega + \Gamma_b(\omega) - 1}{1 - \frac{1}{z_2}\Gamma_a(\omega)\omega + \Gamma_b(\omega) - 1}$$

The above may be written as,

$$b(\omega) = \frac{z_2\Gamma_a(\omega) - z_2\beta_1^{m_1} - \Gamma_a(\omega)\omega + z_2\Gamma_b(\omega)}{-\Gamma_a(\omega)\omega + z_2\Gamma_b(\omega)}$$

Dividing both the denominator and numerator by $\Gamma_b(\omega)$ in the above equation gives,

$$b(\omega)=\frac{(z_2-\omega)\Gamma_a(\omega)/\Gamma_b(\omega)-z_2\beta_1^{m_1}/\Gamma_b(\omega)+z_2\,\Gamma_b(\omega)/\Gamma_b(\omega)}{z_2\,\Gamma_b(\omega)/\Gamma_b(\omega)-\omega\Gamma_a(\omega)/\Gamma_b(\omega)}$$

From (B.7) in Appendix B, we have,

$$b(\omega)=\frac{(z_2-\omega)\Gamma_1(\omega)-z_2\beta_1^{m_1}/\Gamma_b(\omega)+z_2}{z_2-\omega\Gamma_1(\omega)} \tag{C.12}$$

Substituting (C.11, C.12) back into (C.7) gives,

$$A_1=\frac{1-z_2}{z_2}\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12}^2)^i(C_{22}\lambda_{22}^2)^{m_2-i}\omega^2}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\left[\frac{z_2\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}-\beta_1^{m_1}\right] \tag{C.13}$$

- Derivation of A_2

Exchanging the order of summations for A_2 in (C.4), we have,

$$A_2=-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j}a_{k-j-r_2}B_2(k+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{k+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\sum_{\ell_2=1}^{k-j}\sum_{r_2=\ell_2}^{k-j}a_{k-j-r_2}B_2(k+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\omega^{k+1}$$

Letting $n=k-j$ in the above equation, we get,

$$A_2=-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell_2=1}^{n}\sum_{r_2=\ell_2}^{n}a_{n-r_2}B_2(n+j+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{n+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell_2=1}^{n}\sum_{r_2=\ell_2}^{n}a_{n-r_2}B_2(n+j+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\omega^{n+j+1}$$

Exchanging the order of summations in the above equation again, we have,

$$A_2=-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{n=\ell_2}^{\infty}\sum_{r_2=\ell_2}^{n}a_{n-r_2}B_2(n+j+1-\ell_2)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{n+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{n=\ell_2}^{\infty}\sum_{r_2=\ell_2}^{n}a_{n-r_2}B_2(n+j+1-\ell_2)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\omega^{n+j+1}$$

Letting $i=n-\ell_2$ in the above equation, we get,

$$A_2=-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{i=0}^{\infty}\sum_{r_2=\ell_2}^{i+\ell_2}a_{i+\ell_2-r_2}B_2(i+j+1)\overline{B}_1^{(r_2+1-\ell_2)}(r_2+2-\ell_2)Q_{\ell_2}(1,1,0,0)\omega^{i+\ell_2+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{i=0}^{\infty}\sum_{r_2=\ell_2}^{i+\ell_2}a_{i+\ell_2-r_2}B_2(i+j+1)\Re_{r_2+1-\ell_2}Q_{\ell_2}(0,0,0,0)\omega^{i+\ell_2+j+1}$$

Letting $k = r_2 - \ell_2$ in the above equation, we get,

$$A_2 = -\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i} a_{i-k}B_2(i+j+1)\overline{B}_1^{(k+1)}(k+2)Q_{\ell_2}(1,1,0,0)\omega^{i+\ell_2+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i} a_{i-k}B_2(i+j+1)\Re_{k+1}Q_{\ell_2}(0,0,0,0)\omega^{i+\ell_2+j+1}$$

Exchanging the order of summations, we have,

$$A_2 = -\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty} a_{i-k}B_2(i+j+1)\overline{B}_1^{(k+1)}(k+2)Q_{\ell_2}(1,1,0,0)\omega^{i+\ell_2+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty} a_{i-k}B_2(i+j+1)\Re_{k+1}Q_{\ell_2}(0,0,0,0)\omega^{i+\ell_2+j+1}$$

Letting $h = i - k$ in the above equation, we get,

$$A_2 = -\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} a_h B_2(h+k+j+1)\overline{B}_1^{(k+1)}(k+2)Q_{\ell_2}(1,1,0,0)\omega^{h+k+\ell_2+j+1}$$
$$+\frac{1-z_2}{z_2}\sum_{j=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} a_h B_2(h+k+j+1)\Re_{k+1}Q_{\ell_2}(0,0,0,0)\omega^{h+k+\ell_2+j+1}$$

From (3.23, 3.24), the above equation may be written as,

$$A_2 = -\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} a_h B_2(h+k+j+1)\overline{B}_1^{(k+1)}(k+2)\omega^{h+k+j+1}$$
$$+\frac{1-z_2}{z_2}[Q(0,0,0,0,\omega)-1]\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} a_h B_2(h+k+j+1)\Re_{k+1}\omega^{h+k+j+1}$$

Expanding $B_2(h+k+j+1)$ by using (2.26) in the above equation, we get,

$$A_2 = -\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\binom{m_2}{i}\left(C_{12}\lambda_{12}^{h+k+j+1}\right)^i\left(C_{22}\lambda_{22}^{h+k+j+1}\right)^{m_2-i}$$
$$\cdot a_h\overline{B}_1^{(k+1)}(k+2)\omega^{h+k+j+1}$$
$$+\frac{1-z_2}{z_2}[Q(0,0,0,0,\omega)-1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\binom{m_2}{i}\left(C_{12}\lambda_{12}^{h+k+j+1}\right)^i\left(C_{22}\lambda_{22}^{h+k+j+1}\right)^{m_2-i} a_h\Re_{k+1}\omega^{h+k+j+1}$$

$$A_2 = -\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\binom{m_2}{i}C_{12}^{i}C_{22}^{m_2-i}\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{h}a_h\cdot\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{j} \cdot\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k+1}\overline{B}_1^{(k+1)}(k+2)$$
$$+\frac{1-z_2}{z_2}[Q(0,0,0,0,\omega)-1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\binom{m_2}{i}C_{12}^{i}C_{22}^{m_2-i}\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{h}a_h\cdot\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{j} \cdot\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k+1}\Re_{k+1} \quad \text{(C.14)}$$

Because of,

$$\sum_{k=0}^{\infty}\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k+1}\overline{B}_1^{(k+1)}(k+2) = \sum_{k=1}^{\infty}\overline{B}_1^{(k)}(k+1)\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k}$$

From (B.6) in Appendix B we have,

$$\sum_{k=0}^{\infty}\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k+1}\overline{B}_1^{(k+1)}(k+2) = \sum_{k=0}^{\infty}\left(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega\right)^{k}\overline{B}_1^{(k)}(k+1) - \overline{B}_1^{(0)}(1)$$
$$= \Gamma_a(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega) - \beta_1^{m_1}$$

Substituting the above result in (C.14) and noting (C.8) gives us,

$$A_2 = -\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}C_{12}^{i}C_{22}^{m_2-i}\cdot a(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)\cdot\frac{\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega} \cdot[\Gamma_a(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)-\beta_1^{m_1}]$$
$$+\frac{1-z_2}{z_2}[Q(0,0,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}C_{12}^{i}C_{22}^{m_2-i}\cdot a(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega)\cdot\frac{\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega}{1-\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega} \cdot\Re(\lambda_{12}^{i}\lambda_{22}^{m_2-i}\omega) \quad \text{(C.15)}$$

where $\Re(\omega) = \sum_{k=0}^{\infty}\Re_{k+1}\omega^{k+1}$ (C.16)

Next let us determine $\Re(\omega)$.

From (3.5, 3.6 and 3.8), we have,

$$\Re_{k+1} = (z_2-1)f_{k+1}$$
$$\Re_{k+1} = (z_2-1)(\frac{1}{z_2}g_k - g'_{k+1})$$
$$\Re_{k+1} = (z_2-1)\left[\frac{1}{z_2}\overline{B}_1^{(k)}(k+1) - \overline{B}_1^{(k+1)}(k+1)\right]$$

Substituting the above result in (C.15), we have,

$$\Re(\omega)=\frac{z_2-1}{z_2}\sum_{k=0}^{\infty}\overline{B}_1^{(k)}(k+1)\omega^{k+1}-(z_2-1)\sum_{k=0}^{\infty}\overline{B}_1^{(k+1)}(k+1)\omega^{k+1}$$

From (B.6) in Appendix B,

$$\Re(\omega)=\frac{z_2-1}{z_2}\omega\Gamma_a(\omega)-(z_2-1)[\Gamma_b(\omega)-1] \tag{C.17}$$

Let us substitute (C.11, C.17) in (C.15), then we obtain,

$$\begin{aligned}A_2=&-\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\&\cdot\frac{z_2\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)-\beta_1^{m_1}z_2/\Gamma_b(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}\\&-\frac{(z_2-1)^2}{z_2}[Q(0,0,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\\&\cdot\frac{\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)-z_2+z_2/\Gamma_b(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}\end{aligned} \tag{C.18}$$

The relation between $Q(1,1,0,0,\omega)-1$ and $Q(0,0,0,0,\omega)-1$ may be found through the following:

Substituting $z_1=y_1=0$ in (3.1), we have,

$$Q_k(0,0,0,0)=\beta_1^{m_1}Q_k(1,1,0,0)\quad,\quad k\geq 1 \tag{C.19}$$

If $k=0$, from the zero-initial assumption, we have $Q_0(0,0,0,0)=1=Q_0(1,1,,0,0)$.

From the above results and noting (3.23, 3.24), it is easy to find,

$$Q(0,0,0,0,\omega)-1=\beta_1^{m_1}[Q(1,1,0,0,\omega)-1] \tag{C.20}$$

Substituting (C.20) in (C.18) and canceling out the identical terms, we have,

$$\begin{aligned}A_2=&-\frac{(z_2-1)^2}{z_2}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega z_2\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]}\\&+\frac{(z_2-1)^2}{z_2}\beta_1^{m_1}[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}\end{aligned} \tag{C.21}$$

- Derivation of A_3

Exchanging the order of summations for A_3 in (C.5), we have,

$$A_3=(z_2-1)\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}B_2(j)Q_{k+1-j}(1,1,0,0)\omega^{k+1}-\frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}B_2(j)Q_{k+1-j}(0,0,0,0)\omega^{k+1}$$

Letting $n = k - j$, we have,

$$A_3 = (z_2 - 1)\sum_{j=1}^{\infty}\sum_{n=0}^{\infty} B_2(j) Q_{n+1}(1,1,0,0)\omega^{n+j+1} - \frac{(z_2-1)^2}{z_2}\sum_{j=1}^{\infty}\sum_{n=0}^{\infty} B_2(j) Q_{n+1}(0,0,0,0)\omega^{n+j+1}$$

From (3.23, 3.24) we have,

$$A_3 = (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{j=1}^{\infty} B_2(j)\omega^j - \frac{(z_2-1)^2}{z_2}[Q(\omega) - 1]\sum_{j=1}^{\infty} B_2(j)\omega^j$$

Expanding $B_2(j)$ from (2.26),

$$A_3 = (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\binom{m_2}{i}(C_{12}\lambda_{12}^j)^i (C_{22}\lambda_{22}^j)^{m_2-i}\omega^j$$
$$-\frac{(z_2-1)^2}{z_2}[Q(0,0,0,0,\omega) - 1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\binom{m_2}{i}(C_{12}\lambda_{12}^j)^i (C_{22}\lambda_{22}^j)^{m_2-i}\omega^j$$

$$A_3 = (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\binom{m_2}{i}C_{12}^i C_{22}^{m_2-i}(\lambda_{12}^i \lambda_{22}^{m_2-i}\omega)^j$$
$$-\frac{(z_2-1)^2}{z_2}[Q(0,0,0,0,\omega) - 1]\sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\binom{m_2}{i}C_{12}^i C_{22}^{m_2-i}(\lambda_{12}^i \lambda_{22}^{m_2-i}\omega)^j$$

$$A_3 = (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i (C_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i \lambda_{22}^{m_2-i}\omega}$$
$$-\frac{(z_2-1)^2}{z_2}[Q(0,0,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i (C_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i \lambda_{22}^{m_2-i}\omega}$$

From (C.20), the above equation may be written as,

$$A_3 = (z_2 - 1)[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i (C_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i \lambda_{22}^{m_2-i}\omega}$$
$$-\frac{(z_2-1)^2}{z_2}\beta_1^{m_1}[Q(1,1,0,0,\omega) - 1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i (C_{22}\lambda_{22})^{m_2-i}\omega}{1 - \lambda_{12}^i \lambda_{22}^{m_2-i}\omega} \quad \text{(C.22)}$$

Adding A_0, A_1, A_2, A_3 together from (C.6, C.13, C.21, C.22), we finally obtain,

$$
\begin{aligned}
&Q(1,1,z_2,y_2,\omega) \\
&= 1+\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega} \\
&\quad -(z_2-1)\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\
&\quad -(z_2-1)^2[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)}{(1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)[z_2-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega\Gamma_1(\lambda_{12}^i\lambda_{22}^{m_2-i}\omega)]} \\
&\quad +(z_2-1)[Q(1,1,0,0,\omega)-1]\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(C_{12}\lambda_{12})^i(C_{22}\lambda_{22})^{m_2-i}\omega}{1-\lambda_{12}^i\lambda_{22}^{m_2-i}\omega}
\end{aligned}
\tag{C.23}
$$

Appendix D

Simplification of $Q(1_{n-1}, 1_{n-1}, z_n, y_n, \omega)$ Given in (5.5)

In this appendix, the simplification of transform $Q(1_{n-1}, 1_{n-1}, z_n, y_n, \omega)$ given in (5.5) is presented. The special case of this transform is also needed in section 4.1.2 and 4.2.1. First, (5.5) is repeated here,

$$
\begin{aligned}
&Q(1_{n-1}, 1_{n-1}, z_n, y_n, \omega) \\
&= 1 + \sum_{k=1}^{\infty} B_n(k)\omega^k + (z_n - 1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^k \\
&- (z_n - 1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j)B_n(k)\omega^k \qquad \text{(D.1)} \\
&- (z_n - 1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^k \\
&- \frac{(z_n - 1)^2}{z_n}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(0_{n-1}, 0_{n-1}, 0_n, 0_n)\omega^k
\end{aligned}
$$

In the above equation, let us define A_0 - A_4 to correspond to the separate parts of the RHS, thus,

$$A_0 = 1 + \sum_{k=1}^{\infty} B_n(k)\omega^k \qquad \text{(D.2)}$$

$$A_1 = (z_n - 1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^k \qquad \text{(D.3)}$$

$$A_2 = -(z_n - 1)\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum_{r=1}^{k-j}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j)B_n(k)\omega^k \qquad \text{(D.4)}$$

$$A_3 = -(z_n - 1)^2\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j-h+1)B_n(k-h)Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^k \qquad \text{(D.5)}$$

$$A_4 = -\frac{(z_n - 1)^2}{z_n}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} B_n(j) Q_{k-j}(0_{n-1}, 0_{n-1}, 0_n, 0_n)\omega^k \qquad \text{(D.6)}$$

Next, each of the above expressions will be determined.

- Derivation of A_0

Expanding $B_n(k)$ in A_0 using Binomial theorem from (2.26), we have,

$$A_0 = 1 + \sum_{k=1}^{\infty} B_n(k)\omega^k$$

$$= 1 + \sum_{i=0}^{m_n}\sum_{k=1}^{\infty}\binom{m_n}{i}(C_{1n}\lambda_{1n}^k)^i (C_{2n}\lambda_{2n}^k)^{m_n-i}\omega^k \tag{D.7}$$

$$= 1 + \sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i (C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}$$

- Derivative of A_1

A_1 in (D. 3) may be written as

$$A_1 = (z_n - 1)\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} B_n(j)Q_{k-j}(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^k$$

Letting $h = k - j$

$$A_1 = (z_n - 1)\sum_{j=1}^{\infty}\sum_{h=1}^{\infty} B_n(j)Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n)\omega^{h+j}$$

$$A_1 = (z_n - 1)[Q_h(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1]\sum_{j=1}^{\infty} B_n(j)\omega^j$$

where $Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) = \sum_{k=0}^{\infty} Q_k(1_2, 1_2, 0_3, 0_3)\omega^k$

Substituting for $B_n(j)$ from (2.26),

$$A_1 = (z_n - 1)[Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1]\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\binom{m_n}{i}(C_{1n}\lambda_{1n}^j)^i (C_{2n}\lambda_{2n}^j)^{m_n-i}\omega^j$$

$$A_1 = (z_n - 1)[Q(1_{n-1}, 1_{n-1}, 0_n, 0_n, \omega) - 1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i (C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \tag{D.8}$$

- Derivation of A_2

A_2 in (D.4) may be written as,

$$A_2 = -(z_n - 1)\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\sum_{r=1}^{k-j}\frac{1}{z_n^r}\varphi_r^{(n-1)}(k-j)B_n(k)\omega^k$$

Letting $\ell = k - j$

$$A_2 = -(z_n - 1)\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell)B_n(\ell + j)\omega^{\ell+j}$$

Substituting for $B_n(\ell + j)$ from (2.26)

$$A_2 = -(z_n - 1)\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell}\binom{m_n}{i}(C_{1n}\lambda_{1n}^{\ell+j})^i(C_{2n}\lambda_{2n}^{\ell+j})^{m_n-i}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell)\omega^{\ell+j}$$

From equation (A.1)

$$A_2 = -(z_n - 1)\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\binom{m_n}{i}(C_{1n}\lambda_{1n}^{\ell+j})^i(C_{2n}\lambda_{2n}^{\ell+j})^{m_n-i}\varphi^{(n-1)}(\ell)\omega^{\ell+j}$$

From equation (A.3)

$$A_2 = -(z_n - 1)\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\binom{m_n}{i}(C_{1n}\lambda_{1n}^{j})^i(C_{2n}\lambda_{2n}^{j})^{m_n-i}\omega^j\Phi^{(1)}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)$$

From equation (A.5) finally we have,

$$A_2 = -(z_n - 1)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n - \lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]} \qquad \text{(D.9)}$$

- Derivation of A_3

Exchanging the order of summations in A_3 in (D.5), we have,

$$\sum_{k=3}^{\infty}\sum_{j=1}^{k-2}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1} \Rightarrow \sum_{j=1}^{\infty}\sum_{k=j+2}^{\infty}\sum_{h=1}^{k-j-1}\sum_{r=1}^{k-j-h+1}$$

Letting $\ell = k - j$, we have,

$$A_3 = -(z_n - 1)^2\sum_{j=1}^{\infty}\sum_{\ell=2}^{\infty}\sum_{h=1}^{\ell-1}\sum_{r=1}^{\ell-h+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell - h + 1)B_n(\ell + j - h)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{\ell+j}$$

Exchanging the order of summations, we have,

$$\sum_{\ell=2}^{\infty}\sum_{h=1}^{\ell-1} \Rightarrow \sum_{h=1}^{\infty}\sum_{\ell=h+1}^{\infty}$$

Letting $m = \ell - h$, we have,

$$A_3 = -(z_n - 1)^2\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{m+1}\frac{1}{z_n^r}\varphi_r^{(n-1)}(m+1)B_n(m + j)Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{m+h+j}$$

Because $\sum_{m=1}^{\infty}\sum_{r=1}^{m+1} \Rightarrow \sum_{m=0}^{\infty}\sum_{r=1}^{m+1} - \sum_{m=0}^{0}\sum_{r=1}^{1}$

$$A_3 = -(z_n - 1)^2 \left\{ \sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=1}^{m+1} \frac{1}{z_n^r} \varphi_r^{(n-1)}(m+1) B_n(m+j) Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{m+h+j} \right.$$
$$\left. - \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \frac{1}{z_n} \varphi_1^{(n-1)}(1) B_n(j) Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{h+j} \right\}$$

Letting $\ell = m+1$ and substituting for $B_n(m+j)$ from (2.26)

$$A_3 = -(z_n - 1)^2 \left\{ - \sum_{j=1}^{\infty}\sum_{h=1}^{\infty} \frac{1}{z_n} \varphi_1^{(n-1)}(1) B_n(j) Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{h+j} \right.$$
$$\left. + \sum_{i=0}^{m_2}\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell} \binom{m_n}{i} \frac{1}{z_n^r} \varphi_r^{(n-1)}(\ell)(C_{1n}\lambda_{1n}^{\ell-1+j})^i (C_{2n}\lambda_{2n}^{\ell-1+j})^{m_2-i} Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{\ell-1+h+j} \right\}$$

Next, $\varphi_1^{(n-1)}(1)Q_h(1_{n-1},1_{n-1},0_n,0_n)$ will be expressed in terms of $Q_h(0_{n-1},0_{n-1},0_n,0_n)$.

From the definition of $\varphi_1^{(n-1)}(1)$ in Appendix A, we have,

$\varphi_1^{(n-1)}(1)$ =Prob (multiplexer-($n-1$) has a single busy period during one slot),

Then from the definition of a busy period in Appendix A, we have,

$\varphi_1^{(n-1)}(1)$ = Prob (multiplexer-($n-1$) has a zero duration period),

But this is equivalent to,

$\varphi_1^{(n-1)}(1)$ = Prob (multiplexer-($n-1$) has two consecutive idle slots)

As explained before, $Q_h(1_{n-1},1_{n-1},0_n,0_n)$ is the probability that the n'th multiplexer is idle at h'th slot; but given $Q_h(1_{n-1},1_{n-1},0_n,0_n)$ it is known that the (n-1)'st multiplexer must be idle at (h-1)'st slot and the n'th multiplexer is idle at h'th slot. Thus from the preceding explanation, it is easy to conclude that $\varphi_1^{(n-1)}(1)Q_h(1_{n-1},1_{n-1},0_n,0_n)$ is the probability that both the (n-1)'st and n'th multiplexers are idle at h'th slot, which is $Q_h(0_{n-1},0_{n-1},0_n,0_n)$.

Then we have,

$$\varphi_1^{(n-1)}(1)Q_h(1_{n-1},1_{n-1},0_n,0_n) = Q_h(0_{n-1},0_{n-1},0_n,0_n) \qquad \text{(D.10)}$$

Substituting (D.10) in the above A_3 gives,

$$A_3 = -(z_n - 1)^2 \left\{ -\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\frac{1}{z_n}B_n(j)Q_h(0_{n-1},0_{n-1},0_n,0_n)\omega^{h+j} \right.$$

$$\left. + \sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\sum_{h=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell}\binom{m_n}{i}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell)(C_{1n}\lambda_{1n}^{\ell-1+j})^i(C_{2n}\lambda_{2n}^{\ell-1+j})^{m_n-i}Q_h(1_{n-1},1_{n-1},0_n,0_n)\omega^{\ell-1+h+j} \right\}$$

Substituting from (5.4) in the above,

$$A_3 = \frac{(z_n-1)^2}{z_2}[Q(0_{n-1},0_{n-1},0_n,0_n,\omega)-1]\sum_{j=1}^{\infty}B_n(j)\omega^j$$

$$-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell}\binom{m_n}{i}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell)(C_{1n}\lambda_{1n}^{\ell-1+j})^i(C_{2n}\lambda_{2n}^{\ell-1+j})^{m_n-i}\omega^{\ell-1+j}$$

From equation (A.1), and substituting for $B_n(j)$ from (2.26)

$$A_3 = \frac{(z_n-1)^2}{z_n}[Q(0_{n-1},0_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}$$

$$-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell}\binom{m_n}{i}\frac{1}{z_n^r}\varphi_r^{(n-1)}(\ell)(C_{1n}\lambda_{1n}^{\ell-1+j})^i(C_{2n}\lambda_{2n}^{\ell-1+j})^{m_n-i}\omega^{\ell-1+j}$$

From equation (A.3)

$$A_3 = \frac{(z_n-1)^2}{z_n}[Q(0_{n-1},0_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}$$

$$-(z_2-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\sum_{j=1}^{\infty}\binom{m_n}{i}(C_{1n}\lambda_{1n}^{j-1})^i(C_{2n}\lambda_{2n}^{j-1})^{m_n-i}\Phi^{(n-1)}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)\omega^{j-1}$$

From equation (A.5)

$$A_3 = \frac{(z_n-1)^2}{z_n}[Q(0_{n-1},0_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega}$$

$$-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{1n}\lambda_{1n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]}$$

(D.11)

- Derivation of A_4

Following the same steps as in derivation of A_1, , we have,

$$A_4 = -\frac{(z_n-1)^2}{z_n}[Q(0_{n-1},0_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \quad \text{(D.12)}$$

Finally, substituting $A_0 - A_4$ in (D.1) and canceling out the two identical terms, we have,

$$
\begin{aligned}
&Q(1_{n-1},1_{n-1},z_n,y_n,\omega) \\
&=1+\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \\
&\quad+(z_n-1)[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega}{1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega} \\
&\quad-(z_n-1)\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]} \\
&\quad-(z_n-1)^2[Q(1_{n-1},1_{n-1},0_n,0_n,\omega)-1]\sum_{i=0}^{m_n}\binom{m_n}{i}\frac{(C_{1n}\lambda_{1n})^i(C_{2n}\lambda_{2n})^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)}{(1-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)[z_n-\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega\Gamma_{n-1}(\lambda_{1n}^i\lambda_{2n}^{m_n-i}\omega)]}
\end{aligned}
\tag{D.13}
$$

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