GROUPS, GRAMMARS AND DESIGNS

A Mathematical Analysis of Artifactual Patterns

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Abstract

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In keeping with other developments in the natural sciences, more and more researchers use mathematics to analyze the geometric and constructive principles hidden in works of art and then apply these principles to generate new patterns. This thesis is in line with trend. It develops tools for the analysis of the geometric patterns hidden in Bakuba textiles and Zillij mosaics. As such, it contributes to the interdisciplinary *Generative Design* project, which explores the mathematical structure of artifacts using groups, shape grammars, and other techniques. The thesis consists of six parts:

1. A brief introduction to background of the thesis. After introducing the project and geometric patterns in Bakuba textiles and Zillij mosaics, we discuss the structure of the *Generative Design* systems, discuss results the project team has obtained, and explore the possibility of applying groups to implement our goals.

2. An introduction to wallpaper groups, related theorems and results. After introducing the *Crystallographic Restriction* theorem, we provide a proof that establishes that there are precisely seventeen groups for creating wallpaper patterns from given geometric patterns. To understand these seventeen groups and to explain the relationship between groups and Bakuba textiles and Zillij mosaics, we illustrate these groups using motifs extracted from Bakuba and Zillij.

3. An illustration of the mathematical reconstruction of Bakuba and Zillij motifs using groups. After analyzing patterns in Bakuba Textiles and Zillij Mosaic, we show that we can use certain groups to reconstruct patterns with different motifs involving various parameters. This brings us closer to our goal. The groups considered at this stage are not necessarily wallpaper groups.

4. An illustration of the mathematical reconstruction of Bakuba and Zillij motifs using grammars. In this part, we use shape grammars defined by Stiny to reconstruct patterns in Bakuba and Zillij by defining suitable grammars and applying appropriate sequences of shape grammar rules.

5. A new method of creating new pattern. In this section, we compare and discuss the methods of group theory and shape grammars to extract motifs from classical artifacts and to generate new patterns. Then we develop a new method that enhances the two classical methods by generating new patterns from Bakuba and Zillij motifs using wallpaper groups.

6. The thesis ends with appendices in which technical definitions and basic proofs are given to make the thesis self-contained. In particular, the proof that there are precisely seventeen wallpaper groups is presented in detail for completeness and accessibility.

Acknowledgments

The end of my studies for a Master's degree in Mathematics is now in sight. In the three years in which I worked towards this goal, I achieved a great deal. I believe that all these achievements should be attributed to the help I received from many people.

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Chapter 1

Introduction

This thesis is a contribution to the *Generative Design* project. It is therefore necessary to provide a brief introduction to the project and describe its goals and background. I will consider four aspects of the project: its purpose, its history, its progress, and my involvement in it. In Bhakar [3], the project is described as follows:

"Advances in information technology now provide a variety of digital tools for the mathematical investigation of the visual complexity of textile patterns and decorative designs. In this paper, we report on innovative applications of this technology to the geometric analysis of Kuba cloth and Zillij mosaics. From our perspective, these objects present distinctly different analytical challenges, and typify problematic aspects of the classification and generation problems of artistic design. Mathematical considerations led us to use neural networks, shape grammars, and related technologies to approach these problems. Our ultimate goal is to use our methods, samples, and peripherals to build an interactive database for the study of historical pattern and the generation of contemporary designs."

This thesis is a contribution to this goal. As is pointed out in this paper: "Geometric patterns in African Kuba cloth are fractured and reiterated in an astounding display of improvisation that connects the viewer to the artisan in a special dialogue. Women embroider, stitch and cut raffia pile into velvet patterns on a woven raffia base prepared by the men. Confounding traditional notions of geometric symmetry and repetition, Kuba women design their textiles freely as they work, without drawings, over a period of months and sometimes years (see Adams [1]). These distinctive monochromatic velvets come from the Kasai-Sankuru river region in central Africa, where western anthropologists, impressed with the cultural richness of carvings and textiles, have visited the multi-ethnic Kuba kingdom for research and study since the end of nineteenth century.

Fractal symmetry, including recursion, scaling, self similarity, infinity and fractal dimension, is displayed in the Kuba designs through juxtapositions of linear embroidery, velvet forms and contrasting color that becomes counterpunctual in the composition. According to Eglash [9], fractals are part of African numerical systems as evidenced in their village planning, decorative motifs and textiles. He specifically describes Kuba designs in the computational terms of a complexity spectrum:

These Kuba designs tend to show periodic tilings along one axis, and aperiodic tiling-often moving from order to disorder-along the other. Similar geometric visualizations of the spectrum from order to disorder have been used in computer science. As far as I can tell, the Bakuba weavings never reach more that halfway across the spectrum—they are typically moving between 1 and 1.5, that is, from periodic to fractal, rather than stretching all the way to pure disorder.

Kuba women create patterns of mathematical complexity and beauty by deviating from static geometric structures with their own personal logic of improvisation. Henri Matisse and Gustav Klimt were both inspired by Kuba design motifs, transforming and referencing the hypergeometric patterns in their paintings. Analogous to the African American idiom of improvisational Jazz, improvisation in Kuba cloth makes their design motifs unique to Western textiles."

The methods studied and developed in this thesis are designed to explore the geometric features of these artifactual patterns. What, then, is a Bakuba cloth? Here is an example that will guide us in our quest for new descriptive and generative techniques to explore the geometric richness of these artifacts:

The second class of artifactual patterns studied in the *Generative Design* project are patterns found in Zillij mosaics in North Africa. What are Zillij mosaics? In Bhakar [3], these patterns are described as follows:

"The traditional Zillij mosaics have been traced back to the 12th century and reached a zenith in Fez in the 17th century. Some of the best examples are in the madrasahs (universities) in the old Medina. In a small open fronted shop it is possible to still see three generations involved in the task of making the mosaics: the son making the squares, the father cutting the specific shapes and the grandfather supervising. The mosaics are fired, then glazed in colors derived mainly from metals, and then fired again. It is said that to make a Zillij mosaic panel in the traditional way requires great skill and the integration of hand, eye, heart, and mind. Muslims develop their memory and their heart by learning the Qur'ān by heart. A Zillij panel is made by laying each piece in place upside down and then plaster is poured over the back and the pattern is only seen

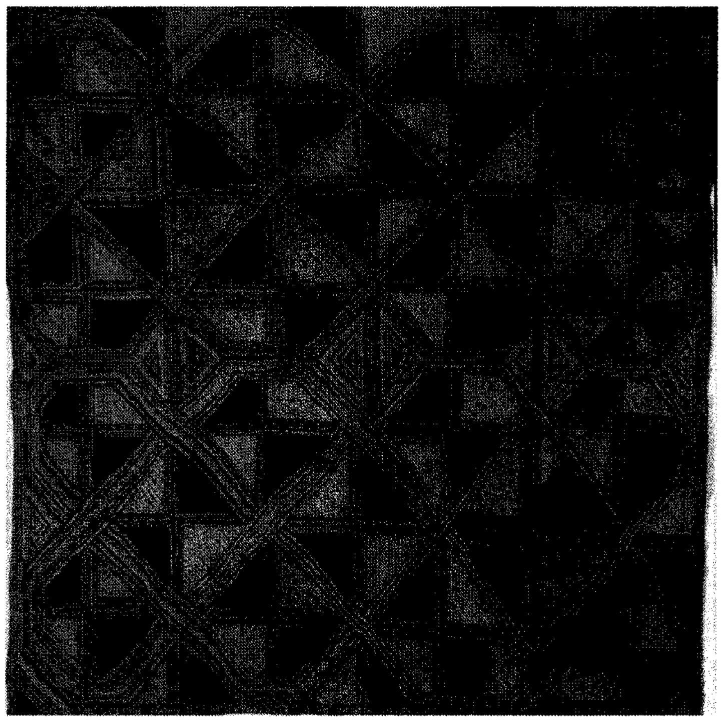


Figure 1.1: A Bakuba Cloth (Photo: Hansjorg Mayer)

when it is dry. Traditionally, the pattern is conceived by the Master and held in the mind as he lays each piece without seeing the pattern develop.

The geometric figures on which the patterns are based have a symbolic component. All the patterns are based on a circle the symbol of the infinite without beginning or end and where every part of the line is the same. Patterns are constructed starting with the first theorem of Euclid (see Courant and Robbins [5]). After a circle is drawn, the radius is maintained and the compass point is moved to any point on the circumference and a second circle is draw creating an almond shape in the middle, and the circumference of each circle touches the centre of the other. This can represent the first division of the cell, the beginning of life and of polarization, and the number two. The compass point can then be situated on the point on the circumference where the two circles overlap and another circle can be draw. This can go on until six circles fit exactly within the circumference —six circles around a central circle, and two sets of three, as in the petals of many flowers. Three is the simplest polygon. Patterns based on four can be the symbol of materiality—the four directions, and the four states of matter. The pentagon has the ratio of the golden section (the only ratio in which the smaller section is to the greater as the greater section is to the whole), which was referred to as "the divine proportion" by Renaissance writers.

The patterns for the Zillij mosaics are for the most part based on geometric systems which are progressions and multiples of figures with three, four and five equal sides. These in turn have a parallel in music. In the patterns the circle from which all are derived is still evident in the rotational and reflexive symmetry of the center figures. It is this symbolism in the transcendent, geometric patterns of Zillij mosaics that suggest a comprehensive cosmology (Critchlow 1976)."

Here is an example of a Zillij mosaic that will guide us in our exploration of the geometric properties of such patterns in this thesis:

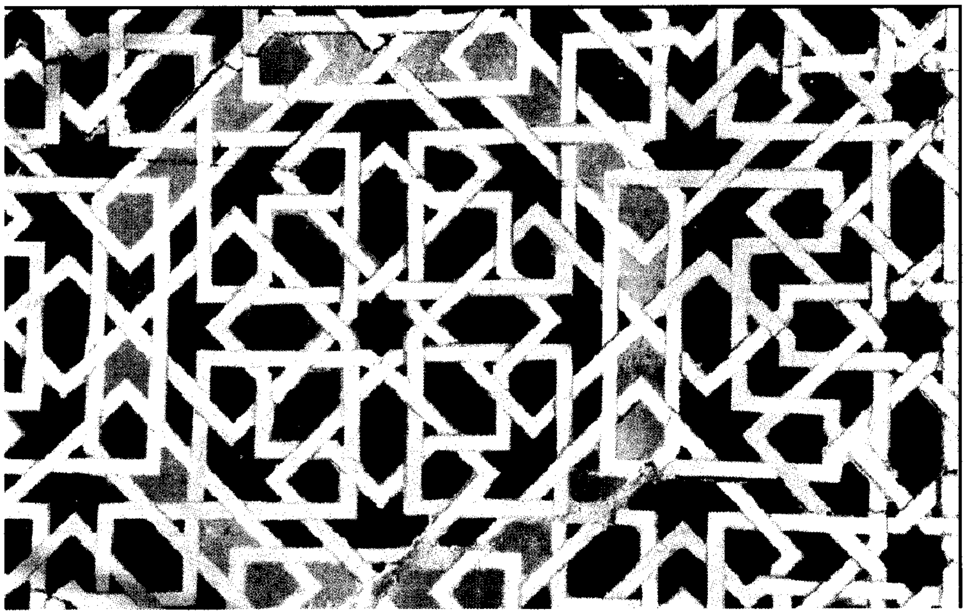


Figure 1.2: A Zillij Mosaic (Photo: Sharman 1994)

1.1 Digital Tools

Since starting the project in 2001, the research team has done a great deal of work and has made substantial progress. In this section, we will sketch some salient features of this work relevant to my thesis: image segmentation, object classification, the structure of shape grammar systems, and recent results.

1.1.1 Image segmentation

In order to identify appropriate digital tools for the analysis of the geometric content of Bakuba cloths and Zillij mosaics, the Concordia Center for Pattern Recognition and Machine Intelligence carried out a number of experiments to determine the suitability of different pattern recognition tools for the pattern analysis of such textiles. At first, interesting objects or shapes had to be segmented for successive applications. Therefore, various edge detection techniques were used. By converting color images to gray images, we used traditional methods, like Sobel, Canny, and Marr, to detect edges and then segmented shapes or interesting objects from background. We also worked on binary images. In this way, we converted color images to gray images; then we used threshold method, in which the values of threshold were carefully calculated and varied according to neighbor pixels, to binarize the image to the binary images; lastly we got shapes or objects. We also attempted to use color information in images to detect edges. Of course, we also made some experiments by used some software packages on edge detection, but, it is hard to code their parameters and embed to our systems, we had to study various methods and write ourselves code as discussed in Dudek [8].

1.1.2 Object classification

After segmenting interesting objects from images, we classified all objects according to their shapes or features. These works are very helpful for designing shape grammars in the successive steps. For choosing and locating groups of shapes in real-world images of artifacts, we had enlisted the help of the Centre for Pattern Recognition and Machine Intelligence at Concordia in assembling a group of systems which identify groups of similar shapes in raster images. We integrated these systems in an upcoming rule-suggesting system, which will look at the clusters found and attempt to discern a few basic types of arrangement and symmetry.

One of classification methods is to treat an image as a matrix, use algebraic methods to extract its features such as SV and then work on these algebraic features by using traditional methods (see Huang [16], Huang [18] Huang [19], and Guo [12]). At the beginning of the project, we used this method in our work, but the results were not suitable for more complex images since it was was very difficult to separate objects from their background Huang [17]. Another method of clustering shapes used information of size, position, rotation and normalized Fourier descriptors. We also attempted to use neural networks for classification (see Dudek [8] and Huang [17]).

1.2 Shape Grammars

Shape grammars are designed explicitly to free the design process from being caught using excessively high-level "symbols" or "atoms". Various attempts at making fairly general shape grammar interpreters have gone before us, and we operate on the assumption that a judicious choice of simplifications and specializations will make creating an interpreter for Bakuba cloths and Zillij mosaics patterns a fairly tractable problem, both at design time and at run time.

To understand shape grammars well, some members of our team attempted to write our systems to implement shape grammar (see Hortop [14]). First they planned to develop a shape grammar interpreter in Flash. The interpreter was defined by James Gips in [10]. They wanted to explore shape grammar applications as a part of our project and assess what was required for ultimate goal and the processes of studying and implementing our project. This approach was later supplanted by the creation of a shape grammar interpreter written in Java. Currently, the interface to this interpreter is being refined with the goal of making it user-friendly and accessible to non-specialists.

After studying and analyzing these methods, I believe that the goal of the *Generative Design* project can be achieved more elegantly and more effectively by dividing the development of a the ultimate system required for *Generative Design* into the following components: a rule extraction tool, a generalized shape grammar interpreter, a rule generator, a pattern generator, and a searchable data bank of images. This thesis is mainly a contribution to the formulation of a "generalized" rule generator using groups.

1.2.1 Rule extraction

In which the current system developed by the textitGenerative Design group, shape grammar rules are partially generated automatically or by users' help. Users design shape grammar rules through their understanding. First, they should analyze some principles hiding in artifacts, and then they may design some appropriate rules. During these processes, shapes or objects are segmented from their background automatically or manually. To reduce the complexity of algorithms, we may use clues for segmentation. Clustering or classifying shapes and objects are helpful for extracting rules automatically or manually. This is a key step.

1.2.2 Grammar interpretation

The interpreter is used to understand the shape grammar rules. It retrieves a group of rules about a specific topic in database, and generates a graph according to the description of the group rules.

The biggest problem we faced, and one faced by most shape grammar researchers, is the subshape problem—finding groups of faces in a design that match single faces in a rule, something which is required in shape grammar theory (stated as a mandatory ability to ignore overlapping features) and that allows for emergent behaviour to take place. Some implementations are very general on content while being weak on the subshape problem. The approach in the *Generative Design* project used was to pare down the range of designs we worked with in order to be better able to deal with the subshape problem in our domain of interest (see Hortop [14]).

1.2.3 Pattern generation

After rule extraction, it is necessary to formulate new rules based on the extracted ones and create new patterns with existed motifs. At this point, new methods and new concepts can be introduced that are faithful to their origin, but that have not been used before. For example, we can introduce symmetries that exists in most patterns of Bakuba cloths and Zillij mosaics. Several authors, among them D. Washburn [35], who have used symmetry as a tool for classifying and analyzing ornaments. Stiny (see [33]) also used symmetry properties to generate artifactually-inspired patterns by computer using "shape grammars"Leonardo Da Vinci and Dürer used the idea of superimposing geometric grids on images to visualize their content, and Descartes pioneered the use of geometric grids to describe geometric content with algebraic equations (see Huang [16]). Besides, there are many researchers who used structural information like geometric features of patterns and their affine transformations in their work (see Martin [24], Knight [22], Levy [23], Bix [4], McDowell [27], and Grünbaum [11]). Therefore, it is appropriate to study the possibility of applying some concepts of symmetry in our project, and I will discuss it in this thesis.

1.2.4 Pattern storage

Databases are used to store data that may be shapes, objects, motifs, shape grammar rules, and others. In the future, we may retrieve these data to modify, delete, or generate graphics (patterns). In particular, the database will be accessible on the Internet and later in a gallery setting, and allow users to contribute their works by uploading files in some open format. Users will also be able to request the generation of "synthetic" patterns, based on constraints they supply. With this mixed input stream, the database will develop a behaviour similar to that of a genetic algorithm where users' visits, contributions and requests to the system influence the available vocabulary and generative tendencies of the system (see Dudek [7] and Dudek [8]).

1.3 Wallpaper Groups

We mentioned that we needed to explore the possibility of using the concept of symmetry to generate new patterns based on shape grammars. Let us take a closer look at what is involved.

Symmetry exists everywhere: in nature, in life and in the arts. Human beings have known about symmetry and applied it from the earliest times. From artifacts to modernistic art, from ornaments to building decorations, symmetrical reflection has been present (see Jablan [20]). Human beings have studied symmetry both from an artistic and an aesthetic point of view, and applied it in their daily lives in ornamental art for years. In fine art, "symmetry" has played a key role for a long time, indicating harmony, accord, and regularity, and was identified with mirror symmetry in the more narrow sense. As such, the descriptive languages used in most discussions on ornaments drifted apart from the exact language of geometry. As the result, even when talking about the same object, these languages are used in quite different senses.

As the natural sciences (crystallography, chemistry, physics, and so on) evolved, symmetric structures attracted more and more the attention of mathematicians (see Sharman [31]), and became an important area of geometric studies. From a mathematical point of view, the key ideas involved in "symmetry" are transformation groups, invariance, and isometry. This view of symmetry has led to the development of ornamental art (see Speiser [32]).

Symmetry is very helpful for discovering patterns and provides natural stopping conditions for objects called "shape grammar generators," the central structures used in this thesis for the mathematical analysis of artifactual patterns. We combine shape grammars with methods of group theory to set a finite bound on the number of types of symmetry to be considered. As a result, our goal becomes achievable since it limits the search for symmetry-based rules.

The techniques used in this thesis evolved from our study of the geometric structures of patterns of Bakuba cloths and Zillij mosaics when we discovered that group structures exist in some artifactual patterns. For example, we were able to use group theory to explain the intrinsic relationship between patterns created by audio frequencies in a liquid medium and the geometric patterns of traditional Morocco Zillij mosaics discovered by Dr. Sharman in her study of the geometric structures and mathematical progressions in the traditional mandala and tessellation patterns in the art of several cultures. For these reasons we have focused our attention on groups. Wallpaper groups, in particular, because of the fact that building larger images from motifs is akin to tiling a planar surface according to certain principles.

In order to carry out our task, we needed to take a closer look at the geometric principles involved in wallpaper groups. The proof that there are only seventeen of them provides some of the relevant clues. We therefore spend considerable time in this thesis on the proof that classifies these groups and the geometric ideas involved.

We then show how the ideas involved in the study of wallpaper groups are relevant to the design of shape grammars. As is clear from our previous publications (see Huang [16] and [17]), "group-enriched" shape grammars are appropriate for the study of the certain artifactual patterns and are therefore our main tool for building new images from the geometric motifs extracted from Bakuba cloths and Zillij mosaics.

Chapter 2

Wallpaper Groups

We begin this thesis with a brief discussion of wallpaper groups and the steps required to prove that in view of the crystallographic restriction and its related theorems, there are only seventeen such types of groups. To understand these groups well and to find the relationships between wallpaper groups and Bakuba textiles and Zillij mosaics, we illustrate each group with examples. In particular, we use group theory to analyze the group structures of Bakuba textiles and Zillij mosaics. After that, we use shape grammars to analyze our interesting objects. We end with the idea of combining shape grammars and symmetry groups to study Bakuba and Zillij patterns.

Our first objective is to clarify and illustrate the well-known fact that there are only seventeen different kinds of wallpaper groups. What does this mean? Since every such group consists of affine transformations of symmetry, this means that the possible combination of translation, rotation, reflection, and glide reflection generate all wallpaper groups.

Wallpaper groups are also called plane symmetry groups or plane crystallographic groups. They constitute a mathematical classification tool for two-dimensional repetitive patterns. The concepts required to understand wallpaper groups can be found in Appendix A. Especially, we need to mention the *crystallographic restriction*. It provides the key to why there are only seventeen wallpaper groups. We therefore deal in depth with the crystallographic restriction as we proceed.

After observing the fact that the rotational symmetries of a crystal in chemistry and mineralogy are limited to 2-fold, 3-fold, 4-fold, and 6-fold rotations, mathematicians had idea that same restrictions might apply to symmetries in mathematics, i.e., if P is an n-center of a wallpaper group, then n should equal $2, 3, 4, 6$. This assumption is called the *crystallographic restriction.* In mathematics, a crystal is generated by independent finite translations, and is modeled as a discrete lattice. Because discreteness requires that the spacing between lattice points have a lower bound, the group of rotational symmetries of the lattice at any point must be a finite group. Here, we limit our attention to the plane and to the crystallographic restriction on plane patterns. The classification of wallpaper groups is achieved by imposing on these groups, first of all, the crystallographic restriction. The key concepts used in the classification proofs are therefore the n-centers in combination with the crystallographic restriction.

The proof involves two steps: First, we prove that all wallpaper groups satisfy the crystallographic restriction (the details can be found in Appendix B). We then use this fact to explore the consequence of this restriction using the idea of n-centers (the details can be found in Appendix C). This means that we identify all possible wallpaper groups. To facilitate our understanding what is involved in the is classification, we illustrate each wallpaper group with a figure.

The details are as follows:

2.1 The Crystallographic Restriction

The proof of the crystallographic restriction theorem is built around the ideas of a translation lattice and a unit cell.

Definition 1 *The **translation lattice** for ϖ determined by point A is the set of all images of A under the translations in ϖ.*

Definition 2 *A **unit cell** for ϖ with respect to point A and generating translations τ_1 and τ_2 is a quadrilateral region with vertices A_{ij}, $A_{i+1,j}$, $A_{i.j+1}$, and $A_{i+1,j+1}$.*

Based on these ideas, the crystallographic restriction is proved in the following steps:

Theorem 3 *If σ_l is in wallpaper group ϖ, then l is parallel to a diagonal of a rhombic unit cell for ϖ or else l is parallel to a side of a rectangular unit cell for ϖ.*

Theorem 4 *If wallpaper group ϖ contains a glide reflection, then ϖ have a translation lattice that is rhombic or rectangular.*

Theorem 5 *If a glide reflection in wallpaper group ϖ fixes a translation lattice for ϖ, then ϖ contains a reflection.*

Theorem 6 *For given n, if point P is an n-center for group G of isometries and G contains an isometry that takes P to Q, then Q is an n-center for G. If l is a line of symmetry for a figure and the symmetry group for the figure contains an isometry that takes l to m, then m is a line of symmetry for the figure.*

Lemma 7 (Proved by Xiu Wu Huang) *If T is the center of square $\square PQRS$, $\sigma_P, \sigma_Q, \sigma_R$, are in $< \sigma_{\overleftrightarrow{PQ}}, \sigma_S, \sigma_T >$.*

Lemma 8 (Proved by Xiu Wu Huang) $< \rho_{A,60}, \sigma_{\overleftrightarrow{CG}} >=< \sigma_{\overleftrightarrow{AG}}, \sigma_{\overleftrightarrow{CG}}, \sigma_{\overleftrightarrow{AB}} >$ *if G is the center of equilateral $\triangle ABC$.*

Theorem 9 *If $\rho_{A,360/n}$ and $\rho_{P,360/n}$ with $P \neq A$ and $n > 1$ are in wallpaper group ϖ, then $2AP$ is not less than the length of the shortest nonidentity translation in ϖ.*

Lemma 10 *We shall use Theorem 9 immediately to show that the possible values of n such that there is an n-center in a wallpaper group are rather restricted.*

This proves that wallpaper groups satisfy the crystallographic restriction:

Theorem 11 *If point P is an n-center for a wallpaper group, and then n is one of 2, 3, 4, or 6.*

The details of these proofs may be found in Appendix B.

2.2 Classification

We start with the following theorem:

Theorem 12 *If a wallpaper group contains a 4-center, then the group contains neither a 3-center nor a 6-center.*

We shall find all possible wallpaper groups ϖ, beginning with those groups that contain an n-center. By the crystallographic restriction it is sufficient to consider only the values 6, 3, 4, and 2 for n.

2.2.1 6-centers

We begin by stating the following theorem, which shows the abundance of symmetry required to support a 6-center.

Theorem 13 *Suppose A is a 6-center for wallpaper group ϖ. There is no 4-center for ϖ. Further, the center of symmetry nearest to A is a 2-center M, and A is the center of a regular hexagon whose vertices are 3-centers and whose sides are bisected by 2-ceners. All the centers of symmetry for ϖ are determined by A and M.*

After proving the theorem above, we explore all possible wallpaper groups with a 6-center and get the following result: A wallpaper pattern having a 6-center has a symmetry group ϖ_6 or ϖ_6^1.

1. *A wallpaper pattern with symmetry group ϖ_6 has a 6-center but no line of symmetry;*

2. A *wallpaper pattern with symmetry group ϖ_6^1 has a 6-center and a line of symmetry.*

2.2.2 3-centers, but no 6-center

We now turn our attention to wallpaper groups with a 3-center but no 6-center and state the following analogue to the previous theorem.

Theorem 14 *If A is a 3-center for wallpaper group ϖ and there are no 6-center for ϖ, then every center of symmetry for ϖ is a 3-center and A is the center of a regular hexagon whose vertices are 3-centers. All the centers of symmetry for ϖ are determined by A and a nearest 3-center.*

It follows from this theorem that wallpaper groups that contain only 3 centers have ϖ_3, ϖ_3^1, ϖ_3^2.

1. *A wallpaper group ϖ_3 has a 3-center, has no 6-center, and has no line of symmetry.*

2. *A wallpaper group ϖ_3^1 has a 3-center, has no 6-center, and every 3-center is on a line of symmetry.*

3. *A wallpaper group ϖ_3^2* has a 3-center off a line of symmetry but no 6-center.

2.2.3 4-centers

The following theorem regarding 4-centers in a wallpaper group is analogous to that concerning 6-centers. Here is the statement of the theorem.

Theorem 15 *Suppose A is a 4-center for wallpaper group ϖ. Then, there are no 3-centers for ϖ and there are no 6-centers for ϖ. Further, the center of symmetry nearest to A is a 2-center M, and A is the center of a square whose vertices are 4-centers and whose sides are bisected by 2-centers. All the centers of symmetry for ϖ are determined by A and M.*

By the theorem, we have the following result: Wallpaper groups that contain 4 centers have ϖ_4, ϖ_4^1, ϖ_4^2.

1. *A wallpaper pattern with symmetry group ϖ_4 has a 4-center and no line of symmetry.*

2. *A wallpaper pattern with symmetry group ϖ_4^1 has a line of symmetry on a 4-center.*

3. *A wallpaper pattern with symmetry group ϖ_4^2 has a 4-center and a line of symmetry off all 4-centers.*

2.2.4 2-centers

Now suppose a wallpaper group has a 2-center A and every center of symmetry is a 2-center. If this is the case, wallpaper groups have ϖ_2, ϖ_2^1, ϖ_2^2, ϖ_2^3, ϖ_2^4.

1. A wallpaper group ϖ_2 has a 2-center, every center of symmetry is a 2-center, and is not fixed by an odd isometry.

2. A wallpaper group ϖ_2^1 has a 2-center, every center of symmetry is a 2-center, and some but not all 2-centers are on a line of symmetry.

3. A wallpaper group ϖ_2^2 has a 2-center, every center of symmetry is a 2-center, and every 2-center is on a line of symmetry.

4. A wallpaper group ϖ_2^3 has a 2-center, every center of symmetry is a 2-center, has a line of symmetry, and all lines of symmetry are parallel.

5. A wallpaper group ϖ_2^4 has a 2-center, every center of symmetry is a 2-center, has no line of symmetry, but is fixed by a glide reflection.

2.2.5 No centers

Finally we come to wallpaper groups that have no center of symmetry. The possible wallpaper groups have ϖ_1, ϖ_1^1, ϖ_1^2, and ϖ_1^3.

1. A wallpaper group ϖ_1 has no center of symmetry and is not fixed by any odd isometry.

2. A wallpaper ϖ_1^1 has no center of symmetry, is fixed by both reflections and glide reflections, but some axes of the glide reflections are not lines of symmetry.

3. A wallpaper group ϖ_1^2 has no center of symmetry, is fixed by both reflections and glide reflections, and all axes of the glide reflections are lines of symmetry.

4. A wallpaper group ϖ_1^3 has no center of symmetry, has no line of symmetry, but is fixed by a glide reflection.

We can combine this analysis into the expected theorem:

Theorem 16 (Wallpaper Group Classification) *If G is a wallpaper group, then there are points and lines such that G is one of the seventeen groups defined above.*

The details of these proofs and their analysis may be found in Appendix C.

2.3 Illustration

The various planar patterns can be classified by the transformation groups that leave them invariant, their symmetry groups. A mathematical analysis of these groups shows that there are exactly 17 different plane symmetry groups. This means that there are 17 types of such groups, corresponding to 17 essentially distinct ways to tile the plane in a doubly-periodic pattern.

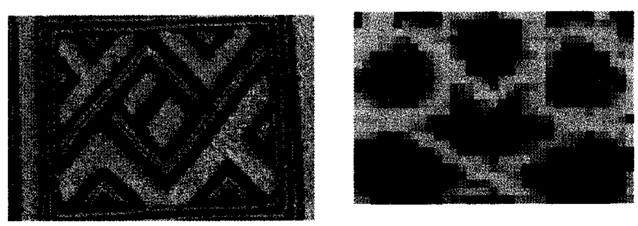
In this section we illustrate each of seventeen wallpaper groups based on our understanding of what is involved. The illustration combines ideas from Martin [24] M. McCallum [26], D.E. Joyce [21] and F. E. Szabo [34]. Moreover, we use some motifs with features of Bakuba and Zillij to implement some symmetry groups with Mathematica [36].

Table 2.1: Wallpaper Groups 1 to 17

Symmetry group	IUC notation	Martin notation	Definition
1	$p6$	ϖ_6	has rotations of order six (60°), rotations of order three, and rotations of order two, but no reflections
2	$p6m$	ϖ_6^1	has rotations of order six (60°), rotations of order three, rotations of order two, and reflections
3	$p3$	ϖ_3	has rotations of order three (120°) but no reflections or glide reflections
4	$p3m1$	ϖ_3^1	has rotations of order three (120°), and no reflections
5	$p31m$	ϖ_3^2	has rotations of order three (120°), and no reflections
6	$p4$	ϖ_4	has rotations of order four (90°) and no reflections or glide reflections
7	$p4m$	ϖ_4^1	has rotations of order four (90°), and reflections
8	$p4g$	ϖ_4^2	has rotations of order four (90°), and reflections
9	pgg	ϖ_2^4	contains rotations of order two (180°), glide reflections, and no reflections.
10	$p2$	ϖ_2	contains rotations of order two (180°) but no reflections or glide reflections
11	cmm	ϖ_2^1	has reflections in two perpendicular directions, and a rotation of order two (180°)
12	pmm	ϖ_2^2	has reflections, and no rotations or glide reflections
13	pmg	ϖ_2^3	has rotations of order two (180°), and reflections
14	$p1$	ϖ_1	contains only translations
15	cm	ϖ_1^1	contains reflections and glide reflections with parallel reflection axes, but no rotations.
16	pm	ϖ_1^2	contains reflections with parallel axes and no rotations
17	pg	ϖ_1^3	contains only glide reflections with parallel axes no rotations or reflections

Table 2.1 lists all seventeen groups. The IUC notation for these groups is the notation for symmetry groups adopted by the International Union of Crystallography in 1952. In our description, some ideas are adapted from Martin [24], Joyce [21], and an article in Wikipedia [37].

To show the relationships between wallpaper groups and Bakuba cloths as well as Zillij mosaics, we generate examples with wallpapers groups that are based on motifs extracted directly from Bakuba textiles and Zillij mosaics (shown in Figure 2.1, or on motifs derived from features of Bakuba cloths and Zillij mosaics.



Motif in a Bakuba Cloth Motif in a Zillij Mosaic

Figure 2.1: Two Sample Motifs

We start our illustration of each of the seventeen wallpaper groups in the order adopted in Table 2.1.

2.3.1 6-centers

The Group ϖ_6 ($p6$)

This group contains 60-° rotations, that is, rotations of order 6. It also contains rotations of orders 2 and 3, but no reflections.

Example 17 *To illustrate the principle of the group ϖ_6, we design an equilateral triangle containing some features of Bakuba cloths and Zillij mosaics. In this example, the fundamental region is the central pattern in Figure 2.2. The rotation center of the rotation of order 6 is at the lower vertex (designated in the figure by the symbol $\bigcirc$). The rotation centers of the rotations of order 3 are at the other two vertices (designated in the figure by the symbol $\bigtriangledown$). The rotation center of the rotation of order 2 is at the midpoint of the side connecting the rotations of order 3 (designated in the figure by the symbol $\times$).*

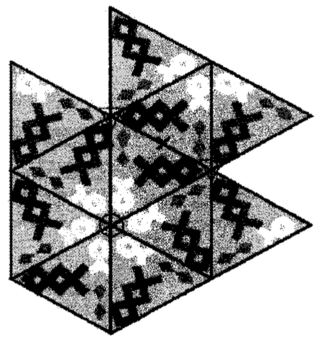
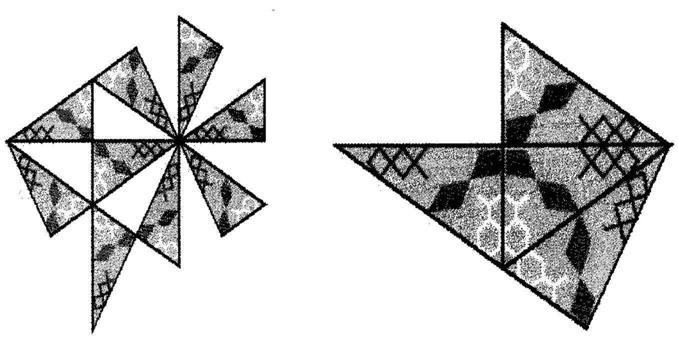


Figure 2.2: A Pattern Generated by the Group ϖ_6 $(p6)$

The group ϖ_6^1 $(p6m)$

This most complicated group $(p6m)$ has rotations of order 2, 3, and 6, as well as reflections. The axes of reflection meet at all the centers of rotation.

Example 18 *To illustrate the principle of the group* (ϖ_6^1)*, we design a* 30^0*-*60^0*-*90^0 *triangle containing some of the features of Bakuba cloths and Zillij mosaics. In this example, we generate two patterns. In the left pattern, the rotation of order 6 is at the 30-degree vertex, a rotation of order 3 is at the 60-degree vertex, and the rotation of order 2 is at the 90-degree vertex. In the right pattern, we show three reflections that use the three sides of the triangle as reflection mirrors.*



Triangle with Three Rotations Three Reflections

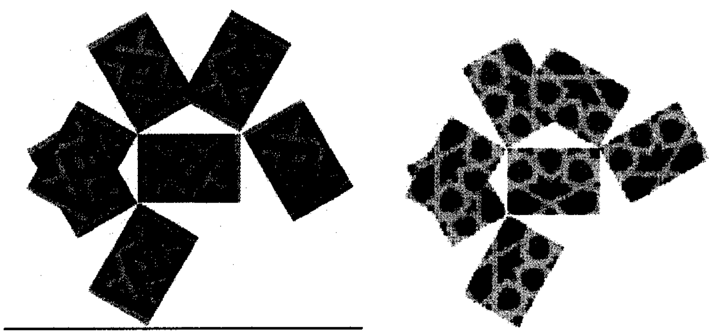
Figure 2.3: A Pattern Generated by the Group $\varpi_6^1(p6m)$

2.3.2 3-centers

The group ϖ_3 $(p3)$

This is the simplest group containing a 120° rotation, that is, a rotation of order 3.

Example 19 *Here are two examples to illustrate the principle of the symmetry group ϖ_3 $(p3)$. In the examples, the fundamental region is a rectangle and the three vertices of the rectangles are rotation centers of 120 degrees.*



Tiling of a Bakuba Cloth Motif   Tiling of Zillij Mosaic Motif

Figure 2.4: Two Patterns Generated by of the Group $\varpi_3(p3)$

The group ϖ_3^1 $(p3m1)$

This group is similar to the last one in that it contains reflections and order 3 rotations. The axes of the reflections are inclined at 60° to one another, but for this group, all of the centers of rotation lie on the reflection axes.

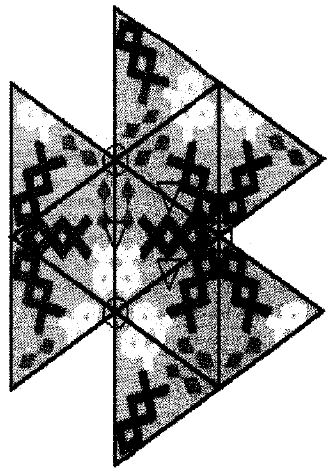


Figure 2.5: A Pattern Generated by the Group $\varpi_3^1(p3m1)$

Example 20 *We illustrate the principle of the symmetry group* ϖ_3^1 $(p3m1)$. *In the examples, the fundamental region is an equilateral triangle with each side a reflection (designated by the symbol* $\bigtriangledown$*) and each vertex a rotation of order 3 (designated by the symbol* $\bigcirc$*).*

The group ϖ_3^2 $(p31m)$

This group contains reflections (whose axes are inclined at 60° to one another) and rotations of order 3. Some of the centers of rotation lie on the reflection axes, and some do not.

Example 21 *To illustrate the principle of the symmetry group* ϖ_3^2 $(p31m)$, *we use a picture from [25]. In the examples, the fundamental region is an isosceles triangle with a vertex of 120 degrees. There are three distinct rotations,* ρ_1 *is at the 120-degree vertex,* ρ_2 *is at the 30-degree vertices and* ρ_3 *is a reflection of* ρ_1 *about* σ_{l_1}, *the horizontal mirror. The other two mirrors are rotations of* σ_{l_1} *by 120 and 240 degrees.*

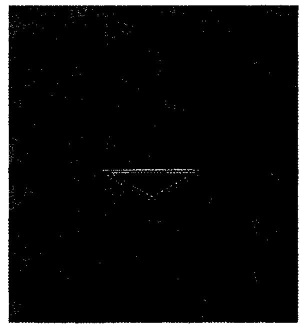


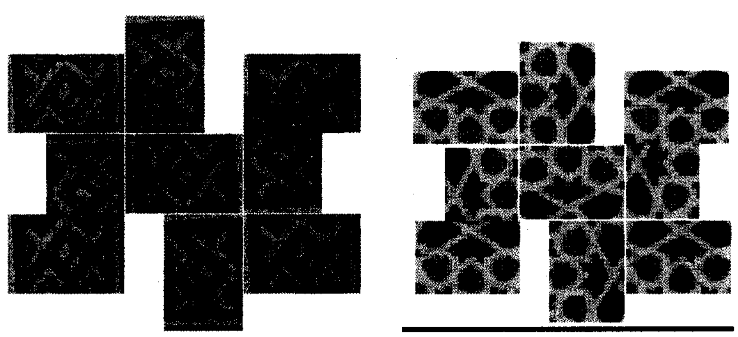
Figure 2.6: A Pattern Generated by the Group $\varpi_3^2(p31m)$

2.3.3 4-centers

The group ϖ_4 $(p4)$

This is the first group with a 90° rotation, that is, a rotation of order 4. It also has rotations of order 2. The centers of the order-2 rotations are midway between the centers of the order-4 rotations. There are no reflections.

Example 22 *Figure 2.7 contains two examples illustrating the principle of the symmetry group* ϖ_4 $(p4)$. *In the examples, the rectangle in the middle is the fundamental region; the upper left vertex and lower right vertex of the fundamental region are rotation centers of 90 degrees; The 180 degree centers of rotation are at the other two vertices of the fundamental region.*



Motif in a Bakuba Cloth — Motif in a Zillij Mosaic

Figure 2.7: Two Patterns Generated by the Group $\varpi_4(p4)$

The group ϖ_4^1 $(p4m)$

Example 23 *Here is an example for the group ϖ_4^1 $(p4m)$ in Figure 2.8 that has a triangular fundamental region with all edges being reflection mirrors [25]. The fundamental region is an isosceles right triangle with a 180-degree rotation center at the 90-degree vertex and 90-degree rotation centers at the other two vertices.*

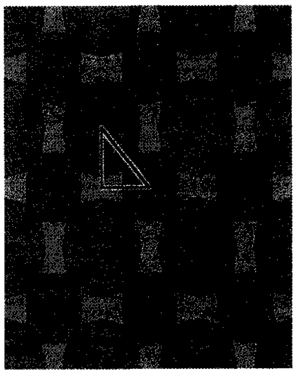


Figure 2.8: A Pattern Generated by the Group ϖ_4^1 $(p4m)$

The group ϖ_4^2 $(p4g)$

This group ϖ_4^2 $(p4g)$ also contains reflections and rotations of orders 2 and 4. But the axes of reflection are perpendicular, and none of the rotation centers lie on the reflection axes. Again, the lattice is square, and an eighth of a square fundamental region of the translation group is a fundamental region for the symmetry group.

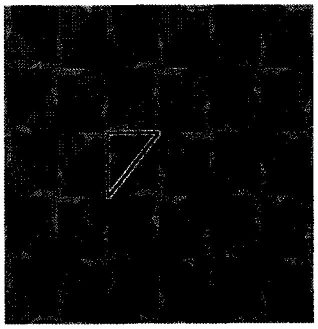


Figure 2.9: A Pattern Generated by the Group ϖ_4^2 $(p4g)$

Example 24 *This pattern has an isosceles right triangle for a fundamental region [25]. Both the horizontal and vertical glide reflections are readily apparent here. The second rotation of order 4 can be seen as well (shown in Figure 2.9).*

2.3.4 2-centers

The group ϖ_2 $(p2)$

This group contains 180° rotations, that is, rotations of order 2.


3 × 3 Tiling of a Bakuba Cloth Motif

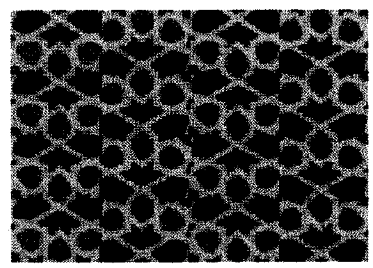

4 × 4 Tiling of a Zillij Mosaic Motif

Figure 2.10: Two Patterns Determined the Group $\varpi_2(p2)$

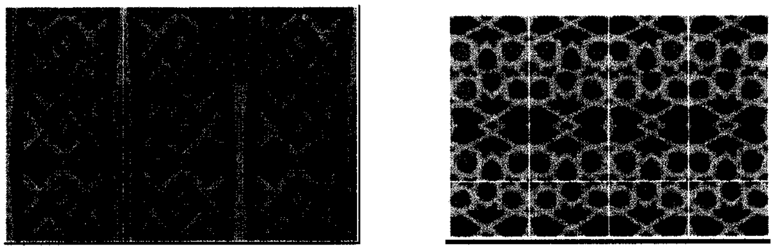
As in all symmetry groups, there are translations, but there are neither reflections nor glide reflections. The two translations axes may be inclined at any angle to each other and the lattice is a parallelogrammatic.

Example 25 *Here are two examples in Figure 2.10 that illustrate the principle of the symmetry group* ϖ_2 $(p2)$. *In the examples, two translation axes are perpendicular to each other and the fundamental region is a rectangle. The centers of rotation are at the four vertices and the midpoints of the vertical sides.*

The group ϖ_2^1 (cmm)

This group has perpendicular reflection axes, but it also has rotations of order 2. The centers of the rotations do not lie on the reflection axes.

Example 26 *Here are two examples in Figure 2.11 that illustrate the principle of the symmetry group* ϖ_2^1 (cmm). *In the examples, the fundamental region is a rectangle. Starting from the patterns in the middle, it is easy to see the three reflections that two are vertical and one horizontal, and the three rotations of* 180^0 *degrees. If we take a closer look, we can also see glide reflections.*



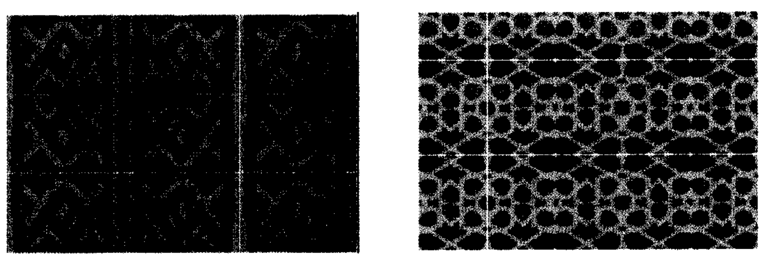
3 × 3 Tiling of a Bakuba Cloth Motif   4 × 4 Tiling of a Zillij Mosaic Motif

Figure 2.11: Two Patterns Generated by the Group ϖ_2^1 (cmm)

The group ϖ_2^2 (pmm)

There are no glide reflections or rotations, the lattice is rectangular, and a rectangle can be chosen for the fundamental region of the translation group.

Example 27 *Here are two examples in Figure 2.12 that illustrate the principle of the symmetry group* ϖ_2^2 (pmm)*. In the examples, the fundamental region is a rectangle; all of the edges of the rectangle are mirrors; each vertex of the rectangle is the center of a rotation of 180 degrees.*



3 × 3 Tiling of Bakuba Cloth Motif   5 × 5 Tiling of Zillij Mosaic Motif

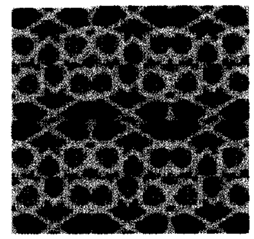
Figure 2.12: Two Patterns Generated by the Group ϖ_2^2 (pmm)

The group ϖ_2^3 (pmg)

This group ϖ_2^3 (pmg) contains both a reflection and a rotation of order 2. The centers of rotations do not lie on the axes of reflection. The lattice is rectangular.

Example 28 *Here are two examples in Figure 2.13 that illustrate the principle of the symmetry group* ϖ_2^3 (pmg)*. In the examples, the fundamental region is a rectangle. Starting from the patterns in the middle, it is easy to see the reflections, glide reflections, and the rotations.*

3 × 3 Tiling of a Bakuba Cloth Motif

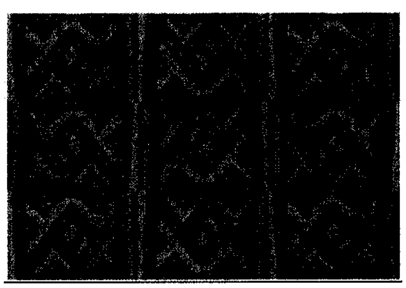


4 × 3 Tiling of a Zillij Mosaic Motif

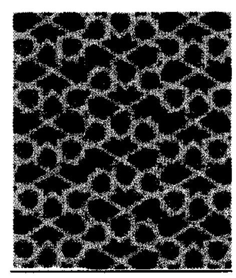
Figure 2.13: Two Patterns Generated by the Group $\varpi_2^3(pmg)$

The group ϖ_2^4 (pgg)

This group ϖ_2^4 (pgg) contains no reflections, but it has glide-reflections and half-turns. There are perpendicular axes for the glide reflections, and the centers of the rotations do not lie on these axes.



3 × 3 Tiling of a Bakuba Cloth Motif



5 × 3 Tiling of a Zillij Mosaic Motif

Figure 2.14: Two Patterns Generated by the Group $\varpi_2^4(pgg)$

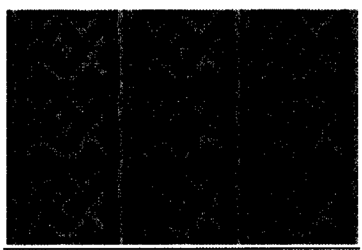
Example 29 *Here are two examples in Figure 2.14 that illustrate the principle of the symmetry group* ϖ_2^4 (pgg). *In the examples, the two translation axes are perpendicular to each other, and each has length twice the length of their corresponding parallel edges; the fundamental region is a rectangle. Starting from the central patterns, it is easy to see the glide reflections and the rotations.*

2.3.5 No centers

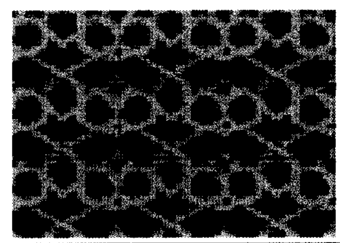
The group ϖ_1 $(p1)$

This is the simplest symmetry group. It consists only of translations. There are neither reflections, nor glide-reflections, and no rotations. The two translation axes may be inclined at any angle to each other. Its lattice is parallelogrammatic, so a fundamental region for the symmetry group is the same as that for the translation group, namely, a parallelogram.

Example 30 *Here are two examples in Figure 2.15 that illustrate the principle of symmetry group* ϖ_1 $(p1)$. *In Figure, two translation axes are perpendicular to each other, and motifs are only translated. Therefore, the motifs in the tiling patterns are parallel to each other.*



3×3 Tiling of a Bakuba Cloth Motif



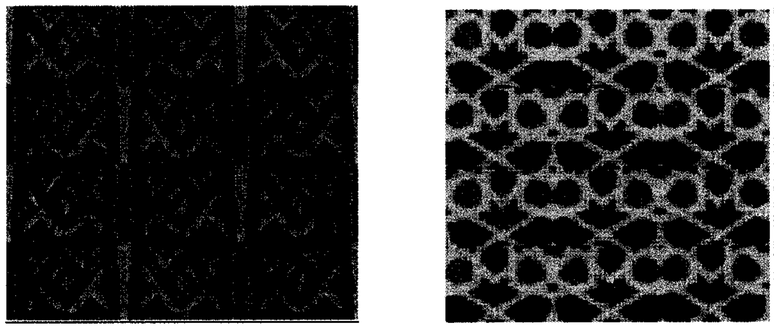
3×3 Tiling of a Zillij Mosaic Motif

Figure 2.15: Two Patterns Generated by the Group ϖ_1 $(p1)$

The group ϖ_1^1 (cm)

This group ϖ_1^1 (cm) contains reflections and glide reflections with parallel axes. There are no rotations in this group. The translations may be inclined at any angle to each other, but the axes of the reflections bisect the angle formed by the translations, so the fundamental region for the translation group is a rhombus. A fundamental region for the symmetry group is half a rhombus.

Example 31 *Here are two examples in Figure 2.16 that illustrate the principle of the symmetry group* ϖ_1^1 (cm). *In the examples, two translation axes are perpendicular to each other, and the reflection has two vertical mirrors with two vertical edges of the fundamental region as axes. The glide reflection has a vertical mirror through the midpoints of the horizontal edges.*

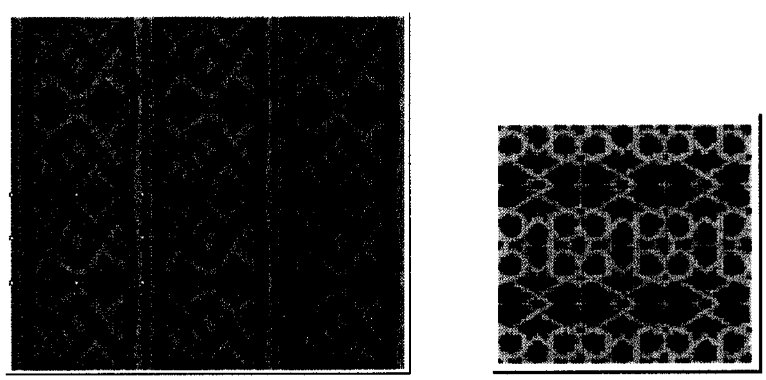


4 × 3 Tiling of a Bakuba Cloth Motif   4 × 3 Tiling of a Zillij Mosaic Motif

Figure 2.16: Two Patterns Generated by the Group $\varpi_1^1(cm)$

The group ϖ_1^2 (pm)

This group ϖ_1^2 (pm) contains reflections. The axes of reflection are parallel to one axis of translation and perpendicular to the other axis of translation. A fundamental region for the translation group is a rectangle, and one can be chosen that is split by an axis of reflection so that one of the half rectangles forms a fundamental region for the symmetry group.



4 × 3 Tiling of Bakuba Cloth Motif   4 × 3 Tiling of Zillij Mosaic Motif

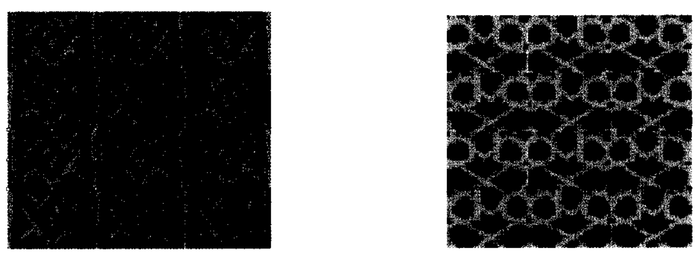
Figure 2.17: Two Patterns Generated by the Group $\varpi_1^2(pm)$

Example 32 *There are two examples in Figure 2.17 that illustrate the principle of the symmetry group ϖ_1^2 (pm). In the examples, two translation axes are perpendicular to each other, and the reflection has two horizontal mirrors with two horizontal edges of the fundamental region as axes.*

The group ϖ_1^3 (pg)

The direction of the glide reflection is parallel to one axis of translation and perpendicular to the other axis of translation. The lattice is rectangular, and a rectangular fundamental region for the translation group can be chosen that is split by an axis of a glide reflection so that one of the half rectangles forms a fundamental region for the symmetry group.

Example 33 *Here are two examples in Figure 2.18 that illustrate the principle of the symmetry group ϖ_1^3 (pg). In the examples, two translation axes are perpendicular to each other, and the glide reflection has a vertical mirror through the midpoints of the horizontal sides.*



4×3 Tiling of a Bakuba Cloth Motif 4×3 Tiling of a Zillij Mosaic Motif

Figure 2.18: Two Patterns Generated by the Group $\varpi_1^3(pg)$

2.3.6 Conclusion

After illustrating the seventeen wallpaper groups with motifs most of which have the features of Bakuba textiles and Zillij mosaics, we can find some clues that link the wallpaper groups to the objects studied in the *Generative Design* project. To describe these clues clearly, we analyze some examples and provide our ideas in the following sections.

Chapter 3

Groups and Bakuba Textile Patterns

After introducing the required theorems about wallpaper groups and illustrating the seventeen wallpaper groups, we now establish a relationship between these groups and Bakuba textiles. To this end, we study two different Bakuba textile patterns.

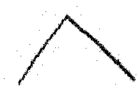
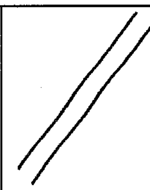
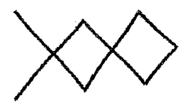
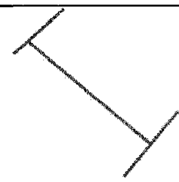
3.1 Analysis of a Bakuba Cloth

First we consider a pattern as shown in Figure 3.1:

Figure 3.1: A Bakuba Cloth

We can extract some patches from Figure 3.1 as shown in Table 3.1:

Table 3.1: A List of Patches in Figure 3.1

					
Patch 1	Patch 2	Patch 3	Patch 4	Patch 5	Patch 6
					
Patch 7	Patch 8	Patch 9	Patch 10	Patch 11	Patch 12

Next we use symmetry operations on the patches in Table 3.1 and get Figure 3.2:

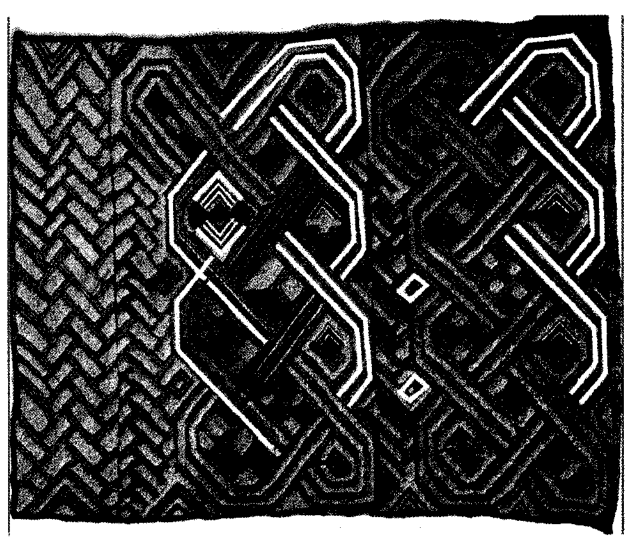


Figure 3.2: A Pattern in a Bakuba Cloth Generated by a Symmetry Group

From our analysis of Figure 3.2 we get the information listed in Table 3.2.

Table 3.2: Symmetry Operations on the Patches in Table 3.1

Patch name	Operation on patch		Patch name	Operation on patch
Patch 1	translation τ_1		Patch 2	translation τ_2
Patch 3	translation τ_3		Patch 4	translation τ_4
Patch 5	no operation		Patch 6	translation τ_6
Patch 7	translation τ_7		Patch 8	translation τ_8
Patch 9	reflection σ_m with horizontal axes m		Patch 10	translation τ_{10}
Patch 11	translation τ_{11}		Patch 12	translation τ_{12}

Here, τ_1, τ_2, τ_3, τ_4, τ_6, τ_7, τ_8, τ_{10}, τ_{11}, τ_{12} are different translations generating the patterns shown in Figure 3.2 by operating on patches. Since τ_1 is a vertical translation and τ_{11} is a horizontal one, τ_1 and τ_{11} are not parallel. Therefore these translations can be expressed by the combination of τ_1 and τ_{11}. In other words, they are elements of the groups $< \tau_1, \tau_{11} >$. Although these translations cannot be a group, since this pattern is only a piece of Bakuba textiles,we can see from the above example that group theory is useful for the analysis of Bakuba textiles.

Certainly these translations act on different patches and generate patterns that are part of Figure 3.2. Moreover, if we combine the reflections and translations in the wallpaper group ϖ_1^2 (pm) with these patches, more complicated patterns are generated that are not a part of Figure 3.2.

3.2 A Typical Pattern in Figure 1.1

Let us now examine a specific component of Figure 1.1.

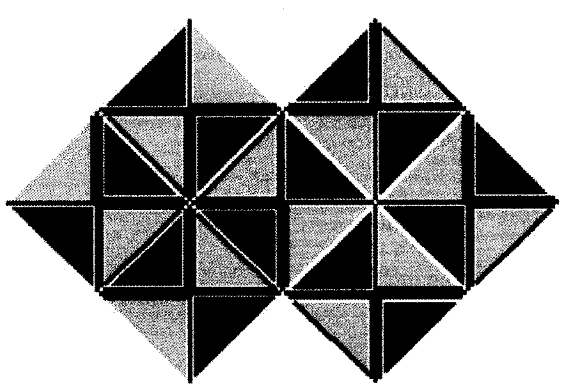
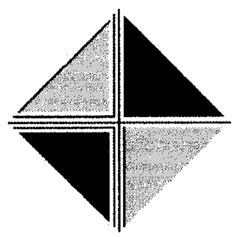
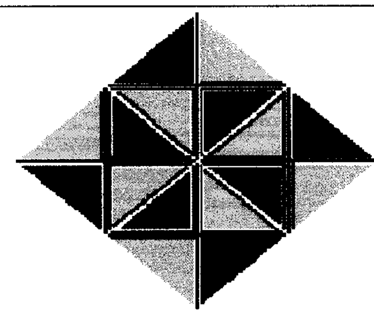
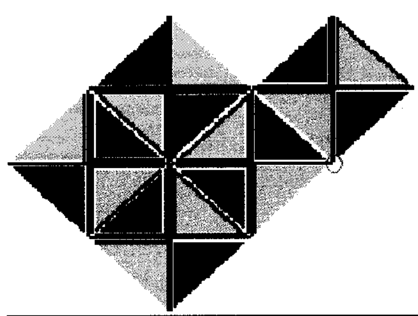
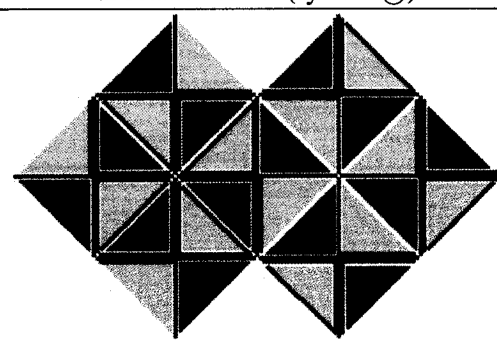


Figure 3.3: Pattern in Figure 1.1

After analyzing the structure of this component of Figure 3.3, we find that we can construct these components by group $G_1 =< \rho_{C,360/4}, \rho^4_{C,360/4} = E >$ operating on the patch shown in Table 3.3, where $\rho_{C,360/4}$ is a rotation of order 4, and C is a rotation center. It should be noted that in the process of generating these components, we must use different rotation centers.

Table 3.3: Generating Figure 3.3 on a given Patch

	
Patch	The group G_1 acts on the patch with right vertex of the patch as rotation center (symbol ◯)
	
The group G_1 acting on the right-most patch of the pattern generated by last step with right vertex as rotation center (symbol ◯)	The pattern is generated by the group G_1 with two different rotation centers

3.3 Conclusion

To generate the mentioned patterns in the two examples above, we use the group of rotations of order 4, translation, reflection on different patches with various parameters like rotation centers. If we choose the patches appropriately, some of the patterns in the given figure can be generated by symmetry operations on the patches. Therefore, symmetry groups are relevant to the study of Bakuba textiles.

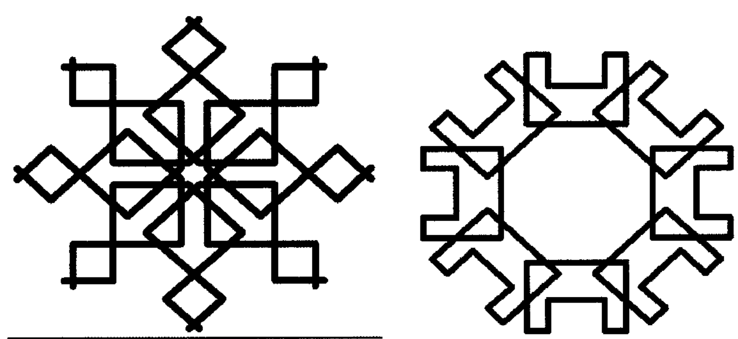
Chapter 4

Groups and Zillij Mosaic Patterns

We want to discuss the group structure of Zillij mosaics using the following examples.

4.1 A Group Structure in Weaving

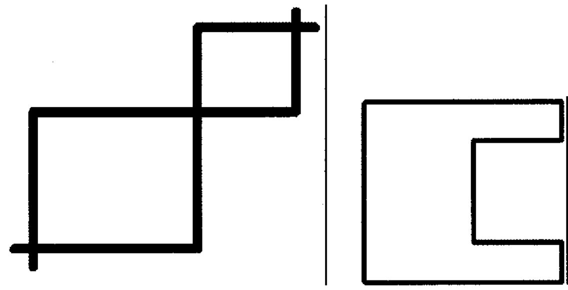
Think about the sample mentioned in Figure 1.2 and decompose it into the following parts:



Pattern 1 of Figure 1.2 Pattern 2 of Figure 1.2

Figure 4.1: Patterns in Figure 1.2

Then consider the following two patches in (Figure 4.2):



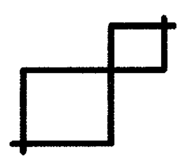
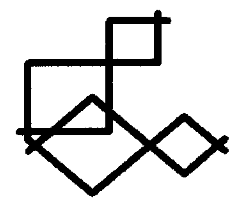
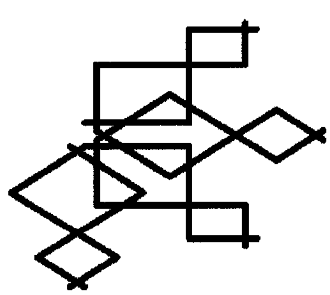
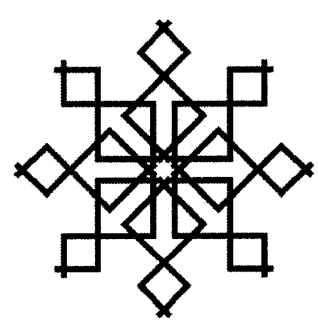
Patch 1 Patch 2

Figure 4.2: Patches for Pattern 1 and Pattern 2

Now we think about the group $G_2 =< \rho_{C,360/8}, \rho^8_{C,360/8} = E >$, where $\rho_{C,360/8}$ is a rotation of order 8, with C being a rotation center.

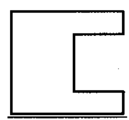
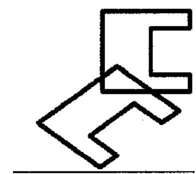
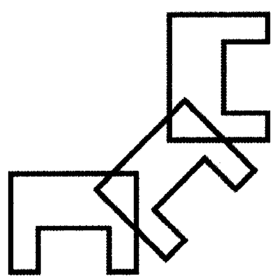
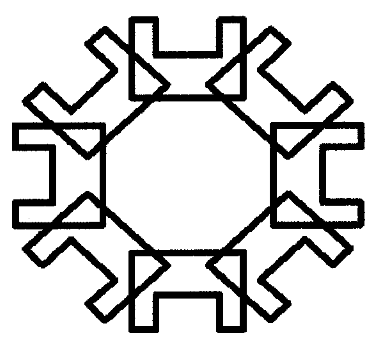
We can construct the patterns in Figure 4.1 with the group G_2 acting on Patches 1 and 2, but the rotation centers are different. For Patch 1, the center of the rotation is on the left bottom vertex of a fundamental region. The details are described in Table 4.1.

Table 4.1: Generating Process for Pattern 1

	
E acting on Patch 1	$\rho_{C,360/8}$ acting on Patch 1
	
$\rho^2_{C,360/8}$ acting on Patch 1	The pattern generated by G_2 acting on Patch 1

Now let G_2 act on Patch 2 as shown in Table 4.2. The center of the rotation is more complicated since it lies on the axis that is perpendicular to the left edge of Patch 2, goes through its midpoint, and is away from it by "the length of the left edge".

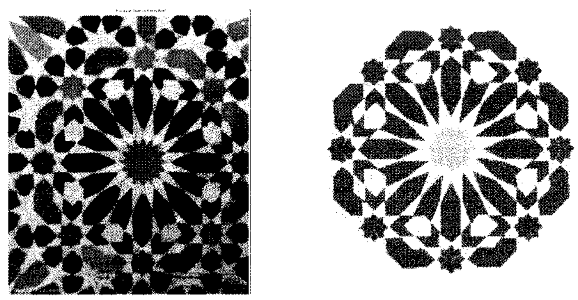
Table 4.2: Generating Process for Pattern 2

	
E acting on Patch 2	$\rho_{C,360/8}$ acting on Patch 2
	
$\rho^2_{C,360/8}$ acting on Patch 2	The pattern generated by G_2 acting on Patch 2

From Tables 4.1 and 4.2 we see that the subgroup of rotations of order 8 may generate different patterns by acting on a different patch with an appropriate rotation center.

4.2 The Group Structure of Zillij Patterns

Let us consider a pattern like the one shown in Figure 4.3 (a).

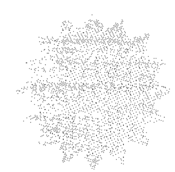
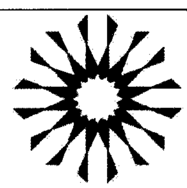
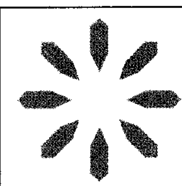
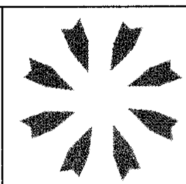
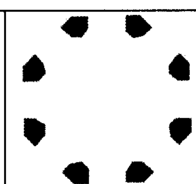
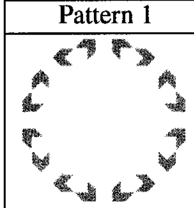
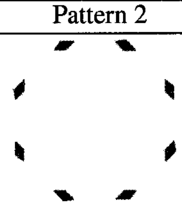
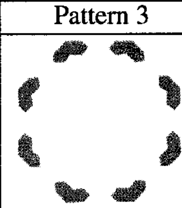
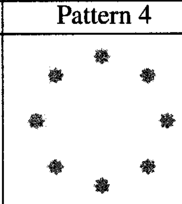
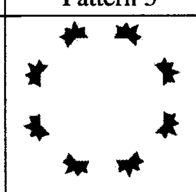


(a) An Original Image b) A Similar Image

Figure 4.3: Another Sample of Zillij Mosaics

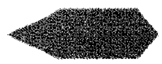
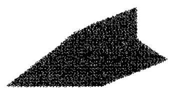
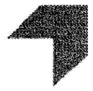
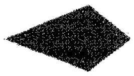
To simplify our study, we omit some of the background from Figure 4.3 (a) and get a figure similar to Figure 4.3 (b). From Figure 4.3 (b) it is easy to see that this pattern can be decomposed into the patterns shown in Table 4.3.

Table 4.3: Patterns in Figure 4.3 (b)

				
Pattern 1	Pattern 2	Pattern 3	Pattern 4	Pattern 5
				
Pattern 6	Pattern 7	Pattern 8	Pattern 9	Pattern 10

Next we focus our attention on the patterns in Table 4.3, analyze them and obtain the patches shown in Table 4.4.

Table 4.4: Patches for Table 4.3

				
Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
				
Patch 6	Patch 7	Patch 8	Patch 9	Patch 10
				
Patch 11	Patch 12	Patch 13	Patch 14	

We can also use the rotation group of order 8 mentioned above to generate Patterns 1 to 10, acting on different patches with different rotation centers. We list the relationship between patterns and patches in in Table 4.3 and the Patches in Table 4.4.

Table 4.5: Relationship between Patterns and Patches

Pattern name	Patch name	Group act on Patches
Pattern 1	Patch 1	$< \rho_{C,360/16}, \rho^{16}_{C,360/16} = E >$
Pattern 2	Patch 13	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 3	Patch 3	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 4	Patch 4	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 5	Patch 5	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 6	Patch 14	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 7	Patch 7	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 8	Patch 8	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 9	Patch 12	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$
Pattern 10	Patch 11	$< \rho_{C,360/8}, \rho^{8}_{C,360/8} = E >$

Whereas Patch 14 is created by a horizontal reflection on Patch 6, Patch 13 is generated by a horizontal reflection, and Patch 12 is generated by the above group G_2 acting on Patch 9 and Patch 10 with appropriate rotation centers.

4.3 The General Group Structure of Zillij Mosaics

After analyzing the above two samples of Zillij mosaic, we find that we can use symmetry groups to generate a Zillij mosaic pattern. The group may consist of rotation, reflection, translation, and identity, with the order of rotations not limited to 2,3,4,6. Sometimes we need combine these operations with a scaling that expands or contracts an object, as in Patch 12, which is generated by the group G_2 acting on Patch 9 and Patch 10, and is contracted to the appropriate size in the pattern.

4.4 Conclusion

Although the order of rotation in the groups mentioned in Table 4.5 is not 2,3,4, or 6 in the two examples considered, we can nevertheless conclude that if we choose our motifs appropriately and construct the groups carefully, groups can be used to analyze the structure of Zillij mosaic.

Chapter 5

Grammars for Artifactual Patterns

In the 1950s, a combination of logic, mathematics, and linguistics was being developed to describe a family of artifacts and architectural structures, and the concept of a shape grammar was defined. "Shape Grammars, though, were one of the earliest, most complete, and most successful generalizations of the formal grammars for one-dimensional logical or natural languages, to formal grammars for two- or three-dimensional spatial languages" [22]. Shape grammars manipulate shapes, while grammars in logic and related fields manipulate symbols. The processes of defining a shape grammar to describe a known style contain four steps. First, assign a vocabulary of shapes and a set of spatial relations; second, design a set of shape rules that define the occurrences of spatial relations; then, assign a set of initial shapes that begin the generations of designs; last, specify a set of shape grammars in term of shape rules and initial shapes.

5.1 Shape Grammars

In [33], Stiny defines a shape grammar as a quadruple: $SG =< V_T, V_M, R, I >$, where

1. V_T is a finite set of shapes. A finite arrangement of an element or elements of V_T in which any element of V_T can be used multiple times with the desired Euclidean transformations (translation, rotations, scale and reflection), gives the set V_T^*. Also we define $V_T^+ = V_T^* - \phi$.

2. V_M is a finite set of shapes such that shapes in the nonempty sets V_T^+ and V_M^+ are distinguishable.

3. R is a finite set of shape rules of the form $u \rightarrow v$, where u and v are shapes formed by the shape union

of shapes in V_T^* and V_M^*. The shape u must have at least one sub-shape that is a shape in V_M^+. The shape v may be the empty shape.

4. I is the shape formed by the shape union of shapes in the power sets V_T^* and V_M^*. I must have at least one sub-shape that is a shape in V_M^+.

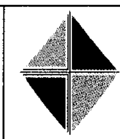
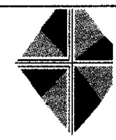
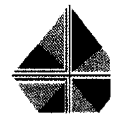
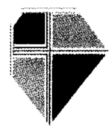
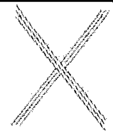
Shapes in the sets V_T^* or V_T^+ are called terminal shapes (or terminals). Shapes in the sets V_M^+ or V_M^* are called non-terminals (or markers). Terminals and markers are distinguishable, i.e., given a shape formed by the shape union of terminals and markers, the terminals occurring in the shape can be uniquely separated from the markers occurring in the shape. No shape in V_T^+ is a sub-shape of any shape in V_M^+ and vice versa. I is called the initial shape. The shape consisting of all the markers in I has at least one sub-shape that is a marker. Euclidean transformations could be applied to any of the above shapes to facilitate creation and orientation. But only those that are applied to markers in a rule are applied to the subsequent generations employing that marker.

A shape is generated from a shape grammar by beginning with the initial shape and recursively applying the shape rules. The shape generation process is terminated when no shape rule in the shape grammar can be applied. The sentential set of a shape grammar, $SS(SG)$, is the set of shapes (sentential shapes) which contains the initial shape and all shapes which can be generated from the initial shape using the shape rules. The language of a shape grammar $L(SG)$, is the set of sentential shapes containing only terminals. In other words, the language defined by a shape grammar $L(SG)$ is the set of shapes generated by the grammar that do not have any sub-shapes which are markers. The language of a shape grammar may be a finite or an infinite set of shapes. The next section demonstrates examples.

5.2 A Shape Grammar for Bakuba Textiles

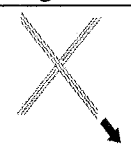
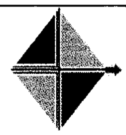
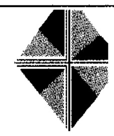
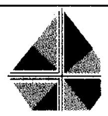
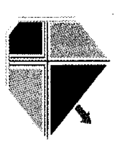
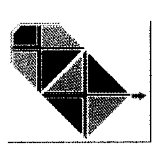
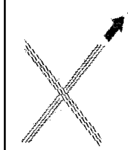
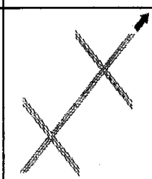
Let us return to the Bakuba textile in Figure 1.1 and think about a shape grammar for this design. We begin by listing the required terminals and markers in Table 5.1.

Table 5.1: Terminals and Marker (V_T and V_M) for Figure 1.1

Terminal 1		Terminal 2		Terminal 3	
Terminal 4		Terminal 5		Marker	

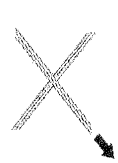
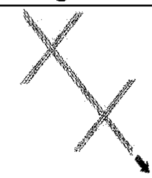
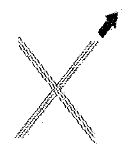
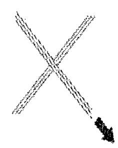
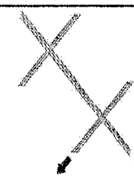
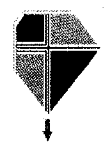
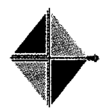
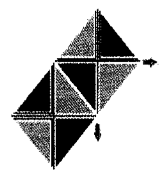
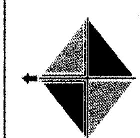
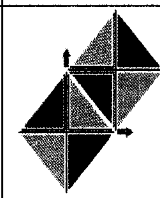
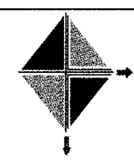
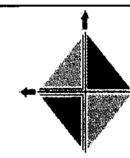
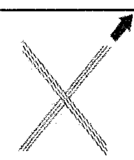
Next we create a set of shape grammar rules for Figure 1.1 (1), separated into Tables 5.2 and 5.3.

Table 5.2: Shape Grammar Rules for Figure 1.1 (1)

	Left Side		Right Side		Left Side		Right Side
1		→|		2		→|	
3		→|		4		→|	
5		→|		6		→|	
7		→|		8		→|	

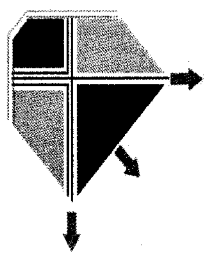
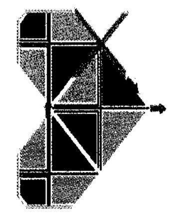
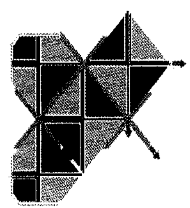
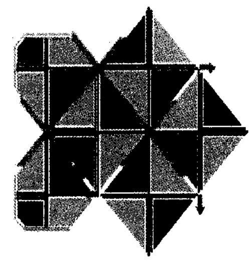
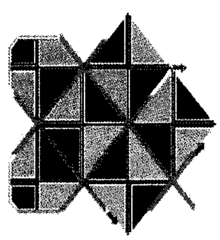
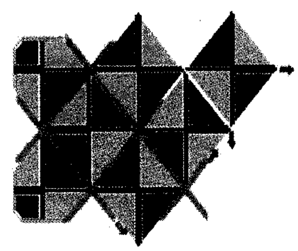
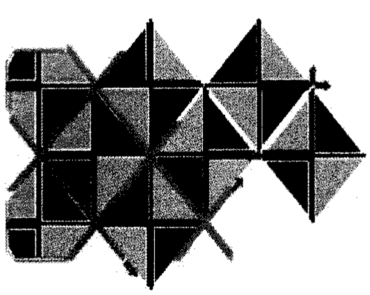
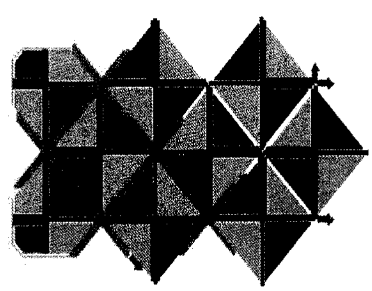
Here is the second table.

Table 5.3: Shape Grammar Rules for Figure 1.1 (2)

	Left Side		Right Side		Left Side		Right Side
9		→		10		→	
11		→		12		→	
13		→		14		→	
15		→		16		→	
17		→					

Then we assign an initial shape I as shown in the figure in the Table 5.4.

Table 5.4: Application of a Shape Grammar (Figure 1.1) (1)

	
Step 1: Initial Shape	Step 2: Using Rule 2, Rule 7, and Rule 12
	
Step 3: Rule 1,Rule 13, and Rule 9	Step 4: Twice Rule 13 and Once Rule 6
	
Step 5: Two times of using Rule 6 using Rule 9 and Rule 10	Step 6: Using Rule 15
	
Step 7: Using Rule 16	Step 8: Using Rule 14

We summarize this analysis in our definition of a shape grammar for Figure 1.1:

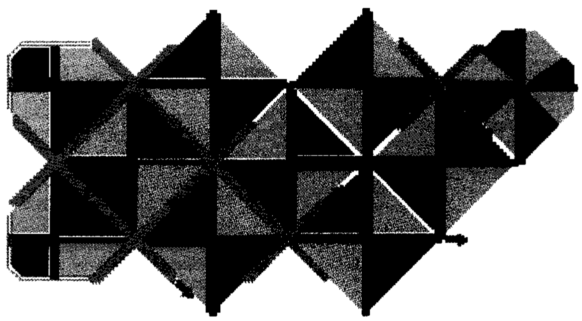
$$SG = < V_T, V_M, R, I >$$

where V_T, V_M, R, I are defined as above.

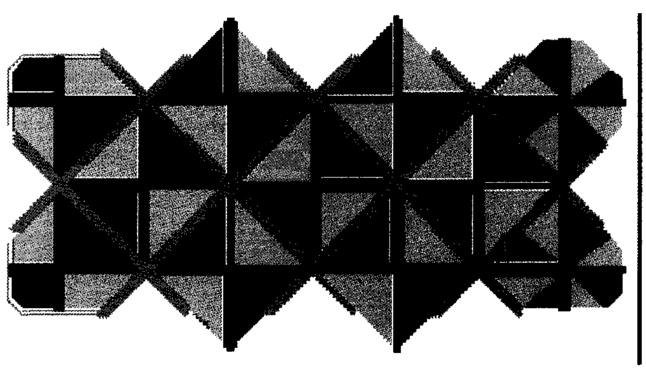
Now we apply the rules in Table 5.2 and Table 5.3 to generate a pattern similar to that at the top of the Figure 1.1.

The generating processes involved are summarized in Table 5.4 and Table 5.5.

Table 5.5: Application of a Shape Grammar (Figure 1.1) (2)



Step 9: Rule 1, Rule 4, and Rule 6

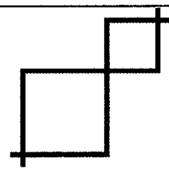
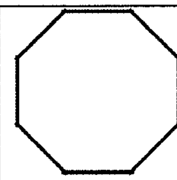
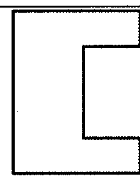
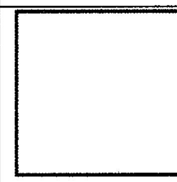


Step 10: Rule 5, Rule 10, Rule 8, Rule 11, and Rule 8
Then using Rule 6 to erase the arrow

5.3 A Shape Grammar for Zillij Mosaics

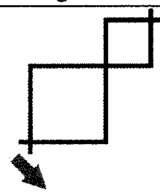
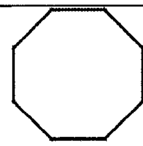
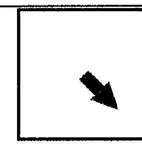
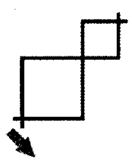
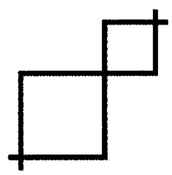
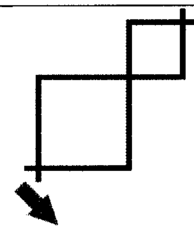
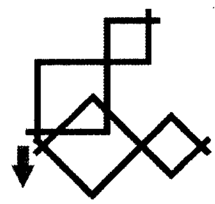
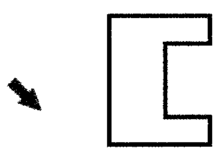
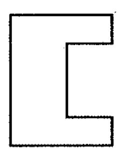
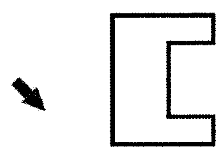
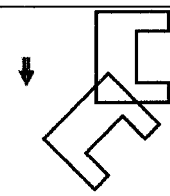
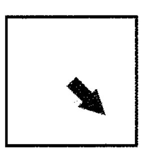
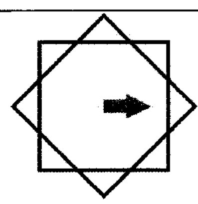
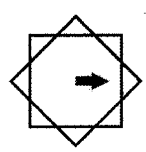
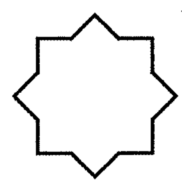
We now define a shape grammar for Zillij mosaics. We illustrate the construction of the grammar using the patterns in Figure 1.2. First we list terminals and markers as in Table 5.6.

Table 5.6: Terminals and Marker for Figure 1.2

Terminal 1		Terminal 2			
Terminal 3		Terminal 4		Marker	

Next we form the set of shape grammar rules as shown in Table 5.7. In Tables 5.8, 5.9 we show the results of applying these rules.

Table 5.7: Shape Grammar Rules for Figure 1.2

	Left Side		Right Side		Left Side		Right Side
1		→|		2		→|	
3		→|		4		→|	
5		→|		6		→|	
7		→|		8		→|	
9		→|		10		→|	

In addition, we assign an initial shape I as shown in Figure 5.1.

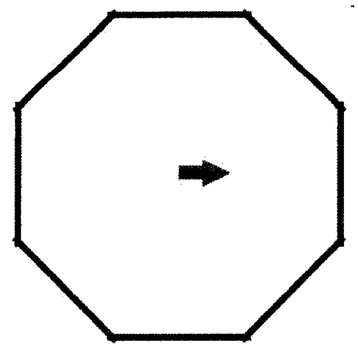


Figure 5.1: An Initial Shape I for Figure 1.2

We now have all required components for a shape grammar and define the grammar for Figure1.2 as

$$SG =< V_T, V_M, R, I >$$

where V_T, V_M, R, I are the components as defined.

If we apply the rules in Table 5.7, we can see that they generate a pattern similar to that in the chosen component of Figure 1.2.

The generating processes are summarized in Table 5.8 and Table 5.9. From these tables we can see that by using the defined grammar, we can reproduce either the entire figure or, at worst, a significant part of it.

Table 5.8: Application of a Shape Grammar (Figure 1.2) (1)

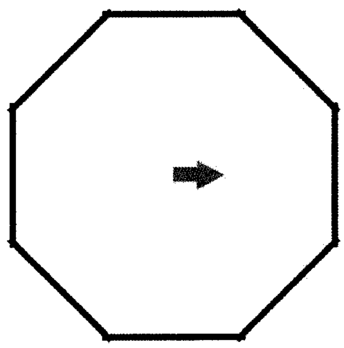
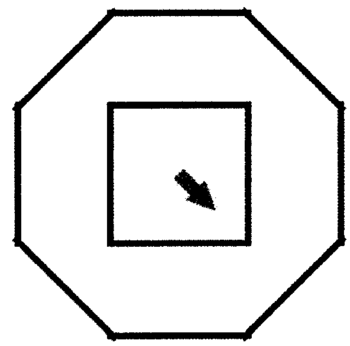
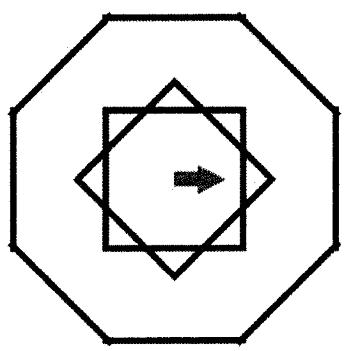
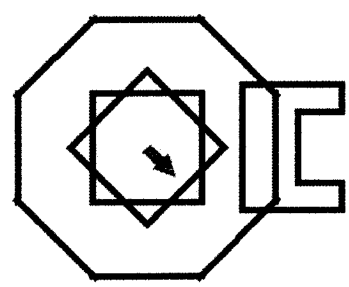
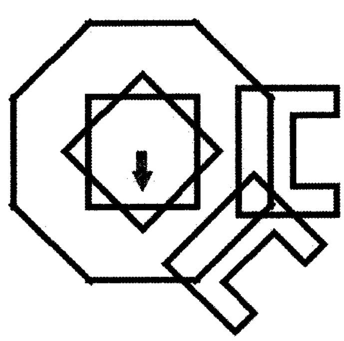
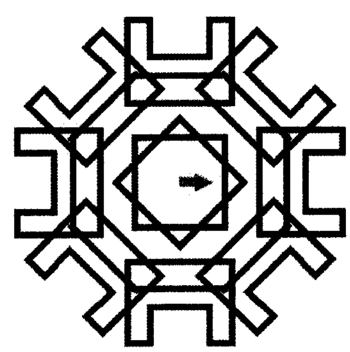
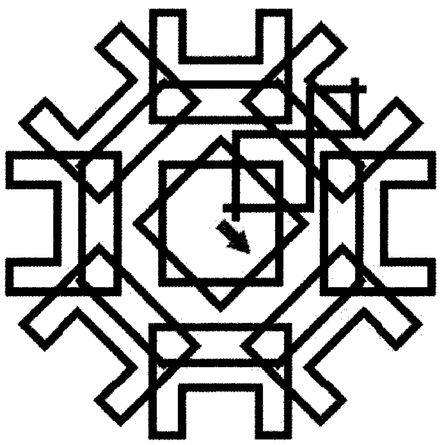
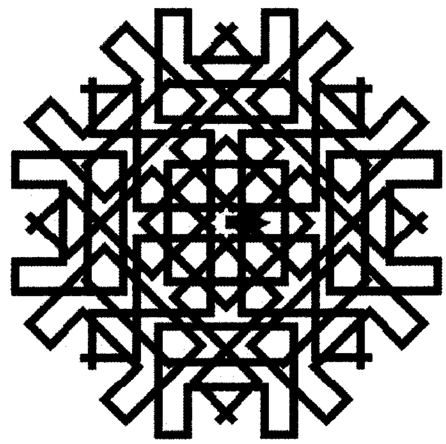
 Step 1: Initial Shape	 Step 2: Using Rule 4
 Step 3: Using Rule 9	 Step 4: Using Rule 2
 Step 5: Using Rule 8	 Step 6: Repeat 6 times of Rule 8

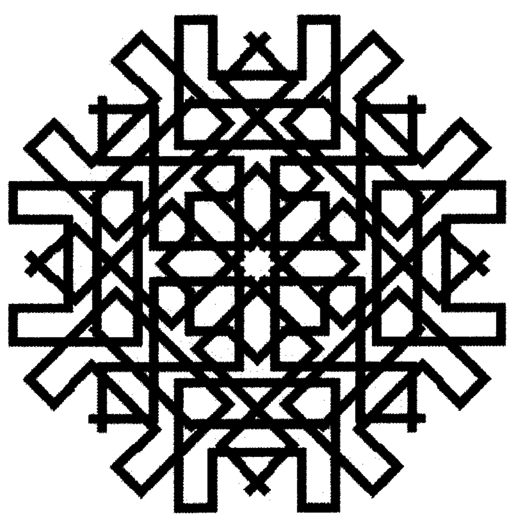
Table 5.9: Application of a Shape Grammar (Figure 1.2) (2)



Step 7: Using Rule 1



Step 8: Repeat Rule 6 Seven Times



Step 9: The Result After Using Rule 10

Chapter 6

A New Method for Generating Patterns

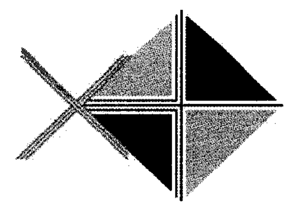
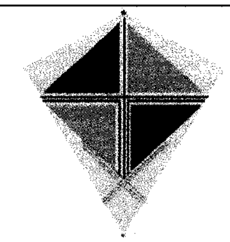
6.1 The Idea behind the New Method

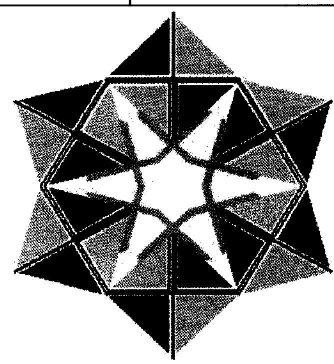
So far we have used symmetry groups and shape grammars to analyze Bakuba textiles and Zillij mosaics, and have found that our method is limited since Bakuba textiles and Zillij mosaics are closer to art than mathematics. Moreover, the process of designing shape grammar rules is more complicated if the pattern consists of many patches. We therefore extends our methods and tools by adding symmetry groups, i.e., wallpaper groups, to our analytical techniques. We illustrate this idea with the following two examples.

6.1.1 First illustration

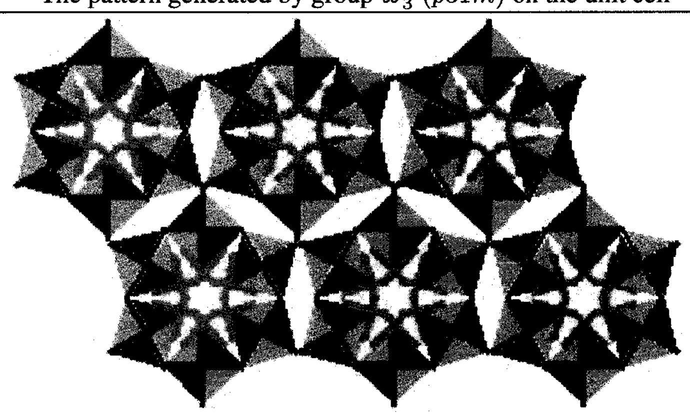
In Table 6.1, we use the patch in Figure 1.1 and the wallpaper group $\varpi_3^2(p31m)$ to create a new pattern. The unit cell for the group is a graphics of the fundamental domain (gray polygon), control points that have been used to form the domain (red points), and the graphical content of the unit cell. The downward oriented red point in the unit cell is the rotation center.

Table 6.1: Creating a New Pattern with Symmetry Group ϖ_3^2 $(p31m)$

	
A patch with Bakuba features	The unit cell for the group ϖ_3^2 $(p31m)$



The pattern generated by group ϖ_3^2 $(p31m)$ on the unit cell

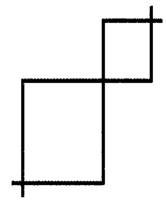
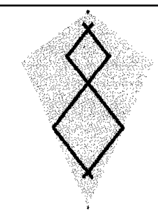
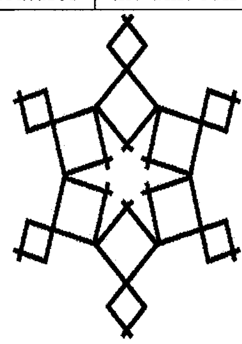
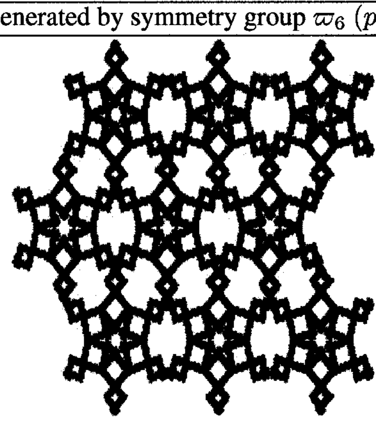


The tiling of the above pattern (2X3)

6.1.2 Second illustration

In Table 6.2, we use the patch in Figure 1.2 and the wallpaper group ϖ_6($p6$) to create new pattern, with a control path used to indicate the rotation center.

Table 6.2: Creating a New Pattern with Symmetry Group ϖ_6 ($p6$)

<table>
<tr><td></td><td></td></tr>
<tr><td>A patch with Zillij features</td><td>the unit cell for the group $\varpi_6(p6)$</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td colspan="2">The pattern generated by symmetry group ϖ_6 ($p6$) on the patch</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td colspan="2">The tiling of the above pattern (3X3)</td></tr>
</table>

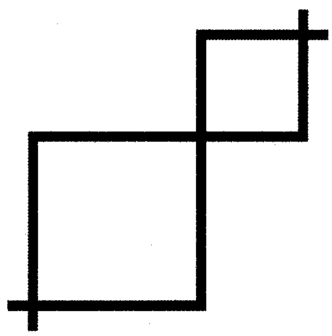
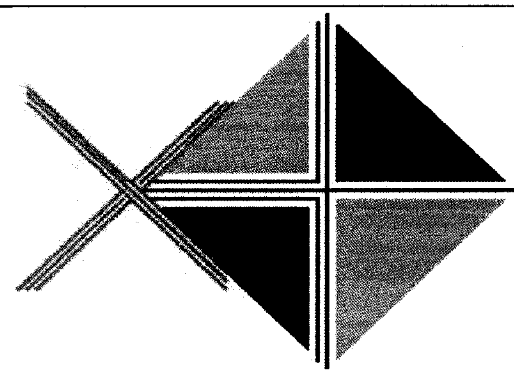
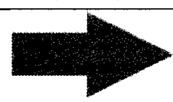
The examples above show that it makes sense to enrich our analytical tools by adding wallpaper groups to shape grammars. When describing shape grammar rules, we can therefore use one of the seventeen wallpaper groups and replace shape grammar rules in an original way: way can simplify the rules and improve the way of creating new patterns.

6.2 The New Method

Since it is not easy to describe our new method in words, we describe it by an example.

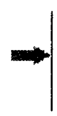
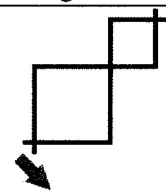
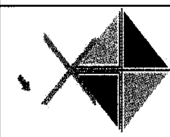
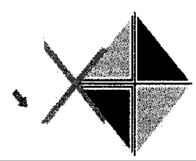
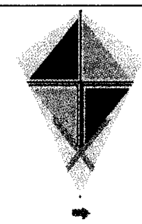
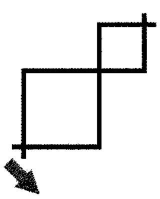
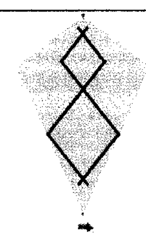
First we define terminals and markers, i.e., the set of V_T and V_M as Table 6.3. To simplify the description, we only define two terminals and one marker.

Table 6.3: Terminals and Marker (V_T and V_M)

	
Terminal 1	Terminal 2
	
Marker	

Next we define a set of shape grammar rules using a symmetry group as shown in Table 6.4.

Table 6.4: Combining Shape Grammar Rules and Symmetry Groups

	Left Side		Right Side	Symmetry group
1		→|		
2		→|		
3		→|		ϖ_3^2 ($p31m$)
4		→|		ϖ_6 ($p6$)
5		→|		

Now we assign an initial shape I as follows:

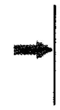
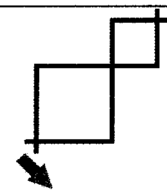
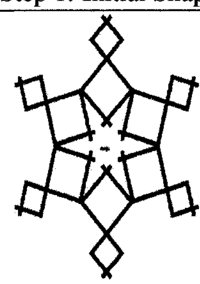
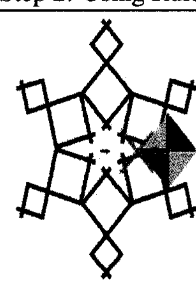
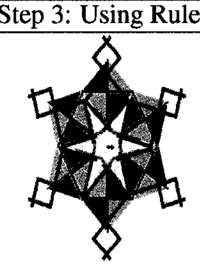
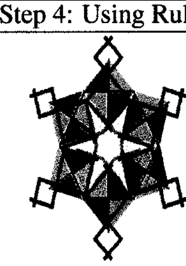


Figure 6.1: Initial Shape I for New Method

This completes the definition of the components of the shape grammar $SG =< V_T, V_M, R, I >$, where V_T, V_M, R, I is defined as the above.

We now apply the rules in Table 6.4 and show the result in Table 6.5.

Table 6.5: Combined Application of Shape Grammar Rules and Symmetry Groups

	
Step 1: Initial Shape	Step 2: Using Rule 1
	
Step 3: Using Rule 4	Step 4: Using Rule 2
	
Step 5: Using Rule 3	Step 6: Using Rule 5

From the above example we can see that the main difference between the new and old methods lies in how shape grammar rules are defined and how they are applied.

6.2.1 How to define shape grammar rules

Based on the original way of defining shape grammar rules, we add some wallpaper groups to shape grammar rules. When doing so, we just need to define unit cells that define the operation areas and rotation centers for the groups, and indicate which group is used.

6.2.2 How to apply shape grammar rules

When building shape grammar rules without the use of wallpaper groups, as in the case of rotation rules, for example, we often had to repeat a rules many times. With the help of wallpaper groups, we need to use the rule just once. This not only simplifies the process of applying shape grammars, it also avoids the possibilities of producing incomplete patterns because the number of repetitions is inadvertently incompletely defined.

6.3 Conclusion

We have shown that wallpaper groups can be combined effectively with shape grammars to study patterns in Zillij mosaics and Bakuba textiles. This follows, in particular, from the fact that although wallpaper groups involve glide reflections, as shown in Section 2.3, these operations are simply product of translations and reflections, and we know that these operations are also needed for the use of shape grammars in the study of the patterns considered in this thesis. Not only do these groups provide a way of simplifying the specification of grammars that involve iterations of operations, therefore, they also provide an easy method for validating the completeness of sequences of operations.

In the future, we hope to combine the method of tilings with the use of wallpaper groups to generate bigger patterns based on motifs derived from Zillij mosaics and Bakuba textiles. For example, we can use the patterns in Table 6.1 and Table 6.2 and appropriate tilings to generate the pattern shown in Figure 6.2.

On the other hand, because the order of rotation in wallpaper groups is limited in 2,3,4,6, but the order of rotation occurred in Bakuba textiles and Zillij mosaics may be 8, 16, and is out of the range of 2,3,4,6, we have to extend the order of rotations when combining wallpaper groups with shape grammars.

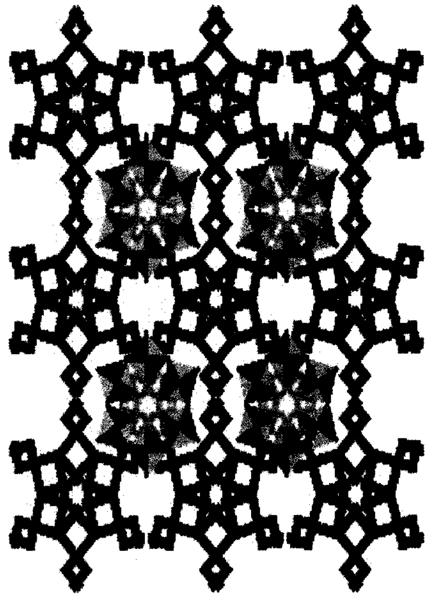


Figure 6.2: Tiling of Two Patterns

As the above discussion shows, the use of both wallpaper groups and tilings are a promising line of research that still needs to be explored. Not only do we need to generalize the idea of a shape grammar to include wallpaper groups and tilings, we also need to broaden the class of wallpaper groups itself for this purpose. Figure 6.2 shows that we can xgenerate beautiful patterns with this extension of our methods.

Appendix A

Geometry, Symmetry and Groups

A.1 Concepts from Geometry

Suppose we are familiar with the Cartesian plane.

Definition 34 *A **point** is an ordered pair* (x, y) *of numbers.*

Definition 35 *A line is the set of points* (x, y) *that satisfy an equation* $ax + by + c = 0$ *where* a, b, c *are number with* $a^2 + b^2 > 0$.

Definition 36 *The **distance** between points* (x_1, y_1) *and* (x_2, y_2) *is the value given by*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Definition 37 *If all points of a set of points lie on the same line, we call the points of this set **collinear points.***

Definition 38 *If two lines meet at a right angle, they are called **perpendicular lines**.*

Definition 39 *We say that a line is a **perpendicular bisector** of a line segment if the line is perpendicular to the line segment and passes through the midpoint of the line segment.*

A.2 Symmetry Transformations

Definition 40 *A **transformation** on the plane a one-to-one correspondence from one set of points in the plane to other set of points in the plane.*

Definition 41 *For any two transformations τ_1, τ_2 in the same space, the* ***product*** *$\tau_1\tau_2$ of two transformations τ_1, τ_2, is the composition of the transformations: $\tau_1\tau_2(P) = \tau_2(\tau_1(P))$.*

Moreover, we use a symbol τ^n for the composition $\tau...\tau$, where τ occurs n times, i.e. the n-th power of the transformation τ.

Definition 42 *The* ***order*** *of the transformation τ is the minimal $n(n \in N)$ for which $\tau^n = E$ holds.*

Remark 43 *If there is no finite number n which satisfies the given relation, then the transformation T is called* ***a transformation of infinite order****. If $n = 2$, then the transformation τ is called* ***an involution****. If transformations τ_1 and τ_2 are such that $\tau_1\tau_2 = E$, then τ_1 is called* ***the inverse of*** *τ_2, We use the symbol τ_2^{-1} to express it; and vice versa, i.e. $\tau_2^{-1} = \tau_1$ and $\tau_1^{-1} = \tau_2$. Moreover, for the product of two transformations $(\tau_1\tau_2)^{-1} = \tau_2^{-1}\tau_1^{-1}$ holds.*

Definition 44 *Theidentity* ***transformation*** *of a space is the transformation E under which every point of the space is invariant, i.e. $E(P) = P$ holds for each point P of the space.*

The identity transformation is symmetry of any given figure. Any figure in which set of symmetries consists only of the identity transformation E is called asymmetric; any other figure is called symmetric.

Definition 45 *A* ***translation*** *$\tau_{A,B}$ is a mapping having equations of the form*

$$\begin{cases} x\prime = x + a \\ y\prime = y + b \end{cases}$$

where a and b are numbers, and $A(x, y)$, $B(x', y')$.

Definition 46 *A* ***reflection*** *σ_m in line m is the mapping defined by*

$$\sigma_m(P) = \begin{cases} P & \text{if point } P \text{ is on } m \\ Q & \text{if } P \text{ is off } m \text{ and } m \text{ is the perpendicular bisector of } \overline{PQ}. \end{cases}$$

Definition 47 *If a and b are distinct lines perpendicular to line c, then $\sigma_c\sigma_b\sigma_a$ is called a* ***glide reflection*** *with axis c.*

Definition 48 *A **rotation** about point C through directed angle of θ is the transformation $\rho_{C,\theta}$ that fixes C and otherwise sends a point P to the point $P\prime$ where $CP\prime = CP$ and θ is the directed angle measure of the directed angle from $\overrightarrow{CP}$ to $\overrightarrow{CP\prime}$.*

A.3 Symmetry Groups

A figure g is any non-empty subset of points of space.

Definition 49 *A figure g is called **invariant** with respect to a transformation τ if $\tau(g) = g$, where the transformation τ is called a **symmetry** of the figure g.*

Definition 50 *The set of points invariant with regard to all the powers of a given symmetry τ is called the **element of symmetry** of the figure g.*

As a binary operation $*$ we understand any rule which assigns to each ordered pair (a, b) a certain element c written as $a * b = c$, or in the short form, $ab = c$.

Definition 51 *A structure $(G, *)$ formed by a set G and a binary operation $*$ is a **group** if it satisfies the axioms:*

1) (Closure): for all a_1, $a_2 \in G$, $a_1 a_2 \in G$ is satisfied;

2) (Associativity): for all a_1, a_2, $a_3 \in G$, $(a_1 a_2)a_3 = a_1(a_2 a_3)$ is satisfied;

3) (Existence of a neutral element): there exists $e \in G$ that for each $a_1 \in G$ the equality $a_1 e = a_1$ is satisfied;

4) (Existence of inverse elements): for each $a_1 \in G$ there exists $a_1^{-1} \in G$ so that $a_1^{-1} a_1 = e$ is satisfied.

5) (Commutativity): for all a_1, $a_2 \in G$, $a_1 a_2 = a_2 a_1$ is satisfied, the group is commutative or Abelian.

Definition 52 *The **order** of a group G is the number of elements of the group.*

Remark 53 *we distinguish finite and infinite groups by the order of groups. The power and the order of a group element are defined analogously to the definition of the power and the order of a transformation.*

Definition 54 *A figure g is said to be an **invariant** of the group of transformations T if it is invariant with respect to all its transformations, i.e. if $\tau(g) = g$ for any $\tau \in T$.*

Definition 55 *All symmetries of a figure g form a group, that we call the* ***group of symmetries*** *of g and denote by* T_g.

Definition 56 *A subset* H *of group* T*, which by itself constitutes a group with the same binary operation, is called a subgroup of group* T *if and only if for all* $a_1, a_2 \in H, a_1 a_2^{-1} \in H$.

Definition 57 *An* ***isometry*** *is a distance-preserving transformation of the geometry objects in the plane consisting of translations, rotations, reflections, and glide reflections.*

Definition 58 *A* ***wallpaper group*** ϖ *is a group of isometries whose translations are exactly those in* $< \tau_1, \tau_2 >$ *where if* $\tau_1 = \tau_{A,B}$ *and* $\tau_2 = \tau_{A,C}$ *then* A, B, C *are noncollinear points.*

Definition 59 *A* ***cyclic subgroup*** *of a wallpaper group is a subgroup generated by a single element of the group.*

Definition 60 *For a set s of points, if* $\sigma_P(s) = s$*, we call point* P *is a* ***point of symmetry***

Definition 61 *A point* P *is an* n***-center*** *for a group* G *of isometries if the rotations in* G *with center* P *form a finite cyclic group* C_n *with* $n > 1$.

Definition 62 *A* ***figure*** *is a nonempty set of points. If point* P *is an* n*-center for the symmetry group for a figure, then* P *is also called an* n***-center*** *for the figure.*

Definition 63 *A* ***center of symmetry*** *is an* n*-center for some* n.

Remark 64 *A point of symmetry is not necessarily an* n*-center, and vice versa. For example, if point* P *is a 4-center for some figure, then* P *is a point of symmetry for that figure since* $\sigma_P = \rho^2_{P,90}$*. In this case, point* P *is a point of symmetry but* P *is not a 2-center. Also note that if point* Q *is a 3-center for some figure then* Q *is not a point of symmetry for that figure.*

Definition 65 *The* ***crystallographic restriction*** *says that if point* P *is an* n*-center of a wallpaper group, then* n *is one of 2,3,4,or 6.*

Appendix B

Proof of the Restriction Theorem

Unless otherwise stated, the details of the proof are taken from MARTIN [24].

For proving the crystallographic restriction theorem, we need some basic concepts that are introduced in Appendix A.

The **ornamental groups** of the plane are the rosette groups, the frieze groups, and the wallpaper groups. We pay our attention on the last of the ornamental groups of the plane by considering wallpaper groups whose subgroup of translations is generated by two translations.

Definition 66 *The* ***translation lattice*** *for ϖ determined by point A is the set of all images of A under the translations in ϖ. Since every translation in ϖ wallpaper group is of the form $\tau_2^j\tau_1^i$, then all points A_{ij} form a translation lattice where $A_{ij} = \tau_2^j\tau_1^i(A)$.*

See Figure B.1.

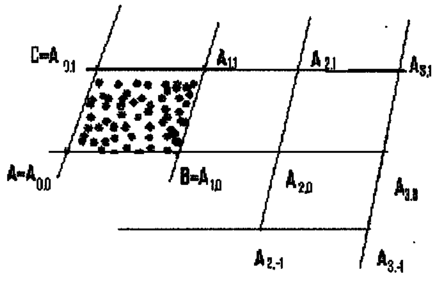


Figure B.1: Illustration of the Translation Lattice

Definition 67 *A **unit cell** for ϖ with respect to point A and generating translations $\tau_1$and τ_2 is a quadrilateral region with vertices A_{ij}, $A_{i+1,j}$, $A_{i.j+1}$, and $A_{i+1,j+1}$.*

A unit cell is always a quadrilateral region determined by a parallelogram. A translation lattice with a rectangular unit cell is called **rectangular**; a translation lattice with a rhombic unit cell is called **rhombic**.

Our first task is to show that a translation lattice is necessarily rhomic or rectangular when contains odd isometries.

Theorem 68 *If σ_l is in wallpaper group ϖ, then l is parallel to a diagonal of a rhombic unit cell for ϖ or else l is parallel to a side of a rectangular unit cell for ϖ.*

Proof

Suppose σ_l is in wallpaper group ϖ. We wish to show that l is parallel to a diagonal of a rhombic unit cell or that l is parallel to a side of a rectangular unit cell. Let A be a point on l. Let $\tau_{A,P}$ be a shortest nonidentity translation in ϖ.

There are two cases.

Case 1: Neither $\overleftrightarrow{AP} = l$ nor $\overleftrightarrow{AP} \perp l$. let $Q = \sigma_l(P)$. Then $\tau_{A,Q}$ is in ϖ as $\tau_{A,Q} = \sigma_l \tau_{A,P} \sigma_l^{-1}$. See Figure B.2. Since $AP = AQ$ and points A, P, Q are not collinear, then$<\tau_{A,P}, \tau_{A,Q}>$ is the group of all translations in ϖ and l contains a diagonal of a rhombic unit cell.

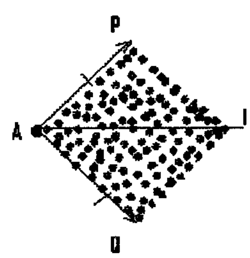


Figure B.2: Illustration of Case 1

Case 2: $\overleftrightarrow{AP} = l$ or $\overleftrightarrow{AP} \perp l$.

Let a be perpendicular to $\overleftrightarrow{AP}$ at A;

let m be the perpendicular bisector of $\overleftrightarrow{AP}$;

and let $n = \sigma_a(m)$.

See Figure B.3.

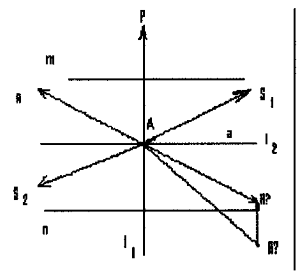


Figure B.3: Illustration of Case 2

Let $\tau_{A,R}$ be a shortest translation in ϖ that is not in $<\tau_{A,P}>$.

Then R is on m, on n, or between m and n, as otherwise $\tau_{A,P}^{\pm 1}\tau_{A,R}$ is shorter than $\tau_{A,R}$.

Further, considering both $\tau_{A,R}$ and its inverse, we may suppose without loss of generality that $\tau_{A,R}$ is such that R is on m, on a, or between m and a.

Let $S = \sigma_l(R)$. Assume R is between m and a.

If $l = \overleftrightarrow{AP}$. then $\tau_{A,S}\tau_{A,R}$ is a translation in ϖ shorter than $\tau_{A,P}$.

If $l \perp \overleftrightarrow{AP}$, then $\tau_{S,A}\tau_{A,R}$ is a translation in ϖ shorter than $\tau_{A,P}$.

Therefore we must have R is on m or a.

If R is on m, then $<\tau_{A,R}, \tau_{A,S}>$ is the same as $<\tau_{A,P}, \tau_{A,R}>$ since $\tau_{A,S}\tau_{A,R} = \tau_{A,P}$.

Thus l is parallel to a diagonal of a rhombic unit cell ($\square ARPS_1$ in Figure B.3) with respect to point A and generating translations $\tau_{A,R}$ and $\tau_{A,S}$.

On the other hand, if R is on a, then $<\tau_{A,P}, \tau_{A,R}>$ is the group of all translations in ϖ and l is parallel to a side of a rectangular unit cell for ϖ.

This proves the theorem. ■

Theorem 69 *If wallpaper group ϖ contains a glide reflection, then ϖ has a translation lattice that is rhombic or rectangular.*

Proof

First, we suppose wallpaper group ϖ contains no reflections but does contain a glide reflection with axis line l.

By the Theorem of the Frieze Groups [24], we see that the smallest group containing the glide reflection and the translations in ϖ that fix l is a frieze group $\mathcal{F}_l^3$ generated by glide reflection γ with axis l.

Hence, we may suppose γ^2 is a shortest translation fixing l.

Let A be a point on l.

Let a be perpendicular to l at A, $m = \gamma(a), p = \gamma^2(a)$, and $P = \gamma^2(A)$.

So $\tau_{A,P}$ is a shortest translation in $<\gamma>$.

Let $\tau_{A,B}$ be a shortest translation in ϖ that is not in $<\gamma^2>$.

Since $\tau_{A,P}^{\pm 1}\tau_{A,B}$ cannot be shorter than $\tau_{A,B}$,we may suppose without loss generality that B is on a or lies between a and p.

If B is on a, then ϖ has a rectangular translation lattice and l is parallel to a side of a rectangular unit cell. Suppose B is between a and p.

See Figure B.4.

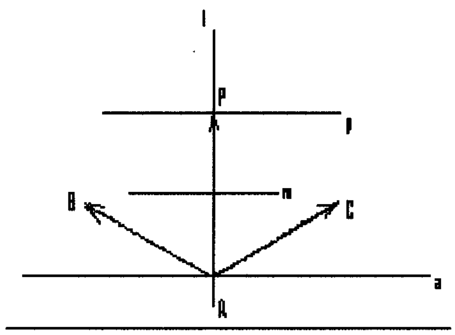


Figure B.4: Illustration of B between a and p

Let $C = \sigma_l(B)$.

Then $\tau_{A,C}$ is in ϖ as $\tau_{A,C}$=$\gamma\tau_{A,\,B}\gamma^{-1}$.

So $\tau_{A,C}\tau_{A,\,B}$=γ^2 and B is on m.

So $\Box ABPC$ is a rhombic unit cell with l containing a diagonal.■

Theorem 70 *If a glide reflection in wallpaper group ϖ fixes a translation lattice for ϖ, then ϖ contains a reflection.*

Proof

If glide reflection γ takes point A to point P in the translation lattice determined by A for wallpaper group ϖ, then γ followed by $\tau_{P,\,A}$ must be a reflection as the product is an odd isometry fixing point A.■

This finishes the preliminary results needed about odd isometries in a wallpaper group.

Theorem 71 *For given n, if point P is an n-center for group G of isometries and G contains an isometry that takes P to Q, then Q is an n-center for G. If l is a line of symmetry for a figure and the symmetry group for the figure contains an isometry that takes l to m, then m is a line of symmetry for the figure.*

Proof

First, for a given n, the set of n-centers must be fixed by every isometry in the group. To see this, suppose $\alpha(P) = Q$ for some isometry α in group G. Form the equations $\alpha\rho_{P,\Theta}\alpha^{-1} = \rho_{Q,\pm\Theta}$ and $\alpha^{-1}\rho_{Q,\Phi}\alpha = \rho_{P,\pm\Phi}$, we see that Q is an n-center iff P is an n-center (for the same n).

Similarly, if $\alpha(l) = m$, and σ_l is in G, since $\sigma_m = \alpha\sigma_l\alpha^{-1}$, therefore, σ_m is in G.■

Lemma 72 (Xiu Wu Huang) *If T is the center of square $\square PQRS$, σ_P, σ_Q, σ_R, are in $< \sigma_{\overleftrightarrow{PQ}}, \sigma_S, \sigma_T >$.*

Proof

According to the theorem about half-turns, we have:

$\sigma_P = \sigma_{\overleftrightarrow{PQ}}.\sigma_{\overleftrightarrow{PS}}$,

$\sigma_Q = \sigma_{\overleftrightarrow{PQ}}.\sigma_{\overleftrightarrow{QR}}$

$\sigma_R = \sigma_{\overleftrightarrow{QR}}.\sigma_{\overleftrightarrow{RS}}$

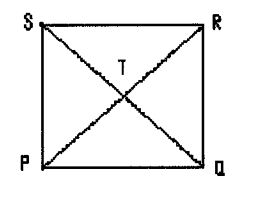


Figure B.5: Illustration of Lemma 74

Moreover, from the condition, we have:

$\overleftrightarrow{RS} = \sigma_T(\overleftrightarrow{PQ})$,

$\overleftrightarrow{PS} = \sigma_S(\overleftrightarrow{RS})$,

$\overleftrightarrow{QR} = \sigma_T(\overleftrightarrow{PS})$

Therefore, we have:

$\sigma_{\overleftrightarrow{RS}} = \sigma_T\sigma_{\overleftrightarrow{PQ}}\sigma_T$

$\sigma_{\overleftrightarrow{PS}} = \sigma_S\sigma_{\overleftrightarrow{RS}}\sigma_S = \sigma_S\sigma_T\sigma_{\overleftrightarrow{PQ}}\sigma_T\sigma_S$

$\sigma_{\overleftrightarrow{QR}} = \sigma_T\sigma_{\overleftrightarrow{PS}}\sigma_T = \sigma_T\sigma_S\sigma_T\sigma_{\overleftrightarrow{PQ}}\sigma_T\sigma_S\sigma_T$

That is $\sigma_P, \sigma_Q, \sigma_R$ are in $< \sigma_{\overleftrightarrow{PQ}}, \sigma_S, \sigma_T >$.■

Lemma 73 (Xiu Wu Huang) $< \rho_{A,60}, \sigma_{\overleftrightarrow{CG}} >=< \sigma_{\overleftrightarrow{AG}}, \sigma_{\overleftrightarrow{CG}}, \sigma_{\overleftrightarrow{AB}} >$ *if* G *is the center of equilateral* $\triangle ABC$.

Proof

Let $G_1= <\rho_{A,60}, \sigma_{\overleftrightarrow{CG}}>$, $G_2= <\sigma_{\overleftrightarrow{AG}}, \sigma_{\overleftrightarrow{CG}}, \sigma_{\overleftrightarrow{AB}}>$ and $G_3= <\rho_{A,60}, \sigma_{\overleftrightarrow{CG}}, \rho_{B,60}, \sigma_{\overleftrightarrow{BG}}, \sigma_{\overleftrightarrow{AB}}, \sigma_{\overleftrightarrow{AG}}>$.

Suppose $\rho_{A,60}(B) = C$ for orientation.

Since $B = \sigma_{\overleftrightarrow{CG}}(A)$, and $\overleftrightarrow{BG} = \rho_{A,60}(\overleftrightarrow{CG})$, so, we have:

$\rho_{B,60} = \sigma_{\overleftrightarrow{CG}} \rho_{A,60}^{-1} \sigma_{\overleftrightarrow{CG}}$

(we can checked it by using A, B, C)

since $\overleftrightarrow{AG} = \rho_{B,60}(\overleftrightarrow{CG})$, $\sigma_{\overleftrightarrow{AB}} = \sigma_{\overleftrightarrow{AG}}^{-1} \rho_{A,60}$, and Theorem 71,

so $\rho_{B,60}$ and $\sigma_{\overleftrightarrow{BG}}$ in $G1$, so $G_1 = G_3$.

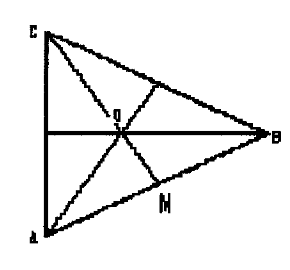


Figure B.6: Illustration of Lemma 75

Now, think G_2 and G_3: Since $\rho_{A,60} = \sigma_{\overleftrightarrow{AG}} \sigma_{\overleftrightarrow{AB}}$, $\overleftrightarrow{BG} = \sigma_{\overleftrightarrow{AG}}(\overleftrightarrow{CG})$, the above conclusion about $\rho_{B,60}$, and Theorem 71, thus, $G_2 = G_3$. That is $<\rho_{A,60}, \sigma_{\overleftrightarrow{CG}}>= <\sigma_{\overleftrightarrow{AG}}, \sigma_{\overleftrightarrow{CG}}, \sigma_{\overleftrightarrow{AB}}>$.■

Theorem 74 *If* $\rho_{A,360/n}$ *and* $\rho_{P,360/n}$ *with* $P \neq A$ *and* $n > 1$ *are in wallpaper group* ϖ*, then* $2AP$ *is not less than the length of the shortest nonidentity translation in* ϖ.

Proof

Suppose rotations $\rho_{A,360/n}$ and $\rho_{P,360/n}$ with $P \neq A$ and $n > 1$ are in wallpaper group ϖ.

Then ϖ contains the product $\rho_{P,360/n}\rho_{A,-360/n}$, which is a nonidentity translation $\tau_2^j \tau_1^i$ for some i and j by the Angle-addition Theorem.

So

$$\rho_{P,360/n} = \tau_2^j \tau_1^i \rho_{A,360/n}$$

and

$$\rho_{P,360/n}(A) = \tau_2^j \tau_1^i \rho_{A,360/n}(A) = A_{i,j}.$$

Hence, either P is the midpoint of A and $A_{i,j}$ (when $n = 2$) or else $\triangle APA_{i,j}$ is isosceles.

In either case, $2AP = AP + PA_{i,j} \geq AA_{i,j} > 0$ by the triangle inequality.

Therefore, $2AP$ is not less than the length of any nonidentity translation in ϖ.■

Theorem 74 shows that certain centers of symmetry cannot be arbitrarily close. This conclusion is very useful for the following proof.

The theorem above more precisely states that no two n-centers (same n) can be "too close".

Lemma 75 *The possible values of n such that there is an n-center in a wallpaper group are rather restricted.*

Proof

Suppose point P is an n-center of wallpaper group.

Let Q be an n-center (same n) at the least possible distance from P with $Q \neq P$.

The existence of point Q is assured by the previous two theorems.

1. Let $R = \rho_{Q,360/n}(P)$. Then R is an n-center and $PQ = QR$.
2. Let $S = \rho_{R,360/n}(Q)$. Then S is an n-center and $RQ = RS$.
3. If $S = P$, then n=6. See Table B.7.
4. If $S \neq P$, then we must have $SP \geq PQ = RQ$ by the choice of Q.

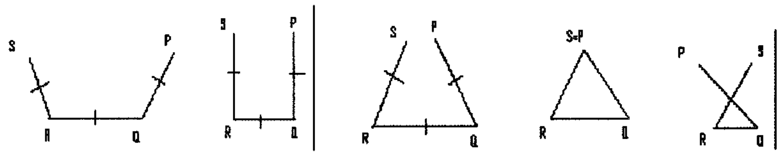


Figure B.7: Figures for Case 3

Hence, if $S \neq P$, then $n \leqslant 4$. So n is one of 2,3,4,or 6.■

We have proved the crystallographic restriction theorem:

Theorem 76 (Crystallographic Restriction Theorem) *If the point P is an n-center for a wallpaper group, then n is one of 2,3,4,or 6.*

Appendix C

Proof of the Classification Theorem

Unless otherwise stated, the details of the calculations below may be found in [24].

We begin by observing that from Theorem 76 in Appendix B, we can get the following result:

Theorem 77 *If a wallpaper group contains a 4-center, then the group contains neither a 3-center nor a 6-center.*

Proof

We suppose there is a wallpaper group containing a 3-center P, i.e. $\rho_{P,120}$, and a 4-center Q, i.e. $\rho_{Q,90}$.

Therefore, the product $\rho_{P,120}\rho_{Q,-90}$, i.e. a rotation of $30^0(n = 12)$ about some point, is also in the wallpaper group. By the crystallographic restriction theorem (Theorem 76), that is impossible. Thus, a wallpaper group cannot contain both a 4-center and a 3-center at the same time.

Since $\rho^2_{P,60} = \rho_{P,120}$, according to the above conclusion, it is impossible that a wallpaper group contains a 3-center and a 6-center.■

Based on the theorems above, we shall now find all possible wallpaper groups ϖ, beginning with the groups that contain an n-center.

By the crystallographic restriction it is sufficient to consider only values 6,3,4, and 2 for n.

C.1 6-Centers

We start our process of exploring all possible wallpaper groups with 6-center, and we have the following theorem.

Theorem 78 *Suppose A is a 6-center for wallpaper group ϖ. There are no 4-center for ϖ. Further, the center of symmetry nearest to A is a 2-center M, and A is the center of a regular hexagon whose vertices are 3-centers and whose sides are bisected by 2-ceners. All the centers of symmetry for ϖ are determined by A and M.*

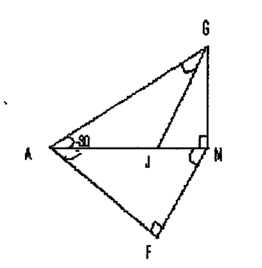


Figure C.1: Illustration of the Proof of Theorem 78 (1)

Proof

According to Theorem 77, ϖ can contain no 4-centers since ϖ contains the 6-center A.

Let M be an n-center nearest to A. If M were a 3-center or a 6-center, then there would be a center F closer to A than M, where $\rho_{M,120}\rho_{A,60} = \rho_{F,180}$. See Figure C.1 above. Hence, M must be a 2-center. Define point G by the equation $\rho_{M,180}\rho_{A,-60} = \rho_{G,120}$. So G is either a 3-center or a 6-center. However, G cannot be a 6-center as then there would be a center J between A and M, where J is defined by the equation $\rho_{G,60}\rho_{A,60} = \rho_{J,120}$. Hence G must be a 3-center. The images of G under the action of $\rho_{A,60}$ are the vertices of the hexagon in the statement of the theorem. With $B = \sigma_M(A)$ and $C = \rho_{A,60}(B)$, then B and C are 6-centers for ϖ. The centers of symmetry determined by the 6-center A and the 2-center M are as arranged in Figure C.2. With $N = \rho_{A,60}(M)$, then N is a 2-center for ϖ. Also, since 6-center A must go to a 6-center under an element of ϖ, then $\sigma_M\sigma_A$ and $\sigma_N\sigma_A$ are shortest translations in ϖ. Hence $\tau_{A,B}$ and $\tau_{A,C}$ must generate the translation subgroup of ϖ.■

We have our first specific wallpaper group:

$$\varpi_6 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,60} >=< \rho_{A,60}, \sigma_M >$$

where $\triangle ABC$ is equilateral and M is the midpoint of $\overline{AB}$. A smallest polygonal region t such that the plane is covered by $\{\alpha(t)|\alpha \in \varpi\}$ is called a **polygonal base** for wallpaper group ϖ.

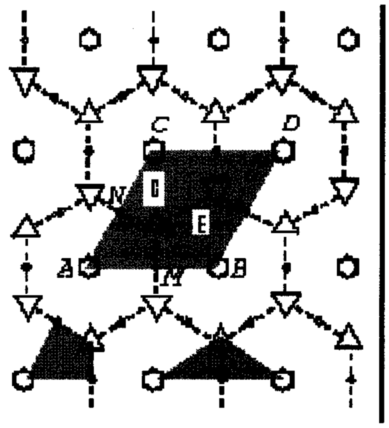
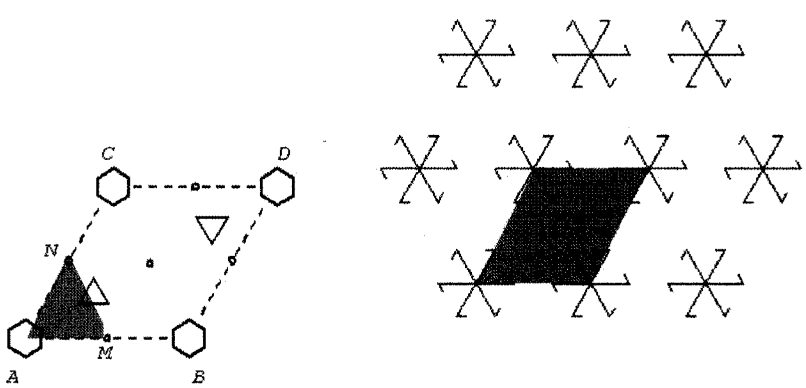


Figure C.2: Illustration of the Proof of Theorem 78 (2)

The bases may be used to create wallpaper patterns having a given wallpaper group as symmetry group. If t' is a figure with identity symmetry group in base t, then the union of all images $\alpha(t')$ with α in ϖ is a figure with all the symmetries in ϖ.

This figure is said to have **motif** t'. In producing a wallpaper pattern, once motif t' is picked for base t, that part of the pattern in a unit cell is determined and this is then just translated throughout the plane to give the wallpaper pattern.

For each of the seventeen wallpaper groups, we shall have a figure such as Figure C.3.



$$\varpi_6 = < \tau_{A,B}, \tau_{A,C}, \rho_{A,60} >$$
$$= < \rho_{A,60}, \sigma_M >$$
$$= p6$$

Figure C.3: Illustration of the Wallpaper Group ϖ_6

These will have a pattern with the check motif, a unit cell with a base and symmetries indicated, two sets of generators for the group, and two designations for the group.

In addition to the notation of Fejes Tóth involving "ϖ", the *short international form* used by crystallographers will be given.

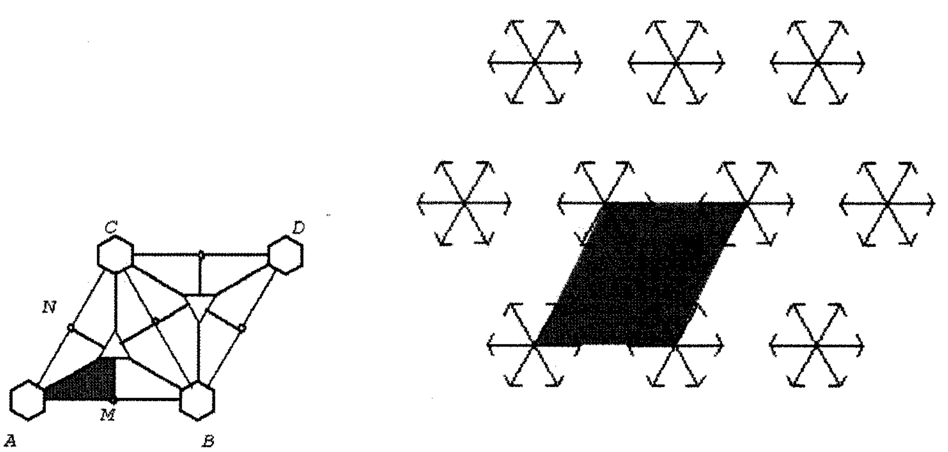
For example, the group ϖ_6 is designated $p6$ by crystallographers.

Consider extending ϖ_6 to ϖ.

Since the rhombic translation lattice of 6-centers determined by 6-center A must be fixed by any isometry in ϖ, then by Theorem 70 in Appendix B, any extensions of ϖ_6 are obtained by adding reflections that fix this translation lattice.

However, because of the richness of ϖ_6, adding any one of the possible reflections requires introducing all of the possible reflections.

See Figure C.4.



$\varpi_6^1 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,60}, \sigma_{\overleftrightarrow{MC}} >$
$=< \rho_{A,60}, \sigma_{\overleftrightarrow{MC}} >$
$= p6m$

Figure C.4: Illustration of the Wallpaper Group ϖ_6^1

Let

$\varpi_6^1 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,60}, \sigma_{\overleftrightarrow{MC}} >$.

From the result on ϖ_6, then

$\varpi_6^1 =< \rho_{A,60}, \sigma_M, \sigma_{\overleftrightarrow{MC}} >$.

Since

$\sigma_{\overleftrightarrow{GM}} = \sigma_{\overleftrightarrow{MC}}$, $\rho_{A,60} = \sigma_{\overleftrightarrow{AG}}\,\sigma_{\overleftrightarrow{MA}}$,

and

$\sigma_M = \sigma_{\overleftrightarrow{GM}}\sigma_{\overleftrightarrow{MA}}$;

moreover, we can rewrite the last two equalities as:

$\sigma_{\overleftrightarrow{AG}} = \rho_{A,60}\sigma^{-1}_{\overleftrightarrow{MA}}$, $\sigma_{\overleftrightarrow{MA}} = \sigma^{-1}_{\overleftrightarrow{GM}}\sigma_M = \sigma^{-1}_{\overleftrightarrow{MC}}\sigma_M$.

So

$\varpi_6^1 =< \sigma_{\overleftrightarrow{AG}}, \sigma_{\overleftrightarrow{GM}}, \sigma_{\overleftrightarrow{MA}} >$

This means that ϖ_6^1 is generated by the three reflections in the three lines that contain the side of a 30^0,60^0,90^0 triangle. by Lemma 73 in Appendix B, we get:

$\varpi_6^1 =< \rho_{A,60}, \sigma_{\overleftrightarrow{MC}} >$

A wallpaper pattern with symmetry group ϖ_6 has a 6-center but no line of symmetry; a wallpaper pattern with symmetry group ϖ_6^1 has a 6-center and a line of symmetry. A wallpaper pattern having a 6-center has a symmetry group ϖ_6 or ϖ_6^1.

C.2 3-Centers, but no 6-Centers

After exploring all possible wallpaper groups with 6-center, we continue our way and explore wallpaper groups containing 3-centers, but do not have 6-center. We have the following theorem:

Theorem 79 *If A is a 3-center for wallpaper group ϖ and there are no 6-center for ϖ, then every center of symmetry for ϖ is a 3-center and A is the center of a regular hexagon whose vertices are 3-centers. All the centers of symmetry for ϖ are determined by A and a nearest 3-center.*

Proof

Since the fact that $\rho_{A,-120}\rho_{P,180}$ cannot be in ϖ for any point P since ϖ contains no 6-centers, that every center P for ϖ must be a 3-center follows.

Let G be a nearest 3-center to A.

Let J be such that $\rho_{G,120}\rho_{A,120} = \rho_{J,240}$.

Then J is a 3-center and $\triangle AGJ$ is an equilateral triangle.

The images of G and J under the action of $\rho_{A,120}$ are the vertices of the hexagon in the statement of the theorem.

Repetition of the argument for each 3-center shows that all the 3-centers are arrayed as in Figure C.5.

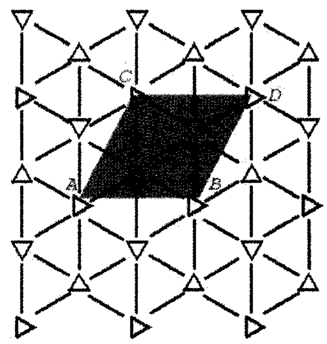


Figure C.5: Illustration of the Proof of Theorem 79

This finishes the proof of the theorem. ■

Further, from Figure C.6,

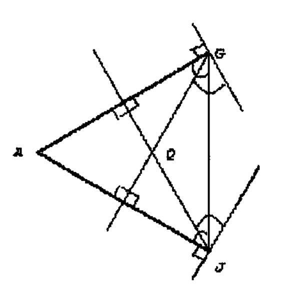


Figure C.6: Advanced Illustration of Theorem 79

we see that each of $\rho_{G,120}\tau_{A,G}$ and $\rho_{J,120}\tau_{A,J}$ is $\rho_{Q,120}$ where Q is the centroid of $\triangle AGJ$.

Hence neither $\tau_{A,G}$ nor $\tau_{A,J}$ is in ϖ as otherwise Q would be a 3-center nearer to A than G.

So if $\tau_{A,B}$ is a shortest translation in ϖ, then 3-center B is not a vertex of the hexagon of nearest 3-centers to A.

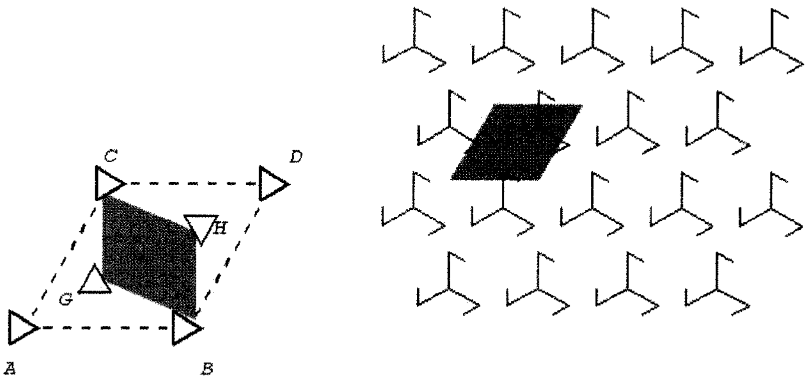
Let B and C be defined by $\tau_{A,B} = \rho_{G,120}\rho_{A,-120}$ and $\tau_{A,C} = \rho_{G,-120}\rho_{A,120}$.

Then $\tau_{A,B}$ and $\tau_{A,C}$ are shortest translations in ϖ and take A to next nearest 3-centers.

Hence, $\tau_{A,B}$ and $\tau_{A,C}$ generate the translation group of ϖ.

Let $\varpi_3 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,120} >=< \rho_{A,120}, \rho_{G,120} >$.

If ϖ contains no odd isometries then ϖ must be ϖ_3.



$\varpi_3 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,120} >$
$=< \rho_{A,120}, \rho_{G,120} >$
$= p3$

Figure C.7: Illustration of the Wallpaper Group ϖ_3

See Figure C.7.

Consider extending ϖ_3 to a wallpaper group ϖ without reflections by adding glide reflections.

Then by Theorem 70 in Appendix B, ϖ must have a glide reflection that takes 3-center A to a 3-center that is not in the translation lattice determined by A.

By composing this glide reflection with a translation and possibly a rotation about A, we may assume ϖ contains a glide reflection that takes A to either G or J.

Suppose γ is a glide reflection in ϖ that takes A to G.

Then $\gamma = \sigma_z \sigma_Z$ where Z is the midpoint of A and G and z is some line through G.

Since σ_Z fixes the set of all 3-centers, then σ_z must also fix the set of all 3-centers.

See Figure C.8.

By composing σ_z with a rotation about G, we may suppose without loss of generality that z is either the perpendicular bisector of $\overline{JB}$ or $z = \overleftrightarrow{GJ}$.

The first is impossible as otherwise $\overleftrightarrow{AG}$ is the axis of γ and $\tau_{B,A}\gamma^2$ is a translation in ϖ of length AG and shorter than $\tau_{A,B}$.

So $z = \overleftrightarrow{GJ}$.

However, then $\rho_{G,-120}\gamma = \sigma_{\overleftrightarrow{AG}}\sigma_z\sigma_z\sigma_Z = \sigma_{\overleftrightarrow{ZJ}}$ and ϖ contains the reflection in the perpendicular bisector of $\overline{AG}$.

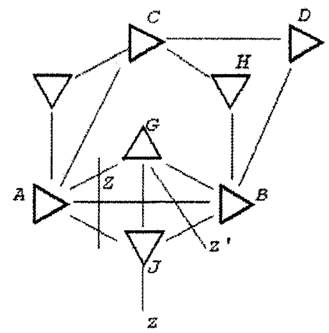
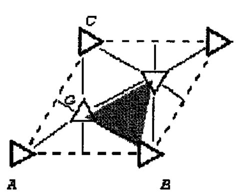


Figure C.8: Illustration of the Wallpaper Group ϖ_3^1

Likewise, the presence of a glide reflection taking A to J implies the reflection in the perpendicular bisector of $\overline{AJ}$ is in ϖ.

In any case, ϖ must contain a reflection if ϖ is an extension of ϖ_3 and contains an odd isometry.



$$\begin{aligned}\varpi_3^1 &=< \tau_{A,B}, \tau_{A,C}, \rho_{A,120}, \sigma_{\overleftrightarrow{A,G}} > \\ &=< \rho_{A,120}, \rho_{G,120}, \sigma_{\overleftrightarrow{A,G}} > \\ &= p3m1\end{aligned}$$

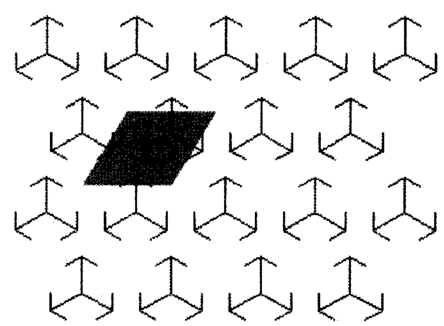


Figure C.9: Illustration of the Wallpaper Group ϖ_3^1

All extensions of ϖ_3 to a group with no 6-centers by adding odd isometries are obtained by adding reflections.

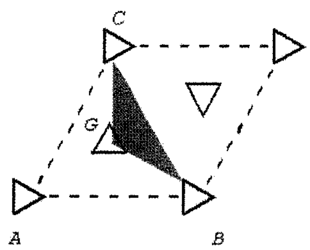
If ϖ_3 is extended by adding σ_l, then line l must be a line of symmetry for the set of 3-centers.

Since such a line must pass through at least one 3-center, we suppose l is a line through 3-center A.

Let ϖ_3^1=<$\tau_{A,B}, \tau_{A,C}, \rho_{A,120}, \sigma_{\overleftrightarrow{A,G}}$> and ϖ_3^2=<$\tau_{A,B}, \tau_{A,C}, \rho_{A,120}, \sigma_{\overleftrightarrow{A,B}}$>.

See the above Figure C.9 and the following Figure C.10.

The above shows that ϖ_3^1 is generated by the three reflections in the three lines containing the sides of an equilateral triangle.



$\varpi_3^2 = < \tau_{A,B}, \tau_{A,C}, \rho_{A,120}, \sigma_{\overleftrightarrow{A,B}} >$
$= < \rho_{G,120}, \sigma_{\overleftrightarrow{A,B}} >$
$= p31m$

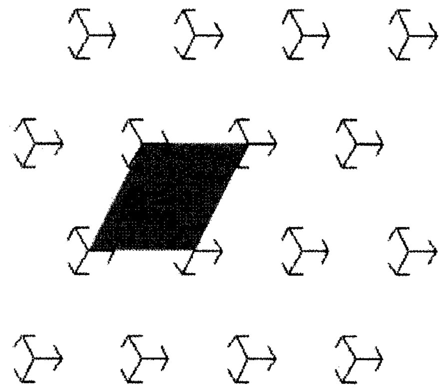


Figure C.10: Illustration of the Wallpaper Group ϖ_3^2

Group ϖ_3^1 and ϖ_3^2 are obtained by adding to ϖ_3 the reflection in one of the diagonals of the rhombic unit cell determined by A.

Adding the reflections in both diagonals would introduce a halfturn and a 6-center.

So any wallpaper pattern containing only three centers has one of ϖ_3,ϖ_3^1, ϖ_3^2 or as its symmetry group.

1. A wallpaper pattern with symmetry group ϖ_3 has a 3-center, has no 6-center, and has no line of symmetry.
2. A wallpaper pattern with symmetry group ϖ_3^1 has a 3-center, has no 6-center, and every 3-center is on a line of symmetry.
3. A wallpaper pattern with symmetry group ϖ_3^2 has a 3-center off a line of symmetry but no 6-center.

Remark 80 *The short international forms p3m1 and p31m have often been interchanged in the mathematical literature and caution is advised whenever these are encountered.*

C.3 4-Centers

Now we turn our attention to the wallpaper groups with 4-centers.

The following theorem regarding 4-centers in a wallpaper group is analogous to that about 6-centers. The proof follows the statement of the theorem. See Figure C.11.

Theorem 81 *Suppose A is a 4-center for wallpaper group ϖ. Then, there are no 3-centers for ϖ and there are no 6-centers for ϖ. Further, the center of symmetry nearest to A is a 2-center M, and A is the center of*

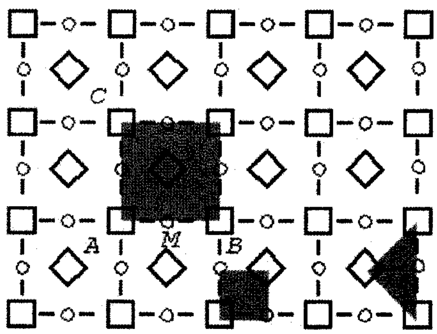


Figure C.11: Illustration of Theorem 81

a square whose vertices are 4-centers and whose sides are bisected by 2-centers. All the centers of symmetry for ϖ are determined by A and M.

Proof

By Theorem 79 of the crystallographic restriction, if A is a 4-center for wallpaper group ϖ, then every center of symmetry for ϖ is either a 2-center or a 4-center.

Let M be a center of symmetry nearest to A.

If M were a 4-center, then K would be a center of symmetry closer to A than M where K is given by

$\rho_{M,90}\rho_{A,90} = \sigma_K$.

See Figure C.12.

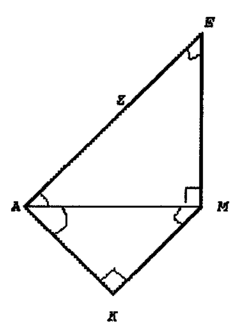


Figure C.12: Illustration of the Proof of Theorem 81

So M must be a 2-center.

Then E is a 4-center where

$\rho_{M,180}\rho_{A,-90} = \rho_{E,90}$.

The images of E and M under the action of $\rho_{A,90}$ are, respectively, the vertices and midpoints of the square in the statement of the theorem.

Translation $\tau_{A,E}$ is not in ϖ as otherwise Z is a center of symmetry closer to A than M, where $\tau_{A,E}\sigma_A = \sigma_Z$.

With $N = \rho_{A,90}(M)$,$\tau_{A,B} = \sigma_M\sigma_A$, and $\tau_{A,C} = \sigma_N\sigma_A$, then $\Box NAME$ is a square and $\tau_{A,B}$ are shortest translations in ϖ and generate the translation subgroup.

Thus there is no more room for more centers of symmetry than those already encountered (Theorem 83).

The centers of symmetry for ϖ are as arranged in the above Figure C.12. ■

Let $\varpi_4 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,90} >$ where E is the center of square $\Box ABDC$. If ϖ contains no odd isometries then ϖ must be ϖ_4.Now we use three noncollinear points A, B, C to check $\tau_{A,B} = \rho_{E,90}\rho_{A,-90}$:

See Figure C.13

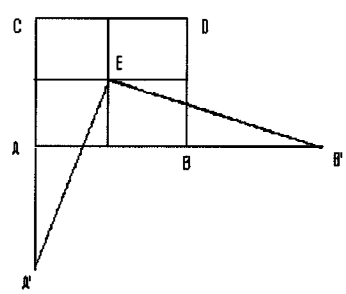


Figure C.13: Illustration of the Wallpaper Group ϖ_4

where $\overline{AA\prime} = \overline{AB} = \overline{AC} = \overline{BB\prime}$.

$\tau_{A,B}(A) = B$, $\tau_{A,B}(B) = B'$, $\tau_{A,B}(C) = D$

and

$\rho_{E,90}\rho_{A,-90}(A) = \rho_{E,90}(A) = B$

since the angle $A\prime EB\prime = 90^0$, $\rho_{E,90}\rho_{A,-90}(B) = \rho_{E,90}(A') = B'$

$\rho_{E,90}\rho_{A,-90}(C) = \rho_{E,90}(B) = D$

Therefore, $\tau_{A,B}(A) = \rho_{E,90}\rho_{A,-90}(A), \tau_{A,B}(B) = \rho_{E,90}\rho_{A,-90}(B), \tau_{A,B}(C) = \rho_{E,90}\rho_{A,-90}(C)$.

Similarly, we can use A,B,C to check $\tau_{A,C} = \rho_{E,-90}\rho_{A,90}$.

So, $\varpi_4 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,90} >=< \rho_{A,90}, \rho_{E,90} >$.

See Figure C.14.

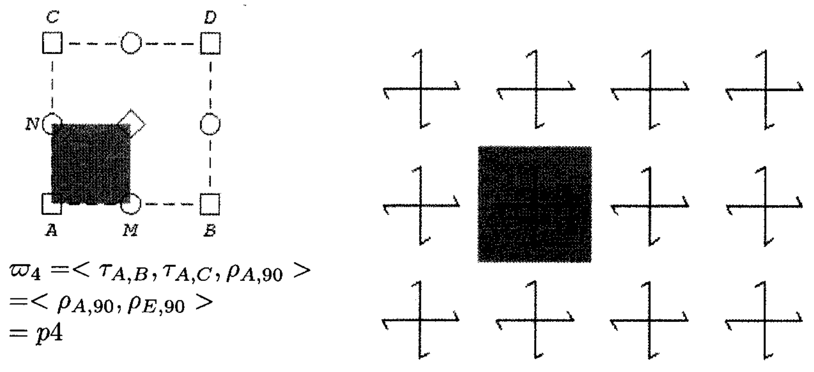


Figure C.14: Illustration of the Wallpaper Group ϖ_4

Consider extending ϖ_4 to wallpaper group ϖ by adding odd isometries. If σ_l is in ϖ, then l must be a line of symmetry for the set of all 4-centers in ϖ.

Because of the abundance of rotations in ϖ_4, it is sufficient to consider adding either a reflection in a line of symmetry through a 4-center or else a reflection in a line off all the 4-centers.

Lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{MN}$ will serve our purpose.

First, let $\varpi_4^1 = <\tau_{A,B}, \tau_{A,C}, \rho_{A,90}, \sigma_{\overleftrightarrow{AE}}>$.

See Figure C.15.

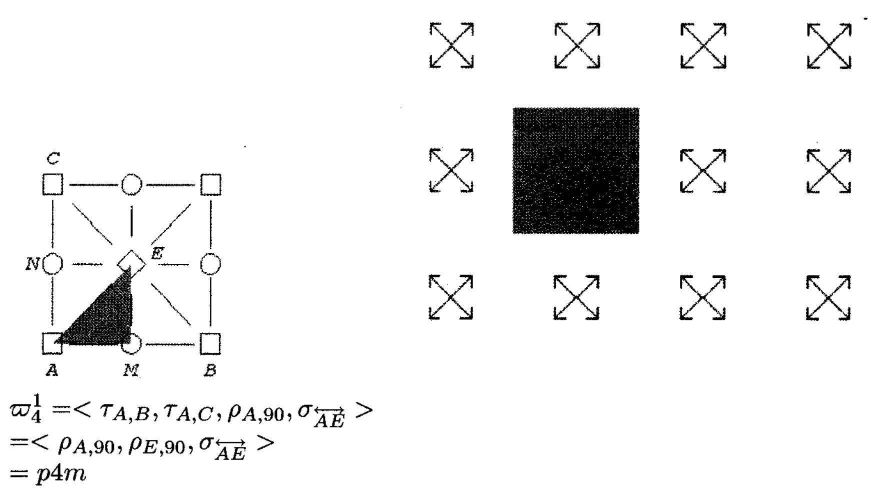


Figure C.15: Illustration of the Wallpaper Group ϖ_4^1

Since

$$\rho_{A,90} = \sigma_{\overleftrightarrow{AE}}\sigma_{\overleftrightarrow{AB}},\ \rho_{E,90} = \sigma_{\overleftrightarrow{ME}}\sigma_{\overleftrightarrow{AE}},$$

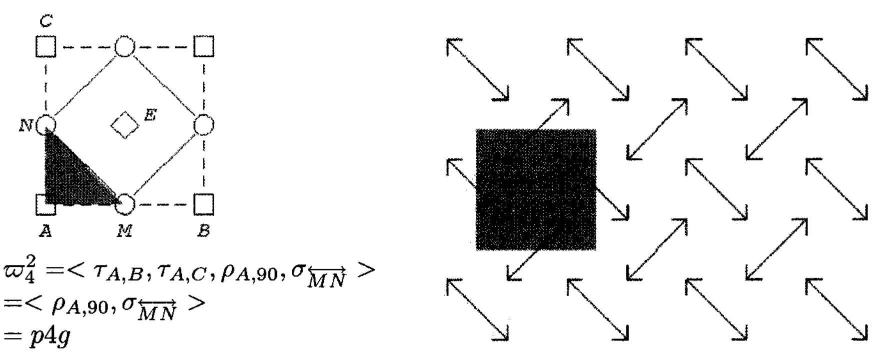
we have

$\varpi_4^1 =< \rho_{A,90}, \rho_{E,90}, \sigma_{\overleftrightarrow{AE}} >=< \sigma_{\overleftrightarrow{AB}}, \sigma_{\overleftrightarrow{ME}}, \sigma_{\overleftrightarrow{AE}} >$

this means that ϖ_4^1 is also generated by the three reflections in the three lines that contain the sides of an isosceles right triangle.

Secondly, let $\varpi_4^2 = <\tau_{A,B}, \tau_{A,C}, \rho_{A,90}, \sigma_{\overleftrightarrow{MN}}>$.

See Figure C.16.



$\varpi_4^2 =< \tau_{A,B}, \tau_{A,C}, \rho_{A,90}, \sigma_{\overleftrightarrow{MN}} >$
$=< \rho_{A,90}, \sigma_{\overleftrightarrow{MN}} >$
$= p4g$

Figure C.16: Illustration of the Wallpaper Group ϖ_4^2

Both $\sigma_{\overleftrightarrow{AE}}$ and $\sigma_{\overleftrightarrow{MN}}$ cannot be added to ϖ_4 without introducing a center of symmetry closer to A than M.

The group ϖ_4^1 differs from the above group ($p4$) in that it also has reflections.

To consider the possibility of extending ϖ_4 to a wallpaper group ϖ without reflections by adding odd isometries, it is sufficient (Theorem 70 in Appendix B) to suppose ϖ contains a glide reflection taking 4-center A to a 4-center that is not in the translation lattice determined by A.

By composing this glide reflection with an appropriate translation, we may suppose ϖ contains a glide reflection γ taking A to E. With Z the midpoint of A and E, then $\gamma = \sigma_z \sigma_Z$ for some line z through E.

Since γ must fix the set of all 4- centers, then z must be one of $\overleftrightarrow{ME}$,$\overleftrightarrow{BE}$, or $\overleftrightarrow{NE}$.

However, γ followed, respectively, by $\rho_{E,90}$, $\rho_{E,180}$,or $\rho_{E,270}$ gives the reflection in $\overleftrightarrow{MN}$.

Extending ϖ_4 by odd isometries leads only to ϖ_4^1 or ϖ_4^2.

1. A wallpaper pattern with symmetry group ϖ_4 has a 4-center and no line of symmetry.

2. A wallpaper pattern with symmetry group ϖ_4^1 has a line of symmetry on a 4-center.

3. A wallpaper pattern with symmetry group ϖ_4^2 has a 4-center and a line of symmetry off all 4-centers.

C.4 2-Centers

Now, we focus our attention on the wallpaper groups with 2-center.Suppose wallpaper group ϖ has a 2-center A and every center of symmetry for ϖ is a 2-center.

So σ_A is in ϖ.

With $< \tau_{A,B}, \tau_{A,C} >$ the translation subgroup of ϖ, let $\sigma_M = \tau_{A,B}\sigma_A$, $\sigma_N = \tau_{A,C}\sigma_A$, and $\sigma_E = \sigma_N\sigma_A\sigma_M$.

Points M, N, E are 2-centers and we have our usual notation with $\square ABDC$ defining a unit cell.

See Figure C.17.

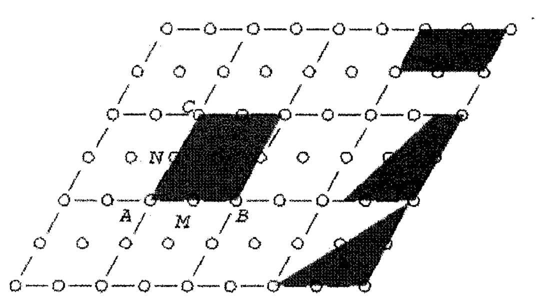


Figure C.17: Illustration of 2-Centers

Every point A_{ij} in the translation lattice determined by A is a 2-center as well as the midpoint of any two such lattice points.

There can be no more centers of symmetry than these.

Let $\varpi_2 =< \tau_{A,B}, \tau_{A,C}, \sigma_A >$.

If contains no odd isometries, then $\varpi = \varpi_2$.

Since, we have:

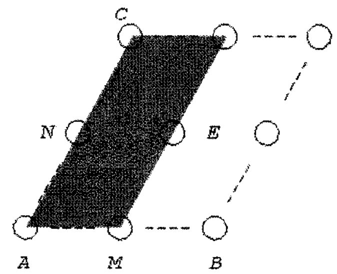
$\sigma_M = \tau_{A,B}\sigma_A, \sigma_N = \tau_{A,C}\sigma_A$

so

$\varpi_2 =< \tau_{A,B}, \tau_{A,C}, \sigma_A >=< \sigma_M, \sigma_E, \sigma_N >$.

This means that ϖ_2 is generated by $\tau_{A,B}$, $\tau_{A,C}$, and σ_E and also by σ_M, σ_E and σ_N.

See Figure C.18.



$\varpi_2 =< \tau_{A,B}, \tau_{A,C}, \sigma_A >$
$=< \sigma_M, \sigma_E, \sigma_N >$
$= p2$

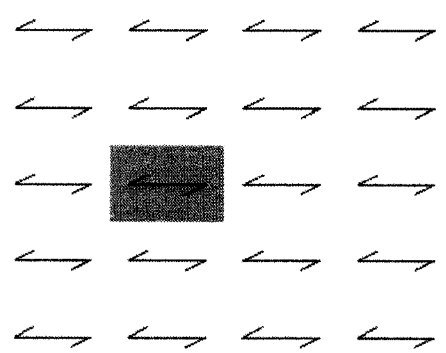


Figure C.18: Illustration of the Wallpaper Group ϖ_2

Consider extending ϖ_2 to wallpaper group ϖ by adding only odd isometries.

Suppose σ_l is in ϖ.

Then ϖ has a rhombic or rectangular translation lattice (Theorem 68 in Appendix B).

In the nonrectangular rhombic case, line l is parallel to a diagonal of a unit cell and so must pass through a 2-center.

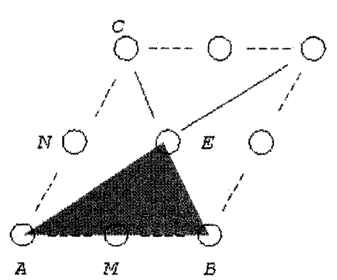
In this case, we may suppose A to be a 2-center on l.

Then l contains a diagonal unit cell of the translation lattice determined by A.

However, adding the reflection in one diagonal of a unit cell necessitates adding the reflection in the other diagonal as the center of the unit cell is a 2-center.

Let $\varpi_2^1 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AE}}, \sigma_{\overleftrightarrow{BE}} >$.

See Figure C.19.



$\varpi_2^1 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AE}}, \sigma_{\overleftrightarrow{BE}} >$
$=< \sigma_{\overleftrightarrow{AE}}, \sigma_{\overleftrightarrow{BE}}, \sigma_M >$
$= cmm$

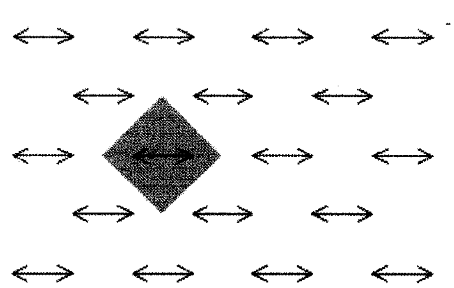


Figure C.19: Illustration of the Wallpaper Group ϖ_2^1

Therefore, in the nonrectangular case, we have only the one possibility that $\varpi = \varpi_2^1$.

If a rhombic unit cell is rectangular, then the unit cell is square and is a special case of the general rectangular case considered next.

An extension ϖ of ϖ_2 cannot have a reflection in a diagonal of a unit cell unless the unit cell is rhombic and cannot have reflections in both a diagonal and a line parallel to a side since every n-center is a 2-center.

Thus, to extend ϖ_2 with reflections there remains to consider only the case where a unit cell is rectangular (possibly square) and σ_l is in ϖ with l parallel to a side of unit cell defined by $\square ABDC$.

There are the two possibilities that either l passes through a 2-center or else l passes between two adjacent rows of 2-centers.

In the first case, introducing the reflection in one of the lines that contains a side of $\square NAME$ requires the introduction of the reflection in each of these lines.

In this case, ϖ is ϖ_2^2 defined by

$\varpi_2^2 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AM}}, \sigma_{\overleftrightarrow{AN}} >$.

Since, we have:

$\tau_{A,B}\sigma_{\overleftrightarrow{AN}}\sigma_{\overleftrightarrow{AM}} = \sigma_{\overleftrightarrow{AM}}\sigma_{\overleftrightarrow{ME}}$ and $\tau_{A,C}\sigma_{\overleftrightarrow{AN}}\sigma_{\overleftrightarrow{AM}} = \sigma_{\overleftrightarrow{AN}}\sigma_{\overleftrightarrow{NE}}$

so,

$\varpi_2^2 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AM}}, \sigma_{\overleftrightarrow{AN}} >==< \sigma_{\overleftrightarrow{AM}}, \sigma_{\overleftrightarrow{ME}}, \sigma_{\overleftrightarrow{AN}}, \sigma_{\overleftrightarrow{NE}} >$

See Figure C.20.

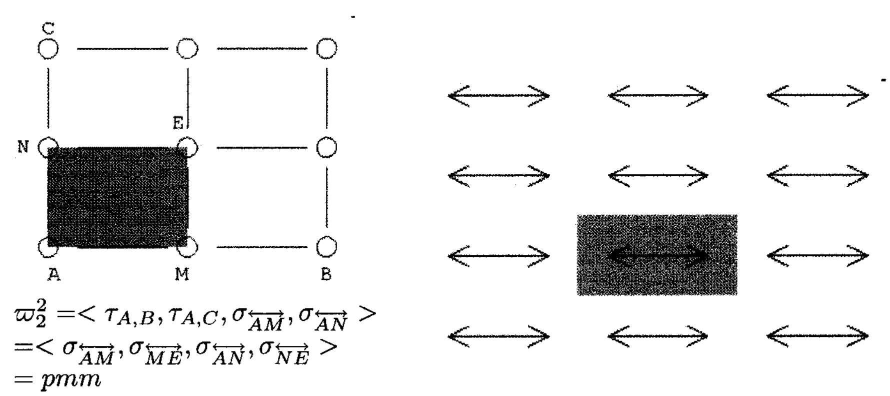


Figure C.20: Illustration of the Wallpaper Group ϖ_2^2

In the second case, where l passes between two adjacent rows of 2-centers, we may suppose without loss of generality that l is parallel to $\overleftrightarrow{AN}$.

In this case, ϖ is ϖ_2^3 defined by $\varpi_2^3 =< \tau_{A,B}, \tau_{A,C}, \sigma_A, \sigma_p >$ where p is the perpendicular bisector of $\overline{AM}$.

See Figure C.21.

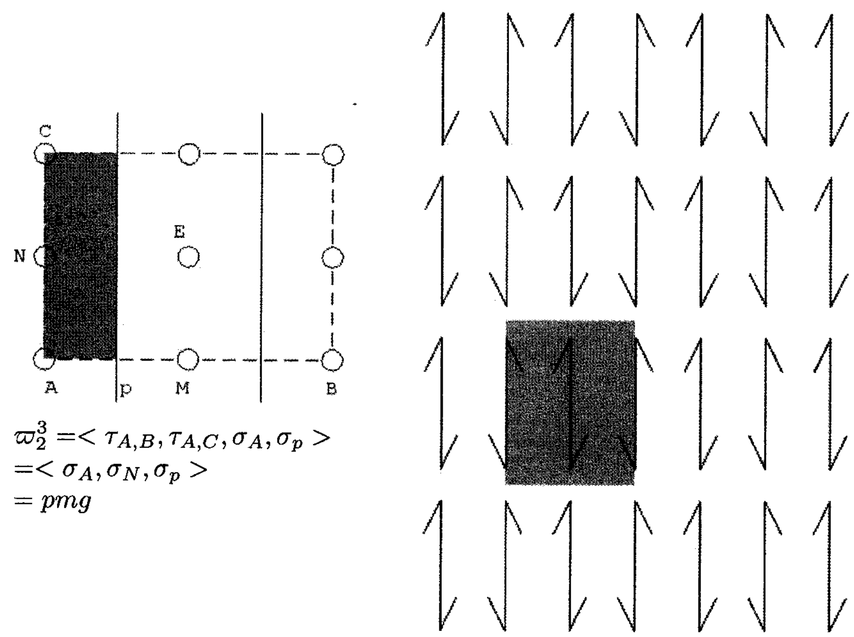


Figure C.21: Illustration of the Wallpaper Group ϖ_2^3

We have finished extending ϖ_2 by adding only reflections.

Consider extending ϖ_2 to a wallpaper group ϖ by adding a glide reflection γ such that no reflections are introduced a glide reflection with an axis that passes through a 2-center necessitates the introduction of a reflection.

Hence, the axis of γ must pass between two adjacent rows of 2-centers.

This requires that the parallelogram unit cell be rectangular.

We shall see that one choice of axis for γ introduces glide reflections whose axes consist of all possible candidates.

Let p be the perpendicular bisector of $\overline{AM}$; let q be the perpendicular bisector of $\overline{AN}$.

A glide reflection with axis p taking 2-center M to 2-center C followed by $\tau_{C,A}$ produces the undesired reflection σ_p.

So let γ be the glide reflection taking M to N and A to E.

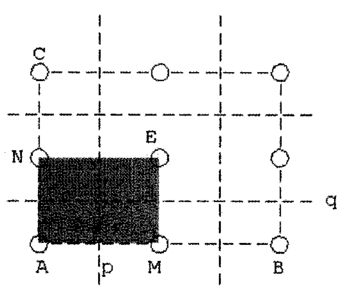
Let ε be the glide reflection taking N to M and A to E.

The axis of γ is p and γ^2=$\tau_{A,C}$; the axis of ε is q and $\varepsilon^2 = \tau_{A,B}$.

Check that $\gamma\sigma_A = \varepsilon$. Let ϖ_2^4=$<\tau_{A,B}, \tau_{A,C}, \sigma_A, \gamma>$.

Then ϖ_2^4 not only contains both γ and ε -and hence all possible glide reflections whose presence does not also require reflections—but ϖ_2^4 is even generated by γ and ε.

See Figure C.22, where the axes of glide reflections are indicated by broken lines.



$$\varpi_2^4 =< \tau_{A,B}, \tau_{A,C}, \sigma_A, \gamma >$$
$$=< \gamma, \varepsilon >$$
$$= pgg$$

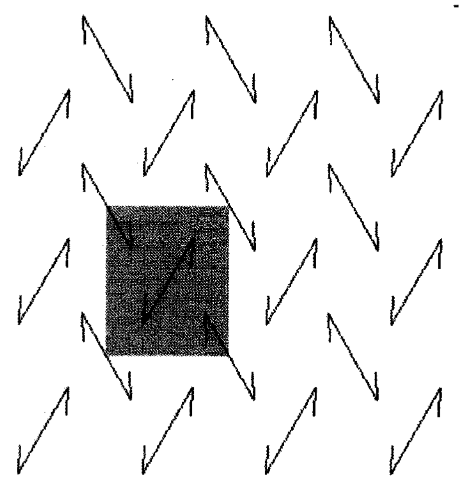


Figure C.22: Illustration of the Wallpaper Group ϖ_2^4

So far, we have:

1. A wallpaper pattern with symmetry group ϖ_2 has a 2-center, every center of symmetry is a 2-center, and is not fixed by an odd isometry.

2. A wallpaper pattern with symmetry group ϖ_2^1 has a 2-center, every center of symmetry is a 2-center, and some but not all 2-centers are on a line of symmetry.

3. A wallpaper pattern with symmetry group ϖ_2^2 has a 2-center, every center of symmetry is a 2-center, and every 2-center is on a line of symmetry.

4. A wallpaper pattern with symmetry group ϖ_2^3 has a 2-center, every center of symmetry is a 2-center, has a line of symmetry, and all lines of symmetry are parallel.

5. A wallpaper pattern with symmetry group ϖ_2^4 has a 2-center, every center of symmetry is a 2-center, has no line of symmetry, but is fixed by a glide reflection.

C.5 No Centers

Finally we come to the wallpaper groups ϖ that have no center of symmetry.

If ϖ contains σ_l, suppose A is on l.

If ϖ contains no reflection but does have a glide reflection, suppose A is on the axis of a glide reflection in ϖ.

There is the case where ϖ contains no isometries other than translations. In this case, the $\varpi_1 = \varpi$ where $\varpi_1 =< \tau_{A,B}, \tau_{A,C} >$ with A arbitrary and A, B, C noncollinear.

See Figure C.23.

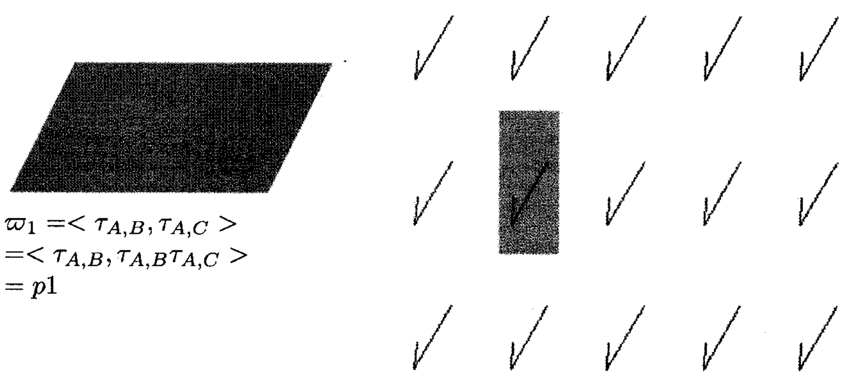


Figure C.23: Illustration of the Wallpaper Group ϖ_1

Consider extending ϖ_1 to wallpaper group ϖ by adding only odd isometries.

If σ_l is in ϖ with A on l, then, by Theorem 1, there is a rhombic unit cell defined by $\square ABDC$ with $l = \overleftrightarrow{AD}$ or else there is a rectangular unit cell defined by $\square ABDC$ with $l = \overleftrightarrow{AC}$.

In case $\square ABDC$ is a square, ϖ cannot have reflections in both $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$ because ϖ contains no rotations.

Let $\varpi_1^1 = <\tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AD}}>$ and $\varpi_1^2 = <\tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AC}}>$.

If ϖ contains a reflection then ϖ is one of ϖ_1^1 or ϖ_1^2.

Since, we have

$\sigma_{\overleftrightarrow{AD}}(B) = C, \tau_{A,C} = \sigma_{\overleftrightarrow{AD}}\tau_{A,B}\sigma_{\overleftrightarrow{AD}}$

so

$\varpi_1^1 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AD}} >=< \tau_{A,B}, \sigma_{\overleftrightarrow{AD}} >$

See Figure C.24 and note that $\overleftrightarrow{NK}$ is not a line of symmetry for ϖ_1^1.

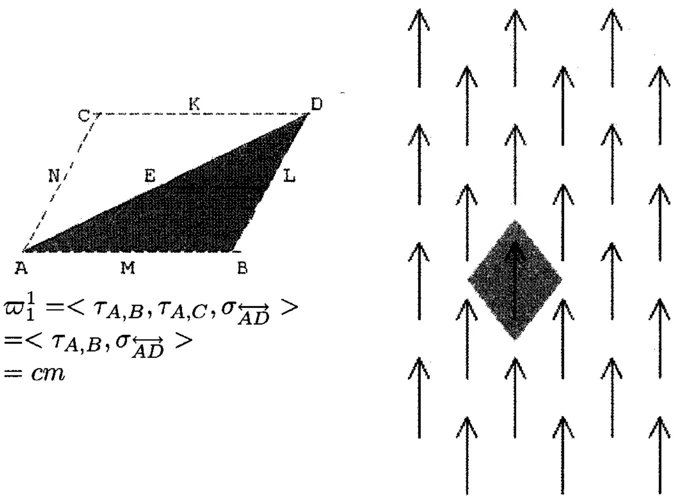


Figure C.24: Illustration of the Wallpaper Group ϖ_1^1

On the other hand, all the glide reflections in ϖ_1^2 are of the form $(\sigma_{\overleftrightarrow{AC}}\tau_{A,B}^j)\tau_{A,C}^i$ with $i \neq 0$ and have axes that are also lines of symmetry for ϖ_1^2.

This property can be used to distinguish patterns with symmetry group ϖ_1^1 and ϖ_1^2. See the above Figure C.24, and the following Figure C.25.

Finally, consider extending ϖ_1 with glide reflections only.

The axes of the glide reflections must be parallel because ϖ_1 contains no rotations.

We have the one further case where one of the generating translations is the square of a glide reflection in ϖ. Let $\varpi_1^3 =< \tau_{A,B}, \tau_{A,C}, \gamma >$ where γ is the glide reflection with axis $\overleftrightarrow{AB}$ that takes A to M.

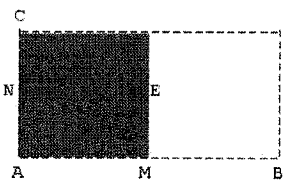
See Figure C.26.

So $\gamma^2 = \tau_{A,B}$.

Clearly ϖ_1^3 is generated by $\tau_{A,C}$ and γ.

Let $\varepsilon = \tau_{A,C}\gamma$.

Check that ε is the glide reflection with axis $\overleftrightarrow{NE}$ that takes N to E and that ϖ_1^3 is generated by γ and ε.



$\varpi_1^2 =< \tau_{A,B}, \tau_{A,C}, \sigma_{\overleftrightarrow{AC}} >$
$=< \tau_{A,C}, \sigma_{\overleftrightarrow{AC}}, \sigma_{\overleftrightarrow{ME}} >$
$= pm$

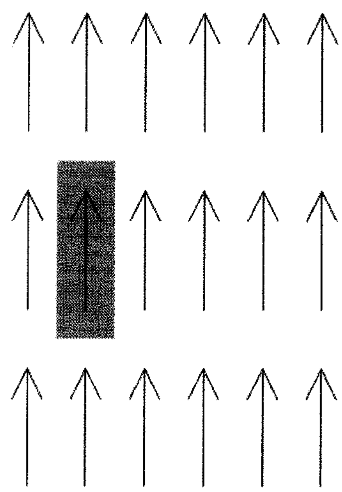


Figure C.25: Illustration of the Wallpaper Group ϖ_1^2

So ϖ_1^3 contains all possible glide reflections.

The extension of ϖ_1 to a wallpaper group by adding only odd isometries gives one of ϖ_1^1, ϖ_1^2, or ϖ_1^3.

1. A wallpaper pattern with symmetry group ϖ_1 has no center of symmetry and is not fixed by any odd isometry.

2. A wallpaper pattern with symmetry group ϖ_1^1 has no center of symmetry, is fixed by both reflections and glide reflections, but some axes of the glide reflections are not lines of symmetry.

3. A wallpaper pattern with symmetry group ϖ_1^2 has no center of symmetry, is fixed by both reflections and glide reflections, and all axes of the glide reflections are lines of symmetry.

4. A wallpaper pattern with symmetry group ϖ_1^3 has no center of symmetry, has no line of symmetry, but is fixed by a glide reflection.

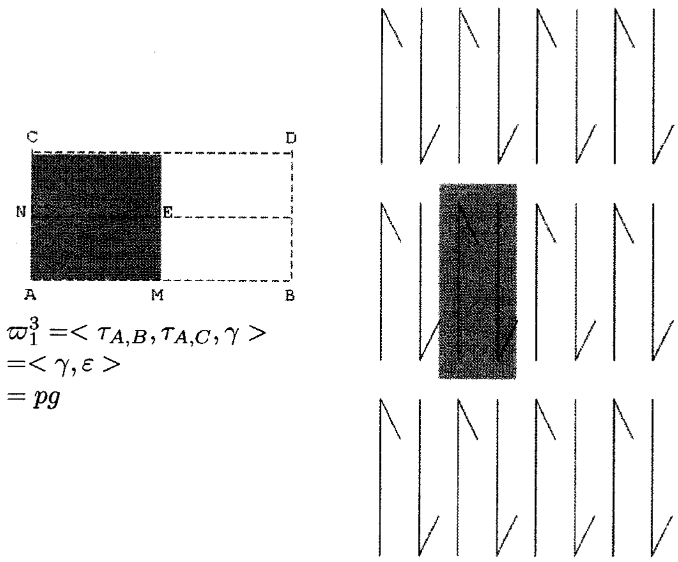


Figure C.26: Illustration of the Wallpaper Group ϖ_1^3

C.6 Conclusion

We have finished the search for all possible wallpaper groups. Therefore the corollary of crystallographic restriction theorem, mentioned at the beginning of this thesis, can be stated as the following theorem.

Theorem 82 *If G is a wallpaper group, then there exist points and lines such G that is one of the seventeen groups defined above.*

Appendix D

Notation

AB is the distance from point A to point B.

$\overleftrightarrow{AB}$ is the line passing through the points A and B.

$\overline{AB}$ is the line segment of the line $\overleftrightarrow{AB}$ between A and B.

$\triangle ABC$ is the triangle determined by the points A, B, C.

$\Box ABDC$ is the parallelogram determined by the points A, B, D, C.

$\overrightarrow{CP}$ is the ray from C to P consisting of all points of $\overleftrightarrow{CP}$ between P and C.

ϖ_i^j is the $(j+1)$th wallpaper group with rotation of order i for $i > 1$ and no rotation for $i = 1$.

$B(a, b)$ is the point B with coordinate (a, b).

σ_m is the reflection with line m as the reflection axis.

$< \tau_1, \tau_2, ..., \tau_n >$ is the group generated by transformation $\tau_1, \tau_2, ..., \tau_n$.

$\overleftrightarrow{AP} \perp l$ states that the lines $\overleftrightarrow{AP}$ and l are perpendicular.

$\rho_{A,\theta}$ is the rotation with rotation center A and rotation angle is θ.

σ_M is the rotation with rotation center M and with a rotation angle of 180^0 degree.

$N = \tau(M)$ is the image of M under the transformation τ.

E or e is the identity transformation or the identity element of a group.

Remark 83 *In the case of transformation groups, we follow the usual practice of denoting the identity element "e" by the capital letter "E".*

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