APOS Analysis of Students' Understanding of Logarithms

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Abstract

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This research is aimed at analyzing students' understanding of the concept of logarithms. Traditionally students are introduced to logarithmic functions as the inverse of exponential functions, even though logarithms were invented independently of exponents.

In this study, a review of the historical genesis of logarithms, the literature review on the understanding of exponential and logarithmic functions, and a review of several textbooks from different times and geographical places are presented. These reviews allow the identification of two notions of logarithms: arithmetic and functional. Even though current curriculum introduces the functional notion of logarithms, attained curriculum focuses on the arithmetic notion of logarithms. In addition, the importance of logarithms, that is, their use in converting multiplication of numbers into addition of logarithms of numbers, has not been highlighted.

I conducted six clinical interviews with students from prerequisite mathematics courses and a core mathematics course. This data is analyzed within the context of APOS theory to identify students' difficulties in understanding the concept of logarithms. The results reveal that most students' understanding of the arithmetic notion of logarithms does not go beyond the "process" level, since understanding the arithmetic notion of logarithms as undoing what exponentiation does require an understanding of exponentiation with real exponents.

Acknowledgments

My sincere thanks to Dr. Sierpinska for her tremendous help and support. I could not really complete my thesis without her ideas and support. I also want to thank Dr. Gooya for providing Iranian textbooks which broadened my outlook on teaching logarithms. Finally, I greatly appreciate Dr. Hillel's and Dr. Byers' time in reviewing my thesis and providing me with comments.

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Chapter 1: Introduction

My interest in studying the teaching and learning of logarithms originated in my teaching experience in the Math Help Center at Concordia University. As I was helping students with prerequisite mathematics courses, I became aware of the difficulties that students had in understanding the concept of logarithm and decided to study the nature and reasons of these difficulties.

I assumed that students' understanding of logarithms depends on their prior knowledge, the way they were taught the concept and the mental operations they engage in when working on problems involving logarithms. Students' beliefs about what it means to do mathematics may also have an impact on their understanding.

Today's students are introduced to logarithms as functions that are inverses of exponential functions. This is quite different from the way logarithms were first perceived in the history of mathematics. John Napier (1550-1617), the inventor of logarithms, was motivated to reduce the labor of multiplication and division of huge numbers by converting these operations into addition and subtraction of logarithms of these numbers. Logarithms as a computational device were used in the teaching of mathematics until 1960s. As a basis of the slide rule, they also served the computational needs of engineers and businessmen. Today, scientific calculators have eliminated this use of logarithms. However, students are required to learn logarithms because of their applications in sciences (e.g., biology, chemistry (e.g. pH scale), earth sciences (e.g. the Richter scale), acoustics (the decibel scale)), computer science (especially information theory), finance (models of capital growth) and industry. A common application of logarithms in any domain is the logarithmic scale in graphing relationships. A semi-log plot can be used to graph and manage a wide range of values when it is impossible to show all actual values on one graph. Where $x \geq 1$, and $a > 0$, a log-log plot can be employed to convert the equation $y = ax^b$ into a linear equation $\log y = \log a + b \log x$, with slope b and intercept $\log a$. The linear equation allows more values of the function to be represented in a single graph (see Figure 1). This application of logarithms has not been stressed in academic mathematics teaching, and thus students are often unprepared for interpreting logarithmic scales in the workplace (Williams and Wake 2007). In a case study, Williams et al. (2001) found that interpreting the results of a semi-log plot in the context of an experiment can be problematic for students.

The advancement of technology made the computational and graphing uses of logarithms less important and this may have had an impact on the place of logarithms in school mathematics and in mathematics education research. A brief survey of papers published in the journal *Mathematics Teacher* between 1924 and 2006 revealed a peak of interest in logarithms in the decades of 1960-1970 and 1980-1990. Between 1960 and 1970 the focus was on the notion of the inverse function and between 1980 and 1990 most authors focused on applications of the exponential and logarithmic functions. These foci of interest reflect the prevailing reform movements of the times: the New Math reforms of the 1960s focused on the abstract and formal foundational concepts of mathematics such as sets, relations and functions, and the 1980s which tried to restore the meaning of school mathematics through stressing its applications. In the New Math approach, logarithms were merely an example of a general concept of function and an illustration of the notion of inverse function.

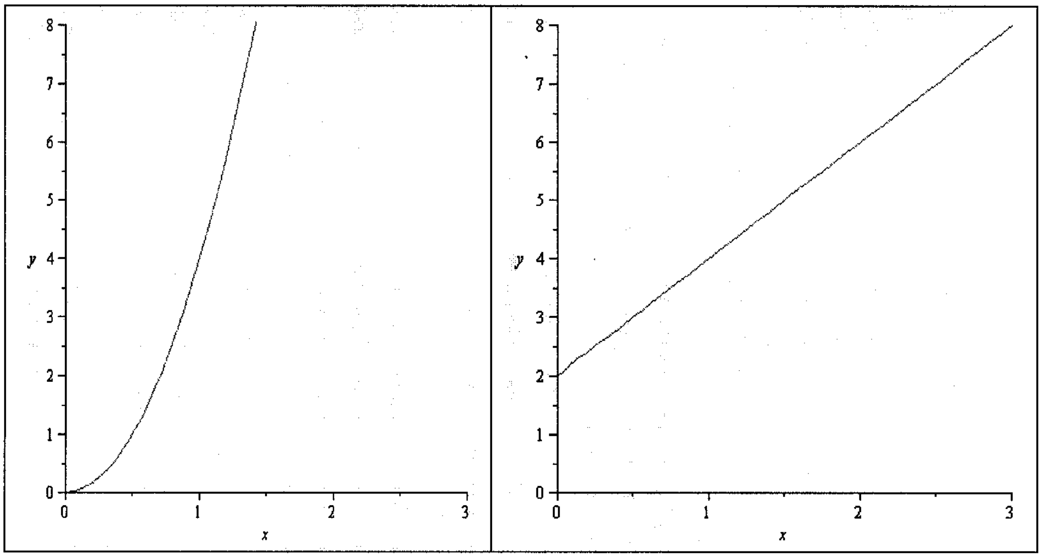

Figure 1. The left-hand side graph represents the function $y = 4x^2$ in the window x=0..3, y=0..8. Only the values of the function for x up to $\sqrt{2}$ could be shown. The right-hand side graph represents the function $y' = 2 + 2x'$ in the same window. If the numbers on the axes in the right-hand side graph are interpreted as logarithms in base 2 of numbers x or y, and y' and x' are interpreted as $y' = \log_2 y$ and $x' = \log_2 x$, then in such log-log scales, the graph represents the function $y = 4x^2$ for values of x between 1 and 8.

Although the New Math approaches have been abandoned, the notion of logarithm as a computational tool has not returned to school curricula because technological advances made this tool obsolete. Students are still introduced to logarithmic functions as inverse of exponential functions, because the notion of function has become the main concept of secondary school mathematics; it is only now introduced less formally than in New Math and the focus is on the idea of functions as models of relationships between variable quantities. But, to cope with this notion of logarithm, the student needs to understand exponentiation, the concept of function, exponential functions, and the notion of the inverse of a function. Explanation of each of these concepts is based on the use of other concepts such as exponent, variable, operation, and so forth, which further complicates the process of understanding the notion of logarithm.

According to Weber (2002), who analyzed understanding of exponentiation using Action-Process-Object-Schema (APOS) framework, students construct a meaningful understanding of concepts of exponents and logarithms when they understand exponentiation as a mathematical *process* and exponential expressions as mathematical *objects*. Therefore, understanding the logarithmic function at even the modest process level presumes almost the most sophisticated understanding of the exponential function. Obviously, this approach to logarithms requires understanding the notion of function, and Dubinsky and Harel's research on how students develop an understanding of the concept of function (1992) shows that this is a very demanding concept as well. Since students are introduced to different classes of functions by their defining formulas, the majority identify functions with algebraic expressions, where numbers can be substituted for variables to obtain other numbers. This understanding does not support well the functional notion of logarithms which involves understanding the general notion of the inverse of a function. Students learn to find inverses of functions through a mechanical procedure based on manipulation of algebraic expressions. This procedure does not apply to finding the inverse of the exponential function and students' understanding fails at this point.

There have been attempts at designing teaching to help students develop a deeper understanding of functions that would also be more naturally constructed by them based on their previous knowledge. In particular, Confrey and Smith (1995), have described the process of construction by students of a covariation conception of functions and, based on that, an understanding of exponential and logarithmic functions (details of this research will be presented in Chapter 3).

Still, I did not find Confrey & Smith's research helpful in guiding my students towards a better understanding of logarithms. Therefore I engaged in this study in the hope of finding some explanation of students' difficulties with logarithms for myself.

I looked at how logarithms have been invented in the history of mathematics, how they then have been taught in different times and geographical places (19^{th} century Iran and present day Iran, where I have learned how to compute the logarithm of a number, using logarithmic tables in 1990, and present day Canada) based on a study of textbooks, and how they have been learned by a few Canadian students who volunteered to be interviewed about their understanding of logarithms. This thesis gives an account of this research.

This thesis is organized in 7 chapters, including this Introduction. Chapter 2 is a review of the historical genesis of logarithms from Napier's until Euler's time. Chapter 3 reviews approaches to logarithms in a few textbooks. Chapter 4 summarizes three relevant studies on students' understanding of exponential and logarithmic functions conducted by Kastberg (2002), Weber (2002), and Confrey and Smith (1995). Chapter 5 presents the methodology of my empirical study of students' understanding of logarithms. The results of my study are presented in chapter 6. I discuss both the results of students' understanding from my empirical research and the theoretical analysis of relevant literature and textbooks to propose some recommendations for teaching logarithms in chapter 7.

Chapter 2: The Meaning of Logarithms in the History of Mathematics

This chapter summarizes the historical invention of logarithms and reveals how the arithmetic notion of logarithms (a tool which allows to calculate products by means of addition and radicals by means of division), invented by John Napier, was changed to the functional notion in the 18th century. The chapter reviews the early history of logarithms, and then describes the computational and theoretical aspects of Napier's discovery. Improvement to Napier's work, and the invention of the slide rule to visualize and compute logarithms up to the fourth decimal place, are explained. Then the functional notion of logarithms is described.

2.1 Early History of Logarithms

Historians trace the history of logarithms back to the Babylonian clay tablets (2000 to 600 B.C). According to archaeologists, these tablets include table texts and problem texts. One such table text consist of the first ten successive powers for numbers: 3, 4, 10 and 15 in base sixty which was the Babylonian number base (Boyer 1968). Besides these earliest known examples of exponential tables, a text problem which can be interpreted as "to what power must a certain number be raised in order to yield a given number?" shows that the first appearance of logarithms was not very far from exponential operations.

Other evidence of the use of exponents was revealed in the work of Archimedes (287-212 B.C) when he tried to estimate the number of grains of sand in the universe. He used the term "orders" as exponents in an exponential expression with base one hundred million, and noticed that the addition of "orders" corresponds to the product of the terms, which is known as the first law of exponents (Boyer 1968; Cooke 2005).

2.2 Before Napier

Seventeen hundred years after the previously mentioned events, Nicolas Chuquet (1445-1500) worked on algebraic concepts and exponentiation. He may have been the first mathematician to recognize zero and negative numbers as exponents. He listed the first 20 powers of 2 and pointed out that when two such numbers are multiplied, their indices (powers) are added. This shows that he had a clear idea of the laws of integer exponents (Cooke 2005). By this time, Michael Stifel (1487-1567) was doing research on arithmetic and algebra. He worked on negative exponents and extended the table for powers of 2 from 0 to 20 to include $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$, $2^{-3} = \frac{1}{8}$. However, the notations which he used for those negative powers in his book, *Arithmetica Integra*, were abbreviations of German words (Boyer 1968). He compared arithmetic and geometric progressions and noticed a relationship between the terms of a geometric progression and their corresponding exponents.

2.3 The John Napier Era

The work by Stifel (1487-1567) shows that he considered logarithms as the inverse of exponents and his focus was only on the integer exponents. Napier's idea (1550-1617) was to extend this relationship to real numbers and fill the large gaps between the terms of the geometric progression. His motivation was to reduce the labor of calculation and he was introduced to the prosthaphaeresis method employed by astronomers. Prosthaphaeresis, from the Greek word meaning "addition and subtraction", is the method of computation based on trigonometric identities, which were used since trigonometric tables accurate to fifteen decimal places existed (Boyer 1968). The prosthaphaeresis method which existed in the sixteenth century utilized the trigonometric identities, the product of two trigonometric expressions such as $\sin a \times \cos b$, to calculate the product of numbers. For example, to find the product of 4226×9396 by the prosthaphaeresis method, the trigonometric identity $\sin a \times \cos b = \dfrac{\sin(a+b)+\sin(a-b)}{2}$ is used. The two given numbers are replaced by $\sin a = 0.4226$ and $\cos b = 0.9396$, placing the decimal point to accommodate the range of sine and cosine values. Then $a = 25^{\circ}$ and $b = 20^{\circ}$; substituting these angles in the trigonometric formula, we will have $0.4226 \times 0.9396 = \dfrac{\sin(45^{\circ})+\sin(5^{\circ})}{2} \approx 0.3970$ which by multiplying 0.3970 by 10^8, $4226 \times 9396 \approx 39{,}700{,}000$. This method made the multiplication operation easier and less error-prone. Napier's knowledge of the relationship between the terms of arithmetic and geometric progressions, and the prosthaphaeresis method triggered his efforts to develop a computational aid. His work consists of two parts: computational and theoretical. The theoretical part which is based on a continuous geometric model was published two years after his death in 1619, under the title of *Mirifici Logarithmorum Canonis Constructio*. The computational part titled *Mirifici Logarithmorum Canonis Descriptio*, published in 1614, was based on a discrete approximation of the continuous model (Cooke 2005).

2.4 The Computational Aspects of Napier's Work

In the computational part, Napier considered a unit divided into 10^7 parts, then subtracted from the unit its 10^7th part, $(1-1/10^7 = 0.9999999)$, to get a number close to

1 and small enough to be a base so its powers would grow slowly (Maor 1994, Pierce 1977). The powers of this number $(1-1/10^7 = 0.9999999)$ generate terms of a geometric sequence in which the gap between the terms is diminished. The first several terms are:

$(0.9999999)^0 = 1$, $(0.9999999)^1 = 0.9999999$, $(0.9999999)^2 = 0.99999981$, $(0.9999999)^3 = 0.99999971$, ...

These examples illustrate the slow decrease of the progression. Further, he multiplied each power by 10^7 to avoid decimals, and he considered $N = 10^7(1-1/10^7)^l$, where l is Napier's logarithm of the number *N*. However, as illustrated by $(1-1/10^7)^{10^7} \approx 1/e$, Napier had no concept of a base for a system of logarithms. He filled his first table starting with 10^7 and iterations of $10^7(1-1/10^7)^l$ consisted of hundred one entries. Some entries were:

$10^7 = 10{,}000{,}000$, $10^7(1-1/10^7) = 9{,}999{,}999$, $10^7(1-1/10^7)^2 = 9{,}999{,}998$ .... $10^7(1-1/10^7)^{100} = 9{,}999{,}900$

In the second table, he started with 10^7 and considered the ratio of the last term to the first term of the first table ($9999900/10000000 = 0.99999$) as proportional to the numbers of the second table. Several terms were:

$10^7 = 10{,}000{,}000$, $10^7(1-1/10^5) = 9{,}999{,}900$, $10^7(1-1/10^5)^2 = 9{,}999{,}800$, ... $10^7(1-1/10^5)^{50} = 9{,}995{,}001$

It contained fifty-one entries. A third table contained twenty-one rows and sixty-nine columns. He started the first column with 10^7 and considered the proportion of the last term to the fist term of the second table ($9995001/10000000 = 0.9995001$). A few terms from first column were:

$10^7 = 10{,}000{,}000$, $\quad 10^7(1-5/10^4) = 9{,}995{,}000$, $\quad 10^7(1-5/10^4)^2 = 9990002.5$, $\ldots 10^7(1-5/10^4)^{20} = 9900473.5$.

Then from each entry in this first column he created sixty-eight additional entries using the proportion of the last entry to the first entry of the first column ($9900473/10000000 \approx .99$). A few entries from the first and second columns follow (for more details, see Appendix A):

$$\begin{array}{ll}
10^7 = 10{,}000{,}000 & 10^7(1-1/10^2) = 9900000 \\
10^7(1-5/10^4) = 9{,}995{,}000 & 9995000(1-1/10^2) = 9{,}895{,}050 \\
10^7(1-5/10^4)^2 = 9990002.5 & 9990002.5(1-1/10^2) = 9890102 \\
\vdots & \vdots \\
10^7(1-5/10^4)^{20} = 9900473.5 & 9900473.5(1-1/10^2) = 9801468.8
\end{array}$$

2.5 The Theoretical Aspects of Napier's Work

The theoretical part is based on a geometric model which is explained by considering two moving points, P and O, along two straight lines (see Figure 2.1). P starts from A and moves geometrically along AB with decreasing velocity in proportion to its distance from B. Point O moves arithmetically along a second line CD at a constant velocity generating a number line. These two points start at the same time and begin moving at the same speed. However, point P slows down and takes infinitely longer to reach B, and at B its velocity would be zero. Point O continues to move at a constant speed (Cooke 2005). Suppose P is at the distance y from B at some instant in time t, while point O reaches the position x from C. If x is the Napierean logarithm of y then

$x = Nap\log(y)$. While point P moves to a new position, the division between the two positions in the geometric model is mirrored by a subtraction in the arithmetic model in the corresponding positions; thus, the diagram changes division into subtraction (Cooke 2005).

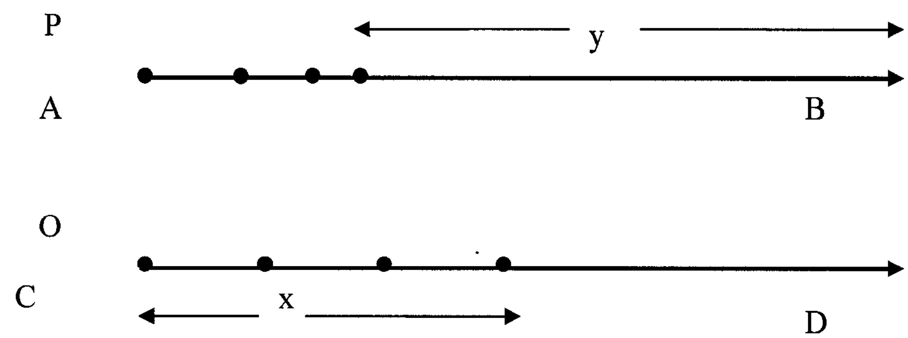

Figure 2.1. Presents a model of Napier's work

2.6 After Napier

The improvements added by Henry Briggs (1561-1630) after Napier's death, such that logarithm of 1 is 0 and the logarithm of 10 is 1, are the basic ideas for what we now call common logarithms. As a consequence, logarithms to base 10 came to be known as Briggsian Logarithms. In 1624, Briggs published a table of logarithms to base 10 for all integers from 1 to 20,000 and from 90,000 to 100,000, with an accuracy of fourteen decimal places, under the title *Arithmetica Logarithmica.* The gap from 20,000 to 90,000 was filled by Vlacq (1600-1667) to an accuracy of ten decimal places and added to the second edition of the *Arithmetica Logarithmica* in 1628.

At almost the same time that Napier invented logarithms, Burgi (1552-1632) constructed a table of logarithms with a number slightly greater than one ($1+10^{-4}$) and instead of multiplying powers of this number by 10^7, he multiplied by 10^8. However, Napier published his work prior to Burgi (Boyer 1968).

2.7 Slide Rule Invention

Shortly after the invention of logarithms, the need to visualize the logarithms of any given number without looking it up in the tables inspired Gunter (1620) to draw a two foot long line with the whole numbers spaced at intervals proportional to their respective logarithmic values (Cajori 1908). Figure 2.2. shows his effort.

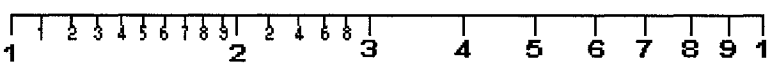

Figure 2.2. Gunter's design to visualize logarithms of numbers

In 1632, Oughtred combined two of Gunter's rules to make a device that is known as the slide rule. He placed these two rules opposite each other and showed that by sliding them back and forth, one can do calculations, the result of which is correct up to the fourth decimal place. Figure 2.3. shows one design of a slide rule.

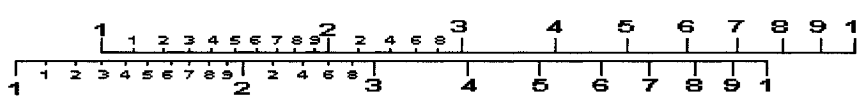

Figure2.3. a design of a slide rule

Over the decades different types of slide rules were designed and were in use before the pocket calculator.

Oughtred also stated the logarithmic laws: $\log a + \log b = \log ab$, $\log a - \log b = \log \frac{a}{b}$, and $\log a^m = m \log a$ in word form rather than as formulas in his publication in 1652 (Cajori 1913). In 1618, an anonymous appendix (probably written by William Oughtred) in the reprinted translated edition of *Mirifici Logarithmorum Canonis Descriptio* contained the table of natural logarithms; Napier's logarithms were not natural logarithms (Mitchell & Strain 1936). The development of natural logarithms was delayed for a few years since Briggs introduced common logarithms in 1624. Common logarithms were used to simplify the calculations. However, the natural logarithms' distinctiveness in that they can be defined by a simple integral of a parabola or a series makes the use of natural logarithms more practical. Furthermore, to model exponential growth and decay problems, base e is used more often than base 10.

2.8 Graphical Representation of Logarithmic Functions

At the beginning of the 17th century, a functional relation between physical quantities became apparent and, as a consequence, the analytical representation of a function which introduces the formula of the function emerged (Youschkevitch, 1976). Fifteen to twenty years after the invention of logarithms, Descartes (1596-1650) began applying algebraic methods of representation to geometry, and gradually the graphical representation of function developed. During the 17th century, the theoretical viewpoint of logarithms was extended by the graphical representation, both in rectangular and polar coordinates (Cajori, 1913). The first inventor of the logarithmic curve has not been ascertained, but it was found in *Geometriae pars universalis* by James Gregory in 1667. In addition, hyperbola $xy = 1$ was published by Gregory St. Vincent in 1647 (Cajori, 1913).

Mercator expanded the hyperbola $y=\frac{1}{1+x}$ into infinite series, $\frac{1}{1+x}=1-x+x^2-x^3+...$, and then integrated it $\int_0^x \frac{dx}{1+x}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....$ He published his work in *Logarithmotechnia* in 1668 (Boyer 1968). He did not write down the logarithmic series $x-\frac{x^2}{2}+\frac{x^3}{3}-...=\ln(1+x)$; instead he calculated the numerical value of the first few terms of the series by taking different values for x such as $x=0.1$, $x=0.21$ (Cajori 1913). He called the obtained results natural logarithms (Boyer 1968) and used the work of St. Vincent and Gregory to relate the result to logarithms (Cajori 1913).

2.9 Defining Logarithms Using Integration

In 1676, Gottfried W. Leibniz (1646-1716) derived the integral $\int\frac{dx}{x}$ and concluded that it is a logarithmic curve (Cajori 1913). Leibniz pondered on the logarithm of a negative number and argued that negative numbers do not have real logarithms, despite Jean Bernoulli's (1667-1748) belief that $\log(-n)=\log(+n)$ (Boyer, 1968). However, Leonhard Euler (1707-1783), a student of Bernoulli, used the formula $e^{i\theta}=\cos\theta+i\sin\theta$ and showed that logarithms of negative numbers are not real numbers. He considered $\theta=\pi$, and therefore $e^{i\pi}=-1$ and $i\pi=\ln(-1)$ (Boyer 1968). After all these efforts, Euler is the one who defined logarithm in terms of exponent $y=\log_b x \Leftrightarrow x=b^y$ such that $b>0, b\neq 1$ (Freudenthal 1973).

Logarithms as computational aids to simplify multiplication and division of huge numbers by converting these operations to addition and subtraction were used until the late 1970s (Maor 1994), and the slide rule was manufactured for the last time in 1975 in United States (see Figure 2.4.)

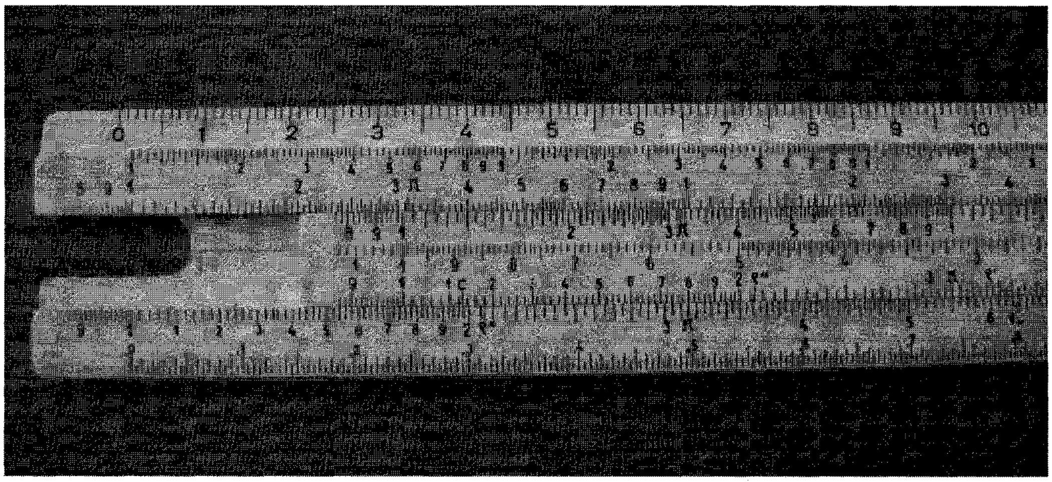

Figure 2.4. A slide rule produced in the 1960s in Europe

A five-inch pocket portable slide rule made the calculation easy; however, in the early 1970s the electronic calculator replaced the slide rule. The HP-35, the first scientific calculator with logarithmic and trigonometric functions, was introduced in 1972 (Waits & Demana 2000). It was the world's first electronic slide rule and it terminated the use of mechanical slide rule in calculation. With the development of powerful electronic calculators, logarithmic tables disappeared from school textbooks; and since the 1970s logarithms have been taught as the inverse of exponential functions (Maor, 1994).

Chapter 3: The Meaning of Logarithms in the Teaching of Mathematics

In this chapter, I review a few textbooks to analyze teaching approaches of logarithms in different times and geographical places. Two of the textbooks are used in prerequisite mathematics courses offered at a large urban North American university. The rest are the current Iranian high school curriculum textbooks, some Iranian textbooks from the 1940s and 1970s, and application problems from an Iranian textbook published in 1895.

I also included a review of Freudenthal's theory (1973), the founder of Realistic Mathematics Education (RME), which is used to determine a global perspective on teaching logarithms. Freudenthal's theory (1973) of how logarithms should be taught was used as a model in reviewing textbooks and evaluating the actual teaching approaches to logarithms.

3.1 Logarithms in Present Canadian Textbooks

3.1.1 College Algebra 4th edition, by Stewart, Redlin and Watson

College Algebra (Stewart et at., 2004) is used as a textbook for a prerequisite mathematics course offered at a large urban North American university. The prerequisite course, which we label here "course A", is offered for students who lack a subject requirement for taking a first calculus course labeled as "course B".

College Algebra, 4th edition, is organized in 10 chapters and was published in 2004. Chapter 3 discusses the concept of function, graphs of functions, behavior of a function in its domain, one-to-one functions and the notion of the inverse of a function.

The concept of function is defined as "a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B" (p.215). The different types of functions such as polynomial and rational functions are introduced prior to exponential and logarithmic functions.

Chapter 5 is structured in 5 sections whose aim is to introduce exponential and logarithmic functions. Section 2 recalls the concept of the inverse function and applies a horizontal line test to $f(x) = a^x$, $a > 1$ to conclude that $f(x) = a^x$ is a one-to-one function and that it has the inverse function. Yet, a highlighted definition of the logarithmic function does not explicitly mention the notion of inverse function:

> Let a be a positive number with $a \neq 1$. The logarithmic function with base a, denoted by $\log_a$ is defined by $\log_a x = y \Leftrightarrow a^y = x$ (Stewart et al., 2004, p.398).

A further explanation of logarithms is given: "$\log_a x$ is the exponent to which the base a must be raised to give x" (p. 398). The above two definitions introduce the "functional" and the "arithmetic" notion of logarithms, respectively. The functional notion of logarithms introduces the logarithmic function and involves the notion of function and characteristics of a function. In contrast, the arithmetic notion of logarithms introduces logarithms as undoing exponentiations. The arithmetic notion of logarithms encompasses the computational notion of logarithms in which the arithmetic notion of logarithms is employed as a computational tool.

Follow examples of evaluating logarithms of given numbers by converting the logarithmic form into the corresponding exponential form. Knowledge of natural, negative and fractional numbers as exponents is required to understand the examples.

There is no example of evaluating logarithms of positive irrational numbers to extend the domain of a logarithmic function to positive real numbers.

Properties of inverse functions are employed to infer properties of logarithms. For example, consider $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, where f: $R \to R^+$ and $f^1 : R^+ \to R$. Therefore,

a) $f^{-1}(f(x)) = x$ which implies $\log_a a^x = x$ for $x \in R$

b) $f(f^{-1}(x)) = x$ which implies $a^{\log_a x} = x$ for $x > 0$

A student can apply the definition of the composition of functions in part a) and obtain $f^{-1}(f(x)) = f^{-1}(a^x) = \log_a(a^x)$, but an algebraic manipulation does not apply to determine the equality $\log_a a^x = x$. At this point, an understanding of logarithms, as undoing what exponentiation does is required (i.e. $\log_a a^x$ can be interpreted as the logarithm in base a undoes the exponentiation of a to power x, resulting in $\log_a a^x = x$).

As well, in part b) to show that the equality $a^{\log_a x} = x$ holds, $a^{\log_a x}$ can be interpreted as: exponentiation in base a undoes the logarithm of x in the base a, resulting in $a^{\log_a x} = x$.

Therefore, a mechanical procedure based of applying algebraic manipulation to find a composition of functions fails in the context of exponential and logarithmic functions. When an understanding of a mathematical concept does not take place in their mind, students try to memorize the properties of the concept and consequently they will not recall them correctly after a while.

A graphical representation of the logarithmic function $y = \log_a x$ is introduced by reflecting the graph of $y = a^x, a > 1$ about line $y = x$, followed by identifying the asymptote of the graph. Graphs of different logarithmic functions such as $\log_2 x$, or $\log_{10}(x-3)$ are represented respectively by point plotting or transforming the graph of the initial function $\log_{10} x$ (i.e. horizontal shifting the graph of $\log_{10} x$ to the right 3 units presents the graph of $f(x) = \log_{10}(x-3)$). The point plotting procedure can be more applicable in plotting common logarithmic and natural logarithmic functions, since the different values of these functions can be computed by a calculator.

An application of a common logarithmic function in evaluating the subjective intensity of the stimulus, $s = k\log(\frac{I}{I_0})$ in which k is a constant, I is the physical intensity and I_0 is the threshold of physical intensity, follows the common logarithms topics.

The property of natural logarithm, $\ln e = 1$, may cause a misunderstanding of ln as a number. Since students are introduced to the natural number e , ln can be interpreted as a reciprocal of e rather than a function. $\ln e = 1$ can be understood in a way that ln undoes what e does on 1. When students are presented ready-made properties they generally try to memorize them rather than reason and reinvent the properties. For example, $e^{\ln x} = x$ and $\ln e^x = x$ which are extracted from the inverse properties of the natural logarithmic and natural exponential functions, need a justification and explanation since they cannot be determined by algebraic manipulations.

Section 3 of chapter 5 introduces the product, quotient and power laws of logarithms. The proof of each law is derived from the law of exponents. Examples on expanding and combining logarithmic expressions, as well as warning examples about employing incorrect laws of logarithms such as $\frac{\log 6}{\log 2} \neq \log(\frac{6}{2})$, $(\log_2 x)^3 \neq 3\log_2 x$, and $\log_a(x+y) \neq \log_a x + \log_a y$ are presented.

The change of base law, $\log_b x = \frac{\log_a x}{\log_a b}$, is introduced to convert the logarithm of numbers from one base to another base. The proof of the change of base law and of a particular formula, $\log_b a = \frac{1}{\log_a b}$, which is derived by substituting $x = a$ in the change of base of law, are presented. The importance of the change of base law becomes apparent as it allows the calculation of logarithms in any base by converting logarithms into common or natural logarithms and then employing calculators.

Section 4 of chapter 5 presents different examples of solving logarithmic and exponential equations. In some of the examples a graphical solution is presented along with an algebraic solution. The section includes compound interest problems to illustrate the application of exponential functions in modeling, and computational aspects of logarithms in finding the growth time of an investment.

Section 5 of chapter 5 emphasizes the convenience of logarithmic scales in managing a large range of values. The logarithmic scales such as pH, Richter, and decibel scales are explained and formulas to measure acidity of a solution, the intensity of an earthquake and the loudness of a sound are presented.

The chapter ends with extra modeling examples of exponential or power functions titled, *Focus on Modeling, Fitting Exponential and Power Curves to Data* (p.445-457). To distinguish which model fits the data points (x, y), logarithms are used to linearize the data. If the data fit an exponential function $y = ce^{kx}$, the function is linearized as $\ln y = \ln c + kx$, therefore data points $(x, \ln y)$ will lie on a straight line. Otherwise, when data fit a power function $y = ax^n$, and the function is linearized as $\ln y = \ln a + n \ln x$, the data points $(\ln x, \ln y)$ will lie on a straight line.

College Algebra includes a wide variety of resources to enhance different instructional approaches, but it does not provide a ground for reinventing and exploring the mathematical concepts. The extra modeling problems are not considered in the curriculum of course A and they may not even be noticed by students or highlighted by the teacher. Reviewing a final examination of course A shows that the attained curriculum focuses on computational aspects of logarithms.

3.1.2 Calculus for Business, Economics, Life Sciences, and Social Sciences 10th edition, by Barnett, Zingler, and Byleen, 2005

This textbook is a popular textbook for the prerequisite Calculus course offered in the same university. The prerequisite course, "course B", is offered for students who want to obtain admission into undergraduate programs such as Business, Finance, and Life Sciences. The textbook is organized into 9 chapters. The first two chapters include a quick review of the concept of function and different types of functions to prepare students for calculus. A function is introduced as "a rule that produces a correspondence between two sets of elements such that to each element in the first set there corresponds one and only one element in the second set" (p.7). Different types of function such as polynomial, rational, exponential and logarithmic functions are presented in chapter 2.

Section 3 of chapter 2, *Logarithmic Functions*, begins by defining one-to-one functions and the inverse of a function. As an example, the exponential function $y = 2^x$

is considered to describe how to find a logarithmic function. The inverse of $y = 2^x$ is found by interchanging the variables $x = 2^y$, and stating "$x = 2^y$ if and only if $y = \log_2 x$" (p.111). A mechanical procedure in finding an inverse function (e.g. to find an inverse function of $f(x) = 2x$, students learn to interchange x and y, and solve the equation for y in terms x to obtain the inverse function $f^{-1}(x) = \frac{x}{2}$) does not apply to finding the inverse of the exponential function. Therefore, stating "$x = 2^y$ if and only if $y = \log_2 x$" may confuse students. After this example, the definition of a logarithmic function follows as:

> The inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$, logarithmic form $y = \log_b x$ is equivalent to exponential form $x = b^y$. The log to the base b of x is the exponent to which b must be raised to obtain x (p.111).

The above definition includes both functional and arithmetic notions of logarithms. However, the examples on conversion of exponential form to logarithmic form and vice versa, indicate that the intended curriculum of introducing the concept of logarithms in this section is centered on the arithmetic notion of logarithms. Even though the functional notion of logarithms is introduced, there is no example discussing the behavior of a logarithmic function, domain and range of a logarithmic function and modeling problems with logarithmic functions.

Further, the properties of logarithms such as $\log_b 1 = 0$ and $\log_b b = 1$ which can be inferred from the definition of logarithms are listed along the laws of logarithms in theorem 1 (p.113):

If b, M and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1) $\log_b 1 = 0$
2) $\log_b b = 1$
3) $\log_b b^x = x$
4) $b^{\log_b x} = x, x > 0$
5) $\log_b MN = \log_b M + \log_b N$
6) $\log_b \frac{M}{N} = \log_b M - \log_b N$
7) $\log_b M^p = p \log_b M$
8) $\log_b M = \log_b N \Leftrightarrow M = N$

Theorem 1 lists the properties of logarithms without justifying them. Only the proof of the property 5 is presented and the rest of properties are practiced through examples. It is noteworthy that there exist *Explore & Discuss* activities to discuss the relationship between $\log_b M + \log_b N$ with $\log_b (M + N)$ and $\log_b MN$. However, the allotted class time may not permit students to participate in discussions and explorations. Since it is assumed that students have been introduced to logarithms in a pre-calculus course, only two class sessions (each equal to 75 minutes) are devoted to review exponential functions, logarithmic functions and applications of exponential functions in modeling compound interest problems. The arithmetic notion of logarithms is employed in finding the time of investment in compound interest problems. For example, a problem, "If interest is 5% compounded continuously, how long will it take for money invested to double?", taken from one of a final examination in course B, shows that the attained curriculum focuses on the computational notion of logarithms.

The arithmetic notion of logarithms is highlighted by employing a calculator to compute common logarithms and natural logarithms of numbers under the title of *Calculator Evaluation of Logarithms*. A calculator is used to find the natural logarithm or common logarithm of a given number and vice versa. Finding the number of which a common logarithm or natural logarithm is given is not a trivial procedure like the former one. For example, to estimate the number whose natural logarithm is 3 ($\ln x = 3$), the logarithmic equation needs to be converted to the exponential equation ($e^3 = x$); then a calculator can be helpful.

A few examples are presented to show how to solve an exponential equation. For example, to solve $e^x = 3$, after taking the natural logarithm of both sides of the equation, the equation becomes $\ln e^x = \ln 3$. Of course, $\ln 3$ can be evaluated by a calculator but justifying that $\ln e^x$ is equal to x requires an understanding of the logarithm as undoing what exponentiation does ($\ln$ undoes what e does to x).

Furthermore, the textbook introduces derivatives of logarithmic functions in chapter 5. The concept of derivative is tied to the concept of function, therefore presentation of functional notion of logarithms is essential at this point. However, a review of final examinations in course B shows an emphasis on knowing the rules of differentiation of logarithms and exponential functions rather than on modeling problems with a logarithmic function and analyzing the behavior of the function. For example, "Find $\frac{dy}{dx}$ for given $y = [\ln(x+2) - e^{(x^2-4x)}]^2$" is a typical question to ask in a final examination in course B.

Calculus for Business, Economics, Life Sciences, and Social Sciences includes a quick review of the concept of logarithms in comparison to *College Algebra* which provides a more detailed instructional approach. Both of the textbooks present the functional notion of logarithms while the attained curriculum focuses on the computational aspects of logarithms. They both fail to provide students with guided lessons in which students can develop their own understanding of the concept of logarithms.

3.2 Logarithms in the Iranian Textbooks

The Iranian high school curriculum design varies by stream of studies, whether in Humanities, Natural Science or Math/Physical Science. Based on their Grade Point Average (GPA) from junior high school, students enroll in a senior high school or a vocational high school. The senior high school offers mostly programs in Humanities, Natural Science and Math/Physics. Natural Science and Math/Physics programs have the same curriculum in the first year of senior high school. In second year, students pursue their interest either in Natural Science or Math/Physics. Figure 3.1. illustrates the school system in Iran.

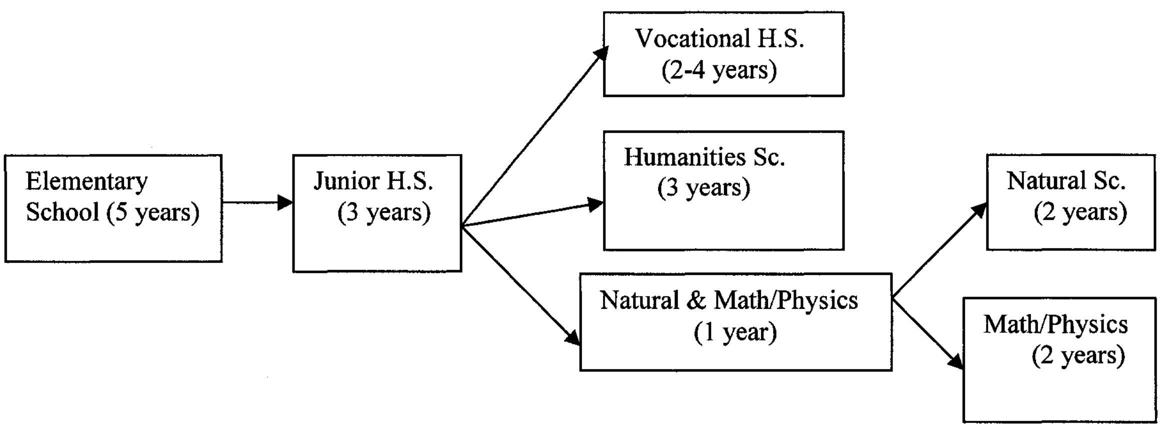

Figure 3.1. Illustration of the current Iranian schooling system

3.2.1 Mathematics 2, Ministry of Education, 2003

Mathematics 2 is a current high school curriculum textbook for second year students (around 15-16 years old) in Natural Science, Math/Physics and also in vocational high schools. *Mathematics 2* is organized in 8 chapters and totals 177 pages. It introduces a variety of topics such as relations and functions, matrices and systems of linear equations, graphs of functions, sequences and series, trigonometry, vectors, and combinations and probability. In this textbook, a function is defined "as a relation in which two distinct pairs of numbers (x, y) do not have same x-coordinate. If two pairs have equal x-coordinates then their y-coordinates need to be equal" (p.21). Topics such as onto functions, one-to-one functions and the inverse of a function are introduced after students learn how distinguish a function from a relation. Different types of functions such as trigonometric, ceiling, exponential and logarithmic functions are introduced in chapter 4.

Exponential and logarithmic functions are presented in section 3 of chapter 4. The section starts with the definition of positive and negative rational exponents. To extend the domain of exponents to real numbers, it poses the question "what does $2^{\sqrt{2}}$ mean?". To find $2^{\sqrt{2}}$, the graph of $y = 2^x$ is sketched by point plotting and $2^{\sqrt{2}}$ is estimated by interpolation. After students have constructed a meaning for real exponents, the definition of an exponential function, $y = a^x$, $a > 0, a \neq 1$ follows: "If x is a real number, a^x satisfies all of the laws of exponents. For example, $a^{x+y} = a^x . a^y$" (p.87). The graph of $y = a^x$ for $a > 1$ and $0 < a < 1$ follows the definition of the exponential function. Since the graphs indicate that the exponential function is an onto and one-to-one function, the logarithmic function $y = \log_a x$ is introduced as its inverse function. The relation between $y = a^x$ and $y = \log_a x$ is emphasized by sketching the graph of both functions in a unique coordinate system and indicating that the graph of $y = \log_a x$ is a reflection of $y = a^x$ about the line $y = x$; therefore $x = a^y \Leftrightarrow \log_a x = y$. A few examples are presented to show how to convert an exponential equation to a logarithmic equation and vise versa.

The product and quotient laws of logarithms are extracted from the laws of exponents and practiced through examples. The product law is verbalized as "logarithm of the product of numbers is the sum of the logarithms of the numbers" (p.89). The power law of logarithms, $\log_a b^x = x \log_a b$ for $a, b > 0, a \neq 1$, is mentioned as an obvious property of logarithms without proof and justification. It does not seem be an obvious property, unless it can be resulted from the product law. Furthermore, natural logarithms and common logarithms of numbers, and evaluating them with a calculator are discussed.

The lesson ends by emphasizing the importance of invention of logarithms in the history of mathematics to reduce the computational labor by converting multiplication of numbers to addition of logarithms of numbers and division operation of numbers to subtraction of logarithms of numbers.

Mathematics 2 illustrates a functional notion of logarithms within 4 pages, however most of the given examples and exercises are centered on the computational notion of logarithms. There is no application problem to be modeled with a logarithmic function or an exponential function. In contrast, abstract mathematics problems are designed particularly for students in Math/Physics program as follows:

1. If $y = \log_e x - \log_e (x+1), x > 0$ show that $x = \dfrac{e^y}{1 - e^y}$
2. Sketch the graph of $2\log y = \log 2 + \log(x+1)$

I translated the section from its original language and included in Appendix B to provide more details on the teaching logarithms in Iran.

3.2.2 Fundamentals of Mathematics, Ministry of Education, 2004

Fundamentals of Mathematics, the pre-university textbook for the Humanities stream, introduces computational aspects of logarithms. It has a total of 150 pages in 5 chapters. Chapters include topics on mathematical induction, sequences of numbers, logarithms, mathematical modeling, and basic probability. Chapter 2, the chapter which precedes the one on logarithms, presents a sequence of numbers and then introduces different types of sequences. Arithmetic and geometric sequences are employed in chapter 3 to reveal the relationship between corresponding terms of these sequences and define the arithmetic notion of logarithms. Chapter 3 in total includes 6 sections as follows: genesis of logarithms, common logarithms, logarithms and scientific notations, computation with logarithms, proof of properties of logarithms, and the application of logarithms in measuring the magnitude of an earthquake and intensity of a sound wave.

Section 1, *Genesis of Logarithms*, starts with modeling the growth rate of bacteria. The time of growth is modeled by an arithmetic sequence and the number of bacteria is modeled by a geometric sequence (see Table 3.1).

x :Time (day)	0	1	2	3	4	5	6	7	8	9	...
y :Number of Bacteria	1	2	4	8	16	32	64	128	256	512	...
y :Number of Bacteria	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Table 3.1. Sequences of numbers used to model the growth rate of bacteria

The table shows that the product of two terms of the geometric sequence such as 16 and 32 can be calculated by adding the indices of the terms, in this case, 4 and 5, respectively, which are terms of an arithmetic sequence, and then finding the corresponding term for 9 in the geometric sequence, i.e. 512 in this case. The third row of the table indicates that geometric terms can be written in base 2 to the power of corresponding arithmetic terms. Thus $y = 2^x \Leftrightarrow x = \log_2 y$, such that y is a number in geometric sequence and x is the logarithm of this number in base 2.

Another non-standard example of logarithms presented in this textbook is:

Numbers	16	$\times$	32	=	512
	$2\times2\times2\times2$	$\times$	$2\times2\times2\times2\times2$	=	$2\times2\times2\times2\times2\times2\times2\times2\times2$
Logarithms	4	+	5	=	9

This example reveals the most significant property of logarithms, converting multiplication of numbers to addition of the logarithms of the numbers.

After some practice exercises and examples, the textbook gives the following definition of logarithms: "$\log_b y$ is a number [representing the power] to which b needs to be raised in order to obtain y, so $\log_b y = x \Leftrightarrow b^x = y, y > 0, b > 0, b \neq 1$"(p.62).

Section 2, *Common Logarithms*, presents an algorithm to construct a table of logarithms in base 10 for numbers between 1 and 10. This table is employed to provide a rough estimate of the common logarithm of numbers. Section 3, *Logarithms and Scientific Notations*, shows that the integer part of the common logarithm of a number can be obtained easily from the scientific notation of the number and the decimal part can be estimated form the constructed table of logarithms for numbers from 1 to 10. For example to estimate the logarithm of 200, 200 can be expressed in scientific notation 2×10^2 and, by referring to the table, $2 \approx 10^{0.301}$ so $2\times10^2 \approx 10^{0.301}\times10^2$ and $\log 200 \approx 2.301$, in which the integer part of 2.301 is the power of 10 and the decimal part is found in the table. This textbook contains a rarely-seen-today common logarithmic table of numbers from 1 to 10000 up to four decimal digits.

Section 5 introduces the laws of logarithms. The product, division, and power laws of logarithms are proved and practiced through examples. Furthermore, the change of base of law of logarithms is introduced to facilitate finding logarithms in bases different from 10.

Applications of logarithms to measuring the magnitude of an earthquake and the intensity of a sound wave are presented in the last section. The Richter scale and decibel scale are explained and it is emphasized that logarithmic scales are more convenient scales in comparing a wide range of numbers.

In this textbook, the arithmetic notion of logarithms is presented through examples and modeling problems in a way which is compatible with the historical genesis of logarithms. For example, the presented bacteria growth rate problem establishes a connection between geometric terms and corresponding arithmetic terms to introduce the concept of logarithms. The table of common logarithms for numbers from 1 to 10 is constructed to provide students with a concrete computational knowledge of logarithms. Furthermore, the lessons do not overwhelm students with logarithmic symbols since the natural logarithm of numbers is not presented. I translated the chapter 3 of *Fundamentasl of Mathematics* and included in Appendix C to provide more details on the teaching logarithms in Iran.

3.2.3 Logarithms in several Iranian Textbooks from the 1940s and 1970s and Application Problems from 1895

A review of several Iranian textbooks from the 1940s and 1970s reveals that the arithmetic notion of logarithms is defined either by emphasizing the relationship between corresponding terms in arithmetic and geometric sequences to conclude that an arithmetic term is the logarithm of the corresponding geometric term, or by introducing logarithms as the inverse of exponentiation.

In the textbooks from the 1970s, logarithms are defined as the inverse of exponentiation. A high school curriculum textbook from 1972 ("Algebra",1972) defines logarithms as follows: "the number x , $x = \log_a A$, is called the logarithm of A to base a ($a > 0$), when $a^x = A$. A is called the antilogarithm of x to base a , $A = ant\log_a x$".

Pezeshk (1942), an editor of curriculum textbooks, defines logarithms as follows: Given two sequences A and G such that G is a geometric sequence with the initial term 1 and q as a common ratio, and A is an arithmetic sequence with the initial term 0 and r as a common difference. A and G form a system, $\begin{cases} G: ...q^{-n},....,q^{-2},q^{-1},1,q,q^2,...q^n,... \\ A: ..-rn,...,-2r,-r,0,r,2r,...,nr,... \end{cases}$, such that each term of the arithmetic sequence is the logarithm to base q^r of the corresponding term in the geometric sequence and each term of the geometric sequence can be obtained by raising $q^{\frac{1}{r}}$ to the corresponding term in the arithmetic sequence.

Pezeshk (1942) considers two consecutive terms of a geometric sequence, and respectively their corresponding terms of an arithmetic sequence to prove a theorem. The theorem shows that the difference between any two consecutive terms in each sequence approaches zero by inserting terms between these two sequential terms. Based on this theorem he concludes that a) any term of a geometric sequence has a corresponding term of an arithmetic term which is its logarithm and b) any positive number has a logarithm. Furthermore, he proves the uniqueness of the logarithm of a number in any given base. Further details are included in Appendix D.

In addition, a translation of some finance problems where logarithms are used in simplifying exponential expressions and solving the problems is included in Appendix E. These problems are taken from *Mizanoalhesab* written by Mir Krinish in 1895. One of the interesting problems and its solution follows:

A barrel is filled with syrup. The total volume of the barrel, a, is equal to the volume of 100 pitchers and a pitcher of this syrup costs $c = 36$ tomans. Let b be the volume of the pitcher. Suppose a person wants to reduce the concentration of syrup so that the cost of a pitcher of the diluted drink decreases to 1 toman. He dilutes the syrup with water by repeatedly taking out one pitcher of syrup from the barrel and pouring in one pitcher of water instead. He repeats this process until the value of each pitcher of syrup becomes $d = 1$ toman. Find out how many times this person needs to repeat this process to decrease the value of each pitcher to $d = 1$ toman.

Solution: It is clear that the ratio of the volume of syrup in the barrel to the volume of syrup in the pitcher is equal to the ratio of the volume fraction of the syrup in the barrel to the volume fraction of the syrup in the pitcher. By taking out the first pitcher of syrup and pouring in a pitcher of water, the volume of the syrup is $a-b$, so $\frac{a}{b} = \frac{a-b}{x}$ and $x = \frac{b}{a}(a-b)$ which x indicates the volume of the syrup in a pitcher, so by taking out the second pitcher from the barrel the volume of syrup in the barrel is

$(a-b)-\frac{b}{a}(a-b)=\frac{(a-b)^2}{a}$ and the equality of ratios is $\frac{a}{b}=\frac{\frac{(a-b)^2}{a}}{x}$ which $x=\frac{b(a-b)^2}{a^2}$. Repeating this process and taking out n pitchers, the volume of the syrup in the barrel becomes $\frac{(a-b)^n}{a^{n-1}}$ which is worth $c=36$ tomans per pitcher. Therefore, the value of syrup in the barrel is $\frac{(a-b)^n}{a^{n-1}}\times c$. On the other hand, the value of each pitcher of syrup is decreased to $d=1$, so ad indicates the value of diluted syrup in the barrel. By solving $\frac{(a-b)^n}{a^{n-1}}\times c=ad$, n indicates the number of times that the person needs to take out pitchers of syrup to decrease the value of each pitcher to $d=1$ toman, so

$$n\log(a-b)+\log c-(n-1)\log a=\log a+\log d$$

$$n=\frac{\log c-\log d}{\log a-\log(a-b)}=\frac{\log 36-\log 1}{\log 100-\log 99}=356$$

A review of current Iranian textbooks indicates two distinct approaches — functional and arithmetic— in teaching the concept of logarithms. The functional notion of logarithms is introduced to students who study Natural Science, Math/Physics or in vocational high schools. The arithmetic notion of logarithms is presented for students in Humanities programs. Students' fields of study influence the instructional approach of logarithms. Furthermore, the arithmetic notion of logarithms is presented through examples and modeling problems in a way which provides a grounding for students' reinvention and exploration of logarithmic concept.

3.3 Realistic Mathematics Education

Freudenthal's theory in teaching exponential and logarithmic functions

Freudenthal's idea is that logarithm and exponential functions need to be introduced to students through real-life problems (Freudenthal, 1973). Once they have been introduced, the activities which help students reinvent the exponential and logarithmic laws need to be employed. Students reinvent the mathematical concepts through a learning process that passes through a variety of sequential learning activities (Freudenthal, 1973). These activities compel the students to invent the mathematical concepts, to recognize the common aspects of the concepts, to formulate and then apply them in other situations. The teachers' role is to help students invent and formulate the concepts (Freudenthal, 1973). Freudenthal (1991) stresses that there is no need to teach the knowledge that students can invent by themselves (Freudenthal, 1999). Further, he explains a teaching approach to logarithms which does not use the term logarithm and logarithmic notation. This approach was applicable in 1970 because of existing logarithmic tables in textbooks. The logarithmic tables were used in a way that allowed students to write a number a in the form of 10^p. To compute ab, students use a logarithmic table and write $a = 10^p$ and $b = 10^q$. They multiply ab and apply the exponential law ($z^x \times z^y = z^{x+y}$), thus $ab = 10^{p+q}$. Then, they can compute the logarithmic value of 10^{p+q} from the logarithmic table (Freudenthal 1973). Although, this approach was limited to logarithms to base 10, it did not necessarily require learning the laws of logarithms. Rather the exponential laws were applied.

According to Freudenthal (1973, page 375), functions and mapping concepts arise intuitively through real-life based examples, such as interest as a function of time, temperature as a function of time, price as a function of quality. He argues for the need for mathematical activities to help students concretize the concept of mapping. From his point of view, concretizing a mapping is categorized in two ways: displacement and transfiguration (Freudenthal, 1973, page 383). In displacement, the focus is on the place of the objects not on the nature of the objects. The object x is displaced by rule f to fx. In contrast, in transfiguration the focus is on the objects. The object x is transfigured into fx by rule f.

To grasp the concept of mapping, students should be able to perform at least one method of concretization since a complete concretization of the mapping concept with respect to object and place is not easy to form (Freudenthal, 1973, page 384). However, logarithmic functions require a complete concretization, displacement and transfiguration. Displacement can explain the mapping of the geometrical sequence to the corresponding arithmetical sequence. Transfiguration happens in mapping x to $\log x$. For example, consider $f(x) = \log_3 x$ and $x = 9$, then $f(9) = \log_3 9 = \log_3 3^2 = 2$. This implies that 9 is transfigured to $\log_3 9$ and displaced to 2. Displacement concretization requires an understanding of logarithmic functions as the inverse of exponential functions. In the above example of displacement concretization, $\log_3 9$ needs to be interpreted as to what power 3 must be raised to be equal to 9 ($3^x = 9$). The development of both types of concretization of a logarithmic function might not happen simultaneously for all students.

Freudenthal (1973), the founder of Realistic Mathematics Education (RME), has a global perspective on teaching mathematical concepts. His idea is that mathematics education needs to be implemented through realistic context problems. Realistic context problems provide the opportunity for students to develop a basis for their informal knowledge, make connections between informal knowledge and formal mathematics and reinvent the mathematical concepts. The context of the problems is not necessarily limited to real situations, rather the situations provide grounds to experience mathematical concepts (Freudenthal, 1973 & Van den Heuvel-Panhuizen, 2003). Through their experiences, students recognize the common aspects of the concepts and organize their own mathematical activities. Organizing matters from reality or a mathematical perspective is called "mathematizing"(Freudenthal 1973, 1991). Mathematizing is the core aim of mathematics education (Van den Heuvel-Panhuizen, 2003). Two ways of mathematizing—horizontal mathematizing and vertical mathematizing—in an educational context are proposed by Treffers (Van den Heuvel-Panhuizen, 2003). Horizontal mathematizing refers to the mathematical tools to solve realistic context problems, while vertical mathematizing refers to the students' own mathematical activities, such as finding strategies, discovering concepts, and making connections between concepts (Van den Heuvel-Panhuizen, 2003). Students' mathematical knowledge develops through the process of progressive mathematization, engaging horizontal and vertical mathematization to schematize a mathematical concept.

Gravemeijer and Doorman (1999) analyze the role of context problems in calculus from an instructional perspective. As an example, the historical development of calculus provides a modeling perspective about velocity and distance problems starting from discrete functions. Gravemeijer & Doorman (1999) emphasize that the RME approach "transcends the dichotomy" between formal mathematics and informal knowledge (Gravemeijer & Doorman, 1999).

RME may offer a helpful perspective for teaching logarithms by providing opportunities in which students can reinvent logarithms and the laws of logarithms. In the design of context problems, the historical genesis of logarithms can be taken into account, as it was done in the Iranian books reviewed here. It is noteworthy that solving such problems may require more than mathematical facts and formulas: for example, the ability to decode a contextual problem, apply a proper mathematical concept and interpret the result in the context of the problem. Furthermore, the teachers' role in RME is different than their role in traditional mathematics education. They need to know the objectives of RME as well as multidisciplinary applications of mathematical concepts. In their study, Wubbels and others discovered that student teachers generally fail to provide opportunities for student-centered learning and discoveries in RME (Wubbels, Korthagen & Broekman, 1997).

Reflection

The review of the introduction of the concept of logarithm in current Canadian textbooks shows that the instructional approach focuses on the functional notion of logarithms. Regardless of their field of study, Canadian students are required to learn the functional notion of logarithms. The instructional approaches provide facts, formulas, and procedures to gain competency in solving typical examination questions. These approaches were completely different from Freudenthal's theory which suggests the use of realistic context problems to introduce the mathematical concepts.

In contrast, current Iranian textbooks' approaches are compatible with students' fields of study. The textbook for the Humanities stream introduces the arithmetic notion of logarithms in the vein of Freudenthal's theory. The lessons started with modeling the growth rate of bacteria to establish a connection between bacterial growth rate (a geometric sequence) and time of growth (an arithmetic sequence) to introduce the concept of logarithms. The construction of a table of common logarithms, relation between a number in scientific notation and its common logarithms, and the importance of logarithmic scales were discussed to provide students with a concrete computational knowledge of logarithms.

Chapter 4: Literature Review on the Learning of Logarithms

This chapter presents a review of relevant studies on the learning of logarithmic and exponential functions. First is Kastberg's theory about understanding a mathematical concept and her empirical study (2002) on understanding logarithmic functions. Second is Weber's empirical study on how students can extend their understanding of natural exponents to real exponents. Third is Smith and Confrey's theoretical study on understanding logarithmic functions. They extend Confrey's theory of "splitting" in understanding exponentiation, and emphasize that both splitting and covariation approach to functions are central in understanding logarithmic functions.

4.1 Kastberg's Study on Students' Understanding of Logarithmic Functions

Kastberg (2002) develops a theory of understanding and applies her theory as a framework to analyze how students understand logarithmic functions. She suggests that a student's understanding of a mathematical concept can be perceptible from the student's collection of beliefs about the concept. Evidence of the students' beliefs are their ideas and feelings about the concept (conceptions), the way that they represent the concept (representations), how they make connection between the different representation forms (connections) and how they apply the concept to solve a problem (applications).

To probe students' understanding of logarithmic functions, Kastberg (2002) conducted a case study over three instructional phases (pre-instructional, instructional and post-instructional). She designed nine interview protocols including a variety of activities and interviewed four students who enrolled in college algebra course at a rural community college in the United States. Activities included phenomenological questions, and standard and non-standard logarithmic problems. In the pre-instructional phase, students were asked to define a function, logarithm, and logarithmic function, list properties of a logarithmic function, and solve standard logarithmic problems. The instructional phase did not require students to do any logarithmic problem. In the post-instructional phase, non-standard problems were presented. For example, one of the problems used the relations between sequences, another introduced the function $f(AB) = f(A) + f(B)$ and $f(2) = 1$ then asked to evaluate the function for some values. There also existed two table-completion problems which asked for the numerical approximation of logarithms of a few numbers.

Based on this study, she concludes that students understood the logarithmic function as a "problem to do" in which the problem has four categories: level of difficulty, type of the problem, tools to solve the problem and characteristics of the logarithmic function.

In the pre-instructional phase, she states that students' understanding was "speculative". Students' beliefs about the level of difficulty of logarithmic functions were associated with their performance in solving activities; therefore, they described logarithmic functions as easy or hard. Since students in the pre-instructional phase were given standard logarithmic problems, they thought that logarithmic problems include evaluating, simplifying, graphing, making tables of values and solving exponential equations. Tools such as facts, formulas, a calculator and a procedure were needed to be able to solve these types of problems. Furthermore, they characterized the logarithmic function as a symbols or a notation such as $\log_{base} \#$.

The instructional phase was concurrent with class instruction on logarithmic functions. Only one interview was conducted after they had been taught and it did not require solving logarithmic problems. Students believed that logarithmic functions are easy to solve. They tried to absorb facts, formulas, and procedure to solve these problems. Evaluating logarithms of numbers and converting logarithmic form to exponential form and vice versa were the types of problems which were emphasized in the class instruction. Since students were introduced to logarithmic functions as the inverse of exponential functions, they characterized logarithmic functions as related to exponential functions.

In the post-instructional phase, students were given non-standard problems based on logarithms without mentioning logarithmic function. Students could not recognize that these problems were based on logarithms. They tried to manipulate each problem with a method that they used before and it made sense for them. Kastberg (2002) categorizes their attempt to solve non-standard problems: recognition the pattern of numbers in the relation between sequences problems, successive approximation in the table-completion problems, linear interpolation in the evaluating values of function $f(AB) = f(A) + f(B)$, and awareness of inconsistencies to eliminate them.

Kastberg compared students' performance on standard logarithmic problems in the pre-instructional phase with their performance on non-standard logarithmic based problems in the post-instructional phase. Since the instructional phase was based on a traditional approach, it might improve students' performance on standard problems rather than non-standard problems.

Some of her observations can be summarized as follows: students focused on standard problems which were presented in their classes since they understood logarithmic functions as particular problems to do. The traditional instructional approach failed in providing grounds to stimulate students' cognitive development. Furthermore, it provided resources of facts, formulas and procedures to employ in standard mathematics problems. Presenting facts and formulas without providing any rationale for their needs and uses accounts for memorization and incorrectly recalling from a memory. As an example, the product law of logarithms $\log_c AB = \log_c A + \log_c B$ can be distorted by the distributive property of numbers and incorrectly result in $\log_c (A+B) = \log_c A + \log_c B$.

4.2 Weber's Study on Students' Understanding of Exponential and Logarithmic Functions

In a case study, Weber (2002) adapts Action-Process-Object-Schema (APOS) theory to analyze how students develop an understanding of exponents and logarithms as reverse of exponentiation. His theory is that students' understanding of exponents takes place in four stages (adapted from APOS theory): exponentiation as an action, exponentiation as a process, exponential expressions as the result of a process, and generalization. Generalization indicates the ability of justifying real numbers as exponents, whereas exponentiation as an action indicates the ability of computing exponential functions only for positive integer exponents. Within the context of this theory, he has conducted an empirical research to analyze students' understanding of exponents and logarithms. He interviewed fifteen students who enrolled in a pre-calculus course three weeks after they were introduced to exponential and logarithmic functions. Some of interview problems were:

1) What does the function $f(x) = a^x$ mean to you?

2) Is 5^{17} an even or an odd number?

3) How would you compute $\log_5 78125$ without using a calculator?

Analyzing the students' responds for question 3 showed that none of students solved $\log_5 78125$, even though four out of fifteen students convert $\log_5 78125$ to $5^x = 78125$. Presenting logarithms as reverse of exponentiations require a process conception of exponentiation.

Based on his observations, he concludes that students construct a meaningful understanding of concepts of exponents and logarithms when they understand exponentiation as a mathematical process and exponential expressions as mathematical objects (Weber, 2002). He has developed instructional activities according to his theory's stages and implemented in an experimental pre-calculus course for further research.

In enhancing students' understanding of exponentiation as a process, he asks students to write a program to perform exponentiation with a graphing calculator and answers the question such as "why $(-1)^x$ negative when x is odd". In exponential expressions as the result of a process, for example, students are asked to demonstrate that $2^3 2^4 = 2^7$. In the generalization stage, students discuss "what it means to be a half factor of 2".

4.3 Confrey's and Smith's Research on Students' Understanding of Exponential and Logarithmic Functions

Confrey and Smith (1995) criticized the current curriculum approach to multiplication as repeated addition; such an approach results in an underdeveloped understanding of multiplication in exponential growth and decay (Confrey & Smith, 1995). They propose splitting as a primitive model for multiplication and division (Confrey & Smith, 1995). In splitting, a particular quantity splits into equivalent multiple versions with respect to growth rate. As an example of splitting, try folding a sheet of paper symmetrically, the paper splits in two similar rectangles (one two-splits, one indicates level of split). Fold the paper one more time so that there are 4 rectangles (two two-splits, two indicates the level of split) and so on. The connection between the levels of splitting and the geometric view of the results (similar rectangles) distinguishes splitting from counting (Confrey, 1994). Hence, splitting is a metaphor for exponentiations.

Juxtaposing the level of splits and the result of splits makes the isomorphism between counting (arithmetic sequences) and splitting (geometric sequences) become apparent. A covariation approach of this isomorphism is required to develop an understanding of exponential functions (Confrey, 1994). Thus, a covaried isomorphism between counting (repeated addition) and splitting (repeated multiplication) provides a basis for the understanding of exponential functions (Confrey & Smith, 1995). The covariation approach to functions allows a student to conceive the elements of domain in relation to the corresponding elements of range. Through this approach the student can consider the entire process as happening to all values at once and describe the situation in terms of the rate of change.

In contrast, the correspondence approach considers the equation of a function as a producer, such that substituting x from the domain in the equation produces exactly one y in the range. The current curriculum approach to the concept of functions relies on the correspondence view which may create difficulties in understanding the notion of function (Confrey & Smith, 1995). Students experience functional situations in real-life based examples in which one quantity varies with or depends upon another such as price as a function of quality. Such real-life based examples are compatible with a covariation approach to function, but curriculum design stresses the correspondence approach to function.

Smith and Confrey (1994) review the theoretical aspects' of Napier's work to extend their research to logarithmic functions. They conclude that Napier's consideration of a covaried isomorphism— between (a) the position of a geometrically moving point on a line $AB = 10^7$ units with the velocity proportional to its distance of B and (b) the position of an arithmetically moving point on a number line— is basis for the invention of logarithms (see Figure 2.1.). So, they categorize the historical development of logarithms in four steps:

1. The development of arithmetic and geometric sequences
2. The juxtaposition of arithmetic and geometric series
3. The development of continuous geometric worlds
4. The cogeneration of continuous arithmetic and continuous geometric worlds (Smith & Confrey 1994)

and summarize splitting and covariation approach to function as bases for understanding of logarithmic functions (Smith & Confrey, 1994).

Students initially develop a correspondence view (action level) of functions (Sfard, 1989), which is considering a function as a machine to compute numeric values for a given input. A covariation view (process level) of a function happens when a student can generalize the concept of function as a set of inputs and outputs instead of considering a function formula to produce an output from an input. The current curriculum approach to the concept of function relies on the correspondence view since it presents different types of functions such as polynomial, quadratic, absolute value, exponential and logarithms by emphasizing their equations and the properties of each type of function. So the covariation view of a function might not be developed at the time that students are introduced to logarithmic functions. For example, $f(x) = a^x$ represent a covariation isomorphism between splitting and counting. Splitting happens in base a and creates a geometric set $\{a^0, a^1, a^2, a^3, ...\}$ with a corresponding arithmetic set $\{0,1,2,3,...\}$ that indicates the level of splits. This function can be viewed as a covariation isomorphism between two sets when the domain of the function includes only positive integers. If the domain is extended to real sets, how can a^{-1}, $a^{\frac{1}{2}}$, $a^{\sqrt{2}}$ be explained by splitting? Does it make sense to say a^{-1} as negative one a-splits? The splitting conjecture based on repeated multiplication has limitations in extending exponents to real numbers.

Chapter 5: An Empirical Study of Students' Understanding of Logarithms

This chapter describes my own study of students' understanding of logarithms. I start by explaining the theoretical framework, APOS theory, that I have adapted to describe the levels of understanding of the concept of logarithms in the students I have interviewed. APOS theory has been used as a guideline for designing the study, as well as for analyzing and interpreting the data.

5.1 Theoretical Framework: APOS

APOS theory has been developed by Dubinsky (1991) as an expansion of Piaget's concept of "reflective abstraction" to speak about levels of understanding among undergraduate mathematics students (originally this concept was part of a model of early childhood cognitive development, Piaget 1958). Dubinsky (1991) hypothesizes that learning a mathematical concept takes place in an individual's mind through constructing mental actions, processes, objects, and organizing them in schemas to apply to problem situations. This model of mental construction of concepts is called APOS theory (Dubinsky 1991), where the letters in the acronym stand for "action", "process", "object" and "schema"

At the action level of understanding a mathematical concept, an individual refers to instructions and follows them step-by-step to perform operations related to the concept. For example, understanding the concept of function at the action level means to be able to calculate the values of a function given by an algebraic expression for concrete values of the independent variable.

By repeating such actions and reflecting on the actions, an individual may interiorize them and no longer need to be told what to do to perform operations related to the concept in appropriate situations (e.g. the individual calculates several well chosen values of a function to figure out how the values of the function change, whether they grow or not, changes from negative to positive, etc.). When the actions have thus become a kind of generalized procedure for the individual, Dubinsky proposes to say that the individual's understanding of the concept is at the *process level*. Thus, at the process level, an individual is able to perform the operations implicitly through constructed mental perception without referring to instructions.

At some point, the individual may want to isolate the process as a theoretical object in its own right, that is, as a concept, which deserves a name and a definition; Dubinsky says that the process is *encapsulated* into a cognitive *object* in an individual's mind. At this point in the construction of the concept of function the individual has a general notion of function and can think of functions as a new kind of variables that can operated upon (they can be added, multiplied by scalars, combined, reversed in some cases).

Interiorization and encapsulation are therefore the main cognitive mechanisms involved in understanding mathematical concepts. Encapsulation is a form of synthesis of a certain knowledge and know-how. Once the individual's understanding is at the level of objects, he or she can further synthesize the various actions, processes, and mental objects to form a coherent entity, called a schema. For example, the schema level of understanding functions would include knowing the general notion of function and how it is different from, say, the notions of relation, curve, equation; knowing what kind of questions can be asked about functions (domain, range, continuity, differentiability, integrability, extrema, maxima and minima, etc.) and sets of functions (rings of polynomials, normed spaces of continuous functions on closed intervals, etc.).

Normally, in the process of learning mathematics, an individual's mind constructs and holds distinct local schemas associated with a given mathematical concept and retrieves proper schemas depending on problem situations.

APOS theoretical framework has been used to both organize learning mathematical concepts and analyze actual mental operations employed by individuals trying to understand mathematical concepts, mostly at the undergraduate level. In the context of this research, APOS theory has been used to analyze a few students' understanding of the concept of logarithm.

5.2 Adaptation of APOS Theory for Analyzing Data

In chapter 4, I identified two notions of logarithm: arithmetic and functional. At a given moment of solving a problem or reflecting on a mathematical question, a student may use one or the other of these notions at different levels suggested by APOS theory. In this section, I describe what I mean by saying that a student thinks at the action, process, object or schema level in relation to each of the two logarithm notions.

Levels of thinking about the arithmetic notion of logarithms

A_{al}: Action level: guessing the logarithm ($\log_b a$) of a given number, a, by raising the base b of the logarithm to powers n and checking if $b^n = a$; for example: to find $\log_2 8$, trying 2^2, 2^3 and deciding that 3 is $\log_2 8$.

P_{al}: Process level: at this level, the guessing described above is treated as a method; by repeating the actions above and reflecting on them, the student develops some more general rules of proceeding when faced with the problem of finding logarithms; he or she is able to describe this method and justify his or her evaluation of a logarithm by saying that the base to this value is the number of which the logarithm is sought; at this level, for example, the student may find $\log_x x$ and justify the answer by saying that $x^1 = x$.

O_{al}: Object level: at this level, the processes described above are treated as objects in themselves (they are "encapsulated" into wholes), which can be operated upon; for example, added, subtracted, compared; for example, computing $\log_4 7 + \log_4 5$ and comparing with other objects such as $\log_4 12$.

S_{al}: Schema level: understanding the objects of the object level as a system, where the elements are built through conceptual relationships, example: extracting the laws of logarithms from the laws of exponents.

Levels of thinking about the functional notion of logarithms

A_{fl}: Action level: given a graph of an exponential function $y = a^x$ and a concrete value of y, finding the corresponding x.

P_{fl}: Process level: systematically repeating the action of finding x's for concrete y's, obtaining a table of pairs (y, x), drawing a graph of the inverse of $y = a^x$ from the table (for concrete a)

O_{fl}: Object level: realizing that the process of sketching the graph of x as dependent on y gives a new function which can be given a special name, and has some special properties, one of which is that it undoes what the exponential function does.

S_{fl}: Schema level: conceptualizing the relationships between exponentiations and logarithms, example: applying log and ln schemas in solving exponential equations.

5.3 Research Procedures

In this section, I describe subject recruitment, data collecting techniques, and procedures used to analyze collected data.

The six subjects were volunteers to participate in the research, recruited from among students registered in three different levels of mathematics courses in a large, urban, North American university. Two of the courses were pre-university level courses; the first one (course A) was a pre-requisite for the second one (course B), and the third (course C) was one of the core courses for mathematics major students.

Participants of these six case studies included two female students from course A, one male and one female from course B and one male and one female from course C.

To collect data I used a clinical interview technique and a written questionnaire consisting of general questions, a review example on logarithms and a set of problems or "activities" on logarithms. The general questions asked about subjects' background, attitude toward the concept of function and their prior experience with logarithms. The review example (Figure 5.1.) was intended to help subjects recall the concept of logarithm. The clinical interview provided me with a flexible method to analyze the mental operations underlying the subjects' understanding of the concept of logarithms. While subjects engaged in activities, I observed their behavior, asked questions to reveal their thought processes, and clarified any misunderstanding concerning the logarithmic problems. I recorded the interviews and transcribed each interview. It is noteworthy that students did not have access to a calculator during the interviews. Transcripts and written questionnaires are included in Appendix F.

<u>Review:</u> $\log_3 9 = 2$ means the same as $3^2 = 9$. What does $\log_2 8$ mean?

Figure 5.1. The review example given in the questionnaire

In designing the six problems on logarithms, I was trying to categorize them according to APOS theory developed in section 5.2. That is, each question was supposed to require a certain minimal level of thinking about logarithms, and I wanted to have questions requiring at least the action level, at least the process level, etc., for each of the two notions of logarithm (arithmetic and functional). I describe the problems below.

The first problem, labeled Activity 1 (Figure 5.2.), emphasizes the relationship between exponents and logarithms.

Activity1. Complete the table row by row.

Logarithms	Exponentials
$\log_4 16 = ?$	$4^2 = ?$
$\log_4 256 = ?$	$4^4 = ?$
$\log_4 4 = ?$	$4^? = 4$
$\log_4 1 = ?$	$4^? = 1$
$\log_4 \frac{1}{16} = ?$	$4^? = \frac{1}{16}$
$\log_4 (-16) = ?$	$4^? = -16$

Figure 5.2. Activity 1

The first three rows in the table of Activity 1 (Figure 5.2) – to find $\log_4 16$, $\log_4 256$ and $\log_4 4$ – do not require anything more than the Action level of the arithmetic notion of logarithm (A_{al}): the answers can be obtained by performing multiplication of natural numbers.

The next two rows – $\log_4 1$ and $\log_4 \frac{1}{16}$, – require knowledge of definitions of the zero and negative exponents which goes beyond performing the ordinary arithmetic operations. At this point, the subject must have generalized the guessing behavior into a method applicable to exponents that are not only whole numbers but arbitrary integers. Thus, at least the process level P_{al} of understanding the arithmetic notion is necessary here.

The last row – $\log_4(-16)$ – requires the object level O_{al}, because the subject must be aware of the conditions of existence of the logarithm of a number, which presumes thinking about logarithms as objects in their own right (and not as computational exercises).

In the second problem, Activity 2 (Figure 5.3.), subjects were asked to estimate the value of the exponential function $f(x) = 4^x$ at the irrational number $\sqrt{2}$ from a graph of the function, and to find the logarithm $\log_4 4^{\sqrt{2}}$. The action level of the functional notion of logarithms A_{fl} is sufficient to answer the first question, since the graph $f(x) = 4^x$ is given. Since the exponent is irrational but concrete, answering the second question ($\log_4 4^{\sqrt{2}}$) requires no more than the P_{al} level of the arithmetic notion of logarithms and can be solved independently of the graph and the first question. The expected answer would be $\sqrt{2}$. If the graph is used to answer the second question (tracing back the value of exponent from the value of the power), the action level of the functional notion A_{fl} is sufficient. In this case the student's answer could be a decimal approximation of $\sqrt{2}$, say, "about 1.4".

Activity2.The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and estimate $\log_4 4^{\sqrt{2}}$.

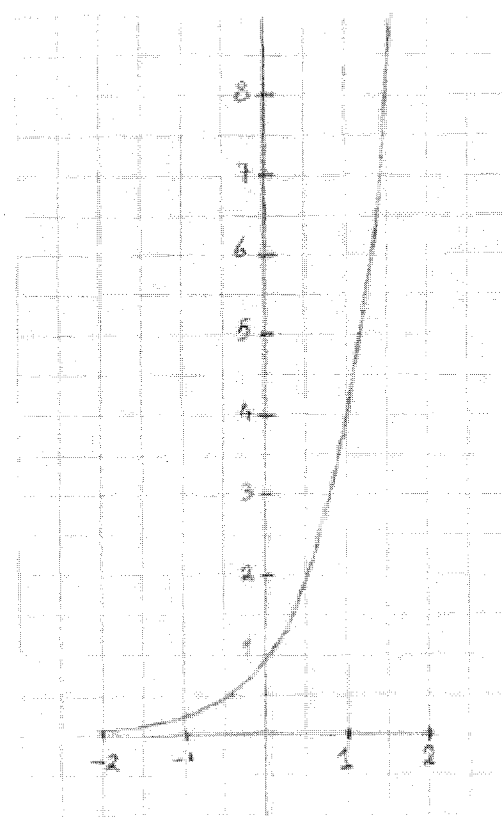

Figure 5.3. Activity 2

The third problem, Activity 3 (Figure 5.4.), may be conceived of at a process or an object level of the arithmetic notion of logarithm. At the object level O_{al} of understanding the arithmetic notion of logarithm, the first expression, $\log_4 7 + \log_4 5$ is calculated by applying a law of operations on logarithms, treated as objects in its own right. For example, if the subject know the laws of logarithms correctly, he or she may find the answer by estimating $\log_4 35$. At the process level the subject may evaluate each logarithm individually and add the results. The action level is not sufficient here because fractional exponents are involved. As regards the second expression – $\log_3 27 \times \log_{27} 3$ – similar behaviors may be expected. At the object level, the subject would be expected to think about the laws of operations on logarithms; at the process level, the subject would evaluate each term individually without mentioning the applicability of laws of operations on logarithms in this case. Evaluating $\log_{27} 3$ requires knowledge of fractional numbers as exponents and the definition of fractional exponents, therefore the action level is not sufficient.

Activity 3. Evaluate the expressions.

$\log_4 7 + \log_4 5 = ?$

$\log_3 27 \times \log_{27} 3 = ?$

Figure 5.4 .Activity 3

In Activity 4 (Figure 5.5.), subjects are asked to reflect on the validity of an identity involving logarithms of arbitrary numbers: $\log_{10} x^3 = (\log_{10} x)^3$. It is suggested that they substitute a value for x. Answering this question requires the object level O_{al} of understanding the arithmetic notion of logarithms: it is a question about laws governing operations on logarithms treated as objects in their own right.

Activity 4.Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

Figure 5.5. Activity 4

Activities 5 and 6 (Figure 5.6.) require the schema level S_{al} of understanding the arithmetic notion of logarithm, since logarithms with a variable basis x appear in the questions. Here, logarithms have to be conceived as constituting a system, governed by definitions and laws. These problems were addressed mainly to subjects from course C, a core course for mathematics majors. It was not expected to be accessible for students of the pre-university level courses.

Activity 5.
Find the value(s) of x such that $\log_x(x-1) + \log_x(x) = 1$.

Activity 6.
Consider the inequality $\log_x |x^2 - 1| < 2$ what do you think about this inequality.

Figure 5.6. Activities 5 & 6

5.4 Analysis of Data

In order to analyze each subject's understanding of logarithms (according to the categories adapted from APOS theory as described in the previous sections), I identified significant statements from each subject's transcripts and/or her/his solution and categorized them as shown in Table 5.1.

1	2	3	4	5	6
Activities	Subject's statements	APOS category	Explanation of subject's behaviors	Justification of APOS analysis	Comment s
....					
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	174. Okay, 2 to power 2 is 4, um--- 2 to power 3 is 8. so therefore log 2 , 8 is 3.	A_{al}	.	She computes $\log_2 8 = ?$ with finding $2^3 = 8$	
....					

Table 5.1.The data analysis procedure

Column 1 contained the set of problems (activities). Column 2 contained corresponding statements from the subject. Column 3 categorized student's level of thinking by using APOS theory developed in section 5.2 . Column 4 contained the notes that I took during the clinical interview. I justified my categorization in column 5 and added comments in column 6. This procedure helped me identify the variation and stabilization of a subject's mental operations according to APOS theory. I represented the variation/stabilization in a graph for each subject, where the $x-axis$ was scaled by the progression of the transcripts and the $y-axis$ was scaled by A_{al}, P_{al}, O_{al}, and S_{al} or the corresponding levels of understanding the functional notion of logarithms. I then explained the results of each case study by introducing each subject and describing her/his remarkable efforts in solving logarithmic problems.

Chapter 6: Results of Six Case Studies of Students' Understanding of Logarithms

In this chapter, I briefly introduce the subjects of my case studies who were students registered in three different levels of mathematics courses in a large, urban, North American university. I explain some of their particular approaches to solving logarithmic problems. Based on my adaptation of APOS theory explained in section 5.2, I tried to identify their thought processes, and summarized the levels of thought processes of each students in a graph, where the $x-axis$ was scaled by the progression of the transcripts (the transcripts were consecutively numbered and included in Appendix F) and the $y-axis$ was scaled 1(A_{al}), 2(P_{al}), 3 (O_{al}) and 4 (S_{al}).

Case1: Student 1 Enrolled in a Pre-Calculus Course (Course A)

The subject was an independent female mature student from course A, a secondary school graduate from 14 years ago. Her interest in Finance motivated her to start a university program. She believed that the notion of function has applications in economy and provides a better picture of a given situation. She did not recall seeing logarithms before or working with the log key on a calculator, therefore she found the review example helpful. Despite her lack of knowledge she was interested to work on the problems. She interpreted natural exponents as an abbreviation for multiplication, since she interpreted $3^2 = 9$ as "how many times 3 to get 9" (line 13, Appendix F). She could not compute $\log_4 1 = ?$ and $\log_4 \frac{1}{16} = ?$ because she had a lack of knowledge of the definition of zero, and negative numbers as exponents. She guessed that $\log_4(-16) = ?$

would be -2 and she referred to $\log_4 16 = 2$. She was puzzled with $\sqrt{2}$ as an exponent, and when given the approximation of $\sqrt{2}$, she computed $4^{1.41}$ as 5.64. This reveals her misconception about raising a number to a power: she interprets $4^{1.41}$ as 4×1.41. She did not extend the domain of exponents to real numbers. She did not know how to read the given graph in Activity 2. She reflected on her actions to evaluate $\log_4 7 + \log_4 5$ because first she evaluated each logarithm separately and added the results then she suggested a law such that $\log_4 7 + \log_4 5 = \log_4 12$. To verify her suggestion she compared the value of $\log_4 7 + \log_4 5$ with $\log_4 12$ and rejected her suggestion. Her understanding advanced to O_{al} level while she experienced a stabilization at P_{al} level (see Figure 6.1.).

In Activity 5, she substituted some values for x and verified whether $\log_x(x-1) + \log_x(x) = 1$ holds, but she did not rearrange the equation by applying the definition and the laws of logarithms.

Lines 1 to 95 in Appendix F are a transcript from the interview with her and analysis of data is represented in Appendix G.

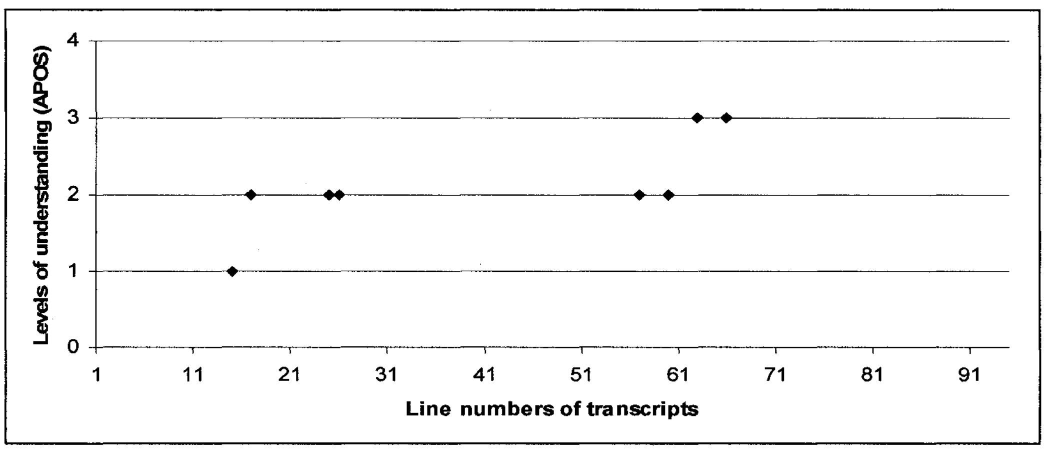

Figure 6.1.Represents student 1'levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Case 2: Student 2 Enrolled in a Pre-Calculus Course (Course A)

At the time of the interview, Student 2 (female, non-mature) was enrolled in the Education program but was planning to switch to Marketing and was therefore taking course A as a prerequisite for admission. She said she studied logarithms at school the year before and admitted that she did not like logarithms. She believed that the concept of function is useful in marketing and statistics. She solved the first four questions in Activity 1 correctly, justifying in each case her answers to the log questions by making correct exponential statements. But in the fifth question of Activity 1, she proposed that the solution to $4^{?} = \frac{1}{16}$ is $\frac{1}{2}$ "because $4^2 = 16$". She apparently did not remember the definitions of negative and fractional exponents. She solved the questions in Activity 2 correctly. In Activity 3, to evaluate $\log_4 7 + \log_4 5$, she converted each part into exponential forms and tried to find exponents and add them. She had the potential to reach the O_{al} level, since she experienced stabilization at P_{al} level, but a lack of knowledge of the definition of negative and fractional numbers as exponents influenced her progress in understanding the concept of logarithms. Figure 6.2. represents the variation/stabilization of her thought processes.

She rejected the equality in Activity 4 as soon as she read the problem; however, she tried different values for x, such as 2 and 10 to justify her answer. She probably selected 2 because 2 is a small natural number, but she was not able to estimate $\log_{10}(2)^3 = \log_{10} 8$. By the given hint (select x as one of powers of 10) she selected x=10 to show that the equality does not hold. To solve $\log_x(x-1) + \log_x(x) = 1$, she tried each term separately, so that the equation was simplified to $\log_x(x-1)=0$. Then by referring to the review example she converted $\log_x(x-1)=0$ to $x^0=(x-1)$. She modified the inequality in Activity 6 to $\log_x\left|x^2-1\right|\geq 0$, since she believed logarithms are always positive. She appeared to confuse the domain and the range of logarithms.

Lines 96 to 167 in Appendix F are a transcript from the interview with her and analysis of data is represented in Appendix G.

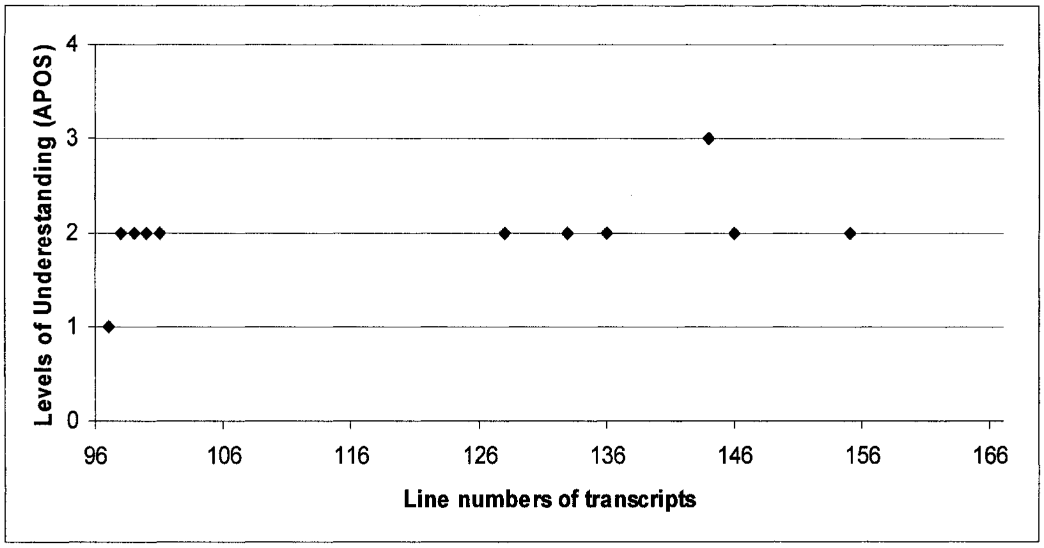

Figure 6.2. Represents student 2's levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Case 3: Student 3 Enrolled in a Calculus Course (Course B)

This student (female, non-mature) was a student interested in the Business program who registered in course B to fulfill the Business program requirements. She said that she never learned about logarithms in her secondary school because her teacher ran out of time. She did not believe that the concept of function would help her in the future unless she studied mathematics. Even though it was her first time working on logarithms, she figured out the relationship between exponentials and logarithms from the review example. In Activity 1, first she evaluated the exponentials column of the table and then followed the pattern between exponentials and logarithms to evaluate the logarithms column. She was the only subject influenced by the distributive law in evaluating the expression $\log_4 7 + \log_4 5$ and believed that $\log_4 7 + \log_4 5 = \log_4 12$. To evaluate $\log_3 27 \times \log_{27} 3$, she converted each term to exponential forms but she could not evaluate $27^x = 3$. Lack of the extension of the domain of exponents from natural numbers to real numbers and lack of knowledge of the definition of negative and fractional numbers as exponents hindered her progression in understanding logarithms. Furthermore, she had a misconception of powers/roots as multiplication/division. Her understanding level varied between A_{al} and P_{al} represented in Figure 6.3. She believed that learning mathematics would be easier if only concrete numbers were used rather than variables. In addition, she asked if she could use a calculator.

Line 168 up to 275 in Appendix F presents a transcription of her clinical interview and Appendix G presents analysis of data.

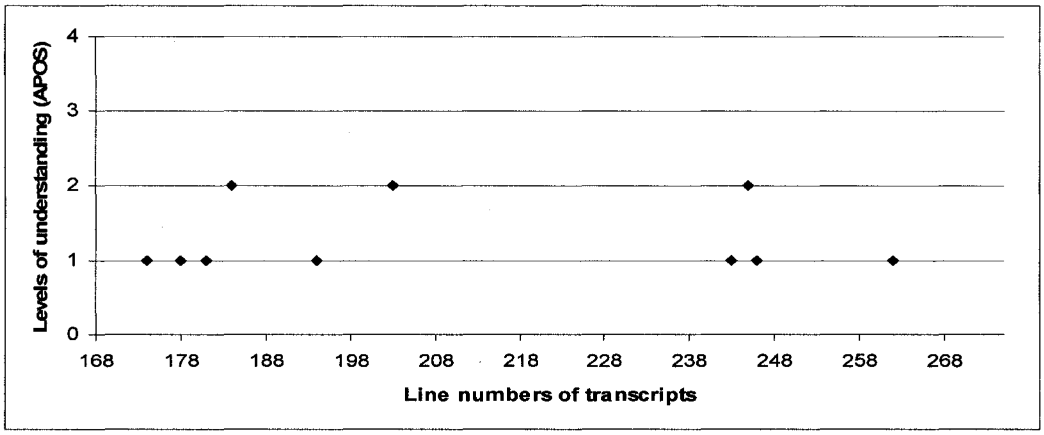

Figure 6.3. Represents student 3's levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Case 4: Student 4 Enrolled in a Calculus Course (Course B)

This (male, non-mature) student was interested in studying commerce. He was taking course B as a prerequisite and had been taught logarithms in course A. Further, he had a review on logarithms a few hours before this interview in the course. He did not believe that the concept of function is useful for his area of work. He mentioned that he hates fractions, and gets nervous when he sees fractions. He referred to the review example several times while he was completing Activity 1. He remembered the definition of negative and fractional exponents but he had difficulty with $4^{?} = -16$. He was puzzled with $\sqrt{2}$ in Activity 2 due the lack of practice with irrational exponents and the lack of extension of the domain of exponents to real numbers. He applied the product law of logarithms to evaluate $\log_4 7 + \log_4 5$ and wrote $\log_4 35$. To evaluate $\log_3 27 \times \log_{27} 3$, he tried to suggest a law such that numbers can be added when bases are common but he could not apply it since bases were not common. To evaluate $\log_3 27 \times \log_{27} 3$, he confused the operation between logarithmic terms in the product law and suggested that to evaluate the product of logarithmic terms, we add the numbers when the bases are common. Since bases were not common in $\log_3 27 \times \log_{27} 3$, he could not apply his suggested law. Therefore, he converted $\log_3 27$ and $\log_{27} 3$ to exponential forms. However, he had difficulty in finding a power to raise 27 and obtain 3 ($27^{?} = 3$). In Activity 5, he applied the product law and wrote $\log_x (x^2 - x) = 1$; however, to convert it to exponential form he had difficulty since the problem was not a numerical problem and involved understanding variables. He again referred to the review example and converted the given inequality in Activity 6 to $x^{<2} = |x^2 - 1|$ and he could not solve it any further. His understanding of the concept of logarithms did not advance further than A_{al} level, despite his knowledge of formulas. He had a misconception of powers/roots as multiplication/division.

Lines 276 to 389 in Appendix F are a transcript of the interview with this student and analysis of data is presented in Appendix E.

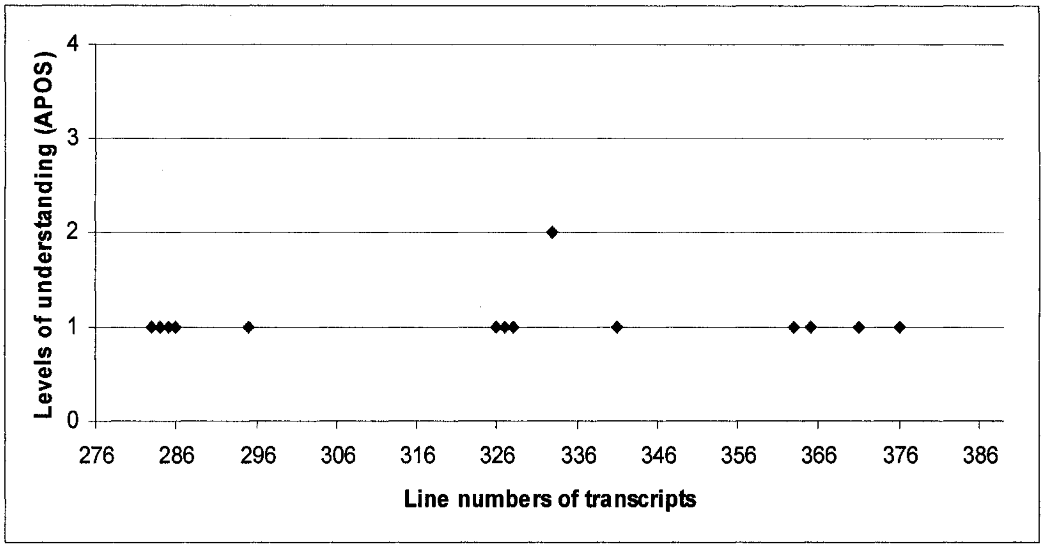

Figure 6.4. Represents student 4's levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Case 5: Student 5 Enrolled in a Core Mathematics Course (Course C)

The subject was a mature female student studying Pure and Applied Mathematics enrolled in course C. She mentioned that the concept of function is useful in everything. She did not remember logarithms and forgot how to read logarithmic notations; she had been taught logarithms five years before. She did not pay enough attention to the review example and wrote $\log_2 8 = 2^3$. She did not want to try $\log_4(-16)$ and $4^? = -16$ in Activity 1. She used the given graph in Activity 2 to estimate $4^{\sqrt{2}}$, but the value $\log_4 4^{\sqrt{2}}$

was not apparent to her. To evaluate $\log_4 7 + \log_4 5$, she converted each part to exponential forms and estimated them by the given graph in Activity 2. She evaluated $\log_3 27 \times \log_{27} 3$ by converting each term to exponential forms and finding the product of the results. She did not immediately reject the equality $\log_{10} x^3 = (\log_{10} x)^3$ in Activity 4, rather she tried to find an example to verify it. She was frustrated by the fact that all of the problems asked about logarithms. She could not solve the problems in Activity 5 and 6 and kept blaming herself, saying "I did really bad, I do not remember log" (line 489, Appendix D). In activity 6, she converted the inequality into $x^? = |x^2 - 1| < 2$, but could not simplify it any further. She advanced to P_{al} level of understanding of the arithmetic notion of logarithms. Furthermore, she had a potential to reach P_{fl} level since she had an understanding of the concept of function and experience with the graphical representation of functions. Since she could not conceptualize the relationship between exponentials and logarithms, her understanding of both arithmetic and functional notions of logarithms varied between action and process level and did not stabilize at process level. She had difficulty in conceptualizing logarithms as undoing what exponentiations do.

Lines 390 to 498 in Appendix F present a transcript of her clinical interview and analysis of this data is indicated in Appendix G.

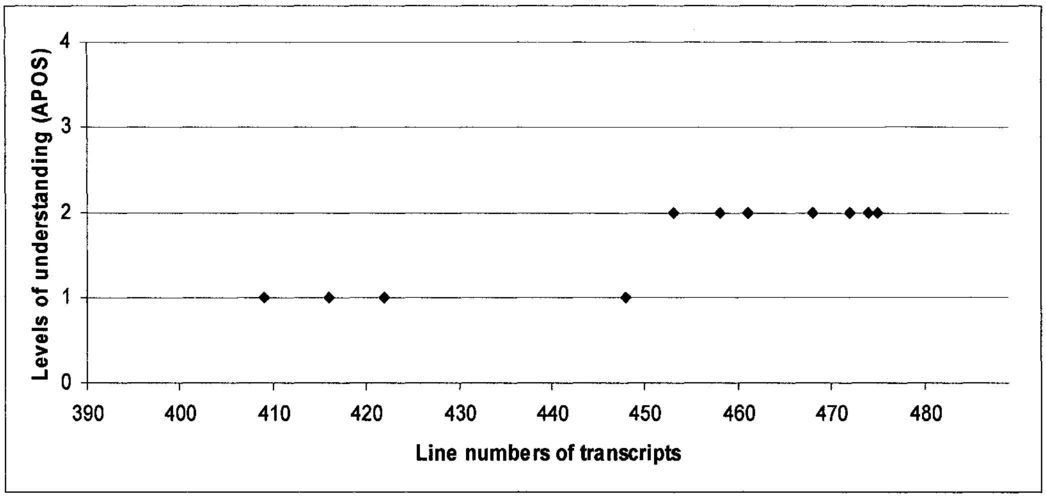

Figure 6.5. Represents student 5's levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Case 6: Student 6 Completed a Core Mathematics Course (Course C)

The subject was a mature male student completed course C. He was studying in a graduate program. He had seen logarithms recently in one of his courses. He strongly believed that the concept of function is useful in day to day calculations. He was the only subject who remembered logarithms and the laws of logarithms. He applied the power law of logarithms ($\log_b a^c = c\log_b a$) to evaluate $\log_4 4^{\sqrt{2}}$ in Activity 2. He used the product law of logarithms ($\log_b(ac) = \log_b a + \log_b c$) in Activity 3 and wrote $\log_4 7 + \log_4 5 = \log_4 35$. Then he referred to the given graph in Activity 2 to verify that $\log_4 7 + \log_4 5 = \log_4 35$ holds. In evaluating $\log_3 27 \times \log_{27} 3$, he thought that there is a law to find a product of logarithmic terms. Since the bases were not common he evaluated each term separately and found the product of the results. In Activity 5, he incorrectly recalled the change of base law of logarithms as $\log_x(x-1)=\frac{\log_x x}{\log_x 1}$. In Activity 6, he used the definition of logarithms ($y=\log_b x \Leftrightarrow x=b^y$) and extended it such that $\log_b x<y$ implies $b^y<x$, and then he solved the problem. He simplified the problem to inequality $x^2<-(x^2-1)$ and concluded that $x<\pm\sqrt{\frac{1}{2}}$. He advanced to S_{al} level of understanding of the arithmetic notion of logarithms and had a potential to reach S_{fl}. His knowledge of formulas enabled him to have a different approach in some of the activities; for example, his knowledge of the power law of logarithms helped him to simplify the logarithmic expressions. Since he applied the laws of logarithms in solving problems and he had previously developed an understanding of the concept of logarithms, his interview did not provide me with significant data about his cognitive development during his engagement in the activities.

Lines 490 to 534 present a transcript of his clinical interview in Appendix F and analysis of his interview are included in Appendix G.

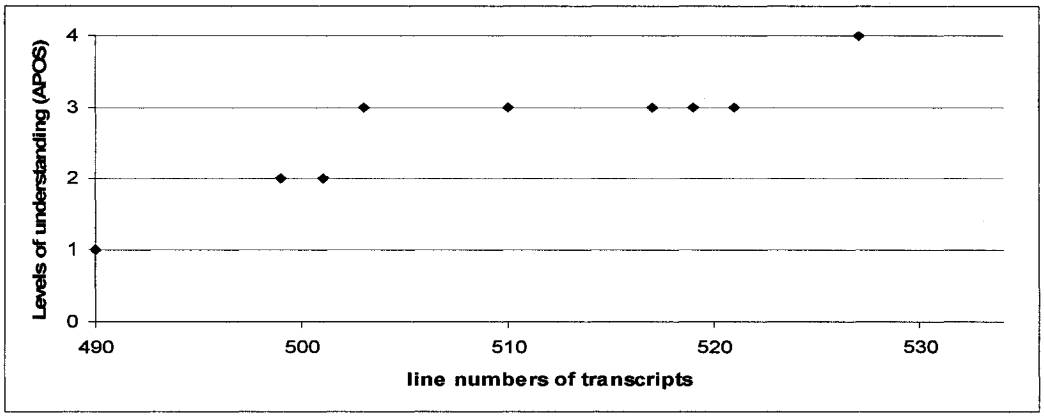

Figure 6.6. Represents student 6's levels of understanding: 1=Action, 2=Process, 3=Object, 4=Schema

Summary

Cases 1 through Case 4 focused on students from prerequisite mathematics courses who wanted to obtain admission into undergraduate programs such as Business, Finance, and Commerce. The results of their interviews showed that three students out of four had a lack of knowledge of the definition of negative and fractional numbers as exponents. Even though my study did not focus on exponentiation, my observations confirm Weber's study in the sense that students' understanding of exponents was limited to natural exponents. When the given exponent was not a natural number to be interpreted as an abbreviation for multiplication such as $4^{1.41}$, I became aware of my students' misconception of exponents as times, i.e. computing $4^{1.41}$ as 5.64. As I assumed in the beginning of my study, students' prior knowledge of some mathematical facts and laws influenced their understanding of logarithms. For example, the distributive law ($ab + ac = a(b + c)$) influenced one of my student's approach in evaluating $\log_4 7 + \log_4 5$ as $\log_4 12$. When students are presented with mathematical facts (i.e. logarithms are not defined for negative numbers) without providing grounds for reinvention and reasoning they may not recall them correctly after a while. For example, one of my subjects thought that logarithms are always positive. None of my four subjects were able to answer and justify their answer to $\log_4(-16) = ?$ without my help.

Furthermore, I noticed that facts and laws presented in mathematical notations can become complicated for students to remember and recall correctly rather than the facts and laws presented in words. For example, remembering that the logarithm of a product of numbers is equal to the addition of logarithms of the numbers can be easier than memorizing the formula $\log AB = \log A + \log B$ for some of students.

Prerequisite mathematics courses students (Case 1 through Case 4) did not demonstrate any understanding of the functional notion of logarithms, even though it is part of the current curriculum. I observed that they had difficulties with understanding variables and solving the given problem for x. One of my subjects tried to substitute different values for x to verify if the given equality holds, instead of simplifying the equation and solving.

Case 5 and Case 6 focused on students who registered for a core mathematics course. The student in Case 5 did not remember logarithms since she had been taught logarithms five years ago. I realized her anxiety while she engaged in activities. I think her anxiety did not allow her to reflect on her actions and develop an understanding of logarithms. She kept referring to the given review example more than other subjects and she could not conceptualize the relationship between logarithms and exponentials. In contrast, the student in Case 6 tried to employ the laws of logarithms if they were applicable. He even thought of a law to evaluate a product of logarithmic terms. Even though he remembered and recalled the laws of logarithms, he did not grasp the importance of logarithms in converting multiplication and division of numbers into addition and subtraction of logarithms of these numbers since he incorrectly recalled the quotient law of logarithms as $\log_x(x-1)=\frac{\log_x x}{\log_x 1}$. These two subjects did not provide me with significant data on their mental development during engagement in activities, but Case 5's interview revealed how affective issues influence development of mental operations and Case 6's showed that remembering and recalling laws did not indicate the level of understanding of mathematical concepts.

Chapter7: Conclusions and Discussion

This chapter presents the results of my study on students' understanding of the concept of logarithm. I also discuss and compare Katsberg's observations on students' understanding of logarithms with my observations through this research.

Conclusions

I identified two notions of logarithms – arithmetic and functional – by reviewing the historical genesis of logarithms and a few textbooks from different times and geographical places. Current curriculum approach focuses on the functional notion of logarithms which requires knowledge of exponentiations, the concept of function and exponential function, and the notion of the inverse of a function. In contrast, to present the arithmetic notion of logarithms knowledge of exponentiation is sufficient. Logarithms as undoing what exponentiation does may also provide a broader view of exponentiation with real exponents. The historical genesis of logarithms can provide modeling perspectives to design an arithmetic instructional approach.

The functional notion of logarithms can be presented when students advance to an object level of understanding of the arithmetic notion of logarithms and develop an understanding of the concept of function and exponential function. Furthermore, my review of textbooks showed that the functional notion of logarithms is generally introduced to provide a tool for solving exponential equations. However, the functional notion of logarithms is not necessary at this point.

As Freudenthal (1973) points out, logarithms can be introduced through context problems in which they appear as a tool for calculation or measurement. The arithmetic notion of logarithms can be employed as a proper instructional approach to emphasize the relationship between exponentials and logarithms and provide a computational tool.

> Arithmetic instruction shows us the correct way. If mathematics fraught with relations should be taught, it should be tied to the other member of the relation, to start with and again and again, whether the other part of the relation be mathematics, physics or everyday life. ..., logarithms should start with slide rule or with air pressure, or with the hyperbola if it should be applied there,... (p.133)

I analyzed empirical data, gathered through clinical interviews, using APOS theory to describe students' understanding of the concept of logarithms; however, mental operations of an individual do not necessarily follow the linear progress of levels of the APOS theory. The results of four case studies with students from prerequisite mathematics courses showed that the understanding of exponentiation, extending the domain of exponents to real numbers and conceptualizing the abstract definition of negative and fractional exponents influenced students' understanding of the arithmetic notion of logarithms. My study confirms Weber's emphasis, that a process of understanding of exponentiation is required prior to presenting logarithms. His study's focus on students' understanding of exponents indicates that most of the students' understanding of exponents did not progress beyond an action level (Weber, 2002).

Discussion

Students' difficulties in understanding the concept of logarithms has been studied theoretically and empirically. Smith & Confrey's theoretical study (1994) reveal that splitting and covariation approach to function are bases for understanding of logarithmic functions. My review of their study indicated that the splitting conjecture based on repeated multiplication has limitations in extending exponents to real numbers and the covariation view of a function might not be developed at the time that students are introduced to logarithmic functions.

I also reviewed Kastberg's empirical study (2002) on students' understanding of logarithmic functions within the context of her theory of understanding. Her model of understanding is developed based on students' beliefs about mathematical concepts and mathematics. She considers four categories of evidence (conceptions, representations, connections and applications) as indicators of students' beliefs about a mathematical concept. On the other hand, my research analyzed students' level of understanding of the concept of logarithms within the context of APOS theory. In my research questionnaire, the given review example helped students realize the relationships between logarithms and exponentials in order to engage in activities. Then, I monitored their progressions and possible constraints in their understanding of the concept of logarithms.

As Kastberg points out the current mathematics curriculum approach provides facts, formulas, and procedures to gain some formal mathematical knowledge and apply them in typical mathematics examinations. This instructional approach influences students' beliefs about logarithmic functions as a collection of problems to do. According to Freudenthal (1973), ready-made laws prevent invention by students and do not allow students to develop an understanding of mathematical concepts. Furthermore, application problems such as measuring the magnitude of an earthquake, provide a formula, $M = \log\frac{I}{S}$ (in which I is the intensity of an earthquake and S is the intensity of a standard earthquake) and all that is required is to substitute some values and calculate. These kinds of application problems do not provide grounds to realize the importance of employing logarithms and logarithmic scales. Therefore, students try to memorize laws, facts and formulas without experiencing them through realistic context problems. The RME approach seems promising in providing realistic context problems in which students reinvent mathematical concepts. Realistic context problems may motivate students and shape their beliefs about mathematics as well as enhance their understanding of mathematical concepts.

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Appendix A: A logarithmic table written by Napier

1st column	2nd column	⋯	68th column	69th column
10 000 000.000 000 0	9 900 000.000 000	⋯	5 099 857.462 496	5 048 858.887 871
9 995 000.000 000 0	9 895 050.000 000	⋯	5 097 307.533 764	5 046 334.458 427
9 990 002.500 000 0	9 890 102.475 000	⋯	5 094 758.879 998	5 043 811.291 198
9 985 007.498 750 0	9 885 157.423 763	⋯	5 092 211.500 558	5 041 289.385 552
9 980 014.995 000 6	9 880 214.845 051	⋯	5 089 665.394 807	5 038 768.740 859
9 975 024.987 503 1	9 875 274.737 628	⋯	5 087 120.562 110	5 036 249.356 489
9 970 037.475 009 4	9 870 337.100 259	⋯	5 084 577.001 829	5 033 731.231 811
9 965 052.456 271 9	9 865 401.931 709	⋯	5 082 034.713 328	5 031 214.366 195
9 960 069.930 043 7	9 860 469.230 743	⋯	5 079 493.695 971	5 028 698.759 012
9 955 089.895 078 7	9 855 538.996 128	⋯	5 076 953.949 123	5 026 184.409 632
9 950 112.350 131 2	9 850 611.226 630	⋯	5 074 415.472 149	5 023 671.317 427
9 945 137.293 956 1	9 845 685.921 017	⋯	5 071 878.264 413	5 021 159.481 768
9 940 164.725 309 1	9 840 763.078 056	⋯	5 069 342.325 280	5 018 648.902 028
9 935 194.642 946 5	9 835 842.696 517	⋯	5 066 807.654 118	5 016 139.577 577
9 930 227.045 625 0	9 830 924.775 169	⋯	5 064 274.250 291	5 013 631.507 788
9 925 261.932 102 2	9 826 009.312 781	⋯	5 061 742.113 166	5 011 124.692 034
9 920 299.301 136 1	9 821 096.308 125	⋯	5 059 211.242 109	5 008 619.129 688
9 915 339.151 485 6	9 816 185.759 971	⋯	5 056 681.636 488	5 006 114.820 123
9 910 381.481 909 8	9 811 277.667 091	⋯	5 054 153.295 670	5 003 611.762 713
9 905 426.291 168 9	9 806 372.028 257	⋯	5 051 626.219 022	5 001 109.956 832
9 900 473.578 023 3	9 801 468.842 243	⋯	5 049 100.405 912	4 998 609.401 853

Appendix B: Translation of exponential and logarithmic functions from the Iranian Textbook *Mathematics 2* (p. 85-93)

4.3 Exponential and Logarithmic Functions

a^n is defined for a positive real a and an integer n , and its preliminary properties are formulated in *Mathematics 1*. It can be proved that there exists one and only one positive real number b such that $b^n = a$. This definition implies that $b = a^{\frac{1}{n}}$, or $b = \sqrt[n]{a}$ and b is called the n^{th} root of a. Therefore, $(\sqrt[n]{a^n}) = a$.

For every positive number a, $a^0 = 1$. If $\frac{m}{n}$ is a positive rational number, $a^{\frac{m}{n}}$ is defined as: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Examples:

$$a^{\frac{1}{5}} = \sqrt[5]{a}$$
$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$$
$$3^{\frac{2}{5}} = (\sqrt[5]{3})^2 = \sqrt[5]{3^2} = \sqrt[5]{9}$$

If $\frac{m}{n}$ is a negative rational number ($n \geq 2$, $n \in N$, $m < 0$), then $a^{\frac{m}{n}} = \frac{1}{(\sqrt[n]{a})^{-m}}$, and note that the denominator is defined since $-m > 0$.

Example: $125^{-\frac{2}{3}} = \frac{1}{(\sqrt[3]{125})^2} = \frac{1}{\sqrt[3]{(125)^2}} = \frac{1}{\sqrt[3]{(5^3)^2}} = \frac{1}{5^2} = \frac{1}{25}$

The laws of exponents, $\begin{cases} a^r.a^s = a^{r+s} \\ \dfrac{a^r}{a^s} = a^{r-s} \\ (a^r)^s = (a^s)^r = a^{rs} \end{cases}$, $a \neq 0$, which are studied in *Mathematics 1* hold for any rational exponents such as r and s .

How irrational exponents can be defined? For example, what does $2^{\sqrt{2}}$ mean? The graph of $y = 2^x$ is sketched by point plotting.

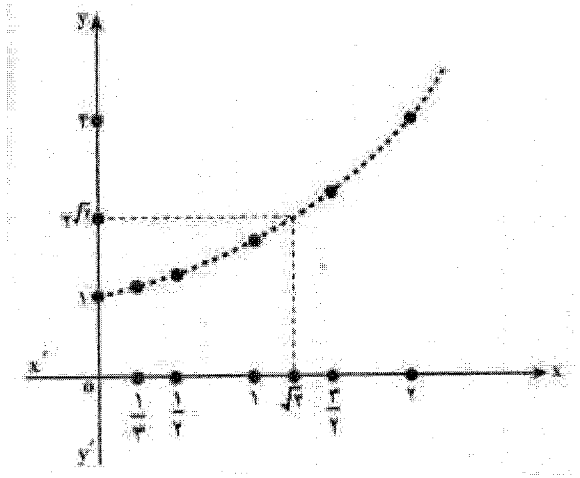


Figure B.1 shows the graph of $y = 2^x$ taken from the textbook

x	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{3}{2}$	2
y	1	1.26	1.41	1.56	2	2.83	4

Table B.1 is a table of values of $y = 2^x$

The black dots on the graph indicate the values of the function at the given positive rational points (see the Table B.1). To find $2^{\sqrt{2}}$ on the graph, indicate $\sqrt{2}$ on x-axis and find $2^{\sqrt{2}}$ on y-axis. It is obvious that 2^x can be defined for negative real numbers. For example, $x = -\frac{1}{2}$ implies that $2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}} = \frac{1}{1.4142}$, and note that 2^x is always positive. Therefore, the graph of $y = 2^x$ with the real number set as domain is presented in Figure B.2.

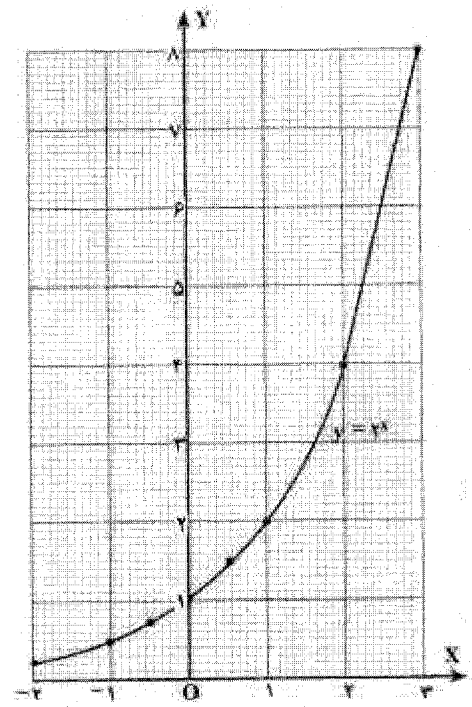


Figure B.2 the graph of $y = 2^x$ taken from the textbook

Example: Suppose a type of bacteria become doubles each day. The number of bacteria after x days can be modeled by an exponential function $y = 2^x$.

x	1	2	3	4
y	2	2^2	2^3	2^4

Exponential Function $y = a^x$

If x is a real number a^x satisfies all of the laws of exponents. For example, $a^{x+y} = a^x \cdot a^y$.

The graph of $y = a^x$ is sketched for $a > 1$ and $0 < a < 1$.

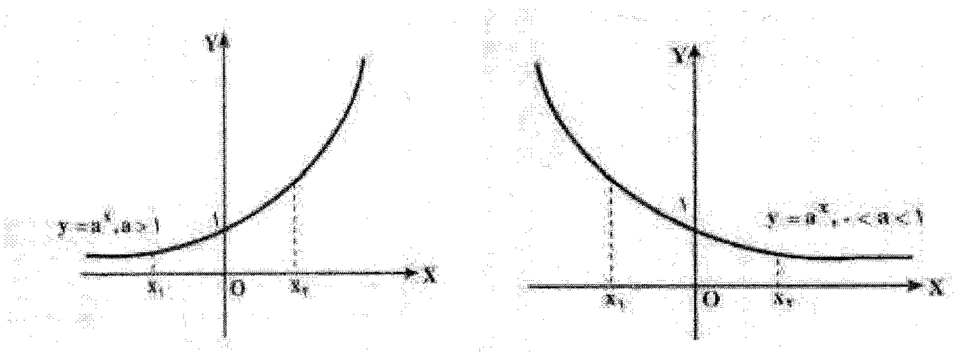


Figure B.3 presents the graphs of $y = a^x$

Logarithmic Function

The above graphs indicate that $y = a^x$ is an onto and one-to-one function with domain R and range $(0, \infty)$. Therefore, $y = a^x$ ($a \neq 1$, $a > 0$) has an inverse. The inverse function is called a logarithmic function and its notation is $y = \log_a x$ where a is the base of logarithm. The figure B.4 shows that the graph of $y = \log_a x$ is a reflection of the graph of $y = a^x$ about the line $y = x$, therefore $x = a^y \Leftrightarrow \log_a x = y$. For example, $32 = 2^5 \Leftrightarrow 5 = \log_2 32$ and $1000 = 10^3 \Leftrightarrow 3 = \log_{10} 1000$. Note that if $a = 10$ then $\log_{10} x$ is called the common logarithm.

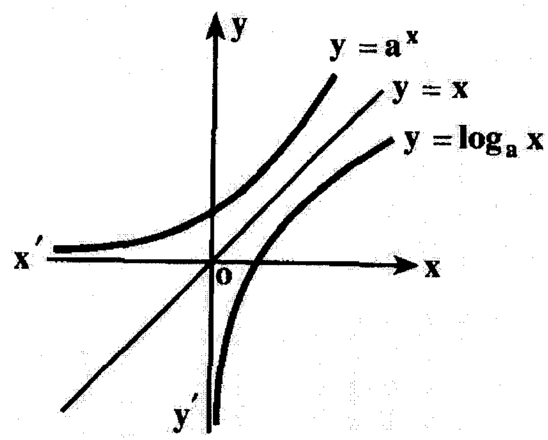


Figure B.4 shows the graph of $y = a^x$ and $y = \log_a x$

Example 1: Convert the given equations, $\log_7 1 = 0$ and $\log_3(\frac{1}{9})$, into the exponential forms.

$\log_7 1 = 0 \Leftrightarrow 1 = 7^0$

$\log_3(\frac{1}{9}) = -2 \Leftrightarrow \frac{1}{9} = 3^{-2}$

Example 2: Evaluate the given expressions $x = \log_2 8$, $x = \log_{10} 100$, $x = \log_5 1$, and $x = \log_7 7$.

$x = \log_2 8$ implies that $2^x = 8$, $x = 3$

$x = \log_{10} 100$ implies that $10^x = 100$, $x = 2$

$x = \log_5 1$ implies that $5^x = 1$, $x = 0$

$x = \log_7 7$ implies that $7^x = 7$, $x = 1$

Properties of Logarithms

1. Let a, u, and v are three real numbers and $a > 0$, $a \neq 1$. Consider the law of exponents, $a^u \times a^v = a^{u+v}$ and suppose that $x = a^u$ and $y = a^v$, thus $x \cdot y = a^{u+v}$ by using the relationship between exponential forms and logarithmic forms, we obtain $\log_a xy = u + v$. Now substitute $x = a^u$, $y = a^v$ and conclude:

   Theorem 1: $\log_a(xy) = \log_a x + \log_a y$ means that logarithm of the product of numbers is the sum of the logarithms of the numbers.

   Example: Given that $\log_{10} 2 \approx 0.3010$, $\log_{10} 3 \approx 0.4771$ and $\log_{10} 5 \approx 0.6990$. Compute $\log_{10} 15$ and $\log_{10} 6$.

$$\log_{10} 6 = \log_{10}(3\times 2) = \log_{10} 3 + \log_{10} 2 = 0.4771 + 0.3010 = 0.7781$$
$$\log_{10} 15 = \log_{10}(3\times 5) = \log_{10} 3 + \log_{10} 5 = 0.4771 + 0.6990 = 1.1761$$

2. Let a, u, and v are three positive numbers and $a \neq 1$. Let $\frac{x}{y} = t$, then Theorem 1,

$\frac{x}{y} = t \Rightarrow x = yt \Rightarrow \log_a x = \log_a y + \log_a t$ and $\log_a t = \log_a x - \log_a y$. Therefore,

   Theorem 2: $\log_a(\frac{x}{y}) = \log_a x - \log_a y$

   Example: $\log_{10}(\frac{2}{3}) = \log_{10} 2 - \log_{10} 3$

3. It is obvious that if $a,b > 0$, $a \neq 1$ then,

   Theorem 3: $\log_a b^x = x \log_a b$

   Example: $\log_{10} \sqrt[3]{100} = \log_{10}(100)^{\frac{1}{3}} = \frac{1}{3}\log_{10}(100) = \frac{1}{3}\times 2 = \frac{2}{3}$

John Napier, the inventor of logarithms, considers the irrational number $e \cong 2.71828$ as a base of logarithms. $\log_e$ is called the natural logarithm. Logarithm in base 10 is generally written without writing base 10 and is the called common logarithm. Nowadays in developed countries, students use calculators to evaluate logarithms, like this;

number	log key	answer
5	log	0.69897010
3.25	log	0.51188336

The invention of logarithms is one of the greatest discoveries in the history of mathematics since logarithms reduced the labor of multiplication and division of huge numbers by converting these operations into addition and subtraction of logarithms of these numbers.

Example 1: Show that $7^{\log_7 3} = 3$.

let $x = \log_7 3 \Leftrightarrow 7^x = 3$, therefore by substituting $7^{\log_7 3} = 3$

Note that $a^{\log_a b} = b$ always holds.

Example 2: Solve the given equation $\log(x+3) + \log x = 1$.

$$\log(x+3) + \log x = 1 \Rightarrow \log x(x+3) = \log 10$$
$$x(x+3) = 10 \Rightarrow x^2 + 3x - 10 = 0 \Rightarrow (x+5)(x-2) = 0 \Rightarrow x = 2 \text{ and } x = -5.$$

$x = -5$ is not acceptable. Why?

Note that $0 < a = b \Rightarrow \log a = \log b$

Example 3: Solve the given system $\begin{cases} 3^{x-y} = 27 \\ 4^{x+y} = 64 \end{cases}$.

First method: $\begin{cases} 3^{x-y} = 27 \\ 4^{x+y} = 64 \end{cases} \Rightarrow \begin{cases} 3^{x-y} = 3^3 \\ 4^{x+y} = 4^3 \end{cases} \Rightarrow \begin{cases} x-y=3 \\ x+y=3 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=0 \end{cases}$

Second method: using logarithms: $\begin{cases} x-y=\log_3 27 \\ x+y=\log_4 64 \end{cases} \Rightarrow \begin{cases} x-y=3 \\ x+y=3 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=0 \end{cases}$

Exercise: [Some of the exercises are as follows]

1. What are the domain and range of $y=\log_{10} x$?

2. Solve the given system of equations using logarithms $\begin{cases} 2^{x+y} = 16 \\ 2^{x-y} = 4 \end{cases}$

3. Solve the given equations for x.

$\log_5(x+1)+\log_5(x-1)=1$
$\log x - \log 15 = \log 0.02$
$\log_6 x = \log_6 121$

4. Show that the given equalities $\begin{array}{l} a) \log 5 + \log 5 + \log 5 = \log 5^3 \\ b) \log 3^2 + \log 3^{-2} = 0 \end{array}$ hold.

5. Is $\log_{10} x$ defined for $x<0$? Is it defined for $x=0$?

6. Consider matrix $A=\begin{bmatrix} \log a & \log b \\ \log b & \log a \end{bmatrix}$. Show that $|A| = \log ab \times \log \frac{a}{b}$.

7. Does the equality $a^{\log_a y} = y$ hold? Why?

Appendix C: Translation of sections on logarithms from the Iranian Textbook *Fundamentals of Mathematics* (p.58-82)

Chapter 3: Logarithms

3.1 Genesis of Logarithms

A certain bacterial culture doubles every day. After a day two bacteria would duplicate and there would be four bacteria. Consider the table below which shows the number of bacteria in a week.

Time (day)	0	1	2	3	4	5	6	7	8	9	...
Number of Bacteria	1	2	4	8	16	32	64	128	256	512	...

Table C.1 shows bacteria growth rate

A careful consideration of Table C.1 shows that the numbers of bacteria in the second row represent a square sequence (a geometric sequence with ratio 2) and the numbers in the first row (Time) represent an arithmetic sequence.

At the beginning of the 17^{th} century John Napier discovered that when we multiply two terms of a give geometric sequence, the product is a geometric term. For example, $4\times16=64$, where 4, 16, 64 are terms of the geometric sequence and correspond to 2, 4, 6 respectively in the arithmetic sequence. Therefore the multiplication of numbers in the geometric sequence corresponds to the addition of numbers in the arithmetic sequence.

Example 1: Consider the given table, find 8×128.

0	1	2	3	4	5	6	7	8	9	10	…
1	2	4	8	16	32	64	128	256	512	1024	…

Table C.2 shows a geometric and an arithmetic sequences

To find 8×128, the numbers corresponding to 8 and 128 can be obtained from the first row of the above table which are 3 and 7. We know that $8\times128=1024$ where 1024 corresponds to 10. Note that $3+7=10$. Therefore, instead of multiplying numbers in the geometric sequence, the corresponding terms in the arithmetic sequence can be added and then the corresponding term for 10, which is 1024, can be obtained. Therefore, numbers in the arithmetic sequence are logarithms of the corresponding numbers in the geometric sequence.

x: Logarithms	0	1	2	3	4	5	6	7	8	9	10	…
y: Numbers	1	2	4	8	16	32	64	128	256	512	1024	…

Table C.3. shows numbers and their logarithms

As you realize, the geometric terms can be written as powers of the number 2.

x: Logarithms	0	1	2	3	4	5	6	7	8	9	10	…
y: Numbers	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	…

Table C.3.1 shows numbers and logarithms of numbers in base 2

In fact, terms in the geometric sequence can be represented by number 2 to the power corresponding to the arithmetic terms, i.e. $y=2^x \Leftrightarrow x=\log_2 y$, such that y is a geometric term and x is the logarithm of y in base 2.

Example 2:

Numbers	16	$\times$	32	$=$	512
	$2\times2\times2\times2$	$\times$	$2\times2\times2\times2\times2$	$=$	$2\times2\times2\times2\times2\times2\times2\times2\times2$
Logarithms	4	$+$	5	$=$	9

Exercise 1: By using a calculator, find the powers of number 2 up to 2^{26} and fill out the Table C.3.1, and then answer the questions below.

Find out the multiplications below by using Table C.3.1. Do not multiply numbers, find an addition operation which indicates the result of each multiplication.

a) 256×128

b) 2048×1024

c) 131072×32

d) 4096×16

Note that logarithms can be employed in computing the powers of a number.

Example 3. Number: $32\times32\times32\times32=(32)^4=1048576$

Logarithm: $5+5+5+5 \;=4\times5=20$

Exercise 2. Compute the expressions below by using the above method and Table C.3.1.

a) $(256)^3$

b) $(64)^5$

c) $(1024)^0$

Logarithm of y in base 2, $\log_2 y$, is the number to which 2 should be raised to obtain y. This implies that exponential forms can be converted to logarithmic forms.

Example 4. If $2^x = 32$, determine the value of x by using the definition of logarithms.

$2^x = 32 \Leftrightarrow x = \log_2 32$ and by referring to Table C.3.1 $x = 5$.

Table C.4 contains a geometric sequence and its corresponding arithmetic sequence.

x: Logarithms	0	1	2	3	4	5	...
y: Numbers	1	5	25	125	625	3125	...

Table C.4. shows a geometric and an arithmetic sequence of numbers

Numbers in the geometric sequence can be written as powers of number 5.

x: Logarithms	0	1	2	3	4	5	...
y: Numbers	5^0	5^1	5^2	5^3	5^4	5^5	...

Table C.4.1 shows the logarithm of numbers in base 5

Therefore, the geometric terms can be represented by number 5 to the power of the corresponding arithmetic terms, $y = 5^x \Leftrightarrow x = \log_5 y$.

Example 5. According to Table C.4.1, $5^3 = 125$ is equal to $\log_5 125 = 3$

In general, $\log_b y$, is the number to which b is raised to obtain y, and $\log_b y = x \Leftrightarrow b^x = y$, $y > 0$, $b > 0, b \neq 1$.

3.2 Common Logarithms

Since counting numbers are written in base 10, logarithmic tables in base 10 are more applicable than other bases. Thus, this section focuses on logarithms in base 10.

x : Logarithms	0	1	2	3	4	5	6	7	8	9	10	...
y : Numbers	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}	...

Table C.5. shows the logarithms in base 10 of numbers

The relationship between the above two sequences (see Table C.5) is $y = 10^x$ where y represents a number and x represents the logarithm of this number in base 10. Logarithm of numbers (numbers are powers of number 10) in base 10 can be easily computed. For example $\log 1000000000$ is equal to 9 because $1000000000 = 10^9$. Furthermore, the number of zeros in the product of two numbers which are powers of number 10, is equal to the sum of the numbers of zeros in both numbers; therefore, there is no need to apply logarithms to find the product of numbers of powers of number 10. How can we find logarithms of numbers which are not powers of 10? For example, how can the logarithm of 2 in base 10 be evaluated? We need to estimate the natural numbers from 1 to 10 by fractional powers of number 10 (see Table C.6).

x : Logarithms	0								1
y : Numbers	1	2	3	4	5	6	7	8	9	10
y : Numbers	10^0	$10^?$	$10^?$	$10^?$	$10^?$	$10^?$	$10^?$	$10^?$	$10^?$	10^1

Table C.6. show the estimation of numbers from 1 to 10 by fractional powers of 10

By the definition of logarithms, logarithms of numbers which are integer powers of number 10 are integer numbers. Since any number can be expressed in scientific notation $a \times 10^n$, $1 \le a < 10$, we design an algorithm to evaluate the logarithm of a, $1 \le a < 10$.

An algorithm to compute common logarithms of numbers

1. let a be a small number greater than 1 and list all natural powers of a:

$a^1, a^2, ..., a^{24}, a^{25}, ...$ The sequence is a geometric sequence.

To find the product of two numbers such as x and y, estimate them by one of the terms of the above geometric sequence.

Example 6. If $x \cong a^{17}, y \cong a^8$ then $x \cdot y \cong a^{17} \cdot a^8 = a^{17+8} = a^{25}$

Example 7. If $x \cong a^{17}, y \cong a^8$, then $\frac{x}{y} \cong \frac{a^{17}}{a^8} = a^{17-8} = a^9$

Example 8. If $x \cong a^{17}$, then $x^{20} \cong (a^{17})^{20} = a^{340}$

2. To make this idea compatible with decimal powers, let a be a number which is a fractional power of the number 10.

Example 9. Consider $a = 10^{\frac{1}{32}}$ implies $a^{32} = 10$ (why?). Find the powers of a from 1 to 32 and fill out the table below.

n	Number: $a^n = 10^{\frac{n}{32}}$	Log: $\frac{n}{32}$
1	1.074	0.0312
2	1.154	0.0625
....	.....	.....
32	10	1

Table C.6.1 shows the algorithm of finding common logarithms of numbers

Exercise. Consider Table C.6.1 and verify the equation $\log(2 \times 3) = \log 2 + \log 3$.

Figure C.1 shows a sketch of a graph in which the $x-axis$ represents $\frac{n}{32}$ and $y-axis$ represents $a^n = 10^{\frac{n}{32}}$ (x and y values are taken from Table C.6.1).

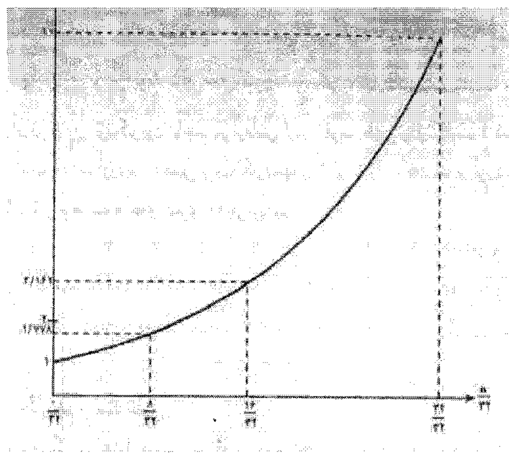


Figure C.1 presents a graph of values indicated in Table C.6.1

Example10. Estimate $\log 2$ by using the above graph.

Indicate 2 on $y-axis$, draw a line parallel to the $x-axis$ to intercept the graph. From the interception point draw a perpendicular line to intercept the $x-axis$, x interception indicates $\log 2 = \frac{965}{3200} = 0.301$. $Log2$ can be also estimated form Table C.6.1., since 2 is between 1.911 and 2.053 and respectively $\frac{9}{32}$ and $\frac{10}{32}$ are logarithms of 1.911 and 2.053 in base 10. Therefore, the estimation of $\log 2$ is $\frac{10}{32}$.

$Log3$, $\log 4$,..., $\log 9$ can be obtained by the same method. In other words, natural numbers from 2 to 9 are expressed as fractional exponents of number 10, thus Table C.6 can be completed as follows:

x : Logarithms	0	0.301	0.477	0.602	0.698	0.778	0.845	0.903	0.954	1
y : Numbers	1	2	3	4	5	6	7	8	9	10
y : Numbers	10^0	$10^{0.301}$	$10^{0.477}$	$10^{0.602}$	$10^{0.698}$	$10^{0.778}$	$10^{0.845}$	$10^{0.903}$	$10^{0.954}$	10^1

Table C.6.2 shows the logarithms of natural numbers from 1 to 10 in base 10

3.3 Logarithms and Scientific Notation

The common logarithms of numbers are more applicable than the logarithms in any other bases because of a relationship between a scientific notation of numbers and their common logarithms. Table C.7 shows the relationship between common logarithms of a few numbers and their scientific notation.

Numbers	Scientific notation of numbers	Numbers represented as powers of number 10		Common logarithms of numbers
20	2×10^1	$10^{0.301}\times10^1$	$10^{1.301}$	1.301
200	2×10^2	$10^{0.310}\times10^2$	$10^{2.301}$	2.301
2000	2×10^3	$10^{0.301}\times10^3$	$10^{3.301}$	3.301
20000	2×10^4	$10^{0.301}\times10^4$	$10^{4.301}$	4.301

Table C.7. indicates the relationship between common logarithms and scientific notation

Note that you can find the logarithm of 20000 by using a calculator, type 20000, and press Log. The calculator displays 4.301.

Exercise 1. Compute $\log 2000000000$ and $\log(2\times10^{17})$ using Table C.7.

Exercise 2. Fill out Table C.8.

a) Find a number which its logarithm is 17.699.

b) Find a number which its logarithm is 28.699.

Numbers	Scientific notation of numbers	Numbers represented as powers of number 10		Logarithms of numbers
50000	5×10^4	$10^{0.699}\times10^4$	$10^{4.699}$	4.699
500000	5×10^5	$10^{0.699}\times10^5$	$10^{5.699}$	5.699
5000000				
50000000				

Table C.8. presents common logarithms and scientific notation of numbers

In this section, a table of logarithms in base 10 is constructed (see Table C.6.1). Using Table C.6.1, the logarithm of numbers which are expressed in scientific notation can be estimated since the whole part of logarithms is the power of 10 and the decimal part can be obtained from Table C.6.1 .

3.4 Computation of Logarithms

[This section includes a picture of a Chinese wooden logarithmic table and a logarithmic table for numbers from 1 to 1000 in base 10]

3.5 Proof of the Laws of Logarithms

The definition of common logarithms implies $10^x = y \Leftrightarrow x = \log_{10} y$. In previous sections the product law of logarithms is practiced through exercises. This section presents the proof of the product law of logarithms.

Theorem 1: for any two positive real numbers, a and b, $\log_{10} ab = \log_{10} a + \log_{10} b$.

Proof: let $\log_{10} a = x_1$ and $\log_{10} b = x_2$ then

1) $a = 10^{x_1}$

2) $b = 10^{x_2}$.

By multiplying 1) and 2), we obtain $ab = 10^{x_1} \times 10^{x_2} = 10^{x_1 + x_2}$ and by the definition of logarithms, $\log_{10} ab = x_1 + x_2$. If we substitute $\log_{10} a = x_1$ and $\log_{10} b = x_2$ we obtain, $\log ab = \log a + \log b$.

Theorem 2: $\log a^n = n \log a$ for $a > 0$. (n is a positive integer; but this theorem can be proved for any real exponent)

Proof: Theorem 2 is an extension of Theorem 1, because $\log a^n = \log aaa...a = \log a + \log a + ... + \log a$ (n times), therefore $\log a^n = n \log a$.

Exercise 1: Prove Theorem 2 using the mathematical induction theorem.

Exercise 2: Prove that for any two positive real numbers, a and b , $\log(\frac{a}{b}) = \log a - \log b$.

Example 10: Compute $\log 5 + \log 20$ using Theorem 1.

$$\log a + \log b = \log ab$$
$$\log 5 + \log 20 = \log 5 \times 20 = \log 100 = 2$$

Exercise 3: Prove that Theorem 1 and 2 are hold for all bases of logarithms.

Theorem 3: $\log_c (ab) = \log_c a + \log_c b$

Theorem 4: $\log_c a^n = n \log_c a$

Theorem 5: $\log_c (\frac{a}{b}) = \log_c a - \log_c b$

Example 11: Apply the above theorems to simplify $\log(\frac{x^2 y}{z})$.

By Theorem 5: $\log(\frac{x^2 y}{z}) = \log(x^2 y) - \log z$, and by Theorem 3:

$\log(x^2 y) = \log x^2 + \log y$ and by Theorem 4 $\log x^2 = 2\log x$. Therefore,

$\log(\frac{x^2 y}{z}) = 2\log x + \log y - \log z$.

Example 12: Simplify the given expression $\log(\sqrt[3]{a}\sqrt{b})$.

$$\log(\sqrt[3]{a}\sqrt{b}) = \log(a^{\frac{1}{3}}b^{\frac{1}{2}}) = \log a^{\frac{1}{3}} + \log b^{\frac{1}{2}} = \frac{1}{3}\log a + \frac{1}{2}\log b$$

Example 13: Rewrite the expression below as a unique logarithm.

$A = \log\sqrt{p} - \log\sqrt{4p} + \log(\frac{1}{2}p^2) + \log 4$.

$$A = \log(\frac{\sqrt{p} \times \frac{1}{2}p^2 \times 4}{\sqrt{4p}}) = \log(\frac{p^{\frac{1}{2}} \times \frac{1}{2}p^2 \times 4}{2p^{\frac{1}{2}}}) = \log(\frac{1}{4} \times p^2 \times 4) = \log p^2$$

Example 14: Solve the equation $\log x - \frac{1}{2}\log(pq) = -\frac{1}{2}\log(\frac{p}{q})$, $p, q > 0$.

$$\log x - \frac{1}{2}\log p - \frac{1}{2}\log q = -\frac{1}{2}\log p + \frac{1}{2}\log q$$
$$\log x = \frac{1}{2}\log q + \frac{1}{2}\log q = \log q$$

so $x = q$.

Since common logarithms are more useful than other bases of logarithms, there is a formula to convert logarithms of different bases to common logarithms.

Theorem 6: If $x > 0$ then $\log_a x = \frac{\log x}{\log a}$

Proof: we know that $\log_a x = y$ implies that $a^y = x$, by the definition of logarithms. If $a = b$ then $\log a = \log b$; as a result $\log(a^y) = \log x$; so $y \log a = \log x$. Therefore $y = \dfrac{\log x}{\log a}$ and $\log_a x = \dfrac{\log x}{\log a}$.

Example 15: use a calculator to compute $\log_3 17$.

Exercises: [Some of the exercises are as follows]

1. Simplify the given expressions by using the theorems.

$\log(a^3 b^5)$

$\log(mr^{-2})$

$\log[(a+b)(a-b)]$

$\log(\dfrac{a^2}{b^2 c})$

2. Rewrite the expressions below as a unique logarithm.

$\log pq - \log 2q$

$\dfrac{1}{4}\log(ab) - \dfrac{3}{5}\log(a^2 b)$

3. Solve the given equations for x.

$a) \log 27 = 3 \log x$

$b) \log x + 2 \log 4 = 2 \log 12$

$c) \log(p-q) = \log(p^2 - q^2) - \dfrac{1}{2}\log x$

3.6 Application of Logarithms: Magnitude of an Earthquake and Intensity of a Sound Wave

In 1990, a great earthquake with a magnitude between 7.2 and 7.6 destroyed Rodbar and caused the death of many people in Northwest Iran. Usually, when there is an earthquake report on the news with a magnitude bigger than 5 on the Richter scale, people are afraid of possible casualties and damages. However, an earthquake with a magnitude between 3.5 and 4 on the Richter scale does not terrify people. How does the difference between 3.5 to 7.2 on the Richter scale influence the amount of damages and losses of life? What is the nature of measurement of magnitude of an earthquake that small numeric increases cause on the Richter scale indicate a great increase in damages?

It is interesting to know that an increase of magnitude 1 on the Richter scale corresponds to a tenfold increase in damages. For example, an earthquake of magnitude 6 is 10 times more severe than an earthquake of magnitude 5 and 100 times more severe than an earthquake of magnitude 4.

Exercise 1. Given that an earthquake of magnitude 4 has an average intensity of 1, fill out the table and answer the following questions.

Average Intensity	1					
Magnitude	4	5	6	7	8	9

a) How can the average intensity of an earthquake and the Richter scale measurement of the earthquake be related?

b) How can an earthquake magnitude of 9 be compared to an earthquake magnitude of 7?

Certainly, you notice that the Richter scale is a logarithmic scale.

Exercise 2. Answer the questions below considering that the Richter scale is a logarithmic scale.

a) Which number has a logarithm of zero?

b) An earthquake magnitude of 2 is imperceptible. Which number has a logarithm of 2?

c) An earthquake magnitude of 7.25 will cause a great amount of damage in a crowded area. What number has a logarithm of 7.25?

Richter scale: Earthquakes release a huge amount of energy. Released energy by a great earthquake is 100 billion times more than released energy by a light earthquake. During the past 150 years, different magnitude scales have been used to measure and compare earthquake magnitudes. Richter (1935), an American geologist, developed a logarithmic scale to measure the intensity of earthquakes. This scale is called Richter magnitude scale and is the most popular scale. Since an earthquake magnitude of less than 4.4 is imperceptible, released energy, by an earthquake magnitude less than 4.4 ($E_0 = 10^{4.4}$ joules) is considered as a standard base to measure an earthquake magnitude. The formula $M = \frac{2}{3}\log\frac{E}{E_0}$ evaluates an earthquake magnitude in which E is released energy in joules from an earthquake magnitude of M and $E_0 = 10^{4.4}$.

Classes	Magnitude
Light	M<4.5
Moderate	4.5<M<5.5
Strong	5.5<M<6.5
Major	6.5<M<7.5
Great	M>7.5

Table C.9. shows the magnitude classes of an earthquake

Example 16. In 1906, the San Francisco earthquake released an energy of around 5.96×10^{16} joules and destroyed the streets of the city. Find the magnitude of this earthquake.

By applying the formula, $M=\frac{2}{3}\log\frac{E}{E_0}$,

$$M=\frac{2}{3}\log\frac{5.96\times10^{16}}{10^{4.4}}$$

$$M=\frac{2}{3}\log(5.96\times10^{11.6}) \quad , \quad M=\frac{2}{3}(0.775+11.6)$$

$$M=\frac{2}{3}(\log5.96+\log10^{11.6}) \quad M=8.25$$

Example 17. The Rodbar earthquake had a magnitude of 7.2 to 7.6. Find the amount of released energy in joules.

By applying the formula, $M=\frac{2}{3}\log\frac{E}{E_0}$, $M=7.2, E_0=10^{4.4}$, then

$$7.2=\frac{2}{3}\log\frac{E}{10^{4.4}}$$

$$\log E=15.2$$

$$E=10^{15.2}=10^{15}\times10^{0.2}=1.584\times10^{15}$$

Exercise 3. Repeat Example 17 by considering the earthquake of magnitude 7.6 on the Richter scale.

The intensity of a sound wave (Loudness of a sound wave): A human ear can hear a wide intensity of sounds. The weakest sound intensity that a human ear can hear has a value of 1×10^{-12} $watts/m^2$ and is called the threshold of human hearing, $I_0=1\times10^{-12}$. To compare relative sound intensities, a decibel scale which is a logarithmic scale is employed. The formula $D = 10\log\frac{I}{I_0}$, where I $watts/m^2$ is the intensity of a sound and $I_0 = 1\times10^{-12}$ $watts/m^2$, indicates the level of a sound in decibels.

Intensity of a sound	Source of a sound
1×10^{-12}	Threshold of hearing
5.2×10^{-10}	Whispering
3.2×10^{-6}	Conversation
8.5×10^{-4}	Heavy Traffic
3.2×10^{-3}	Jack-Hammering
1×10^{0}	Threshold of pain
8.3×10^{2}	Jet engine taking off

Table C.10 shows the intensities of sounds

Example 18. Find the level of a sound in decibels of whispering with a 5.2×10^{-10} intensity.

$$D = 10\log\frac{I}{I_0}\ ,\ D = 10\log\frac{5.2\times10^{-10}}{10^{-12}} = 27.16$$

Logarithmic Scale

Since the logarithm of a number in bases different than 1 has a slower growth rate than the number itself, logarithmic scales are more convenient scales for comparing a wide range of numbers.

Appendix D: Translation of parts of a Textbook written by Pezeshk in 1942 (p.5-11)

1. Overview

1. Definition: consider two infinite sequences, a geometric sequence G, with initial term 1 and q as a common ratio, and an arithmetic sequence A, with 0 as the first term and r as a common difference. Write them below each other so that each term of G corresponds to a term in A, and these form a system D,

$$D)\begin{cases} G: \dots q^{-n}, \dots, q^{-2}, q^{-1}, 1, q, q^{2}, \dots, q^{n}, \dots \\ \\ F: \dots -rn, \dots, -2r, -r, 0, r, 2r, \dots, nr, \dots \end{cases}$$

Therefore,

a) Each term of the arithmetic sequence is the logarithm of the corresponding term in the geometric sequence.

b) Each term of the geometric sequence can be obtained by raising $q^{\frac{1}{r}}$ to the corresponding term in the arithmetic sequence.

2. $\log A$ is the notation to indicate the logarithm of number A.

3. Since terms of a geometric sequence are always positive it is concluded that only positive numbers have logarithms.

4. Theorem: consider two consecutive terms of a geometric sequence. You can insert geometric terms between these two terms so that the difference between any two consecutive terms approaches zero.

Proof: consider two consecutive terms q^{m}, q^{m+1} from a geometric sequence,

$1, q, q^2, \ldots, q^m, q^{m+1}, \ldots, q^n, \ldots$.. Insert $(p-1)$ geometric terms between q^m and q^{m+1}. The common ratio of the constructed sequence is $\sqrt[p]{\frac{q^{m+1}}{q^m}} = \sqrt[p]{q} = q^{\frac{1}{p}}$ and consider $q^{\frac{1}{p}} = q'$. The k^{th} term of the sequence is $q^m q'^k$ and the $(k+1)^{\text{th}}$ term of the sequence is $q^m q'^{k+1}$, and the difference between them is $q^m q'^{k+1} - q^m q'^k = q^m q'^k (q'-1) = q^m q'^k (q^{\frac{1}{p}} - 1)$. Their difference varies by $(q^{\frac{1}{p}} - 1)$ and when p becomes larger, $q^{\frac{1}{p}}$ becomes smaller. When p approaches infinity $q^{\frac{1}{p}}$ approaches 1 and $(q^{\frac{1}{p}} - 1)$ approaches 0. (i.e. the difference between the terms $q^m q'^k$ and $q^m q'^{k+1}$ becomes very small.

5. Conclusion 1: since we insert m geometric terms between any two consecutive terms of a geometric sequence, and respectively m arithmetic terms between the two corresponding consecutive terms of an arithmetic sequence, therefore, each term in the arithmetic sequence is the logarithm of the corresponding term in the geometric sequence.

6. Conclusion 2: any positive number a can be inserted between two consecutive terms of a geometric sequence. Consider a and its consecutive term and insert geometric terms such that the difference between a and its following term becomes very small. Therefore, a is a geometric term and there exists the corresponding arithmetic term which is the logarithm of a. Thus, any positive number has a logarithm.

7. Theorem: when a number can be found by inserting $(m-1)$ terms between two consecutive terms in a geometric sequence, and this number can also be found by

inserting $(m'-1)$ terms between two terms of a geometric sequence, in each case the number has the same logarithm.

Proof: consider two sequences $1, q, q^2, ..., q^m$ and $0, r, 2r, ..., mr$. First, insert $(m-1)$ terms between two consecutive terms of both sequences. Second, insert $(m'-1)$ terms between two consecutive terms of both sequences. Suppose A be $(k+1)$ term in the first sequence as well as A be $(k'+1)$ term in the second sequence. Therefore, from the first sequence $A = (\sqrt[m]{q})^k, \log A = \frac{r}{m}k$ and from the second sequence $A = (\sqrt[m']{q})^{k'}, \log A = \frac{r}{m'}k',$. Then

$$\begin{aligned}
&(\sqrt[m]{q})^k = (\sqrt[m']{q})^{k'} \\
&q^{m'k} = q^{mk'} \\
&km' = mk' \qquad \text{implies that } \log A \text{ is unique.} \\
&\frac{k}{m} = \frac{k'}{m'} \\
&r\frac{k}{m} = r\frac{k'}{m'}
\end{aligned}$$

Any number has only one logarithm in each logarithmic base.

8. Definition: in each logarithmic system, a number where logarithm is 1 is called the base of the system.

Since logarithmic systems depend on the values of q (common ratio) and r (common difference), there exists an infinite number of logarithmic systems. The most useful logarithmic system was invented by Henri Briggs and uses common logarithms or logarithms in base 10. John Napier, the inventor of logarithms, introduces the logarithmic system base $e \approx 2.718...$, called the Napierean logarithm.

9. The logarithm of N in base h is denoted by $\log_h N$. Common logarithm of N is written as $\log N$ and Napierean logarithm of N is written as $\ln$.

2. Properties of Logarithms

10. Theorem: the logarithm of the product of numbers is equal to the sum of the logarithms of each number.

a) Consider two numbers A and B, a geometric sequence, and an arithmetic sequence. Insert geometric terms between two consecutive terms of the geometric sequence to obtain A and B, and, respectively, the corresponding terms in the arithmetic sequence are inserted.

$$\begin{cases} \dots\dfrac{1}{\alpha^n},\dots,\dfrac{1}{\alpha^2},\dfrac{1}{\alpha},1,\alpha,\alpha^2,\dots,a^n \\ \\ \dots-n\beta,\dots,-2\beta,-\beta,0,\beta,2\beta,\dots,n\beta \end{cases}.$$

Therefore, suppose that $A=a^p, B=a^q \;\; A\cdot B=a^p\cdot a^q=a^{p+q}$ and $\log A=p\beta$, $\log B=q\beta$, $\log(AB)=(p+q)\beta$ so $\log(A.B)=\log A+\log B$

b) When there are more than 2 numbers such as A, B, C, D,...

$$\log A.B.C.D.\dots=\log A.(B.C.D\dots)=\log A+\log(B.C.D\dots)$$
$$\log B.C.D\dots=\log B.(C.D\dots)=\log B+\log(C.D\dots.)$$
$$\log C.D\dots=\log C+\log D\dots$$
$$\dots\dots$$

therefore by substituting $\log A.B.C.D\dots=\log A+\log B+\log C+\log D+\dots.$

11. Theorem: the logarithm of the quotient of numbers is obtained by subtracting the logarithm of the denominator from the logarithm of the numerator.

Suppose:

$$\frac{A}{B} = C$$
$$A = B.C$$
$$\log A = \log B + \log C$$
$$\log C = \log A - \log B$$

12. The logarithm of the m^{th} power of a number is equal to m times the logarithm of the number.

$$A^m = A.A.A....A \qquad m \text{ times}$$

$$\log A^m = \log A + \log A + \log A + ...\log A = m \log A$$

13. Conclusion: the logarithm of N in base b is the number to which b is raised to obtain N.

By taking the logarithm of both sides of $N = b^a$, we obtain $\log_b N = a \log_b b$. Since b is base and $\log_b b = 1$, thus $\log_b N = a$

14. Logarithm of the m^{th} root of a number is equal to the logarithm of the number divided by m.

$$a = \sqrt[m]{A} = A^{\frac{1}{m}}$$
$$\log a = \frac{1}{m} \log A$$

3. Common Logarithms (logarithms in base 10)

15. A system of common logarithms can be written as follows:

$$\begin{cases} ..., \dfrac{1}{10^n}, ..., \dfrac{1}{10^2}, \dfrac{1}{10}, 1, 10, 10^2, ..., 10^n, ... \\ \\ ..., -n, ..., -2, -1, 0, 1, 2, ..., n... \end{cases}$$

As you see, the common logarithm of a number in base 10 to an integer power is equal to the integer power. For example

$$\log\frac{1}{10^3} = \log 10^{-3} = -3$$
$$\log 10^0 = 0$$

Therefore, the logarithm of a number that is not an integer power of 10 can be found be estimating the number between two consecutive powers of 10 and obtaining the corresponding estimation in an arithmetic sequence. This logarithm includes an integer part and a decimal part. The integer part is called the characteristic and the decimal part is called the mantissa.

16. Theorem: the characteristic of the logarithm of a number greater than 1 is equal to the digits of the number minus 1.

Consider A with $n+1$ digits. A can be estimated between 10^n and 10^{n+1}. $10^n < A < 10^{n+1}$. By taking the common logarithm of the inequality, $n < \log A < n+1$. This implies that the characteristic of $\log A$ is n.

17. The logarithm of a number smaller than 1 has a negative characteristic.

Appendix E: Translation of several Finance Problems taken from *Mizanoalhesab* Written by Mir Krinish in 1895 (p.8-18)

Problem 59. A barrel is filled with syrup. The total volume of a barrel, a, equals to the volume of 100 pitchers and a pitcher of this syrup is worth $c = 36$ tomans. Consider b as the volume of the pitcher. A person wants to reduce the concentration of syrup such that the value of each pitcher decreases to 1 toman. He dilutes the syrup with water as follows: he takes out one pitcher of syrup from the barrel and pours one pitcher of water in the barrel. He mixes syrup and water and repeats this process until the value of each pitcher of syrup becomes $d = 1$ toman. Find out how many times this person needs to repeat this process to decrease the value of each pitcher to $d = 1$ toman.

Answer. It is clear that the ratio of the volume of pure syrup in the barrel to the volume of pure syrup in the pitcher is equal to the ratio of the volume fraction of the syrup in the barrel to the volume fraction of the syrup in the pitcher.

By taking out the first pitcher of syrup and pouring in a pitcher of water, the volume of syrup is $a-b$, so $\frac{a}{b} = \frac{a-b}{x}$ and $x = \frac{b}{a}(a-b)$ where x indicates the volume of the syrup in a pitcher. So by taking out the second pitcher from the barrel, the volume of syrup in the barrel is $(a-b) - \frac{b}{a}(a-b) = \frac{(a-b)^2}{a}$ and the equality of ratios is $\frac{a}{b} = \frac{\frac{(a-b)^2}{a}}{x}$ where $x = \frac{b(a-b)^2}{a^2}$. Repeating this process and taking out n pitchers, the volume of the syrup in the barrel becomes $\frac{(a-b)^n}{a^{n-1}}$ which is worth $c = 36$ tomans per pitcher. So the value of syrup in the barrel is $\frac{(a-b)^n}{a^{n-1}} \times c$. On the other hand, the value of each pitcher of diluted syrup is decreased to $d = 1$, so ad indicates the value of diluted syrup in the barrel. By solving $\frac{(a-b)^n}{a^{n-1}} \times c = ad$, n indicates the number of times that the person needs to take out pitchers of syrup to decrease the value of each pitcher to $d = 1$ toman, so

$$n \log(a-b) + \log c - (n-1)\log a = \log a + \log d$$

$$n = \frac{\log c - \log d}{\log a - \log(a-b)} = \frac{\log 36 - \log 1}{\log 100 - \log 99} = 356$$

The formulas below are employed in computing the compound interest of an investment:

1) $\log s = \log a + n \log p$ is equivalent to $s = ap^n$ [the rest of formulas are obtained from formula 1].

2) $\log a = \log s - n \log p$

3) $\log p = \dfrac{\log s - \log a}{n}$

4) $n = \dfrac{\log s - \log a}{\log p}$

Application problems in using the above formulas

Problem 60. A population of a city is 2,000,000. Suppose each year the population of city will increase by 2%. Find the population of the city after 100 years.

$$a = 2000000, p = 1.02, n = 100$$
$$\log a = \log s - n \log p$$
$$\log s = 6.3010300 + .8600200 = 7.1610500$$
$$s = 14490000$$

After 100 years the city's population is around 14490000.

Problem 61. A person has to pay back his debt in a total of 6,000 tomans. Find out how much he borrowed 4 years ago at 4% compounded interest rate. In other word, how much investment at 4% compound interest rate after 4 years will be 6,000 tomans.

$$s = 6000, p = 1.04, n = 4$$
$$\log a = 3.7781513 - 0.0681332 = 3.7900181$$
$$a = 5128.827$$

Problem 62. A person lends 600 tomans to another person and he asks for 800 tomans after 3 years. Find out the interest rate.

$$a = 600, s = 800, n = 3$$
$$\log p = \frac{2.9040900 - 2.7781513}{3} = .0416462$$
$$p = 1 + \frac{c}{100}, 1.1006 = 1 + \frac{c}{100}, c = 10.06$$

Problem 63. How long will it take an amount of money to double if it is invested at 4% compounded annually?

$$s = 2a, p = 1.04$$
$$n = \frac{\log 2a - \log a}{\log p} = \frac{\log 2}{\log 1.04} = \frac{.3010300}{.0170333} = 17$$

Problem 64. A person borrows $a = 6{,}000$ tomans for $n = 10$ years at 5% compounded annually. In addition, he borrows $b = 500$ tomans at the end of each year. Find out how much debt he will have at the end of 10 years.

Answer. The amount of a tomans is worth ap^n after n years where $p = 1 + \frac{c}{100}$. The amount of b that he receives at the end of the first year is worth bp^{n-1}, the amount of b that he receives at the end of the second year is worth bp^{n-2}…., and so on. These values $bp^{n-1}, bp^{n-2}, ..., bp^2, bp$ are geometric sequences. We know that the sum of sequences can be computed from $s = \frac{aq^{n-1}(q-a)}{q-1}$. Therefore, $\frac{bp(p^{n-1}-1)}{p-1}$ and

$\varphi = ap^n + \frac{bp(p^{n-1}-1)}{p-1}$ is the total amount of his debt.

$$ap^n = 9773.37$$
$$p^{n-1} = 1.551328$$
$$p^{n-1} - 1 = 0.551328$$
$$\frac{bp(p^{n-1}-1)}{p-1} = 5788.94$$
$$\varphi = 15562.31$$

By considering $a = b$ in the above equation,

1) $\varphi = \frac{ap(p^n - 1)}{p-1}$

2) $\log \varphi = \log a + \log p + \log(p^n - 1) - \log(p-1)$

3) $n = \frac{\log[ap + \varphi(p-1)] - \log a}{\log p} - 1$

Application problems on using the above formulas

Problem 65. A businessman invests 4000 tomans each year at 4% compounded annually for 6 years. Find out how much money he will have at the end of 6 years.

$$a = 4000, n = 6, p = 1.04$$
$$p - 1 = .04, p^n = 1.265318$$
$$\log a = 3.6020600$$
$$\log p = 0.0170333$$
$$\log(p^n - 1) = -0.5762333$$
$$\log(p - 1) = -1.3979400$$
$$\log \varphi = 4.4408000$$
$$\varphi = 27593.07 toman$$

Problem 66. A businessman expects to receive 10,000 tomans after 20 years from a person. However, the businessman asks for yearly payments. Consider 4% annually compounded interest rate and compute the amount of yearly payments.

$$\varphi = 10000, n = 20, p = 1.04$$
$$\log a = 2.5090712, a = 322.902$$

Problem 67. An amount of a tomans at $\frac{c}{100}$ annually compounded rate is invested for n years. At the end of each year the amount of b is withdrawn. Find out how much money R will remain in the account after n years.

Answer. ap^n is the amount of money at the end of n years if no money is taken out from the account. If an amount of money b is not withdrawn in the $(n-1)^{\text{th}}$ year it has a value of bp^{n-1}. If the amount of money b is not withdrawn in the

$(n-2)^{\text{th}}$ year it has a value of bp^{n-2}, ..., and if the amount of money b is not withdrawn in the last year it has a value of b. The values form a geometric sequence bp^{n-1}, bp^{n-2}, ..., bp^2, bp, b and the sum of the geometric sequence is $b(\frac{p^n-1}{p-1})$. Therefore,

1) $R = ap^n - \dfrac{b(p^n-1)}{p-1}$

2) $a = \dfrac{b(p^n-1)}{p^n(p-1)} + \dfrac{R}{p^n}$

3) $b = (ap^n - R)(\dfrac{p-1}{p^n-1})$

4) $n = \dfrac{\log[b-(p-1)R] - \log[b-(p-1)a]}{\log p}$

Problem 68. A person invests 30000 tomans at 4% compounded annually, and he receives 800 tomans at the end of each year. Find out what will be the remaining amount after 15 years?

$a = 30000, b = 800, n = 15, c = 4, p = 1.04$
$R = 38009.41$

Problem 69. A person wants to support his son for 6 years by paying 500 tomans each year. If the interest rate is 3.5% how much does he need to invest in order to cover his son expenses?

$b = 500$, $n = 6$, $c = 3.5$, $p = 1.035$; since there is no money left at the end of 6 years, $R=0$. By applying $a=\frac{b(p^n-1)}{p^n(p-1)}+\frac{R}{p^n}$, $a=2664$..

Problem 70. A person wants to sell his property. There are three customers: one wants to pay in cash 34500 tomans, second one wants to buy for 38000 tomans by paying 6000 tomans in cash and the remaining amount in 4 yearly payments of 8000 tomans. The third customer wants to buy for 40000 by paying 4000 tomans in cash and the remaining amount in 6 yearly payments of 6000 tomans. The seller wants to know, with the 5% interest rate, which one of the customers will pay the highest amount?

Answer. The second customer wants to pay 32000 tomans in 4 years. By applying $a=\frac{b(p^n-1)}{p^n(p-1)}+\frac{R}{p^n}$, $a=28367.6$.

Since $b=8000, n=4, p=1.05, R=0$, therefore, the second customer pays 6000+28367.6=34367.6.

Third customer wants to pay 36000 in 6 years. Therefore $a=30454.2$ and he pays $4000+30454.2=34454.2$. The results show that the first customer pays the highest amount.

Problem 71. A person should pay back 1000 tomans in 5 years. What are the yearly payments if interest rate is 5% compounded annually?

$a=1000, n=5, c=5, p=1.05, R=0$
$b=230.97$

Problem 72. A person borrowed 100,000 tomans from a bank at 5% annually compounded. In order to cover his expenses, he spends each year 6000 tomans. How long he will be able to cover the expenses?

$a = 100000, b = 6000, p = 1.05, R = 0, c = 5$

$$n = \frac{\log 6000 - \log 1000}{\log 1.05} = (\log \frac{6000}{1000}) : \log 1.05 = 36$$

He will have money for 36 years, in 37th years he will have $R = 4163.7$ remaining money.

Problem 73. A person borrows 20,000 tomans and puts up his property income of 1500 tomans as collateral. How long will it take to pay back his debt at 5% annually compounded rate?

$a = 20000, b = 1500, p = 1.05, R = 0$

$$n = \frac{\log 1500 - \log 500}{\log 1.05} = \frac{\log 3}{\log 1.05} = 22$$

It will take 22 years to pay back the debt.

Appendix F: Transcripts of Interviews and Students' Written Questionnaires

"I"- Interviewer and "S"- Student

Case 1: The subject was an independent (female, mature) student from course A (pre-calculus course). Interview was conducted in October 2006.

1. I: do you want to go John Molson School?
2. S: I haven't decided yet. I heard some good things about McGill, regard to future job. I am doing research.
3. S: this no [she reads "have you ever seen log notation?" from the questionnaire].
4. I don't remember. It is possible we did in math 536, advance math in high school. I took 536 almost 14 years ago.
5. S: I do not know is that same thing.
6. I: you probably had it in math 536. I know that the new version of 536 has it.
7. S: Everything we are doing now, I remember I did it in high school. And I remember we did cosine, but because of high school I did not pay attention.
8. S: I know this [$3^2 = 9$] but this [$\log_3 9$] I have never seen.
9. I: try to do what you can, if you feel to do.
10. S: okay.
11. I: if you look at the review example you will get an idea.
12. I: it is called log 9 base 3 which is equal to 2.
13. S: like, how many times 3 to get 9.
14. I: ok.
15. S: Here, that means how many times multiply 2 to get 8, 3.
16. I: if you are feeling to do, try them and use review example. It is not about measuring your knowledge, it is how do you learn and understand logarithms.
17. S: here, log 4, 16 that would be 2 because 4 times 4 is 16.
18. I: yes, it is called log 16 base 4.
19. S: oh, yeah okay.
20. I: can you please write down your answers.
21. S: okay.
22. S: [she followed the column order in filling the table], this one do I get to use the calculator?
23. I: no. Can you please complete the table row by row?
24. S: oh, sorry. [She fills out $4^2 = ?$ and then tries to multiply 4s to obtain 256].
25. S: [mumbling] 4 times 4 is 16, 16 times 4 , 64 times4 is 256.
26. S: this would be 1[$4^? = 4$], so 1[$\log_4 4 = ?$].
27. S: this [[$\log_4 1 = ?$] silence um---and jumps to another one, [however to compute $4^? = 1$ she thought of a decimal number 0.25]
28. S: this[$\log_4 \frac{1}{16} = ?$] I am not sure, fractions, fractions I am really bad.

29. S: I guess this [$\log_4(-16) = ?$] would be -2, that is the best I can do.
30. I: continue if you want.
31. S: function of x is equal to 4 to power x, okay then estimate that...
32. S: silence um---
33. I: have you ever used a graph?
34. S: well, we have started to do a bit of graph in math class, but I mean I use in micro macro, I mean it is a bit more complicated, this is just like assumption and this and that to get formula.
35. I: do you know how to find a point on the graph?
36. S: um---, silence um--- 4 power half, the thing I am really bad with fractions, um---- square roots.
37. I: do you remember the estimate of square root of 2.
38. S: no, I do not remember.
39. I: try to guess what it can be.
40. S: okay. Silence
41. I: what is your best guess?
42. S: I consider square root of 2 as 1, 4 to power 1 which is 4.
43. I: can you try on the graph to show me how you can read that from the given graph.
44. S: x is 4 um--- no, 1 is x , 1 on x-axis and silence...
45. I: draw the vertical line from 1 to the graph, what is the y value of this cross point?
46. S: silence
47. I: the vertical line passing through 1 cross the graph on a point, now draw the horizontal line from this cross point to intersect the y-axis, so y is 4.
48. I: square root of 2 is almost 1.41. Please use the graph to find $4^{\sqrt{2}}$.
49. S: okay, that would be the 5.64.
50. I: how did you find it?
51. S: I multiply 4 by 1.41.
52. I: 1.41 is the power?
53. I: since the graph of $f(x) = 4^x$ is given. Try to use the graph and estimate $4^{\sqrt{2}}$.
54. S: okay, 1.41 on x-axis um---. will give me 6 on y-axis.
55. S: now, $\log_4 4^{\sqrt{2}}$ silence ...
56. I: please tell me what you are thinking.
57. S: I know this [$4^{\sqrt{2}}$] is 6 or around 6. I am thinking how many times this [4] goes to give 6, so I am thinking of one point something, so let say um---. I guess it would be 1.41.
58. S: log 7 base 4 plus log 5 base 4. I never done this, I do not know.
59. I: Try what you can, and what do you think.
60. S: so how many times 4 to get 7 plus how many times 4 to get 5, maybe one point something it will give you 2 something, maybe up to 3.
61. S: so let see if I do 12, I can not add those two. Because I was thinking maybe can add log 7 base 4 to log 5, make it log 12, but by doing that it does not give me 2 something.
62. I: by doing what?

63. S: by adding log 7 to log 5 and have log 12 base 4, but 4 times 4 would be 16 it would be over which I do not think I can do that.
64. S: but let say 12, um--- I am just I do not know it work out either so 2 times 2, 4 times 2 is 8. No from what I see I do not think you can add and solve it. it is kind of like bracket.
65. S: so 3 times 3 is 9, 9 times 3 is 27, so [$\log_3 27$] is 3.
66. S: 3 times um--- 27 um--- that is like point something. Let say 0.5. 3 times 0.5 would be 1.5. If we would be um--- I do not know is it possible actually add two log bases together and two numbers together and get the answer.
67. I: no.
68. S: it is impossible. So you need to do it one at the time. Okay.
69. S: [she reads Activity 4] let say this x is 3 would be 9 to 10 , 10 times 10 one hundred, let say x is 10 , to power of 3 is 1000.
70. S: that would be 10, no it is not hold.
71. I: can you please explain why?
72. S: x is 10, this would be 3 and log 10 base 10 is 1. 1 times 3 is 3 um ---now, log 10 base 10 is 1 and 1 power to three is one, and it does not hold.
73. S: find the value of x such that log (x-1) to the power x plus log x to the power, no base x equal to one, so this whole thing here has to be equal to one.
74. S: okay, let's take 2, log (2-1) base 2 plus log 2 base 2 equal 1? So this would give me 2.
75. I: how did you find 2?
76. S: um--- log 1 base 2 plus one, log 1 base 2 .silence um---
77. I: do you know a power that we can raise any number to this power and obtain one.
78. S: no, I do not remember.
79. I: you considered x equal 2. How did you find it?
80. S: it is a guess, x can be any number, I picked the lowest one.
81. S: consider the inequality what do you think about this inequality. Silence…
82. I: do you know this notation?
83. S: absolute value.
84. I: okay
85. S: that can not be one, because it gives 0 , so I will try with two so it can be absolute value of (4-1) and um---
86. S: no , 4 so 16-1 log 15 base 4 um--- Silence…
87. I: what you are trying?
88. S: I am trying to pick a number.
89. S: this is positive, negative
90. I: what is negative, positive?
91. S: x to be negative or positive, and when x is negative this come to positive and base is negative. Assuming that x would be negative number.
92. I: have you been worked with log key on a calculator?
93. S: with log never.
94. I: probably you saw this key on calculator?
95. S: yes, I saw it.

Case 2: Student 2 (female, non-mature) was taking course A at the time of the interview, January 2007.

96. S: I do not like logs.
97. S: If this [$\log_2 8$] is going to be the same as this I can put here 2, um--- . ok, 2 is base to power 3 is 8. Answer of this [$\log_2 8$] is 3.
98. S: complete row by row, I am doing the same as here. 4 to what power gives us 16, it is 2, because 4 times 4 is 16. Answer is 2.
99. S: here, I think it is 4.
100. S: here, 4 equal to 4 it is one.
101. S: for this zero.
102. S: here is one over two, I think so.
103. I: One over two? Why?
104. S: because $4^2 = 16$, [she writes $4^{\frac{1}{2}} = \frac{1}{16}$].
105. I: what is $4^{\frac{1}{2}}$, what other notation we can show it?
106. S: point 5.
107. I: what other notation to show $4^{\frac{1}{2}}$.
108. S: I do not know.
109. I: it means square root of 4.
110. S: oh right, it is gonna be 2.
111. S: well, 4 times um---- because I am looking at um---, if um---.
112. I: what is 4^{-1}?
113. S: -4.
114. I: no, do you know about negative powers.
115. S: no, I do not know.
116. I: $4^{-1} = \frac{1}{4}$ comes by the definition of negative powers.
117. S: so 4 to -2 is one over 16.
118. S: here, is the same. Oh, no. I think it is going to be 16, but -16. It is not the same. No, it can not be -16.
119. S: no it can not be log negative.
120. S: the graph of $f(x)$ is given. We do not know x.
121. I: do you know how to read the graph.
122. S: um--- , yes.
123. S: I think x is $\sqrt{2}$ here.
124. I: true, it is between 1 and 2, consider it as 1.41.
125. S: ok. Estimate log um--- this is the same.
126. I: you find square root of two on the graph but the question wants us to find $4^{\sqrt{2}}$.
127. S: I need y, so I just go like that, it is 7.
128. S: so for $\log_4 4^{\sqrt{2}}$, I do like this [$4 = 4^{\sqrt{2}}$] and it has to be $\sqrt{2}$, [$4^{\sqrt{2}} = 4^{\sqrt{2}}$].
129. I: ok.

130. S: evaluate the equations. I need to see this page.
131. S: $\log_4 7$, what power of 4 gives us 7. Silence…
132. I: what do you think?
133. S: oh, the one that I had before, $4^{\sqrt{2}} = 7$ and for $\log_4 5$, I need a power of 4 to get 5, $4^{\sqrt{1.1}}$. I do not know exactly but $4^{\sqrt{2}} + 4^{\sqrt{1.1}}$.
134. I: why you add them up?
135. S: oh, sorry, $\sqrt{2} + \sqrt{1.1}$.
136. S: I will put this here 3 to power what is 27, 3; now other one, 27 to what power is 3.
137. S: square root, oh no.
138. S: 27 is 9 times 3. 3 times 3 times 3, root 3.
139. I: can you show it as a power.
140. S: one over three.
141. S: so three times one over three is 1.
142. S: does the equality hold? No.
143. I: why do you think it does not hold?
144. S: because it is gonna be different, the answer is different because 3 here only acting on x, but over log is to power 3.
145. I: ok. Can you have an example?
146. S: for example x=2, so 2 powers 3 is 8, but 10 to what power give 8. um--- I do not know. Silence…
147. I: maybe you can find an easier example.
148. S: like 1, I do not know.
149. I: Can you choose a number which is a power of 10.
150. S: ok, thousand, or 10.
151. S: it has to be log 10 times 3, $\log_{10} 1000$, and $10^x = 1000$ power is 3.
152. S: $(\log_{10} 10)^3$, so log 10, 10 is 10 equal to 10 is 1.
153. S: the answers are different.
154. S: find the values of x such as log…, um---.
155. S: values of x, here $\log_x x$ as the same $x^1 = x$ it is one.
156. S: here $\log_x (x-1)$ is zero, because I got one and equal one.
157. If I can get $\log_x (x-1)$ zero it is right. I think it is minus something. It can not be negative.
158. S: um---, $x^0 = (x-1)$ and $x-1=0$ so x is one.
159. S: consider this inequality what do you think? No it is wrong, because the log should be always positive. I mean bigger than two.
160. I: why?
161. S: I know that first log should be positive and second there is here x square minus one should be positive and there two. It can not be negative.
162. S: I do not need a negative here, can I change it?
163. I: yes. You can write it down.

164. S: $\log_x |x^2 - 1| \geq 0$ this is the correct one. It can not be two and above two but I do not know about one. Here if I put one, it is gonna be log zero base 1. um--- I think it has to be equal to zero.
165. I: ok.
166. S: is that right?
167. I: important to me is how you think and solve them. It does not a matter right or wrong.

Case 3: This student (female, non-mature) was a student registered in course B (calculus course). Interview was conducted in October 2006.

168. S: My calculus teacher told us you need logarithms in university, but we never did it.
169. I: it is fine, there is a review example.
170. I: what about exponentiation?
171. S: I have done that. But, logarithms were last chapter of the book and we did not have time to go through it.
172. S: I know that [$3^2 = 9$)]and I understand this [$\log_3 9 = 2$], this [$\log_2 8 = ?$] is asking me what is the power of 2 equal to 8, just like 3 to power 2 is equal to 9. So what is the power of 2 equal to 8, 2 power 2 is 4, um--- 3 power to 2 is 9, so can it be like a fraction?
173. I: no, this is not a fraction.
174. S: okay. Something to the power of 2 has to be equal 8. 2 to power 2 is 4, um--- 2 to power 3 is 8. So therefore log 2, 8 is 3.
175. I: continue if you want and it is not about testing your knowledge, I am interested in how you can solve them.
176. S: okay. What power of 2 um--- [she did not notice that the base of logarithms in Activity 1 is 4].
177. I: follow row by row, please.
178. S: um--- 4 times 4 is 16. Can I write?
179. I: yes of course.
180. S: mumbling 4 by 4, 16 by 4 and 64 by 4, power 4.
181. S: power of one.
182. S: 4 to what power is equal to one? Now, we will go in fraction.
183. I: no, this is not fraction.
184. S: um---to power of zero.
185. I: good.
186. S: okay. Four to the power of what?
187. S: silence, um--- four to the power of what negative two.
188. S: silence, how the negative will get involve? Silence…
189. S: no, actually no, like a negative power, that is not how it works really? Silence…
190. I: what do you think?
191. S: I should know this negative one.
192. I: do you know a power to raise 4 and obtain a negative number?

193. S: no that is impossible. Okay. Do not exist.
194. S: I do not know is that the same? Is that [$\log_4(-16) = ?$] does not exist too?
195. I: good question. What is the relation between exponentials and logarithms?
196. S: that would be the same. $\log_4(-16) = ?$ is impossible.
197. S: okay, here, wait this fraction one.
198. S: what power of 4, I am assuming to be a fraction, 4 to the power of a fraction. That is possible. Is not it?
199. I: what else it can be?
200. S: decimal, fraction , um---
201. S: I did power to 2, 4 , 1 , zero and um---
202. S: negative power! That is not!
203. S: Because, 4 to the power of negative 2, I am assuming that would be negative two.
204. I: have you ever worked with negative powers?
205. S: not really, I am not good it math.
206. S: that is what I think, negative two. I just do not know how that works.
207. I: what do you think would be 4 to power -1?
208. S: -4
209. I: you multiply 4 by negative one, but negative one is power of 4.
210. I: do you remember the definition of negative exponents?
211. S: um--- not really.
212. I: You had a good guess.
213. S: okay. The graph of …[she reads Activity 2].
214. I: have you ever worked with a graph?
215. S: yes. Not a lot.
216. S: estimate $4^{\sqrt{2}}$, um---the square root of 2 is um---silence
217. S: I just have to know what is the square root of 2 is. It is just one, no it is 2 times. Silence …
218. S: I can get all if I had a calculator. That is why now I do not know what the square root of two is. You forget basic knowledge, because you plug into a calculator.
219. I: I see. Can you estimate it?
220. S: the square root of 4 is 2? right, because 2 times 2 is equal 4, so what times what equals 2? I am thinking like a half.
221. S: [She multiplies $\frac{1}{2} \times \frac{1}{2}$ to verify her guess].
222. S: oh, no silence …
223. I: you told that square root of 4 is 2, and what is the square root of 1?
224. S: it is one.
225. I: can you estimate square root of two?
226. S: not really.
227. I: 2 is between one and 4, so what that gives you?
228. S: um---okay.
229. S: um--- I am trying to find what times to what gives me 2, since 2 times 2 is 4, what times to what gives me 2.

230. S: [she tries $\frac{3}{2}$, $\frac{3}{2}\times\frac{3}{2}$] probably I am off of my number, because 9 over 4 is definitely not two. So um--- silence…
231. S: so 1.2 times 1.2 is um--- probably not.
232. S: I am not sure, I stop here and I can not handle this question.
233. I: okay, I will give you square root of two, so you can continue the question. Square root of two is almost 1.41.
234. S: so then estimate that, okay, square root of 2 is power of 4, $f(x)=4^x$ and x is 1.41. Silence...
235. S: [She uses the graph and finds $4^{1.41}$ as 6.5].
236. S: so now estimate um---, how would I estimate $\log_4 4^{\sqrt{2}}$.
237. I: do you think you can use the graph?
238. S: no, because I will put the 4 um--- wait maybe.
239. S: no, let me check that answer goes 4, silence…
240. S: [she checks several times the review example and table in Activity 1, then she finds $\log_4 4^{\sqrt{2}} = 1.41$].
241. I: do you think you can use the graph?
242. S: no, I do not think so. Maybe, there is a relation. Yes, you should be able too.
243. S: the only thing is that y equals, but 1.41. I do not think it make sense.
244. S: I don't know what is that one [$\log_4 7 + \log_4 5 = ?$].
245. S: 3 times 3 is 9, 9 times 3 is 27, um---because 9 times 2 is 18, this is 3.
246. S: silence… so what is the power of 27 equals to three? How is that possible? I do not know how the log works? How 27 it will be like a fraction? How you get a 3, 27 times one.
247. I: why one?
248. S: I mean um---I do not know how 27 get 3 which power. A negative?
249. I: square root of 16 is what?
250. S: is 4.
251. I: how you can show square root of 16, in power notation?
252. S: -2
253. I: how would be possible? How 4 to power -2 can be 16?
254. S: oh, Yeah one half, I know that.
255. I: true. Now, try to figure out how 27 can be 3?
256. S: if I divide by 9.
257. S: I want to know it? Square root of 9 is 2, I feel it should be 9 somewhere. Just because of multiplying um---
258. I: could you please tell me what do you think about the previous problem $\log_4 7 + \log_4 5 = ?$.
259. S: could you do, I do not have any idea about log, but I assume when there is a common base so you add the numbers.
260. S: [she reads Activity 4] equality of that is hold? No I really do not know.

261. S: no, because here [$\log_{10} x^3$] you will figure out the inside first x and then exponent (x^3) and then $\log_{10} x^3$, but there find $\log_{10} x$ and exponent it $(\log_{10} x)^3$.
262. S: find the value of x such that ...[she reads the Activity 5]. I wish I know logs.
263. S: silence, okay, the only thing that I am thinking there is more than one answer.
264. S: I do not know how to rearrange the equation to solve.
265. S: [she reads Activity 6] what do you think about inequality? What do you mean what do I think about it?
266. I: I mean can you solve it? How you can solve it? Do the notations make sense for you?
267. S: does this x is something different of that x.
268. I: no, x is x.
269. S: It is kind of weird. I find it would be easier to learn if I had numbers.
270. I: do you know this sign?
271. S: absolute value.
272. I: okay.
273. S: um--- if log would not be there, I can definitely figure it out.
274. You can not ever um--- you need a value for x, um--- less than 2!
275. S: absolute value is always positive, there is only few options 0,1 and 2, or 1.5, you know if it is less than 2. That is really odd.

Case 4: The subject (male, non-mature) student was taking course B at the time of interview, November 2006.

276. S: [he reads the review example, and writes $2^3 = 8$].
277. I: can you please tell me what are you thinking?
278. S: Okay. This is pretty simple.
279. S: can I use a calculator?
280. I: no, sorry.
281. S: this [Log 256] I know what is it but, I do not know how to get there.
282. I: can you please follow row by row.
283. S: this is 16. Um--- I guess 4 to 4 is 256. That is right.
284. S: that must be one. I get confuse and every time I get confuse I look at the other ones that I already did.
285. S: zero, um--- zero.
286. S: I hate fractions, anytime I see fractions I get nervous. Um--- -2, -2 .
287. I: do you think today lecture is helpful?
288. S: yeah. Of course, yeah. If I did not have it today, last time it was math 206 so I would be very lost.
289. S: -16, um--- half.
290. I: Are you sure is half. 4 to power half is square root of 4.
291. S: square root um---
292. I: what is 4 to power half meaning for you?
293. S: I do not have an idea. I just put it half. I guess it would be not -2? No.
294. I : -2 ?

295. S: here [$4^{-2} = \frac{1}{16}$] is -2, so that can not be -2 there. If 4 to -2 gives one over 16, so -16 um---
296. I: you are looking for a power to raise 4 and get -16.
297. S: 2 gives 16. It can not be -2 because -2 gives one over 16. No, I do not know.
298. I: have you seen a power to raise a positive number like 4 to this number and obtain a negative number?
299. S: actually, may be but I do not remember. To me, it is totally lost.
300. S: estimate um--- this [$4^{\sqrt{2}} = ?$] here, I am lost.
301. I: the graph is given to help you. Try to use the graph. What is asking?
302. S: 4 to root of two.
303. I: look at the graph and estimate 4 to power square root of 2.
304. S: graph increasing; now maybe it gets decreasing.
305. I: which one decreasing?
306. S: x is increasing, but square root um--- maybe it is parallel, no.
307. I: parallel?
308. S: not parallel, sorry symmetric. No it can not be symmetric.
309. I: let try together. We have $4^1 = 4$, x is 1. Can you show it on the graph?
310. S: x is 1 and y is 4 so it is here.
311. I : also for x equal zero, y is 1. You are seeing these from the graph. Now to find $4^{\sqrt{2}}$, what is x?
312. S: x is square root of 2.
313. I: you have x which is square root of 2 then you draw vertical line from this point to intersect the graph and y –coordinate of this intersection will be $4^{\sqrt{2}}$.
314. S: great. But I do not know square root of 2.
315. I: do you have any estimation of it? what is can be? Between which numbers?
316. S: nothing. I think 4 is 2.
317. I: what is the square root of one?
318. S: it is one. Two is between 4 and 1, square root is between 1 and 2, 1.5.
319. I: okay consider estimation of square root of 2 as 1.5.
320. S: here, means 1.5 [he points to y-axis] um---no, x is 1.5, follow it up. It comes up to 8.
321. I: now, try to estimate $\log_4 4^{\sqrt{2}}$.
322. S: I look at what I did before. Now I know square root of 2 is 1.5, so basically log 4, 4 to 1.5, yeah. 4 to 1.5 is 8. here log 4, 4 is 1, log 1, 4 is zero. The 1.5 come in the front. It is just 1.5, no um--- 1.5 is in the front but after what happens for 4, here is 4.
323. I: what you are trying to find?
324. S: so what happens after, 1.5 times 4, no?
325. I: look at the table that you filled out.
326. S: 4 to 1.5 is 8, now log 8, 4 um--- if it is 16 it is 2, now you do 8 um--- 4,4. silence…it is 1.5. It is okay.
327. S: this addition, when you add logs it is multiplying so it is log 35, 4. in the same base. I had this morning.

328. S: we do not have the same base. Multiplication, they add them but the same base. It is still 3, this 3 and 3 um--- so now. You add them in multiplication.
329. I: why you think we add them?
330. S: we multiply them when it was addition in same base. We do not touch bases here we multiply, but not same bases.
331. I: what if you find each log separately, what do you think?
332. S: finding them and multiplying.
333. S: now we have log base 3, 27. 27 is 3 and 3 so multiply log um--- so first when you do that it becomes 3 to 3 , 27 .Here you have log 27, 3 , you do not have this, this means 3 , 1 um---
334. I: Are you trying to find log 27 base 3?
335. S: it is 3, no 9, no not 9. It is 3.
336. I : look your table.
337. S: it is 3 alone. It is just 3. Okay, but now for here you take 27, because you take base and the answer is 3. Now when I find power I find log.
338. I: in which power you can arise 27 and get 3?
339. S: it is gonna be minus.
340. I: minus?
341. S: it can be a third. 27 to the power of one third.
342. I: yeah. One third means third root of 27.
343. S: this [$4^{?} = -16$] can not be a half.
344. I: it is not a half. Think about do we have a power to raise a positive number and get a negative number.
345. S: I have had powers that are negatives like here -2 or when I do derivatives, they could be negative power. But, when I do base to a power that gives me negative no.
346. I: when you look at the graph, you will see the domain and range of function. The range is a set of values for y=f(x), what is the range of y?
347. S: y's are getting small.
348. I: does the range include negative numbers?
349. S: no. so it is between zero and 1.
350. I: x values or the domain of the function includes negative numbers, zero and positive numbers. But y values or the range does not include negative numbers.
351. S: so it does not exist. Yes, when you put -2 it is very small close to zero. Okay.
352. S: equality?
353. I: equality means these two expressions are equal.
354. S: find an example in situation. Um--- no it is not equal.
355. I: why do you think it is not equal?
356. S: because log base of 10 , x to power 3 is equal to log base 10, x times log base 10, x times log base 10, x.
357. S: but this log base of 10, x to 3, that means base is 10 so $\log_{10} x^3$ equals to base of 10 and then 10 to um--- the answer 27, so here the answer would be x^3, but here we don't have it.
358. I: wait a minute. You need to have your own example. I mean consider x as an arbitrary number that you want.
359. S: okay. You can just use a regular number.

360. I: log is on base 10, so you may choose a value for x such that finding log of this value in base 10 be simple.
361. S: I take 10, um--- 10 to um--- 10 times 10, means 10 to the power of 2.
362. I: okay.
363. S: 100, so that means here I have $\log_{10} 100^3$. So answer for that um--- what do I do? If I put 3 here log 10 what happens for 3? When I do log 100, 10 it is easy. What do I do with 3?
364. I: your x is 100 and you want to find $\log_{10} 100^3$. You can write 100^3 in base 10. I mean 100 is 10 to power 2 ,
365. S: oh. Yeah. Then it is 6, so the answer is 6.
366. S: now this is 6. Basically I do log 10, 100 and 3, so you do log 10 to 2 and then 3, so that is the same thing.
367. I: are you sure? Can you check it?
368. S: because you do 3 times two.
369. I: as you said at the first you need to find log 100 base 10 then multiply your answer three times by itself.
370. S: so it is 2 times 2 times 2 which is 8, that is why it does not work.
371. S: okay, the same base, you can just multiply them.(x-1) times x equals 1. $\log(x^2 - x)$ is 1, now this is the problem, no what um---the 1 other side is zero ?
372. I: no.
373. S: x is the same number as the base.
374. I: yes, x is x.
375. S: so if x is the same as the base, that means having log 3,3 .But now um---
376. S: now, x is the base then you have x to 1 equals to $x^2 - x$. Because base is x so you put the base x and 1 comes to power equals to $x^2 - x$. You solve this.
377. I: okay. Go ahead.
378. S: now, this cancels this.
379. I: it is an equation, try to simplify it.
380. S: do you want me find that?
381. I: yes, please.
382. S: bring x over make it zero, take x out and means x is 0 and 2.
383. S: what do you mean by saying what do you think about inequality.
384. I: do you think you can solve it? How you can solve it?
385. S: (he refers to the given review example again), smaller than 2, smaller than two can not be an answer. You can not put here, the base is x here you need and answer so I just put less than two.
386. I: have you seen smaller notation on the power?
387. S: like this $x^{<2} = |x^{2-1}|$
388. I: is it make sense x to power smaller than 2 , but what ?
389. S: um--- so smaller than 2, you see now I am confused. Because here always you had answer and that was easy, and you have absolute value. So it is always positive. So it is between zero and smaller than two.

Case 5: the students (mature, female) enrolled in course C (a core mathematics course) at the time of interview, November 2006.

390. S: do you want me to write it done here?
391. I: please and tell me what you are thinking.
392. S: silence
393. S: [she writes $\log_2 8 = 2^3$]
394. S: can I put the review example there.
395. I: yes.
396. S: so here we have log of 9, how do you call this?
397. I: base 3.
398. S: $\log_3 9 = 2$, base 3 is equal to 2.
399. S: log of 16 base 4 is 4 to the two.
400. S: oh no um--- 2 to the 4.
401. I: the review example shows that log 9 base 3 is 2, what will be log 16 base 4?
402. S: [she writes 4^2]
403. I: why 4 to two.
404. S: 4 to two is 16, right.
405. I: yes, it is 16, but we are looking for log of 16 base 4.
406. S: in the review example the base is 3, my base is 4 here. I am thinking if the log um---
407. S: yeah, yeah
408. I: log and exponents
409. S: so it is two, do you want me cut it off.
410. I: yes.
411. I: please follow row by row to fill out the table.
412. S: do you want me write 4 to 2?
413. I: yes, please.
414. S: so log of 4 is um--- log base 4 of 256, so that is what?
415. S: okay forget it. um---
416. S: can you do it? Silence…
417. I: you can write of the questionnaire.
418. S: so here, I am kind of like, my brain is not seeing this.
419. I: I am not measuring your knowledge, just be relax
420. S: forget about the log.
421. I: [she looks at the second question and she realizes all is about logs]
422. S: one out of 16, that is a good question. I guess I have to look at these and these and try to find the relationship.
423. I: what is the relationship between them?
424. S: silence…
425. S: can we go the second question.
426. I: yes.
427. S: the graph is given, estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$.
428. I: graph is given to help you.
429. S: Okay. Silence…

430. I: what are you thinking?
431. S: here is my graph, square root of 2, where is square of two?
432. S: square of root 2 is 1.7? I do not know.
433. I: it is almost 1.41.
434. S: so square root of 2 is 1.41 so I am gonna look at around 1.41 here, it is around 7.
435. S: estimate log 4 square root of 2.
436. S: so it is gonna be…
437. S: um--- good question. What about relationship.
438. S: silence…
439. S: okay, one more other thing um---so log um---
440. I: so tell me what are you thinking?
441. S: I am trying to estimate, wait I will come back, later. [She starts next activity].
442. S: evaluate the equations. It is all about log. Laughing, so I need really understand it.
443. I: [she refers to the review example], it is not a knowledge test be relax.
444. S: laughing, I have better figure it out.
445. S: log 3 of 9 equal to 2, so means as 3 is in the base 3 to two is 9, equal to 2 , my base is here.
446. S: [she starts over from Activity 1] so log 4 to the 16, is it good?
447. S: the base is 4 so 4 to something is equal to 4.
448. S: $\log_4 4^{\sqrt{2}} = ?$ is square root of 2.
449. S: wait what about this one -16.
450. S: but the log function is like this. How come?
451. I: this is exponential function.
452. S: right. This is exponential function.
453. S: so evaluate the equations. This is log 7 to base 4 we find out something here $4^{\sqrt{2}} = 7$, here we have log 4 , 5 equal to 5 so 4 to something is equal to 5, so 4 , 5 at the bottom is 1.2.
454. S: what is this, so this is the log of 7 plus log 5, wait what I did?
455. S: you said square root of 2 is 1.4?
456. S: can this stay like that?
457. I: okay.
458. S: it is 3 to something is 27 , how much so equal to 27 times of log 27 , 3 the base is 27 so 27 is something
459. S: I am just writing like that.
460. I: it is fine. I have your explanation on my recorder.
461. S: so I am taking the base 27 on what power will give me 3, 27 to 1/3. I assume, so what I got, I got here 3, 3 times 1/3 equal to 1.
462. S: Activity 4, does the equation hold?
463. S: Okay. log of 10 , x base 10 to x cube is equal to the log of.
464. S: I swear I do not remember logs.
465. S: what we are saying here, is saying that 10 to the power of something is equal to x cube and here we saying this is this side and here we got log 10 x , so it is 10 to the power of something is equal to x then you cube that. Is that equal, good question?

466. S: silence. Um---
467. I: bring your own example and try.
468. S: let say x is equal to two, 10 log of 10 to the cube what is that? that is equal, silence…to out of 10, 8 how do the log function works, so it is 10 to the something equal to 8 , it is what?
469. I: try an easier example.
470. S: so x is equal to 10, okay x is equal to 10…
471. I: okay
472. S: let say x is equal to 10, log 10 ,10 to the cube, log 10 times10, 10 to 3 is equal, this is equal to three. Right? so what about the other side, the other side I got log of 10. x cube so x is gonna be 10 so log of 10 to the 10 cubed right, so 10 to the power of what gives me 10 so 1 give me 10, the answer to the parenthesis is 1 and do the three is one, therefore they do not equal.
473. S: find the value of x such that log of um---
474. S: okay, you want to find the value of x, so log of x, (x-1) okay plus log of x to x is equal 1. So I got here, I have to do it again, so I got base x to something is equal to (x-1) and I got x to the something is equal to x, I am just writing, not a good notation. So this is should be equal to one okay so what x to the power of what give me (x-1), wow!
475. S: x to the power 1 give me x , um--- so this is 1. What I am gonna do with this, this is the power of zero what is this 1, find the value of x, you have to find the value of x, silence…
476. S: so this gives me 1 right? I do not know.
477. I: do you remember any log laws?
478. S: no.
479. S: consider the inequality log x okay, inequality more than 2 what do you think about this equality?
480. S: what do you mean of what do you think?
481. S: is it true or false?
482. I: how you think you can solve it, can you solve it?
483. S: what I am doing here, I am saying base is x so x to this something is equal to, silence, should be smaller than 2 it that what I am saying
484. I: okay.
485. S: and okay forget it.
486. I: if I give you a calculator do you think it is helpful?
487. S: no
488. I: do you want to check them with a calculator?
489. S: But, laughing, I did really bad. I do not remember log.

Case 6: He was a graduate student who completed course C on January 2006. The interview was conducted in November 2006.

490. S: in the first case we have log 9 base 3 equal 2 which means 3 square equals 9, so in this case um---base 2 square base two to power something is equal to 8 , so log 2 cube base 2 equals 3.
491. S: do you want me fill both of them?

492. I: yes, both.
493. S: are there the same answers, anyway?
494. S: this is the answer 4 square 16.
495. S: that is 4^4 um---
496. I : you can please write down your calculation.
497. S: 4 times 4 is 16, 16 times 4 is 64, 64 times 4 is 256.
498. S: which is 4.
499. S: one.
500. S: base 4, is zero. Um---
501. S: I guess negative, -2.
502. S: this [$\log_4(-16) = ?$] can not be done, complex numbers will help me here.
503. I: no, please consider only real numbers.
504. S: can not be done.
505. S: graph is given estimate 4 to power square root two. Okay, power 1.3. um--- you want me look at the graph some where here.
506. I: how you find it?
507. S: square root of 2 is something like 1.3, and I try to figure out 4 to the power of 1.3.
508. I: I need only estimation.
509. S: okay, $4^1 = 4$, $4^2 = 16$ so 4 to the power of root 2 is somewhere between, bigger than 4 less than 16. So it is 6 point something, 6.3.
510. S: estimate the log, in this case you can put this ($\sqrt{2}$) in front so $\sqrt{2}\log_4 4, \log_4 4$ is one and $\sqrt{2}$. Laughing…
511. S: same base you multiply inside, right?
512. I: yes, so you remember the laws of logs?
513. S: I remember them. So log of 35 base 4? Do you want me to find an answer?
514. I: yes please, estimation.
515. S: I guess you can do it separately. 4 to the power what, is equal to 7?
516. I: (he tries to use the given graph from Activity 2).
517. S: it doesn't help that much. It is better 1.4 plus 1.2 equal 2.6. I represent logs as exponents and find them from graph. This side is 4 to the power something equal to 35. It is gotta between 2 and 3. All right.
518. S: um--- but this does not work out. But this is the same thing of something else. Log um--- I do not know, you can convert this again! 3 times 3, 3 cube is 27, so this [$\log_3 27$] is equal 3, 27 to power something is 3, one third. But there is another rule to use, multiplication rule.
519. I: no there is no such a rule. Log convert multiplication of numbers into addition of log of numbers.
520. S: there is another way of writing log 27, $\log 3^3$.
521. S: does the equality hold? No. because $\log_{10} x^3$ is not equal to $(\log_{10} x)^3$.
522. S: it is kind of fun.
523. S: no, is that right? Is it subtraction rule? Is $\log_x(x-1)$ equal to $\dfrac{\log_x x}{\log_x 1}$?
524. I: oh, no.

525. S: but there is a rule for log (a-b) is not it?
526. I: no, log converts division operation into subtraction of logarithms of numbers.
527. S: so $\log_x[(x-1)(x)]=1$, $\log_x(x^2-x)=1$, so convert to exponents , how to convert that? $x^1 = x^2 - x$, so x=2.
528. S: is this only answer or I did wrong?
529. I: in quadratic equation you canceled out one of roots.
530. S: oh, right. Interesting.
531. S: what do you think, what do I think about inequality.
532. I: Are you converting it into exponents?
533. S: yes, it is easy for me, or it seems easier. $x^2 < +(x^2-1)$, $0 < -1$, no it is not true, it does not make sense.
534. S: oh there is absolute value, right it is minus and plus. When it is plus does not work. When it is negative x is less than positive negative square root of half. [He writes $x^2 < -(x^2-1)$, $x^2 < -x^2+1)$, … $x < \pm\sqrt{\frac{1}{2}}$]

Student 1's written questionnaire

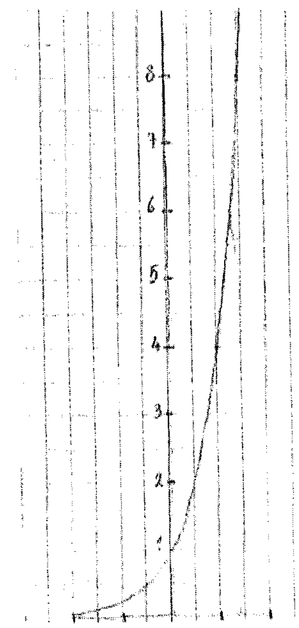
Activity1. Complete the table row by row.

Logarithms	Exponentials
$\log_4 16 = ?$ 2	$4^2 = ?$ 16
$\log_4 256 = ?$ 4	$4^4 = ?$ 256
$\log_4 4 = ?$ 1	$4^{?} = 4$ 1
$\log_4 1 = ?$	$4^{?} = 1$
$\log_4 \frac{1}{16} = ?$	$4^{?} = \frac{1}{16}$
$\log_4 (-16) = ?$ -2	$4^{?} = -16$ -2

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and ~~5.74~~ 6 estimate $\log_4 4^{\sqrt{2}}$.

4^6

1.41



Activity 3. Evaluate the equations.

$\log_4 7 + \log_4 5 = ?$

12

3 .5 1.5 1

$\log_3 27 \times \log_{27} 3 = ?$

Activity 4. Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

10^3 1000

1 10

Activity 5.
Find the values of x such that $\log_x(x-1) + \log_x(x) = 1$.

$\log_2(2-1) + \log_2(2) = 1$

$\log_2(1) + \log 1 =$

Activity 6.
Consider the inequality $\log_x|x^2 - 1| < 2$ what do you think about this inequality.

2

$2\ |4-1|$

16-1 15

4

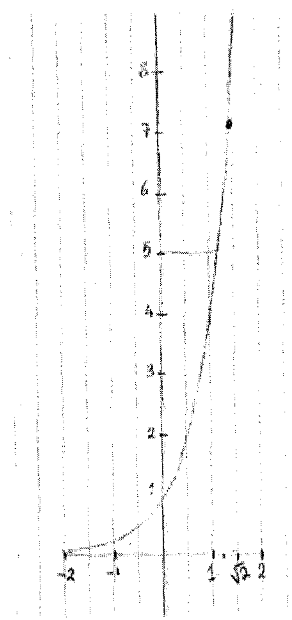
Student 2's written questionnaire

Activity1. Complete the table row by row.

Logarithms	Exponentials
$\log_4 16 = ?$ 2 $4^2 = 16$	$4^? = ?$ 16
$\log_4 256 = ?$ 4 $4^4 = 256$	$4^4 = ?$ 256
$\log_4 4 = ?$ 1 $4^1 = 4$	$4^? = 4$
$\log_4 1 = ?$ 0 $4^0 = 1$	$4^? = 1$ 0
$\log_4 \frac{1}{16} = ?$ $4^{-2} = \frac{1}{16}$	$4^? = \frac{1}{16}$ -2
$\log_4 (-16) = ?$ $4^{2} = -16$ X	$4^? = -16$ X

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ 7 and estimate $\log_4 4^{\sqrt{2}}$.

$4^{\sqrt{2}} = 4^{\sqrt{2}}$



Activity 3. Evaluate the equations.

$\log_4 7 + \log_4 5 = ?$

$4^{\sqrt{1.1}} = 5$

$4^{\sqrt{2}} = 7$

$\sqrt{2} + \sqrt{1.1} =$

$\log_3 27 \times \log_{27} 3 = ?$

$3^3 = 27$

$27^x = 3$

3^3

$\sqrt[3]{27} = 3 \times 3 \times 3$

$27^{1/3} = 3$

$3 \times \frac{1}{3} = 1$

Activity 4. Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

$x = 2$ $x = 10$

$\log_{10} 2^3 = \log_{10} 8$

$\log_{10} 10^3 = \log_{10} 1000$

$10^{3} = 1000$

$(\log_{10} 10)^3 = (1)^3 = 1$

$3 \neq 1$

Activity 5.

Find the values of x such that $\log_x(x-1) + \log_x(x) = 1$.

$\log_x (x-1) = 0$

$\boxed{x^0 = (x-1)}$

$x - 1 = 0$

$x = 1$

$\log_x (x) =$

$x^1 = x$

Activity 6.

Consider the inequality $\log_x|x^2 - 1| < 2$ what do you think about this inequality.

$\log_x |x^2-1| \geq 0$

Thank you so much for completing the activities

Best wishes

Student 3's written questionnaire

Activity1. Complete the table row by row.

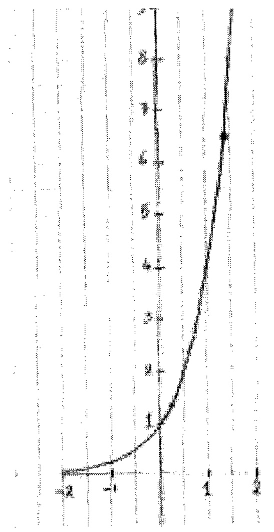
Logarithms	Exponentials
$\log_4 16 = ?$ 2	$4^? = ?$ 16
$\log_4 256 = ?$ 4	$4^? = ?$ 256
$\log_4 4 = ?$ 1	$4^? = 4$ = 1
$\log_4 1 = ?$ 0	$4^? = 1$ = 0
$\log_4 \frac{1}{16} = ?$ = -2	$4^? = \frac{1}{16}$ = -2
$\log_4(-16) = ?$ DNE	$4^? = -16$ DNE

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and estimate $\log_4 4^{\sqrt{2}} = ?$ $4^? = 4^{\sqrt{2}}$

$\sqrt{2} \approx 1.41$

$4^{1.4} = 6.5$

$\log_4 4^{\sqrt{2}} = 1.41$



Activity 3. Evaluate the equations.

$\log_4 7 + \log_4 5 = ?$

$= \log_4 12 = ?$

$\log_3 27 \times \log_{27} 3 = ?$

$(3^? = 27) \times (27^? = 3)$

$= 3 \times \_\_ = \_\_$

Activity 4. Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

no because you must calculate exponents first before log.

$\log_{10} x^3 =$

$\log_{10} 2^3 = \log_{10} 8 =$ answer

Activity 5.
Find the values of x such that $\log_x(x-1) + \log_x(x) = 1$.

- multiple answers because the word "values"
- try re-arrange equation if possible

Activity 6.
Consider the inequality $\log_x|x^2-1| < 2$ what do you think about this inequality.

Student 4's written questionnaire

Activity1. Complete the table row by row.

Logarithms	Exponentials
$\log_4 16 = ?$ 2	$4^2 = ?$ 16
$\log_4 256 = ?$ 4	$4^4 = ?$ 256
$\log_4 4 = ?$ 1	$4^? = 4$ 1
$\log_4 1 = ?$ 0	$4^? = 1$ 0
$\log_4 \frac{1}{16} = ?$ −2	$4^? = \frac{1}{16}$ −2
$\log_4 (-16) = ?$ ∅	$4^? = -16$ ∅

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and estimate $\log_4 4^{\sqrt{2}}$.

8

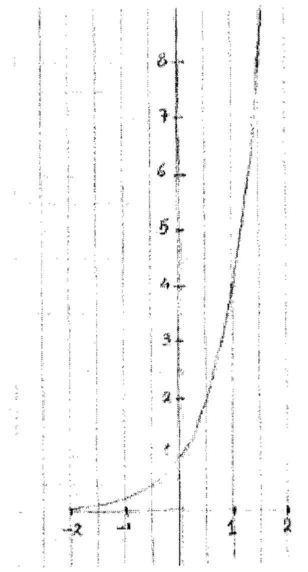
$\log_4 4^{1,5} = 1,5$

$\log_4 8 = 1,5$

$\sqrt{4} = 2$

$\sqrt{1} = 1$

$\sqrt{2} = 1,5$



Activity 3. Evaluate the equations.

$\log_4 7 + \log_4 5 = ?$

$\log_4 (7 \cdot 5)$

$\log_4 35$

$\log_3 27 \times \log_{27} 3 = ?$ 3^3

$\log_3 27 \times \log_{27} 3$

$3^3 = 27 \times 27 = 3$

$3 \times \frac{1}{3}$

1

Activity 4. Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

$(\log_{10} x)^3 = (\log_{10} x)(\log_{10} x) \ldots$

$\log_{10} x^3 = 10 = x^3$

$\log_{10} 100^3 \quad 10^2 = 100 \quad 2$

$\log_{10} 100^3 \Rightarrow \log_{10} (10^2)^3 = \log_{10} 10^6 = 6$

$(\log_{10} 100)^3 \Rightarrow (\log_{10} 10^2)^3$

$(\log_{10} 10^2)(\log_{10} 10^2) \ldots = 8$

Activity 5.

Find the values of x such that $\log_x (x-1) + \log_x (x) = 1$.

$\log_x (x-1)(x) = 1$

$\log_x x^2 - x = 1$

$x^1 = x^2 - x$

x^1

$x = x^2 - x$

$0 = x^2 - 2x$

$-x(x-2)$

$x = 0$

$x = 2$

Activity 6.

Consider the inequality $\log_x |x^2 - 1| < 2$ what do you think about this inequality.

$\log_x |x^2 - 1| < 2$

$x^2 = |x^2 - 1|$

Student 5's written questionnaire

Activity1. Complete the table row by row.

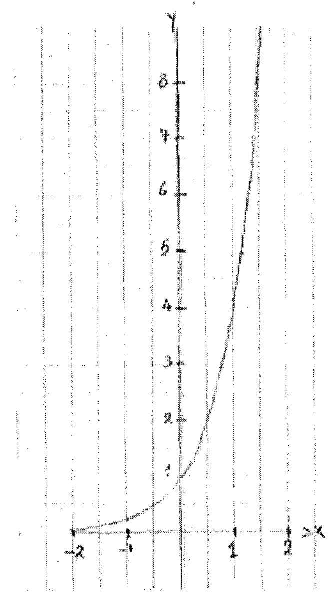
Logarithms	Exponentials
$\log_4 16 = ?$ 4^2 2	$4^2 = ?$ 16
$\log_4 256 = ?$ 4	$4^4 = ?$ $\frac{16 \cdot 4}{64}$ $\frac{64 \cdot 4}{256}$
$\log_4 4 = ?$ 1	$4^? = 4$ $? = 1$
$\log_4 1 = ?$ 0	$4^? = 1$ $4^0 = 1$
$\log_4 \frac{1}{16} = ?$ -2	$4^? = \frac{1}{16}$ 4^{-2}
$\log_4 (-16) = ?$	$4^? = -16$

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ 7 and estimate $\log_4 4^{\sqrt{2}}$ $= \sqrt{2}$

$\sqrt{2} = 1.41$

$4^? = 4^{\sqrt{2}}$

$\sqrt{2}$



Activity 3. Evaluate the equations.

$\log_4 7 + \log_4 5 = ?$

$4^? = 7 \quad 4^? = 5$

$4^{\sqrt{2}} = 7 \quad 4^{1.2} = 5$

$\sqrt{2} + 1.2 =$

$\log_3 27 \times \log_{27} 3 = ?$

$3^? = 27$

$3^3 = 27 \times 27^? = 3$

$27^{1/3} = 3$

$3 \times \frac{1}{3} = 1$

Activity 4. Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.

$\log_{10} x^3 = (\log_{10} x)^3$

$10^? = x^3 = (10^? = x)^3$

$x = 2$

$\log_{10} 2^3 = \log_{10} 8 =$

$10^? = 8$

$(\log_{10} x)^3$

$(\log_{10} 10)^3$

10^1

$1^3 = 1$

$x = 10$

$\log_{10} 10^3 = \log_{10} 1000 =$

$10^3 = 100$

3

$\neq$

Activity 5.

Find the values of x such that $\log_x(x-1) + \log_x(x) = 1$.

$\log_x (x-1) + \log_x (x) = 1$

$\underbrace{x^? = x-1}_{} + \underbrace{x^? = x}_{x^1 = x} = 1$

Activity 6.

Consider the inequality $\log_x|x^2 - 1| < 2$ what do you think about this inequality.

$x^? = |x^2 - 1| < 2$

x

Student 6's written questionnaire

Activity1. Complete the table row by row.

Logarithms	Exponentials
$\log_4 16 = ?$ 2	$4^2 = ?$ 16
$\log_4 256 = ?$ 4	$4^4 = ?$ 256
$\log_4 4 = ?$ 1	$4^? = 4$ 1
$\log_4 1 = ?$ 0	$4^? = 1$ 0
$\log_4 \frac{1}{16} = ?$ −2	$4^? = \frac{1}{16}$ −2
$\log_4(-16) = ?$ can't be done	$4^? = -16$ "

$$\begin{array}{r} 16 \\ \times\ 16 \\ \hline 256 \end{array}$$

$4^1 = 4$

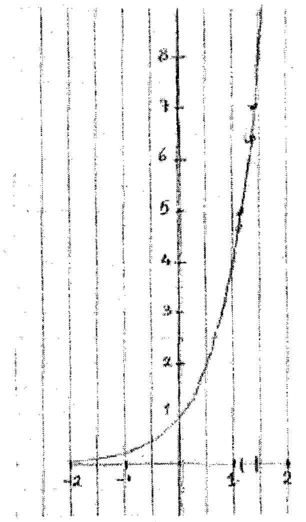
$4^2 = 16$ $\quad 4^{\sqrt{2}}$

Activity2. The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ 6.3 and estimate $\log_4 4^{\sqrt{2}}$. = $\sqrt{2}$

$4^{\sqrt{2}} = \sqrt{2}$

$\sqrt{2}$ $\quad \underbrace{\log_4 4}_{1}$

$\sqrt{2}$



Activity 3. Evaluate the equations.

$4^x=7$ ~~1.4+1.2=2.6~~ $1.4+1.2=2.6$

$\log_4 7+\log_4 5=?$

$\log_4(7\times 5)$

$\log_4(35)=$ $4^x=35$

4 2-3

$\log_3 27\times\log_{27} 3=?$ 1

$3^x=27$ $27^{x}=3$

log

$\log_{27} 3=\log_{3^3} 3^{1}$

$3\times 3\times 3$

Activity 4. Does the equality $\log_{10} x^3=(\log_{10} x)^3$ hold? Find an example and explain your reason.

NO

$\log_{10} x^3 = 3\log_{10} x$

$\neq(\log_{10} x)^3$

Activity 5.

Find the values of x such that $\log_x(x-1)+\log_x(x)=1$.

$\log_x[(x-1)(x)]=1$

$\log_x(x^2-x)=1$

$x^1=(x^2-x)$

$x^2-2x=0$

$x(x-2)=0$

$x=0, x=2$

$2x=x^2$

$2=x$

Thank you so much for completing the activities

Best wishes

Activity 6.

Consider the inequality $\log_x|x^2-1|<2$ what do you think about this inequality.

$x^2<(x^2-1)$

$0<-1$ X

$x^2<-(x^2-1)$

$x^2<-x^2+1$

$2x^2<1$

$x^2<1/2$

$x<\pm\sqrt{1/2}$ ✓

Appendix G: Tables of Analysis of Data

Student 1's interview analysis

Activities	*Interviewee's statements with line numbers*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	8. I know this [$3^2 = 9$] but this I have never seen [$\log_3 9$].			She does not remember because she took math 536, 14 years ago.	
	13. like, how many times 3 to get 9.			Says " times" not " to the power"	Power = times
	15. here, that means how many times multiply 2 to get 8, 3.	A_{al}		She performs multiplication and finds $\log_2 8$	Power = times
Activity 1: $\log_4 16 = ?$	17. here, log 4, 16 that would be 2 because 4 times 4 is 16	P_{al}		She says, "because" she reflects on her answer.	
$4^2 = ?$			She fills this after finding log.		
$4^4 = ?$	25. [mumbling] 4 times 4 is 16, 16 times 4 , 64 times 4 is 256.				
$\log_4 256 = ?$		P_{al}	She multiplies 4 by itself 4 times to find 256, then she follows the relation between log and exponential to find the answer.	She is following the relation between log and exponential.	
$4^? = 4$			She finds the power.		
$\log_4 4 = ?$	26. this would be 1($4^? = 4$), so 1($\log_4 4 = ?$)	P_{al}	She follows the relation between log and exponential.		
$4^? = 1$			She tries decimal power for 4 like 0.25 .	She does not know definition of zero as an exponent.	She has lack of knowledge on exponentiation.
$\log_4 1 = ?$	27. ($\log_4 1 = ?$ silence		She skips this question.		She does not know the definition of zero as an exponent.
$4^? = \frac{1}{16}$			She skips this question also, without any effort to solve.		She doesn't know about zero and negative exponents.
$\log_4 \frac{1}{16} = ?$	28. this[$\log_4 \frac{1}{16} = ?$] I am not sure, fractions, fractions I am really bad.		She jumps to next question.		She does not like fractions.

$4^{?} = -16$			She calculate powers -2.	Since $4^2 = 16$ she thinks $4^{-2} = -16$	She does not extend the domain of exponents to real numbers. Misconception of exponentiation.
$\log_4(-16) = ?$	29. I guess this [$\log_4(-16) = ?$] would be -2.		She looks at $\log_4 16 = ?$ and guesses that $\log_4(-16) = ?$ is -2.		
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$	42. I consider square root of 2 as 1, 4 to power 1 which is 4.		She considers $\sqrt{2} = 1$.	She does not know the estimation of square root of 2.	I told her estimation of square root of two.
	49. okay, that would be the 5.64.		She tries to estimate $4^{\sqrt{2}}$.		She does not how to read a graph.
	51. I multiply 4 by 1.41.			Instead of raising to power, she multiplies the base by the power.	Power = times Lack of knowledge on exponentiation.
	54. okay, 1.41 on x-axis um--- will give me 6 on y-axis.		She uses he graph to estimate $4^{\sqrt{2}}$		
	57. I know this ($4^{\sqrt{2}}$) is 6 or around 6. I am thinking how many times this(4) goes to give 6, so I am thinking of one point something, so let say um--- I guess it would be 1.41.	P_{al}	She estimates $\log_4 4^{\sqrt{2}}$ as 1.41.	She justifies her estimation.	Despite of having lack of exponentiation knowledge and misconception of powers and times, she reasons her acts in solving log problems.
Activity 3: $\log_4 7 + \log_4 5 = ?$	60. so how many times 4 to get 7 plus how many times 4 to get 5, maybe one point something it will give you 2 something, maybe up to 3.	P_{al}	She estimates $\log_4 7$ and $\log_4 5$ by converting them into exponential and estimating the exponents.	Her strategy to calculate $\log_4 7 + \log_4 5 =$, shows that she is reflected on her answer and understanding.	
	63. by adding log 7 to log 5 and have log 12 base 4, but 4 times 4 would be 16 it would be over which I do not think I can do that.	O_{al}	She verifies her answer. She thinks of applying commutative law then she realizes that does not work with logs.	She rejects this approach by relying on log definition.	
$\log_3 27 \times \log_{27} 3 = ?$	66. 27 um--- that is like point something. Let say 0.5. 3 times 0.5 would be 1.5. If we would be um--- I do not know	O_{al}	She finds $\log_3 27$ easily then she tries to estimate	She guesses $\log_{27} 3$ as 0.5, however when she verifies her answer (0.5^3), she	She confuses power with times. Power=times

	is it possible actually add two log bases together and two numbers together and get the answer.		$\log_{27} 3$ by guessing it as 0.5.	multiplies 0.5 by 3.	
Activity 4: Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.	72. *x* is 10, this would be 3 and log 10 base 10 is 1. 1 times 3 is 3 um--- now, log 10 base 10 is 1 and 1 power to three is one, and it does not hold.	O_{al}	She justifies her answer by considering $x = 10$ as an example.		
Activity 5: Find the value(s) of x such that $\log_x(x-1) + \log_x x = 1$	74. okay, let's take 2 , log (2-1) base 2 plus log 2 base 2 equal 1, so this would give me 2.		She guesses $x = 2$ since she says " x can be any number, I pick the lowest one"		she plugs a value for x to verify equality instead of rearranging the equality and solving.
	76. S: um--- log 1 base 2 plus one, log 1 base 2. silence um---			Since she does not know the definition of the zero exponent she can not solve it out.	She does not know $a^0 = 1$
Activity 6: Consider the inequality $\log_x \lvert x^2 - 1 \rvert < 2$ what do you think about this inequality.	85. that can not be one, because it gives 0 , so I will try with two so it can be absolute value of (4-1) and um---		She plugs different values in inequality.		
	91. *x* to be negative or positive, and when x is negative this come to positive and base is negative . Assuming that *x* would be negative number.				She can not manipulate the smaller sign (<) and convert log to exponentials.

Students2'interview analysis

Activities	*Interviewee's statements with line numbers*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	97. If this [$\log_2 8$] is going to be the same as this [exponent]I can put here 2, um--- . ok, 2 is base to 3 is 8.	A_{al}	She follows the given review example.	She raises 2 to power 3 to get 8.	
Activity 1: $\log_4 16 = ?$	98. I am doing the same as here. 4 to what power gives us 16, it is 2, because 4 times 4 is 16. Answer is 2.	P_{al}	She writes exponential form of and tries to find exponent (power).	She says, "because" she is reflected on her answer.	
$4^2 = ?$			She fills $4^2 = ?$ after finding $\log_4 16 = ?$.		
$\log_4 256 = ?$	99. here, I think it is 4.	P_{al}		She follows the relation between logarithmic and exponential form.	
$4^4 = ?$			She writes 256 without calculation by referring to logarithmic form.		
$4^? = 4$			She knows the definition of zero as an exponent.		
$\log_4 4 = ?$	100. here, 4 equal to 4 it is one.	P_{al}	She writes exponential form of log.	She follows the relation between log and exponential.	
$\log_4 1 = ?$	101. for this zero.	P_{al}	She converts log to exponential form and finds exponent.		
$4^? = 1$			She writes zero.		
$\log_4 \frac{1}{16} = ?$	102. here is one over two, I think so.		because of $4^2 = 16$ she writes $4^{\frac{1}{2}} = \frac{1}{16}$		She does not have concrete knowledge on negative exponent. I used an example $4^{-1} = \frac{1}{4}$ to remind her.
	117. so 4 to -2 is one over 16.		She recalls the definition of negative powers.	She recalls the definition of negative powers.	
$4^? = \frac{1}{16}$			She says -2 after I showed her $4^{-1} = \frac{1}{4}$.		

$\log_4(-16) = ?$	118. here, is the same. Oh, no. I think it is going to be 16, but -16. It is not the same. No, it can not be -16.		She looks at $\log_4 \frac{1}{16} = ?$ and says that $\log_4(-16) = ?$ is the same. She compares them several times and says "log can not be negative".	It seems she gets confuse with negative exponents. She does not extend the domain of exponents to real numbers.	She writes $4^2 = -16$ and understands that it is impossible. She thinks range of log function is positive.
$4^? = -16$			Since she finds that $\log_4(-16) = ?$ is not possible, she writes that $4^? = -16$ is impossible also.		She writes $4^2 = -16$ and realizes that it is impossible.
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$	123. I think *x* is $\sqrt{2}$ here.				She thinks question asks square root of 2. I told her to find $4^{\sqrt{2}} = ?$.
	127. I need *y*, so I just go like that, it is 7.		She estimates $4^{\sqrt{2}}$ as 7.		She knows how to read the graph.
	128. so for $\log_4 4^{\sqrt{2}}$, I do like this ($4 = 4^{\sqrt{2}}$) and it has to be $\sqrt{2}$, ($4^{\sqrt{2}} = 4^{\sqrt{2}}$).	P_{al}	She writes exponential form of log and find the exponent, $\sqrt{2}$.	She writes exponential form of log and find the exponent, $\sqrt{2}$.	It seems she has practiced irrational numbers, because she does not get confused by $\sqrt{2}$.
Activity 3: $\log_4 7 + \log_4 5 = ?$	133. oh, the one that I had before, $4^{\sqrt{2}} = 7$ and for $\log_4 5$, I need a power of 4 to get 5, $4^{\sqrt{1.1}}$. I do not know exactly but $4^{\sqrt{2}} + 4^{\sqrt{1.1}}$	P_{al}	She estimates $\log_4 7$ by referring to the previous activity and tries to estimate $\log_4 5$ by converting it into exponential form.	She finds each term by converting them into their corresponding exponential forms and then adds the results.	
	135. oh. Sorry, $\sqrt{2} + \sqrt{1.1}$				
$\log_3 27 \times \log_{27} 3 = ?$	136. I will put this here 3 to power what is 27, 3; now other one, 27 to what power is 3.	P_{al}	She finds $\log_3 27$ easily then she tries to find a power to raise 27 and get 3.	She computes each term separately and multiplies the results.	

	137. square root, oh no.		She thinks of square root, immediately she realizes that it can not be correct.		
	138. 27 is 9 times 3. 3 times 3 times 3, root 3.		She performs calculation on 27 and finds out that 3 is third root of 27.		
Activity 4: Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.	144. because it is gonna be different, the answer is different because 3 here only acting on *x*, but over log is to power 3.	O_{al}		She considers logs as an object.	
	146. for example *x*=2, so 2 powers 3 is 8, but 10 to what power give 8. um--- I do not know. Silence...		She takes $x=2$, then realizes it is not going to be easy to find $\log_{10} 8$.		I told her to choose x one of the powers of ten. She chooses $x=10$ and justifies her answer.
Activity 5: Find the value(s) of x such that $\log_x(x-1)+\log_x x =$	155. values of *x*, here $\log_x x$ as the same $x^1 = x$ it is one.		She finds $\log_x x = 1$, by converting it into exponential form, then she tries to show that $\log_x(x-1)$ is zero.		
	158. um---, $x^0 = (x-1)$ and $x-1=0$ so *x* is one.		She tries to find *x* in $\log_x(x-1)=$ by converting log into exponential form.		She makes a mistake in solving $x^0 = (x-1)$, so she gets wrong answer.
Activity 6: Consider the inequality $\log_x\lvert x^2-1\rvert < 2$ what do you think about this inequality.	159. Consider this inequality what do you think? No it is wrong, because the log should be always positive. I mean bigger than two.		She thinks range of logarithmic functions is positive.		Misconception of the domain of logs as the range of logs
	164. $\log_x\lvert x^2-1\rvert \geq 0$ this is the correct one. It can not be two and above two but I do not know about one. Here if I put one, it is gonna be log zero base 1. um-- I think it has to be equal to zero.		She thinks $\log_x\lvert x^2-1\rvert =$		

Student 3's interview analysis

Activities	*Interviewee's statements*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	172. I know that $(3^2 = 9)$ and I understand this ($\log_3 9 = 2$). This ($\log_2 8 = ?$) is asking me what is the power of 2 equal to 8.		She has not seen log before.		
	174. Okay, 2 to power 2 is 4, um--- 2 to power 3 is 8. so therefore log 2 , 8 is 3.	A_{al}		She computes $\log_2 8 = ?$ with finding $2^3 = 8$	
Activity 1: $4^2 = ?$	178. 4 times 4 is 16.		She first finds exponential form.		
$\log_4 16 = ?$		A_{al}	By referring to the given review example, she finds $\log_4 16 = 2$.	She calculates powers of 4 and concludes that $\log_4 16 = 2$.	
$4^4 = ?$	180. mumbling 4 by 4, 16 by 4 and 64 by 4, power 4.		She multiplies 4, 4 times by itself.	She computes the exponentiation.	
$\log_4 256 = ?$		A_{al}	She finds the exponential form $4^4 = ?$ first. Then she refers to the review example to find log.	She calculates powers of 4 and by referring to the review example she finds log.	
$4^? = 4$ $\log_4 4 = ?$	181. power of one.	A_{al}		She first finds the exponent then log.	
$4^? = 1$	182. 4 to what power is equal to one? Now, we will go in fraction.		She thinks power is a fraction number.	She says power is fraction number since she can not recall the definition of zero power.	She has lack of knowledge on the definition of zero as exponent.
$\log_4 1 = ?$	184. um---to power of zero.	P_{al}		She finds the exponent first then log.	
$4^? = \frac{1}{16}$	198. what power of 4? I am assuming be a fraction, 4 to the power of fraction. That is possible. Is not it?		She skips this first and after completing the last row of the table she goes back to this question.		She does not remember the definition of negative exponents.
	203. 4 to the power of negative 2, I am assuming that would be negative two.		She says I assume.	She guesses it is -2, since she had positive, zero exponents in preceding problems.	she does not know the definition of negative powers.

$\log_4 \frac{1}{16} = ?$		P_{al}	After she finds the power, she calculates $\log_4 \frac{1}{16} = -2$	After she finds the power, she calculates $\log_4 \frac{1}{16} = -2$	
$4^? = -16$	193. no that is impossible. Okay. Do not exist.		She finds out that such a power does not exist.		I gave her a hint by saying that "do you know a power to raise 4 and obtain a negative number?"
$\log_4(-16) = ?$	194. I do not know is that the same? Is ($\log_4(-16) = ?$) that does not exist too?	P_{al}	She tries to mach the relation between log and exponents and says $\log_4(-16) = ?$ is impossible	She finds the answer by referring to the relation between exponentials and logarithms.	
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$	217. I just have to know what is the square root of 2 is. It is just one, no it is 2 times.			She does not know the approximation of square root of 2.	
	220. the square root of 4 is 2? right, because 2 times 2 equal 4, so what times what equals 2? I am thinking like a half.		She guesses half can be square root of two, so she multiplies half by half.	She tries to find square root of 2 by squaring numbers such as half, third half.	
	230. Probably I am off of my number, because 9 over 4 is definitely not two.		She guesses that $\frac{3}{2}$ is square root of two, to verify she multiplies $\frac{3}{2} \times \frac{3}{2}$.		
	243. the only thing is that *y* equals, but 1.41.	A_{al}	She estimates $4^{1.41} = 6.5$ by using the given graph.	She finds $\log_4 4^{\sqrt{2}} = 1.41$ by referring to the review example and Activity 1.	I told her approximation of square root of 2.
Activity 2: $\log_4 7 + \log_4 5 = ?$	245. I don't know what is that $\log_4 7 + \log_4 5 = ?$		She skips this question.		
	260. could you do, I do not have any idea about log, but I assume when there is a common base so you add the numbers.		She writes $\log_4 12$	Her reasoning probably is because of distributive law in algebra.	

$\log_3 27 \times \log_{27} 3 = ?$	245.3 times 3 is 9 , 9 times 3 is 27 , um---because 9 times 2 is 18 . This is 3 .	P_{al}	She writes $\log_3 27$ as $3^3 = 27$.	She calculates powers of 3 and concludes that $\log_3 27 = 3$.	
	246. so what is the power of 27 equals to three? How is that possible? I do not know how the log works? How 27 it will be like a fraction? How you get a 3, 27 times one.	A_{al}	She writes $\log_{27} 3$ as $27^? = 3$.	She does not extend the domain of exponents to real numbers.	
	248. I mean um---I do not know how 27 get 3 which power. A negative?			She has lack of knowledge on fractional exponents.	
	256. if I divided by 9.		She wants to divide 27 by 9, to find the exponent of $27^? = 3$.	She divides 27 by 9 instead of taking root.	Misconception of roots as division.
Activity 4: Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ h old? Find an example and explain your reason.	261. no, because here ($\log_{10} x^3$) you will figure out the inside first (*x*) and then exponent (x^3) and then $\log_{10} x^3$, but there find $\log_{10} x$ and exponent it $(\log_{10} x)^3$.	A_{al}	She considers $x = 2$ as an example. Then she refers to review example and writes $\log_{10} 2^3 = \log_{10}$	Her intuition is correct, but she can not complete her example $\log_{10} 2^3 = \log_{10}$	
Find the value(s) of x such that $\log_x (x-1) + \log_x x =$	264. I do not know how to rearrange the equation to solve.		She does not know how to solve the problem when variables are involved.		
Consider the inequality $\log_x \|x^2 - 1\| < 2$ what do you think about this inequality.	268. Does this *x* is something different of that *x*.				Lack of understanding of variables
	275. Absolute value is always positive, there is only few options 0,1 and 2, or 1.5, you know if it is less than 2. That is really odd.				

Student 4's interview analysis

Activities	*Interviewee's statements*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$			Reads the review example and writes $2^3 = 8$.		
Activity 1: $4^2 = ?$	283. This is 16.		He finds 16 by multiplying 4 by 4.		
$\log_4 16 = ?$		A_{al}	He writes $\log_4 16 = 4$. Then by looking at $4^2 = 16$ he realizes his mistake.	He focuses of relation between logarithms and exponentials forms in each row in order to complete the table.	
$4^4 = ?$	283. Um--- I guess 4 to 4 is 256. That is right.		He guesses and looks at $\log_4 256 = ?$, then says that is right.	He does not perform multiplication to find 4 to power 4.	
$\log_4 256 = ?$		A_{al}	He first finds the power. Then he follows the relation between log and exponential form.		
$4^? = 4$	284. That must be one. I get confuse and every time I get confuse I look at the other ones that I already did.		He checks the previous ones and follows the relation.		
$\log_4 4 = ?$		A_{al}		He writes $\log_4 4 = ?$ by referring to exponential form.	
$4^? = 1$	285. zero, um--- zero.				
$\log_4 1 = ?$		A_{al}		He finds the power, then refers to the review example to find logs.	
$4^? = \frac{1}{16}$	286. I hate fractions, anytime I see fractions I get nervous. Um--- -2, -2.		He thinks few second and says -2.		He hates fractions.

$\log_4 \frac{1}{16} = ?$		A_{al}	After he finds the power, he calculates $\log_4 \frac{1}{16} = -2$ by referring to review example.	After he finds the power, he calculates $\log_4 \frac{1}{16} = -2$	
$4^? = -16$	289. -16, um--- half.				Times=power
	293. I do not have an idea. I just put it half. I guess it would be not -2? No		Since he had -2 power in previous one, he says " it would not be -2".		
$\log_4(-16) = ?$	295. here ($4^{-2} = \frac{1}{16}$)is -2, so that can not be -2 there ($4^? = -16$). If 4 to -2 gives one over 16, so -16 um--- . If 4 to -2 gives one over 16, so -16 um---	A_{al}			Since he does not have concrete knowledge on exponents and he does not know that range of exponential function is positive real numbers.
	345. I have had powers that are negatives like here -2 or when I do derivatives, they could be negative power. But, when I do base to a power that gives me negative no.		He answers based on my hints, but he is not reflected on his answer.		I helped him by asking have you seen a power to raise a positive number like 4 to this number and obtain a negative number?
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$	300. estimate um--- this ($4^{\sqrt{2}} = ?$)here, I am lost.		He is not used to work with irrational exponents.		
	312. x is square root of 2 .		He can not understand the question easily.	.	He does not know how to read the graph.
	318. it is one. Two is between 4 and 1, square root is between 1 and 2, 1.5.		He estimates square root of 2.		
	320. here, means 1.5, um---no, x is 1.5, follow it up. It comes up to 8.		He shows 1.5 on y-axis instead of considering it as a x value.		He does not know how to find a point on a graph and confuses x and y axis.

$\log_4 4^{\sqrt{2}} = ?$	326. 4 to 1.5 is 8, now log 8, 4 um--- if it is 16 it is 2, now you do 8 um-- 4,4. silence…it is 1.5. It is okay.	A_{al}	He goes back to table and checks previous ones that he writes 1.5.		
Activity 3: $\log_4 7 + \log_4 5 = ?$	327. this addition, when you add logs it is multiplying so it is log 35, 4. in the same base	A_{al}	Since he had log in the day of interview, he remembers the law. He multiplies 7 by 5, and writes $\log_4 35$.		He interprets the product law of logs as "add logs it is multiplying"
$\log_3 27 \times \log_{27} 3 = ?$	328. we do not have the same base. Multiplication, they add them but the same base. It is still 3, this 3 and 3 um--- so now. You add them in multiplication.	A_{al}		He writes exponential forms of each terms by referring to the given table in Activity 1, and finds $\log_3 27 = 3$.	He confuses the product law of logs and says "Multiplication, they add them but the same base"
	330. we multiply them when it was addition in same base. We do not touch bases here we multiply, but not same bases.		He struggles to retrieve a law to multiply logs.		Misconception of logs' addition law.
	333. now we have log base 3, 27. 27 is 3 and 3 so multiply log um--- so first when you do that it becomes 3 to 3 , 27 .Here you have log 27, 3 , you do not have this, this means 3 , 1 um---	P_{al}	He writes $\log_3 27$ as $3^3 = 27$.	He calculates power of 3 , and concludes that $\log_3 27 = 3$..	
	337. it is 3 alone. It is just 3. Okay, but now for here you take 27, because you take base and the answer is 3. Now when I find power I find log.		He writes 27=3 and tries to find a power for 27.		
	339. it is going to be minus.				
	341. it can be a third. 27 to the power of one third.	A_{al}		He is not reflected on his actions. He keeps referring to previous activities.	
Activity 4: Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.	357. but this log base of 10, x to 3, that means base is 10 so $\log_{10} x^3$ equals to base of 10 and then 10 to um--- the answer 27, so here the answer would be x^3, but here we don't have it.		He does not why the equality does not hold and has difficulty in understanding $\log_{10} x^3$.		Since variable x is involved in this activity, he does not

	363. 100, so that means here I have $\log_{10} 100^3$. So answer for that um--- what do I do? If I put 3 here log 10 what happens for 3? When I do log 100, 10 it is easy. What do I do with 3?	A_{al}	He takes $x=100$ as an example. However, he has difficulty to calculate $\log_{10} 100^3$..		
	365. Yeah. Then it is 6, so the answer is 6.	A_{al}	He refers to previous ones to answer $\log_{10} 10^6$		I told him to write 100 as 10^2, so he was able to find $\log_{10} 100^3$
	370. so it is 2 times 2 times 2 which is 8, that is why it does not work.		He computes $\log_{10} 100 = 2$, then finds $(\log_{10} 100)^3$.	In computing $(\log_{10} 100)^3$, he times 2 by 3 instead of raising 2 to power 3.	Power=times
Activity 5: Find the value(s) of x such that $\log_x(x-1)+\log_x x=1$	371. okay, the same base , you can just multiply them. (x-1) times x equals 1. now, $\log_x x^2 - x$ is 1.Now this is the problem, no what um---the 1 other side is zero ?	A_{al}	He applies the product law of logs.	He is not reflected on his actions , since he refers to the table in Activity 1 to write the exponential form of $\log_x(x^2-x)=1$	
	373. x is the same number as the base.				Lack of understanding of variables; x is not an mathematical object for him.
	376. now, *x* is the base then you have *x* to 1 equals to $x^2 - x$. Because base is *x* so you put the base *x* and 1 comes to power equals to $x^2 - x$. You solve this.	A_{al}		He is not reflected on his actions , since he refers to the table in Activity 1 to write the exponential form of $\log_x(x^2-x)=1$	
Activity 6: Consider the inequality $\log_x \lvert x^2-1 \rvert < 2$ what do you think about this inequality.	385. smaller than 2, smaller than two can not be an answer. You can not put here, the base is *x* here you need and answer so I just put less than two.		he refers to the given review example, he writes $x^{<2} = \lvert x^{2-1} \rvert$	He tries to follow the relation between exponential and logarithms, but he is confused with smaller sign.	

	389. um--- so smaller than 2, you see now I am confused. Because here always you had answer and that was easy, and you have absolute value. So it is always positive. So it is between zero and smaller than two.				

Student 5's interview analysis

Activities	*Interviewee's statements*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	392. silence		She writes $\log_2 8 = 2^3$. While she does not understand the review example.		She does not pay enough attention.
Activity 1: $\log_4 16 = ?$	399. log of 16 base 4 is 4 to the two.		she writes $\log_4 16 = 4^2$.		She realizes that all is about log, and then she tries to understand the review example and the relation between log and exponents.
	409. So it is two, do you want me cut it off.	A_{al}	She finds the answer after several times referring to the review example.	She follows the review example.	
$4^2 = ?$	412. do you want me write 4 to 2?		She writes 16.		
$\log_4 256 = ?$	414. so log of 4 is um--- log base 4 of 256 , so that is what?				
	416. Can you do it ($\log_4 256 = ?$)?	A_{al}	After finding $4^4 = 256$, she looks the relation between log and exponents and writes $\log_4 256 = 4$.	She uses the review example to find log.	
$4^4 = ?$			She multiplies 4 , 4 times.		
$4^? = 4$			This is very simple and intuitive for her.		
$\log_4 4 = ?$		A_{al}	She writes the power and then follows relation to find log.		
$4^? = 1$			This is very simple and intuitive for her.		
$\log_4 1 = ?$		A_{al}	She writes the power and then follows relation to find log.		
$4^? = \frac{1}{16}$			She finds $4^{-2} = \frac{1}{16}$ easily.		

$\log_4 \frac{1}{16} = ?$	422. one out of 16, that is a good question. I guess I have to look at these ($4^? = \frac{1}{16}$) and theses ($\log_4 \frac{1}{16} = ?$) and try to find the relationship.	A_{al}	She writes the power and then follows relation to find log.		
$4^? = -16$					She has difficulty with finding power.
$\log_4(-16) = ?$	425. Can we go to the second question?		She skips $\log_4(-16) = ?$ and start Activity 2.		
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and estimate $\log_4 4^{\sqrt{2}} = ?$	432. square root of 2 is 1.7? I do not know.		She thinks square root of 2 is 1.7.		I told her that consider square root of 2 as 1.41. she knows how to read the graph.
	434. so square root of 2 is 1.41 so I am gonna look at around 1.41 here , it is around 7.		She uses the graph to estimate $4^{\sqrt{2}}$ and she concludes it is around 7.		
estimate $\log_4 4^{\sqrt{2}} = ?$	437. um--- good question. What about relationship.		First she skips $\log_4 4^{\sqrt{2}} = ?$, and goes to next activity.		
	448. $\log_4 4^{\sqrt{2}}$ is square root of 2.	A_{al}	She refers to the review example.	She uses the review example to find log.	
$\log_4 7 + \log_4 5 = ?$	442. evaluate the equations. It is all about log. Laughing, so I need really understand it.		She goes back to the review example and tries to understand.		
	453. so evaluate the equations. This is log 7 to base 4 we find out something here $4^{\sqrt{2}} = 7$, here we have log 4 , 5 equal to 5 so 4 to something is equal to 5, so 4 , 5 at the bottom is 1.2.	P_{al}	She uses previous activity results and converts $\log_4 7$ to $4^{\sqrt{2}} = 7$, also $\log_4 5$ to $4^{1.2} = 5$.	She says " 4 to something is equal 5" in finding $\log_4 5$.	

$\log_3 27 \times \log_{27} 3 = ?$	458. it is 3 to something is 27 , how much so equal to 27 times of log 27 , 3 the base is 27 so 27 is something	P_{al}	She finds out $\log_3 27$ by converting it to exponential.	She does not refer to the review example.	
	461. so I am taking the base 27 on what power will give me 3, 27 to 1/3. I assume, so what I got, I got here 3, 3 times 1/3 equal to 1.	P_{al}	She finds $\log_{27} 3$ by converting it to exponential.		
Activity 4: Does the equality $\log_{10} x^3 = (\log_{10} x)^3$ hold? Find an example and explain your reason.	468. let say x is equal to two, 10 log of 10 to the cube what is that? that is equal, silence…to out of 10, 8 how do the log function works, so it is 10 to the something equal to 8 , it is what?	P_{al}	First she considers $x = 2$, and she tries to find $\log_{10} 2^3 = \log$ with converting it to $10^? = 8$	She is reflected on her actions, since she is not referring to the review example any more.	
	472. let say x is equal to 10, log 10 ,10 to the cube, log 10 times10, 10 to 3 is equal, this is equal to three. Right? so what about the other side, the other side I got log of 10. x cube so x is gonna be 10 so log of 10 to the 10 cubed right, so 10 to the power of what gives me 10 so 1 give me 10, the answer to the parenthesis is 1 and do the three is one, therefore they do not equal.	P_{al}	She considers $x = 10$ and shows that equality does not hold.	She is reflected on her actions, since she is not referring to the review example any more.	
Activity 5: Find the value(s) of x such that $\log_x (x-1) + \log_x x = 1$	474. okay, you want to find the value of x, so log of x, (x-1) okay plus log of x to x is equal 1. So I got here, I have to do it again, so I got base x to something is equal to (x-1) and I got x to the something is equal to x, I am just writing, not a good notation. So this is should be equal to one okay so what x to the power of what give me (x-1), wow!	P_{al}	She converts logs to exponentials, $\log_x (x-1)$ to $x^? = x - 1$ and $\log_x x$ to $x^? = x$.		

	475. x to the power 1 give me x , um--- so this is 1. What I am gonna do with this, this is the power of zero what is this 1, find the value of x, you have to find the value of x, silence…	P_{al}	She says $x^{?} = x$ power is 1, then she considers $x^{?} = x - 1$ and she does not how to solve it.		
Activity 6: Consider the inequality $\log_x \lvert x^2 - 1 \rvert < 2$ what do you think about this inequality.	483. what I am doing here, I am saying base is x so x to this something is equal to, silence, should be smaller than 2 it that what I am saying.		she writes $x^{?} = \lvert x^2 - 1 \rvert <$. She gives up.		

Student 6's interview analysis

Activities	*Interviewee's statements with line numbers*	*Analysis with APOS*	*Explanation of subjects' behaviors*	*Justification of APOS analysis*	*Comments*
Review example: $\log_3 9 = 2$ means the same as $3^2 = 9$. What means $\log_2 8 = ?$	490. in the first case we have log 9 base 3 equal 2 which means 3 square equals 9, so in this case um---base 2 square base two to power something is equal to 8 , so log 2 cube base 2 equals 3.	A_{al}			
Activity 1: $\log_4 16 = ?$		A_{al}	He finds the exponent and then finds log.		
$4^2 = ?$	495. this is the answer 4 square 16.				
$4^4 = ?$	498. 4 times 4 is 16, 16 times 4 is 64 , 64 times 4 is 256.		He performs calculation to find 4 to power 4.		
$\log_4 256 = ?$	499. which is 4.	P_{al}		He is reflected on his actions.	
$4^? = 4$	500. one .				
$\log_4 4 = ?$		P_{al}	He writes one after finding the exponent.		
$4^? = 1$	501. base 4 , is zero.				
$\log_4 1 = ?$		P_{al}	He is reflected on his actions.		
$4^? = \frac{1}{16}$	502. I guess negative, -2.	P_{al}	.		
$\log_4 \frac{1}{16} = ?$			He writes after finding the exponent.		
$4^? = -16$			He knows that this is impossible.		
$\log_4(-16) = ?$	503. this ($\log_4(-16) = ?$) can not be done., so complex numbers will help me here.	O_{al}, A_{fl}	He has concrete knowledge on the domain and range of logarithmic and exponential functions.	He knows about the domain and rang of logarithmic functions.	
Activity 2: The graph of $f(x) = 4^x$ is given. Estimate $4^{\sqrt{2}} = ?$ and $\log_4 4^{\sqrt{2}} = ?$	509. okay, $4^1 = 4$, $4^2 = 16$ so 4 to the power of root 2 is somewhere between, bigger than 4 less than 16. So it is 6 point something, 6.3.				

	510. estimate the log, in this case you can put this ($\sqrt{2}$) in front so $\sqrt{2}\log_4 4$, $\log_4 4$ is one and $\sqrt{2}$. Laughing…	O_{al}	He knows the power law of logarithms and applies it.	He applies power law of logarithms.	
Activity 3: $\log_4 7+\log_4 5=?$	511. same base you multiply inside, right?		He knows the product law of logarithms.		
	517. it doesn't help that much. It is better 1.4 plus 1.2 equal 2.6. I represent logs as exponents and find them from graph. This side is 4 to the power something equal to 35. It is gotta between 2 and 3. All right.	O_{al}	He applies the product law of logs and finds $\log_4 35$. Then he verifies his answer by evaluating each tem separately and adding them.		
$\log_3 27\times\log_{27} 3=?$	519. um--- but this does not work out. But this is the same thing of something else. Log um--- I do not know, you can convert this again! 3 times 3, 3 cube is 27, so this ($\log_3 27$) is equal 3, 27 to power something is 3, one third. But there is another rule to use, multiplication rule.	O_{al}	He thinks of a multiplication law for logs.		Since he recalls the laws of logarithms, identifying his thought process level is not so easy.
Activity 4: Does the equality $\log_{10} x^3=(\log_{10} x)^3$ hold? Find an example and explain your reason.	521. does the equality hold? No. because $\log_{10} x^3$ is not equal to $(\log_{10} x)^3$.	O_{al}		He applies power law of logarithms and shows that the equality does not hold.	
Activity 5: Find the value(s) of x such that $\log_x(x-1)+\log_x x=1$	523. no, is that right? Is it subtraction rule? Is $\log_x(x-1)$ equal to $\dfrac{\log_x x}{\log_x 1}$?			He does not recall the law question law of logarithms properly, since he has learned how to memorize the formulas not verbalization of formulas.	

	527. so $\log_x[(x-1)(x)]=$, $\log_x(x^2-x)=1$, so convert to exponents , how to convert that? $x^1 = x^2 - x$, so x=2.	S_{al}	He applies the product law of logarithms.		
Consider the inequality $\log_x\|x^2-1\|<2$ what do you think about this inequality.	533. yes, it is easy for me, or it seems easier. $x^2<+(x^2-1)$, $0<-1$, no it is not true, it does not make sense.		He tires to convert the inequality into an exponential form.		
	534. oh there is absolute value, right it is minus and plus. When it is plus does not work. When it is negative *x* is less than positive negative square root of half.		He writes $x^2<-(x^2-1$, $x^2<-x^2+1)$, ... $x<\pm\sqrt{\frac{1}{2}}$		